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A Petrov type I and generically asymmetric rotating dust family

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Abstract. The general line element corresponding to the family of algebraically general, gravito-electric, expanding, rotating dust models with one functionally independent zero-order Riemann invariant is constructed. The isometry group is at most one-dimensional but generically trivial. It is shown that the asymmetric ‡ solutions with constant ratio of energy density and vorticity amplitude provide first examples of Petrov type I space-times for which the Karlhede classification requires the computation of the third covariant derivative of the Riemann tensor.

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1. Introduction

Rotating dust solutions in general relativity may serve to describe phenomena on a galactic scale. The metric g_{ab} obeys the field equation

$$R_{ab} - \frac{1}{2} R\, g_{ab} + \Lambda\, g_{ab} = \mu\, u_a\, u_b \qquad (1)$$

with, as usual, R_{ab} the Ricci tensor, R the Ricci scalar, Λ the cosmological constant, u^a the dust 4-velocity field and μ the energy density. For a space-time filled with rotating dust the fluid flow is non-accelerating and the remaining kinematic variables are the expansion scalar $\theta \equiv u^a{}_{;a}$, shear tensor $\sigma_{ab} \equiv u_{(a;b)} - \frac{\theta}{3} h_{ab}$ and vorticity vector $\omega^a \equiv \frac{1}{2}\epsilon^{abc} u_{[b;c]} \neq 0$ of the fluid, with $h_{ab} \equiv g_{ab} + u_a u_b$ and $\epsilon_{abc} \equiv \eta_{abcd} u^d$ the spatial permutation tensor.

Important classes of rotating dust models have been found by assuming some kind of symmetry, or are algebraically special. Respective examples are Winicour's classification [1] of stationary axisymmetric models satisfying the circularity condition (see [2] for examples and further discussion), and the general rotating dust solution admitting time-like conformally flat hypersurfaces with zero extrinsic and constant intrinsic curvature as found by Stephani [3] and later generalized by Barnes for non-zero Λ [4], which depends on seven free functions of one coordinate and which turns out to

‡ without non-trivial isometries

be of Petrov type D. A final and famous example is the homogeneous Petrov type D Gödel universe [5], which can be interpreted as a rotating dust space-time with a negative cosmological constant.

There seems to be, however, a lack of algebraically general, asymmetric solutions §. In a search for such, the class $\mathcal{A}$ of Petrov type I, *gravito-electric* rotating dust models, i.e. for which the Weyl tensor wrt observers comoving with the dust is purely electric,

$$H_{ab} \equiv \frac{1}{2}\epsilon_{acd}C^{cd}{}_{be}\,u^e = 0, \quad E_{ab} \equiv C_{acbd}u^c u^d \neq 0, \tag{2}$$

has been scrutinized [6]. As a main result, it has been proved that the vorticity ω^a of u^a must be a geodesic eigenvector of $E^a{}_b$. Moreover, the corresponding eigenvalue is linearly related to the energy density μ, which cannot be constant, as this would lead to a set of Petrov type D solutions containing the Gödel universe. Hence, if we denote t_m for the number of independent components of the Riemann tensor and its first m covariant derivatives wrt the Weyl principal tetrad ($E^a{}_b$ eigenframe), we have either $t_0 = 1$ or $t_0 = 2$. The $t_0 = 1$ subclass splits into two separate families: the first consists of all non-expanding solutions ($\theta = 0$, for which necessarily $\Lambda < 0$), whilst the second forms a particular set, say $\mathcal{S}$, of expanding models ($\theta \neq 0$), all having $\Lambda > 0$.

In this communication I deduce the general line element for the family $\mathcal{S}$ directly from its invariant description wrt the Weyl principal tetrad. It depends on three free functions of one coordinate. The construction shows that a metric in $\mathcal{S}$ admits at least three, but generically four, functionally independent scalar invariants, and thus either possesses a group G_1 of isometries or is asymmetric. In a next step, the discussion is widened to comprise the non-rotating limit case. Finally, denoting q for the number of covariant derivatives of the Riemann tensor required in the Karlhede invariant classification algorithm [7], it is shown that space-times belonging to a particular subfamily of $\mathcal{S}$ have $q = 3$. To the best of my knowledge, this is the highest value from Petrov type I examples analysed so far, the previous one being trivial ($q = 1$). In combination with the remarkable recent result by Milson and Pelavas [8], who exhibit a set of Petrov type N space-times with the theoretically maximal value $q = 7$ (which turn out to be the unique solutions with this property [9]), this reopens the question whether the upper bound $q = 5$ is sharp in the algebraically general case as well.

