# Transformation formulas for double hypergeometric series related to 9-$j$ coefficients and their basic analogues

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**Abstract**

In a recent paper, Ališauskas deduced different triple sum expressions for the 9-$j$ coefficient of $su(2)$ and $su_q(2)$. For a singly stretched 9-$j$ coefficient, these reduce to different double sum series. Using these distinct series, we deduce a set of new transformation formulas for double hypergeometric series of Kampé de Fériet type and their basic analogues. These transformation formulas are valid for rather general parameters of the series, although a common feature is that all the series appearing here are terminating. It is also shown that the transformation formulas deduced here generate a group of transformation formulas, thus yielding an invariance group or symmetry group of particular double series.

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# I Introduction

The Wigner 9-$j$ coefficients (or 9-$j$ symbols) arise as recoupling coefficients in the coupling (tensor product) of four irreducible representations of $su(2)$, and play an important role in the quantum theory of angular momentum [1, 2, 3]. Although the relation between recoupling coefficients, such as the 3-$j$ coefficient and the 6-$j$ coefficient, and hypergeometric series or (discrete) orthogonal polynomials of hypergeometric type is well understood [3, 4, 5, 6], the 9-$j$ coefficient remains somewhat a mystery in this respect. There are many known expressions for the 9-$j$ coefficient as a multiple hypergeometric series. The most compact formula is the so-called triple sum series, originally derived by457 Ališauskas and Jucys [7], and rederived in [8]. Whether a triple sum expression is really the best one can do for the 9-$j$ coefficient, is not known; specialists in the field still guess that a double sum series might exist [9].

The triple sum series of Ališauskas and Jucys was recognized as a special case [10, 5] of a triple hypergeometric series defined by Srivastava [11]. It was used to speed up the numerical computation of 9-$j$ coefficients [10], and to derive certain summation and reduction formulas for hypergeometric series by using particular classes of 9-$j$ coefficients [12, 13, 14].

Ališauskas and Jucys's triple sum series was recently rederived in two ways. In [15], Rosengren deduced the triple sum series for 9-$j$ coefficients (of $su(1,1)$ rather than of $su(2)$) based upon the use of coupling kernels; in [16], he showed that the same formula can be deduced starting from the classical expansion of the 9-$j$ coefficient in terms of 6-$j$ coefficients and performing Dougall's summation formula [17] for a very well-poised ${}_4F_3(-1)$ series. In a recent paper [18], Ališauskas realized that this technique can be applied for several distinct expansions of the 9-$j$ coefficient in terms of 6-$j$ coefficients. Thus he obtained seven different triple sum formulas for the 9-$j$ coefficient of $su(2)$. At the same time, he showed that this technique has a basic analogue (or $q$-analogue), depending upon a $q$-analogue of Dougall's summation formula [19]. So he also obtained seven triple sum formulas for the 9-$j$ coefficient of $su_q(2)$, i.e. for the $q$-9-$j$ coefficients.

The study of these different triple sum formulas from the point of view of multiple hypergeometric series would be interesting, though rather tedious because of the complicated structure of the formulas. However, when considering the class of singly stretched 9-$j$ coefficients (i.e. one of the arguments in the 9-$j$ coefficient is the sum of two others), most of these triple sum formulas reduce to double sum formulas which are less complicated and easier to handle. Ališauskas actually wrote down these double sum formulas [18, Section IV.B], and used them to derive certain rearrangement formulas of double sum series and their basic analogues.

In the present paper we shall show that the double sum formulas for the singly stretched 9-$j$ coefficient actually give rise to a fairly complete theory of transformation formulas for terminating double hypergeometric series of Kampé de Fériet type. This is particularly interesting because until now not many transformation formulas for multiple hypergeometric series are known, even though transformation formulas for hypergeometric series of a single variable play an important role [17, 20]. The double hypergeometric series appearing in this context are proper Kampé de Fériet functions $F^{p:r;r}_{q:s;s}$ with $q+s=2$ and $p+r=3$. Such functions have been defined in [21, 22], and studied by Srivastava and Karlsson [23], whose notation we follow. This notation is a rather straightforward extension of that for single hypergeometric series, e.g.

$$F^{1:2;2}_{0:2;2}\left[\begin{matrix} e: a,b\,;a',b'\,; \\ :c,d\,;c',d'\,;\end{matrix} x,y\right] = \sum_{j,k=0}^{\infty} (e)_{j+k} \frac{(a)_j(b)_j}{(c)_j(d)_j}\frac{(a')_k(b')_k}{(c')_k(d')_k}\frac{x^j}{j!}\frac{y^k}{k!}, \tag{1}$$

and

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix} e: a,b\,;a',b'\,; \\ d:\ \ c\ \ ;\ \ c'\ \ ;\end{matrix} x,y\right] = \sum_{j,k=0}^{\infty} \frac{(e)_{j+k}}{(d)_{j+k}} \frac{(a)_j(b)_j}{(c)_j}\frac{(a')_k(b')_k}{(c')_k}\frac{x^j}{j!}\frac{y^k}{k!}. \tag{2}$$

Herein, $(a)_k$ is the classical Pochhammer symbol [20, 17],

$$(a)_k = a(a+1)\cdots(a+k-1); \tag{3}$$

$a$, $b$, ... are referred to as the parameters of the series, and $x$, $y$ as the variables. Observe that factors of the form $(d)_{j+k}$ or $(e)_{j+k}$ are responsible for the fact that such double series cannot simply be written as the product of two single hypergeometric series. The Kampé de Fériet series appearing in the context of double sums related to the 9-$j$ coefficients are those of type $F^{1:2;2}_{0:2;2}$, $F^{1:2;2}_{1:1;1}$ and $F^{0:3;3}_{1:1;1}$.

Convergence properties of such Kampé de Fériet series have been considered in [24]. In this paper, however, all the series dealt with are *terminating* series and hence there are no convergence conditions. Note that the termination of Kampé de Fériet series such as (1) or (2) can be assured in two ways :

- a common numerator parameter equals a negative integer : e.g. $e=-n$, with $n$ a positive integer, in (1) or (2) yields a terminating series irrespective of the value of the other parameters;
- two separate numerator parameters are equal to negative integers : e.g. $a=-n$ and $a'=-m$ in (1) or (2), with $m$ and $n$ positive integers.

In both cases the denominator parameters of the Kampé de Fériet series should not be negative integers. If some of the lower parameters are nevertheless negative integers, then they should be smaller (or equal) than the parameters responsible for the termination of the series.

