Geometric aspects of nonholonomic field theories

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February 1, 2006

Abstract

A geometric model for nonholonomic Lagrangian field theory is studied. The multisymplectic approach to such a theory as well as the corresponding Cauchy formalism are discussed. It is shown that in both formulations, the relevant equations for the constrained system can be recovered by a suitable projection of the equations for the underlying free (i.e. unconstrained) Lagrangian system.

1 Introduction

During the past decades, much effort has been devoted to the differential geometric treatment of mechanical systems subjected to nonholonomic (i.e. velocity-dependent) constraints. To a large extent the growing interest in this field has been stimulated by its close connection to problems in control theory (see for instance the recent books [4, 9], which also contain extended bibliographies on the subject). As far as the study of Lagrangian (or Hamiltonian) systems with nonholonomic constraints is concerned, one can essentially distinguish

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between two different approaches. The first one, commonly called "nonholonomic mechanics", is based on d'Alembert's principle and on an additional rule that specifies a bundle of covectors along the constraint submanifold, representing the admissible reaction forces (or, alternatively, a subbundle of the tangent bundle, representing the admissible infinitesimal virtual displacements). The second one is a constrained variational approach, often coined "vakonomic mechanics". As is well-known, the dynamical equations generated by both approaches are in general not equivalent (for that matter, see for instance [1, 10, 21]).

In this paper we will study an extension of nonholonomic mechanics to the treatment of (classical) Lagrangian field theories with external constraints. For the time being, we will confine ourselves to first-order field theories with constraints that depend on the independent variables (say the space-time coordinates), the fields and their first-order partial derivatives only. For the sake of clarity it should be emphasized that we use the term "classical field theory" here in a broad sense, i.e. it essentially refers to any physical (or other) system that can be described by the Euler-Lagrange equations derived from a Lagrangian density. The constraints involved are called 'external' to distinguish them from the type of constraints appearing in gauge theories that find their origin in the possible degeneracy of the Lagrangian.

Constrained field theories have already been studied extensively in the literature but, as far as the case of external constraints is concerned, those treatments are usually based on a field-theoretic analogue of vakonomic mechanics (see e.g. [24]). The nonholonomic approach we are going to discuss here is a continuation of some work by E. Binz *et al.* [2]. We will refer to this theory as "nonholonomic field theory" mainly for its formal analogy with nonholonomic mechanics and also to distinguish it from the constrained variational approach. However, the term nonholonomic should perhaps be used here with some caution. Indeed, as can be inferred from a remark in [24], there does not seem to exist a general agreement yet on the precise meaning of the notion of nonholonomic constraint in field theory. In that respect, a recent paper by O. Krupkova [14] may come to the rescue with a view on straightening out this matter, at least from a purely mathematical point of view.

The mathematical framework for a nonholonomic field theory that has been proposed in [2] involves, among others, a generalization of d'Alembert's principle and of the so-called Chetaev rule that is commonly used in nonholonomic mechanics to characterize the bundle of constraint forms representing the admissible reaction forces. The constrained field equations, as well as an extension of the so-called De Donder-Weyl equations for classical field theories, are then derived in a finite-dimensional multisymplectic setting. The treatment in [2] also briefly deals with the Cauchy formalism for nonholonomic field theory.

The purpose of the present paper is to continue and extend the work described in [2]. First of all, we will relax the Chetaev-type conditions by allowing for constraint forms that need not be determined by the constraints themselves. Next, in the multisymplectic setting we will construct a kind of projection operator that maps solutions of the free (i.e. unconstrained) De Donder-Weyl equations into solutions of the modified (constrained) De Donder-Weyl equations that have been proposed in [2]. This is similar to the situation encountered in nonholonomic mechanics (see e.g. [15]). Finally, we also treat the Cauchy formalism for nonholonomic field theory. In particular, we will show that in case a global 'space-time' splitting of the base manifold can be fixed, the resulting structures and equations on the infinite-dimensional space of Cauchy data are reminiscent of those appearing in the geometric treatment of time-dependent nonholonomic mechanics. This is in full agreement with the results described for the unconstrained case by A. Santamaría [26] (see also [17]).

The scheme of the paper is as follows. In the next section we recall some basic elements from the multisymplectic approach to (unconstrained) Lagrangian field theories on jet bundles, mainly in order to fix some of the notation that will be used. In Sections 3 and 4 we then discuss the construction of a nonholonomic model for first-order Lagrangian field theories with external constraints. The corresponding constrained field equations are given, as well as the modified De Donder-Weyl equations. Next, a projector is constructed that maps solutions of the De Donder-Weyl equations for the unconstrained Lagrangian system into solutions of the constrained De Donder-Weyl equations. In Section 5, as an example to illustrate the theory, we briefly consider the case of incompressible hydrodynamics. Section 6 is devoted to the Cauchy formalism for nonholonomic field theory and in Section 7 we conclude with some general comments.

2 Multisymplectic approach to Lagrangian field theories

There is an extended literature on the multisymplectic approach to classical field theories: see for instance [7, 11, 19], where the interested reader can also find further references. Before reviewing some aspects of that theory, we first briefly recall some basic notions from jet bundle theory for which we mainly rely on the book by D.J. Saunders [27].

2.1 Jet bundles

Consider a fibre bundle $\pi : Y \to X$ of rank m, whose base space X is assumed to be an oriented manifold of dimension $n+1$, equipped with a fixed volume form η. (All manifolds, maps, vector fields and differential forms are assumed to be of class C^∞). The first-order jet bundle $J^1\pi$ is the set of equivalence classes $j^1_x\phi$ consisting of those (local) sections ϕ of π around a point $x \in X$ having the same Taylor expansion up to order one at x. $J^1\pi$ is a $(n+1+m+(n+1)m)$-dimensional manifold, fibred over X with projection $\pi_1 : J^1\pi \longrightarrow X$. In addition, $J^1\pi$ has the structure of an affine bundle over Y of rank $(n+1)m$, modelled on the vector bundle $V\pi \otimes \pi^*(T^*X) \longrightarrow Y$, with $V\pi$ the bundle of π-vertical tangent vectors to Y, and with projection denoted by $\pi_{1,0} : J^1\pi \longrightarrow Y$. In particular, we have $\pi_1 = \pi \circ \pi_{1,0}$.

As far as coordinates on Y are concerned, we will always consider local bundle coordinates adapted to π, written as (x^μ, y^a), $\mu = 1, \dots, n+1; a = 1, \dots, m$, and where the coordinate system (x^μ) on X is taken such that $\eta = d^{n+1}x := dx^1 \wedge \cdots \wedge dx^{n+1}$. The induced bundle coordinates on $J^1\pi$ will then be denoted by (x^μ, y^a, y^a_μ).

Given a vector field ξ on Y, its first jet prolongation to $J^1\pi$ will be written as $\xi^{(1)}$. In coordinates, if

$$\xi = \xi^\mu(x,y)\frac{\partial}{\partial x^\mu} + \xi^a(x,y)\frac{\partial}{\partial y^a}\,,$$

then

$$\xi^{(1)} = \xi^\mu\frac{\partial}{\partial x^\mu} + \xi^a\frac{\partial}{\partial y^a} + \left(\frac{d\xi^a}{dx^\mu} - y^a_\nu\frac{d\xi^\nu}{dx^\mu}\right)\frac{\partial}{\partial y^a_\mu}\,. \tag{1}$$

In terms of the volume form η on X, one can construct a special tensor field S_η on $J^1\pi$, called the *vertical endomorphism* (see [27] for a formal definition). It is a vector-valued $(n+1)$-form that has the following expression in the coordinate system described above:

$$S_\eta = (dy^a - y^a_\nu dx^\nu) \wedge d^nx_\mu \otimes \frac{\partial}{\partial y^a_\mu}\,,$$

where $d^nx_\mu = i_{\frac{\partial}{\partial x_\mu}} d^{n+1}x$. Note in passing that we will not make a notational distinction between the volume form η on X and its pull-back to Y or to $J^1\pi$ under the respective projections.

A 1-form $\theta \in \Lambda^1(J^1\pi)$ is said to be a contact 1-form whenever $(j^1\phi)^*\theta = 0$ for each section ϕ of π. Locally, the module of contact forms is spanned by the 1-forms $\theta^a = dy^a - y^a_\mu\, dx^\mu$.

In this paper, we will frequently deal with connections (in the sense of Ehresmann) on π_1. Such a connection induces in particular a direct sum decomposition $TJ^1\pi = H\pi_1 \oplus V\pi_1$, where $V\pi_1$ is the bundle of π_1-vertical tangent vectors to $J^1\pi$ and $H\pi_1$ is a complementary distribution, called the horizontal distribution. The corresponding horizontal projection operator will always be denoted by $\mathbf{h}$. We recall that a connection on $J^1\pi$ uniquely determines a section Υ of the bundle $(\pi_1)_{1,0} : J^1\pi_1 \to J^1\pi$, called a *jet field*, and that, conversely, each jet field characterizes a unique connection on π_1 (see [27] for details). Therefore, in the sequel we will use the denominations 'connection' (on a jet bundle) and 'jet field' interchangeably.

An *integral section* of a jet field $\Upsilon : J^1\pi \to J^1\pi_1$ is a local section σ of π_1 such that $j^1\sigma = \Upsilon \circ \sigma$. A jet field, or the associated connection, is called *semi-holonomic* if its integral sections σ (if they exist!) are first jet prolongations of sections of π, i.e. $\sigma = j^1\phi$ for some $\phi : X \longrightarrow Y$. Whenever a semi-holonomic jet field admits integral sections, it is called *holonomic* (or integrable). We now recall the conditions, in coordinates, for a jet field to be (semi-)holonomic.

