Journal of Nonlinear Mathematical Physics (2022) 29:731–749
https://doi.org/10.1007/s44198-022-00057-6

RESEARCH ARTICLE

Regularity Criterion for a Two Dimensional Carreau Fluid Flow

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Received: 18 February 2022 / Accepted: 5 April 2022 / Published online: 30 April 2022

Abstract

Carreau fluids are a source of research from both theoretical and applied approaches. They have been considered to model different non-newtonian phenomena such as blood flow, plasma and viscoeslastic materials. The purpose of this study is to develop the global regularity criteria for a Carreau fluid in two dimensions flowing in a strip. Firstly, a regularity criteria is shown for the initial set $\left(u_{10}, u_{20}\right) \in H^1(\Omega)$ where $\Omega = [0, L] \times [0, \infty)$. Secondly, the analysis focuses on a regularity criteria when $\left(u_{10}, u_{20}\right) \in L^4(\Omega)$ and, lastly, similar results are obtained for $\left(u_{10}, u_{20}\right) \in H^2(\Omega)$ while the fluid velocity vertical component, $u_2(x, y)$, is such that $\frac{\partial u_2}{\partial x} \in L^4(\Omega)$ and $\left(\frac{\partial \nabla u_2}{\partial y}, \Delta u_2\right) \in L^2(\Omega)$.

Keywords Carreau fluid · Two Dimensional fluid · Unsteady flow · Regularity criterion

Mathematics Subject Classification 35Q35 · 35B65 · 76D05

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1 Introduction

Most of the fluid conceptions used to model flows in typical media like air, water or oil, are based on a Newtonian description. In an important number of cases, the supposition of Newtonian behavior does not seem to be accurate enough, leading to expand the theory to introduce additional rheological properties that end in non-Newtonian descriptions. As a set of examples that preclude its ubiquity, the non-Newtonian flow is experienced in industries of mines, where sludges and muds are frequently dealt, in lubrication and biomedical flows.

The non-Newtonian fluids are characterized in different types depending on their rheological characteristics. One of such types is known as Carreau fluid (see [1–17] for few relevant studies). More particularly, an interesting research on Carreau fluids is given in [4] where the fluid is analyzed over a shrinking surface in presence of an infinite shear rate viscosity. In addition, the study of Carreau flows in spheres are widely analyzed in [5, 6].

Currently, there is not a wide literature focused on developing the regularity criteria for equations obtained upon a Carreau fluid. Nonetheless, there exists an extensive literature developing the regularity criteria for Navier–Stokes equations (see [18–23]). Motivated by such facts, our intention in this study is to develop global regularity criteria for a magnetohydrodynamic (MHD) flow of Carreau fluid equations. The fluid is considered to flow between two stationary plates. Flow between the plates is induced by an initial velocity profile and a stationary linear law at $y = 0$ to the x-axis velocity component. In addition, a uniform magnetic field is applied in the transverse direction to the flow.

The paper layout is as follows: Firstly, the Carreau fluid model is discussed based on the general theory of fluid dynamics of non-Newtonian flows. In addition, some preliminary required results are introduced. Afterwards the theorems on existence of regular solutions are presented. The paper introduces progressively, in different sections, each of the required proofs for each of the regularity Theorems together with the supporting information required.

2 Model Proposal

We consider the two dimensional incompressible fluid flow equations of a Carreau fluid. The fluid is electrically conducting in the presence of an applied magnetic field B_0. The MHD flow is governed by the following set of equations:

$$\mathbf{V} = \big(u_1, u_2, 0\big), \qquad \nabla \cdot \mathbf{V} = 0, \tag{2.1}$$

$$\rho \frac{\partial \mathbf{V}}{\partial t} = \nabla \cdot \tau + \mathbf{J} \times \mathbf{B}, \tag{2.2}$$

where $\mathbf{V}$ is the velocity field, t the time, ρ is the fluid density, τ is the Cauchy stress tensor, $\mathbf{J}$ is the current density and $\mathbf{B} = \mathbf{B_0} + \mathbf{B_1}$ is the magnetic field for a Carreau fluid, which is given by

$$\tau = -p\mathbf{I} + \eta\mathbf{A}_1, \tag{2.3}$$

with

$$\eta = \eta_\infty + \left(\eta_0 - \eta_\infty\right)\left[1 + \left(\mathbf{\Gamma}\dot{\gamma}\right)^2\right]^{\frac{n-1}{2}}, \tag{2.4}$$

where p is the pressure field, $\mathbf{I}$ the identity tensor, η_0 the zero-shear-rate viscosity, η_∞ the infinite-shear-rate viscosity, $\mathbf{\Gamma}$ a material time constant, and n express the power law index (since it describes the slope of $\frac{\eta_0-\eta_\infty}{\eta_0+\eta_\infty}$ in the power law region). The shear rate $\dot{\gamma}$ is defined by

$$\dot{\gamma} = \sqrt{\frac{1}{2}\sum_i\sum_j \dot{\gamma}_{ij}\dot{\gamma}_{ji}} = \sqrt{\frac{1}{2}\mathbf{\Pi}} = \sqrt{\frac{1}{2}tr\left(\mathbf{A}_1^2\right)}. \tag{2.5}$$

Here $\mathbf{\Pi}$ is the second invariant strain rate tensor and $\mathbf{A}_1$ is given by

$$\mathbf{A}_1 = (\nabla\,\mathbf{V}) + (\nabla\,\mathbf{V})^T. \tag{2.6}$$

Note that $(\cdot)^T$ denotes the transpose of a matrix. From (2.1) to (2.6), the governing equations in the absence of pressure gradient are given by

$$\begin{aligned}\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y} &= \nu\frac{\partial^2 u_1}{\partial y^2}\left[1 + \Gamma^2\left(\frac{\partial u_1}{\partial y}\right)^2\right]^{\frac{n-1}{2}} \\ &\quad + \nu(n-1)\Gamma^2\frac{\partial^2 u_1}{\partial y^2}\left(\frac{\partial u_1}{\partial y}\right)^2\left[1 + \Gamma^2\left(\frac{\partial u_1}{\partial y}\right)^2\right]^{\frac{n-3}{2}} - \frac{\sigma B_0^2}{\rho}u_1,\end{aligned} \tag{2.7}$$

where $\nu = \frac{\eta_0}{\rho}$ is the kinematic viscosity and σ is related with the electrical charges distribution.

Note that the subject boundary conditions are

$$u_1(x,y,t) = Ux,\ u_2(x,y,t) = 0 \quad \text{at } y = 0, \tag{2.8}$$

$$u_1(x,y,t) = 0,\ u_2(x,y,t) = 0 \quad \text{at } y \to \infty, \tag{2.9}$$

together with

$$u_1(x,y,t) = 0, \quad u_2(x,y,t) = 0 \quad \text{at } x = 0,\ x = L. \tag{2.10}$$

Where U is a constant. In addition, the following initial conditions hold

$$u_1(x, y, 0) = u_{10}(x, y) \qquad u_2(x, y, t) = u_{20}(x, y), \tag{2.11}$$

such that $\big(u_{10}(x, y), u_{20}(x, y)\big)$ corresponds to the vector of initial velocities. Note that the domain is given as $\Omega = \{(x, y) \in [0, L] \times [0, \infty)\}$.