The transformation formulas deduced here from the double sum expressions for 9-$j$ coefficients turn out to be of a quite general nature. Apart from the parameter(s) responsible for the termination, the remaining parameters of the series are completely general. Furthermore, the transformation formulas, together with trivial permutation symmetries, are shown to generate a *symmetry group* for the double hypergeometric series. In other words, we shall show that for each of the double hypergeometric series considered here, there exists a whole set of transformation formulas related to a group action on the parameters of the double series.

In the paper of Ališauskas [18], the emphasis is on the $q$-9-$j$ coefficients, i.e. the 9-$j$ coefficients of $su_q(2)$. So it would be interesting to see if the transformation theory developed here could be generalized to the basic analogue (i.e. the $q$-analogue). This is indeed the case. We shall give and prove a set of new transformation formulas for basic double series. For the notation related to $q$-series and single basic hypergeometric series, we refer to the standard book of Gasper and Rahman [25]. The double series appearing in this context, however, are special cases of general basic double series defined by Srivastava and Karlsson [23, p. 349].

## II The stretched 9-$j$ coefficient and double series

Ališauskas [18] considers the stretched 9-$j$ coefficient denoted by

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j_{12}+j_{34} \end{array} \right\}, \tag{4}$$

which is a transformation coefficient connecting two different ways in which four angular momenta $j_1$, $j_2$, $j_3$ and $j_4$ can be coupled. Since they stand for angular momenta, all the arguments in (4) are nonnegative integers or half-integers. In fact, in [18], the $q$-analogues of such 9-$j$ coefficients are considered, but here we first treat the classical case ($q = 1$).

In [18, Section IV.B], a list of double sum expressions is determined for (4). It is not difficult to rewrite these in terms of double hypergeometric series of Kampé de Fériet type. For example, from [18, (4.3d)] one deduces :

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j_{12}+j_{34} \end{array} \right\} = C \times \tag{5}$$

$$F^{1:2;2}_{0:2;2} \left[ \begin{array}{l} -j_1-j_2+j_{12} : -j_2-j_4+j_{24}, -j_2-j_4-j_{24}-1\,; \\ \qquad\qquad\qquad\;\; : \;\; -2j_2, -j_2-j_4+j_{12}+j_{34}-j_{13} \;\; ; \end{array} \right.$$

$$\left.\begin{matrix} j_{13}-j_1+j_3+1, -j_1-j_3+j_{13}\,; \\ -2j_1, j_4-j_1-j_{34}+j_{13}+1 \quad ; \end{matrix} 1,1\right],$$

where $C$ is some constant. Similarly, [18, (4.4b)] yields :

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j_{12}+j_{34} \end{Bmatrix} = C' \times \tag{6}$$

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix} -j_1-j_2+j_{12} & : -j_2-j_4+j_{24}, 1+j_4-j_2+j_{24}\,; \\ 1-j_1-j_{34}+j_{13}-j_2+j_{24}: & -2j_2 \qquad ; \end{matrix}\right.$$
$$\left.\begin{matrix} j_{13}-j_1+j_3+1, -j_1-j_3+j_{13}\,; \\ -2j_1 \qquad ; \end{matrix} 1,1\right],$$

where $C'$ is another constant. Upon equating the rhs's of (5) and (6), using the actual values of $C$ and $C'$, and relabelling the parameters of the series, one finds :

$$F^{1:2;2}_{0:2;2}\left[\begin{matrix} -n : a,b\,;a',b'\,; \\ : c,d\,;c',d'\,; \end{matrix} 1,1\right] = \frac{(d-a+n-1)!(d-1)!}{(d-a-1)!(d+n-1)!} F^{1:2;2}_{1:1;1}\left[\begin{matrix} -n & : a, c-b\,; a',b'\,; \\ d'+a: & c \quad ; \; c' \quad ; \end{matrix} 1,1\right]. \tag{7}$$

Herein, the parameters satisfy $d+d' = 1-n$, so there are in total eight free parameters (as there are in (4)). Since all the parameters in (4) are nonnegative integers or half-integers, the parameters in (7) in first instance all correspond to integers. In particular, $-n$ corresponds to a negative integer (due to triangular conditions satisfied by the angular momentum coefficients). However, once the equation is rewritten in the form (7), with $\frac{(d-a+n-1)!(d-1)!}{(d-a-1)!(d+n-1)!} = \frac{(d-a)_n}{(d)_n}$, it is obvious that this is a *rationial identity* in the remaining parameters $a$, $b$, $c$, $d$, $a'$, $b'$, $c'$ and $d'$, once $-n$ is a fixed negative integer. Therefore, (8a) holds for arbitrary parameters $a$, $b$, $c$, $d$, $a'$, $b'$, $c'$ and $d'$ (but still subject to the constraint $d+d' = 1-n$). As such, we have found a rather general transformation formula between two terminating Kampé de Fériet series. This proves the first formula of the following theorem :

**Theorem 1** *Let $n$ be a nonnegative integer and $a$, $b$, $c$, $d$, $a'$, $b'$, $c'$ and $d'$ arbitrary parameters with $d+d' = 1-n$. Then the following transformation formulas hold :*

$$F^{1:2;2}_{0:2;2}\left[\begin{matrix} -n : a,b\,;a',b'\,; \\ : c,d\,;c',d'\,; \end{matrix} 1,1\right] = \frac{(d-a)_n}{(d)_n} F^{1:2;2}_{1:1;1}\left[\begin{matrix} -n & : a, c-b\,; a',b'\,; \\ d'+a: & c \quad ; \; c' \quad ; \end{matrix} 1,1\right] \tag{8a}$$

$$= \frac{(d-b+b')_n}{(d)_n} F^{1:2;2}_{0:2;2}\left[\begin{matrix} -n: & c-a, b \quad ; \; c'-a', b' \quad ; \\ & : c, d'-b'+b\,; c', d-b+b'\,; \end{matrix} 1,1\right], \tag{8b}$$

*and*

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix} -n : a,b\,;a',b'\,; \\ d \;:\; c \;\;;\;\; c' \;\;; \end{matrix} 1,1\right] = \frac{(d-b-b')_n}{(d)_n} F^{1:2;2}_{1:1;1}\left[\begin{matrix} -n & : c-a, b\,; c'-a', b'\,; \\ 1-n-d+b+b': & c \quad ; \quad c' \quad ; \end{matrix} 1,1\right]. \tag{9}$$

**Proof.** The transformation formula (8b) was deduced recently in a different context [26]. This equation can now also be seen in the context of the stretched 9-$j$ coefficient. In fact, it corresponds to a symmetry of this 9-$j$ coefficient (namely a transposition of the first and second column in (4)), re-expressed by means of (5). Finally, applying (8a) to the rhs of (8b) and equating the resulting expression with the rhs of (8a) yields (9) (after appropriate relabelling of the parameters). $\square$

Observe that in this section all Kampé de Fériet series are terminating because a common numerator parameter equals a negative integer. In the following section we shall consider some transformation formulas, also deduced from the stretched 9-$j$ coefficient, for Kampé de Fériet series that are terminating because of the appearance of two negative integers as separate numerator parameters.