2. Line element and Karlhede classification

For a generic member of $\mathcal{S}$, we write $\mathcal{B} \equiv (\partial_0 \equiv \mathbf{u}, \partial_1, \partial_2, \partial_3)$ for the essentially unique Weyl principal tetrad, and $(\Omega^0, \Omega^1, \Omega^2, \Omega^3)$ for the dual basis of one-forms. According to the results of [6] we may arrange the tetrad such that the invariant description wrt $\mathcal{B}$ reduces to the following:

(i) Curvature variables ($\lambda \in \mathbb{R}$):

$$\Lambda = \lambda^2, \quad \mu = 2\lambda(\theta - \lambda), \tag{3}$$

§ To the best of my knowledge, no such models have been found so far.

$$H_{ab} = 0, \quad E_{12} = E_{13} = E_{23} = 0, \tag{4}$$

$$E_{11} - E_{22} = \lambda^2, \quad E_{22} - E_{33} = -\lambda\theta, \quad E_{33} - E_{11} = \lambda(\theta - \lambda). \tag{5}$$

(ii) Commutator relations:

$$[\partial_1, \partial_0] = [\partial_1, \partial_2] = 0, \quad [\partial_0, \partial_2] = -\lambda\partial_2, \tag{6}$$

$$[\partial_0, \partial_3] = -2\omega\partial_2 - (\theta - \lambda)\partial_3, \quad [\partial_2, \partial_3] = -2\omega\partial_0, \tag{7}$$

$$[\partial_1, \partial_3] = -\beta\partial_3. \tag{8}$$

(iii) Ricci and Bianchi equations:

$$\partial_0\omega = -\theta\omega, \qquad \partial_1\omega = -\beta\omega, \qquad \partial_2\omega = 0, \tag{9}$$

$$\partial_0\theta = -\theta(\theta - \lambda), \quad \partial_1\theta = -\beta(\theta - \lambda), \quad \partial_2\theta = 0, \tag{10}$$

$$\partial_0\beta = -(\theta - \lambda)\beta, \quad \partial_1\beta = -\beta^2 - \lambda\theta, \quad \partial_2\beta = 0. \tag{11}$$

Note from (3-5) that there is only one *algebraically* independent Riemann curvature component, and that $\lambda \neq 0$, $\theta \neq 0$ and $\theta \neq \lambda$ by the Petrov type I assumption. The describing variables $\omega \equiv \omega_1 = \sigma_{23}$, θ and β are invariantly defined scalars, as they are linearly related to commutator coefficients of the geometrically fixed tetrad $\mathcal{B}$. One checks that the formal PDE system (9-11) is consistent with (6). The ∂_3-derivatives of ω, θ and β being undetermined, there will be three free functions in the general solution, which we now construct.

To start with, we note from (6) that ∂_0, ∂_1 and ∂_2 forms a Lie subalgebra, i.e., the vector field ∂_3 is normal. Hence functions F and z exist such that $\Omega^3 = -\mathrm{d}z/F$. F is related to the commutator coefficients $\gamma^3{}_{jk}$ via Cartan's first structure equation for Ω^3, $\mathrm{d}\Omega^3 = -\frac{1}{2}\gamma^3{}_{jk}\Omega^j \wedge \Omega^k$, which is equivalent to $\partial_i F = \gamma^3{}_{i3}F$, $i < 3$, i.e.,

$$\partial_0 F = -(\theta - \lambda)F, \quad \partial_1 F = -\beta F, \quad \partial_2 F = 0. \tag{12}$$