In coordinates, the horizontal projector $\mathbf{h}$ of a connection on π_1 can be written as

$$\mathbf{h} = dx^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + \Gamma^a_\mu \frac{\partial}{\partial y^a} + \Gamma^a_{\mu\nu} \frac{\partial}{\partial y^a_\nu} \right), \tag{2}$$

for some functions $\Gamma^a_\mu(x^\kappa, y^b, y^b_\kappa)$ and $\Gamma^a_{\mu\nu}(x^\kappa, y^b, y^b_\kappa)$, and the associated jet field Υ then reads:

$$\Upsilon : J^1\pi \longrightarrow J^1\pi_1,\ (x^\mu, y^a, y^a_\mu) \mapsto (x^\mu, y^a, y^a_\mu, \Gamma^a_\mu, \Gamma^a_{\mu\nu}).$$

A local section σ of π, with $\sigma(x) = (x^\mu, \sigma^a(x), \sigma^a_\mu(x))$, is an integral section of the jet field Υ if

$$\frac{\partial \sigma^a}{\partial x^\mu} = \Gamma^a_\mu \quad \text{and} \quad \frac{\partial \sigma^a_\mu}{\partial x^\nu} = \Gamma^a_{\mu\nu}.$$

From this expression, it easily follows that the connection will be semi-holonomic if $\Gamma^a_\mu = y^a_\mu$ and holonomic (or integrable) if, in addition, $\Gamma^a_{\mu\nu} = \Gamma^a_{\nu\mu}$.

We wish to emphasize that the previous discussion about connections on jet bundles can be presented in fully intrinsic terms: we refer again to [27] for details. In particular, the condition for a connection (or jet field), with horizontal projector $\mathbf{h}$, to be semi-holonomic, can be expressed by demanding that

$$i_{\mathbf{h}}\theta = 0, \qquad \text{for each contact 1-form } \theta. \tag{3}$$

We will make use of this characterization later on.

2.2 First-order Lagrangian field theories

Given a fibre bundle $\pi : Y \longrightarrow X$ and its associated first-order jet bundle $J^1\pi$, as considered above, we now briefly recall some aspects from the multisymplectic formulation of first-order Lagrangian field theory on $J^1\pi$, where the fields are the (local) sections of π. Consider a Lagrangian density $L\eta$ with L a smooth function (the Lagrangian) defined on $J^1\pi$. We say that L is *regular* if its Hessian matrix is non-degenerate, i.e.

$$\det \left(\frac{\partial^2 L}{\partial y^a_\mu \partial y^b_\nu} \right) \neq 0$$

at each point of $J^1\pi$.

Using the vertical endomorphism S_η we can construct the following $(n+1)$-form on $J^1\pi$:

$$\Theta_L = S_\eta^* dL + L\eta$$

and we then define the $(n+2)$-form $\Omega_L = -d\Theta_L$, called the *Poincaré-Cartan form*. The following coordinate expression for Ω_L will often be convenient:

$$\Omega_L = -\frac{\partial L}{\partial y^a} dy^a \wedge d^{n+1}x - d\left(\frac{\partial L}{\partial y^a_\mu}\right) \wedge (dy^a - y^a_\nu dx^\nu) \wedge d^n x_\mu. \tag{4}$$

If L is regular, which we will always assume in the sequel, the Poincaré-Cartan form is a multisymplectic form according to the following definition.

Definition 2.1 (see [6, 16, 17]) *A closed m-form Ω on a manifold M is called* multisymplectic *if the mapping $v_x \in T_xM \mapsto i_{v_x}\Omega(x) \in \Lambda^{m-1}(T_x^*M)$ is injective for all $x \in M$.*

Note that symplectic forms ($m = 2$) and volume forms ($m = \dim M$) are particular examples of multisymplectic forms and, moreover, these are the only two cases where the mappings in definition 2.1 are surjective as well as injective (assuming M is finite dimensional).

We now give a brief outline of the derivation of the Euler-Lagrange equations for first-order field theories. For more detailed treatments we refer to [3, 19, 27]. The action functional associated to the given Lagrangian density $L\eta$ is defined as

$$\mathcal{S}(\phi) = \int_U (j^1\phi)^* L\eta \tag{5}$$

where U is an open subset of X with compact closure, and ϕ is a section of π defined over U. A section ϕ is an *extremal* of (5) if

$$\frac{d}{dt}\mathcal{S}(\varphi_t \circ \phi)\Big|_{t=0} = 0,$$

for any local flow $\{\varphi_t\}$ on Y, consisting of diffeomorphisms defined on a neighborhood of $\phi(U)$, satisfying $\pi \circ \varphi_t = \pi$, $\varphi_0 = \text{id.}$, and keeping the boundary of $\phi(U)$ fixed. Extremal sections are characterized by the following property.

Proposition 2.2 *A (local) section ϕ of π is an extremal for the action (5) if and only if*

$$(j^1\phi)^*(i_\zeta \Omega_L) = 0, \tag{6}$$

for all vector fields ζ on $J^1\pi$.

Proof: See for instance [3, prop. 7.1.2], [11, thm. 3.1] or [7]. $\boxed{QED}$

In coordinates, equation (6) yields the well-known Euler-Lagrange equations:

$$\frac{\partial L}{\partial y^a} - \frac{d}{dx^\mu}\left(\frac{\partial L}{\partial y^a_\mu}\right) = 0 \quad (a = 1, \ldots, m).$$

A more general problem consists in looking for sections τ of π_1 such that $\tau^*(i_\zeta \Omega_L) = 0$ for all vector fields ζ on $J^1\pi$ (i.e. τ need not be the prolongation of a section of π). This leads to the so-called De Donder equations of Lagrangian field theory. In case of a regular Lagrangian, the De Donder equations are equivalent to the Euler-Lagrange equations (6). In this paper, we will mostly be concerned with a kind of linearized version of these equations that is more easy to handle and is obtained by looking for a connection on π_1 whose integral sections will be extremals of (5). More precisely we have the following important proposition:

Proposition 2.3 *Given a holonomic connection with horizontal projector* $\mathbf{h}$*, then the integral sections of the associated jet field are extremals of (5) if and only if*

$$i_{\mathbf{h}}\Omega_L = n\Omega_L\,. \tag{7}$$

Proof: See [27, thm. 5.5.5] and [18]. *QED*

Moreover, a simple coordinate computation shows that, given a regular Lagrangian L on $J^1\pi$, a connection on π_1 satisfying (7) will automatically be semi-holonomic. Equation (7) is also referred to as the *De Donder-Weyl equation* of Lagrangian field theory.

3 Nonholonomic Lagrangian field theory

We now bring constraints into the picture. Suppose we have a Lagrangian system on $J^1\pi$, with regular Lagrangian L. Let $\mathcal{C} \hookrightarrow J^1\pi$ be a submanifold of $J^1\pi$ of codimension k, representing some external constraints imposed on the system. Although one can certainly consider more general situations (see e.g. [2]), for the sake of clarity, we will confine ourselves to the case that $\mathcal{C}$ projects onto the whole of Y, i.e. $\pi_{1,0}(\mathcal{C}) = Y$, and that the restriction $(\pi_{1,0})_{|\mathcal{C}} : \mathcal{C} \longrightarrow Y$ of $\pi_{1,0}$ to $\mathcal{C}$ is a (not necessarily affine) fibre bundle. In particular the latter is a quite restrictive condition but, with proper caution, one can probably carry out the further analysis under some weaker assumption. Finally, one could still require that $\mathcal{C}$ should not be the first jet bundle of a subbundle of π, but since it does not really affect the present treatment we will not insist on that (i.e. our discussion also covers the case of "holonomic" constraints that do not essentially depend on the derivatives of the fields).

Since $\mathcal{C}$ is a submanifold of $J^1\pi$, one may always find a covering of $\mathcal{C}$ consisting of open subsets U of $J^1\pi$, with $U \cap \mathcal{C} \neq \varnothing$, such that on each $U \in \mathcal{U}$ there exist k functionally independent smooth functions φ_α that locally determine $\mathcal{C}$, i.e.

$$\mathcal{C} \cap U = \{\gamma \in J^1\pi : \varphi_\alpha(\gamma) = 0 \text{ for } 1 \leq \alpha \leq k\}.$$

We remark here that the assumption that $(\pi_{1,0})_{|\mathcal{C}}$ is a fibre bundle implies, in particular, that the matrix $(\dfrac{\partial \varphi_\alpha}{\partial y^a_\mu})(\gamma)$ has maximal rank k at each point $\gamma \in \mathcal{C} \cap U$.

With the given data — a Lagrangian and a constraint submanifold $\mathcal{C}$ — the question now arises how to construct a suitable constrained field theory. As pointed out in the introduction, one possibility is to follow a constrained variational approach as, for instance, in [24]. Inspired by the situation in classical mechanics one may also think of another approach, called "nonholonomic", which involves some additional ingredients (see [2]). For mechanical systems with nonholonomic constraints it is now well-known that the "vakonomic" and the "nonholonomic" equations of motion are not equivalent in general. Equivalence is achieved, however, if the constraint functions can be written as total time derivatives of velocity-independent constraints, i.e. if we are basically dealing with holonomic constraints (see e.g. [10] and references therein).

3.1 The constraint distribution

Making a small digression to nonholonomic mechanics, we recall that the construction of the equations of motion of a mechanical system with nonholonomic constraints is based on the so-called d'Alembert principle and involves, among others, the specification of a suitable bundle of admissible "reaction forces" (and a corresponding bundle of admissible virtual velocities), defined along the constraint submanifold. This choice relies on an additional rule or principle. In nonholonomic mechanics it is quite common to use the so-called Chetaev principle, whereby the bundle of reaction forces is constructed directly in terms of the given constraints. In principle, however, the specification of the appropriate bundle of reaction forces (or virtual displacements), compatible with the given constraints, is problem dependent and need not necessarily be based on Chetaev's rule. For a critical discussion of this matter we refer to [22]; see also [28].