In addition, consider that the shear stress is zero at $y = 0$. As a consequence, the following holds

$$\tau_{xy} = \nu \frac{\partial u_1}{\partial y} \left[1 + \Gamma^2 \left(\frac{\partial u_1}{\partial y} \right)^2 \right]^{\frac{n-1}{2}} = 0,$$

leading to $\frac{\partial u_1}{\partial y} = 0$ at $y = 0$.

3 Preliminaries and Statement of Results

3.1 Previous Results

Consider the well-known norm $\|\cdot\|_{L^p}$ in the Lebesgue functional space $L_p(\Omega)$ together with the usual Sobolev order m functional space defined by

$$H^m(\Omega) = \left\{ u \in L^2(\Omega) : \ \nabla^m(u) \in L^2(\Omega) \right\}$$

with the norm

$$\|u\|_{H^m} = \left(\|u\|_{L^2}^2 + \|\nabla^m u\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

In addition, the following lemma is also needed (refer to Lemma 1 in [23])

Lemma 1 *The following anisotropic Sobolev inequality holds*

$$\iint_{\Omega} |fgh| dxdy \leq C_0 \|f\|_{L^2} \|g\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial g}{\partial x} \right\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial h}{\partial y} \right\|_{L^2}^{\frac{1}{2}}.$$

3.2 Statement of Results

The main results are stated as follows:

Theorem 3.1 *Assuming* $\big(u_{10}, u_{20}\big) \in H^1(\Omega)$, *where* $\big|\nabla u_0\big| = \left|\frac{\partial u_0}{\partial y}\right|$, *the equation* (2.7) *has solutions on* $(0, T]$ *in the defined strip* $\Omega = [0, L] \times [0, \infty)$.

Theorem 3.2 *Assuming* $\big(u_{10}, u_{20}\big) \in L^4(\Omega)$, *then the system* (2.7) *has solutions on* $(0, T]$ *in the defined strip* $\Omega = [0, L] \times [0, \infty)$.

Theorem 3.3 *Assuming* $\left(u_{10}, u_{20}\right) \in H^2(\Omega)$, *then the system* (2.7) *has solutions on* $(0, T]$ *when* $\frac{\partial u_2}{\partial x} \in L^4(\Omega)$ *and* $\left(\frac{\partial \nabla u_2}{\partial y}, \Delta u_2\right) \in L^2(\Omega)$, *in the defined strip* $\Omega = [0, L] \times [0, \infty)$.

4 Proof of Theorem 3.1

Firstly, the following proposition is required to be shown

Proposition 4.1 *Assume* u_1 *is a solution departing from* u_{10} *to the set of the equation and conditions* (2.7) *to* (2.11). *Neglecting the higher powers of* Γ^4, *then* $u_1(x, y, t)$ *satisfies*

$$\sup_{0\le t\le T}\|u_1\|_{L^2}^2 + 2\nu\int_0^T \left\|\frac{\partial u_1}{\partial y}\right\|_{L^2}^2 dt + (n-1)\nu\Gamma^2\int_0^T \left\|\frac{\partial u_1}{\partial y}\right\|_{L^4}^4 dt + \frac{1}{10}(n-3)(2n-5)\nu\Gamma^4\int_0^T \left\|\frac{\partial u_1}{\partial y}\right\|_{L^6}^6 dt \le \tilde{C}_1\|u_{10}\|_{L^2}^2,$$

where $\tilde{C}_1$ *depends on a suitable constant* M (*to be defined in the proof*) *and* T.

Proof Multiplying the Eq. (2.7) by u_1, operating and neglecting higher power of Γ^4, the following holds

$$\iint_\Omega u_1\frac{\partial u_1}{\partial t} + I_1 = \nu\iint_\Omega \frac{\partial^2 u_1}{\partial y^2}u_1 dxdy + \frac{3}{2}(n-1)\nu\Gamma^2\iint_\Omega \frac{\partial^2 u_1}{\partial y^2}\left(\frac{\partial u_1}{\partial y}\right)^2 u_1 dxdy + \frac{1}{4}(n-3)(2n-5)\nu\Gamma^4\iint_\Omega \frac{\partial^2 u_1}{\partial y^2}\left(\frac{\partial u_1}{\partial y}\right)^4 u_1 dxdy - M^2\iint_\Omega u_1^2 dxdy,$$

After using integration by parts

$$\frac{d}{dt}\|u_1\|_{L^2}^2 = 2I_2 - 2\nu\left\|\frac{\partial u_1}{\partial y}\right\|_{L^2}^2 - (n-1)\nu\Gamma^2\left\|\frac{\partial u_1}{\partial y}\right\|_{L^4}^4 - \frac{1}{10}(n-3)(2n-5)\nu\Gamma^4\left\|\frac{\partial u_1}{\partial y}\right\|_{L^6}^6 - 2M^2\|u_1\|_{L^2}^2, \tag{4.1}$$

where

$$I_1 = -\iint_\Omega u_1\left(u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y}\right)dxdy.$$

Integrating by parts

$$I_1 = \frac{1}{2}\iint\limits_{\Omega} u_1^2 \frac{\partial u_2}{\partial y} dxdy.$$

After using Eq. (2.1), the following reads

$$I_1 = -\frac{1}{2}\iint\limits_{\Omega} u_1^2 \frac{\partial u_1}{\partial x} dxdy.$$

Integrating again we get $\left(I_1 = 0\right)$. Introducing the value of I_1 into Eq. (4.1) and after Young's inequality, the following applies

$$\begin{aligned}
&\frac{d}{dt}\|u_1\|_{L^2}^2 + 2\nu\left\|\frac{\partial u_1}{\partial y}\right\|_{L^2}^2 + (n-1)\nu\Gamma^2\left\|\frac{\partial u_1}{\partial y}\right\|_{L^4}^4 \\
&\quad + \frac{1}{10}(n-3)(2n-5)\nu\Gamma^4\left\|\frac{\partial u_1}{\partial y}\right\|_{L^6}^6 = -2M^2\|u_1\|_{L^2}^2 \\
&\leq \left|2M^2\right|\|u_1\|_{L^2}^2.
\end{aligned}$$

The Grownwall's inequality yields

$$\begin{aligned}
&\sup_{0\leq t\leq T}\|u_1\|_{L^2}^2 + 2\nu\int_0^T\left\|\frac{\partial u_1}{\partial y}\right\|_{L^2}^2 dt + (n-1)\nu\Gamma^2\int_0^T\left\|\frac{\partial u_1}{\partial y}\right\|_{L^4}^4 dt \\
&\quad + \frac{1}{10}(n-3)(2n-5)\nu\Gamma^4\int_0^T\left\|\frac{\partial u_1}{\partial y}\right\|_{L^6}^6 dt \leq \tilde{C}_1\|u_{10}\|_{L^2}^2,
\end{aligned}$$

where $\tilde{C}_1$ depends on M and T. □

In addition, the following Proposition is also required.

