

Restless Temporal Path Parameterized Above Lower Bounds

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Abstract

Reachability questions are one of the most fundamental algorithmic primitives in temporal graphs – graphs whose edge set changes over discrete time steps. A core problem here is the NP-hard SHORT RESTLESS TEMPORAL PATH: given a temporal graph \mathcal{G} , two distinct vertices s and z , and two numbers δ and k , is there a δ -restless temporal s - z path of length at most k ? A temporal path is a path whose edges appear in chronological order and a temporal path is δ -restless if two consecutive path edges appear at most δ time steps apart from each other. Among others, this problem has applications in neuroscience and epidemiology. While SHORT RESTLESS TEMPORAL PATH is known to be computationally hard, e.g., it is NP-hard even if the temporal graph consists of three discrete time steps and it is W[1]-hard when parameterized by the feedback vertex number of the underlying graph, it is fixed-parameter tractable when parameterized by the path length k . We improve on this by showing that SHORT RESTLESS TEMPORAL PATH can be solved in (randomized) $4^{k-d}|\mathcal{G}|^{O(1)}$ time, where d is the minimum length of a temporal s - z path.

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1 Introduction

Susceptible-Infected-Recovered. These are the three states of the *SIR-model* – a canonical spreading model for diseases where recovery confers lasting resistance [6, 31, 39]. Here, an individual is at first susceptible (S) to get a certain disease, can devolve to be infected (I), and ends up resilient after recovery (R). We study one of the most fundamental algorithmic questions in this model: given a set of individuals with a list of physical contacts over time, and two individuals s and z , is it possible to have a chain of infections from s to z ? As the timing of the physical contacts is crucial in this scenario, we use a *temporal graph* $\mathcal{G} := (V, (E_i)_{i=1}^{\tau})$ consisting of a set V of vertices and an edge set that changes over discrete time steps described by a chronologically ordered sequence $(E_i)_{i=1}^{\tau}$ of edge sets over V . Here, the number τ denotes the *lifetime* of the temporal graph \mathcal{G} . A temporal path is a path whose edges appear in chronological order. In particular, a sequence $P := ((e_i, t_i))_{i=1}^m$ of time-edges from $\mathcal{E}(\mathcal{G}) := \bigcup_{i=1}^{\tau} E_i \times \{i\}$ is a *temporal s - z path* of length m if $(\bigcup_{i=1}^m e_i, \{e_i \mid i \in [m]\})$ is an s - z path (no vertex is visited twice) and $t_i \leq t_{i+1}$ for all $i \in [m-1]$. If we construct a temporal graph where the vertices are individuals and an edge $e \in E_t$ represents a physical contact of two individuals at time step t , then a chain of infections is represented by a temporal path. However, not every temporal path yields a potential chain of infections, as an infected person might recover before the next individual is met. To represent infection chains in the SIR-model by temporal paths, we restrict the waiting time at each intermediate vertex to a prescribed duration – that is, the time until an individual becomes resilient after infection. These temporal paths are called *restless*. In particular, the temporal s - z path P is δ -*restless* if $t_i \leq t_{i+1} \leq t_i + \delta$ for all $i \in [m-1]$ where $\delta \in \mathbb{N} \cup \{0\}$. Hence, restless temporal paths model



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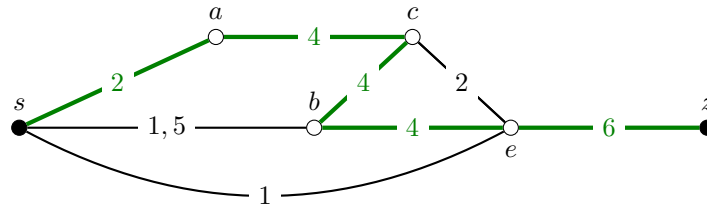
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■ **Figure 1** An illustration of a temporal graph with vertices $s, a, b, c, e,$ and z . The labels on the edges denote at which time steps the edges are present. The time-edges of a 2-restless temporal s - z path in this temporal graph are marked by thick (green) edges. In fact, this is the only 2-restless temporal s - z path in this temporal graph, as we cannot visit a vertex twice and two consecutive time-edges have to be at most two time steps apart.

infection transmission routes of diseases that grant immunity upon recovery [28]. Other applications of restless temporal paths appear in the context of delay-tolerant networking with time-aware routing tables [13], and in the context of finding signaling pathways in brain networks [41]. Consider Figure 1 for an illustration of a temporal graph with a 2-restless temporal s - z path.

The central problem of this work is as follows.

SHORT RESTLESS TEMPORAL PATH

Input: A temporal graph \mathcal{G} , a source vertex $s \in V$, a destination vertex $z \in V$, and integers $\delta, k \in \mathbb{N}$.

Question: Is there a δ -restless temporal s - z path in \mathcal{G} of length at most k ?

Casteigts et al. [13] showed that SHORT RESTLESS TEMPORAL PATH is NP-hard even if $\delta = 1$, $\tau = 3$, every edge appears only once, and the underlying graph has a maximum degree of six. Moreover, they showed that it is W[1]-hard when parameterized by the distance to disjoint paths of the underlying graph¹. Among other things, this presumably excludes exact algorithms for SHORT RESTLESS TEMPORAL PATH running in $f(x) \cdot |\mathcal{G}|^{O(1)}$ time, where x is the distance to disjoint paths of the underlying graph and f is any computable function.

Hence, SHORT RESTLESS TEMPORAL PATH is presumably not fixed-parameter tractable when parameterized by a wide range of well-known parameters of the underlying (static) graph, e.g., feedback vertex number, pathwidth, or cliquewith. However, SHORT RESTLESS TEMPORAL PATH is fixed-parameter tractable when parameterized by k or the treedepth of the underlying graph or the feedback edge number of the underlying graph [13]. Thejaswi et al. [41] showed that for every $p \in \mathbb{R}$ with $0 < p < 1$ there is a randomized $O(2^k k |\mathcal{G}| \delta \log(k \cdot 1/p))$ -time algorithm for SHORT RESTLESS TEMPORAL PATH that has a one-sided error probability of at most p . More precisely, if the algorithm returns *yes*, then the given instance I of SHORT RESTLESS TEMPORAL PATH is a *yes*-instance, and if the algorithm returns *no*, then the probability that I is a *yes*-instance is at most p . They conducted experiments on large synthetic and real-world data sets and showed that their algorithm performs well as long as the parameter k is small. For example, one can solve SHORT RESTLESS TEMPORAL PATH with $k \leq 9$ and a temporal graph with 36 million time-edges in less than one hour with customary desktop hardware. On the data set used in the experiments, k seems to be the

¹ That is, the minimum number of vertices we need to remove from a graph such that the remaining graph consists of a set of vertex-disjoint paths.

only useful parameter for which we know that SHORT RESTLESS TEMPORAL PATH is fixed-parameter tractable; all other known parameters (i.e., timed feedback vertex number [13], treedepth of the underlying graph, and feedback edge number of the underlying graph) are too large to be eligible in practice [41]. Hence, the current algorithms are not satisfactory when it comes to computing long restless temporal paths in real-world temporal networks.

