On the intersection of distance-j-ovoids and subpolygons of generalized polygons

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Abstract

In [5] (see also [4]), a technique was given for calculating the intersection sizes of combinatorial substructures associated with regular partitions of distance-regular graphs. This technique was based on the orthogonality of the eigenvectors which correspond to distinct eigenvalues of the (symmetric) adjacency matrix. In the present paper, we give a more general method for calculating intersection sizes of combinatorial structures. The proof of this method is based on the solution of a linear system of equations which is obtained by means of double countings. We also give a new class of regular partitions of generalized hexagons and determine under which conditions ovoids and subhexagons of order (s', t') of a generalized hexagon of order s intersect in a constant number of points. If the automorphism group of the generalized hexagon is sufficiently large, then this is the case if and only if $s = s't'$. We derive a similar result for the intersection of distance-2-ovoids and suboctagons of generalized octagons.

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1 Sets with left-regular and right-regular partitions

Let X be a nonempty finite set and let $R \subseteq X \times X$ be a relation on X.

A partition $\mathcal{P} = \{X_1, X_2, \ldots, X_k\}$ of X is called *right-regular with respect to R* if there exist constants r_{ij} $(1 \leq i, j \leq k)$ such that for every $x \in X_i$, there are precisely r_{ij} elements $y \in X_j$ such that $(x, y) \in R$. The $(k \times k)$-th matrix whose (i, j)-th entry (i.e. the entry in row $i \in \{1, \ldots, k\}$ and column $j \in \{1, \ldots, k\}$) is equal to r_{ij} is denoted by $R_{\mathcal{P}}$. Let $E^r_{\mathcal{P}}$ denote the multiset whose elements are the complex eigenvalues of $R_{\mathcal{P}}$, the multiplicity of an element λ of $E^r_{\mathcal{P}}$ being equal to the algebraic multiplicity of λ regarded as eigenvalue of $R_{\mathcal{P}}$.

A partition $\mathcal{P} = \{X_1, X_2, \ldots, X_k\}$ is called *left-regular with respect to R* if there exist constants l_{ij} $(1 \leq i, j \leq k)$ such that for every $y \in X_i$, there are precisely l_{ij} elements $x \in X_j$ such that $(x, y) \in R$. The $(k \times k)$-th matrix whose (i, j)-th entry $(1 \leq i, j \leq k)$ is equal to l_{ij} is denoted by $L_{\mathcal{P}}$. Let $E^l_{\mathcal{P}}$ denote the multiset whose elements are the complex eigenvalues of $L_{\mathcal{P}}$, the multiplicity of an element λ of $E^l_{\mathcal{P}}$ being equal to the algebraic multiplicity of λ regarded as eigenvalue of $L_{\mathcal{P}}$.

Notice that a partition $\mathcal{P}$ of X is left-regular (right-regular) with respect to R if and only if $\mathcal{P}$ is right-regular (left-regular) with respect to the inverse $R^{-1} := \{(y, x) \,|\, (x, y) \in R\}$ of R. A partition $\mathcal{P}$ which is right-regular (or equivalently left regular) with respect to a symmetric relation R will be called *regular with respect to R*. In this case, the matrix $R_{\mathcal{P}} = L_{\mathcal{P}}$ will also be denoted by $M_{\mathcal{P}}$.

Given two finite multisets $M = \{\lambda_1, \ldots, \lambda_k\}$ and $M' = \{\lambda'_1, \ldots, \lambda'_{k'}\}$ whose elements are complex numbers, we denote by $O(M, M')$ the multiplicity of 0 as an element of the multiset $\{\lambda_i - \lambda'_j \,|\, 1 \leq i \leq k, 1 \leq j \leq k'\}$.

In Section 3 we will prove the following theorem.

Theorem 1.1 *Let X be a nonempty finite set and let $R \subseteq X \times X$ be a relation on X. Let $\mathcal{P} = \{X_1, X_2, \ldots, X_k\}$ be a partition of X which is right-regular with respect to R and let r_{ij} $(1 \leq i, j \leq k)$ denote the corresponding coefficients. Let $\mathcal{P}' = \{X'_1, X'_2, \ldots, X'_{k'}\}$ be a partition of X which is left-regular with respect to R and let l_{ij} $(1 \leq i, j \leq k')$ denote the corresponding coefficients. Then the following holds:*

(1) $O(E^r_{\mathcal{P}}, E^l_{\mathcal{P}'}) \geq 1$*;*

(2) *If $O(E^r_{\mathcal{P}}, E^l_{\mathcal{P}'}) = 1$, then there exist numbers η_{ij}, $1 \leq i \leq k$ and $1 \leq j \leq k'$, only depending on the numbers r_{mn} $(1 \leq m, n \leq k)$ and l_{mn} $(1 \leq m, n \leq k')$ such that $|X_i \cap X'_j| = \eta_{ij} \cdot |X|$.*

(3) *Suppose the following: (i) R is symmetric; (ii) there exists a $\mu \in$*

$\mathbb{N} \setminus \{0\}$ *such that for any* $x \in X$*, there are precisely* μ *elements* $y \in X$ *for which* $(x, y) \in R$*; (iii)* $O(E^r_{\mathcal{P}}, E^l_{\mathcal{P}'}) = 1$*. Then* $|X_i \cap X_{j'}| = \frac{|X_i| \cdot |X_{j'}|}{|X|}$ *for any* $(i, j) \in \{1, \ldots, k\} \times \{1, \ldots, k'\}$*.*

An alternative proof of Theorem 1.1(3) is implicitly contained in [5] (Lemma 3.3) in the case the matrices $M_{\mathcal{P}}$ and $M_{\mathcal{P}'}$ are diagonizable. (The restriction in [5, Lemma 3.3] to regular partitions associated with distance-regular graphs is not essential; the proof given there also works for arbitrary regular partitions $\mathcal{P}$ and $\mathcal{P}'$ for which $M_{\mathcal{P}}$ and $M_{\mathcal{P}'}$ are diagonizable.) Lemma 3.3 of [5] was used in that paper to determine the intersection sizes of various combinatorial substructures of generalized polygons.