Proposition 4.2 *Assume a solution u_1 to set of the equation and conditions (2.7)–(2.11) departing from u_{10}. Then, neglecting higher powers of Γ^4, $\frac{\partial u_1}{\partial y}$ satisfies*

$$\begin{aligned}
&\sup_{0\leq t\leq T}\left\|\frac{\partial u_1}{\partial y}\right\|_{L^2}^2 + 2\nu\int_0^T\left\|\frac{\partial^2 u_1}{\partial y^2}\right\|_{L^2}^2 dt + 3(n-1)\nu\Gamma^2\int_0^T\left\|\frac{\partial^2 u_1}{\partial y^2}\frac{\partial u_1}{\partial y}\right\|_{L^2}^2 dt \\
&\quad + \frac{1}{2}(n-3)(2n-5)\nu\Gamma^4\int_0^T\left\|\frac{\partial^2 u_1}{\partial y^2}\left(\frac{\partial u_1}{\partial y}\right)^2\right\|_{L^2}^2 dt \leq \tilde{C}_2\left\|\frac{\partial u_{10}}{\partial y}\right\|_{L^2}^2,
\end{aligned}$$

where $\tilde{C}_2$ depends on M and T.

Proof Multiplying (2.7) by $-\frac{\partial^2 u_1}{\partial y^2}$ and integrating by parts:

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u_1}{\partial y}\right\|_{L^2}^2 = \iint\limits_{\Omega} \frac{\partial^2 u_1}{\partial y^2}\left(V\cdot\nabla u_1\right)dxdy - \nu \iint\limits_{\Omega}\left(\frac{\partial^2 u_1}{\partial y^2}\right)^2\left[1+\Gamma^2\left(\frac{\partial u_1}{\partial y}\right)^2\right]^{\frac{n-1}{2}}dxdy$$
$$- \nu(n-1)\Gamma^2 \iint\limits_{\Omega}\left(\frac{\partial^2 u_1}{\partial y^2}\right)^2\left(\frac{\partial u_1}{\partial y}\right)^2\left[1+\Gamma^2\left(\frac{\partial u_1}{\partial y}\right)^2\right]^{\frac{n-3}{2}}dxdy$$
$$- M^2 \iint\limits_{\Omega} u_1\frac{\partial^2 u_1}{\partial y^2}dxdy.$$

Expanding the term on the right side and neglecting the higher powers of Γ^4

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u_1}{\partial y}\right\|_{L^2}^2 = I_2 - \nu \iint\limits_{\Omega}\left(\frac{\partial^2 u_1}{\partial y^2}\right)^2 dxdy - \frac{3}{2}(n-1)\nu\Gamma^2 \iint\limits_{\Omega}\left(\frac{\partial^2 u_1}{\partial y^2}\right)^2\left(\frac{\partial u_1}{\partial y}\right)^2 dxdy$$
$$- \frac{1}{4}(n-3)(2n-5)\nu\Gamma^4 \iint\limits_{\Omega}\left(\frac{\partial^2 u_1}{\partial y^2}\right)^2\left(\frac{\partial u_1}{\partial y}\right)^4 dxdy$$
$$- M^2 \iint\limits_{\Omega}\left(\frac{\partial u_1}{\partial y}\right)^2 dxdy, \tag{4.2}$$

where

$$I_2 = \iint\limits_{\Omega}\frac{\partial^2 u_1}{\partial y^2}\left(V\cdot\nabla u_1\right)dxdy$$
$$= \iint\limits_{\Omega}\frac{\partial^2 u_1}{\partial y^2}u_1\frac{\partial u_1}{\partial x}dxdy + \iint\limits_{\Omega}\frac{\partial^2 u_1}{\partial y^2}u_2\frac{\partial u_1}{\partial y}dxdy.$$

Integrating I_2,

$$I_2 = -\iint\limits_{\Omega} u_1\frac{\partial u_1}{\partial y}\frac{\partial^2 u_1}{\partial x\partial y}dxdy - \iint\limits_{\Omega}\left(\frac{\partial u_1}{\partial y}\right)^2\frac{\partial u_1}{\partial x}dxdy - \frac{1}{2}\iint\limits_{\Omega}\left(\frac{\partial u_1}{\partial y}\right)^2\frac{\partial u_2}{\partial y}dxdy$$
$$= -\iint\limits_{\Omega} u_1\frac{\partial u_1}{\partial y}\frac{\partial^2 u_1}{\partial x\partial y}dxdy - \iint\limits_{\Omega}\left(\frac{\partial u_1}{\partial y}\right)^2\frac{\partial u_1}{\partial x}dxdy + \frac{1}{2}\iint\limits_{\Omega}\left(\frac{\partial u_1}{\partial y}\right)^2\frac{\partial u_1}{\partial x}dxdy$$
$$= -\iint\limits_{\Omega} u_1\frac{\partial u_1}{\partial y}\frac{\partial^2 u_1}{\partial x\partial y}dxdy - \frac{1}{2}\iint\limits_{\Omega}\left(\frac{\partial u_1}{\partial y}\right)^2\frac{\partial u_1}{\partial x}dxdy,$$

where (2.1) has been employed. Now, integrating again we get $I_2 = 0$. Introducing I_2 in Eq. (4.2) and applying Young's inequality, the following holds

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u_1}{\partial y}\right\|_{L^2}^2 \le -\nu\left\|\frac{\partial^2 u_1}{\partial y^2}\right\|_{L^2}^2 - \frac{3}{2}(n-1)\nu\Gamma^2\left\|\frac{\partial^2 u_1}{\partial y^2}\frac{\partial u_1}{\partial y}\right\|_{L^2}^2 - \frac{1}{4}(n-3)(2n-5)\nu\Gamma^4 \iint\limits_{\Omega}\left\|\frac{\partial^2 u_1}{\partial y^2}\left(\frac{\partial u_1}{\partial y}\right)^2\right\|_{L^2}^2 - M^2\left\|\frac{\partial u_1}{\partial y}\right\|_{L^2}^2,$$

which implies that

$$\frac{d}{dt}\left\|\frac{\partial u_1}{\partial y}\right\|_{L^2}^2 + 2\nu\left\|\frac{\partial^2 u_1}{\partial y^2}\right\|_{L^2}^2 + 3(n-1)\nu\Gamma^2\left\|\frac{\partial^2 u_1}{\partial y^2}\frac{\partial u_1}{\partial y}\right\|_{L^2}^2 + \frac{1}{2}(n-3)(2n-5)\nu\Gamma^4 \iint\limits_{\Omega}\left\|\frac{\partial^2 u_1}{\partial y^2}\left(\frac{\partial u_1}{\partial y}\right)^2\right\|_{L^2}^2 \le -2M^2\left\|\frac{\partial u_1}{\partial y}\right\|_{L^2}^2 \le 2M^2\left\|\frac{\partial u_1}{\partial y}\right\|_{L^2}^2.$$

Now, applying Gronwall's inequality

$$\sup_{0\le t\le T}\left\|\frac{\partial u_1}{\partial y}\right\|_{L^2}^2 + 2\nu\int_0^T\left\|\frac{\partial^2 u_1}{\partial y^2}\right\|_{L^2}^2 dt + 3(n-1)\nu\Gamma^2\int_0^T\left\|\frac{\partial^2 u_1}{\partial y^2}\frac{\partial u_1}{\partial y}\right\|_{L^2}^2 dt + \frac{1}{2}(n-3)(2n-5)\nu\Gamma^4\int_0^T\left\|\frac{\partial^2 u_1}{\partial y^2}\left(\frac{\partial u_1}{\partial y}\right)^2\right\|_{L^2}^2 dt \le \tilde{C}_2\left\|\frac{\partial u_{10}}{\partial y}\right\|_{L^2}^2,$$

where $\tilde{C}_2$ depends on M and T. □

Finally, note that the Theorem 3.1 is proved by using Propositions 4.1 and 4.2.

