# An Improved Sufficient Condition for Routing on the Hypercube with Blocking Nodes

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# Abstract

We study the problem of routing between two nodes in a hypercube with blocking nodes using shortest path. This problem has been previously studied by other researchers, they have proposed a few algorithms to solve the problem. Among the work done, one has found several sufficient conditions for such a path to exist. One such condition states that a shortest path between node $0^n$ and $1^n$ exists if the number of blocking nodes is less than $n$ in an $n$-dimensional hypercube. We improve this condition by proposing the condition that if the size of a SDR (system of distinct representatives) for the blocking nodes is less than $n$, then a shortest path between the two nodes $0^n$ and $1^n$ exists. Since the number of blocking nodes can be greater than or equal to $n$, while the size of SDR is less than $n$, thus this result improves the previous sufficient condition.

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# Chapter 1

# Introduction

In interconnection networks, routing is one of the most basic problems. According to the number of source nodes and target nodes, routing problems can be classified into three categories: (1) routing from one node to another node (one-to-one); (2) routing from one fixed node to a set of nodes (one-to-many); (3) routing from a set of nodes to another set of nodes (many-to -many). There are a lot of work done on many well-known interconnection networks for various routing paradigms, especially for the first two categories. There are different versions of routing problems depending on the conditions applied, for example, routing from source node to target node with a few blocking nodes. We can also impose that the path is shortest. Routing with blocking nodes, or fault tolerant routing for the hypercube has been widely studied, significant results were proposed by Chiu and Wu [1], Day et al [2], and Kanedo and Ito [4]. In this project, we focus on routing from one source node to the target node on the hypercube with blocking nodes using shortest path.

As one of the most popular interconnection networks, a hypercube [5,6] of dimension $n$, or an $n$-cube, consists of $2^n$ nodes which can be labeled as 0, 1, 2, $\cdots$ ,$2^n$-1. Each node has a unique binary representation. Two nodes are considered connected if and only if their binary representations differ in exactly one bit. For instance, in a 3-cube, node $a = 5$ whose binary representation is 101, is connected to node $b = 7$ which is 111. Note that the binary representation of a node in an $n$-cube is $v = v_1 v_2 \cdots v_n$ where $v_i \in \{0, 1\}$ and $1 \leq i \leq n$. Fig. 1.1 shows a 3-cube $Q_3$.

One important characteristic of hypercubes is that they are symmetric. This allows us to assume the source node as $0^n$ and target node being $1^n$ without loss of generality. Note that the blocking nodes cannot be the source node nor the target node. Therefore, our problem can be stated as follows.

Given the source node $s = 0^n$ and the target node $t = 1^n$ in an $n$-cube $Q^n$, and $m$

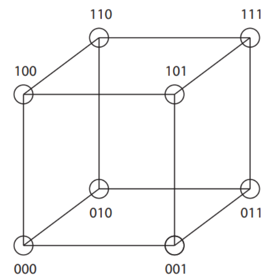


Figure 1.1: 3-Cube

other blocking nodes denoted as $b_i$ where $b_i \neq s$, $b_i \neq t$, $1 \leq i \leq m$, does there exist a shortest path from $s$ to $t$ that does not intersect any blocking node $b_i$?

In a hypercube, a path between two nodes, say node $u$ and $v$, are considered shortest only when its length is equal to the Hamming distance between $u$ and $v$, denoted as $H(u, v)$, namely, the number of positions in which $u$ and $v$ differ. We want to examine the conditions for such a path to exist. Clearly, the shortest path between $s$ and $t$ has to be $H(s, t)$ which is $n$ in this case. Thus, there are $n$+1 nodes on the shortest path, which indicates that, $m$ can be at most $2^n - (n + 1)$ and we still have a shortest path from $0^n$ to $1^n$. In addition, we want to determine the sufficient condition for such a path to exist.

Defined by Kanedo and Ito [4], a nonfaulty node is called unsafe if it is adjacent to at least two faulty nodes or more than two faulty or unsafe nodes, it is safe otherwise. Chiu and Wu [1] showed that a shortest path from $s$ to $t$ not involving any faulty node exists if either $s$ or $t$ is safe. This is clearly a sufficient condition since such a path could still exist even if both $s$ and $t$ are unsafe. Kanedo and Ito [4] introduced the notion of full reachability and found out that a nonfaulty node $u$ is fully reachable if every nonfaulty node that is $h$ Hamming distance away from $u$ is reachable by a path of length $h$. Their algorithm has been proven by the simulation results to perform better than the one from [1]. In [2], there are meaningful results provided, but we consider their problems to be different from ours since their assumptions are different. For example, they assume the number of blocking nodes to be at most $n$-1, node failures occur dynamically, and each node only knows the faulty status of its neighbours. However, in our case, we assume the number of blocking nodes could be greater than $n$-1, and we know all the blocking nodes in advance. Our routing problem has been considered in [3], in which the following lemma is proposed.

