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*Research article*

## **Stability and Hopf bifurcation in an eco-epidemiological system with the cost of anti-predator behaviors**

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**Abstract:** The fear effect is a powerful force in prey-predator interaction, eliciting a variety of anti-predator responses which lead to a reduction of prey growth rate. To study the impact of the fear effect on population dynamics of the eco-epidemiological system, we develop a predator-prey interaction model that incorporates infectious disease in predator population as well as the cost of anti-predator behaviors. Detailed mathematical results, including well-posedness of solutions, stability of equilibria and the occurrence of Hopf bifurcation are provided. It turns out that population density diminishes with increasing fear, and the fear effect can either destabilize the stability or induce the occurrence of periodic behavior. The theoretical results here provide a sound foundation for understanding the effect of the anti-predator behaviors on the eco-epidemiological interaction.

**Keywords:** eco-epidemiological system; fear effect; stability; Hopf bifurcation

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### **1. Introduction**

The effect of disease on eco-epidemiology system is a significant topic from both mathematical and ecological perspectives. The disease factor usually leads to a more complex and diverting dynamics than those in the disease-free system [1, 2]. Within the interactions between predator and prey, the disease could only spread in prey or predator population, also could spread between prey and predator [3–5]. Birds (particularly pelicans) infect vibrio and die by preying on vibrio-infected fish (particularly tilapia) at the Salton Sea in the desert of Southern California [3], which is an example of disease spreads amongst the prey. For the disease in predator, taking fox rabies as an example, foxes (*Vulpis*) infect rabies and transmit to other foxes or their prey rabbits by biting in Europe and North America [6]. More relevant examples could be found in [7]. From the mathematical epidemiology

point of view, one needs much more attention in the dynamics of infected predator to observe whether the presence of the prey allows the survival of a part of the predator population [8].

A variety of diseased predator models have been proposed to study the complex interaction between prey and predator with infected diseases [2, 9, 10] and the reference therein. Most common epidemic model applied in predator-prey interactions is the  $SI$ -type, i.e., the predator population  $Y(t)$  is divided into two sub-classes, namely susceptible predator  $S(t)$  and infected predator  $I(t)$ , respectively [10–12]. The infection term could be mass-action term (bilinear form)  $\beta SI$  or saturation form  $\frac{\beta SI}{S+I}$  [4]. The infected predators usually behave differently with susceptible ones, and suffer an additional death rate. In an epidemic model, the global dynamics are usually determined by the basic reproduction number  $\mathcal{R}_0$ , i.e., the disease will die out in the population when  $\mathcal{R}_0 \leq 1$ , and the disease will persist in the population when  $\mathcal{R}_0 > 1$ . However, the basic reproduction number is no longer a threshold parameter determining the global dynamics in diseased predator models, on the contrary, the dynamics are relatively comprehensive and unexpected.

Predation is the key force in a prey-predator interaction, which could affect the size of prey population by direct hunting [9, 13–15], and elicit a variety of anti-predator responses [16–18]. Consequently, prey tends to alter behaviors in a certain extent, such as change of habitat, foraging activity, vigilance, physiological changes. This anti-predator behavior accelerates the extinction, evolution and development of prey population in the long run. Under the risk of predation, prey may reduce its foraging activity in order to stay alert, leading to starvation which impacts on population growth [19, 20]. Therefore, an immediate result of anti-predator behaviors is the reduction of prey growth rate, which is the cost for prey in prey defense [19, 21–26].

Consider a simple birth-death process of the prey  $X(t)$  with the cost of anti-predator behaviors [27]:

$$\frac{dX}{dt} = [F(k, Y)a]X - dX,$$

where  $X, Y$  represent the density of the prey and predator, respectively.  $a$  is the birth rate of prey,  $d$  is the natural death rate of prey.  $F(k, Y)$  accounts for the cost of anti-predator defence due to fear, the parameter  $k$  reflects the level of fear which drives anti-predator behaviors of prey. The fear factor  $F(k, Y)$  has some specific assumptions under the ecological motions, for details see [20, 27].

To derive a simple diseased predator model incorporating the anti-predator defence due to fear, we adopted the following fear effect term  $F(k, Y)$ :

$$F(k, Y) = \frac{1}{1 + kY} = \frac{1}{1 + k(S + I)}.$$

Based on the results in [4, 9, 11], we can obtain the eco-epidemiological system with cost of anti-predator behaviors as following system of nonlinear differential equations:

$$\begin{cases} \frac{dX}{dt} = \frac{rX}{1 + k(S + I)} - \frac{rX^2}{K} - \frac{aXS}{1 + bX}, \\ \frac{dS}{dt} = \frac{eaXS}{1 + bX} - d_1S - \beta SI, \\ \frac{dI}{dt} = \beta SI - d_2I, \end{cases} \quad (1.1)$$

where  $X, S, I$  represent the density of prey, susceptible predator and infected predator at time  $t$ , respectively.  $r$  is the intrinsic growth rate of prey,  $K$  is the carrying capacity of the prey,  $a$  is the predation coefficient,  $b$  is the predators handling time of a prey,  $e$  is the biomass conversion constant,  $\beta$  is the transmissibility coefficient.  $d_1$  and  $d_2$  are the mortality rates of the susceptible predator and infected predator, and naturally  $d_1 < d_2$ .

This paper consists of six sections. In the next section, we prove the positivity and boundedness of the solution of system (1.1). In Section 3, we provide the existence conditions of the equilibria of the model. We analyze the stability of equilibria and show the occurrence of Hopf bifurcation in Section 4. In Section 5, the correctness of the theoretical proof is illustrated by numerical simulation. Finally, we summarize our results with ecological interpretations in Section 6.

## 2. Positivity and boundedness

In view of the ecological significance, we only consider the solutions  $(X(t), S(t), I(t))$  of system (1.1) on

$$\mathbb{R}_+^3 = \{(X(t), S(t), I(t)) \in \mathbb{R}_+^3 : X(t) \geq 0, S(t) \geq 0, I(t) \geq 0\}.$$

**Theorem 2.1.** *Each solution of system (1.1) with initial value  $(X(0), S(0), I(0)) \in \mathbb{R}_+^3$  is positive and ultimately bounded.*

*Proof.* Since the right-hand side of system (1.1) is completely continuous and locally Lipschitzian on  $\mathbb{R}_+^3$ , the solution  $(X(t), S(t), I(t))$  with initial condition  $(X(0), S(0), I(0)) \in \mathbb{R}_+^3$  exists and is unique on  $\mathbb{R}_+^3$ .