5 Proof of Theorem 3.2

To this end, the following Proposition is required

Proposition 5.1 *Assume u_1 is a solution departing from u_{10} to the set of the equation and conditions (2.7) to (2.11). Neglecting the higher powers of Γ^4, then $u_1(x, y, t)$ satisfies*

$$\sup_{0\le t\le T}\|u_1\|_{L^4}^4 + \nu\int_0^T\left\|u_1\frac{\partial u_1}{\partial y}\right\|_{L^2}^2 dt + (n-1)\nu\Gamma^2\int_0^T\left\|u_1^{\frac{1}{2}}\frac{\partial u_1}{\partial y}\right\|_{L^4}^4 dt + \frac{1}{10}(n-3)(2n-5)\nu\Gamma^4\int_0^T\left\|u_1^{\frac{1}{3}}\frac{\partial u_1}{\partial y}\right\|_{L^6}^6 dt \le \tilde{C}_3\|u_{10}\|_{L^4}^4,$$

where $\tilde{C}_3$ depends on M and T.

Proof Multiplying the Eq. (2.7) by u_1^3, operating, neglecting higher power of Γ^4, and after using integration by parts

$$\iint_{\Omega} u_1^3 \frac{\partial u_1}{\partial t} + I_3 = \nu \iint_{\Omega} \frac{\partial^2 u_1}{\partial y^2} u_1^3 dxdy + \frac{3}{2}(n-1)\nu\Gamma^2 \iint_{\Omega} \frac{\partial^2 u_1}{\partial y^2}\left(\frac{\partial u_1}{\partial y}\right)^2 u_1^3 dxdy + \frac{1}{4}(n-3)(2n-5)\nu\Gamma^4 \iint_{\Omega} \frac{\partial^2 u_1}{\partial y^2}\left(\frac{\partial u_1}{\partial y}\right)^4 u_1^3 dxdy - M^2 \iint_{\Omega} u_1^4 dxdy,$$

Afer using integration by parts

$$\frac{d}{dt}\|u_1\|_{L^2}^4 = 4I_3 - 12\nu\left\|u_1 \frac{\partial u_1}{\partial y}\right\|_{L^2}^2 - 6(n-1)\nu\Gamma^2\left\|u_1^{\frac{1}{2}} \frac{\partial u_1}{\partial y}\right\|_{L^4}^4 - \frac{3}{5}(n-3)(2n-5)\nu\Gamma^4\left\|u_1^{\frac{1}{3}} \frac{\partial u_1}{\partial y}\right\|_{L^6}^6 - 4M^2\|u_1\|_{L^2}^2, \tag{5.1}$$

where

$$I_3 = \iint_{\Omega} u_1^4 \frac{\partial u_1}{\partial x} dxdy + \iint_{\Omega} u_1^3 u_2 \frac{\partial u_1}{\partial y} dxdy.$$

Applying integration by parts on the second term in right side and making use of Eq. (2.1)

$$I_3 = \frac{1}{2} \iint_{\Omega} u_1^4 \frac{\partial u_1}{\partial x} dxdy.$$

Integrating again we get $\left(I_3 = 0\right)$. Introducing the value of I_3 in Eq. (5.1) and after Young's inequality

$$\frac{d}{dt}\|u_1\|_{L^2}^4 + \nu\left\|u_1 \frac{\partial u_1}{\partial y}\right\|_{L^2}^2 + (n-1)\nu\Gamma^2\left\|u_1^{\frac{1}{2}} \frac{\partial u_1}{\partial y}\right\|_{L^4}^4 + \frac{1}{10}(n-3)(2n-5)\nu\Gamma^4\left\|u_1^{\frac{1}{3}} \frac{\partial u_1}{\partial y}\right\|_{L^6}^6 = -2M^2\|u_1\|_{L4}^4 \leq \left|2M^2\right|\|u_1\|_{L^4}^4.$$

The Grownwall's inequality yields

$$\sup_{0\le t\le T}\|u_1\|_{L^4}^4+\nu\int_0^T\left\|u_1\frac{\partial u_1}{\partial y}\right\|_{L^2}^2dt+(n-1)\nu\Gamma^2\int_0^T\left\|u_1^{\frac{1}{2}}\frac{\partial u_1}{\partial y}\right\|_{L^4}^4dt$$
$$+\frac{1}{10}(n-3)(2n-5)\nu\Gamma^4\int_0^T\left\|u_1^{\frac{1}{3}}\frac{\partial u_1}{\partial y}\right\|_{L^6}^6dt\le\tilde{C}_3\|u_{10}\|_{L^4}^4,$$

where $\tilde{C}_3$ depends on M and T. □

Note that the Theorem 3.2 is shown as per the Proposition 5.1 introduced and proved.

6 Proof of Theorem 3.3

The Theorem is shown based on the coming Propositions.

Proposition 6.1 *Assume u_1 is a solution departing from u_{10} to the set of the equation and conditions (2.7) to (2.11). In addition, assume the existence of ∇u_{10} in $L^2(\Omega)$ and that $\frac{\partial u_2}{\partial x}\in L^4(\Omega)$. Neglecting the higher power of Γ^4 then*

$$\sup_{0\le t\le T}\|\nabla u_1\|_{L^2}^2+(2\nu-\epsilon)\int_0^T\left\|\frac{\partial\nabla u_1}{\partial y}\right\|_{L^2}^2dt+3(n-1)\nu\Gamma^2\int_0^T\left\|\frac{\partial u_1}{\partial y}\frac{\partial\nabla u_1}{\partial y}\right\|_{L^2}^2dt$$
$$+\frac{1}{2}(n-3)(2n-5)\nu\Gamma^4\int_0^T\left\|\left(\frac{\partial u_1}{\partial y}\right)^2\frac{\partial\nabla u_1}{\partial y}\right\|_{L^2}^2dt\le\tilde{C}_4\|\nabla u_{10}\|_{L^2}^2,$$

where $\tilde{C}_4$ depends on M and T.