## III Kampé de Fériet series with two negative integers as parameter

Though the transformation formulas with a single common numerator parameter as a negative integer (i.e. Theorem 1) are new, there do exist some transformation formulas for Kampé de Fériet series with two separate numerator parameters as negative integers. One of these formulas is given by Singh [27], and reads :

$$F^{0:3;3}_{1:1;1}\left[\begin{matrix} : -n, a, b\, ; -m, a', b'\, ; \\ d: \quad c \quad ; \quad c' \quad ; \end{matrix} 1, 1\right] = \frac{(c-a)_n (c'-a')_m}{(c)_n (c')_m} \times F^{0:3;3}_{1:1;1}\left[\begin{matrix} : \quad -n, a, b' \quad ; \quad -m, a', b \quad ; \\ d: 1+a-c-n\, ; 1+a'-c'-m\, ; \end{matrix} 1, 1\right], \tag{10}$$

where $n$ and $m$ are nonnegative integers and $b+b'=d$. This, and some other transformation formulas of similar type, can be found in or deduced from [18, Appendix C].

Let us first consider some transformation formulas that express a Kampé de Fériet series of type $F^{1:2;2}_{0:2;2}$ into a series of a different type :

**Theorem 2** *Let $m$ and $n$ be nonnegative integers, and $a$, $b$, $c$, $d$, $a'$, $b'$ and $c'$ be arbitrary parameters with $c+c'=1+d$, then*

$$F^{1:2;2}_{0:2;2}\left[\begin{matrix} d: -n, a\, ; -m, a'\, ; \\ : \; b, c \;\; ; \;\; b', c' \;\; ; \end{matrix} 1, 1\right]$$

$$= \frac{(b-a)_n(1-c)_m}{(b)_n(c')_m} F^{0:3;3}_{1:1;1}\left[\begin{matrix} : -n, a, -d+c-m\,; -m, b'-a', d\,; \\ c-m: \quad -n+a-b+1 \quad ; \qquad b' \qquad ; \end{matrix} 1, 1\right] \tag{11a}$$

$$= \frac{(1-c)_m}{(c')_m} F^{1:2;2}_{1:1;1}\left[\begin{matrix} d \quad : -n, a\,; -m, b'-a'\,; \\ c-m: \quad b \quad ; \qquad b' \qquad ; \end{matrix} 1, 1\right]. \tag{11b}$$

The proof of (11a) follows by comparing equations (4.3c) and (4.3e) of [18], making appropriate relabellings, and using the same rational expression argument as in the proof of Theorem 1. In a similar way, (11b) follows from (4.3b) and (4.4c) of [18].

It is worth mentioning that transformation formulas (8a) and (11b) are formally equivalent (after rewriting the Pochhammer symbols in terms of Gamma functions and using the constraint $1-c = c'-d$ to eliminate $c$ from the Gamma functions in (11b)).

We can now present three results, giving transformation formulas for Kampé de Fériet series of a particular type into series of the same type, for each of the types $F^{1:2;2}_{0:2;2}$, $F^{1:2;2}_{1:1;1}$ and $F^{0:3;3}_{1:1;1}$.

**Theorem 3** *Let $n$ and $m$ be nonnegative integers and $a$, $b$, $c$, $a'$, $b'$, $c'$ and $d$ be arbitrary parameters with $c+c' = d+1$, then*

$$F^{1:2;2}_{0:2;2}\left[\begin{matrix} d: -n, a\,; -m, a'\,; \\ : \ b, c \ ; \ b', c' \ ; \end{matrix} 1, 1\right] = \frac{(-1)^m (d)_n (b-a)_n (a')_m}{(b)_n (c')_m (b')_m (c)_{n-m}} \tag{12a}$$

$$\times F^{1:2;2}_{0:2;2}\left[\begin{matrix} -m-c'+1: \qquad -n, -n-b+1 \qquad ; \qquad -m, -a'+b' \qquad ; \\ : -n+a-b+1, -n-d+1\,; -m-a'+1, -m+n+c\,; \end{matrix} 1, 1\right]$$

$$= \frac{(-1)^m (d)_m (b-a)_n (a')_m}{(b)_n (b')_m (c')_m}$$

$$\times F^{1:2;2}_{0:2;2}\left[\begin{matrix} -m-c'+1: \qquad -n, a \qquad ; \qquad -m, 1-m-b' \qquad ; \\ : -n+a-b+1, c\,; 1-m-a', 1-m-d\,; \end{matrix} 1, 1\right]. \tag{12b}$$

**Proof.** The first formula, (12a), follows by comparing expressions (4.3a) and (4.3d) of [18], and using the rational expression argument. The second formula is derived using (11a) and Singh's formula (10). □

In (12a) the difference $n-m$ might be negative, and then $c_{n-m} = (-1)^{m-n}/(1-c)_{m-n}$, which is the natural extension of the Pochhammer symbol.

Using the above two formulas and (11b) yields :

**Theorem 4** *Let $n$ and $m$ be nonnegative integers and let $a$, $b$, $a'$, $b'$, $c$ and $d$ be arbitrary parameters, then*

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix} c: -n, a\,; -m, a'\,; \\ d: \quad b \quad ; \quad b' \quad ; \end{matrix} 1, 1\right] = \frac{(c)_{n+m}(b-a)_n(b'-a')_m}{(d)_{n+m}(b)_n(b')_m}$$

$$\times F^{1:2;2}_{1:1;1}\left[\begin{matrix} d-c : -n, -n-b+1\,; -m, -m-b'+1\,; \\ -n-m-c+1 : -n+a-b+1\,; -m+a'-b'+1\,; \end{matrix} 1, 1\right] \tag{13a}$$

$$= \frac{(b-a)_n(b'-a')_m}{(b)_n(b')_m} F^{1:2;2}_{1:1;1}\left[\begin{matrix} d-c: \quad -n, a \quad ; \quad -m, a' \quad ; \\ d \quad : -n+a-b+1\,; -m+a'-b'+1\,; \end{matrix} 1, 1\right]. \tag{13b}$$

As a third and final result, we give the transformation formulas for Kampé de Fériet series of type $F^{0:3;3}_{1:1;1}$. The first formula follows from (12a) and (11a); the second is just Singh's formula (10).

