A non-existence result on Cameron-Liebler line classes

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Abstract

Cameron-Liebler line classes are sets of lines in $\mathrm{PG}(3, q)$ that contain a fixed number x of lines of every spread. Cameron and Liebler classified Cameron-Liebler line classes for $x \in \{0, 1, 2, q^2 - 1, q^2, q^2 + 1\}$ and conjectured that no others exist. This conjecture was disproven by Drudge for $q = 3$ [8] and his counterexample was generalised to a counterexample for any odd q by Bruen and Drudge [4]. A counterexample for q even was found by Govaerts and Penttila [9]. Non-existence results on Cameron-Liebler line classes were found for different values of x. In this paper, we improve the non-existence results on Cameron-Liebler line classes of Govaerts and Storme [11], for q not a prime. We prove the non-existence of Cameron-Liebler line classes for $3 \le x < \frac{q}{2}$.

1 Introduction

Cameron-Liebler line classes were introduced by Cameron and Liebler [5] in an attempt to classify collineation groups of $\mathrm{PG}(n, q)$ that have equally many point orbits and line orbits. In their paper, they conjectured which groups these are. It is now known [2] that the conjecture is true when the group is irreducible, but there is no classification yet of Cameron-Liebler line classes. In this paper, new non-existence results are presented.

There are many equivalent definitions for Cameron-Liebler line classes. Following Penttila [15], a *clique* in $\mathrm{PG}(3, q)$ is either the set of all lines through a point P, denoted by $\mathsf{star}(P)$, or dually the set of all lines in a plane π, denoted by $\mathsf{line}(\pi)$. The planar pencil of lines in a plane π through a point P is denoted by $\mathsf{pen}(P, \pi)$.

Definition 1.1 (Cameron and Liebler [5], Penttila [15]) Let $\mathcal{L}$ be a set of lines in $\mathrm{PG}(3, q)$ and let $\chi_{\mathcal{L}}$ be its characteristic function. Then $\mathcal{L}$ is called a *Cameron-Liebler line class* if one of the following equivalent conditions is satisfied.

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1. There exists an integer x such that $|\mathcal{L} \cap \mathcal{S}| = x$ for all spreads $\mathcal{S}$.
2. There exists an integer x such that for every incident point-plane pair (P, π)

$$|\mathsf{star}(P) \cap \mathcal{L}| + |\mathsf{line}(\pi) \cap \mathcal{L}| = x + (q+1)|\mathsf{pen}(P, \pi) \cap \mathcal{L}|. \tag{1}$$

3. There exists an integer x such that for every line l of $\mathrm{PG}(3, q)$

$$|\{m \in \mathcal{L} : m \text{ meets } l, m \neq l\}| = (q+1)x + (q^2-1)\chi_{\mathcal{L}}(l). \tag{2}$$

The parameter x is called the *parameter* of the Cameron-Liebler line class. We note that the first definition implies that $x \in \{0, 1, 2, \ldots, q^2+1\}$. Cameron and Liebler [5] showed that a Cameron-Liebler line class of parameter x consists of $x(q^2+q+1)$ lines and that the only Cameron-Liebler line classes for $x = 1$ are the cliques, i.e., all lines through a point or all lines in a plane, and for $x = 2$ the unions of two disjoint cliques. They also noted that the complement of a Cameron-Liebler line class with parameter x is a Cameron-Liebler line class with parameter q^2+1-x. So, it suffices to study Cameron-Liebler line classes with parameter $x \leq \lfloor (q^2+1)/2 \rfloor$. Thus, the case $q = 2$ was immediately solved. In their paper, Cameron and Liebler conjectured that no other Cameron-Liebler line classes exist.

Penttila [15] shows that for $q \neq 2$ there exist no Cameron-Liebler line classes with parameter $x = 3$ or $x = 4$, with possible exception of the cases $(x, q) \in \{(4, 3), (4, 4)\}$. Bruen and Drudge [3] prove the non-existence of Cameron-Liebler line classes with parameter $2 < x \leq \sqrt{q}$. Drudge [8] excludes the existence of a Cameron-Liebler line class with parameter $x = 4$ in $\mathrm{PG}(3, 3)$, and proves that for $q \neq 2$ there exist no Cameron-Liebler line classes with parameter $2 < x \leq \epsilon$, where $q+1+\epsilon$ denotes the size of the smallest nontrivial blocking sets in $\mathrm{PG}(2, q)$. He also gives a counterexample to the conjecture of Cameron and Liebler: a Cameron-Liebler line class with parameter $x = 5$ in $\mathrm{PG}(3, 3)$, in this way settling the case $q = 3$. Bruen and Drudge [4] then construct a Cameron-Liebler line class with parameter $x = (q^2+1)/2$ for any odd q. In [9], Govaerts and Penttila completed the study of the case $x = 4$ by showing that there exists no Cameron-Liebler line class with parameter $x = 4$ in $\mathrm{PG}(3, 4)$. In [9], Govaerts and Penttila also disproved the conjecture of Cameron and Liebler for q even by showing the existence of a Cameron-Liebler line class with parameter $x = 7$ in $\mathrm{PG}(3, 4)$.

