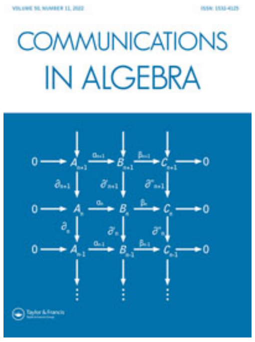

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The isomorphism problem for graph magma algebras

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ABSTRACT
(One-value) graph magma algebras are algebras having a basis $\mathcal{B} = V \cup \{1\}$ such that, for all $u, v \in V$, $uv \in \{u, 0\}$. Such bases induce graphs and, conversely, certain types of graphs induce graph magma algebras. The equivalence relation on graphs that induce isomorphic magma algebras is fully characterized for the class of associative graphs having only finitely many non-null connected components. In the process, the ring-theoretic structure of the magma algebras induced by those graphs is given as it is shown that they are precisely those graph magma algebras that are semiperfect as rings. A complete description of the semiperfect rings that arise in this fashion, in ring theoretic and linear algebra terms, is also given. In particular, the precise number of isomorphism classes of one-value magma algebras of dimension n is shown to be $\sum_{j \leq n} p(j)$ where, for any $i \in \boldsymbol{Z}^+$, $p(i)$ is the number of partitions of i. While it is unknown whether uncountable dimensional algebras always have amenable bases, it is shown here that graph magma algebras do.

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1. Introduction and preliminaries

The use of the expression *the isomorphism problem* in our title aims to be akin to its use in the classic question in the theory of group algebras whether, given a field F and two groups G and K, if the group algebras $F[G]$ and $F[K]$ are isomorphic, it is necessary for G and K to be isomorphic. Following the way that studied in [8], in which certain simple directed graphs induce algebraic structures on their sets of vertices and how these structures induce algebras over arbitrary fields [4], in this paper we aim to characterize when two graphs induce isomorphic algebras. Our best result to that avail accomplishes the goal for the semiperfect case. Interestingly, the answer does not depend on the choice of the field F.

The following definition stems back to [8] but we use a slightly different terminology, inspired by the common use of the expression *magma* to denote a set with a binary operation when no other properties are explicitly assumed, (e.g. [6, 12], etc.) Other expressions that appear in the literature for magma are *binar* and *groupoid*. The term magma was used by Serre in [19] and also appears in Bourbaki's [5].

Definition 1. Given an arbitrary set V, let G be a simple directed graph $G = (V, E)$. Note that, when we say that G is simple, we mean that, for $u, v \in V$, there is at most one edge connecting u to v, and it is, therefore, possible to identify E with a subset of $V \times V$. Consider, in addition, a symbol $0 \notin V$ and an operation on $S = V \cup \{0\}$ via the rule $uv = u$ if $(u, v) \in E$ and $uv = 0$, otherwise. We say that 0 is the annihilator element of S. We will denote this structure *the graph magma* $M(G)$ induced by G.

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As mentioned above, graph magmas were originally introduced in [8], where they were denoted *graph algebras*. The emphasis in [8], translated to our terminology, was to characterize the class of directed graphs with graph magmas isomorphic to syntactic semigroups of languages. Among other results, [8] offers a characterization of graphs such that $M(G)$ is a semigroup (i.e. those whose operation is associative). We state that characterization in Theorem 2. It is reasonable to refer to such graphs, in this context, as *associative graphs*. **Most of the time, unless otherwise clear from the context, our interest here lies solely on associative graphs.**

We reserve the expression *algebra* for an alternative use. Our *graph magma algebras* are the contracted semigroup algebras induced by associative graphs which we define next. The definition of such algebras depends on the notion of a *semigroup algebra*. Semigroup algebras are normally defined in a completely analogous fashion to group algebras. In our context, in order to guarantee that the obtained algebra has an identity element, by the semigroup algebra $F[S]$ of a semigroup S, we always actually mean the semigroup algebra of the monoid obtained by attaching an identity e to the semigroup S. Any possible awkwardness derived from the possibility of attaching an identity to a semigroup that already has one is completely absent in our context because the semigroups we are interested in are monoids only in the most trivial of cases ($M(G)$ is a monoid only when $G = (\{u\}, \{(u, u)\})$). Consequently, we will always add an identity to $M(G)$.

It could be conceivable, perhaps, to create graph-magma algebras that are not associative or lack an identity element. However, we are not pursuing that avenue at this time because our motivation to study them, is to facilitate answering questions about amenable bases over associative algebras with one of infinite dimension as mentioned below. At present time, the study of amenability has been focused solely on associative algebras with identity.

Definition 2.

1. For a field F and a monoid S with an annihilator element $0 \in S$ (i.e. an element such that $0x = 0 = x0$ for all $x \in S$) the *contracted semigroup algebra* over F induced by S is the quotient of the semigroup algebra $F[S]$ modulo the ideal consisting of the scalar multiples of the annihilator $0 \in S$.
2. Given a field F and an associative graph G, the contracted semigroup algebra induced by $M(G)$, denoted $A(G)$, is denoted the *F- graph magma algebra* induced by G.

The interested reader is encouraged to consult [7, 17], or [18], for more information about contracted semigroup algebras.

Recently, interest in graph-magma algebras grew in the context of the study of amenable bases for infinite dimensional algebras. Amenable bases are those that permit a natural extension of the A-module structure of ${}_AF^{(\mathcal{B})}$ (induced by the identification of the vector spaces ${}_FA$ and ${}_FF^{(\mathcal{B})}$) to the vector space $F^{\mathcal{B}}$. A basis $\mathcal{B}$ is amenable if, for all $r \in A$, the infinite column-finite matrix $[l_r]_{\mathcal{B}}$, that represents the F-linear map *left multiplication by r*, is also row-finite. For any F-algebra A (F a field), the multiplication in A is determined by the multiplication of elements of any basis $\mathcal{B}$. It is therefore easy to see that, in order to determine that $\mathcal{B}$ is amenable, it suffices to check that the condition is satisfied for the elements r of any basis, in particular those of $\mathcal{B}$.

With this motivation, in order to stimulate our ability to fabricate manageable examples in response to questions about amenability and related notions, it makes sense to look at algebras having bases over which the operation is as simple as possible. Notice, looking at the equivalent definition of graph-magma algebras given in Definition 3, that these algebras are indeed exemplary of the requirement sought.

The countably infinite dimensions have taken priority in the study of amenability because such algebras are guaranteed, by a theorem in [1], to have amenable bases. On the other hand, in that setting, and also in ours, we are interested in arbitrary infinite dimensions. To emphasize the potential impact of our results, we close the paper with Theorem 8, showing that all graph magma algebras have amenable bases.

An additionally extensive discussion of amenability of bases would be out of the scope of this paper; the interested reader can find more information in [1, 3, 4, 13, 14], and [15], among other references. It suffices to note that graph magma algebras have been a fertile source of examples to answer questions regarding amenability and simplicity of bases. For instance, they have been used in [4] and in [14], to show examples of algebras without simple or projective bases and are actively being used in other ongoing projects such as the consideration of symmetry of the amenability condition.

In a possibly unusual deviation from what is customary, for a graph $G = (V, E)$, we will use $|G|$ to denote $|V|$. The reason for this is that many times V and E are not be explicitly mentioned and, in any case, the contribution of G to the elements of an algebra consists precisely of the elements of V. We are confident that the readers will easily become comfortable with this unorthodox practice.

An alternative (graph-less) perspective on graph magma algebras is that they are algebras having a basis $\mathcal{B} = V \cup \{1\}$ such that, for all $u, v \in V$, $uv \in \{u, 0\}$. A similar notion is obtained by considering the so-called *two-value magma algebras* where the requirement is that there exists a basis $\mathcal{B}$ such that, for all $u, v \in \mathcal{B}$, $uv \in \{u, v\}$. In light of the similarity between these two types of so-called graph magma-algebras, sometimes one refers to our graph magma-algebras as *one-value* magma-algebras to emphasize the contrast. Both, one-valued and two-valued magmas, appear in [11] where it is shown that the families of associative operations of both kinds are submonoids of the magma monoid considered in [10]. Two-value magmas also appear in [9] where they are referred to as *groupoids with the orientation property*.

In summary, the definition of our main object of study can equivalently be phrased in the following way:

Definition 3. Let $G = (V, E)$ be a simple directed graph and F be a field. $A = A[G]$ is a *(one-value) graph magma algebra* if it has $\mathcal{B} = V \cup \{1\}$ as a basis and, for $u, v \in V$,

$$uv = \begin{cases} u & \text{if}(u, v) \in G \\ 0 & \text{otherwise} \end{cases}$$

Notice that, in spite of the similar sounding names, graph-magma algebras do not seem to be related in any way to, say, *path algebras* or other related concepts such as *Leavitt Path Algebras*, etc. In the various types of path algebras, the edges of the graph are elements of the algebra; in graph-magma algebras, it is the vertices of the graph that contribute elements for the algebra while the role of the edges is to encode their arithmetic.