**Theorem 5** *Let $n$ and $m$ be nonnegative integers and let $a$, $b$, $c$, $a'$, $b'$, $c'$ and $d$ be arbitrary parameters such that $b + b' = d$, then*

$$F^{0:3;3}_{1:1;1}\left[\begin{matrix} : -n, a, b\,; -m, a', b'\,; \\ d: \quad c \quad ; \quad c' \quad ; \end{matrix} 1, 1\right] = \frac{(b')_{n+m}(a)_n(c'-a')_m}{(d)_{n+m}(c)_n(c')_m}$$

$$\times F^{0:3;3}_{1:1;1}\left[\begin{matrix} : -n, c-a, 1-n-m-d\,; -m, -c'-m+1, b\,; \\ -n-m+1-b' : \quad -n-a+1 \quad ; \quad -m+a'-c'+1 \quad ; \end{matrix} 1, 1\right] \tag{14a}$$

$$= \frac{(c-a)_n(c'-a')_m}{(c)_n(c')_m} F^{0:3;3}_{1:1;1}\left[\begin{matrix} : \quad -n, a, b' \quad ; \quad -m, a', b \quad ; \\ d: 1+a-c-n\,; 1+a'-c'-m\,; \end{matrix} 1, 1\right]. \tag{14b}$$

# IV Symmetry groups of terminating Kampé de Fériet series

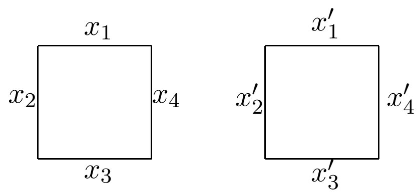
In the previous sections we have determined transformation formulas between (terminating) Kampé de Fériet series of the same type. It is known that transformation formulas of hypergeometric series of a single variable can give rise to a transformation group [28]. This transformation group, known as the symmetry group or invariance group of the series, arises as a finite group acting on the parameters of the series. The existing transformation formulas are then expressed as the invariance of a certain hypergeometric series under the action of group elements on its parameters. For single hypergeometric series (and basic series), this idea has been expanded in [28].

So it would be interesting to see whether there are any invariance groups behind the transformation formulas for double hypergeometric series, as the ones we are dealing with in this paper. One such invariance group for a double series has recently been discussed [26]. This concerns the invariance group related to the transformation formula (8b). Observe that (8b) gives a transformation between two series of the type $F^{1:2;2}_{0:2;2}$. Apart from this transformation, there are also trivial transformations for

$$F^{1:2;2}_{0:2;2}\left[\begin{matrix} -n : a, b\,; a', b'\,; \\ : c, d\,; c', d'\,; \end{matrix} 1, 1\right] \qquad \text{with } d + d' = 1 - n, \tag{15}$$

namely the transposition of $a$ and $b$, or the transposition of $a'$ and $b'$, or the exchange of all primed with the corresponding unprimed parameters. It was shown in [26] that superposing such trivial transformations with (8b) gives rise to a set of 64 transformations for the series $F^{1:2;2}_{0:2;2}$ (with one common numerator parameter equal to $-n$). These 64 transformations correspond to a group $G$ of order 64, that we shall briefly describe because it also plays a role in other transformations considered in this paper.

First, consider the permutation group $S_8$ acting on $(x_1, x_2, x_3, x_4, x'_1, x'_2, x'_3, x'_4)$, and its subgroup $D_8 \times D'_8$. Herein, $D_8$ stands for the dihedral group [29] (sometimes denoted by $D_4$) consisting of the 8 symmetries of the square (i.e. those permutations of $x_1, \ldots, x_4$ that preserve the square whose sides are labelled by $x_1, \ldots, x_4$). Similarly, $D'_8$ is the same dihedral group but acting on the primed labels $x'_1, \ldots, x'_4$. The group $D_8 \times D'_8$ consists of 64 elements; superposing on this group the interchange of primed and unprimed elements yields a group of order 128, denoted by $S_2 \times (D_8 \times D'_8)$. This is the invariance group of two squares whose sides are labelled as follows :




The group $G$ now consists of those 64 elements of $S_2 \times (D_8 \times D'_8)$ that preserve the constraint

$$x_1 + x_3 + x'_1 + x'_3 - x_2 - x_4 - x'_2 - x'_4 = 0, \tag{16}$$

i.e. those elements that map $X = x_1 + x_3 + x'_1 + x'_3 - x_2 - x_4 - x'_2 - x'_4$ into $\pm X$ by permuting the indices. The following proposition [26] then describes the invariance group generated by the transformation (8b) :

**Proposition 6** *Let $x_i$, $x'_i$ $(i = 1, \ldots, 4)$ be arbitrary parameters such that $x_1 + x_3 + x'_1 + x'_3 = x_2 + x_4 + x'_2 + x'_4$ and let $n$ be a nonnegative integer. Then the expression*

$$f_1(x) = (\tfrac{1-n}{2} + x_2 - x'_2)_n \; F^{1:2;2}_{0:2;2}\left[\begin{matrix} -n : & x_2 + x_3, x_1 + x_2 & ; & x'_2 + x'_3, x'_1 + x'_2 & ; \\ & : \sum_i x_i, \tfrac{1-n}{2} + x_2 - x'_2 & ; & \sum_i x'_i, \tfrac{1-n}{2} + x'_2 - x_2 & ; \end{matrix} 1, 1\right]$$

*is (upto a sign) invariant under the action of $G$. The action of an element $g$ of $G$ is by permuting the indices of $x_1, \ldots, x_4'$, and we can write*

$$f_1(g \cdot x) = \epsilon^n f_1(x),$$

*where $\epsilon = \pm 1$ is determined by $g(X) = \epsilon X$.*

When determining the invariance group of the series

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix} -n : a, b \,;\, a', b' \,; \\ d \;:\; c \;;\; c' \;; \end{matrix} 1, 1\right] \tag{17}$$

the following relabelling is appropriate :

$$\begin{aligned} &a = x_2 + x_3, \qquad b = x_1 + x_2, \qquad c = \sum_i x_i, \\ &a' = x_2' + x_3', \qquad b' = x_1' + x_2', \qquad c' = \sum_i x_i', \qquad d = \frac{1-n}{2} + x_2 + x_2'. \end{aligned} \tag{18}$$