The parameter k of SHORT RESTLESS TEMPORAL PATH can be seen as the *solution size* and is thus a natural and well-motivated parameter from a parameterized algorithmics point of view. However, as we observed before, FPT-algorithms regarding the solution size are not necessarily practical, e.g., if all solutions are large. To address this problem, one can investigate *parameterizations above guaranteed lower bounds* [4, 8, 14, 25, 27, 33, 34, 29]: that is, the *difference* between the smallest size of a solution and a guaranteed lower bound for the solution size. We refer to Gutin and Mnich [26] for a survey on graph problems parameterized above and below guarantees.

In the case of SHORT RESTLESS TEMPORAL PATH, we are looking for a guaranteed lower bound d on the length of any δ -restless temporal s - z path such that SHORT RESTLESS TEMPORAL PATH admits an FPT-algorithm when parameterized by $k - d$. Note that if d exceeds k , then this is a *no-instance*. Towards the choice of a lower bound, we see the following three intermediate options.

The distance from s to z : The minimum length of an s - z path in the underlying graph.

This is lower bound for the length of any δ -restless temporal s - z path, as each δ -restless temporal s - z path induces an s - z path in the underlying graph.

The temporal distance from s to z : The minimum length of a temporal s - z path. Clearly, this is a lower bound for the length of any δ -restless temporal s - z path. Note that, the temporal distance from s to z is at least the distance from s to z in the underlying graph. Hence, the parameter $k - d_t$ is *at least as good* as the parameter $k - d_u$ for SHORT RESTLESS TEMPORAL PATH, where d_u is the distance between s and z and d_t is the temporal distance between s and z .

The δ -restless temporal distance from s to z : The minimum length of a δ -restless temporal s - z walk. Herein, a sequence $W := ((e_i, t_i))_{i=1}^m$ of time-edges is a *temporal s - z walk* of length m if the edges $(e_i)_{i=1}^m$ induce an s - z walk and $t_i \leq t_{i+1}$ for all $i \in [m - 1]$. Moreover, W is δ -restless if $m = 1$ or $t_{i+1} - t_i \leq \delta$. As any δ -restless temporal s - z path is also a δ -restless temporal s - z walk, the minimum length of a δ -restless temporal s - z walk is a lower bound for the length of any δ -restless temporal s - z path. Moreover, this lower bound is at least as good as the temporal distance from s to z .

For the sake of brevity, we say for an instance $(\mathcal{G}, s, z, \delta, k)$ of SHORT RESTLESS TEMPORAL PATH that the δ -restless temporal distance from s to z , the temporal distance from s to z , and the distance from s to z are always below $k + 1$, as otherwise $(\mathcal{G}, s, z, \delta, k)$ is a trivial *no-instance*.

Unfortunately, a closer look at the NP-hardness reductions of Casteigts et al. [13] reveals that, unless $P=NP$, there is no $|\mathcal{G}|^{f(k-d_r)}$ -time algorithm for SHORT RESTLESS TEMPORAL PATH, where d_r is the δ -restless temporal distance from s to z and f is any computable function. However, we show that one can determine in polynomial time whether there is a δ -restless temporal s - z path which has the same length as a shortest temporal s - z path. Moreover, we show that SHORT RESTLESS TEMPORAL PATH admits an FPT-algorithms when parameterized above the (temporal) distance from s to z .

Our contributions. We show that SHORT RESTLESS TEMPORAL PATH can be solved in randomized $4^{k-d}|\mathcal{G}|^{O(1)}$ time, where d is the temporal distance from s to z . To the best of our knowledge, this is the first above-lower-bound FPT-algorithm on temporal graphs.

More precisely, we show that for every $p \in \mathbb{R}$ with $0 < p < 1$ there is a randomized $O(4^\ell \cdot \ell^2 |\mathcal{G}|^3 \delta \log^{(k/p(\ell+1))})$ -time algorithm for SHORT RESTLESS TEMPORAL PATH with a one-sided error probability of at most p , where $\ell := k - d$ and d is the temporal distance from s to z . Informally speaking, the parameter ℓ is minimum length of the detour a δ -restless temporal s - z path has to take compared to a (non- δ -restless) temporal s - z path. The main technical contribution behind this is a geometrical perspective onto temporal graphs which seems applicable to other temporal graph problems when parameterized above the temporal distance between vertices. In the resulting algorithm, the only subroutine with a super-polynomial running time is the algorithm of Thejaswi et al. [41] that we employ to find δ -restless temporal path of length at most $2(k - d) + 1$. In fact, this subroutine can be replaced by a deterministic algorithm of Casteigts et al. [13] – this leads to a $2^{O(k-d)} |\mathcal{G}|^3 \delta$ -time deterministic algorithm for SHORT RESTLESS TEMPORAL PATH. The running time overhead induced by our technique is $O(|\mathcal{G}|^2 \ell)$ in the deterministic case and $O(|\mathcal{G}|^2 \ell \log^{(k/(\ell+1)p)})$ if we use the algorithm of Thejaswi et al. [41], where $\ell := k - d$ and d is the temporal distance from s to z . The overhead with the randomized algorithm is larger as we need that the error probability of several calls of the randomized algorithm accumulate to p . Even though the running time overhead of our technique is slightly larger with the randomized algorithm of Thejaswi et al. [41], the exponential part of running time of the overall algorithm for finding restless temporal paths is better than with the algorithm of Casteigts et al. [13].

Further related work. In the literature, waiting-time constraints are studied from various angles. Himmel et al. [7] studied a variant of restless temporal paths where multiple visits of vertices are permitted, i.e., restless temporal *walks*. In contrast to restless temporal paths, they showed that such walks can be computed in polynomial time. Pan and Saramäki [40] empirically studied the correlation between waiting times of temporal paths and the ratio of the network reached in spreading processes. Akrida et al. [1] studied flows in temporal networks with “vertex buffers”, which however pertains to the quantity of information that a vertex can store, rather than a duration.

Algorithmic reachability questions are one of the most thriving research topics in temporal graphs. Bui-Xuan et al. [11] and Wu et al. [42] studied the computation of temporal paths that satisfy certain optimality criteria and show that shortest, fastest, and foremost temporal path can be computed in polynomial time. In the temporal setting, reachability is not an equivalence relation among vertices and the reachability relation between vertices is not even transitive – this makes many problems computationally harder than their counterpart on static graphs. Michail and Spirakis [36] studied the NP-hard question of whether a temporal graph contains a temporal walk that visits each vertex at least once. This problem remains computationally hard even if the underlying graph is a star [3, 12]. If the underlying graph is connected at each time step and the walk can only contain one edge in each time step, then a fast exploration is guaranteed [20, 22, 21]. However, on these so-called always-connected temporal graphs, the decision problem remains NP-hard, even if the underlying graph has pathwidth two [10]. Kempe et al. [30] studied whether there are k vertex-disjoint temporal paths between two given vertices. While the classical analogue of this on static graphs is polynomial-time solvable, it becomes NP-hard in the temporal setting. Moreover, this problem remains NP-hard on a single underlying path, when we are looking for a set of temporal paths which is only pairwise vertex-disjoint at any point in time [32]. Furthermore, the related problem of finding small separators in temporal graphs becomes computationally hard [23, 30], even on quite restricted temporal graph classes [23]. Bhadra and Ferreira [9] showed that finding a maximum temporally connected component is NP-hard. Furthermore, a temporal graph may not have a sparse spanner [5], and computing a spanner with a minimum number of time-edges is NP-hard [2, 35].