Proof Considering the inner product in Eq. (2.7) with Δu_1 and integrating

$$\iint_\Omega\frac{\partial u_1}{\partial t}\Delta u_1dxdy+\iint_\Omega\Delta u_1\left(u_1\frac{\partial u_1}{\partial x}+u_2\frac{\partial u_2}{\partial y}\right)dxdy$$
$$=\nu\iint_\Omega\frac{\partial^2u_1}{\partial y^2}\left[1+\Gamma^2\left(\frac{\partial u_1}{\partial y}\right)^2\right]^{\frac{n-1}{2}}\Delta u_1dxdy$$
$$+\nu(n-1)\Gamma^2\iint_\Omega\frac{\partial^2u_1}{\partial y^2}\left(\frac{\partial u_1}{\partial y}\right)^2\left[1+\Gamma^2\left(\frac{\partial u_1}{\partial y}\right)^2\right]^{\frac{n-3}{2}}\Delta u_1dxdy$$
$$-M^2\iint_\Omega u_1\Delta u_1dxdy.$$

Expanding the term on the right side in the last equation and neglectng the higher powers of Γ^4,

$$\begin{aligned}\iint\limits_{\Omega}\frac{\partial u_1}{\partial t}\Delta u_1 dxdy + I_4 &= \nu\iint\limits_{\Omega}\frac{\partial^2 u_1}{\partial y^2}\Delta u_1 dxdy\\ &+\frac{3}{2}(n-1)\nu\Gamma^2\iint\limits_{\Omega}\frac{\partial^2 u_1}{\partial y^2}\left(\frac{\partial u_1}{\partial y}\right)^2\Delta u_1 dxdy\\ &+\frac{1}{4}(n-3)(2n-5)\nu\Gamma^4\iint\limits_{\Omega}\frac{\partial^2 u_1}{\partial y^2}\left(\frac{\partial u_1}{\partial y}\right)^4\Delta u_1 dxdy\\ &-M^2\iint\limits_{\Omega}u_1\Delta u_1 dxdy.\end{aligned}$$

Aplying interation by parts

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\nabla u_1\right\|_{L^2}^2 - I_4\\ &\quad=-\nu\iint\limits_{\Omega}\left(\frac{\partial\nabla u_1}{\partial y}\right)^2 dxdy-\frac{3}{2}(n-1)\nu\Gamma^2\iint\limits_{\Omega}\left(\frac{\partial u_1}{\partial y}\right)^2\left(\frac{\partial\nabla u_1}{\partial y}\right)^2 dxdy\\ &\qquad-\frac{1}{4}(n-3)(2n-5)\nu\Gamma^4\iint\limits_{\Omega}\left(\frac{\partial u_1}{\partial y}\right)^4\left(\frac{\partial\nabla u_1}{\partial y}\right)^2 dxdy - M^2\left\|\nabla u_1\right\|_{L^2}^2,\end{aligned}$$

which implies that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\nabla u_1\right\|_{L^2}^2+\nu\left\|\frac{\partial\nabla u_1}{\partial y}\right\|_{L^2}^2+\frac{3}{2}(n-1)\nu\Gamma^2\left\|\frac{\partial u_1}{\partial y}\frac{\partial\nabla u_1}{\partial y}\right\|_{L^2}^2\\ &\quad+\frac{1}{4}(n-3)(2n-5)\nu\Gamma^4\left\|\left(\frac{\partial u_1}{\partial y}\right)^2\frac{\partial\nabla u_1}{\partial y}\right\|_{L^2}^2=I_4-M^2\left\|\nabla u_1\right\|_{L^2}^2,\end{aligned}\tag{6.1}$$

where

$$I_4=\iint\limits_{\Omega}\Delta u_1\left(u_1\frac{\partial u_1}{\partial x}+u_2\frac{\partial u_1}{\partial y}\right)dxdy.$$

Now, the intention is to further develop the integral I_4, to this end

$$
\begin{aligned}
I_4 &= -\iint_{\Omega} \nabla u_1 \nabla \left(u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} \right) dxdy \\
&= -\iint_{\Omega} \left(\frac{\partial u_1}{\partial x} \right)^3 dxdy - \iint_{\Omega} u_1 \frac{\partial u_1}{\partial x} \frac{\partial^2 u_1}{\partial x^2} dxdy - \iint_{\Omega} \frac{\partial u_1}{\partial x} \frac{\partial u_1}{\partial y} \frac{\partial u_2}{\partial x} dxdy \\
&- \iint_{\Omega} u_2 \frac{\partial^2 u_1}{\partial x \partial y} \frac{\partial u_1}{\partial x} dxdy - \iint_{\Omega} \left(\frac{\partial u_1}{\partial y} \right)^2 \frac{\partial u_1}{\partial x} dxdy - \iint_{\Omega} u_1 \frac{\partial u_1}{\partial y} \frac{\partial^2 u_1}{\partial x \partial y} dxdy \\
&- \iint_{\Omega} \left(\frac{\partial u_1}{\partial y} \right)^2 \frac{\partial u_2}{\partial y} dxdy - \iint_{\Omega} u_2 \frac{\partial u_1}{\partial y} \frac{\partial^2 u_1}{\partial y^2} dxdy, \\
&= -\frac{1}{2} \iint_{\Omega} \left(\frac{\partial u_1}{\partial x} \right)^3 dxdy - \iint_{\Omega} \frac{\partial u_1}{\partial x} \frac{\partial u_1}{\partial y} \frac{\partial u_2}{\partial x} dxdy \\
&- \frac{1}{2} \iint_{\Omega} \left(\frac{\partial u_1}{\partial y} \right)^2 \frac{\partial u_1}{\partial x} dxdy - \frac{1}{2} \iint_{\Omega} \left(\frac{\partial u_1}{\partial y} \right)^2 \frac{\partial u_2}{\partial x} dxdy.
\end{aligned}
$$

From Eq. (2.1), the following holds

$$
I_4 = -\frac{1}{2} \iint_{\Omega} \left(\frac{\partial u_1}{\partial x} \right)^3 dxdy + \iint_{\Omega} \frac{\partial u_2}{\partial y} \frac{\partial u_1}{\partial y} \frac{\partial u_2}{\partial x} dxdy.
$$

Integrating again

$$
\begin{aligned}
I_4 &= -\frac{1}{2} \iint_{\Omega} \left(\frac{\partial u_1}{\partial x} \right)^3 dxdy - \iint_{\Omega} u_1 \frac{\partial^2 u_2}{\partial y^2} \frac{\partial u_2}{\partial x} dxdy - \iint_{\Omega} u_1 \frac{\partial u_2}{\partial y} \frac{\partial u_2}{\partial x \partial y} dxdy \\
&= -\frac{1}{2} \iint_{\Omega} \left(\frac{\partial u_1}{\partial x} \right)^3 dxdy - \iint_{\Omega} u_1 \frac{\partial^2 u_2}{\partial y^2} \frac{\partial u_2}{\partial x} dxdy + \frac{1}{2} \iint_{\Omega} \left(\frac{\partial u_2}{\partial y} \right)^2 \frac{\partial u_1}{\partial x} dxdy \\
&= -\iint_{\Omega} u_1 \frac{\partial^2 u_1}{\partial x \partial y} \frac{\partial u_2}{\partial x} dxdy,
\end{aligned}
$$

where Eq. (2.1) has been used, therefore I_4 becomes

$$I_4 \leq \|u\|_{L^4} \left\| \frac{\partial \nabla u_1}{\partial y} \right\|_{L^2} \left\| \frac{\partial u_2}{\partial x} \right\|_{L^4}$$
$$\leq \frac{\epsilon}{2} \left\| \frac{\partial \nabla u_1}{\partial y} \right\|_{L^2}^2 + \frac{1}{2\epsilon} \|u_1\|_{L^4}^2 \left\| \frac{\partial u_2}{\partial x} \right\|_{L^4}^2.$$

