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A note on strict passivity

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Abstract

We show that there exists an explicit descriptor state space format which actually describes all strictly passive transfer functions. A key advantage of this explicitly strictly passive descriptor state space format resides in its relation with congruence projection-based reduced order modeling, where the resulting reduced order model is also cast in this same format. Another advantage of the format is that it allows for a simple construction of strictly passive random systems generators.

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1. Introduction

For time-invariant linear dynamical systems, strict passivity guarantees stability and the possibility of synthesis of the transfer function by means of a lossy physical network of resistors, capacitors, inductors and transformers [1]. It is well-known that strict passivity is equivalent with the strict positive reality of the system's transfer function [3]. Hence the strict passivity of a linear system can be checked by determining whether its transfer function is strictly positive real, and this in turn, by the well-known Kalman–Yakubovich–Popov positive-real lemma, implies testing the solvability of certain linear matrix inequalities (LMIs). It is known [3] that there are explicit solutions to LMI problems for only a few very special cases. However, they can be solved numerically by interior point methods.

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In this paper we tackle the strictly positive real problem in another fashion. We show that there exists an explicit descriptor state space format involving positive definite matrices, which actually describes all strictly positive real transfer functions. One of the main advantages of this explicitly strictly passive descriptor state space format resides in its connection with congruence projection-based reduced order modeling, where the resulting reduced order model is cast directly in the same strictly passive state space format. Another advantage is that it allows for a simple construction of a strictly passive random systems generator.

2. Main results

In what follows X^T and X^H , respectively, denote the transpose and Hermitian transpose of a matrix X , and I_m denotes the identity matrix of dimension m . For two

Hermitian matrices X and Y , the matrix inequalities $X > Y$ or $X \geq Y$ mean that $X - Y$ is, respectively, positive definite or positive semidefinite. For the real system with minimal realization

$$\dot{x} = Ax + Bu, \quad (1)$$

$$y = L^T x + Du, \quad (2)$$

where $B \neq 0$ and $L \neq 0$ are $n \times p$ real matrices and A is a $n \times n$ real matrix, to be strictly passive (also called strictly positive real), it is required that the $p \times p$ transfer function

$$H(s) = L^T(sI_n - A)^{-1}B + D \quad (3)$$

is analytic in the open right halfplane $\Re[s] > 0$, such that

$$H(i\omega) + H(i\omega)^H \geq \varepsilon I_p \quad \forall \omega \in \mathbb{R} \quad (4)$$

for some $\varepsilon > 0$. This naturally implies that all the poles of $H(s)$ must be located in the open left halfplane $\Re[s] < 0$, or stated otherwise: A must be stable, i.e. $\Re[Sp(A)] < 0$.

Note that, from requirement (4), it is readily seen that adding a constant $p \times p$ matrix D_0 to a merely passive $H(s)$ results in a strictly passive transfer function $H(s) + D_0$ if and only if $D_0 + D_0^T > 0$. Before proving our main result we need the following

Lemma. *Let*

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \quad (5)$$

be a $(n+p) \times (n+p)$ symmetric matrix partitioned in its $n \times n$, $n \times p$, $p \times n$, $p \times p$ blocks. Then $M > 0$ if and only if there exists a $n \times n$ nonsingular matrix Q and a $n \times p$ matrix W such that

$$\begin{aligned} M_{11} &= QQ^T, \\ M_{12} &= QW, \\ M_{22} &> W^T W. \end{aligned} \quad (6)$$

Proof. Let Q and W satisfy (6). Then M can be written as

$$M = \begin{bmatrix} Q & 0 \\ W^T & I_p \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & M_{22} - W^T W \end{bmatrix} \begin{bmatrix} Q^T & W \\ 0 & I_p \end{bmatrix}. \quad (7)$$

Since $M_{22} - W^T W > 0$, the matrix M is a congruence of a positive definite matrix and hence itself positive definite.

Conversely, if $M > 0$ then $M_{11} > 0$ and hence has a Cholesky factorization $M_{11} = QQ^T$. Now, with $W = Q^{-1}M_{12}$ it is evident that (7) is a block Cholesky factorization of M and hence $M_{22} - W^T W > 0$ must hold. \square

Theorem 1. *Let system (1)–(2) with transfer function*

$$H(s) = L^T(sI_n - A)^{-1}B + D \quad (8)$$

be strictly passive (and hence stable). Then there exists a $n \times n$ matrix $P = P^T > 0$, a $n \times n$ matrix G such that $G + G^T > 0$ and a $n \times p$ matrix R such that $H(s)$ can be written as

$$H(s) = L^T(sP + G)^{-1}R + \frac{1}{2}(L - R)^T(G + G^T)^{-1} \times (L - R) + D_1, \quad D_1 + D_1^T > 0. \quad (9)$$

Conversely, let $P = P^T > 0$ and G such that $G + G^T > 0$. Then the system with transfer function (9) is strictly passive.