It should not be much concern that one does not seem to easily find graph-magma algebras *in nature*. The introduction of graph-magma algebras, as explained above, has been rather utilitarian with the purpose to answer specific questions about amenable bases. We can list next, nonetheless, a handful of graph-magma algebras that appear as algebras of matrices. The nomenclature *bases of vertices* should be intuitive enough to get ahead of ourselves and use it here. However, any doubts that may arise can be resolved by simply looking ahead at the formal terminology introduced in Definitions 4 and 6.

Example 1. The following algebras of matrices are graph-magma algebras:

1. The algebra $A = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix} = T_2(F)$ of upper-triangular two-by-two matrices is a graph-magma algebra with basis of vertices $V = \{e_{12}, e_{22}\}$ and basis $\mathcal{B} = \{1\} \cup V = \{1, e_{12}, e_{22}\}$. A is isomorphic to $A[N_1 \oplus K_1]$.
2. The subalgebra $A = \begin{pmatrix} F & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{pmatrix}$ of $T_3(F)$ is a graph-magma algebra with basis of vertices $V = \{e_{12}, e_{22}, e_{23}, e_{33}\}$. Its basis is $\mathcal{B} = \{1\} \cup V = \{1, e_{12}, e_{22}, e_{23}, e_{33}\}$. A is isomorphic to $A[(N_1 \oplus K_1) \sqcup (N_1 \oplus K_1)]$.

3. In general, for all $n \in \mathbf{Z}^+$, the subalgebra

$$A = \begin{pmatrix} F & F & F & \dots & F \\ 0 & F & 0 & \dots & 0 \\ 0 & 0 & F & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & F \end{pmatrix}$$

of $T_n(F)$ is a graph-magma algebra with basis of vertices

$$V = \{e_{i,i+1} | 1 \le i \le n-1\} \cup \{e_{ii} | 2 \le i \le n\}.$$

A is isomorphic to $A\left[(N_1 \oplus K_1)^n\right]$, where $(N_1 \oplus K_1)^n$ denotes a graph with n connected components each isomorphic to $N_1 \oplus K_1$.

4. An analogous, infinite dimensional example would be an algebra of $\omega \times \omega$ square matrices whose nonzero entries appear only on entries of the first row (finitely many) and on the main diagonal in such a way that the sequence $\{a_{ii}\}$ is eventually constant. A typical element of A is of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & a_{22} & 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & a_{nn} & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & a & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & a & 0 & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

A is a graph-magma algebra with basis of vertices $V = \{e_{1,i+1} | i \ge 1\} \cup \{e_{ii} | i \ge 2\}$ and edges $E = \{(e_{1,i+1}, e_{i+1,i+1}) | i \ge 1\}$. $A = A[G]$ and G denotes a graph with a countably infinite number of connected components each isomorphic to $N_1 \oplus K_1$.

5. The algebra $A = \begin{pmatrix} F & 0 \\ F & F \end{pmatrix} = L_2(F)$ of lower-triangular two-by-two matrices is a graph-magma algebra with basis of vertices $V = \{e_{21}, e_{11}\}$ and basis $\mathcal{B} = \{1\} \cup V = \{1, e_{21}, e_{11}\}$. A is isomorphic to $A[N_1 \oplus K_1]$.

6. The subalgebra

$$A = \bigsqcup_{a \in F} \begin{pmatrix} F & 0 & 0 \\ F & a & 0 \\ F & 0 & a \end{pmatrix}$$

of $L_3(F)$ is a graph-magma algebra with basis of vertices $V = \{e_{11}, e_{21}, e_{31}\}$. Its basis is $\mathcal{B} = \{1\} \cup V = \{1, e_{11}, e_{21}, e_{31}\}$. A is isomorphic to $A[N_2 \oplus K_1]$.

7. In general, for all $n \in \mathbf{Z}^+$, the subalgebra

$$A = \bigsqcup_{a \in F} \begin{pmatrix} F & 0 & 0 & \dots & 0 \\ F & a & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ F & 0 & \dots & \dots & a \end{pmatrix}$$

of $L_n(F)$ is a graph-magma algebra with basis of vertices $V = \{e_{i,1} | 1 \le i \le n\}$. A is isomorphic to $A[N_{n-1} \oplus K_1]$.

8. The infinite dimensional version of the last example is an algebra of $\omega \times \omega$ square matrices whose nonzero entries appear only finitely many entries of the first column and as a constant element (except for the a_{11} value.) A typical element of A is of the form

$$\begin{pmatrix} a_{11} & 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{21} & a & 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{31} & 0 & a & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & 0 & \dots & \dots & a & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & a & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & a & 0 & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

A is a graph-magma algebra with basis of vertices $V = \{e_{i1} | i \geq 1\}$ and edges $E = \{(e_{i1}, e_{11}) | i \geq 1\}$. $A = A[G]$, where $G = N_\omega \oplus K_1$.

With Theorem 7, we manage to reach our goal and characterize all isobraic graphs (that is, those giving rise to isomorphic one-value magma algebras) that have finitely many non-null connected components. Instrumental in the characterization is the realization that the one-value magma algebras for such graphs are precisely the ones that are semiperfect rings. We expect that this realization, in fact, will be a valuable byproduct of our project, since it provides user-friendly strategies to cook up examples of semiprimary basic rings.

We also anticipate that the very concrete constructions of finite dimensional algebras offered in Proposition 7 to facilitate ongoing efforts by several researchers to classify finite rings (c.f. [16, 20], etc.). Example 7 illustrates how the result, using finite fields, provides large collections of pairwise non-isomorphic rings with a given finite cardinality.

A standard reference for semiperfect and basic rings is Chapter 27 of [2]. In particular, one can get what is needed here, by going over Theorem 27.6 and Proposition 27.10 of [2]. The next few paragraphs are our attempt to keep things self-contained as we highlight those aspects that will be needed for our narrative. The basic facts about semiprimary rings that will be needed here are the subject of Exercise 9, in Section 15 of [2], where the definition of a semiprimary ring first appears in that book.

A ring R is semiperfect if every finitely generated R-module has a projective cover. It is known that a semiperfect ring R has a decomposition as a sum of indecomposable projective left modules $R = Re_1 \oplus \dots \oplus Re_n$, where $1 = e_1 + \cdots + e_n$, and, the e_i's are orthogonal idempotents. Every indecomposable projective appears (up to isomorphism) in the family $\{Re_i | \ i = 1 \dots n\}$. If one chooses (renumbering as needed) a subset $\{Re_i | \ i = 1 \dots k\}$ of $\{Re_i | \ i = 1 \dots n\}$ including only one copy of each indecomposable projective left R-module, then $e = e_1 + \cdots + e_k$ is said to be a *basic idempotent*. A subring eRe of R, where e is a basic idempotent is said to be a *basic subring*. It can be shown that any two basic subrings of a semiperfect ring are isomorphic. A semiperfect ring R is said to be *basic* if it equals its basic subring. A ring R, with Jacobson radical $J(R) = J$ is said to be *semiprimary* if R/J is semisimple and J is nilpotent. It is known that semiprimary rings are semiperfect (as they are, in fact, right and left perfect).

Our use of module theoretic terminology is pretty standard and in agreement, for example, with [2]. In particular, every left module has a submodule, its socle, that consists of the sum of all of its simple submodules. A ring has two socles, a *left socle* and a *right socle* depending on whether it is being considered as a left or a right module over itself. Our use of the word module, for the most part, will refer to left modules unless otherwise explicitly stated. When dealing with an F-algebra A and related left A-modules, the use of the word *dimension* will always mean dimension as an F-vector space.

The following theorem highlights two facts that are essential to the understanding of Section 4.

Theorem 1. *(From Theorem 27.6, and Proposition 27.10, of [2])*

For a ring R,

1. *R is semiperfect if and only if R has a complete orthogonal set $e_1, \dots e_n$ of idempotents with each $e_i Re_i$ a local ring.*
2. *As every complete set of orthogonal idempotents contains a basic set, it is important to recognize that $e_1, \dots e_m$ is a basic set of primitive idempotents if and only if $Re_1/Je_1, \dots, Re_m/Je_m$ is an irredundant set of representatives of the simple left R-modules.*

2. Sets and bases of vertices of an algebra

As usual, two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a one-to-one correspondence φ between the sets V_1 and V_2 such that, for $u, v \in V_1$, u is incident to v in G_1 if and only if $\varphi(u)$ is incident to $\varphi(v)$ in G_2.

Theorem 2 identifies those graphs that yield associative operations in terms of their connected components. To facilitate its statement, we start by identifying, in the next definition, the suitable connected components.