Here again, $-n$ is a negative integer and $x_1, \ldots, x_4'$ are arbitrary parameters satisfying (16). Using this relabelling in (17), the transformation (9) corresponds (apart from a factor) to the permutation $g_1 = (x_1\ x_2)(x_3\ x_4)(x_1'\ x_2')(x_3'\ x_4')$. The trivial transposition of $a$ and $b$ in (17) corresponds to the permutation $g_2 = (x_1\ x_3)$. And the interchange of primed and unprimed parameters in (17) corresponds to the permutation $g_3 = (x_1\ x_1')(x_2\ x_2')(x_3\ x_3')(x_4\ x_4')$. It is now easy to see that the elements $g_1$, $g_2$ and $g_3$ generate the group $G$ described earlier. Thus we have the following result :

**Proposition 7** *Let $x_i$, $x_i'$ $(i = 1, \ldots, 4)$ be arbitrary parameters such that $x_1 + x_3 + x_1' + x_3' = x_2 + x_4 + x_2' + x_4'$ and let $n$ be a nonnegative integer. Then the expression*

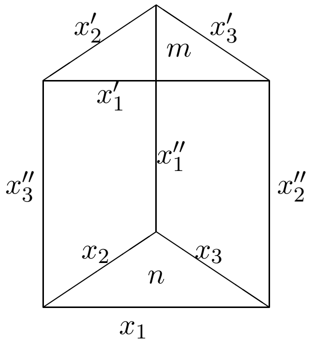
$$f_2(x) = (\tfrac{1-n}{2} + x_2 + x_2')_n\ F^{1:2;2}_{1:1;1}\left[\begin{matrix} -n & : x_2 + x_3, x_1 + x_2 \,;\, x_2' + x_3', x_1' + x_2' \,; \\ \frac{1-n}{2} + x_2 + x_2' : & \sum_i x_i \quad ; \quad \sum_i x_i' \quad ; \end{matrix} 1, 1\right]$$

*is (upto a sign) invariant under the action of $G$, i.e. $f_2(g \cdot x) = \epsilon^n f_2(x)$, where $\epsilon = \pm 1$ is determined by $g(X) = \epsilon X$.*

So the invariance groups of (15) and (17) are the same : both series have 64 symmetries. Moreover, the two non-trivial transformations (8b) and (9) both correspond to the same element, namely $g_1$, of $G$.

Now we shall show that also the transformations with two numerator parameters $-n$ and $-m$ being negative integers give rise to an interesting symmetry group. It will be convenient to first describe the group, and then show that under a certain relabelling of the parameters it is indeed the symmetry group of the transformations given in Theorems 3, 4 and 5.

Consider a prism with an equiangular triangle as basis and edges orthogonal to this basis. The sides of the triangles are labelled by $x_1$, $x_2$, $x_3$ and $x_1'$, $x_2'$, $x_3'$; the three edges are labelled by $x_1''$, $x_2''$, $x_3''$. For convenience we shall also label the basis triangle by $n$ and the opposite triangle by $m$ :




The symmetry group $H$ of this prism is generated by four planes of symmetry : the three planes of symmetry through an edge $x_i''$ $(i = 1, 2, 3)$ and the plane of symmetry parallel with the basis. Let $r_i$ $(i = 1, 2, 3)$ denote the reflection about a plane of symmetry through an edge $x_i''$, and let $r_0$ denote the reflection about the plane of symmetry that is parallel with the basis. These four reflections map the prism into itself, and they generate the symmetry group of the prism. This symmetry group $H$ is a group of order 12, and it is easy to verify that it is isomorphic to the dihedral group $D_{12}$ (i.e. the symmetries of the hexagon). The generating reflections correspond to permutations of $x_1, x_2, \ldots, x_3''$ (and possibly an interchange of $n$ and $m$) :

$$
\begin{aligned}
r_1 &: (x_2\ x_3)(x_2'\ x_3')(x_2''\ x_3''), \\
r_2 &: (x_1\ x_3)(x_1'\ x_3')(x_1''\ x_3''), \\
r_3 &: (x_1\ x_2)(x_1'\ x_2')(x_1''\ x_2''), \\
r_0 &: (x_1\ x_1')(x_2\ x_2')(x_3\ x_3')(n\ m).
\end{aligned}
$$

It turns out that the transformations given in Theorems 3, 4 and 5 all have the same symmetry group, described by $H$. Thus we can state the following :

**Proposition 8** *Let $m$ and $n$ be nonnegative integers, and let $x_i$, $x'_i$, $x''_i$ $(i = 1,2,3)$ be arbitrary parameters such that $\sum_{i=1}^3 x_i = 0$, $\sum_{i=1}^3 x'_i = 0$, $\sum_{i=1}^3 x''_i = 0$. Then the following expressions*

$$g_1(x) = (\tfrac{2(1-n)}{3} - x_1)_n(\tfrac{2-2n+m}{3} - x''_2)_n(\tfrac{2(1-m)}{3} - x'_1)_m(\tfrac{2-2m+n}{3} - x''_3)_m \tag{19}$$
$$\times \quad F^{1:2;2}_{0:2;2}\left[\begin{matrix} \frac{1-n-m}{3} + x''_1 : & -n, \frac{1-n}{3} + x_2 & ; & -m, \frac{1-m}{3} + x'_3 & ; \\ & : \frac{2(1-n)}{3} - x_1, \frac{2-2n+m}{3} - x''_2 & ; & \frac{2(1-m)}{3} - x'_1, \frac{2-2m+n}{3} - x''_3 & ; \end{matrix} 1, 1\right],$$

$$g_2(x) = (\tfrac{2(1-n-m)}{3} + x''_2)_{n+m}(\tfrac{2(1-n)}{3} - x_3)_n(\tfrac{2(1-m)}{3} - x'_3)_m \tag{20}$$
$$\times \quad F^{1:2;2}_{1:1;1}\left[\begin{matrix} \frac{1-n-m}{3} - x''_3 & : -n, \frac{1-n}{3} + x_2 & ; -m, \frac{1-m}{3} + x'_2 & ; \\ \frac{2(1-n-m)}{3} + x''_2 : & \frac{2(1-n)}{3} - x_3 & ; \frac{2(1-m)}{3} - x'_3 & ; \end{matrix} 1, 1\right],$$

$$g_3(x) = (\tfrac{2(1-n-m)}{3} - x''_1)_{n+m}(\tfrac{2(1-n)}{3} + x_2)_n(\tfrac{2(1-m)}{3} + x'_3)_m \tag{21}$$
$$\times \quad F^{0:3;3}_{1:1;1}\left[\begin{matrix} & : -n, \frac{1-n}{3} - x_1, \frac{1-n-m}{3} + x''_2 & ; -m, \frac{1-m}{3} - x'_1, \frac{1-n-m}{3} + x''_3 & ; \\ \frac{2(1-n-m)}{3} - x''_1 : & \frac{2(1-n)}{3} + x_2 & ; \frac{2(1-m)}{3} + x'_3 & ; \end{matrix} 1, 1\right],$$