Related to spreading processes, Enright et al. [18, 19], Deligkas and Potapov [16], and Molter et al. [38] studied restricting the set of reachable vertices via various temporal graph modifications – all described decision problems are NP-hard in rather restricted settings.

2 Preliminaries

We denote by \mathbb{N} and \mathbb{N}_0 the natural numbers excluding and including zero, respectively. By \mathbb{R} , \mathbb{Q} , and \mathbb{Z} we denote the real numbers, rational numbers, and the integers, respectively. Moreover, $[a, b] := \{i \in \mathbb{Z} \mid a \leq i \leq b\}$, $[n] := [1, n]$, $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$, and $\mathbb{Q}_+ := \{x \in \mathbb{Q} \mid x \geq 0\}$. We denote by $\lceil \log(x) \rceil$ the ceiling of the binary logarithm of x ($\lceil \log_2(x) \rceil$), where $x \in \mathbb{R}$.

Let $(a_i)_{i=1}^n := (a_1, a_2, \dots, a_n)$ be a sequence of length n and let $(b_i)_{i=1}^m$ be a sequence of length m . We denote by $x \in (a_i)_{i=1}^n$ that there is an $i \in [n]$ such that $x = a_i$. We denote by $(a_i)_{i=1}^n \subseteq (b_i)_{i=1}^m$ that $(a_i)_{i=1}^n$ is a *subsequence* of $(b_i)_{i=1}^m$. That is, there is an injective function $\sigma: [n] \rightarrow [m]$ such that $a_i = b_{\sigma(i)}$ for all $i \in [n]$ and $\sigma(i) < \sigma(j)$ for all $i, j \in [n]$ with $i < j$. Moreover, for a set S , we denote by $(a_i)_{i=1}^n \setminus S$ the subsequence of $(a_i)_{i=1}^n$ where an element a_i is removed if and only if $a_i \in S$, for all $i \in [n]$. Appending an element x to sequence $(a_i)_{i=1}^n$ results in the sequence $(a_i)_{i=1}^{n+1}$, where $a_{n+1} = x$.

A *randomized (Monte-Carlo) algorithm* has additionally access to an oracle that, given some number $n \in \mathbb{N}$, draws a value $x \in [n]$ uniformly at random in constant time. A (randomized) algorithm with error probability p is a randomized algorithm that returns the correct answer with probability $1 - p$. For a finite alphabet Σ and a language $L \subseteq \Sigma^*$, a (randomized) algorithm for L with a one-sided error probability p is a randomized algorithm that returns for every input $x \in \Sigma^*$ either *yes* or *no*, and one of the following is true:

- If *yes* is returned, then $x \in L$ with probability 1. If *no* is returned, then $x \in L$ with probability p .
- If *yes* is returned, then $x \notin L$ with probability p . If *no* is returned, then $x \notin L$ with probability 1.

We refer to Mitzenmacher and Upfal [37] for more material on randomized algorithms. If it is not stated otherwise, then we use standard notation from graph theory [17]. Graphs are simple and undirected by default.

Temporal graphs. A temporal graph $\mathcal{G} := (V, (E_i)_{i=1}^\tau)$ consists of a set of vertices $V(\mathcal{G}) := V$ and a sequence of edge sets $(E_i)_{i=1}^\tau$. The number τ is the *lifetime* of \mathcal{G} . The elements of $\mathcal{E}(\mathcal{G}) := \bigcup_{i \in [\tau]} E_i \times \{i\}$ are called the *time-edges* of \mathcal{G} . We say that time-edge $(e, t) \in \mathcal{E}(\mathcal{G})$ has time stamp t and is in time step t . The graph (V, E_i) is called *layer i* of temporal graph \mathcal{G} , for all $i \in [\tau]$. The *underlying graph* of \mathcal{G} is the (static) graph $(V, \bigcup_{i=1}^\tau E_i)$. For every $v \in V$ and every $t \in [\tau]$, we denote the *appearance of vertex v at time t* by the pair (v, t) . For a time-edge $(\{v, w\}, t)$ we call the vertex appearances (v, t) and (w, t) its *endpoints*. We assume that the *size* of \mathcal{G} is $|\mathcal{G}| := |V| + \sum_{i=1}^\tau \max\{1, |E_i|\}$, that is, we do not assume to have compact representations of temporal graphs. For a vertex set $X \subseteq V$ of a temporal graph $\mathcal{G} := (V, (E_i)_{i=1}^\tau)$, we denote by $\mathcal{G}[X]$ the temporal graph $(X, (E'_i)_{i=1}^\tau)$, where $E'_i := \{e \in E_i \mid e \subseteq X\}$. Moreover, we denote the temporal graph \mathcal{G} without the vertices X by $\mathcal{G} - X := \mathcal{G}[V \setminus X]$. For a time-edge set Y , we denote by $\mathcal{G} \setminus Y$ the temporal graph where $V(\mathcal{G} \setminus Y) := V(\mathcal{G})$ and $\mathcal{E}(\mathcal{G} \setminus Y) := \mathcal{E}(\mathcal{G}) \setminus Y$.

The set of vertices of the temporal path $P = (e_i = (\{v_{i-1}, v_i\}, t_i))_{i=1}^m$ is denoted by $V(P) = \{v_i \mid i \in [m] \cup \{0\}\}$. We say that P *visits* the vertex v_i at time t if $t \in [t_i, t_{i+1}]$, where $i \in [m-1]$. The *departure* (or *starting*) *time* of P is t_1 and the *arrival time* of P is t_m . A $(\delta$ -restless) temporal s - z path of length m in a temporal graph \mathcal{G} is a *shortest* $(\delta$ -restless) temporal s - z path if each temporal s - z path in \mathcal{G} is of length at least m .

A *solution* of an instance $(\mathcal{G}, s, z, \delta, k)$ of SHORT RESTLESS TEMPORAL PATH is a δ -restless temporal s - z path of length at most k in \mathcal{G} .

Parameterized complexity. Let Σ be a finite alphabet. A *parameterized problem* L is a subset $L \subseteq \Sigma^* \times \mathbb{N}_0$. The size of an instance $(x, k) \in \Sigma^* \times \mathbb{N}_0$ is denoted by $|x|$ and usually we have that $|x| + k \in O(|x|)$. An instance $(x, k) \in \Sigma^* \times \mathbb{N}_0$ is a *yes-instance* of L if and only if $(x, k) \in L$ (otherwise it is a *no-instance*). A parameterized problem L is *fixed-parameter tractable* (in FPT) if there is an (FPT)-algorithm that decides for every input $(x, k) \in \Sigma^* \times \mathbb{N}_0$ in $f(k) \cdot |x|^{O(1)}$ time whether $(x, k) \in L$, where f is some computable function only depending on k . By slightly abusing the FPT-terminology, we sometimes say that a parameterized problem is fixed-parameter tractable even if the FPT-algorithm has a constant one-sided error probability. A parameterized problem L is in XP if for every input (x, k) one can decide in $|x|^{f(k)}$ time whether $(x, k) \in L$, where f is some computable function only depending on k .