2 On the intersection sizes of distance-j-ovoids and subpolygons of generalized polygons

2.1 Definitions

Let $n \geq 2$ and $s, t \geq 1$. A *generalized* $2n$*-gon* of order (s, t) is a partial linear space $\Gamma = (X, \mathcal{L}, \mathrm{I})$, $\mathrm{I} \subseteq X \times \mathcal{L}$, which satisfies the following properties:

(GP1) For every point x and every line L, there exists a unique point on L nearest to x.

(GP2) The maximal distance between two points of Γ is equal to n.

(GP3) If x and y are two points at distance $i \in \{1, \ldots, n-1\}$ from each other, then y is collinear with a unique point which lies at distance $i-1$ from x.

(GP4) Every line is incident with precisely $s+1$ points and every point is incident with precisely $t+1$ lines.

In (GP1), (GP2) and (GP3), distances $\mathrm{d}(\cdot, \cdot)$ are measured in the collinearity graph of Γ. If $x \in X$ and $i \in \mathbb{N}$, then $\Gamma_i(x)$ denotes the set of points of Γ at distance i from x. Two points x and y of Γ are called *opposite* if $\mathrm{d}(x, y) = n$. If $s = t$, then Γ is also called a generalized $2n$-gon of order s. Properties (GP1) and (GP2) imply that Γ is a so-called near $2n$-gon ([1]). The generalized polygons considered in this paper are either generalized hexagons ($n = 3$) and generalized octagons ($n = 4$). A generalized hexagon of order (s, t) contains $(s+1)((st)^2 + st + 1)$ points and $(t+1)((st)^2 + st + 1)$ lines. A generalized octagon of order (s,t) contains $(s+1)((st)^3+(st)^2+st+1)$ points and $(t+1)((st)^3+(st)^2+st+1)$ lines.

Suppose $X' \subseteq X$ and $\mathcal{L}' \subseteq \mathcal{L}$ such that $\Gamma' = (X', \mathcal{L}', \mathrm{I}')$ is a generalized $2n$-gon of order (s', t'), where $s', t' \in \mathbb{N} \setminus \{0\}$ and $\mathrm{I}' := \mathrm{I} \cap (X' \times \mathcal{L}')$. Then Γ' is called a *sub-$2n$-gon* of order (s', t') of Γ. The sub-$2n$-gon Γ' is called *proper* if $(s', t') \neq (s, t)$, *full* if $s' = s$, and *ideal* if $t' = t$. Notice that the distance between two points x and y of Γ' in (the collinearity graph of) Γ' coincides with the distance between x and y in Γ.

Let $j \in \{2, \ldots, n\}$. A distance-j-ovoid of Γ is a set X' of points satisfying

(DO1) $\mathrm{d}(x, y) \geq j$ for every two distinct points x and y of X';

(DO2) for every point a of Γ, there exists a point x of X' such that $\mathrm{d}(a, x) \leq \frac{j}{2}$;

(DO3) for every line L of Γ, there exists a point x of X' such that $\mathrm{d}(L, x) \leq \frac{j-1}{2}$.

If X' is a distance-j-ovoid of Γ with j odd, then for every point a of Γ, there exists a unique point $x \in X'$ such that $\mathrm{d}(a, x) \leq \frac{j-1}{2}$. If X' is a distance-j-ovoid of Γ with j even, then for every line L of Γ, there exists a unique point $x \in X'$ such that $\mathrm{d}(L, x) \leq \frac{j-2}{2}$. A distance-$n$-ovoid of Γ is also called an *ovoid* of Γ. A set X' of points of Γ is a distance-2-ovoid if every line of Γ intersects X' in a unique point.

We will now discuss some known restrictions on the parameters of finite generalized hexagons and generalized octagons. If Γ is a generalized hexagon of order (s, t) with $s, t \geq 2$, then st is a perfect square by Feit & Higman [6] and $s^{\frac{1}{3}} \leq t \leq s^3$ by Haemers & Roos [7]. If Γ' is a proper subhexagon of order (s', t') of a generalized hexagon of order (s, t), then $st \geq s'^2 t'^2$ by Thas [11]. If O is an ovoid of a generalized hexagon of order (s, t), then $s = t$ by Offer [9] and the ovoid O contains precisely $s^3 + 1$ points. If Γ is a generalized octagon of order (s, t) with $s, t \geq 2$, then $2st$ is a perfect square by Feit & Higman [6] and $s^{\frac{1}{2}} \leq t \leq s^2$ by Higman [8]. Restrictions on the orders of suboctagons of generalized octagons were derived in Yanushka [14] and Thas [12]. Restrictions on the parameters of generalized polygons admitting certain distance-j-ovoids were derived in Offer & Van Maldeghem [10], see also De Bruyn [2] for a discussion in the context of general regular near polygons.

2.2 A class of regular partitions of generalized hexagons

A *regular partition of a finite point-line geometry* $\mathcal{S}$ is a partition of the point-set of $\mathcal{S}$ which is regular with respect to the adjacency relation of the collinearity graph of $\mathcal{S}$.

Consider now a generalized hexagon $\Gamma = (X, \mathcal{L}, \mathrm{I})$ of order (s,t) having a proper subhexagon $\Gamma' = (X', \mathcal{L}', \mathrm{I}')$ of order (s', t') such that $st = s'^2 t'^2$. Consider the following subsets of X:

- $X_1 = X'$;
- X_2 consists of those points of $X \setminus X_1$ which are contained on a line of $\mathcal{L}'$;
- X_3 consists of those points of $X \setminus (X_1 \cup X_2)$ which are collinear with a (necessarily unique) point of X';
- X_4 consists of those points of X which are not collinear with a point of X'.