Proof.

- *Direct part of the theorem:* It is known [3] that requirement (4) is satisfied if and only if there exists a $n \times n$ symmetric matrix $P = P^T > 0$ satisfying the LMI

$$\begin{bmatrix} A^T P + PA & PB - L \\ B^T P - L^T & -D - D^T \end{bmatrix} < 0. \quad (10)$$

By the Lemma, this is equivalent with finding P , a $n \times n$ nonsingular matrix Q and a $n \times p$ matrix W such that

$$A^T P + PA = -QQ^T < 0, \quad (11)$$

$$PB - L = -QW, \quad (12)$$

$$D + D^T > W^T W \geq 0. \quad (13)$$

After eliminating Q and W we obtain the inequality

$$D + D^T > -(L - PB)^T \times (A^T P + PA)^{-1}(L - PB). \quad (14)$$

Since the system is strictly passive, a $P = P^T > 0$ satisfying (11) exists. Putting $G = -PA$ and $R = PB$, inequality (14) can be written as

$$D + D^T > (L - R)^T (G + G^T)^{-1} (L - R), \quad (15)$$

where of course $G + G^T = QQ^T > 0$. If we substitute $A = -P^{-1}G$ and $R = PB$ into the transfer function, we obtain

$$H(s) = L^T (sP + G)^{-1} R + D \quad (16)$$

which can be written as

$$H(s) = L^T (sP + G)^{-1} R + \frac{1}{2}(L - R)^T \times (G + G^T)^{-1} (L - R) + D_1 \quad (17)$$

with

$$D_1 = D - \frac{1}{2}(L - R)^T (G + G^T)^{-1} (L - R), \quad (18)$$

where $D_1 + D_1^T > 0$ and $D_1 - D_1^T = D - D^T$.

- *Converse part of the theorem:* Suppose $H(s)$ is of the form (9), i.e.

$$H(s) = L^T (sP + G)^{-1} R + \frac{1}{2}(L - R)^T \times (G + G^T)^{-1} (L - R) + D_1 \quad (19)$$

with $G + G^T > 0$, $P = P^T > 0$ and $D_1 + D_1^T > 0$. Put $A = -P^{-1}G$, $B = P^{-1}R$ and

$$D = \frac{1}{2}(L - R)^T (G + G^T)^{-1} (L - R) + D_1. \quad (20)$$

Let $G + G^T = QQ^T$ and $W = -Q^{-1}(PB - L)$. Then $H(s) = L^T (sI_n - A)^{-1} B + D$ with

$$A^T P + PA = -QQ^T < 0, \quad (21)$$

$$PB - L = -QW, \quad (22)$$

$$D + D^T > W^T W + (D_1 + D_1^T) > WW^T. \quad (23)$$

Thus conditions (11)–(13) are satisfied. \square

Note that the Lyapunov inertia theorem [2] applied to (21) immediately implies that A is stable. Interestingly enough, if we take $L = R$, in [4] it is proved that transfer functions of the form¹

$$H(s) = L^T (sP + G)^{-1} L \quad P \geq 0 \quad G + G^T \geq 0 \\ \det(sP + G) \neq 0 \quad (24)$$

are passive—not necessarily strictly.

Note also that the format (9) is invariant with respect to nonsingular square congruence transforms, i.e. let U be an $n \times n$ nonsingular square matrix and define the modified matrices as

$$\tilde{P} = U^T P U, \quad \tilde{G} = U^T G U, \\ \tilde{L} = U^T L, \quad \tilde{R} = U^T R. \quad (25)$$

Then $\tilde{H}(s) = H(s)$. But we can say more.

Theorem 2. *Let $P = P^T > 0$ and $G + G^T > 0$. Let V be an $n \times r$, $1 \leq r \leq n$ matrix of full rank and define the modified $r \times r$ and $r \times p$ matrices as*

$$\tilde{P} = V^T P V, \quad \tilde{G} = V^T G V, \\ \tilde{L} = V^T L, \quad \tilde{R} = V^T R. \quad (26)$$

Then the transfer function

$$H_2(s) = \tilde{L}^T (s\tilde{P} + \tilde{G})^{-1} \tilde{R} + \frac{1}{2}(\tilde{L} - \tilde{R})^T \times (G + G^T)^{-1} (L - R) + D_1, \quad (27)$$

where $D_1 + D_1^T > 0$, is strictly passive.