Introducing the assessed integral I_4 into Eq. (6.1) and after using Proposition 3 and $\frac{\partial u_2}{\partial x} \in L^4(\Omega)$

$$\frac{d}{dt} \|\nabla u_1\|_{L^2}^2 + (2\nu - \epsilon) \left\| \frac{\partial \nabla u_1}{\partial y} \right\|_{L^2}^2 + 3(n-1)\nu\Gamma^2 \left\| \frac{\partial u_1}{\partial y} \frac{\partial \nabla u_1}{\partial y} \right\|_{L^2}^2$$
$$+ \frac{1}{2}(n-3)(2n-5)\nu\Gamma^4 \left\| \left(\frac{\partial u_1}{\partial y} \right)^2 \frac{\partial \nabla u_1}{\partial y} \right\|_{L^2}^2 \leq C_{16} \|\nabla u_1\|_{L^2}^2.$$

From Grownwall's inequality

$$\sup_{0 \leq t \leq T} \|\nabla u_1\|_{L^2}^2 + (2\nu - \epsilon) \int_0^T \left\| \frac{\partial \nabla u_1}{\partial y} \right\|_{L^2}^2 dt + 3(n-1)\nu\Gamma^2 \int_0^T \left\| \frac{\partial u_1}{\partial y} \frac{\partial \nabla u_1}{\partial y} \right\|_{L^2}^2 dt$$
$$+ \frac{1}{2}(n-3)(2n-5)\nu\Gamma^4 \int_0^T \left\| \left(\frac{\partial u_1}{\partial y} \right)^2 \frac{\partial \nabla u_1}{\partial y} \right\|_{L^2}^2 dt \leq \tilde{C}_4 \|\nabla u_{10}\|_{L^2}^2,$$

where $\tilde{C}_4$ depends on M and T. □

Proposition 6.2 *Assume u_1 is a solution departing from u_{10} to the set of the equation and conditions (2.7) to (2.11). assume that in the strip Ω, there exists $\left(\Delta u_{10}, \frac{\partial \nabla u_2}{\partial y}, \Delta u_2 \right) \in L^2(\Omega)$. Neglecting the higher power of Γ^4, then*

$$\sup_{0 \leq t \leq T} \|\Delta u_1\|_{L^2}^2 + 3(n-1)\nu\Gamma^2 \int_0^T \left\| \frac{\partial u_1}{\partial y} \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 dt$$
$$+ \frac{1}{2}(n-3)(2n-5)\nu\Gamma^4 \int_0^T \left\| \left(\frac{\partial u_1}{\partial y} \right)^2 \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 dt$$
$$+ 2(\nu - 8\epsilon) \int_0^T \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 dt \leq \tilde{C}_5 \|\Delta u_{10}\|_{L^2}^2,$$

where $\tilde{C}_5$ depends on constants and T.

Proof Apply the monotone operator Δ to Eq. (2.7) and take inner product with Δu_1. After integration by parts, the following reads

$$\iint_{\Omega} \Delta u_1 \Delta \frac{\partial u_1}{\partial t} dxdy + \iint_{\Omega} u_1 \Delta \left(u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} \right) \Delta dxdy$$
$$= \nu \iint_{\Omega} \Delta \left(\left(\frac{\partial^2 u_1}{\partial y^2} \right) \left[1 + \Gamma^2 \left(\frac{\partial u_1}{\partial y} \right)^2 \right]^{\frac{n-1}{2}} \right) \Delta u_1 dxdy$$
$$+ \nu(n-1)\Gamma^2 \iint_{\Omega} \Delta \left(\left(\frac{\partial^2 u_1}{\partial y^2} \right) \left(\frac{\partial u_1}{\partial y} \right)^2 \left[1 + \Gamma^2 \left(\frac{\partial u_1}{\partial y} \right)^2 \right]^{\frac{n-3}{2}} \right) \Delta u_1 dxdy$$
$$- \frac{\nu \phi}{k} \iint_{\Omega} \Delta \left(\left[1 + \Gamma^2 \left(\frac{\partial u_1}{\partial y} \right)^2 \right]^{\frac{n-1}{2}} u_1 \right) \Delta u_1 dxdy - M^2 \iint_{\Omega} \Delta u_1 \Delta u_1 dxdy.$$

Expanding the terms on the right side and neglecting the higher powers of Γ^4,

$$\frac{1}{2} \frac{d}{dt} \iint_{\Omega} \left(\Delta u_1 \right)^2 dxdy - I_5 = \nu \iint_{\Omega} \frac{\partial^2 u_1}{\partial y^2} \Delta u_1 dxdy$$
$$+ \frac{3}{2}(n-1)\nu\Gamma^2 \iint_{\Omega} \Delta \left(\frac{\partial^2 u_1}{\partial y^2} \left(\frac{\partial u_1}{\partial y} \right)^2 \right) \Delta u_1 dxdy$$
$$+ \frac{1}{4}(n-3)(2n-5)\nu\Gamma^4 \iint_{\Omega} \Delta \left(\frac{\partial^2 u_1}{\partial y^2} \left(\frac{\partial u_1}{\partial y} \right)^4 \right) \Delta u_1 dxdy - M^2 \iint_{\Omega} \left(\Delta u_1 \right)^2 dxdy.$$

After Integration

$$\frac{1}{2} \frac{d}{dt} \left\| \Delta u_1 \right\|_{L^2}^2 = I_5 - \nu \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 + \frac{3}{2}(n-1)\nu\Gamma^2 I_6 + \frac{1}{4}(n-3)(2n-5)\nu\Gamma^4 I_7 - M^2 \left\| \Delta u_1 \right\|_{L^2}^2, \tag{2.13}$$

where

$$I_5 = - \iint_{\Omega} \Delta \left(u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} \right) \Delta u_1 dxdy,$$
$$I_6 = \iint_{\Omega} \Delta \left(\frac{\partial^2 u_1}{\partial y^2} \left(\frac{\partial u_1}{\partial y} \right)^2 \right) \Delta u_1 dxdy,$$
$$I_7 = \iint_{\Omega} \Delta \left(\frac{\partial^2 u_1}{\partial y^2} \left(\frac{\partial u_1}{\partial y} \right)^4 \right) \Delta u_1 dxdy.$$