By integrating, it follows from system (1.1) that

$$\begin{aligned} X(t) &= X(0) \exp \left\{ \int_0^t \left( \frac{r}{1+k(S(\tau)+I(\tau))} - \frac{rX(\tau)}{K} - \frac{aS(\tau)}{1+bX(\tau)} \right) d\tau \right\} \geq 0, \\ S(t) &= S(0) \exp \left\{ \int_0^t \left( e \frac{aX(\tau)}{1+bX(\tau)} - d_1 - \beta I(\tau) \right) d\tau \right\} \geq 0, \\ I(t) &= I(0) \exp \left\{ \int_0^t (\beta S(\tau) - d_2) d\tau \right\} \geq 0. \end{aligned}$$

Hence, the solution  $(X(t), S(t), I(t))$  of system (1.1) with the initial condition  $(X(0), S(0), I(0)) \in \mathbb{R}_+^3$  remains positive.

From the first equation of (1.1), we can obtain

$$\frac{dX}{dt} = \frac{rX}{1+k(S+I)} - \frac{rX^2}{K} - \frac{aXS}{1+bX} \leq rX - \frac{rX^2}{K} = rX \left( 1 - \frac{X}{K} \right),$$

then

$$\limsup_{t \rightarrow \infty} X(t) \leq K.$$

Let  $N(t) = eX(t) + S(t) + I(t)$ , we can get

$$\begin{aligned} \frac{dN}{dt} &= \frac{erX}{1+k(S+I)} - \frac{erX^2}{K} - d_1S - d_2I \\ &\leq erX - \frac{erX^2}{K} - d_1S - d_2I \\ &\leq erX \left(1 - \frac{X}{K}\right) + ed_1X - d_1N \\ &\leq \frac{eK(r+d_1)^2}{4r} - d_1N, \end{aligned}$$

then

$$\limsup_{t \rightarrow \infty} N(t) \leq \frac{eK(r+d_1)^2}{4rd_1}.$$

This ends the proof. □

*Remark 2.2.* From Theorem 2.1, we know that all positive solutions of system (1.1) with initial conditions  $(X(0), S(0), I(0)) \in \mathbb{R}_+^3$  are defined in the following positive bounded invariant:

$$\Gamma := \left\{ (X(t), S(t), I(t)) \in \mathbb{R}_+^3 : 0 \leq X(t) \leq K, 0 \leq eX(t) + S(t) + I(t) \leq \frac{eK(r+d_1)^2}{4rd_1} \right\}.$$

### 3. Existence of the equilibria

System (1.1) possesses at most three boundary equilibria:

- (i) Trivial equilibrium:  $E_0 = (0, 0, 0)$ ;
- (ii) Axial equilibrium:  $E_1 = (K, 0, 0)$ ;
- (iii) Planar equilibrium:  $E_2 = (X_2, S_2, 0)$  exists if  $ea - bd_1 > 0$  and  $K > \frac{d_1}{ea - bd_1}$ , where

$$\begin{aligned} X_2 &= \frac{d_1}{ea - bd_1}, \\ S_2 &= \frac{-[K(ea - bd_1)^2 + rd_1ke] + \sqrt{[K(ea - bd_1)^2 - rd_1ke]^2 + 4K^2kre(ea - bd_1)^3}}{2Kk(ea - bd_1)^2}. \end{aligned} \quad (3.1)$$

For epidemic models, the most critical problem is the threshold property for the extinction and persistence of the disease, which is generally governed by the basic reproduction number  $\mathcal{R}_0$ . The basic reproduction number can be interpreted as the expected number of secondary cases produced, in a completely susceptible population, by a typical infected individual during its entire period of infectiousness. Following [28], we define the basic reproduction number for the predator population in the system (1.1) by

$$\mathcal{R}_0 := \frac{\beta S_2}{d_2},$$

where  $S_2$  is given by (3.1).

Next, we mainly focus on the existence of positive equilibrium  $E_3 = (X_3, S_3, I_3)$  of system (1.1). The coordinates  $X_3, S_3, I_3$  are positive solutions to the following system of equilibrium equations:

$$\begin{cases} \frac{r}{1+k(S_3+I_3)} - \frac{rX_3}{K} - \frac{aS_3}{1+bX_3} = 0, \\ e\frac{aX_3}{1+bX_3} - d_1 - \beta I_3 = 0, \\ \beta S_3 - d_2 = 0. \end{cases}$$

Thus,

$$S_3 = \frac{d_2}{\beta}, \quad I_3 = \frac{X_3(ea - bd_1) - d_1}{(bX_3 + 1)\beta},$$

and  $X_3$  is the positive root of (3.2) in  $(X_2, +\infty)$ :

$$Q(X) = m_3X^3 + m_2X^2 + m_1X + m_0 = 0, \quad (3.2)$$

where

$$m_3 := -b\beta r(k(ea - bd_1) + b(kd_2 + \beta)),$$

$$m_2 := \beta r(Kb^2\beta - k(ea - bd_1) - 2b(kd_2 + \beta) + bkd_1),$$

$$m_1 := (-kd_2a(ea - bd_1) - b(akd_2^2 + a\beta d_2 - 2\beta^2 r))K - \beta r(-kd_1 + kd_2 + \beta),$$

$$m_0 := -K(-ad_2(kd_1 - kd_2 - \beta) - \beta^2 r).$$

If  $ea - bd_1 > 0$  and  $r > \frac{ad_2(\beta + k(d_2 - d_1))}{\beta^2}$ , we have

$$m_3 < 0, \quad m_0 > 0.$$

By Descartes' rule of signs, system (1.1) has at least one positive equilibrium  $E_3$ .

Hence, we have the following results on the existence of the positive equilibrium. It is worthy to note that the positive equilibrium is not unique due to the impact of fear effect  $k$ .

**Theorem 3.1.** *If  $ea - bd_1 > 0$  and  $r > \frac{ad_2(\beta + k(d_2 - d_1))}{\beta^2}$ , then system (1.1) has at least one positive equilibrium  $E_3 = (X_3, S_3, I_3)$ , where  $S_3 = \frac{S_2}{\mathcal{R}_0}$ ,  $I_3 = \frac{X_3(ea - bd_1) - d_1}{(bX_3 + 1)\beta}$  and  $X_3$  is the positive root of (3.2) in  $(X_2, +\infty)$ .*

## 4. Stability analysis

### 4.1. Local stability

Regarding the local stability of trivial equilibrium  $E_0$  and axial equilibrium  $E_1$ , we have the following results. The proof is standard, so we omit it here.