The parameterized analogues of NP and NP-hardness are the W-hierarchy $\text{FPT} \subseteq \text{W}[1] \subseteq \text{W}[2] \subseteq \dots \subseteq \text{W}[P] \subseteq \text{XP}$ and W[t]-hardness, where $t \in \mathbb{N} \cup \{P\}$ and all inclusions are conjectured to be strict. If some W[t]-hard parameterized problem is in FPT, then $\text{FPT} = \text{W}[t]$. We refer to Cygan et al. [15] for more material on parameterized complexity.

3 The Algorithm

In this section, we show that SHORT RESTLESS TEMPORAL PATH can be solved in $4^{k-d} \cdot |\mathcal{G}|^{O(1)}$ time with a constant one-sided error probability, where d is the minimum length of a temporal s - z path. More precisely, we show the following.

► **Theorem 1.** *For every $p \in \mathbb{R}$ with $0 < p < 1$, there is a randomized $O(4^\ell \cdot \ell^2 |\mathcal{G}|^3 \delta \log^{(k/p(\ell+1))})$ -time algorithm for SHORT RESTLESS TEMPORAL PATH, where $\ell := k - d$ and d is the minimum length of a temporal s - z path. If this algorithm*

- *returns yes, then the given instance is a yes-instance, and if it*
- *returns no, then with probability of at least $1 - p$ the given instance is a no-instance.*

The formal proof of Theorem 1 is deferred to Section 3.3. In a nutshell, we only check for δ -restless temporal subpaths whose length are upper-bounded by $2(k - d) + 1$ and then puzzle them together to ultimately find a δ -restless temporal s - z path, where d is the minimum length of a temporal s - z path. To detect δ -restless temporal paths of some given length, we employ the algorithm of Thejaswi et al. [41].

► **Proposition 2** ([41]). *For every $p \in \mathbb{R}$ with $0 < p < 1$ there is a randomized $O(2^k \cdot k |\mathcal{G}| \delta \log(k \cdot 1/p))$ -time algorithm that takes as input a temporal graph \mathcal{G} , two vertices s, z , and two integers δ, k . If this algorithm*

- *returns yes, then there is a δ -restless temporal s - z path of length exactly k in \mathcal{G} , and if it*
- *returns no, then with probability at least $1 - p$ there is no δ -restless temporal s - z path of length exactly k in \mathcal{G} .*

In our algorithm, Proposition 2 can be replaced by any algorithm to find δ -restless temporal paths of length k . For example, with the deterministic $2^{O(k)} \cdot |\mathcal{G}| \delta$ -time algorithm of Casteigts et al. [13] instead of Proposition 2, we would end up with a $2^{O(k-d)} \cdot |\mathcal{G}|^3 \delta$ -time algorithm for SHORT RESTLESS TEMPORAL PATH that is deterministic. Here, the exponential part of the running time is worse than with the randomized subroutine Proposition 2.

First, we set up the geometric perspective on temporal graph based on the temporal distance between vertices. This is the main contribution of this paper and might be of independent interest, as the ideas seem to be transferable to other path finding problems

in temporal graphs where the distance between two vertices is a lower bound. Afterwards, we design a dynamic program to solve SHORT RESTLESS TEMPORAL PATH in $4^{k-d} \cdot |\mathcal{G}|^{O(1)}$ time, where d is the minimum length of a temporal s - z path.

3.1 Geometric Perspective on Temporal Graphs Based on the Temporal Distance

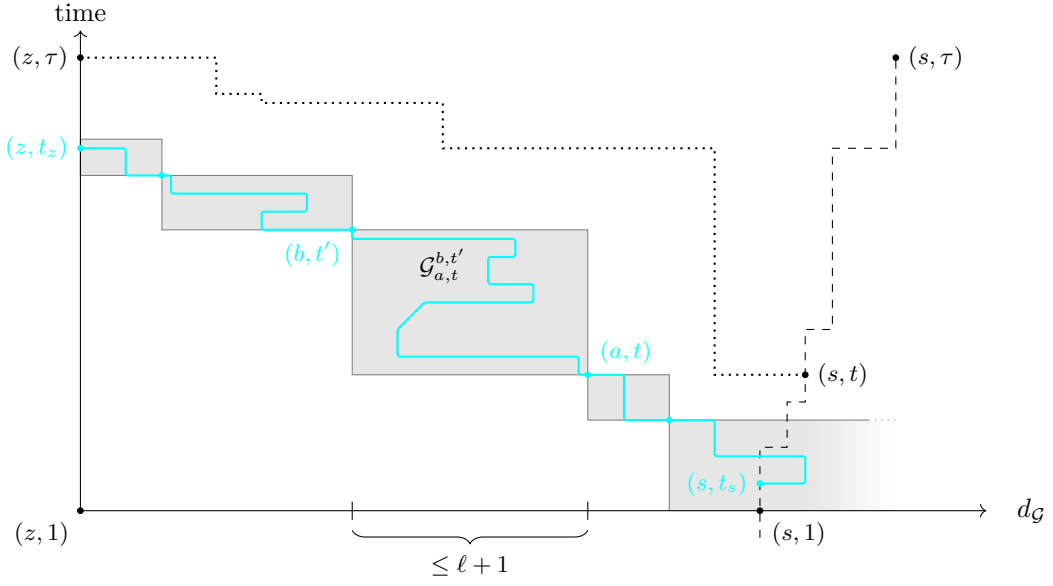
In this section, we present the key ideas of the algorithm behind Theorem 1. To this end, we need some notation. Let $\mathcal{G} := (V, (E_i)_{i=1}^{\tau})$ be a temporal graph with two distinct vertices $s, z \in V(\mathcal{G})$ and $\delta, k \in \mathbb{N}$. We define the distance function $d_{\mathcal{G}}: V(\mathcal{G}) \times [\tau] \rightarrow \mathbb{N}_0 \cup \{\infty\}$ which maps a vertex $v \in V(\mathcal{G})$ and time $t \in [\tau]$ to the length of a shortest temporal v - z path in \mathcal{G} that departs at a time at least t . If such a temporal path does not exist, then $d_{\mathcal{G}}(v, t) = \infty$. We drop the subscript \mathcal{G} if it is clear from the context.