Returning to the case of first-order field theory with external constraints, we now introduce a special subbundle F of rank k of the bundle of exterior $(n+1)$-forms on $J^1\pi$ defined along the constraint submanifold $\mathcal{C}$, where we recall that k is the codimension of $\mathcal{C}$. This bundle, which we will simply refer to as the bundle of constraint forms, will play a role similar to that of the bundle of reaction forces in nonholonomic mechanics. We define this bundle by considering a submodule $\mathcal{F}$ of rank k of the module of $(n+1)$-forms Φ, defined on a neighborhood of $\mathcal{C}$, and which are n-horizontal and 1-contact, i.e. Φ vanishes when contracted with any two

π_1-vertical vector fields and $(j^1\phi)^*\Phi = 0$ for any section ϕ of π. In particular, one can find an open cover $\mathcal{U}$ of $\mathcal{C}$ such that on each open set $U \in \mathcal{U}$, the module $\mathcal{F}$ is generated by k independent $(n+1)$-forms Φ_α that locally read

$$\Phi_\alpha = (C_\alpha)^\mu_a(dy^a - y^a_\nu dx^\nu) \wedge d^n x_\mu = (C_\alpha)^\mu_a \theta^a \wedge d^n x_\mu \,, \tag{8}$$

for some smooth functions $(C_\alpha)^\mu_a$ on U. Independence of the forms Φ_α clearly implies that the $(k \times (n+1)m)$-matrix whose elements are the $(C_\alpha)^\mu_a$, has constant maximal rank k. The *bundle of constraint forms* is then defined by

$$F = \bigcup_{\gamma \in \mathcal{C}} F_\gamma \quad \text{with } F_\gamma = \{\Phi(\gamma) \,|\, \Phi \in \mathcal{F}\} \,.$$

At this point, the reason for selecting a constraint bundle of the type described above is primarily based on the analogy with nonholonomic mechanics.

Remark 3.1 *In [2] the authors have constructed the bundle of constraint forms by considering a natural extension of the Chetaev-principle that is commonly used in mechanics when dealing with nonlinear nonholonomic constraints. More precisely, they define the local generators Φ_α of the bundle of constraint forms by putting*

$$\Phi_\alpha := S^*_\eta(d\varphi_\alpha) \,,$$

where the φ_α are the local constraint functions (see the beginning of Section 3). One easily verifies that these Φ_α are indeed of the form (8), with $(C_\alpha)^\mu_a = \frac{\partial \varphi_\alpha}{\partial y^a_\mu}$. In the case we are considering, the independence of these Φ_α is guaranteed by our initial assumption that $\mathcal{C}$ should have a fibre bundle structure over Y.

As we will now show, the constraint bundle F gives rise to a distribution D along $\mathcal{C}$, called the *constraint distribution.* As above, consider an open cover $\mathcal{U}$ of $\mathcal{C}$ such that on each $U \in \mathcal{U}$, the module $\mathcal{F}$ is generated by k independent $(n+1)$-forms Φ_α of the form (8).

Proposition 3.2 *For each α, there exists a unique vector field $\zeta_\alpha \in \mathfrak{X}(U)$ such that*

$$i_{\zeta_\alpha}\Omega_L = -\Phi_\alpha. \tag{9}$$

Proof: Take ζ_α to be a $\pi_{1,0}$-vertical vector field on U, i.e. $\zeta_\alpha = (\zeta_\alpha)^a_\mu \frac{\partial}{\partial y^a_\mu}$. Herewith, equation (9) reduces to

$$(\zeta_\alpha)^a_\mu \frac{\partial^2 L}{\partial y^a_\mu \partial y^b_\nu} = (C_\alpha)^\nu_b, \tag{10}$$

which determines the $(\zeta_\alpha)^a_\mu$ uniquely, as L is supposed to be regular. This already proves the existence of a solution of (9). Uniqueness then follows from the fact that Ω_L is multisymplectic.

QED

The vector fields ζ_α span a k-dimensional distribution D_U on U. It is not difficult to check that for any two open sets $U, V \in \mathcal{U}$ with nonempty intersection, and for each $\gamma \in U \cap V$, $D_U(\gamma) = D_V(\gamma)$. Indeed, assume $\mathcal{F}$ is generated on U by k independent forms Φ_α and on V by k independent forms $\overline{\Phi}_\alpha$, then there exists a nonsingular matrix of functions r^β_α on $U \cap V$ such that

$$\Phi_\alpha = r^\beta_\alpha \overline{\Phi}_\beta \,.$$

If we denote the corresponding generators of D_U by ζ_α and those of D_V by $\overline{\zeta}_\alpha$, it readily follows from the previous proposition that

$$\zeta_{\alpha|U\cap V} = r^\beta_\alpha \overline{\zeta}_{\beta|U\cap V} \,,$$

which proves that $D_U = D_V$ on $U \cap V$. Consequently, the local distributions described in the previous proposition induce a well-defined (global) distribution D along the constraint submanifold $\mathcal{C}$, whose sections are $\pi_{1,0}$-vertical vector fields. Moreover, using a similar argument as above one easily verifies that this distribution does not depend on the initial choice we made for an open cover $\mathcal{U}$ of $\mathcal{C}$.

3.2 The nonholonomic field equations

Summarizing the above, we are looking for a field theory built on the following data: (i) a Lagrangian density $L\eta$ with regular Lagrangian $L \in C^\infty(J^1\pi)$; (ii) a constraint submanifold $\mathcal{C} \subset J^1\pi$ that can be locally represented by equations of the form $\varphi_\alpha(x^\mu, y^a, y^a_\mu) = 0$, for $\alpha = 1, \ldots, k$ and where the matrix $(\partial\varphi_\alpha/\partial y^a_\mu)$ has maximal rank k; (iii) a bundle F of constraint forms and an induced constraint distribution D, both defined along $\mathcal{C}$, whereby F is locally generated by k independent $(n+1)$-forms (8), and D is defined according to the construction described in Proposition 3.2.

To complete our model for nonholonomic field theory, we now have to specify what the field equations are. Proceeding along the same lines as in [2] we introduce the following definition, using a generalization of d'Alembert's principle.

Definition 3.3 *A (local) section σ of $\pi : Y \longrightarrow X$, defined on an open set $U \subset X$ with compact closure, is a solution of the constrained problem under consideration if $j^1\sigma(U) \subset \mathcal{C}$ and*

$$\int_U (j^1\sigma)^* \mathcal{L}_{\xi^{(1)}} L\eta = 0 \,,$$

for all π-vertical vector fields ξ on Y that vanish on the boundary of $\sigma(U)$ and such that

$$i_{\xi^{(1)}}\Phi = 0 \quad (*)$$

for all sections Φ of the bundle F of constraint forms.

Putting $\xi = \xi^a(x,y)\partial/\partial y^a$ and taking into account the expression (1) for the prolonged vector field $\xi^{(1)}$, it is easily seen that the condition $(*)$ translates into

$$(C_\alpha)^\mu_a \xi^a = 0\,,$$

where the $(C_\alpha)^\mu_a$ are the coefficients of the constraint forms introduced in (8). One can then verify that if $\sigma(x) = (x^\mu, \sigma^a(x))$ is a solution of the constrained problem, then the functions $\sigma^a(x)$ satisfy the following system of partial differential equations

$$\frac{\partial L}{\partial y^a} - \frac{d}{dx^\mu}\left(\frac{\partial L}{\partial y^a_\mu}\right) = \lambda^\alpha_\mu (C_\alpha)^\mu_a \quad (a = 1,\ldots,m)\,, \tag{11}$$

$$\varphi_\alpha(x^\mu, \sigma^a(x), \frac{\partial \sigma^a}{\partial x^\mu}(x)) = 0 \quad (\alpha = 1,\ldots,k)\,. \tag{12}$$

As usual, the (a priori) unknown functions λ^α_μ play the role of 'Lagrangian multipliers'. The equations (11) are called the *nonholonomic field equations* for the constrained problem. Note that if the bundle F of constraint forms is defined according to a Chetaev-type prescription (see Remark 3.1), then we recover the nonholonomic field equations derived in [2].

Let $\mathcal{I}(F)$ be the ideal of differential forms, defined along $\mathcal{C}$, generated by the constraint forms: i.e any element of $\mathcal{I}(F)$ is of the form $\sum_i \lambda_i \wedge \Phi^i$, for some $\Phi^i \in \mathcal{F}$ and arbitrary differential forms λ_i. Again proceeding along the same lines as in [2] we can formulate the following modification of the De Donder-Weyl problem for nonholonomic Lagrangian field theory: find a connection on $\pi_1 : J^1\pi \longrightarrow X$ with horizontal projector $\mathbf{h}$ such that along the constraint submanifold $\mathcal{C}$

$$i_{\mathbf{h}}\Omega_L - n\Omega_L \in \mathcal{I}(F) \quad \text{and} \quad \mathrm{Im}\,\mathbf{h} \subset T\mathcal{C}\,. \tag{13}$$

For simplicity we will refer to (13) as the *nonholonomic De Donder-Weyl equation.* In coordinates, if we represent $\mathbf{h}$ by (2) one can easily check that the relation on the left of (13) leads to the following set of equations for the connection coefficients of the connection we are looking for:

$$(\Gamma^b_\nu - y^b_\nu)\left(\frac{\partial^2 L}{\partial y^a_\mu \partial y^b_\nu}\right) = 0\,,$$

$$\frac{\partial L}{\partial y^a} - \frac{\partial^2 L}{\partial x^\tau \partial y^a_\tau} - \Gamma^b_\tau \frac{\partial^2 L}{\partial y^b \partial y^a_\tau} - \Gamma^b_{\tau\nu}\frac{\partial^2 L}{\partial y^b_\tau \partial y^a_\nu} + (\Gamma^b_\nu - y^b_\nu)\frac{\partial^2 L}{\partial y^a \partial y^b_\nu} = \lambda^\alpha_\tau (C_\alpha)^\tau_a\,,$$

for $a = 1, \ldots, m$ and $\mu = 1, \ldots, n+1$ and some Lagrangian multipliers λ^α_τ. This should still be supplemented by the requirement that for any $\gamma \in \mathcal{C}$ and any $v \in T_\gamma J^1\pi$, $\mathbf{h}(v) \in T_\gamma \mathcal{C}$. This is equivalent to requiring that $\mathbf{h}(v)(\varphi_\alpha) = 0$ for all $v \in T_{\mathcal{C}} J^1\pi$, where φ_α $(\alpha = 1, \ldots, k)$ are the (local) constraint functions. If, locally, $\mathbf{h}$ is written in the form (2), then the previous condition translates into the following additional equations for the connection coefficients in points of $\mathcal{C}$:

$$\frac{\partial \varphi_\alpha}{\partial x^\mu} + \Gamma^b_\mu \frac{\partial \varphi_\alpha}{\partial y^b} + \Gamma^b_{\mu\nu} \frac{\partial \varphi_\alpha}{\partial y^b_\nu} = 0 \quad \text{for all} \quad \mu = 1, \ldots, n+1;\ \alpha = 1, \ldots, k.$$

One can prove that in case of a regular Lagrangian, integral sections of a connection satisfying (13) will be 1-jet prolongations of solutions of the nonholonomic field equations (see [2] for details).

