Anti-Newtonian universes do not exist

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Abstract. In a paper by Maartens, Lesame and Ellis (1998) it was shown that irrotational dust solutions with vanishing electric part of the Weyl tensor are subject to severe integrability conditions and it was conjectured that the only such solutions are FLRW spacetimes. In their analysis the possibility of a cosmological constant Λ was omitted. The conjecture is proved, irrespective as to whether Λ is zero or not, and qualitative differences with the case of vanishing *magnetic* Weyl curvature are pointed out.

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1. Introduction

Irrotational dust (ID) spacetimes are perfect fluid solutions of the general relativistic field equations characterized by vanishing pressure ($p = 0$) and vorticity vector ($\omega^a = 0$). They serve as potential models for the late universe [1, 2] and gravitational collapse [3]. In an initial-value problem formulation for such models, the Ricci-identity for the fluid 4-velocity and the Bianchi equations (incorporating the field equations via the Ricci tensor) may be covariantly split into propagation and constraint equations [4]. In a streamlined setting [5] Maartens showed that these constraint equations are consistent with each other (i.e., not overdetermined on an initial hypersurface) and are preserved under evolution, generically. However, when *further* external conditions are imposed their consistency is not a priori guaranteed, and rather unexpected conclusions may come out.

In this respect, one of the most promising and natural further restrictions one can make is to set the gravito-magnetic tensor H_{ab} equal to zero (a cosmological motivation for doing so was given in [6]), with the remaining gravito-electric tensor $E_{ab} \neq 0$. As the latter is the relativistic generalization of the tidal tensor in Newtonian theory, the corresponding 'purely electric' ID models were called *Newtonian-like* in [7]. One might therefore expect to find a rich class of solutions, at least incorporating translates of classical Newtonian models. However, in two independent papers [8, 9] a non-terminating chain $\mathcal{C}_E$ of consistency conditions for the dynamical variables was deduced, which made the authors conjecture that the class was unlikely to extend beyond the known spatially homogeneous members and the Szekeres [10] subfamily, which exhausts Petrov type D [11].

The natural counterpart to the above is to set E_{ab} equal to zero, with $H_{ab} \neq 0$. These purely magnetic ID models were studied in [7] and were logically called 'anti-Newtonian'. Some parallels and differences between Newtonian-like and anti-Newtonian universes were pointed out. In both cases no spatial derivatives occur in the remaining propagation equations; hence the flowlines emerging from an initial hypersurface evolve separately from each other, and solutions may therefore be called 'silent' [7], although in the literature this terminology is normally used and was introduced [12] for the purely electric case only. Also, the chain $\mathcal{C}_E$ for the Newtonian-like case has a direct analogue $\mathcal{C}_{H,1}$ in the anti-Newtonian case. However, whereas $\mathcal{C}_E$ is identically satisfied in the linearized theory (with Friedman-Lemaitre-Robertson-Walker (FLRW) background), leading to a *linearization instability* for Newtonian-like ID models (in the sense that there are consistent linearized solutions which are not the limit of any consistent solutions in the full, nonlinear theory [8]), it was shown by the primary integrability condition in $\mathcal{C}_{H,1}$ that there are no linearized anti-Newtonian ID models at all. On top of this, a second chain $\mathcal{C}_{H,2}$ of integrability conditions exists for the magnetic case, which has no analogue in the electric case. Both facts have lead the authors of [7] to conjecture the *non-existence of purely magnetic ID models*, but a proof for it within the full, nonlinear theory was not found until now.

It is usually believed that the presence of a cosmological constant Λ is of minor or no importance to the qualitative outcome of consistency analyses for particular conditions, and in this line Λ was omitted in [8, 9] and [7]. For Newtonian-like silent models however, one of the special cases where one can show that the constraints with $\Lambda \leq 0$ are inconsistent (namely the case where one of the eigenvalues of the expansion tensor is 0), surprisingly turns out to admit two new and explicitly constructable families of spatially *in*homogeneous Petrov type I metrics when $\Lambda > 0$ [13]. This result points out that a cosmological constant should preferably be incorporated in consistency analyses.

In the present paper and in this respect, the generalization to arbitrary cosmological constant Λ of the above Maartens *et al* conjecture for purely magnetic ID models is proved, by showing that the combined conditions of vanishing pressure p, vorticity vector ω^a and gravito-electric tensor E_{ab} are only consistent for FLRW spacetimes. The structure of the paper is as follows. In section 2.1 the 1+3 covariant system of propagation and constraint equations for general irrotational dust is written down as in [7], with a cosmological constant added. In section 2.2 the origin of the two chains of integrability conditions for the purely magnetic case is resumed and it is pointed out how a third chain is coming into play. Comparison with the purely electric case is made in Section 2.3, and the conjecture for Newtonian-like ID models [8, 9] is properly restated in function of the sign of Λ. The actual proof (in two steps) for the inconsistency of anti-Newtonian universes is the content of Section 3. A natural continuation of the analysis is pointed out in Section 4.

2. Magnetic versus electric ID

2.1. Basic setting for general ID

Einstein's field equation with cosmological constant Λ and perfect fluid source term $T_{ab} = \mu u_a u_b + p\, h_{ab}$ can be covariantly written as $(8\pi G = 1 = c)$:

$$R_{ab} = \frac{1}{2}(\rho + 3p)u_a u_b + \frac{1}{2}(\rho - p)h_{ab} - \Lambda g_{ab}. \tag{1}$$

It provides an algebraic definition of the Ricci tensor R_{ab} in terms of the matter density ρ, pressure p and normalized timelike four-velocity field u^a $(u^a u_a = -1)$ of the perfect fluid; the corresponding pair of complementary idempotent operators $(-u_a\, u^b, h_a{}^b := g_a{}^b + u_a\, u^b)$ projects along and orthogonal to u^a, respectively.

The introduction of the following 1+3 covariant projections and operations (see [5, 15]) considerably compactifies expressions. Herein round (square) brackets denote symmetrization (anti-symmetrization) and $\mathrm{tr}(S)$ is written for $S^a{}_a$; for 2-tensors U_{ab} and one-forms X_a the 'dot-free' notation $(SU)_{ab}$ for $S_a{}^c U_{cb}$, resp. $(SX)_a$ for $S_a{}^b X_b$, is used throughout the paper.