Proof. Since $P = P^T > 0$, $G + G^T > 0$ and V is of full rank, we know that $V^T P V$ and $V^T (G + G^T) V$ are both positive definite. Hence $\tilde{P} > 0$ and $\tilde{G} + \tilde{G}^T > 0$ and consequently, by Theorem 1, $H_1(s)$ defined as

$$H_1(s) = \tilde{L}^T (s\tilde{P} + \tilde{G})^{-1} \tilde{R} + \frac{1}{2}(\tilde{L} - \tilde{R})^T (\tilde{G} + \tilde{G}^T)^{-1} \times (\tilde{L} - \tilde{R}) + D_1 \quad (28)$$

is strictly passive. There remains to be proved that

$$(\tilde{L} - \tilde{R})^T (\tilde{G} + \tilde{G}^T)^{-1} (\tilde{L} - \tilde{R}) \\ \leq (L - R)^T (G + G^T)^{-1} (L - R). \quad (29)$$

¹ $\det(sP + G) \neq 0$ means that $sP + G$ is a regular matrix pencil, i.e. $\det(sP + G) = 0$ has a finite number of s values as solutions.

Putting $G + G^T = S$, this will be the case when

$$S^{-1} - V(V^T S V)^{-1} V^T \geq 0. \quad (30)$$

Taking the Cholesky decomposition $S = \Delta \Delta^T$, inequality (30) can be transformed into

$$F = I_n - \Delta^T V \left(V^T \Delta \Delta^T V \right)^{-1} V^T \Delta \geq 0. \quad (31)$$

It is readily verified that F is an orthogonal projector, i.e. $F^2 = F$ and $F^T = F$, implying $F \geq 0$. \square

3. Applications

3.1. Strictly passive reduced order modeling

The theorems of the preceding section have interesting applications in terms of reduced order modeling. To see this, we first write the Laurent–Taylor expansion of $H(s)$ in the vicinity of $s = \infty$. We have

$$\begin{aligned} H(s) &= L^T (sP + G)^{-1} R + D \\ &= D + \sum_{k=0}^{\infty} (-1)^k s^{-k-1} L^T (P^{-1}G)^k P^{-1} R. \end{aligned} \quad (32)$$

Putting $P^{-1}G = \Omega = -A$ and $P^{-1}R = B$, this can be written as

$$\begin{aligned} H(s) &= D + \sum_{k=0}^{\infty} (-1)^k s^{-k-1} L^T \Omega^k B \\ &= \sum_{k=-1}^{\infty} (-1)^k s^{-k-1} \mathcal{M}_k. \end{aligned} \quad (33)$$

The coefficients $\mathcal{M}_k = L^T \Omega^k B$, $k \geq 0$ and $\mathcal{M}_{-1} = -D$ are known as the Markov moments of $H(s)$ at $s = \infty$. Next consider the $n \times r$ Krylov matrix ($r = pq$)

$$\mathcal{K} = [B, \Omega B, \Omega^2 B, \dots, \Omega^{q-1} B] \quad (34)$$

and consider choosing an orthonormal basis for the columns of \mathcal{K} , which is equivalent to performing the ‘thin’ SVD of the Krylov matrix as $\mathcal{K} = U \Sigma V^T$, where the $n \times r$ matrix U is column-orthogonal. Putting

$$\begin{aligned} \tilde{P} &= U^T P U, \quad \tilde{G} = U^T G U, \quad \tilde{R} = U^T R, \\ \tilde{L} &= U^T L, \quad \tilde{\Omega} = \tilde{P}^{-1} \tilde{G}, \quad \tilde{B} = \tilde{P}^{-1} \tilde{R} \end{aligned} \quad (35)$$

the new Markov moments are given by

$$\begin{aligned} \tilde{\mathcal{M}}_{-1} &= \mathcal{M}_{-1} = -D, \\ \tilde{\mathcal{M}}_k &= \tilde{L}^T \tilde{\Omega}^k \tilde{B} \quad k = 0, 1, \dots \end{aligned} \quad (36)$$

We are now in a position to prove (see also [4]).

Theorem 3. *With the choice of U as above, the Markov moments are equal up to order $q - 1$, i.e. $\tilde{\mathcal{M}}_k = \mathcal{M}_k$ for $k = 0, 1, \dots, q - 1$.*

Proof. Since we have constructed an orthonormal basis for the columns of \mathcal{K} , we can write $\Omega^k B = U W_k$, $k = 0, \dots, q - 1$, where W_k is a suitable $r \times p$ matrix. Note that we have $R = P B = P U W_0$ and $\tilde{R} = U^T R = U^T P U W_0 = \tilde{P} W_0$ and hence $\tilde{B} = \tilde{P}^{-1} \tilde{R} = W_0$. Next consider the $n \times n$ matrix

$$Z = U \tilde{P}^{-1} U^T G. \quad (37)$$

By induction, it is easy to prove that $Z^k U = U \tilde{\Omega}^k$ for $k = 0, \dots, q - 1$ and hence

$$\begin{aligned} \tilde{\mathcal{M}}_k &= \tilde{L}^T \tilde{\Omega}^k \tilde{B} = L^T Z^k U W_0 \\ &= L^T Z^k B \quad k = 0, \dots, q - 1. \end{aligned} \quad (38)$$