In Section 4, we prove that $\{X_1, X_2, X_3, X_4\}$ determines a regular partition of Γ. More precisely, the following holds.

(I) Suppose $s = s'$ and $t \neq t'$. Then $X_2 = X_4 = \emptyset$ and $\mathcal{P} := \{X_1, X_3\}$ is a regular partition of Γ. In this case,

$$M_{\mathcal{P}} = \begin{bmatrix} s(t'+1) & s(t-t') \\ 1 & s-1+st \end{bmatrix}$$

and the eigenvalues of $M_{\mathcal{P}}$ are equal to $s(t+1)$ and $s-1+\sqrt{st}$.

(II) Suppose $t = t'$ and $s \neq s'$. Then $X_3 = \emptyset$ and $\mathcal{P} := \{X_1, X_2, X_4\}$ is a regular partition of Γ. In this case

$$M_{\mathcal{P}} = \begin{bmatrix} s'(t+1) & (t+1)(s-s') & 0 \\ s'+1 & s-s'-1 & st \\ 0 & t+1 & (t+1)(s-1) \end{bmatrix}$$

and the eigenvalues of $M_{\mathcal{P}}$ are equal to $-(t+1)$, $s(t+1)$ and $s-1+\sqrt{st}$.

(III) Suppose $s \neq s'$ and $t \neq t'$. Then $\mathcal{P} := \{X_1, X_2, X_3, X_4\}$ is a regular partition of Γ. In this case

$$M_{\mathcal{P}} = \begin{bmatrix} s'(t'+1) & (t'+1)(s-s') & (t-t')s & 0 \\ s'+1 & s-s'-1 & 0 & st \\ 1 & 0 & s-1+ts' & t(s-s') \\ 0 & t'+1 & (t-t')(s'+1) & (t'+1)(s-1)+(t-t')(s-s'-1) \end{bmatrix}$$

and the eigenvalues of $M_{\mathcal{P}}$ are equal to $-(t+1)$, $s(t+1)$ and $s-1+\sqrt{st}$ (multiplicity 2).

2.3 On the intersection of ovoids and subhexagons of generalized hexagons

Suppose Γ is a generalized hexagon of order s admitting an ovoid O. By [4, Chapter 6], every full subhexagon of Γ intersects O in precisely $s+1$ points and every ideal subhexagon of Γ intersects O in precisely 2 points. (A very special case of this result was also contained in [3, Lemma 2.4].) These two results are special cases of the following more general result:

Theorem 2.1 *Let* $\Gamma = (X, \mathcal{L}, \mathrm{I})$ *be a generalized hexagon of order* s *admitting an ovoid* O *and a subhexagon* $\Gamma' = (X', \mathcal{L}', \mathrm{I}')$ *of order* (s', t'). *If* $s = s't'$, *then* O *intersects* Γ' *in precisely* $s'+1$ *points.*

Proof. By Section 2.2, there is a regular partition $\mathcal{P}$ associated with the subhexagon Γ' of Γ. In section 2.2, we also listed the eigenvalues corresponding to the partition $\mathcal{P}$. With the ovoid O, there is associated another regular partition $\mathcal{P}' := \{O, X \setminus O\}$ of Γ. The eigenvalues associated with $\mathcal{P}'$ are -1 and s^2+s. Now, Theorem 1.1(3) applies and we can conclude that $|X' \cap O| = \frac{|X'|\cdot|O|}{|X|} = \frac{(s'+1)(s'^2t'^2+s't'+1)\cdot(s^3+1)}{(s+1)(s^4+s^2+1)} = s'+1$. ■

Example. The split-Cayley hexagon $H(q^2)$ ([13]) of order q^2 has subhexagons isomorphic to $H(q)$. Every ovoid of $H(q^2)$ intersects each such subhexagon in precisely $q+1$ points.

One can ask whether it is possible to prove the converse statement of Theorem 2.1:

$(*)$ If O intersects Γ' in precisely $s'+1$ points, then $s = s't'$.

One intuitively feels that statements of the form $(*)$ will be hard to prove (if not impossible). But one might be more successful if one assumes that $(*)$ not only holds for one subhexagon Γ' (for given O) but for every member of a sufficiently large class of subhexagons. Similarly, one might expect to be more successful if one assumes that $(*)$ holds for every member of a sufficiently large class of ovoids O for a given subhexagon Γ' of order (s', t'). In Section 5, we will prove the following theorems.

Theorem 2.2 *Let Γ be a generalized hexagon of order s. Let O be an ovoid of Γ and let $\mathcal{F}$ be a nonempty family of proper subhexagons of order (s', t') of Γ. Suppose (i) the number of elements of $\mathcal{F}$ through a given point of Γ is a constant and (ii) the number of elements of $\mathcal{F}$ through two given opposite points of Γ is a constant. Then the following are equivalent:*
(1) *$s = s't'$;*
(2) *every element of $\mathcal{F}$ intersects O in a constant number of points.*

Theorem 2.3 *Let Γ be a generalized hexagon of order s. Let Γ' be a proper subhexagon of order (s', t') of Γ and let $\mathcal{F}$ be a nonempty family of ovoids of Γ. Suppose (i) the number of elements of $\mathcal{F}$ through a given point of Γ is a constant and (ii) the number of elements of $\mathcal{F}$ through two given opposite points of Γ is a constant. Then the following are equivalent:*
(1) *$s = s't'$;*
(2) *every element of $\mathcal{F}$ intersects Γ' in a constant number of points.*

Theorems 2.2 and 2.3 are especially interesting in the case the generalized hexagon under consideration admits a group which is transitive on the set of points and the set of unordered pairs of opposite points. In this case, we can take for $\mathcal{F}$ any nonempty union of isomorphism classes of proper subhexagons of order (s', t') (Theorem 2.2) or ovoids (Theorem 2.3) of Γ.