- The *spatial part* of a tensor $S^{a_1\ldots a_r}{}_{b_1\ldots b_s}$ is
$$ {}^sS^{a_1\ldots a_r}{}_{b_1\ldots b_s} = h^{a_1}{}_{c_1}\cdots h^{a_r}{}_{c_r} h_{b_1}{}^{d_1}\cdots h_{b_s}{}^{d_s} S^{c_1\ldots c_r}{}_{d_1\ldots d_s}. \tag{2}$$
A tensor is called *spatial* iff ${}^sS^{a_1\ldots a_r}{}_{b_1\ldots b_s} = S^{a_1\ldots a_r}{}_{b_1\ldots b_s}$. The spatial, symmetric and tracefree part $S_{\langle ab\rangle}$ of a 2-tensor S_{ab} is
$$S_{\langle ab\rangle} = {}^sS_{(ab)} - \frac{1}{3}\mathrm{tr}({}^sS)h_{ab}. \tag{3}$$
- The *spatial permutation tensor* is $\epsilon_{abc} = \epsilon_{[abc]} = \eta_{abcd}\, u^d$, where $\eta_{abcd} = \eta_{[abcd]}$ is the spacetime Levi-Civita permutation tensor. One then defines for 2-tensors S_{ab}, U_{ab}:
$$[S, U]_a = \epsilon_{abc}(SU)^{bc}. \tag{4}$$
When S_{ab} and U_{ab} are spatial and symmetric, one can easily show the identity (see also $(A3) of [5]$)
$$[S^2, U]_a = 2\,[S, \widehat{SU}]_a = \mathrm{tr}(S)\,[S, U]_a - (S\,[S, U])_a, \tag{5}$$
with $\widehat{SU}_{ab} = (SU)_{\langle ab\rangle}$. In this case $[S, U]_a = 0$ is equivalent to $(SU)_{ab}$ being symmetric, which is still equivalent to $(SU - US)_{ab} = 0$, whence $S_a{}^b$ and $U_a{}^b$ have a common eigenframe.
- The *electric part* E_{ab} and *magnetic part* H_{ab} of the Weyl tensor C_{abcd} are
$$E_{ab} = E_{\langle ab\rangle} = C_{acbd}\, u^c\, u^d, \quad H_{ab} = H_{\langle ab\rangle} = \epsilon_{cda} C^{cd}{}_{be}\, u^e. \tag{6}$$
- Let ∇ be the spacetime covariant derivative attached to the unique metric connection. The *(covariant) time derivative* T and the *(covariant) spatial derivative* D are defined by:
$$T(S)^{a_1\ldots a_r}{}_{b_1\ldots b_s} = u^c \nabla_c S^{a_1\ldots a_r}{}_{b_1\ldots b_s}, \tag{7}$$
$$D_c\, S^{a_1\ldots a_r}{}_{b_1\ldots b_s} = {}^s\nabla_c S^{a_1\ldots a_r}{}_{b_1\ldots b_s}. \tag{8}$$

Normally the notation $\dot{S}^{a_1\ldots a_r}{}_{b_1\ldots b_s}$ is used instead of $T(S)^{a_1\ldots a_r}{}_{b_1\ldots b_s}$, but the latter emphasizes on the action of T as a (degree 0 derivative) operator and this is more convenient for what follows. Definition (8) tells that D maps each (r,s)-tensor $S^{a_1\ldots a_r}{}_{b_1\ldots b_s}$ to the spatial part of the $(r,s+1)$-tensor $\nabla_c S^{a_1\ldots a_r}{}_{b_1\ldots b_s}$. It gives rise to the covariant spatial divergence (div) and curl operator, acting on vectors V^a and 2-tensors S_{ab} as:

$$\mathrm{div}\, V := D_a V^a, \qquad \mathrm{curl}\, V_a := \epsilon_{abc} D^b V^c \tag{9}$$

$$\mathrm{div}\, S_a := D^b S_{ab}, \quad \mathrm{curl}\, S_{ab} := \epsilon_{cd(a} D^c S_{b)}{}^d; \tag{10}$$

For spatial 2-tensors S_{ab}, U_{ab} one has the identity

$$\mathrm{div}[S,U] = \mathrm{tr}(U\,\mathrm{curl} S) - \mathrm{tr}(S\,\mathrm{curl} U). \tag{11}$$

- The splitting of ∇u into different kinematical quantities [4]:

$$\begin{aligned} u_{a;b} =&\; D_b u_a - \dot{u}_a u_b \\ :=&\; \sigma_{ab} + \frac{1}{3}\theta h_{ab} + \epsilon_{abc}\omega^c - \dot{u}_a u_b. \end{aligned} \tag{12}$$

Herein $\sigma_{ab} = u_{\langle a;b\rangle}$ is the shear tensor, $\theta = u^a{}_{;a}$ the volume-expansion scalar, $\omega^a = \frac{1}{2}\epsilon^{abc} u_{b;c}$ the vorticity vector, and $\dot{u}^a$ the acceleration vector, which are all spatial.

For this introduction, the abstract index notation [14] was most transparent, but an index-free notation turns out to be practical for calculation purposes. In particular, $E_{ab}, H_{ab}, \sigma_{ab}, h_{ab}, \omega^a, \dot{u}^a$ are denoted $E, H, \sigma, \omega, \dot{u}$ from now on. Moreover, since all appearing 1-forms and 2-tensors will be spatial the symbol h will be dropped everywhere. The notation for $S_{\langle ab\rangle}$ becomes $\hat{S}$ (cf. [18]), but when S is symmetric this is written as $S - \frac{1}{3}\mathrm{tr} S$ for computational convenience. In this notation, the $\mathbb{R}$-linear operators T and D satisfy in particular the Leibniz rules

$$T(XY) = T(X)\,Y + X\,T(Y), \quad D(fg) = (Df)\,g + f\,Dg \tag{13}$$

for 2-tensors or scalar functions X, 2-tensors, 1-forms or scalar functions Y, and scalar functions f, g. In the present paper we deal with *non-vacuum irrotational dust (ID)*, i.e. solutions of (1) for which

$$p = 0, \quad \rho \neq 0, \quad \omega = 0, \quad \dot{u} = 0, \tag{14}$$

where $\dot{u} = 0$ is a mere consequence of $p = 0, \rho \neq 0$ and the second contracted Bianchi identity. With $\dot{u} = 0$ one has

$$T[S,U] = [T(S),U] + [S,T(U)]. \tag{15}$$

A commutator rule for T and D, written in appropriate form and for the action on scalar functions f, is also needed. With $\omega = \dot{u} = 0$ it reads [5]:

$$T(Df) = D(T(f)) - (\sigma + \frac{1}{3}\theta)Df. \tag{16}$$

For perfect fluids in general the Ricci identity for u and the Bianchi-identities, hereby substituting (1) for the Ricci tensor, are basic equations to be satisfied. They may be covariantly split into a set of propagation equations (involving the covariant time derivative) and a set of constraint equations [4, 5]. In the case of non-vacuum ID (14) the sets of non-identical equations read:

- Propagation equations:

$$T(\rho) = -\rho\theta, \tag{17}$$
$$T(\theta) = -\frac{1}{3}\theta^2 - \mathrm{tr}(\sigma^2) - \frac{1}{2}\rho + \Lambda, \tag{18}$$
$$T(\sigma) = -\frac{2}{3}\theta\sigma - \sigma^2 + \frac{1}{3}\mathrm{tr}(\sigma^2) - E \tag{19}$$
$$T(E) - \mathrm{curl}H = -\theta E + 3\widehat{\sigma E} - \frac{1}{2}\rho\sigma, \tag{20}$$
$$T(H) + \mathrm{curl}E = -\theta H + 3\widehat{\sigma H} \tag{21}$$

- Constraint equations:

$$\mathcal{C}^1 \equiv \mathrm{div}\sigma - \frac{2}{3}D\theta = 0, \tag{22}$$
$$\mathcal{C}^2 \equiv \mathrm{curl}\sigma - H = 0, \tag{23}$$
$$\mathcal{C}^3 \equiv \mathrm{div}E - \frac{1}{3}D\rho - [\sigma, H] = 0, \tag{24}$$
$$\mathcal{C}^4 \equiv \mathrm{div}H + [\sigma, E] = 0. \tag{25}$$

On considering (19) and (23), (20), (22) and (24), one sees that within the class of non-vacuum ID spacetimes, the FLRW case

$$D\rho = D\theta = 0, \quad \sigma = E = H = 0. \tag{26}$$

is covariantly characterized by vanishing shear tensor ($\sigma = 0$) or, equivalently, by Petrov type 0 ($E = H = 0$).