**Lemma 1.** *In $Q_n$, $n \geq 2$, if the number of blocking nodes is less than $n$, then there* *exists a shortest path from $s$ to $t$ that does not intersect any of the blocking nodes.*

**Definition 1.** *([5]) Let $(A_1, A_2, \cdots, A_m)$ be a collection of subsets of a set $A = \{a_1, a_2, \cdots, a_n\}$, $m \leq n$. An ordered set of distinct elements $[a_{i_1}, a_{i_2}, \cdots, a_{i_m}]$ is called a system of distinct representatives SDR if $a_{i_j} \in A_j$, for $1 \leq j \leq m$.*

That is, for example, if $A = \{1,2,3,4\}$ and $A_1 = \{2,3\}$, $A_2 = \{1,4\}$ and $A_3 = \{1,3\}$, then [2,1,3] is an SDR for $A_1$, $A_2$ and $A_3$, and [2,4,1] is another. While if $A_1 = \{3\}$, $A_2 = \{1,3\}$ and $A_3 = \{1\}$ then an SDR does not exist for $A_1$, $A_2$ and $A_3$.

To create a SDR for a given set of $m$ blocking nodes denoted as $u_1, u_2, \cdots, u_m$, we need to convert their binary representations into sets of integers as follows. If $u_{i_j} = 1$, then set $U_i$ contains $j$. For example, for the following 4 block nodes in $Q_4$:

$$u_1 = 1100$$
$$u_2 = 0110$$
$$u_3 = 1010$$
$$u_4 = 0001$$

We have:

$$U_1 = \{1,2\}$$
$$U_2 = \{2,3\}$$
$$U_3 = \{1,3\}$$
$$U_4 = \{4\}$$

One SDR for the four sets is [2,3,1,4]. The size of a given SDR, denoted as |SDR| is simply the number of elements in it. By the original definition of SDR, |SDR| of $m$ blocking nodes must be $m$. However, in case a SDR does not exist, we seek the largest possible SDR for a subset of blocking nodes. For example, blocking nodes 100, 101, 001 correspond to sets {1}, {1,3}, {3}, clearly, no SDR exists, while the largest SDR: [1,3] has size 2. Note that the original definition of SDR indicates that $m \leq n$, while in our case, $m$ can be greater than $n$. Thus, we will seek the max matching of the $m$ blocking nodes and use its size as the size of the SDR. [6] introduces how to construct the bipartite graph based on a given set of nodes, and find the max matching of it. For example, given 4 blocking nodes in $Q_3$, $u_1 = 100, u_2 = 010, u_3 = 110$ and $u_4 = 111$, their coresonping sets are $U_1 = \{1\}$, $U_2 = \{2\}$, $U_3 = \{1,2\}$ and $U_4 = \{1,2,3\}$, the bipartite graph is shown in Fig.1.2. From now on, whenever we talk about SDR, we mean the SDR with the largest cardinality.

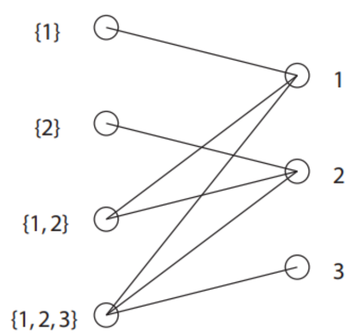


Figure 1.2: A Bipartite Graph for Four Blocking Nodes in $Q_3$

# Chapter 2

# Main Result

In addition to the symmetric property, the hypercube has another important property which is recursive property: An $n$-cube can be decomposed into two disjoint ($n$-1)-cubes along dimension $i$, for any $0 \leq i \leq n-1$, in the way that all nodes whose $i$-th bits are 0's form one ($n$-1)-cube, denoted as $Q_{n-1}^L$, similarly, all the other nodes with $i$-th bits being 1's form the other ($n$-1)-cube denoted as $Q_{n-1}^R$. Thus, $Q_{n-1}^L$ always contains the $0^n$ node.

It is trivial that provided a $Q_n$, the $m$ blocking nodes and their SDR with size $n$-1, $Q_n$ can always be decomposed into $Q_{n-1}^L$ and $Q_{n-1}^R$ such that the |SDR| for $Q_{n-1}^L$ is $n$-1, and $Q_{n-1}^R$ contains no blocking nodes if |SDR| of the given $m$ blocking nodes is $n$-1. Otherwise, |SDR| of $Q_n$ would be $n$. An example of such a decomposition is that if SDR = $\{0, 1, 2, \cdots, n-2\}$, then $Q_n$ can be decomposed along dimension $n$-1. In general, if SDR = $\{0, 1, \cdots, i-1, i+1, \cdots, n-1\}$ for $0 \leq i \leq n-1$, we can decompose $Q_n$ along dimension $i$.