In this paper, new bounds on x for the non-existence of Cameron-Liebler line classes with parameter x are obtained. We improve the results of Govaerts and Storme for q not prime. They proved the following two theorems and corollary [11].

Theorem 1.2 *In $\mathrm{PG}(3, q)$, q prime, $q > 2$, there exist no Cameron-Liebler line classes with parameter $2 < x \leq q$.*

Theorem 1.3 (1) *In $\mathrm{PG}(3, q)$, q square, there exist no Cameron-Liebler line classes with parameter $2 < x \leq \min(\epsilon', q^{3/4})$, where $q+1+\epsilon'$ denotes the size of the smallest nontrivial blocking sets in $\mathrm{PG}(2, q)$ not containing a Baer subplane.*

(2) *Let $q = p^{3h}$, $p \geq 7$ prime, $h \geq 1$ odd, and let $q+1+\epsilon''$ denote the size of the smallest nontrivial blocking sets in $\mathrm{PG}(2, q)$ containing neither a minimal blocking set of*

size $q+p^{2h}+1$*, nor one of size* $q+p^{2h}+p^h+1$*. In* $\mathrm{PG}(3,q)$*, there exist no Cameron-Liebler line classes with parameter* $2 < x \leq \min(\epsilon'', q^{5/6})$*.*

(3) *Let* $q = p^{3h}$*,* $p \geq 7$ *prime,* $h > 1$ *even, and let* $q + 1 + \epsilon''$ *denote the size of the smallest nontrivial blocking sets in* $\mathrm{PG}(2,q)$ *containing neither a Baer subplane, nor a minimal blocking set of size* $q + p^{2h} + 1$*, nor one of size* $q + p^{2h} + p^h + 1$*. In* $\mathrm{PG}(3,q)$*, there exist no Cameron-Liebler line classes with parameter* $2 < x \leq \min(\epsilon'', q^{3/4})$*.*

Corollary 1.4 (1) *Let* q *be a square,* $q = p^h$*,* p *prime.*

1. *If* $q > 16$*, then there exist no Cameron-Liebler line classes in* $\mathrm{PG}(3,q)$ *with parameter* $2 < x \leq c_p q^{2/3}$*, where* c_p *equals* $2^{-1/3}$ *when* $p \in \{2,3\}$ *and* 1 *when* $p \geq 5$*.*
2. *If* $p > 3$ *and* $h = 2$*, then there exist no Cameron-Liebler line classes in* $\mathrm{PG}(3,q)$ *with parameter* $2 < x \leq q^{3/4}$*.*

(2) *Let* $q = p^3$*,* $p \geq 7$ *prime, then there exist no Cameron-Liebler line classes in* $\mathrm{PG}(3,q)$ *with parameter* $2 < x \leq q^{5/6}$*.*

(3) *Let* $q = p^6$*,* $p \geq 7$ *prime, then there exist no Cameron-Liebler line classes in* $\mathrm{PG}(3,q)$ *with parameter* $2 < x \leq q^{3/4}$*.*

We improve these results for q not prime. Theorem 4.2 gives a new improved bound for general $q \neq 2$, q not prime.

This theorem will be proven by studying how the lines of the Cameron-Liebler line class with parameter x correspond with x-tight sets on $\mathrm{Q}^+(5,q)$ and $\{x(q^2+q+1), x(q+1); 5, q\}$-minihypers contained in the Klein quadric $\mathrm{Q}^+(5,q)$. Using properties of the associated $\{x(q^2+q+1), x(q+1); 5, q\}$-minihyper combined with the fact that this minihyper lives on $\mathrm{Q}^+(5,q)$, gives us new non-existence results on Cameron-Liebler line classes.

2 Definitions and preliminary results

Let $v_{n+1} = (q^{n+1}-1)/(q-1)$ denote the number of points of $\mathrm{PG}(n,q)$.