4 The nonholonomic projector

The purpose of the present section is to show that for a nonholonomic first-order field theory in the sense described above, one can construct, under an appropriate additional condition, a projection operator which maps solutions of the De Donder-Weyl equation (7) for the free (i.e. unconstrained) Lagrangian problem into solutions of the nonholonomic De Donder-Weyl equation (13).

Given a constrained problem as described in the previous section, with regular Lagrangian L, constraint manifold $\mathcal{C} \subset J^1\pi$ and constraint distribution D, we now impose the following *compatibility condition*: for each $\gamma \in \mathcal{C}$

$$D(\gamma) \cap T_\gamma \mathcal{C} = \{0\}. \tag{14}$$

If $\mathcal{C}$ is locally defined by k equations $\varphi_\alpha(x^\mu, y^a, y^a_\mu) = 0$ and if D is locally generated by the vector fields ζ_α (see subsection 3.1), a straightforward computation shows that the compatibility condition is satisfied iff

$$\det\left(\zeta_\alpha(\varphi_\beta)(\gamma)\right) \neq 0\,,$$

at each point $\gamma \in \mathcal{C}$. Indeed, take $v \in T_\gamma \mathcal{C} \cap D(\gamma)$. Then $v = v^\alpha \zeta_\alpha(\gamma)$, for some coefficients v^α. On the other hand, $0 = v(\varphi_\beta) = v^\alpha \zeta_\alpha(\varphi_\beta)(\gamma)$. Hence, if the matrix $(\zeta_\alpha(\varphi_\beta)(\gamma))$ is invertible, we may conclude that $v = 0$ and the compatibility condition holds. The proof of the converse is similar.

We now have the following result.

Proposition 4.1 *If the compatibility condition (14) holds, then at each point $\gamma \in \mathcal{C}$ we have the decomposition*

$$T_\gamma J^1\pi = T_\gamma \mathcal{C} \oplus D(\gamma).$$

Proof: The proof immediately follows from (14) and a simple counting of dimensions: $\dim T_\gamma \mathcal{C} = \dim T_\gamma J^1\pi - k$ and $\dim D(\gamma) = k$. $\boxed{QED}$

The direct sum decomposition of $T_\mathcal{C} J^1\pi$ determines two complementary projection operators $\mathcal{P}$ and $\mathcal{Q}$:

$$\mathcal{P} : T_\mathcal{C} J^1\pi \to T\mathcal{C} \quad \text{ and } \quad \mathcal{Q} = I - \mathcal{P} : T_\mathcal{C} J^1\pi \to D\,,$$

where I is the identity on $T_\mathcal{C} J^1\pi$. We will call $\mathcal{P}$ *the nonholonomic projector* associated to the given constrained problem.

Given a connection on π_1 such that the associated horizontal projector $\mathbf{h}$ is a solution of the free De Donder-Weyl equation (7), we will prove that the operator $\mathcal{P} \circ \mathbf{h}_{|T_\mathcal{C} J^1\pi}$ satisfies the constrained De Donder-Weyl equation (13). Note that this operator is only defined along $\mathcal{C}$ and, therefore, strictly speaking it is not the horizontal projector of a connection on π_1. However, one can show that its restriction to $T\mathcal{C}$ induces a genuine connection on the restricted bundle $(\pi_1)_\mathcal{C} : \mathcal{C} \longrightarrow X$, and so the constrained De Donder-Weyl equation still makes sense for this kind of map.

Lemma 4.2 *The map $\mathcal{P} \circ \mathbf{h}_{|T_\mathcal{C} J^1\pi} : T_\mathcal{C} J^1\pi \longrightarrow T\mathcal{C}\,(\subset T_\mathcal{C} J^1\pi), v \longmapsto \mathcal{P}(\mathbf{h}(v))$ is a projector whose restriction $\mathbf{h}_\mathcal{P}$ to $T\mathcal{C}$ induces a connection on $(\pi_1)_{|\mathcal{C}} : \mathcal{C} \longrightarrow X$.*

Proof: First of all, we check that for each $\gamma \in \mathcal{C}$ the map $\mathcal{P}_\gamma \circ \mathbf{h}_\gamma$ is a projector. Indeed, taking into account that $\mathrm{Im}\mathcal{Q} = D$ is $\pi_{1,0}$-vertical, it follows that for all $v \in T_\gamma J^1\pi$

$$(\mathbf{h}_\gamma \circ \mathcal{P}_\gamma)(v) = \mathbf{h}_\gamma(v) - (\mathbf{h}_\gamma \circ \mathcal{Q}_\gamma)(v) = \mathbf{h}_\gamma(v).$$

and therefore

$$(\mathcal{P}_\gamma \circ \mathbf{h}_\gamma)^2 = \mathcal{P}_\gamma \circ \mathbf{h}_\gamma.$$

The restriction $\mathbf{h}_\mathcal{P}$ of $\mathcal{P} \circ \mathbf{h}_{|T_\mathcal{C} J^1\pi}$ to $T\mathcal{C}$ obviously is still a projector. The key point we now have to prove is that $\mathrm{Im}(\mathbf{h}_\mathcal{P})$ is a complementary bundle to $V(\pi_1)_{|\mathcal{C}}$ in $T\mathcal{C}$, i.e.

$$\mathrm{Im}(\mathbf{h}_\mathcal{P}) \oplus V(\pi_1)_{|\mathcal{C}} = T\mathcal{C}\,. \tag{15}$$

For that purpose we start by observing that along $\mathcal{C}$ we have $T\mathcal{C} \cap V\pi_1 = V(\pi_1)_{|\mathcal{C}}$. In view of Proposition 4.1 one can then easily derive the following direct sum decomposition:

$$V(\pi_1)_{|\mathcal{C}} \oplus D = V\pi_1 \quad (\text{along } \mathcal{C}). \tag{16}$$

Next, by taking into account the fact that the constraint distribution D is vertical, and therefore that $\mathbf{h}_\mathcal{P}(T_\gamma\mathcal{C}) = (\mathcal{P} \circ \mathbf{h})(T_\gamma J^1\pi)$ for every $\gamma \in \mathcal{C}$, it is a routine exercise to verify that

$$\dim(\mathcal{P} \circ \mathbf{h})(T_\gamma J^1\pi) = \dim \mathbf{h}(T_\gamma J^1\pi)\,. \tag{17}$$

We now prove the direct sum decomposition (15). Take any $v \in T\mathcal{C}$ with $v \in \mathrm{Im}(\mathbf{h}_{\mathcal{P}}) \cap V(\pi_1)_{|\mathcal{C}}$, then there exists a vector $w \in T\mathcal{C}$ such that $v = \mathcal{P}(\mathbf{h}(w)) = \mathbf{h}(w) - \mathcal{Q}(\mathbf{h}(w))$. Since v is π_1-vertical, we conclude that $\mathbf{h}(w) = 0$ and, hence, $v = 0$. This already implies that $\mathrm{Im}(\mathbf{h}_{\mathcal{P}}) \cap V(\pi_1)_{|\mathcal{C}} = 0$. The equality (15) now follows from a simple dimensional argument. Indeed, relying on Proposition 4.1 as well as on (16) and (17), we have at each point $\gamma \in \mathcal{C}$:

$$\begin{aligned} \dim(\mathbf{h}_{\mathcal{P}}(T_\gamma\mathcal{C})) + \dim V_\gamma(\pi_1)_{|\mathcal{C}} &= \dim(\mathbf{h}(T_\gamma J^1\pi)) + \dim V_\gamma\pi_1 - \dim D(\gamma) \\ &= \dim(T_\gamma J^1\pi) - \dim D(\gamma) \\ &= \dim T_\gamma\mathcal{C}\,. \end{aligned}$$

This concludes the proof that $\mathbf{h}_{\mathcal{P}} = \mathcal{P} \circ \mathbf{h}_{|T\mathcal{C}}$ is the horizontal projector of a connection on $(\pi_1)_{|\mathcal{C}}$. *QED*

Although $(\pi_1)_{|\mathcal{C}} : \mathcal{C} \longrightarrow X$ is not a first-order jet bundle, we will say that a connection on $(\pi_1)_{|\mathcal{C}}$, with associated horizontal projector $\hat{\mathbf{h}}$, is *semi-holonomic* if for each contact 1-form θ on $J^1\pi$

$$i_{\hat{\mathbf{h}}} j^*\theta = 0, \tag{18}$$

where $j : \mathcal{C} \hookrightarrow J^1\pi$ is the canonical injection. Suppose $\tau : X \longrightarrow \mathcal{C}$ is an integral section of a connection on $(\pi_1)_{|\mathcal{C}}$, in the sense that $T\tau(T_xX) \subset \hat{\mathbf{h}}(T_{\tau(x)}\mathcal{C})$ for all $x \in \mathrm{Dom}\,\tau$. Then, if the given connection is semi-holonomic one can verify that, locally, τ can be written as the first jet prolongation of a (local) section of π.