*are (upto a sign) invariant under the action of $H$, the symmetries of the prism, i.e. $g_1(h \cdot x) = (-1)^{l_0(n+m)} g_1(x)$, $g_2(h \cdot x) = (-1)^{l(n+m)} g_2(x)$ and $g_3(h \cdot x) = (-1)^{l_0(n+m)} g_3(x)$, where $l$ is the number of reflections $r_1$, $r_2$, $r_3$ in the expression of $h$ and $l_0$ is the number of reflections $r_0$, $r_1$, $r_2$, $r_3$ in the expression of $h$.*

**Proof.** Consider (19). Equation (12a) of Theorem 3 expresses that $g_1(h_1 \cdot x) = g_1(x)$, with $h_1 = (x_1\ x_3\ x_2)(x'_1\ x'_3\ x'_2)(x''_1\ x''_3\ x''_2)$. Similarly, equation (12b) of Theorem 3 expresses that $g_1(h_2 \cdot x) = (-1)^{m+n} g_1(x)$, with $h_2 = (x_1\ x_3)(x'_1\ x'_3)(x''_1\ x''_3)$. Apart from the two transformations given in Theorem 3, there is of course also the trivial transformation interchanging $-n, a, b, c$ with $-m, a', b', c'$; this expresses that $g_1(h_3 \cdot x) = g_1(x)$ with $h_3 = (x_1\ x'_1)(x_2\ x'_3)(x_3\ x'_2)(x''_2\ x''_3)(n\ m)$. It is now easy to verify that $h_1$, $h_2$ and $h_3$ generate $H$, i.e. the same group as generated by $r_i$ $(i = 0,1,2,3)$. Thus the symmetry statement for (19) follows. The remaining cases (20) and (21) follow in a similar way from Theorems 4 and 5. □

# V Basic analogues of some transformation formulas

In this section we shall be dealing with the basic analogues (or $q$-analogues) of some of the transformation formulas for double hypergeometric series considered in sections II and III. For a general introduction and background to basic hypergeometric series, see [25], whose notation we follow : thus $q$ is a parameter with $|q| < 1$; $(a;q)_n$ is the $q$-shifted factorial; $(a,b,c;q)_n$ stands for $(a;q)_n(b;q)_n(c;q)_n$; ${}_{p+1}\Phi_p$ is the common notation for a basic hypergeometric series in one variable; etc.

The double basic hypergeometric series appearing in the present context is a special case of general double basic series defined by Srivastava and Karlsson [23, p. 349]. So we use their notation to define the series

$$\Phi^{1:2;2}_{0:2;2}\left[\begin{array}{l} e : a, b \,;\, a', b' \,;\, q; x, y \\ \ \ : c, d \,;\, c', d' \,;\, \lambda, \mu, \nu \end{array}\right] = \\ \sum_{j,k=0}^{\infty} q^{\frac{\lambda}{2}j(j-1)+\frac{\mu}{2}k(k-1)+\nu jk}(e;q)_{j+k}\frac{(a;q)_j(b;q)_j}{(c;q)_j(d;q)_j}\frac{(a';q)_k(b';q)_k}{(c';q)_k(d';q)_k}\frac{x^j}{(q;q)_j}\frac{y^k}{(q;q)_k}; \qquad (22)$$

the definition of $\Phi^{1:2;2}_{1:1;1}$ and $\Phi^{0:3;3}_{1:1;1}$ is completely analogous. For double basic series such as (22), $\nu$ is usually taken to be 0, in which case this is a straightforward double series analogue of the basic series ${}_3\Phi_2$. However, also the cases with $\nu = +1$ or $\nu = -1$ appear in the literature [30, 27], and will play a role in the transformation formulas given here.

The main purpose of this section is to show that the different expressions of $q$-9-$j$ coefficients of [18], in the singly stretched case, give rise to new transformation formulas for double basic hypergeometric series of the type $\Phi^{1:2;2}_{0:2;2}$, $\Phi^{1:2;2}_{1:1;1}$ and $\Phi^{0:3;3}_{1:1;1}$. Ališauskas actually realized that his expressions gave rise to "rearrangement formulas of double sums" (see [18, Appendix C]), but he did not write them as transformation formulas of series of the type (22). Furthermore, he did not recognize that some of these formulas allow for a set of very general parameters.

In this section we shall discuss some of the $q$-analogues of theorems given in sections II and III. Rather than derive these $q$-analogues from the different double series expressions of Ališauskas [18], a direct proof is given. It turns out that the direct proofs of such transformation formulas are fairly easy, and all rely on the same technique.

We know of two genuine transformation formulas for double basic hypergeometric series that have appeared in the literature (by a genuine transformation formula, we mean a formula expressing a basic double series of a particular type into another series of the same type). One of these was given by Singh [27] : if $m$ and $n$ are nonnegative integers, and $a, b, c, a', b', c'$

and $d$ arbitrary parameters with $bb' = d$, then

$$\Phi^{0:3;3}_{1:1;1}\left[\begin{matrix} : q^{-n}, a, b \,;\, q^{-m}, a', b' \,;\, q; cdq^n/ab, c'dq^m/a'b' \\ d : \quad c \quad ; \quad c' \quad ; \quad 0,0,1 \end{matrix}\right] =$$
$$\frac{(c/a;q)_n (c'/a';q)_m}{(c;q)_n (c';q)_m}\, \Phi^{0:3;3}_{1:1;1}\left[\begin{matrix} : q^{-n}, a, b' \,;\, q^{-m}, a', b \,;\, q; q, q \\ d : q^{1-n}a/c \,;\, q^{1-m}a'/c' \,;\, 0,0,0 \end{matrix}\right]. \tag{23}$$

The other was the topic of a recent paper [26] : if $n$ is a nonnegative integer, and $a, b, c, d, a', b', c'$ and $d'$ are arbitrary parameters with $dd' = q^{1-n}$, then

$$\Phi^{1:2;2}_{0:2;2}\left[\begin{matrix} q^{-n} : a, b \,;\, a', b' \,;\, q; cdq^n/ab, c'd'q^n/a'b' \\ : c, d \,;\, c', d' \,; \quad 0,0,-1 \end{matrix}\right] =$$
$$\frac{(d'b/b';q)_n}{(d';q)_n} b^{-n}\, \Phi^{1:2;2}_{0:2;2}\left[\begin{matrix} q^{-n} : \ c/a, b \,;\, c'/a', b' \,;\, q; q, q \\ : c, d'b/b' \,;\, c', db'/b \,;\, 0,0,0 \end{matrix}\right]. \tag{24}$$

Observe that (23) is the basic analogue of (14b), and (24) is the basic analogue of (8b).