Now for I_5

$$
\begin{aligned}
I_5 &= -\iint_{\Omega} \Delta\left(u_1\frac{\partial u_1}{\partial x}+u_2\frac{\partial u_1}{\partial y}\right)\Delta u_1 dxdy\\
&= -\iint_{\Omega}\left(\frac{\partial u_1}{\partial x}\left(\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2}\right)+\frac{\partial u_1}{\partial y}\left(\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial^2 u_2}{\partial y^2}\right)\right)\Delta u_1 dxdy\\
&\quad -\int\int\left(2\frac{\partial u_1}{\partial x}\frac{\partial^2 u_1}{\partial x^2}+2\frac{\partial u_1}{\partial y}\frac{\partial^2 u_1}{\partial x\partial y}+2\frac{\partial u_2}{\partial x}\frac{\partial^2 u_1}{\partial x\partial y}+2\frac{\partial u_2}{\partial y}\frac{\partial^2 u_1}{\partial y^2}\right)\Delta u_1 dxdy\\
&\quad -\iint_{\Omega}\left(u_1\frac{\partial^3 u_1}{\partial x^3}+u_1\frac{\partial^3 u_1}{\partial y^2\partial x}+u_2\frac{\partial^3 u_1}{\partial x^2\partial y}+u_2\frac{\partial^3 u_1}{\partial y^3}\right)\Delta u_1 dxdy.
\end{aligned}
$$

Applying integration by parts and continuity

$$
\iint_{\Omega}\left(u_1\frac{\partial^3 u_1}{\partial x^3}+u_1\frac{\partial^3 u_1}{\partial y^2\partial x}+u_2\frac{\partial^3 u_1}{\partial x^2\partial y}+u_2\frac{\partial^3 u_1}{\partial y^3}\right)\Delta u_1 dxdy=0.
$$

Then

$$
\begin{aligned}
I_5 &= -\iint_{\Omega}\Delta u_1\frac{\partial u_1}{\partial x}\Delta u_1 dxdy-\iint_{\Omega}\Delta u_1\frac{\partial u_1}{\partial y}\Delta u_2 dxdy-2\iint_{\Omega}\Delta u_1\frac{\partial u_1}{\partial x}\frac{\partial^2 u_1}{\partial x^2}dxdy\\
&\quad -2\iint_{\Omega}\Delta u_1\frac{\partial u_1}{\partial y}\frac{\partial^2 u_1}{\partial x\partial y}dxdy-2\iint_{\Omega}u_1\frac{\partial u_2}{\partial x}\frac{\partial^2 u_1}{\partial x\partial y}dxdy-2\iint_{\Omega}\Delta u_1\frac{\partial u_2}{\partial y}\frac{\partial^2 u_1}{\partial y^2}dxdy\\
&= k_1+k_2+k_3+k_4+k_5+k_6.
\end{aligned}
\tag{6.2}
$$

In order to solve the above six terms, the Lemma 1 is employed so that

$$
\begin{aligned}
k_1 &= -\iint_{\Omega}\Delta u_1\frac{\partial u_1}{\partial x}\Delta u_1 dxdy\\
&\le C_0\|\Delta u_1\|_{L^2}\|\Delta u_1\|_{L^2}^{\frac{1}{2}}\left\|\frac{\partial\Delta u_1}{\partial y}\right\|_{L^2}^{\frac{1}{2}}\left\|\frac{\partial u_1}{\partial x}\right\|_{L^2}^{\frac{1}{2}}\left\|\frac{\partial^2 u_1}{\partial x^2}\right\|_{L^2}^{\frac{1}{2}}\\
&= C_0\|\Delta u_1\|_{L^2}\|\Delta u_1\|_{L^2}^{\frac{1}{2}}\left\|\frac{\partial\Delta u_1}{\partial y}\right\|_{L^2}^{\frac{1}{2}}\left\|\frac{\partial u_1}{\partial x}\right\|_{L^2}^{\frac{1}{2}}\left\|\frac{\partial^2 u_2}{\partial x\partial y}\right\|_{L^2}^{\frac{1}{2}}\\
&\le C_0\|\Delta u_1\|_{L^2}\|\Delta u_1\|_{L^2}^{\frac{1}{2}}\left\|\frac{\partial\Delta u_1}{\partial y}\right\|_{L^2}^{\frac{1}{2}}\left\|\frac{\partial u_1}{\partial x}\right\|_{L^2}^{\frac{1}{2}}\left\|\frac{\partial\nabla u_2}{\partial y}\right\|_{L^2}^{\frac{1}{2}}\\
&\le \epsilon\left\|\frac{\partial\Delta u_1}{\partial y}\right\|_{L^2}^{2}+C_\epsilon\left\|\frac{\partial u_1}{\partial x}\right\|_{L^2}^{\frac{2}{3}}\|\Delta u_1\|_{L^2}^{2}\left\|\frac{\partial\nabla u_2}{\partial y}\right\|_{L^2}^{\frac{2}{3}}.
\end{aligned}
\tag{6.3}
$$

$$
\begin{aligned}
k_2 &= -\iint_{\Omega} \Delta u_1 \frac{\partial u_1}{\partial y} \Delta u_2 dxdy \\
&\leq C_0 \|\Delta u_2\|_{L^2} \|\Delta u_1\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial u_1}{\partial y} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial^2 u_1}{\partial x \partial y} \right\|_{L^2}^{\frac{1}{2}} \\
&\leq \epsilon \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 + C_\epsilon \left\| \frac{\partial u_1}{\partial y} \right\|_{L^2}^2 \|\Delta u_1\|_{L^2}^2 \left\| \frac{\partial \nabla u_1}{\partial y} \right\|_{L^2}^2 + C_\epsilon \|\Delta u_2\|_{L^2}^2.
\end{aligned} \tag{6.4}
$$

$$
\begin{aligned}
k_3 &= 2 \iint_{\Omega} \Delta u_1 \frac{\partial u_2}{\partial y} \frac{\partial^2 u_1}{\partial x^2} dxdy \\
&\leq C_0 \left\| \frac{\partial^2 u_1}{\partial x^2} \right\|_{L^2} \|\Delta u_1\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial u_2}{\partial y} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial^2 u_2}{\partial x \partial y} \right\|_{L^2}^{\frac{1}{2}} \\
&\leq C_0 \|\Delta u_1\|_{L^2}^{\frac{3}{2}} \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial u_1}{\partial x} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial \nabla u_2}{\partial y} \right\|_{L^2}^{\frac{1}{2}} \\
&\leq \epsilon \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 + C_\epsilon \left\| \frac{\partial u_2}{\partial y} \right\|_{L^2}^{\frac{2}{3}} \|\Delta u_1\|_{L^2}^2 \left\| \frac{\partial \nabla u_2}{\partial y} \right\|_{L^2}^{\frac{2}{3}}.
\end{aligned} \tag{6.5}
$$