As mentioned at the end of subsection 2.2, the regularity of L together with the fact that $\mathbf{h}$ satisfies the free De Donder-Weyl equation, imply that $\mathbf{h}$ is a semi-holonomic connection on $J^1\pi$. Herewith one can prove the following result.

Lemma 4.3 *The connection on $(\pi_1)_{|\mathcal{C}}$ defined in Lemma 4.2, with horizontal projector $\mathbf{h}_{\mathcal{P}}$, is semi-holonomic.*

Proof: We will use the fact that $\mathbf{h}$ is semi-holonomic and therefore satisfies (3). Let $v \in T_\gamma J^1\pi$ be a $\pi_{1,0}$-vertical vector, then for any contact 1-form θ on $J^1\pi$ we have that $i_v\theta(\gamma) = 0$. Now, for each $v \in T_{\mathcal{C}}J^1\pi$ we have that $(\mathcal{P} \circ \mathbf{h} - \mathbf{h})(v) = -\mathcal{Q}(\mathbf{h}(v)) \in D$ and, hence, $(\mathcal{P} \circ \mathbf{h} - \mathbf{h})(v)$ is $\pi_{1,0}$-vertical. Therefore $i_{\mathcal{P}\circ\mathbf{h}}\theta(v) = i_{\mathbf{h}}\theta(v) = 0$ for any contact 1-form θ and any $v \in T_{\mathcal{C}}J^1\pi$. From this one can readily deduce that $\mathbf{h}_{\mathcal{P}}$ satisfies (18) and so we may conclude that the induced connection on $(\pi_1)_{|\mathcal{C}}$ is indeed semi-holonomic. *QED*

We now arrive at the main result of this section. From now on, for ease of notation, we will use the projector $\mathcal{P} \circ \mathbf{h}$ without further indication of its domain. The latter should be clear from the context.

Theorem 4.4 *Consider a constrained problem of the type described above, with regular Lagrangian L, constraint submanifold $\mathcal{C} \subset J^1\pi$ and bundle of constraint forms F, and assume the compatibility condition (14) holds. Let $\mathbf{h}$ be the horizontal projector of a connection on π_1, satisfying the free De Donder-Weyl equation (7) and let $\mathcal{P}$ be the nonholonomic projector associated to the constrained problem. Then, the projector $\mathcal{P} \circ \mathbf{h}$ determines a solution of the constrained De Donder-Weyl problem (13) and restricts to the horizontal projector of a semi-holonomic connection on $(\pi_1)_{|\mathcal{C}} : \mathcal{C} \longrightarrow X$.*

Proof: Along $\mathcal{C}$ we can rewrite the free De Donder-Weyl equation as

$$i_{\mathcal{P}\circ\mathbf{h}}\Omega_L - n\Omega_L = -i_{\mathcal{Q}\circ\mathbf{h}}\Omega_L.$$

Therefore, in order to prove that $\mathcal{P} \circ \mathbf{h}$ satisfies the constrained De Donder-Weyl equation, we only need to verify that the right-hand side is an element of $\mathcal{I}(F)$.

We can write the projector $\mathbf{h}$ as $\mathbf{h} = dx^\mu \otimes H_\mu$, with $H_\mu = (\partial/\partial x^\mu) + \Gamma^a_\mu(\partial/\partial y^a) + \Gamma^a_{\mu\nu}(\partial/\partial y^a_\nu)$ (cf. (2)). Along $\mathcal{C}$ we can then put $\mathcal{Q}(H_\mu) = \lambda^\alpha_\mu \zeta_\alpha$ for some functions λ^α_μ and with the vector fields ζ_α as defined in Proposition 3.2. Then, at each point $\gamma \in \mathcal{C}$ and for any $v_1, \ldots, v_{n+2} \in T_\gamma J^1\pi$ we obtain

$$\begin{aligned}(i_{\mathcal{Q}\circ\mathbf{h}}\Omega_L)(v_1, \ldots, v_{n+2}) &= \sum_{i=1}^{n+2}(-1)^{i+1}\Omega_L((\mathcal{Q}\circ\mathbf{h})(v_i), v_1, \ldots, \hat{v}_i, \ldots, v_{n+2}) \\ &= \sum_{i=1}^{n+2}(-1)^{i+1}\lambda^\alpha_\mu dx^\mu(v_i)(i_{\zeta_\alpha}\Omega_L)(v_1, \ldots, \hat{v}_i, \ldots, v_{n+2}) \\ &= -\lambda^\alpha_\mu(dx^\mu \wedge \Phi_\alpha)(v_1, \ldots, v_{n+2})\,.\end{aligned}$$

This shows that, along $\mathcal{C}$,

$$i_{\mathcal{Q}\circ\mathbf{h}}\Omega_L = -\lambda^\alpha_\mu dx^\mu \wedge \Phi_\alpha \in \mathcal{I}(F),$$

which completes the proof of the first part of the theorem.

The proof that $\mathcal{P}\circ\mathbf{h}$ induces a semi-holonomic connection on $(\pi_1)_{|\mathcal{C}}$ follows from the previous lemmas 4.2 and 4.3. *QED*

Note that even in case a connection on π_1, with horizontal projector $\mathbf{h}$ satisfying the free De Donder-Weyl equation, is holonomic (or integrable), the 'projected' semi-holonomic connection $\mathbf{h}_{\mathcal{P}} = \mathcal{P} \circ \mathbf{h}$ on $(\pi_1)_{\mathcal{C}}$ need not admit integral sections in general.

5 An example: incompressible hydrodynamics

As an example of a field theory with an external constraint, we consider the case of an incompressible fluid flow. This problem has already been treated for instance in [24], using the constrained variational approach. From the point of view of "nonholonomic field theory" this is perhaps an a-typical example since, as we shall see, the constrained field equations resulting from the nonholonomic approach are essentially the same as those derived in [24]. The reason for this probably stems from the fact that, as we will show, the incompressibility constraint can be written as a divergence. Recall that for a mechanical system with a nonholonomic constraint that arises from a total time derivative of a function on the configuration space, the nonholonomic and the vakonomic equations of motion are equivalent.

5.1 The constrained problem

We will consider a simplified model of incompressible fluid motion in the sense that we will not bother about the technicalities related to precise domain and boundary conditions (see [24] for a more detailed description of the geometric model). We identify the base space X with $\mathbb{R} \times \mathbb{R}^3 \equiv \mathbb{R}^4$, with coordinates $(x^\mu) = (t, x^i)$ representing time $t = x^0$ and the material variables x^i $(i = 1, 2, 3)$. The volume form on X is the standard Euclidean volume $\eta := d^4x = dt \wedge dx^1 \wedge dx^2 \wedge dx^3$. For the total space we take $Y = X \times \mathbb{R}^3$ with coordinates (t, x^i, y^a). The projection $\pi : Y \longrightarrow X$ then reads $\pi(t, x^i, y^a) = (t, x^i)$. In order to preserve some consistency with [24], we will denote the corresponding bundle coordinates on $J^1\pi$ by $(t, x^i, y^a, v_0^a, v_i^a)$.

Remark 5.1 *In continuum mechanics it is common to denote the coordinates on X and Y by (t, X^I) and (t, X^I, x^i), respectively, and the coordinates on $J^1\pi$ by $(t, X^I, x^i, v^i, F_I^i)$. However, we will not follow that convention here.*

In addition, we will equip the fibres $\mathbb{R}^3$ of X (over the time axis) and of Y (over X) with the standard Euclidean metric, although one could replace them by more general Riemannian manifolds (see [23, 24]).

A section $\phi(t, x^i) = (t, x^i, \phi^a(t, x))$ of π can be seen as a map taking a material point x of the fluid and mapping it at each time t onto its position $\phi^a(t, x)$, $(a = 1, 2, 3)$ in space. Following [24] we write the Lagrangian density as

$$L(\gamma)d^4x = \frac{1}{2}\left\|v_0\right\|^2 \rho\, d^4x - W(v_i^a)\rho\, d^4x, \tag{19}$$

where the function $\rho = \rho(x)$ represents the material density, and W is the stored energy function. Note that W depends only on the v_i^a, i.e. the "spatial" jet bundle coordinates. Next, we introduce the function $\mathcal{J} : J^1\pi \to \mathbb{R}$ given by

$$\mathcal{J}(\gamma) = \det\left(v_i^a(\gamma)\right).$$

(note that (v_i^a) is a square matrix). For any section ϕ of π, $\mathcal{J} \circ j^1\phi$ measures the volume change of a small fluid element under the 'flow' represented by ϕ. In particular, the incompressibility requirement can be expressed by the condition $\mathcal{J}(j^1\phi) = 1$, i.e. we have the constraint

$$\varphi(\gamma) := \mathcal{J}(\gamma) - 1 = 0\,, \tag{20}$$

defining the constraint submanifold $\mathcal{C}$.

For the bundle F of constraint forms we take the line-bundle along $\mathcal{C}$, generated by the 4-form

$$\begin{aligned}
\Phi : \; &= S_\eta^*(d\varphi) \\
&= \frac{\partial \varphi}{\partial v_\mu^a}(dy^a - v_\nu^a dx^\nu) \wedge d^3x_\mu \\
&= \mathcal{J}(v^{-1})_a^i (dy^a - v_\nu^a dx^\nu) \wedge d^3x_i\,,
\end{aligned}$$

(i.e. we adopt the generalized Chetaev principle: see Remark 3.1).