2.4 On the intersection of distance-2-ovoids and suboctagons of generalized octagons

Suppose Γ is a generalized octagon of order (s, t) admitting a distance-2-ovoid X. If Γ' is a suboctagon of order (s', t') with $s = s'$, then $\Gamma' \cap X$ is a distance-2-ovoid of Γ' and hence contains precisely $(s't')^3 + (s't')^2 + s't' + 1$ points (cf. Lemma 6.1). If $s \neq s'$, then it is not at all obvious what the size of the intersection $\Gamma' \cap X$ is. We will prove the following theorems.

Theorem 2.4 *Let Γ be a generalized octagon of order (s, t). Suppose Γ admits a suboctagon Γ' of order (s', t') with $s' \neq s$ and a nonempty family $\mathcal{F}$ of distance-2-ovoids satisfying the property that the number of elements of $\mathcal{F}$ through two points x and y at distance $i \in \{0, 2, 3, 4\}$ from each other only depends on i and not on x and y. Then $s \geq s't'$ with equality if and only if Γ' intersects every element of $\mathcal{F}$ in a constant number of points. Moreover,* *if $s = s't'$ then Γ' intersects every element of $\mathcal{F}$ in precisely $(s'+1)(s^2+1)$ points.*

Theorem 2.5 *Let Γ be a generalized octagon of order (s,t). Suppose Γ admits a distance-2-ovoid X and a nonempty family $\mathcal{F}$ of suboctagons of order (s',t'), $s' \neq s$, satisfying the property that the number of elements of $\mathcal{F}$ through two points x and y of Γ at distance $i \in \{0,2,3,4\}$ from each other only depends on i and not on x and y. Then $s \geq s't'$ with equality if and only if X intersects every element of $\mathcal{F}$ in a constant number of points. Moreover, if $s = s't'$ then X intersects every element of $\mathcal{F}$ in precisely $(s'+1)(s^2+1)$ points.*

Corollary 2.6 *Let Γ be a generalized octagon of order (s,t) whose automorphism group acts transitively on the set of points of Γ and the set of unordered pairs of points at distance i from each other, $i = 2,3,4$. Let X be a distance-2-ovoid of Γ and Γ' be a suboctagon of order (s',t') of Γ with $s't' = s$. Then $|X \cap \Gamma'| = (s'+1)(s^2+1)$.*

Proof. If $s = s'$, then $|X \cap \Gamma'| = (s't')^3 + (s't')^2 + s't' + 1 = s^3 + s^2 + s + 1 = (s+1)(s^2+1) = (s'+1)(s^2+1)$. So, suppose $s \neq s'$. Let $\mathcal{F}$ denote the orbit of X under the automorphism group of Γ. Then the conditions of Theorem 2.4 are fulfilled. Hence, Γ' intersects each element of $\mathcal{F}$ in precisely $(s'+1)(s^2+1)$ points. In particular, $|X \cap \Gamma'| = (s'+1)(s^2+1)$. ∎

3 Proof of Theorem 1.1

3.1 A property of eigenvalues of matrices

Let n and m be strictly positive integers, let $\mathbb{K}$ be a field and let $\overline{\mathbb{K}}$ denote the algebraic closure of $\mathbb{K}$.

With every $(m \times m)$-matrix M over $\mathbb{K}$, there is associated an $(nm \times nm)$-matrix $\widetilde{M}$. If we regard $\widetilde{M}$ as an $(m \times m)$-matrix with blocks of size $n \times n$, then the (i,j)-th entry $(1 \leq i,j \leq m)$ of $\widetilde{M}$ is equal to $m_{ij} \cdot I_n$. Here, m_{ij} denotes the (i,j)-th entry of M and I_n denotes the $(n \times n)$-identity matrix. Notice that $\widetilde{I_m} = I_{nm}$ and $\widetilde{M_1 \cdot M_2} = \widetilde{M_1} \cdot \widetilde{M_2}$ for any two $(m \times m)$-matrices over $\mathbb{K}$. Hence, if M is an invertible $(m \times m)$-matrix over $\mathbb{K}$, then $\widetilde{M}$ is also an invertible matrix and $\widetilde{M}^{-1} = \widetilde{M^{-1}}$.

With every $(n \times n)$-matrix N over $\mathbb{K}$, there is associated an $(nm \times nm)$-matrix N'. If we regard N' as an $(m \times m)$-matrix with blocks of size $n \times n$, then the (i, j)-th entry $(1 \leq i, j \leq m)$ of N' is equal to N if $i = j$ and equal to 0 otherwise.

The following property clearly holds: if M is an $(m \times m)$-matrix over $\mathbb{K}$ and N is an $(n \times n)$-matrix over $\mathbb{K}$, then $\widetilde{M}$ commutes with N'.

Now, let A, respectively B, denote an $(n \times n)$-matrix, respectively an $(m \times m)$-matrix, with entries contained in $\mathbb{K}$. We define $[[A, B]] := A' - \widetilde{B}$. Let $E_1 \subseteq \overline{\mathbb{K}}$, $E_2 \subseteq \overline{\mathbb{K}}$, respectively $E_3 \subseteq \overline{\mathbb{K}}$, denote the multiset of size n, m, respectively nm, whose elements are the eigenvalues of A, B, respectively $[[A, B]]$ (taking into account their respective algebraic multiplicities).

Lemma 3.1 $E_3 = \{\lambda_1 - \lambda_2 \,|\, \lambda_1 \in E_1, \lambda_2 \in E_2\}$.