Definition 4. The following graphs will henceforth be referred to as *the connected associative graphs*. They will be denoted, respectively, according to the indicated notation. For $k, p \in \{\infty\} \cup \mathbf{Z}^+$,

1. K_p denotes the complete graph on p vertices (every vertex is incident to every other vertex),
2. N_1 denotes the null graph on a single vertex (one vertex, no edges), and
3. $N_k \oplus K_p$ denotes the direct sum of N_k and K_p (in addition to the edges of K_p, every vertex of N_k is incident to every vertex of K_p. For a graph of this type, we say that the elements of N_k are the *source* vertices and the elements of K_p are the *target* vertices.

Remark 1.
1. For any set X of generating vertices for an algebra A and disjoint subsets $U, V \subset X$, we write $U \sqcup V$ to denote the full subgraph that has $U \cup V$ as vertice, when $UV = VU = 0$. In other words, if there are no edges connecting the vertices of U with those of V in either direction.
2. Consistently with the above notation, we use $\sqcup_k N_1 = N_k$ to denote a null graph on k vertices. Such a graph is referred to as a set of k *disconnected vertices*. Similarly, $K_1^{(m)}$ is called a set of m disconnected loops.
3. For the third type of connected associative graph in the above definition, if we ease the restriction that k be positive and allow $k = 0$ as an option, instead, then the family contains the first one (i.e. $N_0 \oplus K_p = K_p$). This flexibility can come in handy and should be used at will.

The following Theorem, originally from [8], specifies the graphs in which we are interested in this project.

Theorem 2. *For any directed graph $G = (V, E)$, the following conditions are equivalent:*

1. *The graph magma M(G) is associative.*
2. *For all $(x, y) \in E$ and $z \in V$, then $(x, z) \in E$ if and only if $(y, z) \in E$.*
3. *Each connected component of G is isomorphic to one of the, so called, connected associative graphs of Definition 4 (i.e. either N_1, K_m, or $N_k \oplus K_p$).*

We assume for the remainder of the article that all graphs are associative.

Corollary 1. *For any directed graph $G = (V, E)$, the following conditions are equivalent:*

1. *The graph magma of G is associative and commutative.*
2. *Each connected component of G is isomorphic to either N_1 or to K_1.*

Proof. All other connected components, as per part 3 of Theorem 2, yield elements that do not commute.

Example 2. The following are some examples of commutative graph magma algebras. Corollary 1 will be expanded in Proposition 8 and we will see that these examples are archetypes of commutative graph magma algebras.

1. $\frac{F[x]}{<x^2>} \oplus F$,
2. $\frac{F[x,y]}{<x^2,y^2,xy>} \oplus F$, and
3. $\frac{F[x_i|i\in I]}{<x_ix_j|i,j\in I>} \oplus F^m$.

Remark 2. When thinking of a vertex v in an associative graph G as elements of the algebra $A[G]$, it is clear that v is either nilpotent (when $(v,v) \notin E$) or idempotent (when $(v,v) \in E$.) We refer to the elements from a connected component of the form $N_p \oplus K_1$ as *source nilpotents* and *target idempotents* to differentiate them, respectively, from the nilpotent elements coming from copies of N_1 or idempotent elements coming from copies of K_1.

Definition 5. We say that two graphs G and H are *isobraic* if and only if they give rise to isomorphic graph algebras $A[G]$ and $A[H]$. This equivalence relation is expressed by $G \sim H$ and the equivalence class of a graph G is denoted by $[G]$.

The study of the relation of isobraicity will be significantly simplified if we embrace a slightly different, yet equivalent, approach to the subject of one-value magma algebras; we lay down the foundation for this alternative perspective next.

Definition 6. Let A be an algebra and V be a subset of A.

1. We say that V is a *set of vertices* if for every $u,v \in V$, $uv \in \{u,0\}$,
2. A set of vertices V is said to be a *spanning set of vertices* if $1 \notin <V>$, and $V \cup \{1\}$ spans A as a vector space.
3. A spanning set of vertices V is a *base of vertices* if $\mathcal{B} = \{1\} \sqcup V$ is a basis for A.
4. If V is a base of vertices, the graph $G = (V,E)$ where $(u,v) \in E$ if and only if $uv = u$, is called *the graph induced by the base of vertices*. Note that the graph G induces an algebra isomorphic to A.
5. If two bases of vertices induce isomorphic graphs, then we say that the bases of vertices themselves are isomorphic (as graphs.)

Remark 3. The expressions *spanning set of vertices* and *base of vertices* are misnomers in the sense that a set of vertices V of either kind is not really a spanning set of A; one must add the element 1 to V to obtain a spanning set or a basis.

The following facts about bases of vertices are immediate from the definition.

Proposition 1.

1. *A is a one-value graph magma algebra if and only if A has a base of vertices.*
2. *The elements of a base of vertices are either idempotent or nilpotent. Nilpotent elements are either isolated vertices or source elements in a direct sum (a connected associative graph of type 3 in Definition 4). Idempotent elements are isolated loops or target vertices of a direct sum.*
3. *Every base of vertices is a spanning set of vertices.*
4. *Since every spanning set contains a basis, an algebra A is a graph magma algebra if and only if it has a spanning set of vertices.*

The significance of the clearer perspective gained by looking at bases of vertices is highlighted in the following statements.

Proposition 2. *Let G and H be two graphs, then*

1. *Two graphs G and H are isobraic if and only if one can find a basis of vertices in* $A[G]$ *whose induced graph is isomorphic to H.*
2. *G and H are isobraic if and only if an algebra A has two bases of vertices* V_G *and* V_H *which induce graphs isomorphic, respectively, to G and H.*

Example 3. Consider the graph magma algebra A over a field of characteristic 2 induced by the graph G consisting of two isolated vertices $\{v_1, v_2\}$ and two isolated loops at w_1 and w_2. Then $V^* = \{v_1 + v_2, v_2, 1 + w_1, w_1 + w_2\}$ is a base of vertices for A. Notice that V^* induces a graph isomorphic to G; Lemma 1 shows that this is not an accident.

We conclude this section using Proposition 2 and the characterization of commutative one-value magma algebras from Corollary 1 to prove that, in the commutative case, two graphs are isobraic if and only if they are isomorphic as graphs.

Lemma 1. *Given two commutative graph algebras* $A[G]$ *and* $A[H]$, $A[G] \simeq A[H]$ *if and only if the graphs G and H are isomorphic.*

Proof. One direction is obvious; the isomorphism of the graphs G and H extends naturally to an isomorphism of the corresponding magma algebras.

Conversely, if a commutative algebra A that has bases of vertices inducing graphs G and H, by Corollary 1, they both must be of the form $N_k \sqcup K_1^{(m)}$. The Lemma will be proven if we show that the values of k and m are the same for both algebras. We will show, in fact, that any set of nilpotents has to be contained in the subspace spanned by the nilpotent elements of either one of the two bases. It then follows that, in particular, the nilpotent elements in each basis must be a basis for the subspace of A consisting of all nilpotent elements and therefore both sets must have the same cardinality.

To that effect, let $\{v_i\}_{i=1}^k$ and $\{w_j\}_{j=1}^m$ be the nilpotent and idempotent elements of a basis V of vertices for A respectively. Let V^* be another basis of vertices for A. Let n be a nilpotent element in $A[G]$, n can be expressed as

$$\alpha_0 + \sum_{i=1}^{k} \alpha_i v_i + \sum_{j=1}^{m} \beta_j w_j$$

If one evaluates the expression, $n^2 = 0$, one obtains

$$\alpha_0 = 0, \ 2\alpha_0\beta_1 + \beta_1^2 = 0, \ \ldots, \ 2\alpha_0\beta_m + \beta_m^2 = 0,$$

and, therefore, $\beta_1 = \ldots = \beta_m = 0$. In particular, this forces the nilpotent elements of V^* to be contained in the subspace generated by $\{v_i\}_{i=1}^k$. Consequently, V^* has at most k nilpotent elements. One can then repeat the argument but reversing the roles of V^* and V and conclude that both bases have the same number of nilpotent elements.

In the finite dimensional setting, this suffices to see that the number of idempotent elements in the two bases must also be the same, which would conclude our proof. For the infinite case, however, further analysis is needed. First, a word of caution: notice, in contrast, looking at Example 3, that it is not necessarily the case that the set of idempotents is generated by the idempotent elements in either basis. So our proof will have to follow a slightly different path.