In [5] it was shown that the time derivatives of each of the $C^i, i = 1..4$, are differential combinations of the C^j themselves, such that (22)-(25) evolves consistently and doesn't give rise to new constraints, generically. It was also shown that actually, $\mathcal{C}^4$ is a differential combination of $\mathcal{C}^1$ and $\mathcal{C}^2$:

$$\mathcal{C}^4 = \frac{1}{2}\mathrm{curl}\,\mathcal{C}^1 - \mathrm{div}\,\mathcal{C}^2.$$

However, if extra covariant conditions are imposed, new constraints may be generated and their consistency should be investigated.

2.2. Purely magnetic ID

In this respect, the constraint $E = 0$ was studied as the natural counterpart for $H = 0$ in [7]. The remaining variables are the scalars ρ, θ and the tensorial quantities σ, H. As put forward in the introduction, one sees that no spatial derivatives occur in the propagation equations (17)-(19) and (21), such that the purely magnetic ID models are 'silent'. Actually, with $E = 0$, (17)-(19) forms a closed subsystem determining the evolution of the matter variables ρ, θ and σ, from which the evolution equation (21) for the gravito-magnetic field H is decoupled. The latter equation may also be seen as arising from the curl of the algebraic constraint $E = 0$ put on the whole of the system (17)-(25). In the same viewpoint and by virtue of (20) and (24), the time derivative and the divergence of $E = 0$ gives rise to two independent constraints which are not preserved under evolution:

$$\mathcal{C}^5 \equiv \mathrm{curl} H - \frac{1}{2}\rho\sigma = 0, \tag{27}$$

$$\mathcal{C}^{3,H} \equiv -\frac{1}{3}D\rho - [\sigma, H] = 0. \tag{28}$$

It was not noted in [7] that, although the *sum* equation $\mathcal{C}^3 = 0$ of $\mathrm{div}\, E = 0$ and $\mathcal{C}^{3,H} = 0$ is consistent under evolution, this is not necessarily the case for both equations separately. As $E = 0$ has no influence on $\mathcal{C}^1$ and $\mathcal{C}^2$, and as (27) is an identity, the constraints (22), (23) and (25) *do* evolve consistently as in the generic case.

The time derivative of (27), modulo (22) and (27), yields a primary integrability condition $\mathcal{E} = 0$ whose repeated time differentiation leads to a non-terminating chain of new constraints $\mathcal{C}_{H,1}$. The linearization of $\mathcal{E}$ around an FLRW background (26) reads $\frac{1}{6}\rho\theta\sigma$, such that linearized non-vacuum ID models with $E = 0$ are either FLRW or satisfy $\theta = 0, \rho = 2\Lambda$, as is seen from (18). Hence in the case $\Lambda = 0$ *no linearized anti-Newtonian universes exist at all* [7]. Also note that, in the light of the question whether to incorporate a cosmological constant or not, the same conclusion cannot be drawn for general Λ). However, for the purpose of proving inconsistency in the full non-linear theory, the chain $\mathcal{C}_{H,1}$ is even more unmanageable than its purely electric analogue $\mathcal{C}_E$ (see Section 2.3).

Analogously, for the time derivative of $\mathcal{C}^{3,H}$ one finds after a short calculation, on using (16), (15), (19), (21) and (5):

$$T(\mathcal{C}^{3,H}) = \left(\frac{1}{2}\sigma - \frac{5}{3}\theta\right)\mathcal{C}^{3,H} + \frac{1}{3}\mathcal{J},$$

$$\mathcal{J} = \rho D\theta + \left(\frac{1}{2}\sigma - \frac{1}{3}\theta\right)D\rho. \tag{29}$$

Henceforth, $\mathcal{J} = 0$. As $E \equiv 0$, and $\mathcal{C}^{3,H} = 0$ is the translation of $\mathrm{div}\, E = 0$ whereas $\mathcal{C}^5 = 0$ is coming from $\dot{E} = 0$, the same condition should be found on calculating $\mathrm{div}\, \mathcal{C}^5$ (see also (A13) in [5]). Indeed, in [7] it was calculated that ‡

$$\mathrm{div}\, \mathcal{C}^5 = \frac{1}{2}\mathrm{curl}\, \mathcal{C}^4 - (\sigma + \frac{1}{3}\theta)\,\mathcal{C}^3 - \frac{1}{2}\mathcal{C}^1 - \frac{1}{3}\mathcal{J}. \tag{30}$$

The successive time derivatives of $\mathcal{J}^{(0)} := \mathcal{J} = 0$ lead to a second chain of constraints $\mathcal{C}_{H,2} = (\mathcal{J}^{(i)} = 0)_{i=0}^{\infty}$, with $\mathcal{J}^{(i)} := T^i(\mathcal{J})$. In [7] $\mathcal{J}^{(1)}$ and $\mathcal{J}^{(2)}$ were explicitly calculated (for $\Lambda = 0$) and the general structure of $\mathcal{J}^{(i)}$ was given, but apparently the consequence of this chain was overlooked. In Proposition 1 of Section 3, it will be shown (for arbitrary Λ) that $\mathcal{C}_{H,2}$ implies $D\rho = 0$. But this gives rise to a *third* chain $\mathcal{C}_{H,3}$ of constraints, as

‡ In [7], the $-\sigma\mathcal{C}^3$ term was forgotten in the right hand side.

follows. With $D\rho = 0$, (28) reads $[\sigma, H] = 0$. Taking the divergence of this and using (11), (23) and (27), one finds:

$$\mathcal{K} \equiv \mathrm{tr}(H^2) - \frac{1}{2}\rho\,\mathrm{tr}(\sigma^2) = 0, \tag{31}$$

and hence $\mathcal{C}_{H,3} = (\mathcal{K}^{(i)} = 0)_{i=0}^{\infty}$, with $\mathcal{K}^{(i)} := T^i(\mathcal{K})$§. In Theorem 1 of Section 3 it is finally shown that within the class of ID's with $E = 0$, $\mathcal{C}_{H,3}$ can only be satisfied for $\sigma = H = 0$, i.e., by the FLRW subclass.

2.3. Comparison with purely electric ID

Analogous to the purely magnetic case, (17)-(20) with $H = 0$ forms a 'silent' system of evolution equations for the remaining variables ρ, θ, σ, E, but here (20) is coupled to (17)-(19) via the E-term in (19). Now $\dot{H} = 0$ (together with $H = 0$) and $\mathrm{div}\, H = 0$ translate into

$$\mathcal{C}^5 \quad = \mathrm{curl}\, E = 0, \tag{32}$$

$$\mathcal{C}^{4,E} = [\sigma, E] = 0. \tag{33}$$

$\mathcal{C}^1$ is independent of H, and $\mathcal{C}^2, \mathcal{C}^3$ contain H only algebraically; hence the corresponding constraints evolve consistently. In analogy with the $E = 0$ case, the time derivative of (33), modulo the constraints (22)-(24) and (32), gives rise to a new tensorial condition $\mathcal{H} = 0$ and an indefinite chain $\mathcal{C}_E = (T^i(\mathcal{H}) = 0)_{i=0}^{\infty}$ of constraints, identically satisfied for Petrov type D and spatially homogeneous models, but not in general. However, in contrast to the magnetic case, the linearizations of the $T^i(\mathcal{H})$ around an FLRW background are identically zero; hence Newtonian-like silent models are subject to a *linearization instability*, cf. Introduction. The difference is essentially due to the extra term $\frac{1}{2}\rho\,\sigma$ in (27) within the magnetic case, not present in (32). The same absence makes that the divergence of $\mathcal{C}^5$ doesn't lead to a new constraint in the electric case:

$$\mathrm{div}\,\mathcal{C}^5 = \frac{1}{2}\mathrm{curl}\,\mathcal{C}^3 - (\sigma + \frac{1}{3}\theta)\,\mathcal{C}^4. \tag{34}$$