Proof. Let Q be an invertible $(m \times m)$-matrix over $\overline{\mathbb{K}}$ such that $Q^{-1}BQ$ is an upper triangular matrix (e.g. the Jordan normal form of B). If λ_i, $i \in \{1, \ldots, m\}$, denotes the (i, i)-th entry of $Q^{-1}BQ$, then $E_2 = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$. We have

$$\begin{aligned} \widetilde{Q}^{-1}[[A, B]]\widetilde{Q} &= \widetilde{Q}^{-1}(A' - \widetilde{B})\widetilde{Q} \\ &= \widetilde{Q}^{-1}A'\widetilde{Q} - \widetilde{Q}^{-1}\widetilde{B}\widetilde{Q} \\ &= A' - \widetilde{Q^{-1}BQ}. \end{aligned}$$

If we regard $\widetilde{Q}^{-1}[[A, B]]\widetilde{Q}$ as an $(m \times m)$-matrix with blocks of size $n \times n$, then the (i, j)-th entry of $\widetilde{Q}^{-1}[[A, B]]\widetilde{Q}$ is equal to O_n if $j < i$ and equal to $A - \lambda_i I_n$ if $i = j$. (Here, O_n denotes the $(n \times n)$-zero matrix.) Hence, the eigenvalues of $\widetilde{Q}^{-1}[[A, B]]\widetilde{Q}$, i.e. the eigenvalues of $[[A, B]]$, are the elements of the form $\lambda - \lambda_i$, where λ is an eigenvalue of A and $i \in \{1, \ldots, m\}$. This proves the lemma. ∎

3.2 Proof of Theorem 1.1

We continue with the notations introduced in the statement of Theorem 1.1.

Fix an $i \in \{1, \ldots, k'\}$ and a $j \in \{1, \ldots, k\}$. We count in two different ways the number N of all ordered pairs $(x, y) \in R \cap (X_i' \times X_j)$. Since for each $x \in X_i' \cap X_f$ $(f \in \{1, \ldots, k\})$, there are precisely r_{fj} elements $y \in X_j$

such that $(x, y) \in R$, we have

$$N = \sum_{f=1}^{k} r_{fj} \cdot |X_f \cap X'_i|. \tag{1}$$

On the other hand, for each $y \in X'_f \cap X_j$ ($f \in \{1, \ldots, k'\}$), there are l_{fi} elements $x \in X'_i$ such that $(x, y) \in R$. Hence, we also have

$$N = \sum_{f=1}^{k'} l_{fi} \cdot |X_j \cap X'_f|. \tag{2}$$

From (1) and (2), we obtain

$$E(i,j): \qquad \sum_{f=1}^{k} r_{fj} \cdot |X_f \cap X'_i| + \sum_{f=1}^{k'} (-l_{fi}) \cdot |X_j \cap X'_f| = 0. \tag{3}$$

Since equation $E(i,j)$ holds for any $(i,j) \in \{1, \ldots, k'\} \times \{1, \ldots, k\}$, we obtain a linear homogeneous system with kk' equations and kk' unknows. We consider the following ordering of these equations and unknowns: $E(i,j) \prec E(i',j')$ if and only if either $i < i'$ or ($i = i'$ and $j < j'$); $|X_i \cap X'_j| \prec |X_{i'} \cap X'_{j'}|$ if and only if either $j < j'$ or ($j = j'$ and $i < i'$). The matrix of the linear system (3) with respect to these orderings is equal to $[[R^T_{\mathcal{P}}, L^T_{\mathcal{P}'}]]$. By Lemma 3.1 and the fact that $R_{\mathcal{P}}$ and $R^T_{\mathcal{P}}$ (respectively $L_{\mathcal{P}'}$ and $L^T_{\mathcal{P}'}$) have the same eigenvalues with the same algebraic multiplicities, the rank of $[[R^T_{\mathcal{P}}, L^T_{\mathcal{P}'}]]$ is equal to kk' if $O(E^r_{\mathcal{P}}, E^l_{\mathcal{P}'}) = 0$ and equal to $kk' - 1$ if $O(E^r_{\mathcal{P}}, E^l_{\mathcal{P}'}) = 1$.

If $O(E^r_{\mathcal{P}}, E^l_{\mathcal{P}'}) = 0$, then we would have that $|X_i \cap X'_j| = 0$ for any $\{i,j\} \in \{1, \ldots, k\} \times \{1, \ldots, k'\}$. This would contradict the fact that $|X| > 0$. Hence, $O(E^r_{\mathcal{P}}, E^l_{\mathcal{P}'}) \geq 1$. If $O(E^r_{\mathcal{P}}, E^l_{\mathcal{P}'}) = 1$, then the solution of system (3) depends on 1 parameter λ. So, for any $(i,j) \in \{1, \ldots, k\} \times \{1, \ldots, k'\}$, $|X_i \cap X'_j|$ is equal to $\lambda \cdot \eta'_{ij}$, where η'_{ij} only depends on the entries of the matrices $R_{\mathcal{P}}$ and $L_{\mathcal{P}'}$. Since $|X| = \sum_{m,n} |X_m \cap X'_n| > 0$, we necessarily have $\sum_{m,n} \eta'_{mn} \neq 0$. Clearly, we have $|X_i \cap X'_j| = \eta_{ij} \cdot |X|$ where $\eta_{ij} := \frac{\eta'_{ij}}{\sum_{m,n} \eta'_{mn}}$.

Finally, suppose that: (i) R is symmetric; (ii) there exists a $\mu \in \mathbb{N} \setminus \{0\}$ such that for any $x \in X$, there are μ elements $y \in X$ for which $(x,y) \in R$; (iii) $O(E^r_{\mathcal{P}}, E^l_{\mathcal{P}'}) = 1$. By the above discussion, we know that the system (3) in combination with the equation $\sum_{i,j} |X_i \cap X'_j| = |X|$ has a unique solution for the unknowns $|X_i \cap X'_j|$. So, it suffices to show that all these equations are satisfied if we put $|X_i \cap X'_j|$ equal to $\frac{|X_i|\cdot|X'_j|}{|X|}$, $(i,j) \in \{1,\dots,k\} \times \{1,\dots,k'\}$. We have

$$\sum_{i,j} \frac{|X_i| \cdot |X'_j|}{|X|} = \frac{\sum_i |X_i| \cdot \sum_j |X'_j|}{|X|} = \frac{|X| \cdot |X|}{|X|} = |X|.$$