5.2 The nonholonomic field equations

Before proceeding towards the field equations, we make the additional assumption that we are dealing with a *barotropic fluid* which, in particular, implies that W depends on the v_i^a through $\mathcal{J}$, i.e. $W = W(\mathcal{J})$. The nonholonomic field equations (11) for a barotropic fluid with Lagrangian (19), subject to the incompressibility constraint (20) and with constraint form Φ, become

$$\rho\, \delta_{ab} \frac{d}{dt} v_0^b - \frac{\partial}{\partial x^j}\left(\rho W' \mathcal{J}(v^{-1})_a^j\right) = \lambda_i \mathcal{J}(v^{-1})_a^i \quad (a = 1, 2, 3)\,,$$

which should be considered together with the constraint equation $\varphi(t, x^i, y^a, v_0^a, v_i^a) = \mathcal{J} - 1 = 0$. This should be compared with equation (4.8) in [24]. In that paper, the the field equations for an incompressible barotropic fluid were derived by means of a constrained variational approach. Since there is only one constraint equation, this approach gives rise to only one Lagrangian multiplier P, which can be interpreted as pressure. If we put $\lambda_i = \frac{\partial P}{\partial x^i}$, it is seen that, for the present example, the nonholonomic equations and the constrained variational equations are essentially the same. When thinking of the comparison between nonholonomic and vakonomic mechanics, the reason for this is to be found in the fact that the incompressibility constraint is determined by a divergence. More precisely, we have the following property.

Proposition 5.2 *The constraint function φ can be written (locally) as a total divergence, i.e. there exist functions ψ^μ such that $\varphi = \frac{d\psi^\mu}{dx^\mu}$.*

Proof: One can easily verify that

$$\frac{d}{dx^\mu}\left(\frac{\partial\varphi}{\partial v^a_\mu}\right) - \frac{\partial\varphi}{\partial v^a} \equiv 0,$$

i.e. φ is a "null-Lagrangian", which is equivalent to φ being a divergence (see e.g. [25, thm. 4.7]). More directly, if we consider the functions

$$\psi^0 = 0 \quad \text{and} \quad \psi^i = \frac{1}{3}\mathcal{J} y^a (v^{-1})^i_a - x^i \,,$$

with v^{-1} the inverse of the matrix (v^a_i), which are well-defined on a neighborhood of $\mathcal{C}$, a rather tedious but straightforward computation shows that $\varphi = d\psi^\mu/dx^\mu$. QED

A detailed study of the comparison between the constrained variational approach and the nonholonomic approach to constrained field theories will be the subject of forthcoming work.

5.3 The nonholonomic projector

To illustrate some further concepts defined in the preceding sections, we now turn to the explicit form of the nonholonomic projector $\mathcal{P}$ for the example of incompressible fluid (not necessarily barotropic). As there is only one constraint, the bundle of constraint forces D is spanned by a single vector field $\zeta = \zeta^a_\mu \partial/\partial v^a_\mu$. The coefficients of this vector field can be derived from (10) where, in the present case, $C^\mu_a = \partial\varphi/\partial v^a_\mu$:

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{\partial^2 W}{\partial v^a_i \partial v^b_j} \end{pmatrix} \begin{pmatrix} \zeta^b_0 \\ \zeta^b_j \end{pmatrix} = \begin{pmatrix} 0 \\ \mathcal{J}(\gamma)\,(v^{-1})^i_a \end{pmatrix}.$$

If, for brevity, we denote the Hessian matrix of W with respect to the v^a_i by $\mathcal{H}$, then ζ is the vector field along $\mathcal{C}$ given by

$$\zeta = (\mathcal{H}^{-1})^{ab}_{ij} \mathcal{J} (v^{-1})^j_b \frac{\partial}{\partial v^a_i}.$$

Let us consider the function $f := \zeta(\varphi)$, or explicitly

$$f = (\mathcal{H}^{-1})^{ab}_{ij} \mathcal{J}^2 (v^{-1})^i_a (v^{-1})^j_b \,.$$

For each $\gamma \in \mathcal{C}$, $f(\gamma) \neq 0$ from which it follows that the compatibility condition (14) holds. The nonholonomic projector $\mathcal{P}$ is then found to be

$$\mathcal{P} = I - f^{-1} d\varphi \otimes \zeta \,.$$

6 Cauchy formalism for nonholonomic field theory

We will now describe the transition from the multisymplectic (covariant) treatment of nonholonomic field theory, discussed in the previous sections, to the formulation of the problem on the space of Cauchy data. The Cauchy formalism for field theories is an infinite-dimensional analogue of classical dynamics of systems with a finite number of degrees of freedom. Instead of looking for sections of a bundle Y over an $(n+1)$-dimensional space-time manifold X (as in the covariant approach), one starts by introducing a space $\tilde{X}$ of embeddings of a fixed 'Cauchy surface' into X. This space replaces the absolute time from Newtonian mechanics and, under suitable conditions, the system can then be described in terms of a particular vector field on an infinite-dimensional manifold $\tilde{Z}$, called the space of Cauchy data, which is a bundle over $\tilde{X}$.

In our discussion of the Cauchy formalism for nonholonomic field theory, attention will be focussed on the case where the base manifold X admits a global splitting in 'space' and 'time'. It will be shown that, under appropriate assumptions, this Cauchy formalism reveals a close resemblance to the cosymplectic formulation of time-dependent nonholonomic mechanics (see e.g. [5]). This is in agreement with the results described in [26] for unconstrained Lagrangian field theory. Our aim is mainly to present the general idea, without entering into all technical details related to the geometry and analysis on infinite dimensional manifolds.

Finally, it should be noted that only for *hyperbolic* partial differential equations it makes sense to consider initial value problems. In the remainder of this section we will therefore tacitly assume that the field equations we are dealing with, are hyperbolic in some suitable sense. We refer to [8] for a detailed analysis of this matter.

6.1 The space of Cauchy data

We first recall some basic aspects of the Cauchy formalism for Lagrangian field theories. We thereby closely follow the treatments presented in [3, 17, 26], to which we also refer for more details and further references on the subject.

6.1.1 Generalities

As before, we start from a fibre bundle $\pi : Y \to X$ whose base space X is an $(n+1)$-dimensional orientable manifold. Let M be an n-dimensional compact oriented manifold with volume form η_M. The pair (M, η_M) is called a Cauchy surface. A *space of (parametrized) Cauchy surfaces* $\tilde{X}$ is then defined as a smooth manifold of embeddings $\tau : M \hookrightarrow X$.

Remark 6.1 *Usually, X and M are taken to be manifolds with boundary and the embeddings τ belonging to $\tilde{X}$ are then assumed to map the interior, resp. boundary, of M into the interior, resp. boundary, of X. However, for the purpose of the present paper we will leave all considerations related to boundary aspects aside.*

In the sequel we will always assume, without loss of generality, that M has volume one, i.e.

$$\int_M \eta_M = 1\,. \tag{21}$$

Points of M will usually be denoted by u.

Given a space of Cauchy surfaces $\tilde{X}$, the *space of Cauchy data* $\tilde{Z}$ is defined as a (infinite dimensional) manifold of embeddings from M into $J^1\pi$, having the property that for each embedding $\kappa : M \hookrightarrow J^1\pi$, there exists a section ϕ of π and an element τ of $\tilde{X}$ such that $\kappa = j^1\phi \circ \tau$. Finally, we define the *space of Dirichlet data* $\tilde{Y}$ as consisting of those embeddings $\delta : M \hookrightarrow Y$ having the property that there exists an element κ of $\tilde{Z}$ such that $\delta = \pi_{1,0} \circ \kappa$.

It is obvious from the previous definitions that the respective projections $\pi_{1,0} : J^1\pi \to Y$ and $\pi : Y \to X$ induce the following natural projections:

$$\tilde{Z} \xrightarrow{\tilde{\pi}_{1,0}} \tilde{Y} \xrightarrow{\tilde{\pi}} \tilde{X}.$$

We further put $\tilde{\pi}_1 = \tilde{\pi} \circ \tilde{\pi}_{1,0}$: the projection of $\tilde{Z}$ onto $\tilde{X}$.

The spaces $\tilde{X}$, $\tilde{Y}$, and $\tilde{Z}$ can be equipped with the Whitney topology and can be made into (infinite-dimensional) manifolds (see [3]). Because of the compactness of M, it does not matter whether one chooses the strong or the weak Whitney topology. However, as mentioned at the beginning of this section, we will not unduly concern ourselves with technical issues related to the smooth nature of these manifolds and of the mappings and other objects defined on them. For a detailed discussion of the topology and differentiable structure on a space of differentiable mappings between manifolds, we refer to [13].

The tangent space of $\tilde{X}$ has a convenient geometrical interpretation (see [3, 12]): let $\tau \in \tilde{X}$, then $T_\tau\tilde{X}$ can be identified with the space of sections of the pull-back bundle τ^*TX. Equivalently, a vector $V_\tau \in T_\tau\tilde{X}$ can be identified with a vector field along τ, i.e. $V_\tau : M \longrightarrow TX$, $u \longmapsto V_\tau(u) \in T_{\tau(u)}X$. Since τ is a bijection (onto its image), one can still identify V_τ with a vector field on X, defined along $\tau(M)$. Similar interpretations exist for elements of $T\tilde{Y}$ and $T\tilde{Z}$ and will be freely used in the sequel. For instance, given $\kappa \in \tilde{Z}$, we will use the notation W_κ to indicate both an element of the tangent space $T_\kappa\tilde{Z}$ and the corresponding vector field on $J^1\pi$ along $\kappa(M)$.