**Theorem 4.1.** *For system (1.1),*

- (i) *The trivial equilibrium  $E_0 = (0, 0, 0)$  is unstable;*
- (ii) *If one of the following inequalities holds:*

(ii-1)  $ea - bd_1 < 0$ ;

(ii-2)  $ea - bd_1 > 0$  and  $K < \frac{d_1}{ea - bd_1}$ ,

then the axial equilibrium  $E_1 = (K, 0, 0)$  is stable; while  $E_1 = (K, 0, 0)$  is unstable if  $ea - bd_1 > 0$  and  $K > \frac{d_1}{ea - bd_1}$ .

Secondly, we will show the local stability of the planar equilibrium  $E_2$  of system (1.1). For convenience, set

$$\begin{aligned} r_1 &:= \frac{d_2(ea - bd_1)}{\beta e}, & r_2 &:= \frac{d_2(ea - bd_1)(ea + bd_1)}{a\beta e^2}, \\ K_1 &:= \frac{ea + bd_1}{b(ea - bd_1)}, & K_2 &:= \frac{\beta e r d_1}{(ea - bd_1)(\beta e r - d_2(ea - bd_1))}, \\ k_1 &:= \frac{Kb(ea - bd_1)^2(Kb(ea - bd_1) - (ea + bd_1))}{ae^2 r(ea + bd_1)}, \\ k_2 &:= -\frac{\beta((ea - bd_1)(d_2(ea - bd_1) - \beta e r)K + \beta e r d_1)}{(Kd_2(ea - bd_1)^2 + \beta e r d_1)d_2}. \end{aligned} \quad (4.1)$$

**Theorem 4.2.** For system (1.1), assume that  $ea - bd_1 > 0$ . If one of the following inequalities holds:

(I)  $r \leq r_1$  and one of the following inequalities holds:

(I-1)  $\frac{d_1}{ea - bd_1} < K \leq K_1$ ;

(I-2)  $K > K_1$  and  $k > k_1$ ;

(II)  $r_1 < r < r_2$  and one of the following inequalities holds:

(II-1)  $\frac{d_1}{ea - bd_1} < K \leq K_1$ ;

(II-2)  $K_1 < K$  and  $k > \max\{k_1, k_2\}$ ;

(III)  $r > r_2$  and one of the following inequalities holds:

(III-1)  $\frac{d_1}{ea - bd_1} < K \leq K_2$ ;

(III-2)  $K_2 < K$  and  $k > \max\{k_1, k_2\}$ ,

then equilibrium  $E_2$  is stable; otherwise, it is unstable.

*Proof.* The Jacobian matrix of system (1.1) at  $E_2$  is given by

$$J_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & 0 & -\beta S_2 \\ 0 & 0 & \beta S_2 - d_2 \end{pmatrix},$$

where

$$\begin{aligned} a_{11} &:= X_2 \left( -\frac{r}{K} + \frac{abS_2}{(1 + bX_2)^2} \right), \\ a_{12} &:= -\frac{krX_2}{(1 + kS_2)^2} - \frac{aX_2}{1 + bX_2}, \\ a_{13} &:= \frac{-krX_2}{(1 + kS_2)^2}, \\ a_{21} &:= \frac{eaS_2}{(1 + bX_2)^2}. \end{aligned}$$

Hence, the characteristic equation of  $J_2$  is given as

$$f(\lambda)(\lambda - \beta S_2 + d_2) = 0, \quad (4.2)$$

where

$$f(\lambda) := \lambda^2 - a_{11}\lambda - a_{12}a_{21}.$$

Clearly, one can see that  $J_2$  has three eigenvalues  $\lambda_1, \lambda_2$  and  $\lambda_3 = \beta S_2 - d_2$ . Since  $a_{12} < 0, a_{21} > 0$ , then  $-a_{12}a_{21} > 0$ .

From (3.1), we can obtain

$$a_{11} = X_2 \left( -\frac{r}{K} + \frac{abS_2}{(1 + bX_2)^2} \right) = \frac{X_2\Phi}{2Kka^2e^2},$$

where

$$\Phi := ab \sqrt{\left( K(ea - bd_1)^2 - rd_1ke \right)^2 + 4K^2kre(ea - bd_1)^3 - ab(K(ea - bd_1)^2 + rd_1ke) - 2rka^2e^2}.$$

Note that the sign of  $\Phi$  depends on

$$\begin{aligned} \tilde{\Phi} &:= a^2b^2 \left( \left( K(ea - bd_1)^2 - rd_1ke \right)^2 + 4K^2kre(ea - bd_1)^3 \right) \\ &\quad - (ab(K(ea - bd_1)^2 + rd_1ke) + 2rka^2e^2)^2 \\ &= 4a^2kerP(k), \end{aligned}$$

where

$$P(k) := -ae^2r(ea + bd_1)k + Kb(ea - bd_1)^2(Kb(ea - bd_1) - (ea + bd_1)).$$

One can obtain that  $P(k)$  is decreasing with respect to  $k$ . If  $K \leq \frac{ea+bd_1}{b(ea-bd_1)}$  holds, we have  $P(0) \leq 0$ , which means that  $P(k) < 0$  for all  $k > 0$ ; if  $K > \frac{ea+bd_1}{b(ea-bd_1)}$  and  $k > k_1$  hold, we can get  $P(k) < 0$ . Therefore, when one of the following inequalities holds:

- (i)  $K \leq \frac{ea+bd_1}{b(ea-bd_1)}$ ;
- (ii)  $K > \frac{ea+bd_1}{b(ea-bd_1)}$  and  $k > k_1$ ,

we can obtain  $a_{11} < 0$ , which implies that the real parts of  $\lambda_1$  and  $\lambda_2$  are all negative.