For any $(i,j) \in \{1,\dots,k'\} \times \{1,\dots,k\}$, we now calculate

$$\Omega_{ij} := \sum_{f=1}^{k} r_{fj} \frac{|X_f| \cdot |X'_i|}{|X|} + \sum_{f=1}^{k'} (-l_{fi}) \frac{|X_j| \cdot |X'_f|}{|X|}.$$

Counting pairs $(x,y) \in R \cap (X_f \times X_j)$, we find $|X_f| \cdot r_{fj} = |X_j| \cdot r_{jf}$. (Here, we used the symmetry of R.) Similarly, counting pairs $(x,y) \in R \cap (X'_i \times X'_f)$, we find $|X'_f| \cdot l_{fi} = |X'_i| \cdot l_{if}$. So, the expression for Ω_{ij} becomes:

$$\sum_{f=1}^{k} \frac{|X_j| \cdot |X'_i|}{|X|} r_{jf} + \sum_{f=1}^{k'} \frac{|X_j| \cdot |X'_i|}{|X|} (-l_{if}).$$

Now, $\mu = \sum_{f=1}^{k} r_{jf} = \sum_{f=1}^{k'} l_{if}$ and hence $\Omega_{ij} = 0$. This finishes the proof of Theorem 1.1.

4 A class of regular partitions of generalized hexagons

Let $\Gamma' = (X', \mathcal{L}', \mathrm{I}')$ be a subhexagon of order (s', t') of a generalized hexagon $\Gamma = (X, \mathcal{L}, \mathrm{I})$ of order (s,t). We divide the point- and line-set of Γ into 4 subsets.

Points of X belonging to X' are called *points of Type I.* Lines of $\mathcal{L}$ belonging to $\mathcal{L}'$ are called *lines of Type I.* Points of $X \setminus X'$ which belong to a line of $\mathcal{L}'$ are called *points of Type II.* Lines of $\mathcal{L} \setminus \mathcal{L}'$ which contain a point of X' are called *lines of Type II.* The following claims are readily verified if one takes into account the axioms that define the generalized polygons:

(a) If x is a point of Type II, then x is contained in a unique line of $\mathcal{L}'$ and no line through x has type II.

(b) If L is a line of Type II, then L contains a unique point of X' and no point of L has type II.

Every point of $L \setminus X'$, where L is a line of Type II is called a *point of Type III*. Every line of $\mathcal{L} \setminus \mathcal{L}'$ containing a point of Type II is called a *line of Type III*. The following claims are again readily verified:

(c) Every point of Type III is incident with 0 lines of Type I, a unique line of Type II and 0 lines of Type III.

(d) Every line of Type III contains 0 points of Type I, a unique point of Type II and 0 points of Type III.

A point of X is said to be of Type IV if it is not of Type I, II or III. A line of $\mathcal{L}$ is said to be of Type IV if it is not of Type I, II or III. Obviously, we have:

(e) Every point of Type IV is incident with only lines of Type III or IV.

(f) Every line of Type IV is incident with only points of Type III or IV.

Making use of properties (a)–(f), we can easily count the total number of points and lines of each type. The number of points of Type I, II, III, respectively IV, is equal to $N_1 := (s'+1)(s'^2t'^2+s't'+1)$, $N_2 := (t'+1)(s'^2t'^2+s't'+1)(s-s')$, $N_3 := s(t-t')(s'+1)(s'^2t'^2+s't'+1)$, respectively $N_4 := (s+1)(s^2t^2+st+1) - N_1 - N_2 - N_3$. The number of lines of Type I, II, III, respectively IV, is equal to $M_1 := (t'+1)(s'^2t'^2+s't'+1)$, $M_2 := (s'+1)(s'^2t'^2+s't'+1)(t-t')$, $M_3 := t(s-s')(t'+1)(s'^2t'^2+s't'+1)$, respectively $M_4 := (t+1)(s^2t^2+st+1) - M_1 - M_2 - M_3$.

In [11], J. A. Thas proved that $st \geq s'^2t'^2$.

Lemma 4.1 *If $st = s'^2t'^2$, then*
(*g*) *every point of Type IV is contained in precisely $t'+1$ lines of Type III and $t-t'$ lines of Type IV,*
(*h*) *every line of Type IV contains precisely $s'+1$ points of Type III and $s-s'$ points of Type IV.*

Proof. We only need to prove claim (g). Claim (h) is the dual statement of claim (g). We distinguish two cases.

If $s = s'$ (and $st = s'^2 t'^2$), then one can calculate that $N_4 = 0$. Hence, claim (g) trivially holds in this case. (The fact that in this case there are no points of Type IV was already remarked in [11, p. 116, first paragraph].)

If $s \neq s'$, then by the final remark of [11], for every point x of Type IV there are $t' + 1$ lines of Type I which contain a unique point x' collinear with x. The point x' is necessarily of Type II and the line xx' is necessarily of type III. The claim follows. ∎

From statements (a)–(h), it now readily follows that the points of Type I, II, III and IV determine a regular partition of Γ. Moreover, the parameters of this regular partition are as claimed in Section 2.2.

5 The intersection of ovoids and subhexagons of generalized hexagons

Proof of Theorem 2.2

Let Γ be a generalized hexagon of order s. Let O be an ovoid of Γ and let $\mathcal{F}$ be a family of $\mu \geq 1$ proper subhexagons of order (s', t') of Γ such that (i) every point of Γ is contained in precisely δ_1 elements of $\mathcal{F}$ and (ii) every two opposite points of Γ are contained in precisely δ_2 elements of $\mathcal{F}$. If $s = s't'$, then by Theorem 2.1, O intersects every element of $\mathcal{F}$ in a constant number of points.