An i-tight set of a finite generalised quadrangle was introduced by Payne [13, 14] and was generalised to polar spaces of higher rank by Drudge [7].

Definition 2.1 A set of points $\mathcal{T}$ of a finite polar space of rank $r \geqslant 2$ over a finite field of order q is *i-tight* if

$$|P^\perp \cap \mathcal{T}| = \begin{cases} i\frac{q^{r-1}-1}{q-1} + q^{r-1} & \text{if } P \in \mathcal{T} \\ i\frac{q^{r-1}-1}{q-1} & \text{if } P \notin \mathcal{T}. \end{cases}$$

This definition poses restrictions on the intersection of a hyperplane with a point set. This has a lot in common with the concept of the minihypers.

Definition 2.2 An *$\{f, m; n, q\}$-minihyper* is a pair (F, w), where F is a subset of the point set of $\mathrm{PG}(n,q)$ and w is a weight function $w : \mathrm{PG}(n,q) \rightarrow \mathbb{N} : P \mapsto w(P)$, satisfying

1. $w(P) > 0 \Leftrightarrow P \in F$,
2. $\sum_{P \in F} w(P) = f$, and

3. $\min\{\sum_{P\in H} w(P) : H \text{ is a hyperplane}\} = m$.

The weight function w determines the set F completely. When this function has only the values 0 and 1, then (F, w) is determined completely by the set F. In this paper, this will always be the case, so we will not make any further reference to the weight function w.

In this paper, we are interested in the $\{x(q^2+q+1), x(q+1); 5, q\}$-minihypers contained in the Klein quadric $Q^+(5, q)$, and associated with the Cameron-Liebler line classes with parameter x. The following results discuss the intersections of subspaces with these minihypers. They will be very crucial to prove the improved results on the non-existence of Cameron-Liebler line classes. The first theorem is stated as a corollary in [6].

Theorem 2.3 *Let F be a $\{\sum_{i=0}^{n-1} \epsilon_i v_{i+1}, \sum_{i=1}^{n-1} \epsilon_i v_i; n, q\}$-minihyper, where $q > h, 0 \leqslant \epsilon_i \leqslant q-1, 0 \leq i \leq n-1, \sum_{i=0}^{n-1} \epsilon_i = h$.*

Then a plane of $PG(n, q)$ is either contained in F or intersects it in an $\{m_1(q+1) + m_0, m_1; 2, q\}$-minihyper, where $m_1 + m_0 \leqslant h$.

Theorem 2.4 (Hamada [12]) *Let F be a $\{\sum_{i=0}^{n-1} \epsilon_i v_{i+1}, \sum_{i=1}^{n-1} \epsilon_i v_i; n, q\}$-minihyper, where $0 \leqslant \epsilon_i \leqslant q-1$, $i = 0, \ldots, n-1$. Then $|F \cap \Delta| \geqslant \sum_{i=1}^{n-1} \epsilon_i v_{i-1}$ for any $(n-2)$-space Δ in $PG(n, q)$ and $|F \cap G| = \sum_{i=1}^{n-1} \epsilon_i v_{i-1}$ for some $(n-2)$-spaces G in $PG(n, q)$.*

Let $H_j, j = 1, 2, \ldots, q+1$, be the $q+1$ hyperplanes in $PG(n, q)$ that pass through an $(n-2)$-space G intersecting F in $\sum_{i=1}^{n-1} \epsilon_i v_{i-1}$ points. Then $F \cap H_j$ is a

$$\{\delta_j + \sum_{i=1}^{n-1} \epsilon_i v_i, \sum_{i=1}^{n-1} \epsilon_i v_{i-1}; n-1, q\}\text{-minihyper}$$

in H_j for $j = 1, 2, \ldots, q+1$, where the δ_j are some non-negative integers such that $\sum_{j=1}^{q+1} \delta_j = \epsilon_0$.

In the case of a $\{\delta v_{\mu+1}, \delta v_\mu; n, q\}$-minihyper, the parameters in Hamada's theorem become very nice. In the remainder of this article, we will only consider minihypers of this form. The next result of [10] is fundamental for the induction arguments used in the lemmas and theorem which follow.

