Measurement: Sensors 23 (2022) 100394

Contents lists available at ScienceDirect

Measurement: Sensors

journal homepage: www.sciencedirect.com/journal/measurement-sensors

Evaluation of the uncertainty due to dynamic effects in linear measuring devices

Giovanni Battista Rossi *, Francesco Crenna, Marta Berardengo

University of Genoa, Department of Mechanical, Energy, Management and Transportation Engineering, Via Opera Pia 15/A, Genoa, Italy

A R T I C L E I N F O

Keywords:
Dynamic measurement
Measurement modelling
Probability
Uncertainty evaluation

A B S T R A C T

The evaluation of the uncertainty due to systematic dynamic effects is addressed. When high dynamic performance is required, they should be compensated, by solving the associated inverse dynamic problem. When instead they are considered compatible with the target uncertainty, they may be simply included in the uncertainty budget. Furthermore, even in the case of dynamic compensation, a residual uncertainty remains, due to the imperfect compensation, and should be evaluated. Therefore, simple formulas are presented here, applicable to many classes of dynamic phenomena, including periodic, harmonic, transitory impulsive and stochastic stationary ones.

1. Introduction

Dynamic measurement, which is measurement where the measurand value varies over time, is the object of increasing attention today, due to its application importance and to the scientific and technological challenges it still poses [1]. Dynamic measurement can be classified as either direct, where the property to be measured is the time history of the quantity of interest [2], or indirect, where some other characteristic of the quantity is sought such as, most frequently, the spectral distribution of the energy of the phenomenon [3]. Only the case of direct dynamic measurement will be considered here.

In this regard, the scientific and technical debate has developed along four main lines, strictly related to each other, but with a focus on.

- generic modelling [4,11],
- dynamic calibration [12,13],
- dynamic compensation [14–16],
- uncertainty reduction and evaluation [17–20],

where the list above includes just a few examples, among many others.

Concerning generic modelling, dynamic measurement can be considered a part of a generic framework for measurement, which has been a key topic of measurement sciences, over the years. A major concern has been the possibility of developing a common approach between physical and social sciences [4,5,7,10,11]. The specific aspects of dynamic measurement have also been discussed [6–9], and the possibility of a probabilistic framework common to static and dynamic measurement has been addressed [2].

In this context, systematic dynamic effects in the measurement system constitute an important point to improve the quality of the measurement process and to evaluate and declare its uncertainty. In linear measuring devices such effects include a possible amplification or attenuation, and phase shift of the spectral components of the indicated signal, with respect to the original phenomenon. When this effect is non-negligible, typically when operating outside the recommended band for the instrument, dynamic compensation should be applied, as discussed elsewhere [10,12]. When operating within the recommended band, uncertainty due to non-ideal behaviour of the measuring device should be evaluated anyway and included in the uncertainty budget. Such an evaluation should also be made even when dynamic compensation is applied, to account for residual uncertainty remaining after such compensation. Here a simple practical formula to do that is derived and presented. Preliminary results, for periodic and harmonic were proposed in Ref. [20]. Here the study is extended to other important classes of dynamic phenomena, namely transient impulsive and stochastic stationary, thus covering a very wide class of practical applications.

* Corresponding author.
E-mail address: g.b.rossi@unige.it (G.B. Rossi).

https://doi.org/10.1016/j.measen.2022.100394
Received 13 March 2022; Received in revised form 3 July 2022; Accepted 15 July 2022
Available online 21 July 2022

2. Modelling dynamic effects in measuring devices

2.1. The time varying error due to real operating conditions

The dynamic behaviour of a linear measuring device can be modelled through its frequency response:

$$H(f) = k\alpha(f)exp(j\varphi(f)) \tag{1}$$

where f is the frequency, $H(f)$ the (complex) frequency response (FR), k is the sensitivity, $\alpha(f)$ is the (dimensionless) modulus of the FR, $\varphi(f)$ is its phase, and j denotes the imaginary unit. For example, in the case of a simple contact thermometer, the modulus is:

$$\alpha(f) = \left(1 + (2\pi fT)^2\right)^{-1/2} \tag{2}$$

and the phase is [15]:

$$\varphi(f) = \tan^{-1}(2\pi fT) \tag{3}$$

where T is the time constant of the thermometer.