Conversely, suppose that O intersects every member of $\mathcal{F}$ in a constant number of points. We will prove that $s = s't'$. We will first express the constants δ_1 and δ_2 in terms of μ, s, s' and t'. Every subhexagon Γ' of order (s', t') of Γ contains $(s'+1)(s'^2 t'^2 + s't' + 1)$ points. The number of points of Γ' at distance 3 from a given point of Γ' is equal to $s'^3 t'^2$. From a straightforward counting, we then have that

$$\begin{aligned}
\delta_1 &= \mu \frac{(s'+1)(s'^2 t'^2 + s't' + 1)}{(s+1)(s^4 + s^2 + 1)}, \\
\delta_2 &= \mu \frac{(s'+1)(s'^2 t'^2 + s't' + 1) s'^3 t'^2}{(s+1)(s^4 + s^2 + 1) s^5}.
\end{aligned}$$

Put $\alpha := |O \cap \Gamma'|$, where Γ' is an arbitrary element of $\mathcal{F}$. Counting pairs (x, Γ') with $\Gamma' \in \mathcal{F}$ and $x \in \Gamma' \cap O$ yields

$$
\begin{aligned}
\alpha &= \frac{|O| \cdot \delta_1}{\mu} \\
&= (s^3+1)\frac{(s'+1)(s'^2t'^2+s't'+1)}{(s+1)(s^4+s^2+1)} \\
&= \frac{(s'+1)(s'^2t'^2+s't'+1)}{s^2+s+1}.
\end{aligned}
$$

Counting triples (x, y, Γ') with $\Gamma' \in \mathcal{F}$, $x, y \in O \cap \Gamma'$ and $x \neq y$ yields

$$
\begin{aligned}
\mu\alpha(\alpha-1) &= |O|(|O|-1) \cdot \delta_2 \\
&= (s^3+1)s^3 \cdot \mu\frac{(s'+1)(s'^2t'^2+s't'+1)s'^3t'^2}{(s+1)(s^4+s^2+1)s^5} \\
&= \mu\frac{(s'+1)(s'^2t'^2+s't'+1)s'^3t'^2}{(s^2+s+1)s^2}.
\end{aligned}
$$

Hence,

$$
\mu\frac{(s'+1)(s'^2t'^2+s't'+1)}{s^2+s+1} \cdot \frac{(s'+1)(s'^2t'^2+s't'+1)-(s^2+s+1)}{s^2+s+1}
$$

$$
= \mu\frac{(s'+1)(s'^2t'^2+s't'+1)s'^3t'^2}{(s^2+s+1)s^2},
$$

$$
s^2(s'+1)(s'^2t'^2+s't'+1) - s^2(s^2+s+1) = (s^2+s+1)s'^3t'^2,
$$

$$
(s-s't')\Big(s^2(s+s't')+s^2-s'(s+s't')-s'^2t's\Big) = 0. \tag{4}
$$

Clearly,

$$
s^2(s+s't')+s^2-s'(s+s't')-s'^2t's = (s^2-s's)+(s^3-s'(s't'))+ss't'(s-s') > 0
$$

since $s \geq s'$ and $s \geq s't'$ (Thas [11]). Hence, $s = s't'$.

Proof of Theorem 2.3

Let Γ be a generalized hexagon of order s. Let Γ' be a proper subhexagon of order (s', t') of Γ and let $\mathcal{F}$ be a family of $\mu \geq 1$ ovoids of Γ such that (i) every point of Γ is contained in precisely δ_1 elements of $\mathcal{F}$ and (ii) every two opposite points of Γ are contained in precisely δ_2 elements of $\mathcal{F}$. If $s = s't'$, then by Theorem 2.1, Γ' intersects every element of $\mathcal{F}$ in a constant number of points.

Conversely, suppose that Γ' intersects every member of $\mathcal{F}$ in a constant number of points. We will prove that $s = s't'$. From a straightforward counting, we have

$$\begin{aligned}
\delta_1 &= \mu\frac{s^3+1}{(s+1)(s^4+s^2+1)} = \frac{\mu}{s^2+s+1}, \\
\delta_2 &= \mu\frac{(s^3+1)s^3}{(s+1)(s^4+s^2+1)s^5} = \frac{\mu}{(s^2+s+1)s^2}.
\end{aligned}$$

Put $\alpha := |O \cap \Gamma'|$, where O is an arbitrary element of $\mathcal{F}$. With similar counting as before, we have

$$\begin{aligned}
\alpha &= \frac{(s'+1)(s'^2t'^2+s't'+1)}{s^2+s+1}, \\
\mu\alpha(\alpha-1) &= \mu\frac{(s'+1)(s'^2t'^2+s't'+1)s'^3t'^2}{(s^2+s+1)s^2}.
\end{aligned}$$

So, we obtain the same equations as in the proof of Theorem 2.2. Hence, $s = s't'$.

6 The intersection of distance-2-ovoids and suboctagons in generalized octagons

We will only prove Theorem 2.4. Theorem 2.5 is proved in a completely similar way and leads to the same equations as in the proof of Theorem 2.4 (similar situation as in the proofs of Theorems 2.2 and 2.3).

Let Γ be a generalized octagon of order (s,t). Then Γ contains $(s+1)((st)^3 + (st)^2 + st + 1)$ points and $(t+1)((st)^3 + (st)^2 + st + 1)$ lines. The number of points at distance $i \in \{0,1,2,3,4\}$ from a given point of Γ is equal to N_i, where $N_0 := 1$, $N_1 := s(t+1)$, $N_2 := s^2t(t+1)$, $N_3 := s^3t^2(t+1)$ and $N_4 = s^4t^3$. Let Γ' be a suboctagon of order (s',t') of Γ with $s \neq s'$. Let $\mathcal{F}$ be a family of $\mu \geq 1$ distance-2-ovoids of Γ such that every two points x and y of Γ at distance i from each other are contained in precisely λ_i elements of $\mathcal{F}$. Here, λ_i is independent from the chosen points x and y.