Lemma 2.5 (Govaerts and Storme [10]) *Let (F, w) be a $\{\delta v_{\mu+1}, \delta v_\mu; n, q\}$-minihyper satisfying $0 \leqslant \delta \leqslant (q+1)/2, 0 \leqslant \mu \leqslant n-1$, and containing a μ-space π_μ. Then the minihyper (F', w') defined by the weight function w', where*

- *$w'(p) = w(p) - 1$, for $p \in \pi_\mu$, and*
- *$w'(p) = w(p)$, for $p \in PG(n, q) \setminus \pi_\mu$,*

is a $\{(\delta-1)v_{\mu+1}, (\delta-1)v_\mu; n, q\}$-minihyper.

It is easy to see that minihypers are closely related to blocking sets. A $\{\delta v_{\mu+1}, \delta v_\mu; n, q\}$-minihyper is a δv_μ-fold blocking set. We state some useful definitions on blocking sets.

Definition 2.6 A *k-fold blocking set* in $\mathrm{PG}(n,q)$ is a set of points that intersects every hyperplane in at least k points.

A k-fold blocking set is called *minimal* if no proper subset is a k-fold blocking set.

A 1-fold blocking set is simply called a *blocking set*. It is called *trivial* if it contains a line.

Theorem 2.7

- **(Szőnyi [16])** *A 1-fold blocking set B in $PG(2,q)$, of size $|B| < q + \frac{q+3}{2}$, where $q = p^h$, p prime, $h \geqslant 1$, is uniquely reducible to a minimal blocking set B' intersecting every line in $1 \pmod p$ points.*

- **(Szőnyi and Weiner [17])** *A minimal 1-fold blocking set B in $PG(n,q)$, $n \geqslant 3$, $q = p^h$, $p > 2$ prime, $h \geq 1$, of size $|B| < q + \frac{q}{2}$, intersects every line in zero points or in $1 \pmod p$ points.*

3 Minihypers on the Klein quadric

It is our intention to prove the non-existence of Cameron-Liebler line classes of parameter $2 < x < \frac{q}{2}$ in $\mathrm{PG}(3,q)$ by using $\{x(q^2+q+1), x(q+1); 5, q\}$-minihypers F contained in the Klein quadric $\mathrm{Q}^+(5,q)$.

Consider an $\{x(q^2+q+1), x(q+1); 5, q\}$-minihyper F, with $x < \frac{q}{2}$, on $\mathrm{Q}^+(5,q)$. We know that a hyperplane H intersects $\mathrm{Q}^+(5,q)$ in either a parabolic quadric $\mathrm{Q}(4,q)$ or in a tangent cone $\langle R, \mathrm{Q}^+(3,q)\rangle$ with vertex R in $\mathrm{Q}^+(5,q)$ and base a 3-dimensional hyperbolic quadric $\mathrm{Q}^+(3,q)$.

Lemma 3.1 *Let F be an $\{x(q^2+q+1), x(q+1); 5, q\}$-minihyper, with $x < \frac{q}{2}$, contained in the Klein quadric $\mathrm{Q}^+(5,q)$, and let H_0 be a hyperplane in $PG(5,q)$ such that $H_0 \cap \mathrm{Q}^+(5,q) = \langle R, \mathrm{Q}^+(3,q)\rangle$ and such that $H_0 \cap F$ is an $\{x(q+1), x; 4, q\}$-minihyper. Then there exists a solid in H_0, not containing R, intersecting F in exactly x points.*

Proof First of all, $|H_0 \cap F| = x(q+1) < \frac{q^2+q}{2}$. Consider a point R' of $\mathrm{Q}^+(5,q) \cap H_0$ with $R' \notin F$, $R' \neq R$. There are q^3+q^2+q+1 lines in H_0 through R'. At most $\frac{q^2+q}{2}$ of them can contain a point of F, so there exists a line l through R' having an empty intersection with F and not containing R. Similarly, we can find a plane π through l having an empty intersection with F. The $q+1$ solids through π together contain $x(q+1)$ points of F and each one of them contains at least x points of F (Theorem 2.4). This means that every solid through π contains exactly x points of F. Choose one of those solids, not containing R, and this is the desired solid. $\square$

Lemma 3.2 *Let F' be an $\{x(q+1), x; 4, q\}$-minihyper, $x < \frac{q}{2}$, contained in $Q(4,q)$. Then F' is the union of x pairwise disjoint lines.*

Proof For every point $R \in F'$, we find a plane π through R only intersecting F' in R. Then consider all solids through π, they all contain at least $x-1$ other points of F', since every solid contains at least x points of F'. There remain $x(q+1)-1-(q+1)(x-1) = q$ other points of F'. So some hyperplane K_0 through π contains more than x points of F'.

By [10, Corollary 2], $K_0 \cap F'$ is a blocking set with respect to the planes of K_0.