Intuitively, we now arrange all vertex appearances (v, t) in the plane where the x -axis describes the distance (via temporal paths) of v to z at time t and the y -axis describes the time. Thus, (v, t) gets the point $(d(v, t), t)$. Consider Figure 2 for a moment. We want to visualize a temporal s - z path P in this figure. To this end, we say that P visits vertex appearance (v, t) if P visits v in time step t . Hence, we can depict a temporal path P by connecting the vertex appearances which are visited by P in the visiting order. Note that no temporal v - z path or walk moves downwards. Moreover, among all temporal v - z paths that depart at a time t or later, the shortest of them move with each time-edge further towards z (i.e., to the left). For example, the dotted line in Figure 2 depicts the trajectory of a shortest temporal s - z path with a departure time t . The temporal path departs at time t and arrives at time τ . This is not the case for a shortest δ -restless temporal s - z path P – such a temporal path can move to the right or stay at the same point while visiting multiple vertices. For example, the solid (blue) line in Figure 2 depicts the trajectory of a shortest δ -restless temporal s - z path. Let $\ell := k - d(s, 1)$. A crucial observation now is that if P moves “too far” to the right or stays for “too long” at the same spot in the x -axis while visiting multiple vertices, then P would be too far away from z (in terms of temporal paths distance) such that P cannot be of length at most k . This will lead us to the observation that for at least every $(2\ell + 1)$ -st vertex v which is visited by P (at time t), the vertex appearance (v, t) has the following separation property:

- (i) each vertex appearance (u, t') that P visits before v (hence, $v \neq u$) is to the right of (v, t) and thus further away from z than (v, t) , and
- (ii) each vertex appearance (u, t') that P visits after v (hence, $v \neq u$) is to the left of (v, t) and thus closer to z than (v, t) .

Moreover, we will observe that two consecutive vertex appearances which have this separation property, have a similar distance to z – the distances differ by at most $\ell + 1$. In Figure 2, these special vertex appearances are at the top-left and bottom-right corners of each gray area. Our dynamic program tries to guess these vertex appearances and then constructs for each gray area in Figure 2 a temporal graph that contains the δ -restless path from the bottom-right corner to the top-left corner of this area. Since we know that these δ -restless temporal paths have length at most $2\ell + 1$, we can use Proposition 2 to find them.

Another crucial observation we are going to make is that two δ -restless temporal paths from the bottom-right corner to the top-left corner of two distinct gray areas in Figure 2 cannot visit the same vertex (except for their endpoints). This is the case because the distance of a vertex v to z may only increase as time goes by. Thus, if we find for each gray area in Figure 2 a δ -restless temporal path from the bottom-right corner to the top-left corner, then this gives us a δ -restless temporal s - z path. Henceforth the details follow.



■ **Figure 2** Illustration of the idea behind the dynamic programming table which is used to show Theorem 1. The y -axis describes the time. The x -axis describes the distance to z (via temporal paths). In this plane, a vertex appearance (v, t) gets the position $(d(v, t), t)$. The positions of the vertex appearances of s are on the dashed line. A shortest (non- δ -restless) temporal s - z path that departs at time t is depicted by the dotted line. The trajectory of a shortest δ -restless temporal s - z path which departs at time t_s and arrives at time t_z is depicted by the solid (blue) line. Each gray area depicts a temporal subgraph which we use to compute δ -restless paths from the vertex appearance on the bottom-right corner to the vertex appearance on the top-left corner, e.g., the temporal graph $\mathcal{G}_{a,t}^{b,t'}$.

Before we describe the dynamic programming table in Section 3.2, we define the temporal graph that contains all (shortest) δ -restless paths in a gray area of Figure 2. To this end, we first define sets containing all vertex appearances of such a gray area. For vertex appearances $(a, t), (b, t') \in V(\mathcal{G}) \times [\tau]$, we define

$$\begin{aligned}
 \mathcal{A}_{a,t}^{b,t'} &:= \{(w, t^*) \in V(\mathcal{G}) \times [\tau] \mid d(b, t') < d(w, t^*) < d(a, t), t^* \in [t, t']\} \text{ and} \\
 \mathcal{A}^{b,t'} &:= \{(w, t^*) \in V(\mathcal{G}) \times [\tau] \mid \infty > d(w, t^*) > d(b, t'), t^* \leq t'\}.
 \end{aligned}$$

Now, the temporal graph $\mathcal{G}_{a,t}^{b,t'}$ for the gray area between (a, t) and (b, t') with $t \leq t'$ is defined by

$$\begin{aligned}
 \mathcal{E}(\mathcal{G}_{a,t}^{b,t'}) &:= \left\{ \left(\{v, u\}, t^* \right) \in \mathcal{E}(\mathcal{G}) \mid (v, t^*), (u, t^*) \in \mathcal{A}_{a,t}^{b,t'} \right\} \cup \\
 &\quad \left\{ \left(\{a, v\}, t \right) \in \mathcal{E}(\mathcal{G}) \mid (v, t) \in \mathcal{A}_{a,t}^{b,t'} \right\} \cup \\
 &\quad \left\{ \left(\{v, b\}, t^* \right) \in \mathcal{E}(\mathcal{G}) \mid t' - \delta \leq t^*, (v, t^*) \in \mathcal{A}_{a,t}^{b,t'} \cup \{(a, t)\} \right\}
 \end{aligned}$$

and

$$V(\mathcal{G}_{a,t}^{b,t'}) := \left\{ v \in V(\mathcal{G}) \mid \exists (e, t^*) \in \mathcal{E}(\mathcal{G}_{a,t}^{b,t'}) : v \in e \right\}.$$

For the gray area containing s we have to adjust the definition of the corresponding temporal graph slightly. To this end, we define $\mathcal{G}^{b,t'}$ with

$$\mathcal{E}(\mathcal{G}^{b,t'}) := \left\{ (\{v, u\}, t^*) \in \mathcal{E}(\mathcal{G}) \mid (v, t^*), (u, t^*) \in \mathcal{A}^{b,t'} \right\} \cup \left\{ (\{v, b\}, t^*) \in \mathcal{E}(\mathcal{G}) \mid t' - \delta \leq t^*, (v, t^*) \in \mathcal{A}^{b,t'} \right\}$$

and

$$V(\mathcal{G}^{b,t'}) := \left\{ v \in V(\mathcal{G}) \mid \exists (e, t^*) \in \mathcal{E}(\mathcal{G}^{b,t'}) : v \in e \right\}.$$

In the remainder of this paper, we use these definitions to formulate a dynamic programming table and eventually solve SHORT RESTLESS TEMPORAL PATH.

3.2 The Dynamic Programming Table

In this section, we describe the table T which we are going to use for the dynamic programming, and show its correctness.

Intuitively, the table T has for each vertex appearance (u, t') an entry, and if this entry contains a number $p < \infty$, then p is the length of the shortest δ -restless temporal s - u path that only visits vertex appearances which are, in Figure 2, below and to the right of (u, t') .