Conversely, for any solution F of (12) it follows that a function z exists such that $\Omega^3 = -\mathrm{d}z/F$. Fixing one such solution F and defining, for any given scalar invariant S, the invariant S_F by

$$\partial_3 S \equiv F S_F, \tag{13}$$

the commutation relations $[\partial_i, \partial_3]S = \gamma^j{}_{i3}\,\partial_j S$ read

$$F\partial_i S_F = \partial_3\partial_i S + \sum_{j=0}^{2} \gamma^j{}_{i3}\,\partial_j S, \quad i < 3, \tag{14}$$

and the differential $\mathrm{d}S$ can be expanded as

$$\mathrm{d}S = \sum_{i=0}^{2} \partial_i S\,\Omega^i - S_F \mathrm{d}z. \tag{15}$$

For suitable F, the choice of which we postpone at the moment, the function z will serve as one of the coordinates. One observes from (6-8) that the null vector fields $\partial_2 \pm \partial_0$ are also normal and that the normalized vorticity vector ∂_1 is normal and geodesic. However, we will not introduce according coordinates; the deduction of the line element and the further discussion are much more elegant and clear when we proceed by constructing the three remaining coordinates directly from the invariants and their ∂_3-derivatives, hereby exploiting the expansion (15).

Firstly, we see from (9) and (10) that $\partial_i(\omega/(\theta-\lambda))=0$, $i<3$; it then follows from (15) that the ratio $S=\omega/(\theta-\lambda)$ is a function of z, which we henceforth denote by $1/f_3(z)$ or, in view of the non-rotating limit to be considered later, by $g_3(z)$. Next, we define the invariants t and x by

$$\tan x = -\frac{\beta}{\theta}, \quad \cos x\, e^t = \frac{\theta}{\theta-\lambda}. \tag{16}$$

This is equivalent to writing β and θ in a different way,

$$\beta = -\frac{\lambda \sin x\, e^t}{\cos x\, e^t - 1}, \quad \theta = \frac{\lambda \cos x\, e^t}{\cos x\, e^t - 1}, \tag{17}$$

so that the vorticity amplitude and matter density read

$$\omega = \frac{\theta-\lambda}{f_3(z)} = \frac{\lambda}{f_3(z)(\cos x\, e^t - 1)}, \quad \mu = 2\lambda(\theta-\lambda) = \frac{2\lambda^2}{\cos x\, e^t - 1}. \tag{18}$$

Equations (10) and (11) imply the simpler derivatives

$$\partial_0 t = \lambda, \quad \partial_1 t = 0, \quad \partial_2 t = 0, \tag{19}$$

$$\partial_0 x = 0, \quad \partial_1 x = \lambda, \quad \partial_2 x = 0. \tag{20}$$

We now use (14). Putting $S=x$ we get $\partial_i x_F = 0$, $i<3$, whence $x_F = f_1(z)$ by (15) with $S=x_F$. Putting $S=t$ and rewriting $t_F=-2y$ we obtain

$$\partial_0 y = 0, \quad \partial_1 y = 0, \quad \partial_2 y = \frac{\lambda\omega}{F}. \tag{21}$$

This is the point where we make a convenient choice for F. Looking at (9), (12) and (19) we see that we may pick $F=\omega e^t$, such that $\partial_2 y = \lambda e^{-t}$ by (12). Finally, putting $S=y$ in (14) we find

$$\partial_0 y_F = -2\lambda e^{-2t}, \quad \partial_1 y_F = 0, \quad \partial_2 y_F = 2y\lambda e^{-t}, \tag{22}$$

i.e. $\partial_i y_F = \partial_i(y^2+e^{-2t})$, $i<3$, such that from (15) with $S = y_F-(y^2+e^{-2t})$ we derive $y_F = f_2(z)+y^2+e^{-2t}$.

Assembling the above pieces one gets

$$\begin{gathered} \mathrm{d}t = \lambda\Omega^0 + 2y\mathrm{d}z, \quad \mathrm{d}x = \lambda\Omega^1 - f_1(z)\mathrm{d}z, \\ \mathrm{d}y = \lambda e^{-t}\Omega^2 - (f_2(z)+y^2+e^{-2t})\mathrm{d}z, \quad \lambda\Omega^3 = f_3(z)(e^{-t}-\cos x)\mathrm{d}z. \end{gathered} \tag{23}$$