There remains to prove that $Z^k B = \Omega^k B$ for $k = 0, \dots, q - 1$. This is clearly the case for $k = 0$. Next suppose that $Z^k B = \Omega^k B$ for some k . Then

$$P^{-1} G Z^k B = \Omega^{k+1} B = U W_{k+1}. \quad (39)$$

Pre-multiplying by $U^T P$ yields

$$U^T G Z^k B = U^T P U W_{k+1} = \tilde{P} W_{k+1} \quad (40)$$

or

$$W_{k+1} = \tilde{P}^{-1} U^T G Z^k B \quad (41)$$

and hence

$$\begin{aligned} Z^{k+1} B &= U \tilde{P}^{-1} U^T G Z^k B \\ &= U W_{k+1} = \Omega^{k+1} B. \quad \square \end{aligned} \quad (42)$$

Recall that by Theorem 2, the reduced order model is strictly passive, when the original strictly passive transfer function $H(s)$ is provided in the previously defined strictly passive format.

Also, one often wishes to have equal Markov moments calculated about another point than infinity [4],

or else to have Markov moments which are coefficients of a Laguerre expansion [5,6]. All these possibilities can be dealt with by transforming the Laplace variable s by means of a Möbius transformation

$$s = \frac{\alpha u + \beta}{\gamma u + \delta}, \quad \alpha\delta - \beta\gamma \neq 0. \quad (43)$$

The resulting transfer function in the u -domain is

$$(\gamma u + \delta)L^T(u(\alpha P + \gamma G) + (\beta P + \delta G))^{-1}R + D. \quad (44)$$

Now assuming that $\alpha P + \gamma G$ is nonsingular, we can define the matrices

$$\begin{aligned} \hat{B} &= (\alpha P + \gamma G)^{-1}R, \\ \hat{Q} &= (\alpha P + \gamma G)^{-1}(\beta P + \delta G). \end{aligned} \quad (45)$$

After construction of a base \hat{U} of the Krylov matrix

$$\hat{\mathcal{H}} = [\hat{B}, \hat{Q}\hat{B}, \hat{Q}^2\hat{B}, \dots, \hat{Q}^{q-1}\hat{B}] = \hat{U}\hat{\Sigma}\hat{V}^T \quad (46)$$

the reduced matrices are now

$$\begin{aligned} \tilde{P} &= \hat{U}^T P \hat{U}, \quad \tilde{G} = \hat{U}^T G \hat{U}, \\ \tilde{R} &= \hat{U}^T R, \quad \tilde{L} = \hat{U}^T L. \end{aligned} \quad (47)$$

For example, when $\alpha = s_0$, $\beta = \gamma = 1$, $\delta = 0$, we in fact perform a Taylor expansion about s_0 , as in [4], and when $\beta = \alpha$, $\gamma = -1$, $\delta = 1$, we in fact perform a scaled Laguerre expansion with scaling factor $\alpha > 0$, as in [5,6]. Of course, by Theorems 1 and 2, strict passivity is always maintained.

3.2. A random strictly passive system generator

From Theorem 1 we know that a strictly passive transfer function can always be written as

$$\begin{aligned} H(s) &= L^T(sP + G)^{-1}R + \frac{1}{2}(L - R)^T \\ &\quad \times (G + G^T)^{-1}(L - R) + D_1 \end{aligned} \quad (48)$$

with $P = P^T > 0$, $G + G^T > 0$, $D_1 + D_1^T > 0$. We can implement this in MATLAB® by means of the following easily understood steps (epsil is a small positive number):

$$P = \text{randn}(n); P = P * P' + \text{epsil} * \text{eye}(n);$$

$$L = \text{randn}(n, p); R = \text{randn}(n, p);$$

$$G = \text{randn}(n); G = G * G' + \text{epsil} * \text{eye}(n);$$

$$D0 = 0.5 * (L - R)' * G \setminus (L - R);$$

$$Z = \text{randn}(n); Z = Z - Z'; G = G + Z;$$

$$D = \text{randn}(p); D = D * D';$$

$$Z = \text{randn}(p); Z = Z - Z'; D = D + Z + D0;$$

$$\text{sys} = \text{dss}(-G, R, L', D, P).$$

The command `dss` is from the control systems toolbox descriptor system assignment, i.e. the command `sys = dss(A, B, C, D, E)` creates a descriptor system with transfer function

$$H(s) = C(sE - A)^{-1}B + D. \quad (49)$$

Note that, for notational convenience, we have used a normal random number generator, but of course, any random number generator with sufficient range will do the job.

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