We shall now indicate how such formulas, and others, can be derived directly. First, we shall deduce two basic analogues of (8a), namely : if $dd' = q^{1-n}$ then

$$\Phi^{1:2;2}_{0:2;2}\left[\begin{matrix} q^{-n} : a, b \,;\, a', b' \,;\, q; cdq^n/ab, y \\ : c, d \,;\, c', d' \,; \quad 0,0,-1 \end{matrix}\right] = \frac{(d/a;q)_n}{(d;q)_n} \Phi^{1:2;2}_{1:1;1}\left[\begin{matrix} q^{-n} : a, c/b \,;\, a', b' \,;\, q; q, ay \\ ad' : \quad c \quad ; \quad c' \quad ; \ 0,0,0 \end{matrix}\right], \tag{25}$$

and

$$\Phi^{1:2;2}_{0:2;2}\left[\begin{matrix} q^{-n} : a, b \,;\, a', b' \,;\, q; q, y \\ : c, d \,;\, c', d' \,;\, 0,0,0 \end{matrix}\right] = \frac{a^n (d/a;q)_n}{(d;q)_n} \Phi^{1:2;2}_{1:1;1}\left[\begin{matrix} q^{-n} : a, c/b \,;\, a', b' \,;\, q; bq/d, y \\ ad' : \quad c \quad ; \quad c' \quad ; \quad 0,0,0 \end{matrix}\right]. \tag{26}$$

Herein, as usual, $n$ is a nonnegative integer, $a, b, c, d, a', b', c'$ and $d$ are arbitrary parameters (subject to $dd' = q^{1-n}$), and $y$ is an arbitrary variable.

For a proof, expand the lhs $L$ of (25) into a double series :

$$\begin{aligned} L &= \sum_{j,k} \frac{(q^{-n};q)_{j+k}(a,b;q)_j (a',b';q)_k}{(q,c,d;q)_j (q,c',d';q)_k} \left(\frac{cdq^n}{ab}\right)^j y^k q^{-jk} \\ &= \sum_k \frac{(q^{-n},a',b';q)_k}{(q,c',d';q)_k} y^k \, {}_3\Phi_2\left[\begin{matrix} q^{-n+k}, a, b \,; \\ c, d \quad ; \end{matrix} q, cdq^{n-k}/ab\right]. \end{aligned}$$

Now apply Sears' transformation formula [25, (III.13)], and expand again :

$$\begin{aligned} L &= \sum_k \frac{(q^{-n},a',b';q)_k}{(q,c',d';q)_k} y^k \frac{(d/a;q)_{n-k}}{(d;q)_{n-k}} \, {}_3\Phi_2\left[\begin{matrix} q^{-n+k}, a, c/b \,; \\ c, aq^{1-n+k}/d \,; \end{matrix} q, q\right] \\ &= \sum_{j,k} \frac{(q^{-n};q)_{j+k}(a,c/b;q)_j (a',b';q)_k}{(q,c,aq^{1-n+k}/d;q)_j (q,c',d';q)_k} y^k q^j \frac{(d/a;q)_{n-k}}{(d;q)_{n-k}}. \end{aligned} \tag{27}$$

Using $d' = q^{1-n}/d$, and elementary properties of $q$-shifted factorials, there comes

$$\frac{1}{(aq^{1-n+k}/d;q)_j(d';q)_k}\frac{(d/a;q)_{n-k}}{(d;q)_{n-k}} = \frac{a^k}{(aq^{1-n}/d;q)_{j+k}}\frac{(d/a;q)_n}{(d;q)_n}.$$

Plugging this in (27) yields the rhs of (25). The proof of (26) is completely analogous.

Now we can give the basic analogue of (9) :

**Proposition 9** *Let $n$ be a nonnegative integer and $a, b, c, a', b', c'$ and $d$ be arbitrary parameters, then*

$$\Phi^{1:2;2}_{1:1;1}\left[\begin{matrix} q^{-n} : a, b\,;\, a', b'\,;\, q; q, dc'q^n/a'b' \\ d \;:\; c \;;\; c' \;;\; 0,0,0 \end{matrix}\right] =$$
$$\frac{b^n(d/bb';q)_n}{(d;q)_n}\,\Phi^{1:2;2}_{1:1;1}\left[\begin{matrix} q^{-n} & : c/a, b\,;\, c'/a', b'\,;\, q; b'aq/d, q \\ q^{1-n}bb'/d : & c \;;\; c' \;;\; 0,0,0 \end{matrix}\right]. \tag{28}$$

**Proof.** This is now straightforward : apply (25) to the lhs of (24) and (26) to the rhs of (24). Comparing these expressions yields (28). □

With this, we have given basic analogues of all transformation formulas of section II. Also for the transformation formulas with two separate numerator parameters as negative integers, given in section III, the basic analogues can be deduced. The proof of such formulas uses similar steps as illustrated in the proof of (25) :

(a) rewrite the double sum as a single sum over a term containing a ${}_3\Phi_2$ series;

(b) perform one of Sears' transformation formulas on the ${}_3\Phi_2$ and rewrite the result as a double sum;

(c) make certain simplifications, using the constraint (if present) between the parameters;

(d) if necessary, repeat (a), (b) and (c) on the double sum obtained so far, and finally rewrite it in the standard notation of a double basic hypergeometric series.

Detailed proofs of the remaining formulas in this section will not be given, since they all follow the above technique. In fact, we will not even give the basic analogues of all of the formulas of section III, but just list those corresponding to the transformation formulas of Theorems 3, 4 and 5.