Let $I := (\mathcal{G}, s, z, \delta, k)$ be an instance of SHORT RESTLESS TEMPORAL PATH, where $k = d(s, 1) + \ell$. For all $(u, t') \in V(\mathcal{G}) \times [\tau]$ such that there is an $e \in E_{t'}$ with $v \in e$, we define T as follows. If $d(s, 1) - d(u, t') \leq \ell$, then

$$T[u, t'] := \begin{cases} 0, & \text{if } u = s; \\ \ell', & \text{if } u \neq s \text{ and } \ell' \in [2\ell] \text{ is the length of a} \\ & \text{shortest } \delta\text{-restless } s\text{-}u \text{ path in } \mathcal{G}^{u,t'}; \\ \infty, & \text{otherwise.} \end{cases} \quad (1)$$

If $d(s, 1) - d(u, t') > \ell$, then

$$T[u, t'] := \min \left(\left\{ \infty \right\} \cup \left\{ T[v, t] + \ell' \mid \begin{array}{l} t \in [t'], e \in E_t, v \in e, \text{ where} \\ d(v, t) > d(u, t') \geq d(v, t) - \ell - 1 \\ \text{and } \ell' \in [2\ell + 1] \text{ is the length of a} \\ \text{shortest } \delta\text{-restless } v\text{-}u \text{ path in } \mathcal{G}_{v,t}^{u,t'} \end{array} \right\} \right) \quad (2)$$

In the end, we will report that I is a *yes*-instance if and only if there is a $t \in [\tau]$ such that $T[z, t] \leq k$. We will show the correctness of this in the following lemmata. We start with the backwards direction.

► **Lemma 3.** *Let $(\mathcal{G}, s, z, \delta, k)$ be an instance of SHORT RESTLESS TEMPORAL PATH. If $T[z, t_z] \leq k < \infty$ (defined in (1) and (2)), then there is a δ -restless temporal s - z path of length at most k in \mathcal{G} .*

Proof. We show by induction on the distance to z that if $T[u, t'] = k' < \infty$, then there is a δ -restless s - u path of length k' in $\mathcal{G}^{u,t'}$ which arrives at u at some time step in $[t' - \delta, t']$.

Note that all temporal s - u paths in $\mathcal{G}^{u,t'}$ arrive at some time in $[t' - \delta, t']$. By (1), for each vertex appearance (u, t') with $d(s, 1) - d(u, t') \leq \ell$ the induction hypothesis is true – this is our base case.

Now let (u, t') be a vertex appearance with $T[u, t'] = k' < \infty$. Assume that for all vertex appearances (v, t) with $d(v, t) > d(u, t')$ we have that if $T[v, t] = k'' < \infty$, then there is a δ -restless temporal s - v path of length k'' in $\mathcal{G}^{v,t}$ which arrives at v at some time step

in $[t - \delta, t]$. Since $T[u, t'] = k'$, we know by (2) that there is a vertex appearance (v, t) with $T[v, t] = k''$, $t \leq t'$, and $d(v, t) > d(u, t')$. Moreover, there is a δ -restless temporal v - u path P_2 in $\mathcal{G}_{v,t}^{u,t'}$ of length $\ell' = k' - k''$. By the definition of $\mathcal{G}_{v,t}^{u,t'}$, P_2 departs at time t and arrives at some time in $[t' - \delta, t']$. By assumption, there is a δ -restless temporal s - v path P_1 of length k'' in $\mathcal{G}^{v,t}$ which arrives at v at some time step in $[t - \delta, t]$. We now append the time-edges of P_2 to the time-edges of P_1 and claim that the resulting time-edge sequence P is a δ -restless temporal s - u path of length k' which arrives at u at some time in $[t' - \delta, t']$. Observe that P is a δ -restless temporal s - u walk of length $k' = k'' + \ell'$, as P_1 is δ -restless, of length k'' , and arrives at v at some time $t^* \in [t - \delta, t]$, and P_2 is of length ℓ' and departs at time t . Moreover, the arrival time of P is the same as the arrival time of P_2 .

It remains to show that P does not visit a vertex twice. To see this, we show that $V(\mathcal{G}^{v,t}) \cap V(\mathcal{G}_{v,t}^{u,t'}) = \{v\}$. This will complete the proof, since we know that $V(P_1) \subseteq V(\mathcal{G}^{v,t})$, $V(P_2) \subseteq V(\mathcal{G}_{v,t}^{u,t'})$, P_1 ends at vertex v , and P_2 starts at vertex v . By definition, we have that $v \in (V(\mathcal{G}^{v,t}) \cap V(\mathcal{G}_{v,t}^{u,t'}))$. Assume towards a contradiction that there is a vertex $w \in (V(\mathcal{G}^{v,t}) \cap V(\mathcal{G}_{v,t}^{u,t'})) \setminus \{v\}$. Then, there must be time steps t_1, t_2 such that $(w, t_1) \in \mathcal{A}^{v,t} \cup \{(u, t')\}$ and $(w, t_2) \in \mathcal{A}_{v,t}^{u,t'}$. Note that $d(w, t_1) > d(v, t) > d(w, t_2)$ and hence each temporal w - z path in \mathcal{G} that departs not earlier than t_1 is longer than a shortest w - z path in \mathcal{G} that departs not earlier than t_2 . This is a contradiction because $t_1 \leq t \leq t_2$. \blacktriangleleft

To show the forward direction of the correctness, we introduce further notation. Recall from the definition of the dynamic programming table T in (1) and (2) that $\ell = k - d(s, 1)$. Assume the input instance I is a *yes*-instance. Thus there is a δ -restless temporal s - z path $P = ((\{v_{i-1}, v_i\}, t_i))_{i=1}^k$ of length at most $d(s, 1) + \ell$ in \mathcal{G} . Let $s = v_0, v_1, \dots, v_k = z$ be the order in which P visits the vertices in $V(P)$. For simplicity, let $t_0 := 1$ and $t_{k+1} := t_k$. For all $i \in [0, k]$, we say that v_i is a *distance separator* if (i) $d(v_i, t_{i+1}) < d(v_j, t_{j+1})$ for all $j \in [0, i - 1]$, and (ii) $d(v_i, t_{i+1}) > d(v_j, t_{j+1})$ for all $j \in [i + 1, k]$.

Before we show the forward direction of the correctness of the dynamic programming table T , we show that P visits a distance separator on regular basis.

► **Lemma 4.** *For all $i \in [0, k]$ there is a $j \in [0, 2\ell]$ such that v_{i+j} is a distance separator.*

Proof. We show this statement with a reverse induction on the length of P . As z is clearly a distance separator, the claim is true for all values in $[\max\{0, k - 2\ell\}, k]$. This is the base case of our induction.

Let $k - 2\ell > 0$ and let $i \in [0, k - 2\ell - 1]$ and assume that for all $i' \in [i + 1, k]$ there is a $j' \in [0, 2\ell]$ such that $v_{i'+j'}$ is a distance separator. Let $n \in [i, k]$ be the smallest possible number such that v_n is a distance separator. Let $f \in [k]$ be the smallest possible number such that $d(v_f, t_{f+1}) - d(v_n, t_{n+1}) = \ell + 1$. Note that if such an f does not exist, then the claim is true. Hence, we assume that such an f exists. Note that $f \leq i$, otherwise n is not the smallest possible number.