Consider the minimal blocking set B inside $K_0 \cap F'$. Suppose that B is not a line.

Take three non-collinear points $R_1, R_2, R_3 \in B$. Every line intersects B in zero or in 1 (mod p) points (Theorem 2.7). The line $l_1 = \langle R_1, R_2 \rangle$ already contains two points of B, so must contain at least $1 + p \geqslant 3$ points of B. A line containing more than two points of a quadric lies on that quadric. Similarly, the lines $l_2 = \langle R_1, R_3 \rangle$ and $l_3 = \langle R_2, R_3 \rangle$ are lines of $\mathrm{Q}(4, q)$. Consider the plane π spanned by l_1, l_2 and l_3. Since these three lines are lines of $\mathrm{Q}(4, q)$, π is contained in $\mathrm{Q}(4, q)$, which is impossible.

Thus the minimal blocking set B is a line, hence the minihyper F' contains a line l. By Lemma 2.5, we have that $F' \backslash l$ is an $\{(x-1)(q+1), x-1; 4, q\}$-minihyper. Repeating the previous arguments x times gives us that F' is the union of x pairwise disjoint lines. □

Lemma 3.3 *Suppose that F is an $\{x(q^2+q+1), x(q+1); 5, q\}$-minihyper, with $x < \frac{q}{2}$. Suppose that P is a point of F lying on two lines l_1, l_2, completely contained in F. Then the plane $\langle l_1, l_2 \rangle$ is completely contained in F.*

Proof Suppose that the plane $\langle l_1, l_2 \rangle \not\subseteq F$, then $F \cap \langle l_1, l_2 \rangle$ is an $\{m_1(q+1)+m_0, m_1; 2, q\}$-minihyper F', where $m_1 + m_0 \leqslant x < \frac{q}{2}$ (Theorem 2.3). Furthermore, $l_1 \cup l_2 \subseteq F$, implying that $|\langle l_1, l_2 \rangle \cap F| \geqslant 2q + 1$, which implies $m_1 \geqslant 2$. So $\langle l_1, l_2 \rangle \cap F$ is a t-fold blocking set, with $m_1 = t \geqslant 2$. Assume now that $|\langle l_1, l_2 \rangle \cap F| = tq + a$, with $a = m_0 + m_1 \leqslant x$.

Considering the lines l_1 and l_2, and the other $q-1$ lines of $\langle l_1, l_2 \rangle$ on P, we find that $|\langle l_1, l_2 \rangle \cap F| \geqslant 2q + 1 + (q-1)(t-1) = (t+1)q - t + 2$. Hence, $|\langle l_1, l_2 \rangle \cap F| = tq + a \geqslant (t+1)q - t + 2$, implying $a \geqslant q - t + 2$. Now $\langle l_1, l_2 \rangle \cap F$ is a t-fold blocking set of size $tq + a$. Note that $a \leqslant x < \frac{q}{2}$, giving $t \geqslant \frac{q}{2} + 2$, a contradiction since $t < \frac{q}{2}$. We conclude that $\langle l_1, l_2 \rangle \subseteq F$. □

4 Cameron-Liebler line classes and minihypers

We can now prove the following theorem.

Theorem 4.1 *An $\{x(q^2+q+1), x(q+1); 5, q\}$-minihyper, with $x < \frac{q}{2}$, contained in $\mathrm{Q}^+(5, q)$ is the union of x pairwise disjoint planes. So for $x \geqslant 3$, such a minihyper does not exist.*

Proof ¿From Theorem 2.4, we can find a solid Δ which intersects F in x points, and such that the $q+1$ hyperplanes through Δ intersect F in an $\{x(q+1), x; 4, q\}$-minihyper F'. These $q+1$ hyperplanes intersect $\mathrm{Q}^+(5, q)$ in either a tangent cone or in a non-singular parabolic quadric $Q(4, q)$.

We can make sure that at least $q-1$ hyperplanes through Δ intersect $\mathrm{Q}^+(5, q)$ in non-singular parabolic quadrics. If at least one of them intersects $\mathrm{Q}^+(5, q)$ in a tangent cone $\langle R, \mathrm{Q}^+(3, q) \rangle$, Lemma 3.1 says that we can choose Δ in this hyperplane in such a way that Δ intersects $\mathrm{Q}^+(5, q)$ in a 3-dimensional hyperbolic quadric. The polarity of the Klein quadric then implies that only two hyperplanes through Δ intersect $\mathrm{Q}^+(5, q)$ in tangent cones.