Suppose now that the measurand is a simple cosinusoidal process:

$$x(t) = x_0 \cos(2\pi f_0 t + \varphi_0), \tag{4}$$

where $f_0 = T_p^{-1}$, and T_p is the period.

Considering, for now, systematic effects only, i.e., neglecting the noise, the instrument indication will then be

$$y(t) = k\alpha(f_0)x_0 \cos(2\pi f_0 t + \varphi_0 + \varphi(f_0)) \tag{5}$$

The measured signal, with no dynamic compensation [5], will then be:

$$\hat{x}(t) = k^{-1}y(t) = = \alpha(f_0)x_0 \cos(2\pi f_0 t + \varphi_0 + \varphi(f_0)) \tag{6}$$

Hence, the dynamic effect can be expressed by the *error*:

$$e(t) = \hat{x}(t) - x(t). \tag{7}$$

If the (ideal) *non-distortion conditions* hold true, i.e., if $\alpha(f) = 1$ and $\varphi(f) = 0$ for all the frequencies of interest, $\hat{x}(t) = x(t)$, no systematic deviation occurs. Therefore, to discuss the actual behaviour of the system, it is convenient to assume $\alpha(f) = 1 + \delta\alpha(f)$ and $\varphi(f) = 0 + \delta\varphi(f)$. Yet, in a typical practical case the exact values of $\delta\alpha(f)$ and $\delta\varphi(f)$ would be unknown, so it makes sense to model them as probabilistic variables. Lastly, since we do not know the "functions" $\delta\alpha(f)$ and $\delta\varphi(f)$ but only some global figures about them, such as their standard deviations (i.e., σ_α and σ_φ), or their ranges (i.e., $\pm\Delta_\alpha$ and $\pm\Delta_\varphi$), we will neglect their dependence upon frequency, thus definitely setting:

$$\alpha(f) = 1 + \delta\alpha$$

$$\varphi(f) = 0 + \delta\varphi \tag{8}$$

It thus results, for the measured signal:

$$\hat{x}(t) = x_0(1 + \delta\alpha)\cos(2\pi f_0 t + \varphi_0 + \delta\varphi)$$

$$= x_0(1 + \delta\alpha)[\cos(2\pi f_0 t + \varphi_0)\cos\delta\varphi - \sin(2\pi f_0 t + \varphi_0)\sin\delta\varphi]$$

Since $\delta\varphi$ is usually small, let us assume $\cos\delta\varphi \cong 1$ and $\sin\delta\varphi \cong \delta\varphi$. Then, after neglecting second order terms, we ultimately obtain for the error:

$$e(t) = \delta\alpha x_0 \cos(2\pi f_0 t + \varphi_0)$$

$$-\delta\varphi x_0 \sin(2\pi f_0 t + \varphi_0) \tag{9}$$

Therefore, for any given dynamic process $x(t)$, *the error is a stochastic process*, depending on the two random parameters $\delta\alpha$ and $\delta\varphi$, that can be modelled as probabilistic variables, that we will assume zero-mean and uncorrelated. Therefore, the error will also be zero-mean.

Two approaches are then possible, for uncertainty evaluation, closely related to the two perspectives from which a stochastic process can be studied, namely the synchronic (along time) and the diachronic (across time) views, to be developed in the following [21].

2.2. The synchronic view

Let us consider the "power" of the signal and of the error, that is their mean quadratic values. For any positive integer n, and for T being a generic time duration, for the signal we obtain:

$$P_x = \frac{1}{T_p}\int_{-T_p/2}^{+T_p/2} x^2(t)\mathrm{d}t = \frac{1}{nT_p}\int_{-nT_p/2}^{+nT_p/2} x^2(t)\mathrm{d}t$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} x^2(t)\mathrm{d}t = \frac{{x_0}^2}{2} \tag{10}$$