Hence, on using z and the scalar invariants t, x, y as coordinates, we infer immediately from (23) that the line element is given by

$$\begin{aligned} \lambda^2 \mathrm{d}s^2 = & -(\mathrm{d}t - 2y\,\mathrm{d}z)^2 + (\mathrm{d}x + f_1(z)\mathrm{d}z)^2 + (\cos x - e^{-t})^2 f_3(z)^2 \mathrm{d}z^2 \\ & + e^{2t}\left[\mathrm{d}y + \left(f_2(z)+y^2+e^{-2t}\right)\mathrm{d}z\right]^2, \end{aligned} \tag{24}$$

where the invariant scalar fields $f_1(z)$, $f_2(z)$ and $f_3(z)$ are arbitrary functions of their argument. Notice that $\lambda=\sqrt{\Lambda}$ plays the role of a constant scaling factor. The variable

t is time-like and x and y are space-like, whereas z is spacelike, null or timelike in the region where

$$g_{zz} = f_1^2 + e^{2t}(f_2 + e^{-2t} + y^2)^2 + (\cos x - e^{-t})^2 f_3^2 - 4y^2 \tag{25}$$

is greater than, equal to or smaller than zero, respectively. Switching to the null coordinates $u_\pm$ and the space-like coordinate ξ defined by

$$u_\pm \equiv y \pm e^{-t}, \quad \xi \equiv x - \phi(z), \quad \frac{\mathrm{d}\phi}{\mathrm{d}z}(z) \equiv -f_1(z), \tag{26}$$

the metric becomes

$$\lambda^2 \mathrm{d}s^2 = (\Omega^2 + \Omega^0)(\Omega^2 - \Omega^0) + (\Omega^1)^2 + (\Omega^3)^2, \tag{27}$$

$$\Omega^2 \mp \Omega^0 = \frac{2}{u_+ - u_-}[\mathrm{d}u_\pm + (f_2(z) + u_\pm^2)\mathrm{d}z], \tag{28}$$

$$\Omega^1 = \mathrm{d}\xi, \quad \Omega^3 = f_3(z)\left(\cos(\xi + \phi(z)) + \frac{u_- - u_+}{2}\right)\mathrm{d}z. \tag{29}$$

(28) makes the normality of $\Omega^2 \mp \Omega^0$ apparent, as $\mathrm{d}u_\pm + (f_2(z) + u_\pm^2)\mathrm{d}z$ are one-forms on 2-spaces; at the same time it reveals the difficulty when we would have started with coordinates $v_\pm$ and scalar fields $G_\pm(v_+, v_-, \xi, z)$ for which $\Omega^2 \mp \Omega^0 = G_\pm \mathrm{d}v_\pm$.

One observes from (17) and (18) that $\mu > 0$, and at the same time $\theta > 0$, at space-time points with $\cos x > e^{-t}$, while the boundary $\cos x = e^{-t}$ forms a two-surface of curvature singularities. It is an open question whether (24) may generate physically plausible rotating dust models.

The discussion of possible Killing vector fields becomes trivial in the above invariant approach: when at least one of the functions $f_1(z), f_2(z)$ or $f_3(z)$ is non-constant it can be taken as a fourth invariantly defined coordinate replacing z, and there is no symmetry; when all of them are constant there is a group G_1 of isometries generated by $\partial/\partial z$.

By putting ω equal to zero in (3-11) one arrives at the corresponding irrotational dust family $\mathcal{S}^0$, which was integrated in [10] (metric (45)). More generally, we may consider the class $\widetilde{\mathcal{S}} = \mathcal{S} \cup \mathcal{S}^0$. Then the above integration procedure can be applied up to (21), and we make the alternative choice $F = (\theta - \lambda)\, e^t$. Taking further advantage of the normality of e.g. $\partial_2 - \partial_0$ and solving Cartan's first structure equations, one readily obtains the line element

$$\begin{aligned} \lambda^2 \mathrm{d}s^2 = &- (\mathrm{d}t - 2y\,\mathrm{d}z)^2 + (\mathrm{d}x + f_1(z)\mathrm{d}z)^2 + (\cos x - e^{-t})^2 \mathrm{d}z^2 \\ &+ (e^{t-K}\mathrm{d}\eta + \mathrm{d}t - 2y\,\mathrm{d}z)^2, \end{aligned} \tag{30}$$

where now

$$y = y(t, \eta, z) = \frac{1}{2}\frac{\partial K}{\partial z}(\eta, z) - e^{-t} g_3(z) \tag{31}$$

and where $K = K(\eta, z)$ is a solution of

$$e^{K(\eta,z)} \frac{\partial^2 K}{\partial \eta\, \partial z}(\eta, z) = 2 g_3(z). \tag{32}$$