By the analogy of the remark made for the magnetic case, the propagation of the constraint (33) should not lead to a new condition either. Indeed, this can easily be checked on using (15), (19), (20) with $H = 0$, and (5). Moreover, it was checked in [16] that even for general ID the condition $\mathrm{div}\, H = 0$, or equivalently $[\sigma, E] = 0$, is consistent under evolution, this making use of the time derivative of (19) instead of using (20)‖. Hence there is no analogue here for the chain $\mathcal{C}_{H,2}$ obtained in the magnetic case, and the difference may also be traced back to the fact that, as in electro-magnetism, $D\rho$

§ Notice that, without the extra fact $D\rho = 0$, the divergence of $\mathrm{div}\,\mathcal{C}^{3,H}$ already incorporates a Laplacian $D^2(\rho)$, which makes the direct chain $T^i(\mathrm{div}\,\mathcal{C}^{3,H}) = 0$ rather unmanageable.

‖ Essentially, this comes down to showing $[\sigma, T(\sigma)] = [\sigma, T^2(\sigma)] = 0$. The first equation follows trivially from (19) and $[\sigma, E] = 0$; the second one then immediately follows from time differentiation of the first and applying (15), but strangely enough the authors of [16] turned to a much longer but equivalent reasoning in an orthonormal tetrad approach to show this.

influences the divergence of the electric, but not the magnetic field. Hence $\operatorname{div} H = 0$ does not lead to direct information involving $D\rho$.

Finally notice that a counterpart for the third chain $\mathcal{C}_{H,3}$ doesn't exist either: the divergence of (33) leads to an identity, since $\operatorname{curl} E = 0$ and since (23) now reads $\operatorname{curl}\sigma = 0$!

Since $[\sigma, E] = 0$, (17)-(20) gives rise to a autonomous dynamical system $\mathcal{S}$ in ρ, θ, and two Weyl and shear eigenvalues (or combinations thereof). Analyses for $\Lambda = 0$ in an orthonormal Weyl eigentetrad approach [8] or coordinate approach [9] both led to the same conclusion, which remains valid for general Λ: for the spatially inhomogeneous Petrov type I subclass $C_{I,si}$ of the purely electric silent models, the chain $\mathcal{C}_E$ eventually translates into one or more indefinite chains of polynomial constraints on $\mathcal{S}$. In [13], four (compact) relations on the variables $\mathbf{x}$ of $\mathcal{S}$ were found, by which all these chains become identically satisfied. Hereby Λ seems to play the role of a bifurcation parameter for $\mathcal{S}$: for $\Lambda \leq 0$ and real $\mathbf{x}$, these relations imply Petrov type 0 (FLRW), while for $\Lambda > 0$ members of $C_{I,si}$ emerge. Further analysis by computer (details of which will be given elsewhere) strongly indicates that these are the *only* members of $C_{I,si}$ for $\Lambda > 0$, whereas there are probably none for $\Lambda \leq 0$. This is a generalized statement of the conjecture in [8, 9]. Still no definitive proof has been found, mainly because of the massiveness of the polynomials.

3. Anti-Newtonian universes do not exist

As outlined in Section 2.2, it is shown in this section that non-vacuum irrotational dust models with $E = 0$ imply $D\rho = 0$ (Proposition 1) and hence are FLRW (Theorem 1), i.e., that anti-Newtonian universes do not exist. Hereby, all statements and expressions are implicitly assumed to be related to, resp. live on, a small open subset U of a spacetime. Since the whole of the reasoning only involves a finite number of polynomial combinations F_i of tensorial quantities on U, one can always assume U small enough such that for all couples (F_i, F_j) either $F_i(p) = F_j(p), \forall p \in U$ (denoted by $F_i = F_j$) or $F_i(p) \neq F_j(p), \forall p \in U$ (denoted by $F_i \neq F_j$).

It should be remarked here that purely magnetic irrotational *vacua* ($E = 0, \omega = 0, p = \rho = 0$, with or without cosmological constant) were shown to be inconsistent, first by Van den Bergh [17] in an orthonormal tetrad approach, and later in a more transparent 1+3 covariant deduction (thereby generalizing the result from vacua to spacetimes with vanishing Cotton tensor) by Ferrando and Saez [18]. *Hence the vacuum case $\rho = 0$ may be excluded from the subsequent analysis.*

Some additional preparations should still be made. For the first step of the proof of Proposition 1, $\mathcal{J}^{(i)} = T^i(\mathcal{J})$ have to be calculated up to $i = 6$. Denote x_k for $\operatorname{tr}\sigma^k$ ($k \in \mathbb{N}$). By (19) with $E = 0$, $T(\sigma)$ is an element of the commutative (!) polynomial algebra (over spacetime functions) generated by σ itself, such that $T(\sigma^k) = k\sigma^{k-1}T(\sigma)$.

Taking the trace and substituting for $T(\sigma)$ one finds

$$T(x_k) = k\left(-\frac{2}{3}\theta\, x_k - x_{k+1} + \frac{1}{3}x_2 x_{k-1}\right). \tag{35}$$

Now the key remark is that *all* of the x_k for $k \geq 4$ are polynomially dependent of x_2 and x_3 (according to the formulation in [7] w.r.t. these traces, this seems to be exactly what was overlooked). Indeed, on using Newton's identities up to power three, applied on the eigenvalues of σ, the Cayley-Hamilton theorem for σ reads

$$\sigma^3 = \frac{1}{2}x_2\sigma + \frac{1}{3}x_3. \tag{36}$$

On multiplying (36) with σ^k and taking the trace (or just taking Newton's identities for powers higher than three) one gets

$$x_{k+3} = \frac{1}{2}x_2 x_{k+1} + \frac{1}{3}x_3 x_k, \quad k = 1, 2, \ldots. \tag{37}$$

Since $x_1 = \operatorname{tr}\sigma = 0$ this reads $x_4 = \frac{1}{2}{x_2}^2$ for $k = 1$. Hence for $k = 2, 3$ (35) becomes:

$$T(x_2) = -\frac{4}{3}\theta x_2 - 2x_3 \tag{38}$$

$$T(x_3) = -2\theta x_3 - \frac{1}{2}x_2^2. \tag{39}$$

Hence (38)-(39) together with (17)-(19) forms an autonomous dynamical system $\mathcal{S}_H$ in the scalar invariants ρ, θ, x_2, x_3. Starting from (29), it is easily seen that for the computation of any $\mathcal{J}^{(i)}$, one only needs the basic operations (13) and (16), together with (19) and the equations of $\mathcal{S}_H$ as substitution rules for $T(\sigma), T(\rho), T(\theta), T(x_2)$ and $T(x_3)$. All $\mathcal{J}^{(i)}$ will be elements of the module generated by $D\rho, \rho D\theta, \rho D x_2$ and $\rho D x_3$ over the (commutative) polynomial ring $R = \mathbb{Q}[\rho, \theta, x_2, x_3, \Lambda][s]$. In particular, one finds for $\mathcal{J}^{(1)}$ and $\mathcal{J}^{(2)}$:

$$\mathcal{J}^{(1)} = -\rho D x_2 - \left(\frac{3}{2}\sigma + \frac{5}{3}\theta\right)\rho D\theta - \left(\frac{1}{3}(\rho+\Lambda) + \sigma^2 + \frac{2}{3}\theta\sigma - \frac{x_2}{2} - \frac{5}{9}\theta^2\right)D\rho,$$

$$\mathcal{J}^{(2)} = 2\rho D(x_3) + \left(\frac{5}{2}\sigma + \frac{13}{3}\theta\right)\rho D(x_2) + \left(\frac{7}{6}\rho - \frac{4}{3}\Lambda + 4\sigma^2 + \frac{19}{3}\theta\sigma + 2x_2 + \frac{10}{3}\theta^2\right)\rho D(\theta)$$
$$+ \left((\frac{17}{12}\sigma + \frac{19}{18}\theta)\rho + (\frac{14}{9}\theta - \frac{1}{3}\sigma)\Lambda + 3\sigma^3 + 4\theta\sigma^2 - (\frac{x_2}{2} - \theta^2)\sigma - x_3 - \frac{8}{3}\theta x_2 - \frac{10}{9}\theta^3\right)D(\rho).$$

Hence $\mathcal{J} = \mathcal{J}^{(1)} = \mathcal{J}^{(2)} = 0$ can be solved for $\rho D\theta, \rho D x_2$ and $\rho D x_3$, which expresses them as elements of the module generated by $D\rho$ over R ¶

$$\rho D\theta = \left(\frac{1}{3}\theta - \frac{1}{2}\sigma\right)D\rho, \tag{40}$$

$$\rho D x_2 = -\left(\frac{1}{4}\sigma^2 + \frac{1}{3}\theta\sigma - \frac{1}{2}x_2 + \frac{1}{3}(\rho+\Lambda)\right)D\rho \tag{41}$$

$$= -\left(\frac{1}{4}\hat{\sigma}^2 + \frac{1}{3}\theta\sigma - \frac{5}{12}x_2 + \frac{1}{3}(\rho+\Lambda)\right)D\rho, \tag{42}$$

¶ (42) and (44) were added for comparison with the corresponding formulas in the case $\Lambda = 0$ from [7]. Apparently there was a typo regarding the coefficient of $x_2\sigma D\rho$ in (44).

$$\rho D x_3 = \left(\Lambda(\frac{1}{4}\sigma + \frac{1}{6}\theta) - \frac{3}{16}\sigma^3 - \frac{1}{8}\theta\sigma^2 + \frac{1}{8}x_2\sigma - \frac{1}{12}x_2\theta + \frac{1}{2}x_3\right) D\rho \quad (43)$$

$$= \left(\Lambda(\frac{1}{4}\sigma + \frac{1}{6}\theta) - \frac{3}{16}\hat{\sigma}^3 - \frac{1}{8}\theta\hat{\sigma}^2 + \frac{1}{8}x_2\sigma - \frac{1}{8}x_2\theta + \frac{7}{16}x_3\right) D\rho. \quad (44)$$

The result of Proposition 1 is that anti-Newtonian universes necessarily satisfy $D\rho = 0$, and hence $[\sigma, H] = 0$ from (28). As σ and H are both spatial and symmetric, this implies that (a) σH is symmetric, such that one can write (21) as

$$T(H) = -\theta H + 3\sigma H - \mathrm{tr}(\sigma H), \quad (45)$$

and (b) that σ and H commute as (1,1)-tensors. Hence $T(\sigma)$ and $T(H)$ are elements of the commutative polynomial algebra (over spacetime functions) generated by σ and H, such that $T(\sigma^i H^j) = \sigma^{i-1}H^{j-1}(i\, H\, T(\sigma) + j\, \sigma\, T(H))$ for arbitrary i and j. Substituting for $T(\sigma), T(H)$ by (19),(45) and taking the trace yields:

$$T(x_{i,j}) = -(\frac{2i}{3} + j)\theta x_{i,j} + (3j - i)x_{i+1,j} + \frac{i}{3}x_{2,0}x_{i-1,j} - j x_{1,1}x_{i,j-1}, \quad (46)$$

wherein $x_{i,j}$ is written for $\mathrm{tr}(\sigma^i H^j)$. On multiplying (36) with $\sigma^i H^j$ and taking the trace these are found to be related by

$$x_{i+3,j} = \frac{1}{2}x_{2,0}x_{i+1,j} + \frac{1}{3}x_{3,0}x_{i,j}, \quad i, j = 0, 1, 2, \ldots. \quad (47)$$

For the first step of the proof of Theorem 1, $\mathcal{K}^{(i)} = T^i(\mathcal{K})$ needs to be computed up to $i = 5$. Starting from $\mathcal{K} = x_{0,2} - \frac{1}{2}\rho x_{2,0}$, inspection of (46) for $j = 2$ learns that $\mathcal{K}^{(i)}, i = 0\,..\,5$ will be of the form $\frac{6!}{(6-i)!}x_{i,2}+$ other terms. Hence $\mathcal{K}^{(i)} = 0$ determines $x_{i,2}$ as a polynomial in $\rho, \theta, \Lambda, x_{j,2}(j < i)$ and some of the $x_{k,0}, x_{l,1}$. Substitution in a particular order of identities (47) and obtained expressions yields compact equations $u_i = 0, i = 0\,..\,5$, where

$$u_0 = x_{0,2} - \frac{1}{2}\rho x_{2,0} \quad (48)$$

$$u_1 = \frac{1}{6}x_{2,0}\rho\theta + 6x_{1,2} + \rho x_{3,0} \quad (49)$$

$$u_2 = -\frac{2}{3}x_{3,0}\rho\theta + \frac{1}{3}\rho {x_{2,0}}^2 - \frac{1}{12}x_{2,0}\rho^2 + \frac{1}{6}x_{2,0}\rho\Lambda + 30x_{2,2} - 12{x_{1,1}}^2 \quad (50)$$

$$u_3 = \frac{1}{2}\rho^2 x_{3,0} + \frac{1}{6}x_{2,0}\Lambda\rho\theta - 2{x_{2,0}}^2\rho\theta + 6x_{3,0}\rho x_{2,0} - 108x_{1,1}x_{2,1} - x_{3,0}\rho\Lambda \quad (51)$$

$$u_4 = -18{x_{1,1}}^2 x_{2,0} - {x_{2,0}}^3\rho + \frac{3}{4}{x_{2,0}}^2\rho^2 - 12\rho {x_{3,0}}^2 - \frac{4}{3}x_{3,0}\Lambda\rho\theta + 6x_{2,0}\rho\theta x_{3,0}$$
$$+ \frac{2}{9}\Lambda\rho\theta^2 x_{2,0} - \frac{1}{12}x_{2,0}\rho^2\Lambda + \frac{1}{6}x_{2,0}\Lambda^2\rho - \frac{5}{3}{x_{2,0}}^2\rho\Lambda - 216{x_{2,1}}^2 \quad (52)$$

$$u_5 = 36{x_{1,1}}^2 x_{3,0} - \frac{29}{18}{x_{2,0}}^2\Lambda\rho\theta - \frac{1}{3}x_{2,0}\Lambda\rho^2\theta - \frac{2}{3}{x_{2,0}}^3\rho\theta - 36x_{2,0}x_{1,1}x_{2,1}$$
$$- 8{x_{3,0}}^2\rho\theta + 6\rho {x_{2,0}}^2 x_{3,0} - \frac{13}{2}x_{2,0}x_{3,0}\rho^2 + \frac{5}{6}\rho^2\Lambda x_{3,0} - \frac{5}{3}x_{3,0}\Lambda^2\rho$$
$$+ \frac{10}{27}\Lambda x_{2,0}\theta^3\rho - \frac{20}{9}\Lambda\theta^2\rho x_{3,0} + 15x_{2,0}x_{3,0}\rho\Lambda + \frac{5}{6}x_{2,0}\Lambda^2\rho\theta \quad (53)$$

Remark. The calculation of $\mathcal{J}^{(i)}, i = 1\,..\,6$ and $u_i, i = 0..5$ were performed in the standard commutative environment of Maple (version 9.5), as allowed by the index-free notation and in the present situation. The used Maple code which reflects how the u_i were obtained, is explicitly given in Appendix A.