For the error, for each pair $(\delta\alpha, \delta\varphi)$, we obtain:

$$\begin{aligned} P_e(\delta\alpha,\delta\varphi) &= \frac{1}{T_p}\int_{-T_p/2}^{+T_p/2} e^2(t)\mathrm{d}t \\ &= \frac{{x_0}^2}{T_p}\int_{-T_p/2}^{+T_p/2}\left[\delta\alpha^2\cos^2(2\pi f_0 t + \varphi_0) + \delta\varphi^2\sin^2(2\pi f_0 t + \varphi_0)\right. \\ &\quad \left. - 2\delta\alpha\delta\varphi\cos(2\pi f_0 t + \varphi_0)\sin(2\pi f_0 t + \varphi_0)\right]\mathrm{d}t \\ &= \frac{{x_0}^2}{2}\left(\delta\alpha^2 + \delta\varphi^2\right) = P_x\left(\delta\alpha^2 + \delta\varphi^2\right) \end{aligned} \tag{11}$$

Then taking the expected value, $\mathrm{E}(\cdot)$, with respect to the probabilistic parameters $\delta\alpha$ and $\delta\varphi$ and remembering that they have been assumed zero-mean, we lastly obtain:

$$P_e = \mathrm{E}[P_e(\delta\alpha,\delta\varphi)] = P_x\left(\sigma_\alpha^2 + \sigma_\varphi^2\right). \tag{12}$$

Finally, considering the usual notation for standard uncertainty, and denoting by u_d the standard uncertainty due to *dynamical* effects in the measuring device, then $u_d^2 = P_e$ and we obtain, noteworthily:

$$\frac{u_d}{x_{rms}} = \sqrt{\sigma_\alpha^2 + \sigma_\varphi^2} = \sqrt{u_\alpha^2 + u_\varphi^2}. \tag{13}$$

2.3. The diachronic view

We can also look for an expression of uncertainty another way round, i.e., by following the diachronic perspective. Here the error is regarded as a time dependent parametrical probabilistic variable, also depending on the parameters $(\delta\alpha, \delta\varphi)$. Let us now calculate the time dependent variance of the error, remembering that the two variables, $\delta\alpha$ and $\delta\varphi$, are zero-mean and uncorrelated. We obtain:

$$\begin{aligned} \sigma_e^2(t) &= \mathrm{E}\left(e^2(t)\right) \\ &= \mathrm{E}\left[\delta\alpha^2 x_0^2\cos^2(2\pi f_0 t + \varphi_0) + \delta\varphi^2 x_0^2\sin^2(2\pi f_0 t + \varphi_0)\right. \\ &\quad \left. - 2\delta\alpha\delta\varphi x_0^2\cos(2\pi f_0 t + \varphi_0)\sin(2\pi f_0 t + \varphi_0)\right] \\ &= \sigma_\alpha^2 x_0^2\cos^2(2\pi f_0 t + \varphi_0) + \sigma_\varphi^2 {x_0}^2\sin^2(2\pi f_0 t + \varphi_0). \end{aligned} \tag{14}$$

The variance is thus time dependent, and such is also the standard deviation due to dynamic effects. Yet, this is not practical, and a constant global value is rather of interest. To obtain that, time averaging over one period may be considered, yielding:

$$\overline{\sigma}_e^2 = \frac{1}{T_p}\int_{-T_p/2}^{+T_p/2}\sigma_e^2(t)\mathrm{d}t = \frac{1}{T_p}\int_{-T_p/2}^{+T_p/2}\left[\sigma_\alpha^2 {x_0}^2\cos^2(2\pi f_0 t+\varphi_0)+\sigma_\varphi^2 {x_0}^2\sin^2(2\pi f_0 t+\varphi_0)\right]\mathrm{d}t = \frac{{x_0}^2}{2}\left(\sigma_\alpha^2+\sigma_\varphi^2\right) = P_x\left(\sigma_\alpha^2+\sigma_\varphi^2\right). \quad (15)$$

This result is in perfect agreement with Equation (12) and thus it also yields Equation (14). Therefore, the two approaches considered here yield the same, simple and compact, result. Equation (13) establishes a simple, elegant, and practical relation between the relative standard uncertainty due to dynamical effects and the uncertainty on the modulus and the phase of the frequency response of the measuring device.