Let $\{v_i\}$ and $\{w_j\}$ be the nilpotent and idempotent elements of a basis of vertices for A respectively. Let x be an idempotent element in $A[G]$, x can be expressed as

$$\alpha_0 + \sum_{i=1}^{k} \alpha_i v_i + \sum_{j=1}^{m} \beta_j w_j.$$

As we evaluate $x^2 = x$, we obtain:

$$\alpha_0^2 = \alpha_0,\ 2\alpha_0\alpha_1 = \alpha_1,\ \ldots,\ 2\alpha_0\alpha_k = \alpha_k,\ 2\alpha_0\beta_1 + \beta_1^2 = \beta_1,\ \ldots,\ 2\alpha_0\beta_m + \beta_m^2 = \beta_m.$$

The first equation forces $\alpha \in \{0, 1\}$. In both cases we get $\alpha_1 = \ldots = \alpha_k = 0$ and

$$2\alpha_0\beta_1 + \beta_1^2 = \beta_1,\ \ldots,\ 2\alpha_0\beta_m + \beta_m^2 = \beta_m.$$

So, idempotents must be of the form

$$\alpha_0 + \sum_{i=1}^{m} \beta_j w_j.$$

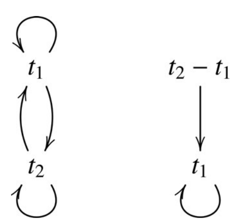
and, in particular, the idempotent elements in V^* must be contained in the subspace generated by $\{1\} \sqcup \{w_j\}_{j=1}^{m}$. However, the space spanned by V^* does not contain 1 (by definition of basis of vertices) and, therefore, that space must be a proper subspace of $<\{1\}\sqcup\{w_j\}_{j=1}^{m}>$. That forces the cardinality of the set of idempotent elements in V^* to be at most m. Repeating the above argument with the roles of V and V^* reversed yields the claimed equality.

3. Isobraic graphs for non-commutative graph magma algebras: the isobraicity class of a connected graph

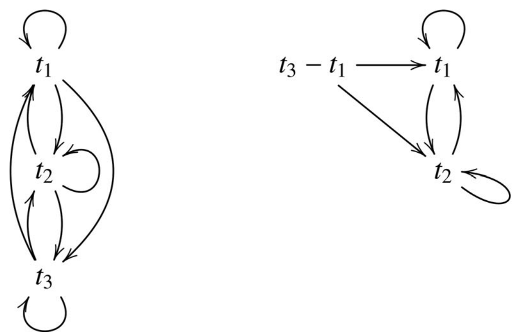
As a motivation, we start out with a couple of examples to illustrate that, when $m > 2$, the isobraicity class of K_m is not simply its isomorphism class.

Example 4.

1. For $m = 2$ the class $[K_2]$ also contains $N_1 \oplus K_1$. We use the labels of the vertices in the original complete graph to refer to the elements of the second basis of vertices. The edges serve to indicate the relations among the vertices; checking that the relations hold is straightforward.



2. For $m = 3$ the class $[K_3]$ also contains $N_1 \oplus K_2$ and $N_2 \oplus K_1$. We label the various bases of vertices in terms of the vertices in the original complete graph. The edges serve to indicate the relations among those vertices. Checking that the relations hold is straightforward.



The examples illustrate clearly the pattern to be followed in the proof of our next result. Notice that K_1 and N_1 are connected graphs with one vertex each which are not isobraic, which explains the need for the hypothesis requiring more than one vertex.

Proposition 3. *Any two connected graphs with more than one vertex are isobraic when they have the same number of vertices.*

Proof. It suffices to show that, when $m \geq 2$, the complete graph K_m is isobraic to every graph of the form $N_k \oplus K_p$ whenever $k+p=m$ with $p \geq 1$. Notice that k, m, and p may be infinite (possibly uncountable.) To that effect, let V be the basis of vertices for $A(K_m)$ consisting of the vertices of K_m and let $K \sqcup P$ be a partition of V such that $|K| = k$ and $|P| = p$. Fix an element $v \in P$ and let $W = \{w - v | w \in K\}$. The set $W \sqcup P$ is a basis of vertices isomorphic to $N_k \oplus K_p$.

Proposition 4. *If two graphs G and H are, respectively, of the forms*

$$\bigsqcup_{i \in I} G_i \text{ and } \bigsqcup_{i \in I} H_i,$$

and, for all $i \in I, G_i$ and H_i are pairwise isobraic connected associative graphs, then G is isobraic to H.

Proof. Consider $A[G]$ and then, following the patterns established in the proof of Proposition 3, find inside each subspace of the form $< G_i >$ a linearly independent set of vertices V_i that spans $< G_i >$, and whose graph is isomorphic to H_i. The (necessarily disjoint) union of the V_i, $i \in I$, is the desired basis of vertices.

We aim next to show that a connected graph cannot be isobraic to a disconnected one, which will complete our characterization of the isobraicity class of K_m for all m (finite or infinite, possibly uncountable.) We will need to consider quotients of various graph magma algebras, therefore the following Lemma is a welcome result.

Lemma 2. *Every quotient A/I of a graph magma algebra A with $I \neq A$ is itself a graph magma algebra.*

Proof. Let A be a graph magma algebra with basis of vertices V and let I be an ideal of A. Then A/I has a spanning set of vertices $\bar{V} = \{x + I | x \in V\}$. Consequently, by virtue or Proposition 1, A/I is a graph magma algebra.

Lemma 3. *For any basis of vertices $V = G \sqcup H$ of a graph magma algebra A, $\langle G \rangle$, the subspace of A spanned by G, is an ideal of A.*

Proof. Straightforward calculations show that this is indeed the case.

Theorem 3.

1. *A graph of the form $N_1 \sqcup G$ is not isobraic to any graph having only idempotent vertices.*
2. *If $m < p$ then $N_m \sqcup G$ is not isobraic to $N_p \sqcup G^\star$ where G is generated by idempotents.*

Proof.

1. We will show that any idempotent element in $A[N_1 \sqcup G]$ belongs to the subspace generated by $G \cup \{1\}$ and, therefore, no set of idempotents can be a spanning set for $A[N_1 \sqcup G]$. Let u be the unique vertex of N_1 and let $e^2 = e \in A[N_1 \sqcup G]$, then $e = \alpha + \beta u + x$, where $x \in < G >$. It follows that $e^2 = \alpha^2 + 2\alpha\beta u + y$, where $y \in < G >$. As $e = e^2$, it follows that $\alpha \in \{0, 1\}$. Since $2\alpha\beta = \beta$, either option for α implies that $\beta = 0$, confirming our claim
2. Suppose $A[X] = A[Y]$, where X and Y are bases of vertices which are isomorphic as graphs, respectively, to $N_m \sqcup G$ and $N_p \sqcup G^\star$. Write $X = N_X \sqcup G_X$ and $Y = N_Y \sqcup G^\star_Y$ in such a way that $N_X \simeq N_m$, $N_Y \simeq N_p$, $G_X \simeq G$ and $G^\star_Y \simeq G^\star$.
   It follows then that
   $$\frac{A[X]}{\langle N_X \rangle} = \frac{A[Y]}{\langle N_X \rangle}.$$
   As mentioned in Lemma 2, the resulting quotient is also a graph magma algebra. Now, the left hand side expression yields a basis of vertices of
   $$\{g + \langle N_X \rangle |\ g \in G_X \cup \{1\}\},$$
   with only idempotent vertices. On the other hand, the right hand side yields the spanning set of vertices
   $$\mathcal{S} = \{a + \langle N_X \rangle |\ a \in N_Y \sqcup G^\star_Y \sqcup \{1\}\}$$
   for $\frac{A[Y]}{\langle N_X \rangle}$. Because $m < p$, at least one of the elements $a + \langle N_m \rangle$ with $a \in N_p$ is non-zero. That implies that $(\mathcal{S})$ contains a basis of vertices $\mathcal{B}$ of the form $N_1 \sqcup H$, which is a contradiction to the first part of the Theorem.

Remember, as pointed out in the introduction, that when dealing with an F-algebra A and related left A-modules, the use of the word *dimension* will always mean dimension as an F-vector space.