The second step of the proofs of both Proposition 1 and Theorem 1 require two independent combinations of the non-zero shear eigenvalues σ_k and of the magnetic Weyl eigenvalues H_k each ($k = 1, 2, 3$). The following choice was made:

$$s_1 := -\frac{3}{2}\sigma_1, \quad t_1 := \frac{1}{2}(\sigma_2 - \sigma_3), \tag{54}$$

$$h_1 := -\frac{3}{2}H_1 \quad k_1 := \frac{1}{2}(H_2 - H_3), \tag{55}$$

with inverse relations

$$\sigma_1 = -\frac{2}{3}s_1, \quad \sigma_2 = \frac{1}{3}s_1 + t_1, \quad \sigma_3 = \frac{1}{3}s_1 - t_1, \tag{56}$$

$$H_1 = -\frac{2}{3}h_1, \quad H_2 = \frac{1}{3}h_1 + k_1, \quad H_3 = \frac{1}{3}h_1 - k_1. \tag{57}$$

Lemma 1. For a purely magnetic ID model (a) $s_1 = 0$ implies FLRW and (b) $t_1 = cs_1$, with c a constant function, implies that the shear tensor is degenerate.

Proof. The shear tensor is a diagonalizable (1,1) tensor. Let P_k be the projector on any eigenvectorfield corresponding to the eigenvalue σ_k. Then it is straightforward to deduce that

$$T(\sigma_k) = \mathrm{tr}(P_k\, T(\sigma)) = -\frac{2}{3}\theta\sigma_k - \sigma_k^2 + \frac{1}{3}\mathrm{tr}(\sigma^2). \tag{58}$$

where the second equation is valid for the specific situation (19). From this it follows that

$$T(s_1) = \; -\frac{2}{3}\theta s_1 + \frac{1}{3}s_1^2 - t_1^2, \tag{59}$$

$$T(t_1) = \; -\frac{2}{3}t_1(s_1 + \theta). \tag{60}$$

(a) If $s_1 = 0$ then also $t_1 = 0$ by (59), such that the ID model is shear-free and hence FLRW. (b) The time derivative of $t_1 - cs_1 = 0$, substituting t_1 for cs_1, yields $c(c-1)(c+1)s_1^2 = 0$. The case $s_1 = 0$ yields FLRW, whereas $c = 0$ or $c = \pm 1$ is exactly saying that the shear tensor is degenerate, according to (56). □

Finally, the following notations and properties concerning multivariate polynomials are used. Consider the polynomial ring $\mathbb{Q}[\mathbf{x}]$, where $\mathbf{x}$ stands for the n-tuple $(x_1, \ldots, x_n)$. For two elements p_1 and p_2 of $\mathbb{Q}[\mathbf{x}]$, $\gcd(p_1, p_2)$ is written for one of the two normalized forms of their greatest common divisor (i.e., for which all coefficients are relatively prime integers); hereby $\gcd(p_1, p_2) \neq 1$, resp. $\gcd(p_1, p_2) = 1$, indicates that p_1 and p_2 have, resp. do not have, a common factor. Write $\mathrm{Res}(p_1, p_2; x_i)$ for the resultant of p_1 and p_2 w.r.t. the variable x_i. It is well known that $\mathrm{Res}(p_1, p_2; x_i)$ is a polynomial over $\mathbb{Q}$ in the variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$, which is an element of the ideal of $Q[\mathbf{x}]$ generated by p_1 and p_2, i.e., there exist polynomials $k_1, k_2 \in Q[\mathbf{x}]$ for which $\mathrm{Res}(p_1, p_2; x_i) = k_1 p_1 + k_2 p_2$. Hence, if $\mathbf{x}^0$ is a root of p_1 and p_2 (i.e., a solution of $p_1(\mathbf{x}) = p_2(\mathbf{x}) = 0$) then $(x_1^0, \ldots, x_{i-1}^0, x_{i+1}^0, \ldots, x_n^0)$ is a root of $\mathrm{Res}(p_1, p_2; x_i)$ (see e.g. [19]). The following properties are perhaps less known but sometimes help to avoid cumbersome computations at the end of a proof by resultants:

Lemma 2. (a) For $p_1, p_2 \in \mathbb{Q}[\mathbf{x}]$ and any x_i, $\mathrm{Res}(p_1, p_2; x_i)$ is the zero polynomial if and only if $\gcd(p_1, p_2) \neq 1$.
(b) When $n = 2$, and if $p_1(x_1, x_2)$ and $p_2(x_1, x_2)$ are homogeneous polynomials of their arguments, then the system $\{p_1 = 0, p_2 = 0\}$

- only has the trivial solution $(x_1, x_2) = (0, 0)$ when $\gcd(p_1, p_2) = 1$;
- is equivalent with the equation $\gcd(p_1, p_2) = 0$ when $\gcd(p_1, p_2) \neq 1$.

Proof. (a) see [19], p 158; (b) is an almost direct consequence of (a). □

With these preparations the main result may now be proved in two steps, as indicated.

Proposition 1. An ID spacetime with $E = 0$ satisfies $D\rho = 0$.

Proof. Calculating $\mathcal{J}^{(3)}, \ldots, \mathcal{J}^{(6)}$, substituting for $D\theta, Dx_2$ and Dx_3 by (40), (41) and (43) one gets four respective equations $p_i\, D\rho = 0$ with $p_i \in R = \mathbb{Q}[\rho, \theta, x_2, x_3, \Lambda][\sigma]$, $i = 1..4$, i.e., $D\rho$ is in the kernel of the (1,1)-tensor p_i. In writing out these equations in their components w.r.t. a shear eigenframe, one gets twelve equations $p_{ij} D_j \rho = 0$, $i = 1..4, j = 1..3$ (no summation over j), where $p_{ij} = p_{ij}(\rho, \theta, s_1, t_1, \Lambda)$ originates from p_i by substituting the j^{th} shear eigenvalue σ_j for σ, ${\sigma_1}^i + {\sigma_2}^i + {\sigma_3}^i$ for x_i and finally the right hand sides of (56) for $\sigma_1, \sigma_2, \sigma_3$.