2.4. The measurand as a stochastic process itself

Lastly, so far, the case of a single signal has been considered, which may be seen as a deterministic approach. Yet in a more general perspective, starting from the deterministic process presented in eq. (4), it is possible to introduce the stochastic stationary process:

$$x(t) = x_0\cos(2\pi f_0 t+\varphi'), \quad (16)$$

where φ' is a random variable in the range $[-\pi, +\pi]$. This a nice "physical" meaning, it may be interpreted as the possibility of accessing at random at the process of eq. (4): if the time origin, $t = 0$, is set at the moment of access, the phase will vary at random, depending on that instant. In this perspective, the probability density of the variable φ is typically assumed as uniform in the range $[-\pi, +\pi]$.

The error is now:

$$e(t) = \delta\alpha x_0\cos(2\pi f_0 t+\varphi') - \delta\varphi x_0\sin(2\pi f_0 t+\varphi'), \quad (17)$$

which is a function of three probabilistic variables, $\delta\alpha$, $\delta\varphi$ and φ'. Its expected value is null, since:

$$E[e(t)] = \int_{-\Delta\alpha}^{+\Delta\alpha}\int_{-\pi}^{+\pi}\delta\alpha x_0\cos(2\pi f_0 t+\varphi')p(\delta\alpha)p(\varphi')\mathrm{d}\delta\alpha\mathrm{d}\varphi' - \int_{-\Delta\varphi}^{+\Delta\varphi} \times \int_{-\pi}^{+\pi}\delta\varphi x_0\sin(2\pi f_0 t+\varphi')p(\delta\varphi)p(\varphi')\mathrm{d}\delta\alpha\mathrm{d}\varphi' = 0, \quad (18)$$

and the variance is:

$$E\left[e^2(t)\right] = \int_{-\Delta\alpha}^{+\Delta\alpha}\int_{-\pi}^{+\pi}\delta\alpha^2 {x_0}^2\cos^2(2\pi f_0 t+\varphi')p(\delta\alpha)p(\varphi')\mathrm{d}\delta\alpha\mathrm{d}\varphi' + \int_{-\Delta\varphi}^{+\Delta\varphi}\int_{-\pi}^{+\pi}\delta\varphi^2 {x_0}^2\sin^2(2\pi f_0 t+\varphi')p(\delta\alpha)p(\varphi')\mathrm{d}\delta\varphi\mathrm{d}\varphi' - 2\int_{-\Delta\alpha}^{+\Delta\alpha}\int_{-\Delta\varphi}^{+\Delta\varphi}\int_{-\pi}^{+\pi}\delta\alpha\delta\varphi x_0^2\cos(2\pi f_0 t+\varphi')\sin(2\pi f_0 t + \varphi')p(\delta\alpha)p(\delta\varphi)p(\varphi')\mathrm{d}\delta\alpha\mathrm{d}\delta\varphi\mathrm{d}\varphi',$$

which in the end yields:

$$\overline{\sigma}_e^2 = \int_{-\Delta\alpha}^{+\Delta\alpha}\delta\alpha^2\left[\frac{1}{2\pi}\int_{-\pi}^{+\pi}{x_0}^2\cos^2(2\pi f_0 t+\varphi')\mathrm{d}\varphi'\right]p(\delta\alpha)\mathrm{d}\delta\alpha + \int_{-\Delta\varphi}^{+\Delta\varphi}\delta\varphi^2\left[\frac{1}{2\pi}\int_{-\pi}^{+\pi}{x_0}^2\sin^2(2\pi f_0 t+\varphi')\mathrm{d}\varphi'\right]p(\delta\varphi)\mathrm{d}\delta\varphi = \frac{{x_0}^2}{2}\left[\int_{-\Delta\alpha}^{+\Delta\alpha}\delta\alpha^2 p(\delta\alpha)\mathrm{d}\delta\alpha + \int_{-\Delta\alpha}^{+\Delta\alpha}\delta\varphi^2 p(\delta\varphi)\mathrm{d}\delta\varphi\right] = P_x\left(\sigma_\alpha^2+\sigma_\varphi^2\right). \quad (19)$$

It appears that this procedure, which is possibly the most elegant and complete of the three examined, yields the same result as the others. In the following, this approach will be applied to important classes of dynamic phenomena, to check and assess its validity.