The $\{x(q+1), x; 4, q\}$-minihypers F' which are the intersection of the other $q-1$ hyperplanes $H_1, \ldots, H_{q-1}$ through Δ with F are contained in non-singular parabolic quadrics and so are the union of x pairwise disjoint lines (Lemma 3.2). Each line of the minihyper $H_i \cap F$ intersects Δ in a point. Suppose that P is a point of $\Delta \cap F$. Then P lies on one line of each minihyper $H_i \cap F$, so P lies on at least two lines of the minihyper F. From Lemma 3.3, we know that the plane π spanned by these lines is completely contained in F. Using Lemma 2.5, we have that $F \backslash \pi$ is an $\{(x-1)(q^2+q+1), (x-1)(q+1); 5, q\}$-minihyper. With $x' = x - 1 < \frac{q}{2}$, we can repeat the previous arguments.

Doing this x times gives us that F is the union of x pairwise disjoint planes. But three planes cannot be pairwise disjoint in $\mathrm{Q}^+(5, q)$. So this minihyper does not exist when $x \geq 3$. □

We now state the new non-existence results on Cameron-Liebler line classes.

Theorem 4.2 *In PG$(3, q)$, $q \geqslant 3$, there exist no Cameron-Liebler line classes with parameter* $2 < x < \frac{q}{2}$.

Proof Let $\mathcal{L}$ be a Cameron-Liebler line class with parameter x. A line l intersects $x(q+1)$ lines of $\mathcal{L}$ if $l \notin \mathcal{L}$ and l intersects $(q+1)x + q^2$ lines of $\mathcal{L}$, including l, if $l \in \mathcal{L}$ (Definition 1.1).

Translated via the Klein correspondence, $\mathcal{L}$ defines a set $\mathcal{T}$ on $\mathrm{Q}^+(5, q)$ such that

$$|P^\perp \cap \mathcal{T}| = \begin{cases} x(q+1) + q^2 & \text{if } P \in \mathcal{T} \\ x(q+1) & \text{if } P \notin \mathcal{T}, P \in \mathrm{Q}^+(5,q). \end{cases}$$

So $\mathcal{T}$ defines an x-tight set on $\mathrm{Q}^+(5, q)$, with $|\mathcal{L}| = \mathcal{T} = x(q^2+q+1)$. So [1, Theorem 12] implies that $\mathcal{T}$ defines an $\{x(q^2+q+1), x(q+1); 5, q\}$-minihyper F on $\mathrm{Q}^+(5, q)$. We only need to check that $\mathcal{T}$ generates PG$(5, q)$.

Since $|\mathcal{T}| \geq 3(q^2+q+1)$, $\dim\langle\mathcal{T}\rangle \geq 4$. If $\dim\langle\mathcal{T}\rangle = 4$, then $\langle\mathcal{T}\rangle \cap \mathrm{Q}^+(5, q) = \mathrm{Q}(4, q)$ since $\mathcal{T}$ is not contained in a tangent hyperplane to $\mathrm{Q}^+(5, q)$.

Since $|\mathcal{T}| < |\mathrm{Q}(4, q)|$, let $R \in \mathrm{Q}(4, q) \setminus \mathcal{T}$. Consider in $T_R(\mathrm{Q}(4, q))$ a plane only intersecting $\mathrm{Q}(4, q)$ in R. This plane then lies in the tangent hyperplane $T_R(\mathrm{Q}(4, q))$ and in q hyperplanes sharing an elliptic quadric $\mathrm{Q}^-(3, q)$ with $\mathrm{Q}(4, q)$.

These elliptic quadrics $\mathrm{Q}^-(3, q)$ define via the Klein correspondence regular spreads of PG$(3, q)$ sharing x lines with $\mathcal{L}$ (Definition 1.1), so these elliptic quadrics contain x points of $\mathcal{T}$. Since $R^\perp$ contains $x(q+1)$ points of $\mathcal{T}$, we find that, in total, $\mathcal{T}$ would contain $x(q+1) + xq = 2xq + x$ points. But this is false, since $|\mathcal{T}| = x(q^2+q+1)$.

So, it is indeed true that $\mathcal{T}$ defines an $\{x(q^2+q+1), x(q+1); 5, q\}$-minihyper F on $\mathrm{Q}^+(5, q)$. But Theorem 4.1 states that this minihyper does not exist, so we conclude that the Cameron-Liebler line classes with parameter $3 \leq x < \frac{q}{2}$ do not exist. □

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