Lemma 4. *If $G = K_{m_1} \sqcup K_{m_2} \ldots \sqcup K_{m_k}$ such that $m_j \geq 2$, for all $j \in \{1, \ldots, k\}$, and $A = A[G]$, then the codimension of $Soc(A)$ (i.e. the dimension of $\frac{A}{Soc(A)}$) equals k.*

Proof. For $1 \leq j \leq k$, label the vertices of K_{m_j} as $B_j = \{v_{j1}, \ldots, v_{j,m_j}\}$. Notice that $B = \{1\} \cup_{j=1}^k B_j$ is a basis for A. Consider, for each value of j, the alternative basis $C_j = \{v_{j1}\} \cup \{v_{j2} - v_{j1}, v_{j3} - v_{j2}, \ldots, v_{j,m_j} - v_{j,m_{j-1}}\}$ for the span of B_j. It follows that $C = \{1\} \cup_{j=1}^k C_j$ is another basis for A. Each C_j is the union of $\{v_{j1}\} \cup S_j$, where $S_j = \{v_{j2} - v_{j1}, v_{j3} - v_{j2}, \ldots, v_{j,m_j} - v_{j,m_{j-1}}\}$, is a basis for the socle of the span of B_j. It is straightforward to confirm that
$$A = A(1 - (v_{11} + v_{21} + \cdots + v_{k1})) \oplus Av_{11} \oplus Av_{21} \oplus \ldots \oplus Av_{k1}$$
and
$$soc(A) = A(1 - (v_{11} + v_{21} + \cdots + v_{k1})) \oplus < S_1 > \oplus < S_2 > \oplus \ldots \oplus < S_k > .$$
Consequently,
$$\frac{A}{SocA} \simeq \frac{Av_{11}}{< S_1 >} \oplus \ldots \oplus \frac{Av_{k1}}{< S_k >}.$$

Corollary 2. *If $K_{m_1} \sqcup K_{m_2} \ldots \sqcup K_{m_k}$ is isobraic to $K_{n_1} \sqcup K_{n_2} \ldots \sqcup K_{n_l}$ and all m_i and n_j are bigger than or equal to 2, then $k = l$*

Proof. By Lemma 4, the socle of an algebra A with bases of vertices isomorphic, respectively, to $K_{m_1} \sqcup K_{m_2} \ldots \sqcup K_{m_k}$ and to $K_{n_1} \sqcup K_{n_2} \ldots \sqcup K_{n_l}$ would have codimension $k+1 = l+1$.

4. Graphs with finitely many non-null connected components

Notice that, unlike in the previous sections where we consistently referred to $A[G]$ as A, to emphasize its algebra structure, in this section we will favor the use of $R = A[G]$ so that the translation from our references in [2], where the results are always about a ring R, is easier. Remember that a one sided ideal I of a ring R is *nilpotent* if $I^k = 0$ for some k. The smallest such value of k is called *the nilpotency* of I.

Our main results will follow after a series of preliminar results:

Lemma 5. *If $V \subset A = R$ is a basis of vertices, whose graph G consists of connected components of the form N_1, K_1, and $N_p \oplus K_1$, and $v, w \in V$ (with p possibly (countably or uncountably) infinite), then:*

1. *If v and w are nilpotent (i.e. coming from a copy of N_1 or a source vertex from a copy of $N_p \oplus K_1$), then $Rv = Fv$ and $Rw = Fw$ are subspaces of R of dimension 1, and consequently, simple left modules. Furthermore, the left modules Rv and Rw are isomorphic and the left ideals Rv and Rw are nilpotent with nilpotency 2.*
2. *If v is a source in a component of the form $N_p \oplus K_1$ and w is its target, then $Rv \subset Rw =< N_p \oplus K_1 >$, the subspace of R having the $p+1$ elements of $N_p \oplus K_1$ as a basis.*
3. *If v, w are isolated idempotent vertices (i.e. they come from two distinct connected components of the form K_1) then $Rv = Fv$ and $Rw = Fw$ are non-isomorphic projective simple left modules of dimension 1.*
4. *If w is an idempotent then wRw is a local ring.*
5. *If v and w are distinct target idempotents then Rv/Jv and Rw/Jw are non-isomorphic simple left modules of dimension 1.*

Proof.
1. A typical element of R is of the form $r = \alpha + \ \{\text{a F}-\mathit{linearcombinationofverticesfrom}\ \}G$. Therefore, $rv = \alpha v$ and $rw = \alpha w$, which confirms our claims.
2. As $vw = v$, $Rv \subset Rw$, confirming the first claim and the inclusion $< N_p \oplus K_1 > \subseteq Rw$. For the converse, observe that, since a typical element r of R equals

$$\alpha + \{\text{a } F\text{-linear combination of vertices from } N_p \oplus K_1\}$$
$$+\{\text{a } F\text{-linear combination of vertices from the other connected components of } G\},$$

   it necessarily follows that $rw = \alpha w + \ aF$-linear combination of vertices from $N_p \oplus K_1$.
3. A typical element of R is of the form $r = \alpha + \beta v + \gamma w + \ aF$-linear combination of the remaining vertices from G. Therefore, $rv = (\alpha + \beta)v$ and $rw = (\alpha + \gamma)w$, which confirms our claims.
4. We saw above that when w is idempotent then $Rw = Fw$ (if $w \in K_1$) or $Rw =< N_p \oplus K_1 >$. In either case, $wRw = FwF$, a field, proving our claim.

5. For any idempotent target z and ring element $r \in R$, as

$$r = \alpha + \gamma_z z + \sum_{\{t \in V | tz = t\}} \gamma_t t$$
$$+ \{\text{a } F\text{-linear combination of vertices from the other connected components of } G\},$$

then

$$rz = (\alpha + \gamma_z)z + \sum_{\{t \in V | tz = t\}} \gamma_t t.$$

It follows that $r(z + Jz) = (\alpha + \gamma_z)(z + Jz)$. If $\varphi : Rv/Jv \to Rw/Jw$ were an isomorphism with $\varphi(v + Jv) = \beta w + Jw$ then, for $r = v$, $r(v + Jv) = v + Jv$ yet $r\varphi(v + Jv) = r(\beta w + Jw) = 0$. Therefore, Rv/Jv and Rw/Jw are not isomorphic.

Proposition 5. *If the graph G is of the form $N_t \sqcup (K_1)^m \sqcup (N_{p_1} \oplus K_1) \sqcup \ldots \sqcup (N_{p_k} \oplus K_1)$, where t may be zero, finite, or (countably or uncountably) infinite, m is finite (possibly zero), k is finite (possibly zero) and each p_i is non-zero but possibly (countably or uncountably) infinite, then*

1. *$R = A[G]$ is the direct sum of $m + k + 1$ mutually non-isomorphic indecomposable projective left modules $Re \oplus Re_1 \oplus \ldots Re_m \oplus Rv_1 \oplus \ldots \oplus Rv_k$, where $\{e_1, \ldots e_m\}$ are the vertices in the m isolated connected components of the form K_1, and $\{v_1, \ldots, v_k\}$ are the k target vertices in the components of the form $N_p \oplus K_1$, and e is given by $e = 1 - (e_1 + \cdots + e_m) - (v_1 + \cdots + v_k)$.*
2. *There is a simple left module S such that $Soc(Re) \simeq S^{(t)}$, and, for each $i = 1, \ldots k$, $Soc(Rv_i) \simeq S^{(p_i)}$. So, the left socle of R is isomorphic to $S^{(t+p_1+\cdots+p_k)} \oplus Re_1 \oplus \ldots Re_m$.*

Proof.

1. A typical element of R is a linear combination of 1 and the elements of the given basis. A different basis may be obtained by replacing 1 with e (as in the statement.) By conditions 2 and 3 in the previous Lemma, the space spanned by any of the connected components may be rewritten as Rv where v is the only idempotent vertex in that component. Consequently, the decomposition $R = Re \oplus Re_1 \oplus \ldots Re_m \oplus Rv_1 \oplus \ldots \oplus Rv_k$ does indeed hold. By condition 4 in Lemma 5, each direct summand, except possibly the first one, is indecomposable, as claimed. It only rests to show that Re is indeed indecomposable; to that effect, we show that its endomorphism ring, eRe, is local. To show that, in fact, eRe is a field (isomorphic to F), we consider an arbitrary element $r = [\alpha e + \text{a linear combination of the vertices in G}]$ in R, then $er = re = ere = \alpha e$.
2. If the graph G has no nilpotent elements, then every Rv is simple, as is $S = Re$. In that case, the result holds as $R = Soc(R) = S^{(1+0+\cdots+0)} \oplus Re_1 \oplus \ldots Re_m$. Otherwise, if there is at least one nilpotent element, v, denote $S = Rv$. From conditions 1 and 2 in Lemma 5, and because of the linear independence of the nilpotent elements in each connected component that has them, for an idempotent vertex w, if Rw is not simple then $Soc(Rw)$ has one vector-space summand per nilpotent vertex in the basis, so the result follows by a dimension considerations.

The following Theorem characterizes one-value magma algebras with finitely many non-null connected components in ring-theoretic terms.

Theorem 4. *Let $R = A[G]$ be a one-value magma-algebra generated by the graph G. Then the following are equivalent:*

1. *G has only finitely many non-null connected components,*
2. *R is a semiperfect ring, and*

3. *R has only finitely many simple left modules up to isomorphism.*

Proof. In light of the characterization of semiperfect rings in Theorem 1, the implication $1 \Rightarrow 2$ follows from condition 1 in Proposition 5. The implication $2 \Rightarrow 3$ is a well-known fact about semiperfect rings. In fact, for a semiperfect ring $R = Re_1 \oplus \ldots \oplus Re_k$, the set of quotients $\{Re_i/Je_i\}_{i=1}^{k}$ contains every simple left module up to isomorphism. One can prove $3 \Rightarrow 1$ by mimicking the proof of condition 5 in Lemma 5.