Lemma 6.1 *Let O be a distance-2-ovoid of Γ and x a point of O. Then $|O| = (st)^3 + (st)^2 + st + 1$, $M_0 := |\Gamma_0(x) \cap O| = 1$, $M_1 := |\Gamma_1(x) \cap O| = 0$, $M_2 := |\Gamma_2(x) \cap O| = st(t+1)$, $M_3 := |\Gamma_3(x) \cap O| = (s^2 - s)t^2(t+1)$ and $M_4 := |\Gamma_4(x) \cap O| = s(s^2 - s + 1)t^3$.*

Proof. If we count in two different ways the number of pairs (x, L) with $x \in O$ and L a line of Γ through x, then we immediately find $|O| = (st)^3 + (st)^2 + st + 1$.

Obviously, $|\Gamma_0(x) \cap O| = 1$ and $|\Gamma_1(x) \cap O| = 0$. Each of the $s(t+1)$ points of $\Gamma_1(x)$ is collinear with precisely t points of $O \cap \Gamma_2(x)$. Conversely, every point of $O \cap \Gamma_2(x)$ is collinear with a unique point of $\Gamma_1(x)$. It follows that $|O \cap \Gamma_2(x)| = st(t+1)$.

We count in two different ways the number of pairs (x_1, x_2) with $x_1 \in \Gamma_2(x)$ and $x_2 \in \Gamma_3(x) \cap O$. Notice that if (x_1, x_2) is such a pair, then $x_1 \in \Gamma_2(x) \setminus O$. We have $|\Gamma_2(x) \setminus O| \cdot t = |\Gamma_3(x) \cap O| \cdot 1$, i.e. $|\Gamma_3(x) \cap O| = (s^2t(t+1) - st(t+1)) \cdot t = (s^2 - s)t^2(t+1)$.

Finally, we have $|\Gamma_4(x) \cap O| = |O| - \sum_{i=0}^{3} |\Gamma_i(x) \cap O| = s(s^2 - s + 1)t^3$. $\blacksquare$

We now express λ_i, $i \in \{0, 1, 2, 3, 4\}$, in terms of μ, s and t. We have

$$\begin{aligned}
\lambda_0 &= \mu \frac{(st)^3 + (st)^2 + st + 1}{(s+1)((st)^3 + (st)^2 + st + 1)} = \frac{\mu}{s+1}, \\
\lambda_1 &= 0, \\
\lambda_2 &= \mu \frac{((st)^3 + (st)^2 + st + 1) \cdot M_2}{(s+1)((st)^3 + (st)^2 + st + 1) \cdot N_2} = \frac{\mu}{s(s+1)}, \\
\lambda_3 &= \mu \frac{((st)^3 + (st)^2 + st + 1) \cdot M_3}{(s+1)((st)^3 + (st)^2 + st + 1) \cdot N_3} = \frac{\mu(s-1)}{s^2(s+1)}, \\
\lambda_4 &= \mu \frac{((st)^3 + (st)^2 + st + 1) \cdot M_4}{(s+1)((st)^3 + (st)^2 + st + 1) \cdot N_4} = \frac{\mu(s^2 - s + 1)}{(s+1)s^3}.
\end{aligned}$$

For every $O \in \mathcal{F}$, put $k_O := |\Gamma' \cap O|$. Summing over all elements O of $\mathcal{F}$, we obtain

$$\begin{aligned}
\sum 1 &= \mu, \\
\sum k_O &= (s'+1)((s't')^3 + (s't')^2 + s't' + 1) \cdot \frac{\mu}{s+1}, \\
\sum k_O(k_O - 1) &= (s'+1)((s't')^3 + (s't')^2 + s't' + 1) \cdot s'^2 t'(t'+1) \frac{\mu}{s(s+1)}
\end{aligned}$$

$$+(s'+1)((s't')^3+(s't')^2+s't'+1)s'^3t'^2(t'+1)\frac{\mu(s-1)}{s^2(s+1)}$$
$$+(s'+1)((s't')^3+(s't')^2+s't'+1)s'^4t'^3\frac{\mu(s^2-s+1)}{(s+1)s^3}.$$

Hence,

$$\begin{aligned}\sum k_O^2 &= \mu(s'+1)((s't')^3+(s't')^2+s't'+1)\cdot\Big(\frac{s'^2t'(t'+1)}{s(s+1)}\\ &\quad+\frac{s'^3t'^2(t'+1)(s-1)}{s^2(s+1)}+\frac{s'^4t'^3(s^2-s+1)}{(s+1)s^3}+\frac{1}{s+1}\Big).\end{aligned}$$

From the Cauchy-Schwartz inequality $(\sum k_O^2)\cdot(\sum 1)\geq(\sum k_O)^2$, we obtain

$$\begin{aligned}&\frac{s'^2t'(t'+1)}{s(s+1)}+\frac{s'^3t'^2(t'+1)(s-1)}{s^2(s+1)}+\frac{s'^4t'^3(s^2-s+1)}{(s+1)s^3}+\frac{1}{s+1}\\ &\qquad\geq\frac{1}{(s+1)^2}\cdot(s'+1)((s't')^3+(s't')^2+s't'+1),\end{aligned}$$

i.e.,

$$\begin{aligned}&s^2(s+1)s'^2t'(t'+1)+s(s^2-1)s'^3t'^2(t'+1)+(s+1)s'^4t'^3(s^2-s+1)+s^3(s+1)\\ &\qquad\geq s^3(s'+1)((s't')^3+(s't')^2+s't'+1).\end{aligned}$$

Bringing all terms to one side and factorizing gives

$$(s-s')(s-s't')(s^2+s'^2t'^2)\geq 0.$$

Since $s>s'$, we find

$$s\geq s't'.$$

By the reasoning above, we know that $s=s't'$ if and only if Γ' intersects every element of $\mathcal{F}$ in a constant number of points. If $s=s't'$, then this constant number of points is equal to

$$\frac{\sum k_O}{\sum 1}=\frac{(s'+1)((s't')^3+(s't')^2+s't'+1)}{(s+1)}=(s'+1)(s^2+1).$$

This proves Theorem 2.4. As told before, the proof of Theorem 2.5 is completely similar.

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