6.1.2 Existence of a global splitting of X

For the further discussion we assume that there exists a global splitting of the base manifold X, induced by a diffeomorphism $\Psi : \mathbb{R} \times M \to X$. In physics, $\mathbb{R}$ is usually associated to time and M to (physical) space. An embedding $\tau : M \hookrightarrow X$ is then called *admissible* if there exists a (necessarily unique) $t \in \mathbb{R}$ such that $\tau(u) = \Psi(t, u)$. We henceforth restrict $\tilde{X}$ to be a manifold of admissible embeddings of M in X. This has the effect of reducing $\tilde{X}$ to a 1-dimensional space diffeomorphic to $\mathbb{R}$, with coordinate function denoted by t. The spaces of Cauchy data $\tilde{Z}$ and of Dirichlet data $\tilde{Y}$, are then restricted accordingly. There is a canonically defined vector field Ξ on $\tilde{X}$, given by

$$\Xi(\tau)(u) = \frac{d}{ds}\Psi(s, u)\Big|_{s=t} \quad \text{for all } u \in M,$$

where t is such that $\tau(\cdot) = \Psi(t, \cdot)$. In particular we have that $\langle \Xi, dt \rangle = 1$. The global time slicing of X allows us to consider a volume form η on X which, with some abuse of notation, can be written as

$$\eta = dt \wedge \eta_M \,.$$

(see e.g. [26]). In the sequel we will always assume that X is oriented in terms of this volume form.

An important property now is that, under the above assumptions (i.e. existence of a global splitting of X and $\tilde{X}$ consisting of admissible embeddings only) one can prove that the space of Cauchy data $\tilde{Z}$ is diffeomorphic to the jet bundle $J^1\tilde{\pi}$ (see [26, par. 5.2.]). To make this diffeomorphism more explicit we note that there is a one-to-one correspondence between sections ϕ of π_1 and sections φ of $\tilde{\pi}_1$:

$$\phi(x) = \varphi(\tau)(u), \quad \text{where } x = \Psi(t, u) \text{ and } \tau(\cdot) = \Psi(t, \cdot).$$

We will frequently switch back and forth between both interpretations without warning, but we will stick to the notation "ϕ" for a section of π_1 and "φ" for the corresponding section of $\tilde{\pi}_1$.

6.2 The unconstrained Lagrangian formalism

Starting from a Lagrangian density $L\eta$ on $J^1\pi$, with regular Lagrangian L, the multisymplectic $(n+2)$-form Ω_L induces a 2-form $\tilde{\Omega}_L$ on the space $\tilde{Z}$ of Cauchy data as follows. Let $\kappa \in \tilde{Z}$, and $W_\kappa, W'_\kappa \in T_\kappa \tilde{Z}$, then put

$$\tilde{\Omega}_L(\kappa)(W_\kappa, W'_\kappa) := \int_M \kappa^*(i_{W_\kappa}\, i_{W'_\kappa}\Omega_L)\,,$$

where on the right-hand side, W_κ and W'_κ are interpreted as vector fields on $J^1\pi$, defined along $\kappa(M)$ (see the end of subsection 6.1.1). Likewise, the $(n+1)$-form η (pull-back of the volume form on X) induces a one-form $\tilde{\eta}$ on $\tilde{Z}$ according to the prescription

$$\tilde{\eta}(\kappa)(W_\kappa) := \int_M \kappa^*(i_{W_\kappa}\eta)$$

for all $\kappa \in \tilde{Z}$, $W_\kappa \in T_\kappa\tilde{Z}$. One can prove that both $\tilde{\Omega}_L$ and $\tilde{\eta}$ are closed forms and, in particular, it turns out that $\tilde{\Omega}_L = -d\tilde{\Theta}_L$, where $\tilde{\Theta}_L$ is the one-form on $\tilde{Z}$ induced by Θ_L (cf. [26] for more details).

As for the jet bundle $J^1\pi$, one can show that the space of Cauchy data $\tilde{Z}$ can be equipped with a 'vertical endomorphism' $\tilde{S}_{\tilde{\eta}}$ (see [26, section 5.2.3]). In the case under consideration, with $\tilde{X}$ being 1-dimensional, $\tilde{S}_{\tilde{\eta}}$ is a vector valued one-form that can be defined as follows. Take any $\kappa \in \tilde{Z}$, with $\kappa = j^1\phi \circ \tau$ for some $\tau \in \tilde{X}$ and section ϕ of π. In view of the identification between $\tilde{Z}$ and $J^1\tilde{\pi}$ (see subsection 6.1.2), we can still represent κ by $j^1_\tau\varphi$. For arbitrary $W_\kappa \in T_\kappa\tilde{Z}$, we then put

$$\tilde{S}_{\tilde{\eta}}(W_\kappa) = \left(T_{j^1_\tau\varphi}\tilde{\pi}_{1,0}(W_\kappa) - T_\tau\varphi \circ T_{j^1_\tau\varphi}\tilde{\pi}_1(W_\kappa)\right)^v , \tag{22}$$

where the superscript 'v' denotes the natural vertical lift operation from $T\tilde{Y}$ into $V\tilde{\pi}_{1,0}$. With the terminology used for vector fields on a first-order jet bundle, we will say that a vector field Γ on $\tilde{Z}$ is *a second-order vector field* (shortly, a SODE) if

$$\tilde{S}_{\tilde{\eta}}(\Gamma) = 0 \quad \text{and} \quad i_\Gamma\tilde{\eta} = 1 . \tag{23}$$

Consider a connection (or jet field) Υ on $\pi_1 : Y \longrightarrow X$, with horizontal projector $\mathbf{h}$. One can then construct a vector field Γ on $\tilde{Z}$ as follows. For $\kappa \in \tilde{Z}$, with $\kappa = j^1\phi \circ \tau$, define the vector $\Gamma(\kappa) \in T_\kappa\tilde{Z}$ by

$$\Gamma(\kappa)(u) = \mathbf{h}\left(Tj^1\phi(\Xi(\tau)(u))\right) , \tag{24}$$

i.e. $\Gamma(\kappa)(u) \in T_{\kappa(u)}J^1\pi$ is the horizontal lift of $\Xi(\tau)(u) \in T_{\tau(u)}X$ under the given connection Υ. We then have the following interesting property.

Proposition 6.2 *If Υ is a semi-holonomic connection on π_1, then the vector field Γ on $\tilde{Z}$, defined by (24) is a second-order vector field.*

Proof: For the contraction of Γ with $\tilde{\eta}$ we find that

$$(i_\Gamma\tilde{\eta})(\kappa) = \int_M \kappa^*(i_{\Gamma(\kappa)}\eta) = \int_M \tau^*(i_{\Xi(\tau)}\eta) = 1 ,$$

where the last equality follows from the normalization assumption (21) and for the second equality we have used the fact that (with previous conventions) $i_{\Gamma(\kappa)}\eta = \pi_1^*\left(i_{\Xi(\tau)}\eta\right)$ and $\pi_1 \circ \kappa = \tau$. Herewith, we have already shown that Γ verifies the second condition of (23).

Next, we investigate the first condition of (23). Since the given connection Υ is semi-holononomic, it is easily checked in coordinates that $\mathbf{h}$ satisfies

$$T_\gamma \pi_{1,0}(\mathbf{h}(v_\gamma)) = T_\gamma(\phi \circ \pi_1)(v_\gamma), \tag{25}$$

where $\gamma = j^1_x\phi$ and $v_\gamma \in T_\gamma J^1\pi$. We now compute $\tilde{S}_{\tilde{\eta}}(\Gamma(\kappa))$. With $W_\kappa = \Gamma(\kappa)$, the first term on the right-hand side of (22) becomes

$$\begin{aligned} T_\kappa \tilde{\pi}_{1,0}(\Gamma(\kappa))(u) &= T_{\kappa(u)}\pi_{1,0}(\Gamma(\kappa)(u)) \\ &= T_{\kappa(u)}\pi_{1,0}(\mathbf{h}(K_{\kappa(u)})), \end{aligned}$$

where $\kappa = j^1_\tau\varphi$ and where, for notational convenience, we have abbreviated $Tj^1\phi(\Xi(\tau)(u))$ by $K_{\kappa(u)}$. Using property (25), we further obtain

$$\begin{aligned} T_\kappa \tilde{\pi}_{1,0}(\Gamma(\kappa))(u) &= T_{\kappa(u)}(\phi \circ \pi_1)(K_{\kappa(u)}) \\ &= T_{\kappa(u)}\phi(\Xi(\tau)(u)), \end{aligned}$$

so that

$$T_\kappa \tilde{\pi}_{1,0}(\Gamma(\kappa)) = T_\tau\varphi(\Xi(\tau)),$$

from which it follows that $\tilde{S}_{\tilde{\eta}}(\Gamma(\kappa)) = 0$, which completes the proof that Γ defines a second-order ODE.

QED

We then arrive at the following important result in the Cauchy formalism for (unconstrained) Lagrangian field theory.

Theorem 6.3 *If* $\mathbf{h}$ *satisfies the De Donder-Weyl equation (7), then the vector field* Γ *on* $\tilde{Z}$*, defined by (24), satisfies the equations*

$$i_\Gamma \tilde{\Omega}_L = 0 \quad \textit{and} \quad i_\Gamma \tilde{\eta} = 1\,.$$

Proof: See [26, chapter 5]. *QED*

6.3 Nonholonomic constraints

We now return to the nonholonomic setting described in sections 3 and 4 and, in particular, we assume that the compatibility condition (14) holds. In order to adapt the Cauchy formalism, discussed in the previous subsection, to the nonholonomic case, we first define a subset $\tilde{\mathcal{C}}$ of $\tilde{Z}$ as follows:

$$\tilde{\mathcal{C}} := \left\{ \kappa \in \tilde{Z} \,|\, \mathrm{Im}\kappa \subset \mathcal{C} \right\}.$$

This set can be equipped with a smooth manifold structure such that $\tilde{\mathcal{C}}$ becomes a (infinite-dimensional) submanifold of $\tilde{Z}$. The tangent space to $\tilde{\mathcal{C}}$ at a point κ is given by $T_\kappa\tilde{\mathcal{C}} = \{W_\kappa \in T_\kappa\tilde{Z} \,|\, W_\kappa(u) \in T_{\kappa(u)}\mathcal{C}\}$.

For each $\kappa \in \tilde{\mathcal{C}}$, let

$$\tilde{D}_\kappa := \left\{ W_\kappa \in T_\kappa\tilde{Z} \,|\, \mathrm{Im}W_\kappa \subset D \right\},$$

where D is the constraint distribution along $\mathcal{C}$. Putting

$$\tilde{D} = \bigcup_{\kappa\in\tilde{\mathcal{C}}} D_\kappa$$

one may verify that $\tilde{D}$ determines a smooth distribution on $\tilde{Z}$ along $\tilde{\mathcal{C}}$.