It follows from system (3.1) that

$$\begin{aligned} \beta S_2 - d_2 &= \frac{-\beta[K(ea - bd_1)^2 + rd_1ke] + \beta \sqrt{[K(ea - bd_1)^2 - rd_1ke]^2 + 4K^2kre(ea - bd_1)^3}}{2Kk(ea - bd_1)^2} - d_2 \\ &= \frac{\Theta}{2Kk(ea - bd_1)^2}, \end{aligned}$$

where

$$\Theta := -K(ea - bd_1)^2(2kd_2 + \beta) - \beta ekrd_1 + \beta \sqrt{\left( K(ea - bd_1)^2 - rd_1ke \right)^2 + 4K^2kre(ea - bd_1)^3}.$$

Note that the sign of  $\Theta$  depends on

$$\begin{aligned} \tilde{\Theta} &:= \beta^2 \left( K(ea - bd_1)^2 - rd_1ke \right)^2 + 4\beta^2 K^2kre(ea - bd_1)^3 - (K(ea - bd_1)^2(2kd_2 + \beta) + \beta ekrd_1)^2 \\ &= -4Kk(ae - bd_1)^2 \left[ (Kd_2(ea - bd_1)^2 + \beta erd_1)d_2k + \beta((ea - bd_1)(d_2(ea - bd_1) - \beta er)K + \beta erd_1) \right]. \end{aligned}$$

Then if one of the following inequalities holds:

- (I)  $ea - bd_1 > 0$  and  $r \leq \frac{d_2(ea - bd_1)}{\beta e}$ ;
- (II)  $ea - bd_1 > 0$ ,  $r > \frac{d_2(ea - bd_1)}{\beta e}$  and one of the following inequalities:
- (II-1)  $K \leq \frac{\beta e r d_1}{(ea - bd_1)(\beta e r - d_2(ea - bd_1))}$ ;
- (II-2)  $K > \frac{\beta e r d_1}{(ea - bd_1)(\beta e r - d_2(ea - bd_1))}$  and  $k > k_2 := -\frac{\beta((ea - bd_1)(d_2(ea - bd_1) - \beta e r)K + \beta e r d_1)}{(K d_2(ea - bd_1)^2 + \beta e r d_1) d_2}$ ,

we have  $\lambda_3 = \beta S_2 - d_2 < 0$ .

Thus, we can arrive at the conclusion.  $\square$

It should be pointed out that another way to state Theorem 4.2 is as follows.

**Remark 4.3.** For system (1.1), assume that  $ea - bd_1 > 0$  and  $\mathcal{R}_0 < 1$ . If one of the following inequalities:

- (I)  $\frac{d_1}{ea - bd_1} < K \leq K_1$ ;
- (II)  $K > K_1$  and  $k > k_1$

holds, then the planar equilibrium  $E_2$  is stable; otherwise, it is unstable.

Next, we will show the local stability of the positive equilibrium  $E_3$  of system (1.1).

The Jacobian matrix of system (1.1) at  $E_3$  is given by

$$J_3 = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & 0 & -d_2 \\ 0 & \beta I_3 & 0 \end{pmatrix},$$

where

$$\begin{aligned} b_{11} &= X_3 \left( -\frac{r}{K} + \frac{abS_3}{(1 + bX_3)^2} \right), \\ b_{12} &= \frac{-krX_3}{(1 + k(S_3 + I_3))^2} - \frac{aX_3}{1 + bX_3} < 0, \\ b_{13} &= \frac{-krX_3}{(1 + k(S_3 + I_3))^2} < 0, \\ b_{21} &= \frac{eaS_3}{(1 + bX_3)^2} > 0. \end{aligned} \quad (4.3)$$

The characteristic equation of  $J_3$  is given as

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0, \quad (4.4)$$

where

$$\begin{aligned} A_1 &= -b_{11}, \\ A_2 &= \beta d_2 I_3 - b_{12} b_{21}, \\ A_3 &= -b_{11} \beta d_2 I_3 - b_{13} b_{21} \beta I_3. \end{aligned} \quad (4.5)$$

Note that if  $A_1 > 0$  holds, then  $b_{11} < 0$ , which means that  $A_3 > 0$ . According to Routh-Hurwitz criterion, the positive equilibrium  $E_3$  is locally asymptotically stable when  $A_1 > 0$  and  $A_1 A_2 - A_3 > 0$ .

Therefore, we can establish the following statement.

**Theorem 4.4.** Assume that  $ea - bd_1 > 0$  and  $r > \frac{ad_2(\beta + k(d_2 - d_1))}{\beta^2}$  hold. The positive equilibrium  $E_3$  of system (1.1) is locally asymptotically stable if  $A_1 > 0$  and  $A_1 A_2 - A_3 > 0$ , where  $A_i, i = 1, 2, 3$  is defined as in (4.5). Otherwise, it is unstable.



*Remark 4.5.* Theorem 4.4 gives a sufficient condition about the stability of the positive equilibrium  $E_3$  for system (1.1). However, the complexity of model (1.1) leads to the failure to theoretically demonstrate how the fear factor affects the stability of the positive equilibrium. This will be discussed later through numerical simulations.

#### 4.2. Hopf bifurcation

In this subsection, we take  $k$  as the bifurcation parameter. The characteristic equation of system (1.1) at  $E_3$  is (4.4), and  $A_i(k)$ ,  $i = 1, 2, 3$  are defined as (4.5).

**Theorem 4.6.** *Hopf bifurcation near the positive equilibrium  $E_3$  for system (1.1) occurs whenever the critical parameter  $k$  attains the value  $k = k_h$  in the domain:*

$$\Omega = \left\{ k_h \in \mathbb{R}^+ : \Delta(k_h) := [A_1(k)A_2(k) - A_3(k)]|_{k=k_h} = 0 \text{ with } A_2(k_h) > 0, \left[ \frac{d\Delta(k)}{dk} \right] \Big|_{k=k_h} \neq 0 \right\}.$$

*Proof.* If  $k = k_h$ , the characteristic Eq (4.4) equals

$$\lambda^3 + A_1(k_h)\lambda^2 + A_2(k_h)\lambda + A_3(k_h) = 0, \quad (4.6)$$

then (4.6) can be factorized as

$$(\lambda^2 + A_2(k_h))(\lambda + A_1(k_h)) = 0. \quad (4.7)$$

Clearly, (4.7) has three roots:  $\lambda_1 = i\sqrt{A_2(k_h)}$ ,  $\lambda_2 = -i\sqrt{A_2(k_h)}$  and  $\lambda_3 = -A_1(k_h)$ . The roots are of the form  $\lambda_1 = p_1(k) + ip_2(k)$ ,  $\lambda_2 = p_1(k) - ip_2(k)$  and  $\lambda_3 = -p_3(k)$ , where  $p_i(k)$  ( $i = 1, 2, 3$ ) are real numbers.