The non-rotating subcase corresponds to $g_3(z) = 0$. Looking at (21), (31) and (32), and after possibly redefining η, we have that $K = K(z)$ is a primitive function of $2y = 2y(z)$. In terms of coordinates z, $\tau \equiv t - K(z)$, $\zeta \equiv \eta - e^{-t}$ and ξ, cf. (26), the non-rotating solutions take the form

$$\lambda^2 \mathrm{d}s^2 = -\mathrm{d}\tau^2 + \mathrm{d}\xi^2 + e^{2\tau}\mathrm{d}\zeta^2 + [e^{-K(z)}e^{-\tau} - \cos(\xi + \phi(z))]^2 \mathrm{d}z^2. \qquad (33)$$

There is at least a G_1 isometry group generated by $\partial_\zeta \sim e^\tau \partial_2$. As the scalar invariants t, x, y and f_1 must be invariant under any isometry, one immediately deduces that there is a group G_2 of motions if and only if both y and f_1 are constant, i.e., if and only if the line element can be transformed to (33) with $K(z) = yz$ and $\phi(z) = f_1 z$, where y and f_1 are constants; the additional Killing vector field is then given by

$$2y\partial_\tau - f_1\partial_\xi - \partial_z \sim 2y\partial_0 - f_1\partial_1 + (e^{-t} - \cos x)\partial_3. \qquad (34)$$

This corrects an inaccuracy in the symmetry discussion of $\mathcal{S}^0$ in [10].

When working wrt the Weyl principal tetrad $\mathcal{B}$ of a Petrov type I space-time, the stop value q within the Karlhede invariant classification algorithm is defined by

$$t_m > t_{m-1},\ 0 \le m < q, \quad t_q = t_{q-1}, \qquad (35)$$

where one formally puts $t_{-1} \equiv -1$. The final value t_{q-1} is the total number of functionally independent scalar invariants and equals $4 - r$, where r stands for the dimension of the isometry group. For members of $\widetilde{\mathcal{S}}$ one has $t_0 = 1$ and $t_{q-1} = 2, 3$ or 4, such that $2 \le q \le 4$ a priori. A direct calculation shows that the components of the first covariant derivative of the Riemann tensor wrt $\mathcal{B}$ are rational functions of the invariants

$$\cos x\, e^t, \quad \sin x\, e^t, \quad 2\, y \cos\, x + \sin x\, f_1(z), \quad g_3(z). \qquad (36)$$

The following possibilities thus arise ($f' = 0$ means that f is constant):

- for metrics in $\mathcal{S}^0$ ($g_3(z) = 0$, $y = y(z)$):
  - (i) $f_1' = y' = 0$. There is a group G_2 of isometries and $t_1 = 2$, such that $t_2 = 2$ and $q = 2$.
  - (ii) $f_1' \neq 0$ or $y' \neq 0$. There is a group G_1 of isometries and $t_1 = 3$, such that $t_2 = 3$ and $q = 2$.
- for metrics in $\mathcal{S}$ ($g_3(z) = 1/f_3(z) \neq 0$, y functionally independent of t, x and z):
  - (i) $f_1' = f_2' = f_3' = 0$. There is a group G_1 of isometries and $t_1 = 3$, such that $t_2 = 3$ and $q = 2$.
  - (ii) $f_3' = 0$, but $f_1' \neq 0$ or $f_2' \neq 0$. There is no symmetry and $t_1 = 3$, such that $t_2 = t_3 = 4$ and $q = 3$.
  - (iii) $f_3' \neq 0$. There is no symmetry and $t_1 = 4$, such that $t_2 = 4$ and $q = 2$.

We conclude that $q = 2$ for all members of $\widetilde{\mathcal{S}}$, except for the asymmetric (whence rotating) models for which the ratio $2\lambda f_3 = \mu/\omega$ of energy density and vorticity amplitude is constant. This special class of models, depending on two free functions of one coordinate and one constant parameter, provides a first example of algebraically general space-times for which $q = 3$.

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