Next, for $\kappa \in \tilde{\mathcal{C}}$ and for each section α of the bundle F of constraint forms along $\mathcal{C}$, we define an element $\tilde{\alpha}_\kappa$ of $T^*_\kappa\tilde{Z}$ by

$$\tilde{\alpha}_\kappa(W_\kappa) = \int_M \kappa^*(i_{W_\kappa}\alpha), \quad \text{for all } W_\kappa \in T_\kappa\tilde{Z}.$$

The set of all such covectors $\tilde{\alpha}_\kappa$ determines a subspace $\tilde{F}_\kappa$ of $T^*_\kappa\tilde{Z}$ and

$$\tilde{F} = \bigcup_{\kappa\in\tilde{\mathcal{C}}} \tilde{F}_\kappa$$

is a codistribution on $\tilde{Z}$ along $\tilde{\mathcal{C}}$.

Since we assume that the given constrained problem satisfies the compatibility condition, we can use the nonholonomic projector $\mathcal{P}$ and the complementary projector $\mathcal{Q} = I - \mathcal{P}$ (cf. Section 4) to define two operators $\tilde{\mathcal{P}}$, $\tilde{\mathcal{Q}} : T_{\tilde{\mathcal{C}}}\tilde{Z} \longrightarrow T_{\tilde{\mathcal{C}}}\tilde{Z}$ as follows. For each $\kappa \in \tilde{\mathcal{C}}$ and $W_\kappa \in T_\kappa\tilde{Z}$, put

$$\tilde{\mathcal{P}}_\kappa(W_\kappa) = \mathcal{P} \circ W_\kappa \; (\in T_\kappa\tilde{Z}), \qquad \tilde{\mathcal{Q}}_\kappa(W_\kappa) = \mathcal{Q} \circ W_\kappa \; (\in T_\kappa\tilde{Z}).$$

Using the properties of $\mathcal{P}$ and $\mathcal{Q}$, it is not hard to check that, for each $\kappa \in \tilde{\mathcal{C}}$, $\tilde{\mathcal{P}}_\kappa$ and $\tilde{\mathcal{Q}}_\kappa$ define complementary projectors in $T_\kappa\tilde{Z}$, i.e.

$$(\tilde{\mathcal{P}}_\kappa)^2 = \tilde{\mathcal{P}}_\kappa, \; (\tilde{\mathcal{Q}}_\kappa)^2 = \tilde{\mathcal{Q}}_\kappa \quad \text{and} \quad \tilde{\mathcal{P}}_\kappa + \tilde{\mathcal{Q}}_\kappa = I_\kappa,$$

with I_κ the identity on $T_\kappa\tilde{Z}$. This implies that $T_\kappa\tilde{Z} = \mathrm{Im}\tilde{\mathcal{P}}_\kappa \oplus \mathrm{Im}\tilde{\mathcal{Q}}_\kappa$. Again relying on the definitions of $\tilde{\mathcal{C}}, \tilde{D}, \tilde{\mathcal{P}}$ and $\tilde{\mathcal{Q}}$, and on the properties of the nonholonomic projector $\mathcal{P}$, one can prove that

$$\mathrm{Im}\tilde{\mathcal{P}}_\kappa = T_\kappa\tilde{\mathcal{C}} \quad \text{and} \quad \mathrm{Im}\tilde{\mathcal{Q}}_\kappa = \tilde{D}_\kappa.$$

Summarizing, we may conclude that under the given conditions we have the following decomposition of $T\tilde{Z}$ along $\tilde{\mathcal{C}}$:

$$T_{\tilde{\mathcal{C}}}\tilde{Z} = T\tilde{\mathcal{C}} \oplus \tilde{D}\,.$$

Let $\mathbf{h}$ be the horizontal projector of a connection Υ on π_1 and let Γ denote the vector field on $\tilde{Z}$ defined by (24). The composition $\tilde{\mathcal{P}} \circ \Gamma_{|\tilde{\mathcal{C}}}$ then determines a vector field on $\tilde{\mathcal{C}}$, shortly denoted by $\tilde{\mathcal{P}}(\Gamma)$, and it is not difficult to see that it is precisely the vector field associated to the induced connection on $(\pi_1)_{|\mathcal{C}}$ with horizontal projector $\mathbf{h}_{\mathcal{P}} = \mathcal{P} \circ \mathbf{h}$ (see Section 4). We now have the following interesting result.

Lemma 6.4 *There exists a section $\tilde{\alpha}$ of $\tilde{F}$, such that*

$$i_{\tilde{\mathcal{P}}(\Gamma)}\tilde{\Omega}_L = i_\Gamma\tilde{\Omega}_L + \tilde{\alpha}. \tag{26}$$

Proof: For $\kappa \in \tilde{\mathcal{C}}$ and $W_\kappa \in T_\kappa\tilde{Z}$, one can deduce from the definition of $\tilde{\Omega}_L$ that

$$(i_{\tilde{\mathcal{P}}(\Gamma)}\tilde{\Omega}_L)(\kappa)(W_\kappa) = \int_M \kappa^*(i_{\tilde{\mathcal{P}}(\Gamma)(\kappa)}i_{W_\kappa}\Omega_L).$$

For the integrand on the right-hand side we have that, with $u \in M$,

$$i_{\tilde{\mathcal{P}}(\Gamma)(\kappa)(u)}i_{W_\kappa(u)}\Omega_L = i_{\Gamma(\kappa)(u)}i_{W_\kappa(u)}\Omega_L - i_{\tilde{\mathcal{Q}}(\Gamma)(\kappa)(u)}i_{W_\kappa(u)}\Omega_L,$$

where $\tilde{\mathcal{Q}}(\Gamma)$ is the vector field associated to $\mathcal{Q} \circ \mathbf{h}$ (note that $\tilde{\mathcal{Q}}(\Gamma)$ is defined along $\tilde{\mathcal{C}}$). Since $\tilde{\mathcal{Q}}(\Gamma)(\kappa)(u)$ is an element of the constraint distribution D, the contraction with Ω_L yields a form $\alpha_{\kappa(u)} \in F_{\kappa(u)}$. Integration over M then gives (26). *QED*

We have now collected all ingredients needed to formulate the main result of this section. Consider a constrained Lagrangian field theory, with regular Lagrangian L, with constraints verifying the appropriate conditions and such that the base manifold X admits a global space-time splitting.

Theorem 6.5 *Let $\mathbf{h}$ be a solution of the unconstrained De Donder-Weyl equation (7) and let Γ be the corresponding second-order vector field on $\tilde{Z}$. Then, the vector field $\tilde{\mathcal{P}}(\Gamma)$ on $\tilde{\mathcal{C}}$ satisfies the following relations:*

$$i_{\tilde{\mathcal{P}}(\Gamma)}\tilde{\Omega}_L \in \tilde{F} \text{ and } \tilde{\mathcal{P}}(\Gamma) \in T\tilde{\mathcal{C}}. \tag{27}$$

Proof: If $\mathbf{h}$ satisfies the De Donder-Weyl equation, then the associated vector field Γ is contained in the kernel of $\tilde{\Omega}_L$ (see proposition 6.3). Expression (26) then proves the first part of (27). The second part follows from the definition of $\tilde{\mathcal{C}}$. *QED*

In addition, we note that $\tilde{\mathcal{P}}(\Gamma)$ is still a vector field of second-order type, due to propositions 4.3 and 6.2.

To conclude, we have shown that under the appropriate assumptions, the Cauchy formalism for nonholonomic field theory leads to a vector field of 'second-order type' on the (infinite dimensional) subspace $\tilde{\mathcal{C}}$ of the space of Cauchy data $\tilde{Z}$, which can be written as a projection of the second-order vector field on $\tilde{Z}$ associated to the free (unconstrained) Lagrangian system.

7 Some final comments

In this paper we have studied various aspects of nonholonomic Lagrangian field theory. Among others, we have shown that both in the multisymplectic approach and in the Cauchy formalism, the equations for the constrained system can be obtained by a projection of the equations for the original unconstrained Lagrangian system.

While finalizing this paper we have come across a recent work of Olga Krupkova ([14]) in which nonholonomic Lagrangian field theory is discussed within the framework of a general study of partial differential equations with differential constraints. This paper — which differs both in purpose and methodology from ours — presents, among others, an interesting analysis of the various types of constraints that one may encounter when dealing with constrained exterior differential systems on fibred manifolds.

The subject of nonholonomic field theory is still in full development. As far as the present study is concerned, there still remains some work to be done concerning the Cauchy formalism, mainly regarding the technicalities related to the infinite-dimensional manifold structure of the spaces $\tilde{Z}$ and $\tilde{\mathcal{C}}$. Another interesting matter that will be treated in future work, concerns the comparison between the constrained variational approach and the nonholonomic approach, i.e. the field theoretic analogue of the comparison between vakonomic and nonholonomic mechanics (see e.g. [10]).

Finally, an important challenge for future work will be the identification of some physically relevant examples to which nonholonomic field theory can be applied and for which, unlike the example of incompressible hydrodynamics treated in Section 5, the nonholonomic and the constrained variational approach are not "equivalent". It is to be expected that interesting examples should come, for instance, from problems in elasticity (such as the rolling without slipping of a deformable body over a surface).

Acknowledgements
J.V and F.C wish to thank the Research Foundation-Flanders (FWO) and the "Bijzonder Onderzoeksfonds" (BOF) of Ghent University, for financial support. They also thank the CSIC (Madrid) for its kind hospitality on the occasion of several research visits. M.dL and

D.MdD wish to thank to MICYT (Spain) Grant MTM2004-7832 for financial support. We would like to thank Marcelo Epstein for valuable discussions.

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