Suppose $D\rho \neq 0$. Then there exists a non-vanishing component $D_j\rho$. Whitout loss of generality we may assume $D_1\rho \neq 0$. Then $p_{i1} = 0$ for $i = 1..4$. One has e.g.:

$$
\begin{aligned}
p_{11} = &-\frac{1}{3}\Lambda^2 + (\frac{8}{9}{s_1}^2 + 2{t_1}^2 + \frac{1}{6}\rho + \frac{4}{9}\theta s_1 - \frac{4}{9}\theta^2)\Lambda + (\frac{1}{3}{s_1}^2 + \frac{7}{6}{t_1}^2)\rho \\
&-4{s_1}^2{t_1}^2 - 4 s_1\theta {t_1}^2 - 3{t_1}^4 \\
p_{21} = &((-\frac{23}{18}\theta + \frac{2}{9}s_1)\rho + \frac{8}{3}{s_1}^3 - 20\theta {t_1}^2 - \frac{68}{9}\theta {s_1}^2 - \frac{52}{9}\theta^2 s_1 + \frac{40}{9}\theta^3 - 4{t_1}^2 s_1)\Lambda \\
&+(\frac{20}{9}\theta - \frac{4}{9}s_1)\Lambda^2 + (-\frac{35}{9}\theta {s_1}^2 + \frac{10}{9}{s_1}^3 - \frac{245}{18}\theta {t_1}^2 + \frac{26}{9}{t_1}^2 s_1)\rho \\
&+16{t_1}^4 s_1 - \frac{16}{3}{s_1}^3{t_1}^2 + \frac{140}{3}\theta^2 s_1 {t_1}^2 + \frac{124}{3}\theta {s_1}^2{t_1}^2 + 40\theta {t_1}^4.
\end{aligned}
$$

Now compute the resultants $q_i(\theta, s_1, {t_1}^2, \Lambda) := \mathrm{Res}(p_{11}, p_{i1}; \rho)$, $i = 2..4$ and next the resultants $r_i(s_1, {t_1}^2, \Lambda) := \mathrm{Res}(q_2, q_i; \theta), i = 3, 4$. One gets equations of the form

$$
r_3(s_1, {t_1}^2, \Lambda) \equiv \Lambda^2(3{t_1}^2 - \Lambda)^4(2{s_1}^2 + 7{t_1}^2 + \Lambda)^4 r_3^*({s_1}^2, {t_1}^2, \Lambda) = 0, \qquad (61)
$$

$$r_4(s_1, {t_1}^2, \Lambda) \equiv \Lambda^2(3{t_1}^2 - \Lambda)^4(2{s_1}^2 + 7{t_1}^2 + \Lambda)^4 s_1 r_4^*({s_1}^2, {t_1}^2, \Lambda) = 0, \quad (62)$$

with $\gcd(r_3^*, r_4^*) = 1$. If s_1 vanished then the spacetime would be FLRW according to Lemma 1 (a), hence $D\rho = 0$, which is a contradiction. Thus we still have to investigate the following four cases:

(i) $\underline{\Lambda = 0}$. One has $q_i(\theta, s_1, t_1, 0) = t_1^2 a_i(\theta, s_1, t_1) = 0, i = 2..4$, with $\gcd(a_i, a_j) = 1$ for $i \neq j$. For $P_i(s_1, {t_1}^2) = \mathrm{Res}(a_2, a_i; \theta), i = 3, 4$ one computes that $\gcd(P_3, P_4) = {t_1}^2(7{t_1}^2 + 2{s_1}^2)^2(35{t_1}^4 + 42{t_1}^2{s_1}^2 + 16{s_1}^4)$. If $t_1 \neq 0$ then s_1 and t_1 must be solutions of $\gcd(P_3, P_4) = 0$ because of Lemma 2 (b); if $t_1 = 0$ then $p_{11}(\rho, \theta, s_1, 0, 0) = \frac{1}{3}\rho {s_1}^2 = 0$. Thus both possibilities imply $\sigma = 0$, which leads to the contradiction $D\rho = 0$, as above.

(ii) $\underline{\Lambda = 3{t_1}^2}$. One has $q_i(\theta, s_1, t_1, 3{t_1}^2) = t_1^2(\theta + s_1)^2 b_i(\theta, s_1, t_1) = 0, i = 2..4$, with $\gcd(b_i, b_j) = 1$ for $i \neq j$. If $t_1 = 0$ then $\Lambda = 0$, impossible since (i) has just been excluded. One has $p_{11}(\rho, -s_1, s_1, t_1, 3{t_1}^2) = \frac{\rho}{3}\rho({s_1}^2 + 5{t_1}^2) = 0$, such that $\theta + s_1 = 0$ again leads to $\sigma = 0$ and contradiction. If $t_1 \neq 0 \neq \theta + s_1$ then $b_i(\theta, s_1, t_1) = 0$, $i = 2..4$. But for $Q_i(s_1, t_1) = \mathrm{Res}(b_2, b_i; \theta), i = 3, 4$ one computes that $\gcd(Q_3, Q_4) = 1$. Hence $\sigma = 0$ by Lemma 2 (b), contradiction.

(iii) $\underline{\Lambda = -2{s_1}^2 - 7{t_1}^2}$. This case is exactly equivalent to the vanishing of the leading coefficient of p_{11}, seen as a polynomial of ρ, i.e., one has $p_{11}(\rho, \theta, s_1, t_1, -2{s_1}^2 - 7{t_1}^2) := F(\theta, s_1, {t_1}^2)$, the exact form of which is compact but not relevant. Set $R_i(\rho, \theta, s_1, {t_1}^2) = p_{i1}(\rho, \theta, s_1, t_1, -2{s_1}^2 - 7{t_1}^2)$, $i = 2..4$, compute the resultants $U_i(\theta, s_1, t_1) = \mathrm{Res}(R_2, R_i; \rho)$ and then the resultants $V_i(s_1, t_1) = \mathrm{Res}(F, U_i; \theta)$, $i = 3, 4$. One finds $\gcd(V_3, V_4) = (7t_1^2 + 2s_1^2)^3(s_1^2 + 5t_1^2)^4$, such that $\sigma = 0$ via Lemma 2 (b), contradiction.

(iv) $\underline{r_3^* = r_4^* = 0}$. Then, according to Lemma 2 (a), ${s_1}^2$ and ${t_1}^2$ are necessarily solutions of the non-trivial homogeneous polynomial equation $\mathrm{Res}(r_3^*, r_4^*; \Lambda) = 0$. Hence the ratio $(t_1/s_1)^2$ must be constant (the case $s_1 = 0$ is excluded), which can only be 1 or 0 by Lemma 1 (b). Now one computes that $\gcd(r_3^*({s_1}^2, {s_1}^2, \Lambda), r_4^*({s_1}^2, {s_1}^2, \Lambda)) = s_1^2(3{s_1}^2 - \Lambda)$ and $\gcd(r_3^*({s_1}^2, 0, \Lambda), r_4^*({s_1}^2, 0, \Lambda)) = s_1^2\Lambda^3$, which should be zero according to Lemma 2 (b); but this is impossible as (i) and (ii) have already been excluded.

We conclude that the assumption $D\rho \neq 0$ was false and this finishes the proof. □

Theorem 1. Any ID spacetime with $E = 0$ is FLRW.