Our main result is given in the following theorem:

**Theorem 1.** *Given $m$ blocking nodes in a $Q_n$, if $|SDR| \leq n$-1 for the blocking nodes, then there exists a shortest path (of length $n$) from $0^n$ to $1^n$ that does not intersect with any of the blocking nodes.*

**Proof:** We apply induction to $n$. When $n = 1$, $Q_1$ is simply an edge between 0 and 1, and $m$ has to be 0, thus, it is trivial that a shortest path always exists between the nodes.

When $n = 2$, if |SDR| $= m = 1$, let's assume the blocking node is 01, then the shortest path is $00 \rightarrow 10 \rightarrow 11$. Similarly, the shortest path exists if the blocking node is 10.

**Induction Hypothesis (I.H.):** Assume that the theorem is true for $n$-1, that is, if $|\text{SDR}| \leq n$-2 for the blocking nodes in $Q_{n-1}$, there exists a shortest path between $0^{n-1}$ and $1^{n-1}$, we show how to find a shortest path between $0^n$ and $1^n$ with m blocking nodes whose $|\text{SDR}| \leq n$-1 in the following way.

We need to consider the following two cases:

**Case1:** $1^n$ has only one free neighbor (meaning it is not a blocking node), without loss of generality, we can assume that this node is $01^{n-1}$. Then we decompose $Q_n$ along dimension 1 and consider the following situations in $Q^L_{n-1}$ that contains $0^{n-1}$ ($0^n$ in $Q_n$):

If |SDR| in $Q^L_{n-1}$ is $n$-2 or less, we are done by the I.H.: shortest path exists from $0^n$ to $01^{n-1}$ within this $Q^L_{n-1}$, then from $01^{n-1}$ to $1^n$ is trivial. The shortest path is:

$$0^n \to \cdots \to 01^{n-1} \to 1^n$$

If |SDR| in $Q^L_{n-1}$ is $n$-1: this is impossible since this SDR union any of the blocking nodes in $Q^R_{n-1}$ (not containing $0^n$) gives a SDR of size $n$ which is impossible as we assume the size of SDR is $n$-1 for $Q_n$. In fact, if $|\text{SDR}| = n$-1 and there is only one free neighbor for $1^n$, then $Q^L_{n-1}$ must have no blocking node for otherwise, |SDR| would be $n$.

**Case2:** $1^n$ has two (or more) free neighbors. As in Case 1, each free neighbor defines a $Q^L_{n-1}$, therefore, there are two (or more) $Q^L_{n-1}$ that contains $0^n$. We now consider the following two situations:

If one of the sub-cubes has $|\text{SDR}| = n$-2 or less, done by I.H. (see Case 1).

If both have $|\text{SDR}| = n$-1, this is impossible since each SDR contains $n$-1 elements from $\{1, 2, \cdots n\}$ and the two SDR's can not be the same, therefore, their union must be a SDR of size $n$, which contradicts the assumption that the size of SDR is $n$-1 for $Q_n$.

An example in $Q_4$ for the above situation is: Two free neighbors are 0111 and 1110, decompose the $Q_4$ along dimension 1 and 4 resulting in two different $Q^L_3$'s. If the SDR's in both $Q^L_3$'s have $|\text{SDR}| = 3$, they must be [2,3,4] and [1,2,3] respectively, thus the union of them are [1,2,3,4] with size 4 which contradicts the assumption.

Therefore, we have proven the theorem. Fig. 2.1 illustrates a case when the number of blocking nodes is greater than $n$-1, yet a shortest path exists between $0^n$ and $1^n$.

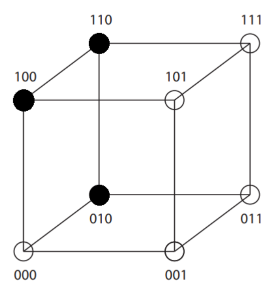


Figure 2.1: An example of $Q_3$ with 3 blocking nodes.

# Chapter 3

# Conclusion

We have considered the problem that given the source node $s = 0^n$ and the target node $t = 1^n$ in an $n$-cube $Q_n$, and $m$ other blocking nodes denoted as $b_i$ where $b_i \neq s$, $b_i \neq t$, $1 \leq i \leq m$, does there exist a shortest path from $s$ to $t$ that does not intersect any blocking node $b_i$. Based on some of the previous work done on this problem and some similar problems, we proposed a sufficient condition for the path to exist which is considered an improvement of Lemma 1 [2] because $m$ could be bigger than $n$-1. There are already efficient algorithms that finds the shortest path if it exists, our future work is to study the possibility of improving these currently existing algorithms and try to improve them.

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