From the characteristic Eq (4.4), we can get

$$\frac{d\lambda}{dk} = -\frac{\lambda^2 A_1' + \lambda A_2' + A_3'}{3\lambda^2 + 2A_1\lambda + A_2}, \quad (4.8)$$

where  $' = \frac{d}{dk}$ . Substituting  $\lambda = i\sqrt{A_2}$  into (4.8), we obtain that

$$\frac{A_3' - A_2A_1' + iA_2'\sqrt{A_2}}{2(A_2 - iA_1\sqrt{A_2})} = -\frac{\frac{d\Delta(k)}{dk}}{2(A_1^2 + A_2)} + i \left[ \frac{\sqrt{A_2}A_2'}{2A_2} - \frac{A_1\sqrt{A_2}\frac{d\Delta(k)}{dk}}{2A_2(A_1^2 + A_2)} \right],$$

which implies that

$$\left[ \frac{d\text{Re}(\lambda)}{dk} \right] \Big|_{k=k_h} = -\frac{\frac{d\Delta(k)}{dk}}{2(A_1^2 + A_2)} \Big|_{k=k_h}.$$

By using monotonicity condition in the real part of the complex root  $\frac{d\text{Re}(\lambda)}{dk} \Big|_{k=k_h} \neq 0$ , the transversality condition  $\frac{d\Delta(k)}{dk} \Big|_{k=k_h} \neq 0$  can be obtained to ensure the existence of Hopf bifurcation.  $\square$

## 5. Numerical simulations

Results from numerical simulations are provided in this section to demonstrate our theoretical results. As we will show, the observations shed lights on the impact of fear factor. We choose the parameters of system (1.1) as follows:

$$r = 0.8, a = 0.2, b = 0.1, e = 0.9, d_1 = 0.05, \beta = 0.1, d_2 = 0.053. \quad (5.1)$$

Then we have

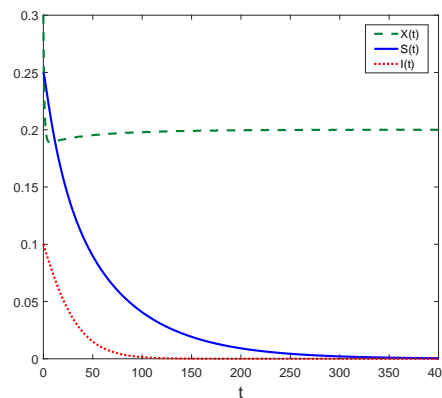
$$ea - bd_1 = 0.175 > 0, \quad \frac{d_1}{ea - bd_1} = 0.286,$$

$$r_1 = \frac{d_2(ea - bd_1)}{\beta e} = 0.103, \quad r_2 = \frac{d_2(ea - bd_1)(ea + bd_1)}{a\beta e^2} = 0.106,$$

$$K_1 = \frac{ea + bd_1}{b(ea - bd_1)} = 10.571, \quad K_2 = \frac{\beta e r d_1}{(ea - bd_1)(\beta e r - d_2(ea - bd_1))} = 0.328.$$

**Example 5.1 (The stability of  $E_1$ ).**

We adopt  $K = 0.2, k = 0.01$ , then system (1.1) has trivial equilibrium  $E_0 = (0, 0, 0)$  and axial equilibrium  $E_1 = (0.2, 0, 0)$ . In this case, one can know that the conditions of Theorem 4.1 are satisfied, which means that  $E_1$  is locally asymptotically stable. The numerical results are shown in Figure 1.



**Figure 1.** Population dynamics of  $X(t)$ ,  $S(t)$  and  $I(t)$  of system (1.1) with  $K = 0.2, k = 0.01$ .

**Example 5.2 (The impacts of  $K$  and  $k$  on the stability of  $E_2$ ).**

In this example, we will choose three values of carrying capacity  $K$  for numerical experiments. We conclude that the carrying capacity and fear effect are other key factors related to the extinction of infected predators, in addition to the basic reproduction number  $\mathcal{R}_0$ .

Firstly, we take  $K = 0.3 < K_2$ , then we have  $k = 0.1$  which yields that  $\mathcal{R}_0 = 0.263 < 1$ . In this case, system (1.1) has trivial equilibrium  $E_0 = (0, 0, 0)$ , axial equilibrium  $E_1 = (0.3, 0, 0)$ , and planar equilibrium  $E_2 = (0.286, 0.139, 0)$ . By Theorem 4.5,  $E_2$  is locally asymptotically stable, see Figure 2(a). Thus, when the carrying capacity of the prey  $K$  is small, no matter what the level of fear  $k$  is, the small size of prey population will lead to the extinction of infected predators.

Secondly, for comparison, we take  $K_2 < K = 15$ , then

$$k_1 = \frac{Kb(ea - bd_1)^2(Kb(ea - bd_1) - (ea + bd_1))}{ae^2r(ea + bd_1)} = 0.149,$$

$$k_2 = -\frac{\beta((ea - bd_1)(d_2(ea - bd_1) - \beta e r)K + \beta e r d_1)}{(Kd_2(ea - bd_1)^2 + \beta e r d_1)d_2} = 10.873.$$

Choosing  $k = 0.1 < \max\{k_1, k_2\}$  which yields  $\mathcal{R}_0 = 5.792 > 1$ , then we have

$$A_1 = 0.25791 > 0, \quad A_1 A_2 - A_3 = 0.00523 > 0.$$

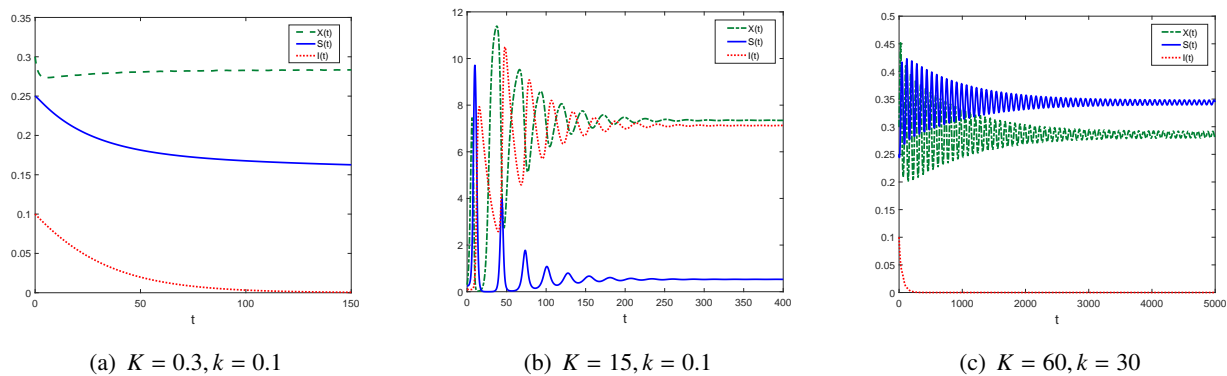
In this case, system (1.1) has trivial equilibrium  $E_0 = (0, 0, 0)$ , axial equilibrium  $E_1 = (15, 0, 0)$ , planar equilibrium  $E_2 = (0.286, 3.070, 0)$ , and positive equilibrium  $E_3 = (7.351, 0.530, 7.125)$ . By Theorem 4.5,  $E_2 = (0.286, 3.070, 0)$  is unstable. On the contrary,  $E_3 = (7.351, 0.530, 7.125)$  is locally asymptotically stable. The numerical simulation is shown in Figure 2(b).