3. Application of the method to stationary processes

3.1. Periodic and harmonic processes

Let us now consider application of the ideas above to several important classes of dynamic phenomena. Stationary processes are considered first, including periodic, harmonic, continuous-spectrum and mixed-spectrum stochastic stationary processes. Let us start from (zero-mean) periodic and harmonic processes. Periodic phenomena, can be modelled by the (limited) Fourier series expansion:

$$x(t) = \sum_{i=1}^{n} c_i\cos(i2\pi f_0 t+\varphi_i), \quad (20)$$

where $x(t)$ is (the time history of) the measurand and.

$f_0 = T_p^{-1}$ is its fundamental frequency.

Harmonic processes can instead be modelled as [22]:

$$x(t) = \sum_{i=1}^{n} c_i\cos(2\pi f_i t+\varphi_i) \quad (21)$$

where the frequencies involved are no longer constrained to be integer multiples of a fundamental frequency, f_0. Since eq. (21) is a generalization of eq. (20), which can be obtained from (21) by putting

$$f_i = i f_0 \quad (22)$$

it is sufficient to discuss the latter.

Firstly, let us calculate the "power" (mean quadratic value) of a harmonic process. Since there is not a fundamental frequency here, the third expression of eq. (10) is appropriate, yielding:

$$P_x = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2}x^2(t)\mathrm{d}t = \sum_{i=1}^{n}\frac{{c_i}^2}{2}. \quad (23)$$

Then, following the approach outlined in Section 2.1, for the instrument indication we obtain:

$$y(t) = k\sum_{i=1}^{n}\alpha(f_i)c_i\cos(2\pi f_i t+\varphi_i+\varphi(f_i)), \quad (24)$$

where k is the sensitivity of the measurement device. The measured signal can thus be derived as:

$$\hat{x}(t) = k^{-1}y(t) = \sum_{i=1}^{n} \alpha(f_i)c_i \cos(2\pi f_i t + \varphi_i + \varphi(f_i)), \tag{25}$$

and for the error due to dynamic effects, still accounting for assumptions (8), we obtain:

$$e(t) = \delta\alpha \sum_{i=1}^{n} c_i \cos(2\pi f_i t + \varphi_i)$$

$$-\delta\varphi \sum_{i=1}^{n} c_i \sin(2\pi f_i t + \varphi_i) \tag{26}$$

Therefore:

$$S_{ee}(f) = S_{yy}(f) - S_{yx}(f) - S_{xy}(f) + S_{xx}(f) \tag{35}$$

Now, remembering that [23]:

$$S_{xy}(f) = H(f)S_{xx}(f), \; S_{yx}(f) = H^*(f)S_{xx}(f) \tag{36}$$

where $H^*(f)$ is the complex conjugate of $H(f)$, and accounting for eq. (35), we obtain:

$$S_{ee}(f) = \left[|H(f)|^2 - H(f) - H^*(f) + 1\right]S_{xx}(f) = \left\{[1 - \mathrm{Re}(H(f))]^2 + [\mathrm{Im}(H(f))]^2\right\}S_{xx}(f) \tag{37}$$

Then, the variance of the error is:

$$\sigma_e^2(t) = \sigma_\alpha^2 \sum_{i=1}^{n} c_i{}^2\cos^2(2\pi f_i t + \varphi_i) + \sigma_\varphi^2 \sum_{i=1}^{n} c_i{}^2\sin^2(2\pi f_i t + \varphi_i). \tag{27}$$

Accounting for eq. (10), the average error variance is now:

$$\overline{\sigma}_e^2 = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} \sigma_e^2(t)dt = P_x\left(\sigma_\alpha^2 + \sigma_\varphi^2\right) \tag{28}$$

and, lastly, we still obtain eq. (13), recalled here for clarity:

$$\frac{u_d}{x_{rms}} = \sqrt{u_\alpha^2 + u_\varphi^2} \tag{29}$$

Therefore, this result, originally obtained for mono-tone processes also turns out to be applicable to the important classes of periodic and harmonic processes.