We now show that $n - f \leq 2\ell + 1$. Assume towards a contradiction that the temporal v_f - v_n path contained in P has length $n - f > 2\ell + 1$. This is a lower bound for the length of P . We get the following.

$$\begin{aligned} d(s, 1) - d(v_f, t_{f+1}) + 2\ell + 1 + d(v_n, t_{n+1}) &< k = d(s, 1) + \ell \\ \implies d(v_n, t_{n+1}) - d(v_f, t_{f+1}) + \ell + 1 &< 0 \implies \ell + 1 < d(v_f, t_{f+1}) + d(v_n, t_{n+1}) \end{aligned}$$

This is a contradiction to $d(v_f, t_{f+1}) - d(v_n, t_{n+1}) = \ell + 1$.

Next, we show that, between v_f and v_{n-1} , P must visit a distance separator. Assume towards a contradiction that $v_{n-j''}$ is not a distance separator, for all $j'' \in [n - f]$. Hence, for all $p \in [d(v_n, t_{n+1}) + 1, d(v_f, t_{f+1})]$, there are two distinct $q, r \in [f, n - 1]$ such that

$d(v_r, t_{r+1}) = d(v_q, t_{q+1}) = p$. Since $[d(v_n, t_{n+1}) + 1, d(v_f, t_{f+1})] = \ell$, we get by the pigeonhole principle that the temporal v_f - v_n path contained in P has length $n - f > 2\ell + 1$ – a contradiction. \blacktriangleleft

Finally, we are set to show the forward direction and Theorem 1 afterwards.

► **Lemma 5.** *Let $(\mathcal{G}, s, z, \delta, k)$ be an instance of SHORT RESTLESS TEMPORAL PATH. If there is a δ -restless temporal s - z path in \mathcal{G} of length at most k with arrival time t_k , then $T[z, t_k] \leq k$ (defined in (1) and (2)).*

Proof. Let $P = ((\{v_{i-1}, v_i\}, t_i))_{i=1}^k$ be a shortest δ -restless temporal s - z path of length at most $k = d(s, 1) + \ell$ in \mathcal{G} . Let $s = v_0, v_1, \dots, v_k = z$ be the order in which P visits the vertices in $V(P)$. For simplicity, let $t_0 := 1$ and $t_{k+1} := t_k$. Moreover, let m be the number of distance separators visited by P and let $\sigma: [m] \rightarrow [0, k]$ be an injective function such that $v_{\sigma(i)}$ is the i -th distance separator which is visited by P (from s to z), for all $i \in [m]$. Note that, the vertex $v_{\sigma(i)}$ is the i -th distance separator visited by P and thus $\sigma(i)$ also describes the length of the δ -restless temporal s - $v_{\sigma(i)}$ subpath contained in P .

We now show that for all $i \in [m]$ we have that $T[v_{\sigma(i)}, t_{\sigma(i)+1}] \leq \sigma(i)$. If $\sigma(1) = 0$, then s is a distance separator and the claim is clearly true, see (1). Otherwise, by Lemma 4, we have $\sigma(1) \leq 2\ell$. Hence, P contains a δ -restless temporal s - $v_{\sigma(1)}$ path of length $\sigma(1) \leq 2\ell$ which is contained in $\mathcal{G}^{v_{\sigma(1)}, t_{\sigma(1)+1}}$. Thus, $T[v_{\sigma(1)}, t_{\sigma(1)+1}] \leq \sigma(1)$.

Now assume that for some $i \in [2, m]$ we have that $T[v_{\sigma(i-1)}, t_{\sigma(i-1)+1}] \leq \sigma(i-1)$. Observe that $t_{\sigma(i-1)+1} \leq t_{\sigma(i)+1}$ and that $d(v_{\sigma(i-1)}, t_{\sigma(i-1)+1}) > d(v_{\sigma(i)}, t_{\sigma(i)+1})$. By Lemma 4, we have that $\sigma(i) - \sigma(i-1) \leq 2\ell + 1$ and that the δ -restless temporal $v_{\sigma(i-1)}$ - $v_{\sigma(i)}$ path Q contained in P is of length $\sigma(i) - \sigma(i-1) \leq 2\ell + 1$. As all of the at most 2ℓ vertices in $V(Q) \setminus \{v_{\sigma(i-1)}, v_{\sigma(i)}\}$ are not distance separators, we have by the pigeonhole principle that $d(v_{\sigma(i-1)}, t_{\sigma(i-1)+1}) - d(v_{\sigma(i)}, t_{\sigma(i)+1}) \leq \ell + 1$. Moreover, note that Q in $\mathcal{G}_{v_{\sigma(i-1)}, t_{\sigma(i-1)+1}}^{v_{\sigma(i)}, t_{\sigma(i)+1}}$, because $v_{\sigma(i-1)}$ and $v_{\sigma(i)}$ are distance separators. Hence, by (2), we have that $T[v_{\sigma(i)}, t_{\sigma(i)+1}] \leq T[v_{\sigma(i-1)}, t_{\sigma(i-1)+1}] + \sigma(i) - \sigma(i-1) \leq \sigma(i)$, as we have $T[v_{\sigma(i-1)}, t_{\sigma(i-1)+1}] \leq \sigma(i-1)$ by assumption.

Since $z = v_k$, we have that k is the only number in $[0, k]$ with $d(v_k, t_{k+1}) = 0$. Hence, v_k is the last distance separator and thus $\sigma(m) = k$. Finally, by our induction, we have that $T[z, t_{k+1}] \leq k$. \blacktriangleleft

3.3 Putting the Pieces Together

In this section, we use the tools from previous sections to show first show Theorem 1. Beforehand, we show for the running the analysis that all necessary values of our distances function $d(\cdot, \cdot)$ are computable in linear time.

► **Lemma 6.** *Given a temporal graph $\mathcal{G} := (V, (E_t)_{t=1}^\tau)$ and a vertex z , one can compute in $O(|\mathcal{G}|)$ time the value $d(v, t)$, for all $v \in V$ and $t \in [\tau]$ where v is not isolated in the graph (V, E_t) .*

Proof. We will construct a directed graph D where each arc has either weight zero or one such that the weight of a shortest z - v_t path equals the value of $d(v, t)$, for all $v \in V$ and $t \in [\tau]$ where v is not isolated in the graph (V, E_t) . Then, a slightly modified breadth-first search will do the task.

We compute the set \mathcal{V} of non-isolated vertex appearances. That is, $\mathcal{V} := \{(v, t) \in V(\mathcal{G}) \times [\tau] \mid \exists e \in E_t : v \in e\}$. Note that this can be done in $O(|\mathcal{G}|)$ time and that $|\mathcal{V}| \leq 2|\mathcal{G}|$. Now we are ready to define D by $V(D) := \{z\} \cup \{v_t \mid (v, t) \in \mathcal{V}\}$ and

$$\begin{aligned}
E(D) := & \{(v_t, u_t), (u_t, v_t) \mid \{v, u\} \in E_t, (v, t), (u, t) \in \mathcal{V} \text{ and } u \neq v\} \cup \\
& \{(v_{t_2}, v_{t_1}) \mid v_{t_1}, v_{t_2} \in V(D) \text{ and } t_2 = \min \{t \mid (v, t) \in \mathcal{V} \text{ and } t > t_1\}\} \cup \\
& \{(z, z_t) \mid z_t \in V(D) \text{ and } t = \max\{t' \mid (z, t') \in \mathcal{V}\}\}.
\end{aligned}$$