Finally, we take  $K_2 < K = 60$ , then we have

$$k_1 = \frac{Kb(ea-bd_1)^2(Kb(ea-bd_1)-(ea+bd_1))}{ae^2r(ea+bd_1)} = 6.629,$$

$$k_2 = -\frac{\beta((ea-bd_1)(d_2(ea-bd_1)-\beta er)K+\beta erd_1)}{(Kd_2(ea-bd_1)^2+\beta erd_1)d_2} = 12.238.$$

Choosing  $k = 30 > \max\{k_1, k_2\}$  which yields that  $\mathcal{R}_0 = 0.649 < 1$ , system (1.1) has trivial equilibrium  $E_0 = (0, 0, 0)$ , axial equilibrium  $E_1 = (60, 0, 0)$ , and planar equilibrium  $E_2 = (0.286, 0.344, 0)$ . By Theorem 4.5,  $E_2$  is locally asymptotically stable, see Figure 2(c). Thus, when the carrying capacity of the prey  $K$  is relatively large, a high level of fear  $k$  will lead to the extinction of infected predators.



**Figure 2.** The impacts of  $K$  and  $k$  on the stability of  $E_2$ .

**Example 5.3 (The impact of  $k$  on the stability of  $E_3$ ).** We adopt  $K = 60$ , then we have  $k_h = 0.26$ . In the next, we will choose three values of  $k$ , corresponding to the local stability of  $E_3$ , Hopf bifurcation, and instability of  $E_3$ , to illustrate the impact of fear factor on the population dynamics.

Firstly, we take  $k = 0.1 < k_h$  which yields that  $\mathcal{R}_0 = 5.881 > 1$ , then system (1.1) has trivial equilibrium  $E_0 = (0, 0, 0)$ , axial equilibrium  $E_1 = (60, 0, 0)$ , planar equilibrium  $E_2 = (0.286, 3.117, 0)$  and a unique positive equilibrium  $E_3 = (24.047, 0.530, 12.213)$ . In this case, we obtain that

$$A_1 = 0.29863 > 0, \quad A_1A_2 - A_3 = 0.00065 > 0,$$

which means that  $E_3$  is local asymptotically stable. The numerical results are shown in Figure 3.

Secondly, we take  $k = 0.26 = k_h$  which yields that  $\mathcal{R}_0 = 4.682 > 1$ , then system (1.1) has trivial equilibrium  $E_0 = (0, 0, 0)$ , axial equilibrium  $E_1 = (60, 0, 0)$ , planar equilibrium  $E_2 = (0.286, 2.481, 0)$  and a unique positive equilibrium  $E_3 = (12.975, 0.530, 9.665)$ . In this case, we obtain that

$$A_1 = 0.14694 > 0, \quad A_1A_2 - A_3 = 0,$$

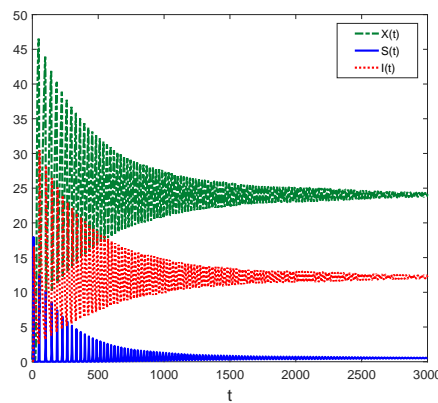
which means that system (1.1) undergoes a Hopf bifurcation and there is a limit cycle around  $E_3$ . The numerical results and the bifurcation diagrams of system (1.1) with respect to the parameter  $k$  are

shown in Figures 4 and 5, respectively. Comparing Figures 3 and 4(a), one can see that there are two different implications induced by the fear factor  $k$ : the first is that the stability of  $E_3$  converts from stable into unstable, and the second is the decrease of values of  $X_3$  and  $I_3$  of  $E_3$ .

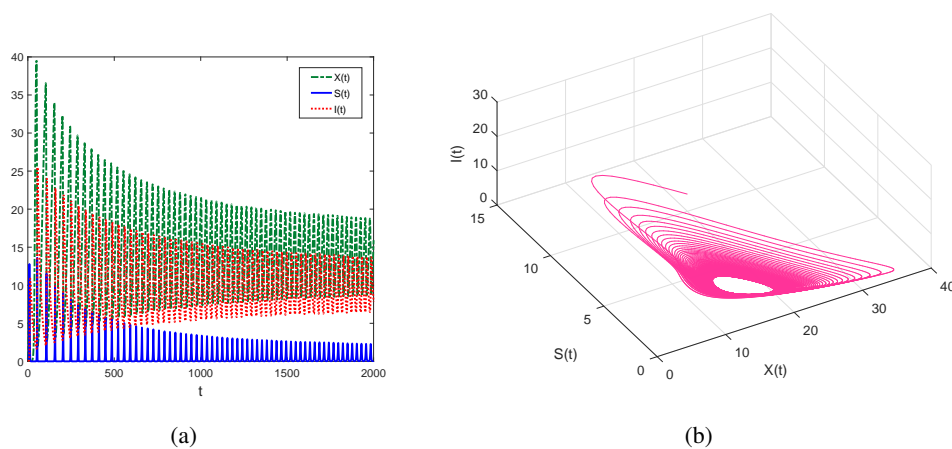
Finally, we take  $k = 0.5 > k_h$  which yields that  $\mathcal{R}_0 = 3.821 > 1$ , then system (1.1) has trivial equilibrium  $E_0 = (0, 0, 0)$ , axial equilibrium  $E_1 = (60, 0, 0)$ , planar equilibrium  $E_2 = (0.286, 2.025, 0)$  and a unique positive equilibrium  $E_3 = (7.685, 0.530, 7.322)$ . In this case, we can obtain that