Proof. From Proposition 1 and $\mathcal{C}^{4,H} = 0$ one has $[\sigma, H] = 0$ and hence the chain $\mathcal{C}_{H,3} = (\mathcal{K}^{(i)} \equiv T^i(\mathcal{K}) = 0)_{i=0}^{\infty}$, with $\mathcal{K}$ given by (31). From the above, the $u_i, i = 0..5$ given by (48)-(53) must vanish. As $x_{2,0} = 0$ is equivalent with $\sigma = 0$, one may divide out factors which are powers of $x_{2,0}$ (or ρ) in equations $F = 0$. For any polynomial p write p^* for the polynomial p from which all such powers are taken out, and write $\mathbf{x}$ for the variable set $(x_{2,2}, x_{3,0}, x_{2,1}, x_{1,2}, x_{2,0}, x_{1,1}, x_{0,2})$. One subsequently computes $O_i(\mathbf{x},\rho,\theta) = \mathrm{Res}^*(u_2,u_i;\Lambda)$, $P_i(\mathbf{x},\rho) = \mathrm{Res}^*(u_1,O_i;\theta)$ and $Q_i(\mathbf{x}) = \mathrm{Res}^*(u_0,P_i;\rho)$ where i runs from 3 to 5. Now in the Q_i perform the substitutions $x_{i,j} = {\sigma_1}^i {H_1}^j + {\sigma_2}^i {H_2}^j + {\sigma_3}^i {H_3}^j$ for the index couples (i,j) appearing in $\mathbf{x}$, followed by the substitutions (56)-(57). This yields polynomials $R_i(s_1,t_1,h_1,k_1)$, $i = 3..5$. The case $s_1 = 0$ leads to FLRW by Lemma 1 (a), so we may assume $s_1 \neq 0$ and scale t_1 with s_1 and h_1, k_1 with s_1^2 to ease the symbolic computations. This boils down to the substitution $s_1 = 1$ in the R_i. Now for $Y_i({t_1}^2,{h_1}^2) := \mathrm{Res}(R_1(1,t_1,h_1,k_1), R_i(1,t_1,h_1,k_1); k_1), i = 2,3$ one computes that $Z := \gcd(Y_1,Y_2) = (3{t_1}^6+75{t_1}^4-15{t_1}^2+1)(3{t_1}^2+1)^{29}{h_1}^{24}$. The second factor is positive definite, and so is the first, as $75{t_1}^4 - 15{t_1}^2 + 1 = 75(({t_1}^2 - \frac{1}{10})^2 + \frac{1}{3000})$. This leaves two cases:

(i) $\underline{h_1 = 0}$. One has $R_i(1,t_1,0,k_1) = {k_1}^4 R_1^*({t_1}^2,{k_1}^2)$ a nd $R_2(1,t_1,0,k_1) = {k_1}^6 R_2^*({t_1}^2,{k_1}^2)$, where $\gcd(R_1^*,R_2^*) = 1$. Thus $\mathrm{Res}(R_1^*,R_2^*;{k_1}^2)$ is a non-trivial polynomial in the scaled variable ${t_1}^2$ by Lemma 2 (a). Hence, if $k_1 \neq 0$ then $(t_1/s_1)^2$ must be a constant, which can only be 1 or 0 by Lemma 1 (b). But one has $R_1^*(1,1,{k_1}^2) \propto {k_1}^4(-5{k_1}^2 + 528)$, $R_1^*(1,1,{k_1}^2) \propto {k_1}^6(-31{k_1}^2 + 2544)$, and $R_1^*(1,0,{k_1}^2) \propto {k_1}^4(-{k_1}^2 + 3)$, $R_2^*(1,0,{k_1}^2) \propto {k_1}^6({k_1}^2 + 3)$, which leads to contradiction. If $k_1 = 0$ then $H = 0$ and the spacetime is FLRW.

(ii) $\underline{h_1 \neq 0}$. Set $Z_i := Y_i/Z, i = 1,2$. Analogously as for $h_1 = 0$, $\mathrm{Res}(Z_1,Z_2;h_1)$ is a non-trivial polynomial in the scaled variable ${t_1}^2$ by Lemma 2(a), and hence $(t_1/s_1)^2$ must be 1 or 0 by Lemma 1 (b). But one computes that $\gcd(Z_1(1,{h_1}^2), Z_2(1,{h_1}^2)) = \gcd(Z_1(0,{h_1}^2), Z_2(0,{h_1}^2)) = 1$, i.e. $Z_1(1,{h_1}^2)$ and $Z_2(1,{h_1}^2)$, resp. $Z_1(0,{h_1}^2)$ and $Z_2(0,{h_1}^2)$, have no common solution, which leads to contradiction.

All possible cases lead to FLRW or a contradiction, and this finishes the proof. □

Remark. The reason why the analysis was first performed in scalar invariants, and not in the variables s_1, t_1, h_1, k_1 from the start, is twofold. Firstly, in Proposition 1 the use of $D(x_2), D(x_3)$ instead of $D(s_1), D(t_1)$ rules out the special case of degenerate shear (the transformation between both sets has a Vandermonde-like determinant). Secondly, and w.r.t. the chain $\mathcal{C}_{H,3}$, the compactness and low degree of the polynomials $u_i, i = 0..5$ is striking: when expressed in s_1, t_1, h_1, k_1 the number of terms increases by a factor three to four. Although eventually this is of no importance here, this may be kept in mind for other consistency analyses, when one has to make a choice for variables to be eliminated.

4. Conclusion and discussion

It was proved in the full, nonlinear theory of general relativity that purely magnetic irrotational dust (ID) models ('anti-Newtonian universes') do not exist, irrespective as to whether a cosmological constant Λ is present or not. This was conjectured in [7] for $\Lambda = 0$. The proof was mainly done in a 1+3 covariant approach, not making use of the Gauss equation for the 3-Ricci curvature of 3-surfaces orthogonal to u (which yields extra information within a tetrad formalism). The analysis for the purely magnetic models was repeated for the purely electric ones, and this summarized the main differences between both cases, as partially pointed out in [7]. The conjecture for spatially inhomogeneous Petrov type I Newtonian-like ID models from [8, 9] was properly restated in function of the sign of Λ.

As a cosmological constant can be interpreted as a negative constant pressure, ID spacetimes are related to the class of irrotational models with an equation of state. In this respect, purely electric or magnetic models for which the flowlines form a *geodesic* congruence are most closely related to the analysis here. This will be dealt with in a forthcoming paper.

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Appendix A.

The following Maple code generates the expressions (48)-(53):

$x[4,0] := 1/2 * x[2,0]^2$:
for i from 3 to 5 do
$x[i,1] := 1/2 * x[2,0] * x[i-2,1] + 1/3 * x[3,0] * x[i-3,1]$:
$CHx2[i] := x[i,2] = 1/2 * x[2,0] * x[i-2,2] + 1/3 * x[3,0] * x[i-3,2]$
od:

$K[0] := x[0,2] - \frac{1}{2} * \rho * x[2,0] : u[0] := K[0]$:
verg:=$x[0,2] = \frac{1}{2} * \rho * x[2,0]$:
for i to 5 do
$K[i]$:=factor($T(K[i-1])$):
$v[i]$:=factor(subs(verg,$K[i]$)):
vgl[i]:=isolate($v[i], x[i,2]$):
if $i >= 3$ then $u[i]$:=factor(subs($CHx2[i]$,verg,$K[i]$)):
else $u[i] := v[i]$
fi:
verg:=verg union {vgl[i]}:
od:

Herein, T is a differential operator which acts on objects $x_{i,j}$ according to (46), which obeys (17)-(19) with $E = 0$ and $\mathrm{tr}\,\sigma^2 = x_{2,0}$, and (45) with $\mathrm{tr}(\sigma H) = x_{1,1}$.

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