Example 5. Case 4 in Example 1 is not semiperfect because it has infinitely many non-isomorphic simple right modules, namely, for $i \geq 2$,

$$S_i = \{\text{row-sequences that are zero everywhere except possibly in the } i\text{-th entry}\}.$$

Our next Theorem characterizes precisely which semiperfect algebras can occur as one-value magma algebras.

Theorem 5. *Let R be an algebra over the field F. Then, the following two conditions are equivalent:*

1. *There exists a graph G with finitely many non-null connected components such that $R = A[G]$, the one-value magma algebra induced by G, and*
2. *R is a basic semiprimary ring with indecomposable decomposition $R = Re_0 \oplus Re_1 \oplus \ldots \oplus Re_m$, where, for each $i = 0, \ldots, m$, d_i denotes $\dim Re_i$, and S_i denotes the simple module $\frac{Re_i}{Je_i}$, such that*
   (a) $J^2 = 0$,
   (b) *for all* $i = 0, \ldots, m$, $\dim S_i = 1$,
   (c) *for all* $i = 1, \ldots, m$, $e_iJe_i = 0$, *while* $e_0Je_0 = Je_0$, *and,*
   (d) *for all* $i = 1, \ldots, m$, *if* $Re_i \neq Je_i \neq 0$, *then* $Je_i = Soc(Re_i) = [S_0]^{(d_i - 1)}$.

Proof. For the implication $1 \Rightarrow 2$, you may assume, without loss of generality, that $G = N_t \sqcup (K_1)^m \sqcup (N_{p_1} \oplus K_1) \sqcup \ldots \sqcup (N_{p_k} \oplus K_1)$, where t may be zero, finite, or (countbaly or uncountably) infinite, m must be finite (possibly zero), k is finite (possibly zero) and each p_i is non-zero but possibly (countably or uncountably) infinite. As in Proposition 5, and Theorem 4, R is semiperfect and has a unique decomposition as a direct sum of indecomposable projectives $R = Re_0 \oplus Re_1 \oplus \ldots \oplus Re_m$. In order to assert that $J^2 = 0$, it suffices to show that each $(Je_i)^2 = 0$.

For the reverse implication, $2 \Rightarrow 1$, the case when each Re_i is simple corresponds to a graph with m copies of K_1, when some Re_i is not simple, one can distinguish Re_0 as the one projective indecomposable whose top is isomorphic to the simples in the socle of Re_i. Consider then the set V consisting of the idempotents $\{e_1, \ldots, e_m\}$ and the union of all bases for the subspaces Je_j, $j = 1, \ldots, m$. Straightforward calculations and dimension considerations confirm that V is indeed a basis of vertices.

5. Further results, examples, and applications

Next, we include a couple of additional results together with some corresponding examples. First, in Proposition 6, we describe the right indecomposable decomposition of a semiperfect one-value magma algebra $R = A[G]$. Our intention in doing so is to point out that the right decomposition of R is not as descriptive of G as its left decomposition is. Then, in Proposition 7, we determine how many pairwise non-isomorphic one-value magma algebras of a given finite dimension exist.

Proposition 6. *Let R be the semiperfect one-value magma algebra over the field F, induced by a graph G with finitely many non-null connected components. Then R is a direct sum of simple right* *modules with dimension 1 plus, possibly, one additional indecomposable projective right module eR with essential homogeneous socle and such that, if* $S = \frac{eR}{Soc(eR)}$, *then* $\dim(S) = 1$ *and every simple submodule of Soc(eR) is isomorphic to S.*

Proof. Let $V = \{e_1, ..., e_m\} \sqcup \{x_1, ...x_{n-m}\}$ where the e_i are idempotent and the x_j are nilpotent. Then, $R = Re \oplus Re_1 \oplus ... \oplus Re_m$, as in the previous theorems. We prove first that for $i \in \{1, ...m\}$, e_iR is a simple right ideal of dimension 1. This follows because a typical element of R is of the form $r = \alpha + \beta_i e_i +$ a linear combination of the other vertices from V. Therefore, $e_ir = (\alpha + \beta)e_i \in < e_i >$, the subspace of A generated by e_i. Hence, $e_iR =< e_i >$.

Furthermore, if $1 \leq i \neq j \leq m$, then e_iR is not isomorphic to e_jR. Otherwise, say $\varphi : e_iR \to e_jR$ is said isomorphism and $\varphi(e_iR) = \gamma e_j$, for some $0 \neq \gamma \in F$. Then $\varphi(e_i) = \varphi\left(e_i^2\right) = (\varphi(e_i))e_i = \gamma e_j e_i = 0$, a contradiction.

Next, we must show that, for every nilpotent vertex x, $xR \subseteq eR$. To that effect, remember that $e = 1 - e_1 - ... - e_m$, and therefore $ex = x \in eR$. In fact, since xR is nilpotent, then $xR \subseteq eJ$.

The fact that xR is simple and congruent to eR/eJ, can be appreciated by comparing the effect of multiplying x with an arbitrary element $r \in R$ and the effect of the same multiplication on $e + eJ \in eR/eJ$. Consider an arbitrary $r = \alpha + \beta_i e_i +$ a linear combination of the vertices from V, then $xr = \alpha x$. Likewise, $(e + eJ)r = \big((e + eJ)(\alpha + \beta_i e_i +$ a linear combination of the ***nilpotent*** vertices from $V\big)$, because e is orthogonal to e_i, for $i = 1, ...m$. But, since the nilpotent vertices are contained in eJ, $(e + eJ)r = \alpha(e + eJ)$.

These results imply that all right ideals of the form vR, v a nilpotent vertex, are of dimension 1, as follows. Since they are all isomorphic, $\dim xR = k$, is shared by all nilpotent vertices and $k = dim(eR/eJ)$, as well. Since eJ contains $\{x_1, ..., x_{n-m}\}$, then $< x_1 > \oplus ... \oplus < x_{n-m} > \subset eJ$. It follows that $\dim(eR) \geq (n - m)k$ and, therefore, $n + 1 = \dim(R) \geq k(n - m) + m$. That implies that $k = 1$ and, because $< x_1 > \oplus ... \oplus < x_{n-m} >= x_1R \oplus ... \oplus x_{n-m}R$, confirms the statement of the theorem.

Theorem 6. *The following conditions are equivalent for a graph G with N vertices such that* $R = A[G]$ *is semiperfect:*

1. *R is right noetherian,*
2. *R is left noetherian,*
3. *R is right artinian,*
4. *R is left artinian,*
5. *R is finite dimensional, and*
6. *G has finitely many vertices.*

Proof. The dimension of R is $N + 1$. So R is finite dimensional if and only if G has finitely many vertices. As we have seen, R is semiperfect if and only if it has finitely many, say m, non-null components. A decomposition of R as a sum of projective indecomposable right or left modules will consist of $m + 1$ summands. Theorem 5 and Proposition 6 describe the lattices of left and right ideals of the ring R in terms of simple left and right ideals. According to these descriptions, considering that the only vertices that might produce principal ideals not in the socle are the idempotent ones, the dimension of either socle of R is at least $N - m - 1$. Therefore, because m is finite, the dimension of R is infinite if and only if either socle (both socles) is (are) infinite dimensional. Since both the right and left socle are made up of simple modules of dimension 1, infinite dimensional implies that the left and right socles of R are direct sums of infinitely many simple modules, proving our claim.

To illustrate the previous results, we look at the left and right decompositions of the pertinent instances listed in Example 1.

Example 6. Numbering the various graph-magma algebras of matrices consistently with Example 1, we offer here, when appropriate, their left and right decompositions as direct sums of projective indecomposables.

1. The left decomposition of $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ is $Re = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}$, $Re_1 = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$.