$$A_1 = 0.07642 > 0, \quad A_1 A_2 - A_3 = -0.00051 < 0,$$

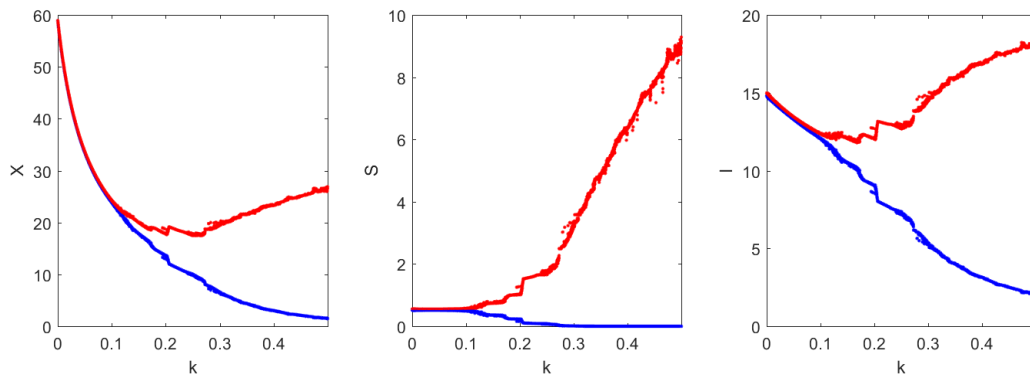
which means that  $E_3$  is unstable. The numerical results are shown in Figure 6. One can find that the difference between Figures 4 and 6 is the decrease of values of  $E_3$  from  $(12.975, 0.530, 9.665)$  to  $(7.685, 0.530, 7.322)$ , which is induced by the impact of the feat factor.



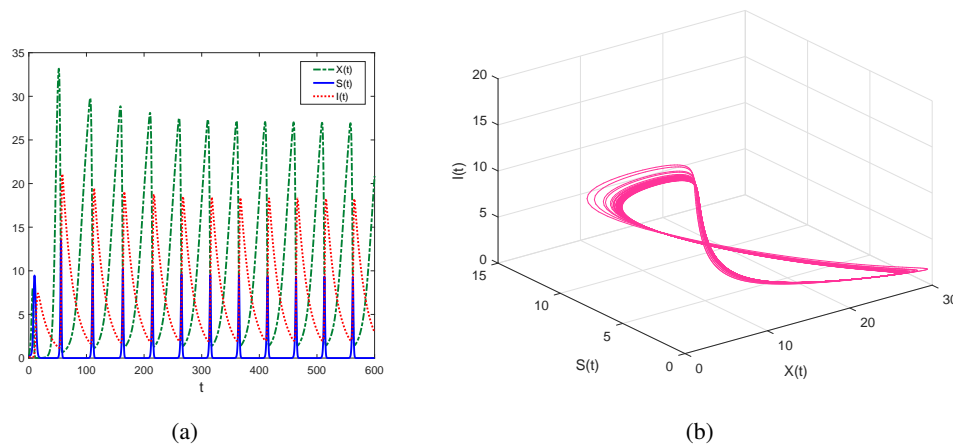
**Figure 3.** Population dynamics of  $X(t)$ ,  $S(t)$  and  $I(t)$  of system (1.1) with  $K = 60, k = 0.1 < k_h$ .



**Figure 4.** Population dynamics of  $X(t)$ ,  $S(t)$  and  $I(t)$  of system (1.1) with  $K = 60, k = 0.26 = k_h$ . (a) Time-series plots; (b) Phase portraits in 3-dimensional space.



**Figure 5.** Bifurcation diagram of the system (1.1) with respect to the parameter  $k$ . Here  $K = 60$ , other parameters are taken as in (5.1).



**Figure 6.** Population dynamics of  $X(t)$ ,  $S(t)$  and  $I(t)$  of system (1.1) with  $K = 60$ ,  $k = 0.5 > k_h$ . (a) Time-series plots; (b) Phase portraits in 3-dimensional space.

## 6. Conclusions

In this paper, we explored a predator-prey model that incorporates infectious disease in predator population and the cost of anti-predator behaviors. The cost of anti-predator behaviors is measured by a fear effect  $k$  leading to an reduction of prey's birth rate. We fulfill a complete stability analysis of equilibria for system (1.1) and show that the system (1.1) exhibits the Hopf bifurcation. Biologically, we focus on the impact of fear effect on the population dynamics. As we will see later, the cost of a high level of fear effect is disastrously. The main findings are summarized in the following.

### 1) Small size of prey population leads to the extinction of infected predators.

If the carrying capacity  $K$  is relatively small, the planar equilibrium  $E_2$  is stable, see Figure 2(a). Thus, no matter what the level of fear effect  $k$  is, a small size of prey population will lead to the extinction of infected predators.

### 2) Low level of the fear effect doesn't impact on the population dynamics.

If the level of fear effect  $k < k_h$ , the positive (coexistence) equilibrium  $E_3$  is stable, see Figure 3. Hence, we conclude that a small fear effect  $k$  is not the key disturbance and does not change the coexistence dynamics of system (1.1). However, the densities of the prey and infected predator gradually decrease as  $k$  increasing.

**3) Certain medium level of the fear effect lead to periodic oscillation.**

If  $k = k_h$ , the fear effect can destabilize the stability of  $E_3$  and will benefit the occurrence of periodic oscillation. In other words, system (1.1) undergoes a limit cycle, see Figures 4 and 5.

**4) High level of the fear effect leads to complex dynamics and the infected predator can go to extinction.**

If  $k > k_h$ ,  $E_3$  is unstable, see Figure 6. Therefore, a large fear effect  $k$  persistently and dramatically influence the population dynamics of prey and predator. Furthermore, if the level of the fear factor  $k$  is extremely high, the planar equilibrium  $E_2$  is stable, see Figure 2(c). The prey will respond to perceived predation risk and show a variety of anti-predator responses, dramatically decreasing the recruitment of susceptible predator, which will lead to an extinction of infected predator.