$$
\begin{aligned}
k_4 &= -2 \iint_{\Omega} \frac{\partial u_1}{\partial y} \frac{\partial^2 u_1}{\partial x \partial y} \Delta u_1 dxdy \\
&\leq C_0 \left\| \frac{\partial u_1}{\partial y} \right\|_{L^2} \|\Delta u_1\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial^2 u_1}{\partial x \partial y} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial^3 u_1}{\partial x^2 \partial y} \right\|_{L^2}^{\frac{1}{2}} \\
&\leq C_0 \left\| \frac{\partial u_1}{\partial y} \right\|_{L^2} \|\Delta u_1\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2} \left\| \frac{\partial \nabla u_1}{\partial y} \right\|_{L^2}^{\frac{1}{2}} \\
&\leq \epsilon \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 + C_\epsilon \left\| \frac{\partial u_1}{\partial y} \right\|_{L^2}^4 \|\Delta u_1\|_{L^2}^2 + C_\epsilon \left\| \frac{\partial \nabla u_1}{\partial y} \right\|_{L^2}^2.
\end{aligned} \tag{6.6}
$$

$$
\begin{aligned}
k_5 &= -2 \iint_{\Omega} \frac{\partial u_1}{\partial x} \frac{\partial^2 u_1}{\partial x \partial y} \Delta u_1 dxdy \\
&\leq C_0 \left\| \frac{\partial u_1}{\partial x} \right\|_{L^2} \|\Delta u_1\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial^2 u_1}{\partial x \partial y} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial^3 u_1}{\partial x^2 \partial y} \right\|_{L^2}^{\frac{1}{2}} \\
&\leq C_0 \left\| \frac{\partial u_1}{\partial x} \right\|_{L^2} \|\Delta u_1\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2} \left\| \frac{\partial \nabla u_1}{\partial y} \right\|_{L^2}^{\frac{1}{2}} \\
&\leq \epsilon \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 + C_\epsilon \left\| \frac{\partial u_2}{\partial x} \right\|_{L^2}^4 \|\Delta u_1\|_{L^2}^2 + C_\epsilon \left\| \frac{\partial \nabla u_1}{\partial y} \right\|_{L^2}^2.
\end{aligned} \tag{6.7}
$$

$$
\begin{aligned}
k_6 &= -2 \iint_{\Omega} \frac{\partial u_2}{\partial y} \frac{\partial^2 u_1}{\partial y^2} \Delta u_1 dxdy \\
&\leq C_0 \|\Delta u_1\|_{L^2} \left\| \frac{\partial u_2}{\partial y} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial^2 u_2}{\partial x \partial y} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial^2 u_1}{\partial y^2} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial^3 u_1}{\partial y^3} \right\|_{L^2}^{\frac{1}{2}} \\
&\leq C_0 \|\Delta u_1\|_{L^2}^{\frac{3}{2}} \left\| \frac{\partial u_1}{\partial x} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial \nabla u_2}{\partial y} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^{\frac{1}{2}} \\
&\leq \epsilon \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 + C_\epsilon \left\| \frac{\partial u_1}{\partial x} \right\|_{L^2}^{\frac{2}{3}} \|\Delta u_1\|_{L^2}^2 \left\| \frac{\partial \nabla u_2}{\partial y} \right\|_{L^2}^{\frac{2}{3}}.
\end{aligned} \tag{6.8}
$$

Integrating I_6, we have

$$
\begin{aligned}
I_6 &= -2 \iint_{\Omega} \frac{\partial u_1}{\partial y} \left(\frac{\partial \nabla u_1}{\partial y} \right)^2 \frac{\partial \Delta u_1}{\partial y} dxdy - \iint_{\Omega} \left(\frac{\partial u_1}{\partial y} \right)^2 \left(\frac{\partial \Delta u_1}{\partial y} \right)^2 dxdy \\
&\leq \epsilon \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 + C_\epsilon \left\| \frac{\partial u_1}{\partial y} \left(\frac{\partial \nabla u_1}{\partial y} \right)^2 \right\|_{L^2}^2 - \left\| \frac{\partial u_1}{\partial y} \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2,
\end{aligned} \tag{6.9}
$$

where we made use of the Young's inequality. Similarly to solve I_7

$$
I_7 \leq \epsilon \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 + C_\epsilon \left\| \left(\frac{\partial u_1}{\partial y} \right)^2 \left(\frac{\partial \nabla u_1}{\partial y} \right)^2 \right\|_{L^2}^2 - \left\| \left(\frac{\partial u_1}{\partial y} \right)^2 \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2. \tag{2.22}
$$

Combining Eq. (2.13) to Eq. (2.22), After using Proposition 1, Proposition 2, Proposition 3 and $\Delta u_2, \frac{\partial \nabla u_2}{\partial y} \in L^2(\Omega)$, we get

$$
\begin{aligned}
&\frac{d}{dt} \|\Delta u_1\|_{L^2}^2 + 3(n-1)\nu\Gamma^2 \left\| \frac{\partial u_1}{\partial y} \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 \\
&\quad + \frac{1}{2}(n-3)(2n-5)\nu\Gamma^4 \left\| \left(\frac{\partial u_1}{\partial y} \right)^2 \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 \\
&\quad + 2(\nu - 8\epsilon) \left\| \frac{\partial \Delta u_1}{\partial y} \right\|_{L^2}^2 \leq C_1 \|\Delta u_1\|_{L^2}^2,
\end{aligned}
$$

where C_1 is suitable constant. Applying Grownwall's inequality again

$$\sup_{0\le t\le T}\|\Delta u_1\|_{L^2}^2+\frac{3}{2}(n-1)\nu\Gamma^2\int_0^T\left\|\frac{\partial u_1}{\partial y}\frac{\partial\Delta u_1}{\partial y}\right\|_{L^2}^2dt$$
$$+\frac{1}{4}(n-3)(2n-5)\nu\Gamma^4\int_0^T\left\|\left(\frac{\partial u_1}{\partial y}\right)^2\frac{\partial\Delta u_1}{\partial y}\right\|_{L^2}^2dt$$
$$+(\nu-8\epsilon)\int_0^T\left\|\frac{\partial\Delta u_1}{\partial y}\right\|_{L^2}^2dt\le\tilde{C}_5\|\Delta u_{10}\|_{L^2}^2,$$

where $\tilde{C}_5$ depends on suitable constants and T. □

Finally, the Theorem 3.3 is shown by using Proposition 6.1 and Proposition 6.2.

7 Conclusion

The proposed Theorems 3.1, 3.2 and 3.3 have been shown in the different supporting propositions. Such Theorems lead to confirm on the existence of regular solutions departing from an initial data generalized under the defined functional spaces. The shown solutions are applicable to the two dimensional Carreau fluid on the defined strip $\Omega=[0,L]\times[0,\infty)$.

Author Contributions José L. Díaz (JL), Saeed Rahman (SR), M. Khan (MK) and Guang-Zhong Yin (GZ). JL, SR and MK conceived the study and the overall manuscript design. JL, SR and MK carried out the analytical studies. SR has performed the specific analytical assessment with the revision of JL. SR, GZ and JL have made a revision of the manuscript. GZ has made the final edition and overall supervision. All authors read and approved the final manuscript.

Funding Not applicable.

Data Availability The authors declare that data can be made available upon request.

Declarations

Conflict of Interest The authors have no competing interests to declare that are relevant to the content of this article.

Consent for Publication Authors declare the consent for manuscript publication

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