   Its right decomposition is $eR = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, $e_1R = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$.
2. The left decomposition of $R = \begin{pmatrix} F & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{pmatrix}$ is $Re = \begin{pmatrix} F & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $Re_1 = \begin{pmatrix} 0 & F & 0 \\ 0 & F & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $Re_2 = \begin{pmatrix} 0 & 0 & F \\ 0 & 0 & 0 \\ 0 & 0 & F \end{pmatrix}$. Its right decomposition is $eR = \begin{pmatrix} F & F & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $e_1R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $e_2R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & F \end{pmatrix}$.
3. For all $n \in \mathbf{Z}^+$, the algebra

$$R = \begin{pmatrix} F & F & F & \dots & F \\ 0 & F & 0 & \dots & 0 \\ 0 & 0 & F & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & F \end{pmatrix}$$

   has left decomposition $Re = \begin{pmatrix} F & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix}$, and, for $1 \le i \le n-1$, Re_i is all zero except possibly for the first and the $i+1$ columns. Its right decomposition is $eR = \begin{pmatrix} F & F & F & \dots & F \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix}$, and, for $1 \le i \le n-1$, e_iR is all zero except possibly for the $(i+1, i+1)$ entry.
4. Proposition 6 does not apply to the fourth item in Example 1 because that ring is not semi-perfect (see Example 5.)
5. The left decomposition of the basic algebra $R = \begin{pmatrix} F & 0 \\ F & F \end{pmatrix}$ consists of $Re_1 = \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$, and $Re = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$. Its right decomposition is $e_1R = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}$, and $eR = \begin{pmatrix} 0 & 0 \\ F & F \end{pmatrix}$.
6. The left decomposition of the basic algebra

$$R = \bigsqcup_{a \in F} \begin{pmatrix} F & 0 & 0 \\ F & a & 0 \\ F & 0 & a \end{pmatrix}$$

   consists of $Re_1 = \begin{pmatrix} F & 0 & 0 \\ F & 0 & 0 \\ F & 0 & 0 \end{pmatrix}$, and $Re = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} | a \in F \right\}$. Its right decomposition is $e_1R = \begin{pmatrix} F & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $eR = \sqcup_{a \in F} \begin{pmatrix} 0 & 0 & 0 \\ F & a & 0 \\ F & 0 & a \end{pmatrix}$.
7. For all $n \in \mathbf{Z}^+$, the left decomposition of the basic algebra

$$R = \bigsqcup_{a \in F} \begin{pmatrix} F & 0 & 0 & \dots & 0 \\ F & a & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ F & 0 & \dots & \dots & a \end{pmatrix} \subset M_n(F)$$

consists of $Re_1 = \begin{pmatrix} F & 0 & 0 & \dots & 0 \\ F & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ F & 0 & \dots & \dots & 0 \end{pmatrix}$, and $Re = \left\{ \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & a & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & a \end{pmatrix} | a \in F \right\}$. Its right decomposition is $e_1R = \begin{pmatrix} F & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix}$, and $eR = \sqcup_{a \in F} \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ F & a & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ F & 0 & \dots & \dots & a \end{pmatrix}$.

8. The eighth item in Example 1 is the infinite analog of item 7 and exhibits similar decompositions. Its left decomposition is $Re_1 = \begin{pmatrix} F & 0 & 0 & \dots & 0 & \dots \\ F & 0 & 0 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ F & 0 & \dots & \dots & 0 & \dots \\ 0 & 0 & 0 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$, and $Re = \left\{ \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & a & 0 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & a & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} | a \in F \right\}$. Its right decomposition is $e_1R = \begin{pmatrix} F & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 0 & \dots \end{pmatrix}$, and $eR = \sqcup_{a \in F} \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ F & a & 0 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ F & 0 & \dots & \dots & a & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$.

Note that this example is not Artinian or noetherian on either side, as explained in Theorem 6.

Another consequence of our characterization of semiperfect graph magma algebras in the previous section is a converse of Proposition 4 that holds for graphs with finitely many non-null connected components.

Theorem 7. *If G has a finite number of non-null connected components and G is isobraic to H then H has a finite number of non-null connected components and there is a one-to-one correspondence between the components of G and H in such a way that the corresponding components are isobraic.*

Proof. For $m \in \mathbf{Z}^+$, and without loss of generality, in light of Propositions 3 and 4, let $G = N_p \sqcup \sqcup_{i=1}^m (N_{p_i} \oplus K_1)$. Likewise, let, $H = N_q \sqcup_{j \in I} (N_{q_j} \oplus K_1)$. Since $R = A[G]$ is semiperfect, by Theorem 1, then $A[H] \cong R$ has only finitely many simple left modules, which implies that I is finite, say $|I| = k$.

Proceeding as in the proof of Theorem 4, $A[G]$ may be decomposed as a sum of $m+1$ indecomposable projective left modules $Re \oplus Re_1 \oplus \dots \oplus Re_m$ and $A[H]$ as a sum of $|I|+1$ indecomposable projective left modules $Rf \oplus Rf_1 \oplus \dots \oplus Rf_k$. But the number of indecomposable projective summands in a semiperfect module is invariant and, therefore, $k = m$. In fact, the uniqueness (up to isomorphism) of such decompositions for semiperfect rings, implies that, upon reordering, the projective indecomposable modules in the G-decomposition are isomorphic to the corresponding ones in the H-decomposition.

When the radical J of R is zero, R is a direct sum of copies of the field F and therefore, as in Proposition 8, for $1 \leq i \leq m$, $p_i = 0$. This, in turn, forces each q_j also to be zero and therefore the result follows.

When $J \neq 0$, Re (and Rf) are distinct in that Re/Je and Rf/Jf embed in JR. So Re and Rf must correspond to one another in the correspondence. For the remaining projective indecomposable summands in the two decompositions of R, since we did not have a preconceived order for I at the beginning of this narrative, for simplicity let us say that, for $1 \leq i \leq m$, $Re_i \cong Rf_i$. But, as seen in the proof of Theorem 4, Re_i is induced by the i-th non-null connected component of G, in such a way that the uniform dimension of the socle of Re_i equals p_i and, likewise, Rf_i is induced by the i-th non-null connected component of H, in such a way that the uniform dimension of $Soc(Rf_i)$ equals q_i. Consequently, $q_i = p_i$. In light of Propositions 3 and 4, this suffices now to arrive at the conclusion of our statement.

The following Proposition calculates the number of graph magma algebras of any given finite dimension up to isomorphism.

Proposition 7. *For an arbitrary $n \in \mathbf{Z}^+$, there exist exactly N isomorphism classes of one-value magma algebras of dimension $n+1$, where $N = 1 + \sum_{j \leq n} p(j)$, and, for any positive integer i, $p(i)$ denotes the number of partitions i.*

Proof. Let $j \leq n$ and $i_1 \leq i_2 \leq \ldots \leq i_k$ be a partition of j and consider the graph

$$G = (N_1)^{n-j} \sqcup \bigsqcup_{t=1}^{k} K_{i_t}.$$

The algebra $R = A[G]$ has dimension $n+1$ and any one-value magma algebra of dimension $n+1$ can be obtained in this way, with the exception of the graph magma algebra generated by $(N_1)^n$, whose absence explains the need to add the 1 in the formula for N. The fact that no two partitions yield the same algebra is a consequence of Theorem 6.

Next, we explore the meaning of Proposition 7 with one example.

Example 7. Let $n = 3$, then the $N = 1 + p(1) + p(2) + p(3) = 7$ isomorphism classes of one-value magma algebras of dimension $n + 1 = 4$ are:

(1) Having zero isolated copies of N_1, allows us to have $p(3) = 3$ choices for graphs with complete connected components. Thus, we obtain the graphs,
   (a) K_3, yielding a basic ring $R = Re \oplus Re_1$, where Re is simple and $Soc(Re_1) \cong (Re)^2$.
   (b) $K_2 \sqcup K_1$, yielding a basic ring $R = Re \oplus Re_1 \oplus Re_2$, where Re and Re_2 are simple and $Soc(Re_1) \cong Re$, and
   (c) $K_1 \sqcup K_1 \sqcup K_1$, where R being commutative and a sum of four pairwise non-isomorphic simple projective left modules $R = Re \oplus Re_1 \oplus Re_2 \oplus Re_3$.
(2) Having one isolated copy of N_1, and, therefore, $p(2) = 2$ choices for graphs with complete connected components, namely,
   (a) $N_1 \sqcup K_2$, yielding a basic ring $R = Re \oplus Re_1$, where both Re and Re_1 have composition length 2, and $Soc(Re) \cong Re/Je \cong Soc(Re_1)$, and
   (b) $N_1 \sqcup K_1 \sqcup K_1$, yielding a basic ring $R = Re \oplus Re_1 \oplus Re_2$, where Re has composition length 2, $Soc(Re) \cong (Re/Je)$ and Re_1 and Re_2 are simple.
(3) Having two copies of N_1 leaves only one possible graph $N_2 \sqcup K_1$ and yields a basic ring $R = Re \oplus Re_1$ with $Soc(Re) \cong (Re/Je)^2$ and Re_1 are both simple. Finally,
(4) the graph N_3 corresponds to the case where all vertices are isolated and to the basic ring R with $J = Soc(R) \cong (R/J)^3$.

Our next proposition enhances the characterization of commutative graph magma algebras in Corollary 1, in the case when the graph has finitely many non-null components. Remember that a ring R is *right duo* if every right ideal is a left ideal.