Now all arcs in $\{(v_t, u_t), (u_t, v_t) \mid (v, t), (u, t) \in \mathcal{V} \text{ and } u \neq v\}$ get weight one, while all the other arcs get weight zero. Note that $V(D) + E(D) \in O(|\mathcal{G}|)$ and that D can be constructed in $O(|\mathcal{G}|)$ time. Observe that for every temporal v - z path P in \mathcal{G} with departure time t there is a z - v_t path in D whose accumulated edge-weight equals the length of P . Hence, if we know the minimum edge-weight of the paths from z to all vertices in D , then we also know the value $d(v, t)$, for all $v \in V$ and $t \in [\tau]$ where v is not isolated in the graph (V, E_t) . Thus, we employ a breadth-first search that starts at z and only explores an arc of weight one if there is currently no arc of weight zero which could be explored instead. At each vertex $v_t \in V(D)$ we store the edge-weight $d(v, t)$ of the path from z to this vertex. Hence, the overall running time of this procedure is $O(|\mathcal{G}|)$ time. \blacktriangleleft

Finally, we are set to show Theorem 1: For every $p \in \mathbb{R}$ with $0 < p < 1$, there is a randomized $O(4^\ell \cdot \ell^2 |\mathcal{G}|^3 \delta \log(k/p(\ell+1)))$ -time algorithm for SHORT RESTLESS TEMPORAL PATH, where $\ell := k - d$ and d is the minimum length of a temporal s - z path. If this algorithm

- returns *yes*, then the given instance is a *yes*-instance, and if it
- returns *no*, then with probability of at least $1 - p$ the given instance is a *no*-instance.

Proof of Theorem 1. Let $I := (\mathcal{G}, s, z, \delta, k)$ be an instance of SHORT RESTLESS TEMPORAL PATH. We perform the following. First, we compute the set \mathcal{V} of non-isolated vertex appearances. That is, $\mathcal{V} := \{(v, t) \in V(\mathcal{G}) \times [\tau] \mid \exists e \in E_t: v \in e\}$. Note that this can be done in $O(|\mathcal{G}|)$ time and that $|\mathcal{V}| \leq 2|\mathcal{G}|$. By Lemma 6, we compute $d(v, t)$ for all $(v, t) \in \mathcal{V}$ in $O(|\mathcal{G}|)$ time. We may assume that there is a temporal s - z path in \mathcal{G} and that a shortest of them has length at most k , otherwise I is clearly a *no*-instance. We set $\ell := k - d(s, 1) = k - d(s, t)$, where $t = \min\{t' \in [\tau] \mid (s, t') \in \mathcal{V}\}$. Note that the table T , defined in (1) and (2), has $O(|\mathcal{G}|)$ entries – one for each element in \mathcal{V} . To compute one entry in T , we consider at most $O(|\mathcal{G}|)$ other entries in T and for each of them we have to check at most $2\ell + 1$ times whether a temporal graph of size $O(|\mathcal{G}|)$ has a δ -restless temporal path of length $\ell' \in [2\ell + 1]$ between two distinct vertices. We answer each of these checks by Proposition 2 with a one-sided error probability of at most p' in $O(4^\ell \cdot \ell |\mathcal{G}| \delta \log(\ell \cdot 1/p'))$ time. How we set the error probability p' will be determined in a moment.

We say that I is a *yes*-instance if and only if there is a $(z, t) \in \mathcal{V}$ such that $T[z, t] \leq k$. If Proposition 2 reports *yes*, then with probability one, there is such a δ -restless temporal path in question. Hence, by Lemma 3, if our overall algorithm reports *yes*, then I is a *yes*-instance – the error probability is zero in this case. If our overall algorithm reports *no*, then the probability that I is a *yes*-instance shall be at most $1 - p$. By Lemma 5, it remains to determine p' . Recall from Lemma 4 that a δ -restless temporal s - z path P of length at most k visits at least every $2\ell + 1$ vertices one distance separator. Hence, we can identify at most $\lceil k/(2\ell + 1) \rceil \in O(k/\ell)$ vertex appearances which are visited by P and thus $O(k/\ell)$ calls of the algorithm behind Proposition 2 such that if these calls are answered correctly then this causes our overall algorithm to report *yes*, as $T[z, t] \leq k$ for some $t \in [\tau]$. Hence, there is a $p' \in O(p^{\ell+1}/k)$ such that we have an error probability of at most p in the case our overall algorithm answers *no*. Thus, we can compute all entries of T in $O(4^\ell \cdot \ell^2 |\mathcal{G}|^3 \delta \log(k/p(\ell+1)))$ time, where $\ell := k - d$ and d is the minimum length of a temporal s - z path. \blacktriangleleft

On a more practical note, one can observe that in order to compute one entry for vertex appearances (u, t) of table T , we only consider table entries of vertex appearances with low temporal distance to (u, t) . Thus, for temporal graphs that are nowhere dense in terms of the temporal distance, it seems reasonable that the presented algorithm does not induce a quadratic running time, in terms of the temporal graph size, on top of running time of Proposition 2 (e.g., in contact networks where mass events are prohibited). Moreover, a δ -restless temporal path of length k has a time horizon of at most $(k - 1)\delta$. Hence, with an overhead of $O(\tau)$ one could guess the departure time t of the δ -restless s - z path and discard all time-edge (e, t') with $t' < t$ or $t' > t + (k - 1)\delta + 1$. This potentially decreases the parameter $k - d$ and thus the exponential part of the running time substantially, where d is the minimum length of a temporal s - z path.

4 Conclusion

We showed that SHORT RESTLESS TEMPORAL PATH can be solved in $4^{k-d} \cdot |\mathcal{G}|^{O(1)}$ time with a one-sided error probability of at most $2^{-|\mathcal{G}|}$, where d is the minimum length of a temporal s - z path. In the corresponding algorithm, we have only one subroutine with a super-polynomial running time: an algorithm to find a δ -restless temporal path of length at most $2(k - d) + 1$. As this is the only subroutine with a non-zero error probability, our algorithm becomes deterministic $2^{O(k-d)} \cdot |\mathcal{G}|^{O(1)}$ -time algorithm, if this subroutine is replaced by the algorithm of Casteigts et al. [13]. Note that the deterministic algorithm does not run in polynomial space and that the running time upper-bound is worse.

We believe that our algorithmic approach opens new research directions to advance further: First, we wonder how good our algorithm performs in an experimental setup comparable to the one of Thejaswi et al. [41]. Second, one could study in detail the temporal subgraphs on which we employ Proposition 2. In principle, Proposition 2, could be replaced with any other algorithm for SHORT RESTLESS TEMPORAL PATH. Do these specific temporal subgraphs admit structural properties which are algorithmically useful? Third, we believe that our geometric perspective can be applied to other temporal graph problems. In particular, for temporal graph problems which ask for specific temporal paths, e.g., temporal paths that obey certain robustness properties [24], or temporal paths that visit all vertices at least once [20, 21, 36] parameterized by the temporal diameter, that is, the length of the longest shortest temporal path between two arbitrary vertices.

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