Here are the basic analogues of (12a) and (12b), given in Theorem 3. The first, (12a), has two basic analogues, namely

$$\Phi^{1:2;2}_{0:2;2}\left[\begin{matrix} d: q^{-n}, a\,; q^{-m}, a'\,; q; bcq^n/ad, b'c'q^m/a'd \\ :\ \ b, c\ \ ;\ \ b', c'\ \ ;\ \ 0, 0, -1 \end{matrix}\right]$$
$$= \frac{(-1)^m (d;q)_n (b/a;q)_n (a';q)_m (b'/a'c)^m (c/d)^n q^{\binom{m+1}{2}-mn}}{(b;q)_n (c';q)_m (b';q)_m (c;q)_{n-m}}$$
$$\times\, \Phi^{1:2;2}_{0:2;2}\left[\begin{matrix} q^{1-m}/c' : \ q^{-n}, q^{1-n}/b\ ;\ q^{-m}, b'/a'\ ; q; aq^{m+1}/c, dq^{n+1}/b' \\ : aq^{1-n}/b, q^{1-n}/d\,; q^{1-m}/a', cq^{n-m}\,;\ \ 0, 0, -1 \end{matrix}\right], \quad (29)$$

and

$$\Phi^{1:2;2}_{0:2;2}\left[\begin{matrix} d: q^{-n}, a\,; q^{-m}, a'\,; q; q, q \\ :\ \ b, c\ \ ;\ \ b', c'\ \ ; 0, 0, 0 \end{matrix}\right] = \frac{(-1)^m (d;q)_n (b/a;q)_n (a';q)_m (d/c)^m a^n q^{\binom{m+1}{2}}}{(b;q)_n (c';q)_m (b';q)_m (c;q)_{n-m}}$$
$$\times\, \Phi^{1:2;2}_{0:2;2}\left[\begin{matrix} q^{1-m}/c' : \ q^{-n}, q^{1-n}/b\ ;\ q^{-m}, b'/a'\ ; q; q, q \\ : aq^{1-n}/b, q^{1-n}/d\,; q^{1-m}/a', cq^{n-m}\,; 0, 0, 0 \end{matrix}\right], \quad (30)$$

where in both formulas $cc' = qd$. The basic analogue of (12b) is

$$\Phi^{1:2;2}_{0:2;2}\left[\begin{matrix} d: q^{-n}, a\,; q^{-m}, a'\,; q; cbq^n/ad, y \\ :\ \ b, c\ \ ;\ \ b', c'\ \ ;\ \ 0, 0, -1 \end{matrix}\right] = \frac{(-1)^m (b/a;q)_n (a', d;q)_m q^{-\binom{m+1}{2}} y^m}{(b;q)_n (b', c';q)_m}$$
$$\times\, \Phi^{1:2;2}_{0:2;2}\left[\begin{matrix} q^{1-m}/c' : \ q^{-n}, a\ ;\ q^{-m}, q^{1-m}/b'\ ; q; q, b'q^{m+2}/a'cy \\ : c, aq^{1-n}/b\,; q^{1-m}/a', q^{1-m}/d\,;\ \ 0, 0, 0 \end{matrix}\right], \quad (31)$$

where again $cc' = qd$.

The basic analogues of the formulas in Theorem 4 are given by :

$$\Phi^{1:2;2}_{1:1;1}\left[\begin{matrix} c: q^{-n}, a\,; q^{-m}, a'\,; q; bdq^n/ac, q \\ d:\ \ b\ \ ;\ \ b'\ \ ;\ \ 0, 0, 0 \end{matrix}\right] = \frac{(b/a;q)_n (b'/a';q)_m (c;q)_{n+m} (a')^m (d/c)^n}{(b;q)_n (b';q)_m (d;q)_{n+m}}$$
$$\times\, \Phi^{1:2;2}_{1:1;1}\left[\begin{matrix} d/c : q^{-n}, q^{1-n}/b\,; q^{-m}, q^{1-m}/b'\,; q; aq^{1-m}/d, q \\ q^{1-n-m}/c :\ \ aq^{1-n}/b\ ;\ a'q^{1-m}/b'\ ;\ \ 0, 0, 0 \end{matrix}\right] \quad (32)$$

and

$$\Phi^{1:2;2}_{1:1;1}\left[\begin{matrix} c: q^{-n}, a\,; q^{-m}, a'\,; q; dbq^n/ac, q \\ d:\ \ b\ \ ;\ \ b'\ \ ;\ \ 0, 0, 0 \end{matrix}\right] = \frac{(a')^m (b/a;q)_n (b'/a';q)_m}{(b;q)_n (b';q)_m}$$
$$\times\, \Phi^{1:2;2}_{1:1;1}\left[\begin{matrix} d/c : \ q^{-n}, a\ ;\ q^{-m}, a'\ ; q; q, cq/b' \\ d\ : aq^{1-n}/b\,; a'q^{1-m}/b'\,;\ \ 0, 0, 0 \end{matrix}\right]. \quad (33)$$

Finally, the basic analogues of the transformation formulas (14a) and (14b) of Theorem 5 are given by

$$\Phi^{0:3;3}_{1:1;1}\left[\begin{matrix} : q^{-n}, a, b\,; q^{-m}, a', b'\,; q; q, q \\ d:\ \ c\ \ ;\ \ c'\ \ ; 0, 0, 0 \end{matrix}\right] = \frac{(a')^m b^n (b';q)_{n+m} (a';q)_n (c'/a';q)_m}{(d;q)_{n+m} (c;q)_n (c';q)_m}$$
$$\times\, \Phi^{0:3;3}_{1:1;1}\left[\begin{matrix} : q^{-n}, c/a, q^{1-n-m}/d\,; q^{-m}, q^{1-m}/c', b\,; q; q, q \\ q^{1-n-m}/b' :\ \ q^{1-n}/a\ \ ;\ \ q^{1-m}a'/c'\ \ ; 0, 0, 0 \end{matrix}\right] \quad (34)$$

and (23), where $bb' = d$ in both formulas.

This completes the list of $q$-analogues of the transformation formulas of Kampé de Fériet series with two nonnegative integers as parameters, as given in Theorems 3, 4 and 5.

## VI Summary

Using the different double sum expressions of Ališauskas [18] for a singly stretched 9-$j$ coefficient of $su(2)$ or $su_q(2)$, we have deduced a set of new transformation formulas for double hypergeometric series of Kampé de Fériet type and their basic analogues. An important observation is that these transformation formulas are valid for quite general parameters, even though the original 9-$j$ coefficients assume only nonnegative integer or half-integer values as arguments. The transformation formulas given here for double hypergeometric series of Kampé de Fériet type are all terminating, which means that either a common numerator parameter, or else two separate numerator parameters are negative integers.

The transformation formulas seem to inherit some of the symmetries of the 9-$j$ coefficient. In particular, we have shown that the given transformation formulas for a double series of a particular type generate a symmetry group, acting on the parameters of the series. These symmetry groups are explicitly determined and described as subgroups of permutation groups, or as symmetry groups of some geometric object.

In the case of basic double hypergeometric series, corresponding to different expressions of 9-$j$ coefficients of $su_q(2)$, the relevant series is a double $q$-series as defined in [23]. Also for these series, the transformation formulas are listed, and we have shown that an independent proof of such transformations is easy.

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