Proposition 8. *For a graph G with finitely many non-null components, the following conditions about the magma algebra $R = A[G]$ that it induces are equivalent:*

1. *R is commutative,*
2. *R is right duo, and*
3. *every simple right ideal of R is a left ideal.*

Under any of the above equivalent conditions, A is of the form $A \cong B \oplus C$, where B is zero or a quotient algebra

$$B = \frac{F[x_i | i \in I]}{< x_i x_j | i, j \in I >}, \text{ and}$$

C is zero or a direct sum of copies of the field F. The case when $B=0$ corresponds to the graph $G=N_p$ (with p arbitrary) and the case when $C=0$ corresponds to the graph $G=(K_1)^m$ (with finite m.)

Proof. The implications (1) implies (2) and (2) implies (3) are trivial. To show that (3) indeed implies that R is commutative, consider $R = Re \oplus Re_1 \oplus \ldots \oplus Re_m$ and $R = eR \oplus e_1R \oplus \ldots \oplus e_mR$, respectively, the left and right decompositions corresponding to the semiperfect ring $R = A[G]$ as per Theorem 1 and Proposition 6. Then e_iR is a simple right ideal of dimension 1 for $i = 1, \ldots m$. It follows that e_iR is also a left ideal and, since its dimension is 1, it must be simple. But then $Re_i \subseteq e_iR$, and, in fact for $i = 1, \ldots m$, $e_iR = Re_i$ is a simple left ideal of dimension 1. This implies that the non-null connected components of G are all of the form K_1 and our result follows from Corollary 1.

Also by Corollary 1, a graph that induces a commutative algebra must be of the form $G = N_p \oplus (K_1)^m$. Let $\{n_i | i \in I\} = N_p$ (where $|I| = p$) and consider the algebra $F[x_i | i \in I] \oplus F \oplus \ldots \oplus F$ (m copies of F). Let φ be the epimorphism obtained by sending the variables x_i $(i \in I)$ to n_i, each unit vector $e_i\, F^m$ to the idempotent element of the same name in $(K_1)^m$, and extending it algebraically due to the universal property of polynomials. The kernel of φ equals the subspace $< x_ix_j | i,j \in I >$, yielding our claim.

A result similar to Proposition 8 holds on the opposite side. Notice, however, that the right-handed version of condition (3) may not be included here.

Remark 4. For a graph G with finitely many non-null components, the magma algebra $R = A[G]$ that it induces A is commutative if and only if it is left duo.

Proof. Of course, it is trivial that (1) implies (2). To see that the converse holds, let $R = Re \oplus Re_1 \oplus \ldots \oplus Re_m$ and $R = eR \oplus e_1R \oplus \ldots \oplus e_mR$, respectively, be the left and right decompositions corresponding to the semiperfect ring $R = A[G]$ as per Theorem 1 and Proposition 6. If A is left duo then the left ideal Re is also a right ideal but then eR must be contained in Re. That means that all nilpotent element of the basis of vertices are contained in eR. That, in turn, implies that there are no nilpotent elements from the graph in any Re_i for $i = 1, \ldots m$. That means that the non-null connected components of the inducing graph are isomorphic to N_1, which, in light of Corollary 1, concludes our proof.

We close with a contribution to the study of amenability of bases for graph-magma algebras. While it is known that all countable dimensional algebras have at least one amenable basis (e.g. [1]), our result shows that all graph magma algebras of any infinite dimension have amenable algebras, regardless of the cardinality of V.

Theorem 8. *For every simple associative graph $G = (V, E)$, the graph magma algebra $A[G]$ has an amenable basis.*

Proof. Say $G = N_m \sqcup \sqcup_{i \in I} G_i$, where each G_i is a non-null connected associative graph. Then, the basis of vertices H obtained by replacing each non-null connected component G_i of G with a set of vertices of the type $N_{p_i} \oplus K_1$, as granted by Theorem 3, yields a basis $V_H \cup \{1\}$ which can easily be seen to be amenable, as follows, by considering the left multiplication maps by elements of $\mathcal{B}$. As $\mathcal{B} = N_p \sqcup \sqcup_{i \in I} (N_{p_i} \oplus K_1)$, if $b \in \mathcal{B}$, one must consider the following cases:

1. If $b \in N_p$, $l_b(x) = b$, if $x = 1$, or 0, otherwise. Therefore, b is the only element of $\mathcal{B}$ used by the outcomes of the elements of $\mathcal{B}$ and is only used once.
2. If $b \in N_{p_i}$ then $l_b(x) \neq 0$, only if $x = 1$ or e_i, the idempotent vertex in $N_{p_i} \oplus K_1$ for the i-th connected component. In both cases, $l_b(x) = b$. Therefore, b is the only element of $\mathcal{B}$ used by the outcomes of the elements of $\mathcal{B}$ and is used exactly twice.

3. Finally, if $b = e_i$, the idempotent vertex in $N_{p_i} \oplus K_1$ for the i-th connected component, then $l_b(x) = b$ when $x = 1$ or e_i and is zero otherwise. Therefore, b is the only element of $\mathcal{B}$ used by the outcomes of the elements of $\mathcal{B}$ and is used exactly twice.

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References

[1] Al-Essa, L. M., López-Permouth, S. R., Muthana, N. (2018). Modules over infinite-dimensional algebras. *Linear Multilinear Algebra* 66(3):488–496. DOI: 10.1080/03081087.2017.1301365.
[2] Anderson, F. W., Fuller, K. R. (1992). *Rings and Categories of Modules*, 2nd. ed. New York: Springer-Verlag.
[3] Aydoğdu, P., Díaz Boils, J., López-Permouth, S. R., Muhammad, R. A. Two value graph magma algebras and amenability. To appear in Proceedings of ICARTA-19. Springer-Verlag.
[4] Aydoğdu, P., López-Permouth, S. R., Muhammad, R. A. (2020). Infinite-dimensional algebras without simple bases. *Linear Multilinear Algebra* 68(12):2390–2407. DOI: 10.1080/03081087.2019.1585414.
[5] Bourbaki, N. (1998). *Elements of Mathematics, Algebra I.* Paris: Springer.
[6] Calderón Martín, A. J. (2017). Extended magmas and their applications. *J. Algebra Appl.* 16(08):1750150. DOI: 10.1142/S021949881750150X.
[7] Dixit, V. N., Singhal, P. K., Weinert, H. J. (1986). On quasi-ideals in contracted semigroup rings. *Semigroup Forum* 33(1):431–439. DOI: 10.1007/BF02573213.
[8] Kelarev, A. V., Sokratova, O. V. (2000). Syntactic semigroups and graph algebras. *Bull. Austral. Math. Soc.* 62(3):471–477. DOI: 10.1017/S0004972700018992.
[9] Kim, H. S., Neggers, J. (2008). The semigroups of binary systems and some perspectives. *Bull. Korean Math. Soc.* 45(4):651–661. DOI: 10.4134/BKMS.2008.45.4.651.
[10] López-Permouth, S. R., Owusu Mensah, I., Rafieipour, A. A Monoid structure on the set of all binary operations over a fixed set. To appear in Semigroup Forum.
[11] López-Permouth, S. R., Owusu Mensah, I., Rafieipour, A. Binary operations induced by graphs. In preparation.
[12] López-Permouth, S. R., Rowen, L. H. (2018). Distributive hierarchies of binary operations. Advances in rings and modules. *Contemp. Math., Amer. Math. Soc.* 715:225–242. DOI: 10.1090/conm/715/14414.
[13] López-Permouth, S. R., Stanley, B. (2020). Topological characterizations of amenability and congeniality of bases. *Appl. Gen. Topol.* 21(1):1–15. DOI: 10.4995/agt.2020.11488.
[14] López-Permouth, S. R., Stanley, B. (2021). Graph magma algebras have no projective bases. *Linear Multilinear Algebra* 69(11):1997–2005. DOI: 10.1080/03081087.2019.1652552.
[15] López-Permouth, S. R., Stanley, B. (2021). On the amenability profile of infinite dimensional algebras. *Commun. Algebra* 49(5):2253–2212. DOI: 10.1080/00927872.2020.1869245.
[16] Marks, G. (2003). A taxonomy of 2-primal rings. *J. Algebra* 266(2):494–520. DOI: 10.1016/S0021-8693(03)00301-6.
[17] Munn, W. D. (1987). A class of contracted inverse semigroup rings. *Proc. Roy. Soc. Edinburgh Sect. A.* 107(1-2):175–196. DOI: 10.1017/S0308210500029449.
[18] Munn, W. D. (1999). Moore-Penrose inversion in complex contracted inverse semigroup algebras. *J. Aust. Math. Soc. A* 66(3):297–302. DOI: 10.1017/S1446788700036624.
[19] Serre, J.-P. (1992). *Lie Algebras and Lie Groups: 1964 Lectures Given at Harvard University*, 2nd ed. New York: W.A. Benjamin.
[20] Szabo, S. (2019). Some minimal rings related to 2-primal rings. *Commun. Algebra* 47(3):1287–1298. DOI: 10.1080/00927872.2018.1503284.