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## Conflicts of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

## References

1. K. P. Das, A mathematical study of a predator-prey dynamics with disease in predator, *ISRN Appl. Math.*, **2011** (2011), 807486. <https://doi.org/10.5402/2011/807486>
2. M. Haque, A predator-prey model with disease in the predator species only, *Nonlinear Anal. Real World Appl.*, **11** (2010), 2224–2236. <https://doi.org/10.1016/j.nonrwa.2009.06.012>
3. D. Greenhalgh, Q. J. A. Khan, J. S. Pettigrew, An eco-epidemiological predator-prey model where predators distinguish between susceptible and infected prey, *Math. Meth. Appl. Sci.*, **40** (2016), 146–166. <https://doi.org/10.1002/mma.3974>
4. A. M. Bate, F. M. Hilker, Complex dynamics in an eco-epidemiological model, *Bull. Math. Biol.*, **75** (2013), 2059–2078.
5. Y. Cai, Z. Gui, X. Zhang, H. Shi, W. Wang, Bifurcations and pattern formation in a predator-prey model, *Int. J. Bifurcation Chaos*, **28** (2018), 1850140. <https://doi.org/10.1142/S0218127418501407>

6. E. Venturino, Epidemics in predator-prey models: disease in the predators, *Math. Med. Biol.*, **19** (2002), 185–205. <https://doi.org/10.1093/imammb/19.3.185>
7. F. Gulland, *The Impact of Infectious Diseases on Wild Animal Populations: A Review*, Cambridge University Press Cambridge, 1995.
8. A. Mondal, A. K. Pal, G. P. Samanta. On the dynamics of evolutionary leslie-gower predator-prey eco-epidemiological model with disease in predator, *Ecol. Genet. Genom.*, **10** (2019), 100034. <https://doi.org/10.1016/j.egg.2018.11.002>
9. I. M. Bulai, F. M. Hilker, Eco-epidemiological interactions with predator interference and infection, *Theor. Popul. Biol.*, **130** (2019), 191–202. <https://doi.org/10.1016/j.tpb.2019.07.016>
10. E. Venturino. Epidemics in predator-prey models: Disease in the predators, *IMA J. Math. Appl. Med.*, **19** (2002), 185–205. <https://doi.org/10.1093/imammb/19.3.185>
11. M. Sieber, H. Malchow, F. M. Hilker. Disease-induced modification of prey competition in eco-epidemiological models, *Ecol. Complex.*, **18** (2014), 74–82. <https://doi.org/10.1016/j.ecocom.2013.06.002>
12. Y. Xiao, L. Chen. Analysis of a three species eco-epidemiological model, *J. Math. Anal. Appl.*, **258** (2001), 733–754. <https://doi.org/10.1006/jmaa.2001.7514>
13. M. A. Aziz-Alaoui, M. D. Okiye, Boundedness and global stability for a predator-prey model with modified Leslie-Gower and Holling-type II schemes, *Appl. Math. Lett.*, **16** (2003), 1069–1075. [https://doi.org/10.1016/S0893-9659\(03\)90096-6](https://doi.org/10.1016/S0893-9659(03)90096-6)
14. A. A. Shaikh, H. Das, An eco-epidemic predator-prey model with Allee effect in prey, *Int. J. Bifurcation and Chaos*, **30** (2020),2050194. <https://doi.org/10.1142/S0218127420501941>
15. S. Yu, Global asymptotic stability of a predator-prey model with modified Leslie-Gower and Holling-type II schemes, *Discrete Dyn. Nat. Soc.*, **2012** (2012), 857–868. <https://doi.org/10.1155/2012/208167>
16. T. Qiao, Y. Cai, S. Fu, W. Wang, Stability and Hopf bifurcation in a predator-prey model with the cost of anti-predator behaviors, *Int. J. Bifurcation Chaos*, **29** (2019), 1950185. <https://doi.org/10.1142/S0218127419501852>
17. S. K. Sasmal, Y. Takeuchi, Dynamics of a predator-prey system with fear and group defense, *J. Math. Anal. Appl.*, **481** (2020), 123471. <https://doi.org/10.1016/j.jmaa.2019.123471>
18. X. Wang, Y. Tan, Y. Cai, W. Wang. Impact of the fear effect on the stability and bifurcation of a Leslie-Gower predator-prey model, *Int. J. Bifurcation and Chaos*, **30** (2020),2050210. <https://doi.org/10.1142/S0218127420502107>
19. X. Meng, J. Li, Dynamical behavior of a delayed prey-predator-scavenger system with fear effect and linear harvesting, *Int. J. Biomath.*, **2021** (2021), 2150024. <https://doi.org/10.1142/S1793524521500248>
20. J. Wang, Y. Cai, S. Fu, W. Wang, The effect of the fear factor on the dynamics of a predator-prey model incorporating the prey refuge, *Chaos*, **29** (2019), 083109. <https://doi.org/10.1063/1.5111121>

21. P. Cong, M. Fan, X. Zou, Dynamics of a three-species food chain model with fear effect, *Commun. Nonlinear Sci. Numer. Simulat.*, **2021** (2021), 105809. <https://doi.org/10.1016/j.cnsns.2021.105809>
22. M. Hossain, N. Pal, S. Samanta, Impact of fear on an eco-epidemiological model, *Chaos Solitons Fractals*, **134** (2020), 109718. <https://doi.org/10.1016/j.chaos.2020.109718>
23. J. Liu, B. Liu, P. Lv, T. Zhang, An eco-epidemiological model with fear effect and hunting cooperation, *Chaos Solitons Fractals*, **142** (2021), 110494. <https://doi.org/10.1016/j.chaos.2020.110494>
24. Y. Tan, Y. Cai, R. Yao, M. Hu, W. Wang, Complex dynamics in an eco-epidemiological model with the cost of anti-predator behaviors, *Nonlinear Dyn.*, **107** (2022), 3127–3141. <https://doi.org/10.1007/s11071-021-07133-4>
25. X. Wang, X. Zou, Modeling the fear effect in predator-prey interactions with adaptive avoidance of predators, *Bull. Math. Biol.*, **79** (2017), 1325–1359. <https://doi.org/10.1007/s11538-017-0287-0>
26. H. Zhang, Y. Cai, S. Fu, W. Wang, Impact of the fear effect in a prey-predator model incorporating a prey refuge, *Appl. Math. Comput.*, **356** (2019), 328–337. <https://doi.org/10.1016/j.amc.2019.03.034>
27. X. Wang, L. Zanette, X. Zou, Modelling the fear effect in predator-prey interactions, *J. Math. Biol.*, **73** (2016), 1179–1204. <https://doi.org/10.1007/s00285-016-0989-1>
28. O. Diekmann, J. A. P. Heesterbeek, J. A. J. Metz, On the definition and the computation of the basic reproduction ratio in models for infectious diseases in heterogeneous populations, *J. Math. Biol.*, **28** (1990), 365–382. [https://doi.org/10.1016/0012-8252\(90\)90054-Y](https://doi.org/10.1016/0012-8252(90)90054-Y)



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