3.2. Stochastic stationary processes

Let us now consider the signal $x(t)$ as a realisation of a stationary stochastic process, having a (continuous) power spectral density (PSD) $S_{xx}(f)$. Then the signal indication, $y(t)$, is again, neglecting any instrument insertion transient and remembering eq. (1), a stationary stochastic process, with PSD:

$$S_{yy}(f) = |H(f)|^2 S_{xx}(f) \tag{30}$$

Let us now consider the dynamic error, $e(t) = k^{-1}y(t) - x(t)$, and assume, to simplify notation,[1] with no impact on the result, $k = 1$. Then $e(t) = y(t) - x(t)$. Let us now consider the autocorrelation of the error:

$$R_{ee}(\tau) = E[e(t)e(t+\tau)] = E[y(t)e(t+\tau) - x(t)e(t+\tau),] = R_{ye}(\tau) - R_{xe}(\tau). \tag{31}$$

Taking the Fourier transform, we obtain:

$$S_{ee}(f) = S_{ye}(f) - S_{xe}(f) \tag{32}$$

Similarly, we obtain:

$$S_{ye}(f) = S_{yy}(f) - S_{yx}(f), \tag{33}$$

and:

$$S_{xe}(f) = S_{xy}(f) - S_{xx}(f) \tag{34}$$

[1] Since k is simply a multiplicative constant, its effect in this procedure is irrelevant.

Now, recalling eq. (8):

$$H(f) = (1+\delta\alpha)\exp(j\delta\varphi) \cong (1+\delta\alpha) + j\delta\varphi \tag{38}$$

for $\delta\alpha$ and $\delta\varphi$ small, where j denotes the imaginary unit.

By combining these two equations, we lastly obtain:

$$S_{ee}(f) = \left(\delta\alpha^2 + \delta\varphi^2\right)S_{xx}(f) \tag{39}$$

which, integrated on the frequency axis, yields the same result as eq. (11), and again, by taking the expected values of $\delta\alpha$ and $\delta\varphi$, the same result as in eq. (13) outcomes.

Therefore, for stochastic stationary process, the same evaluation formula can be applied.

3.3. Mixed-spectrum processes

Another important class of processes include those that are the sum of a continuous-spectrum process with a harmonic process [22,24]. The associated PSD can be expressed as the sum of the two spectra, as (recalling eq. (23)):

$$S(f) = S_c(f) + S_d(f) = S_c(f) + \sum_{i=1}^{n}\frac{c_i^2}{2}\delta(f - f_i) \tag{40}$$

where $S_c(f)$ is the continuous part of the spectrum, and $\delta(\Delta)$ denotes the Dirac delta function. For these processes it is possible to proceed as with the previous ones, by simply taking $S_{xx}(f)$ as in eq. (40). Then eqs. 30–39 are applicable, and the result, expressed by eq. (39) can be obtained.

4. Application of the method to non-stationary processes

4.1. Transient impulsive phenomena

The class of non-stationary processes is wide and difficult to organise in a taxonomy. Here two main classes of processes are considered, transient impulsive [25] and those characterised by slowly varying evolutionary spectra [26,27].

Transient impulsive phenomena occur, for example in shock testing and in the measurement of transit noise.

Signals arising from such phenomena are finite-energy, i.e., they have a finite integral square value:

$$E = \int_{-\infty}^{+\infty} x^2(t)\mathrm{d}t < +\infty \tag{41}$$

Although $x(t)$ may theoretically have an infinite duration, in practice data acquisition may be performed only for a finite time, in such a way as to capture the entire phenomenon. Therefore, a finite duration, T is assumed here. Let us then consider the energy spectral density (ESD):

$$\eta(f) = |X(f)|^2 \tag{42}$$

where $X(f)$ is the Fourier transform of $x(t)$:

$$X(f) = \int_{-\infty}^{+\infty} x(t)\exp(-j2\pi ft)\mathrm{d}t \tag{43}$$

which satisfies Parseval's equality:

$$E = \int_{-\infty}^{+\infty} \eta(f)\mathrm{d}f \tag{44}$$

For finite-energy processes, the ESD closely replaces the PSD that applies to finite-power processes, and eqs. 30–39 still hold, by simply replacing $S(f)$ with $\eta(f)$ [25]. Therefore, we obtain the equivalent to eq. (39), which now reads:

$$\eta_{ee}(f) = \left(\delta\alpha^2 + \delta\varphi^2\right)\eta_{xx}(f) \tag{45}$$

from which

$$E_e = \mathrm{E}[E_e(\delta\alpha, \delta\varphi)] = E_x\left(\sigma_\alpha^2 + \sigma_\varphi^2\right). \tag{46}$$

results. Yet, standard uncertainty is related to the standard deviation of the (dynamic) error, not directly to its energy.

Thus, we may assume:

$$E_e = \overline{\sigma}_e^2 T \tag{47}$$

From which we lastly obtain:

$$\frac{u_d}{\sqrt{E_x/T}} = \sqrt{\sigma_\alpha^2 + \sigma_\varphi^2} = \sqrt{u_\alpha^2 + u_\varphi^2}. \tag{48}$$

which constitutes the equivalent of eq. (13) for finite-energy processes.

4.2. Processes with evolutionary spectra

Finally, let us briefly consider processes with slowly varying evolutionary spectra, although a thorough treatment of them is beyond the scope of this communication [28]. Such processes may be characterised by a time dependent evolutionary power spectral density (EPSD), $S(f,t)$, which is based on describing the behaviour of the process locally through an oscillatory process, in such a way that a local PSD can be estimated. The integral of the EPSD over the frequency domain provides the local power, $P(t)$, of the process.

Therefore, if we consider this local perspective, results like those expressed by eqs. 24–31 can be assumed and we obtain:

$$\frac{u_d(t)}{x_{rms}(t)} = \sqrt{u_\alpha^2 + u_\varphi^2} \tag{49}$$

Although standard uncertainty is time dependent here, relative standard uncertainty is not, and the same estimation formula as above can be applied.

4.3. To sum up

In the treatment above two main classes of signals have been considered: those of finite-power and those of finite-energy.

Finite-power signals are those that satisfy:

$$P_x = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} x^2(t)\mathrm{d}t < +\infty. \tag{50}$$

For them and for signals outcoming from processes slowly varying evolutionary spectra, the following formula can be used:

$$\frac{u_d}{x_{rms}} = \sqrt{u_\alpha^2 + u_\varphi^2} \tag{51}$$

Finite-energy signals instead satisfy:

$$E_x = \int_{-\infty}^{+\infty} x^2(t)\mathrm{d}t < +\infty \tag{52}$$

For them, uncertainty evaluation can be based on:

$$\frac{u_d}{\sqrt{E_x/T}} = \sqrt{u_\alpha^2 + u_\varphi^2}, \tag{53}$$

which differs from eq. (51) only in the way relative uncertainty is expressed.

5. Verification by simulation

In the treatment above the time dependence of $\delta\alpha$ and $\delta\varphi$ on the frequency, f was neglected: see discussion before eq. (8). Although this assumption is based on considering the information typically available, – practical application examples will be provided in Section 6 – a simulation study was performed, to check the potential criticality of such an assumption. A (hypothetical) piezo-electric accelerometer with frequency response:

$$H(f) = k\frac{j2\pi fT}{1 + j2\pi fT}\frac{1}{\left(\frac{jf}{f_0}\right)^2 + 2z\frac{jf}{f_0} + 1}, \tag{54}$$

where k is the sensitivity, $z = 35.5 \times 10^{-3}$ is the damping factor, $f_0 = 43$ kHz is the natural frequency of the seismic sensor and $T = 20$ s is the time constant of the piezo-electric sensor, was considered for the simulation (Fig. 1).

In this case, due to the (deterministic) behaviour of the frequency response, in the band $B = (0.5\text{ Hz}, 10\text{ kHz})$, $\Delta\alpha = 0.06$ and $\Delta\varphi = 1^\circ = 0.0174$ rad, as shown in Figs. 2–3. By applying eq. (51), we obtain a relative standard uncertainty $u_d/x_{rms} = 0.036$.

In each simulation trial, $N = 5000$ signals of the form:

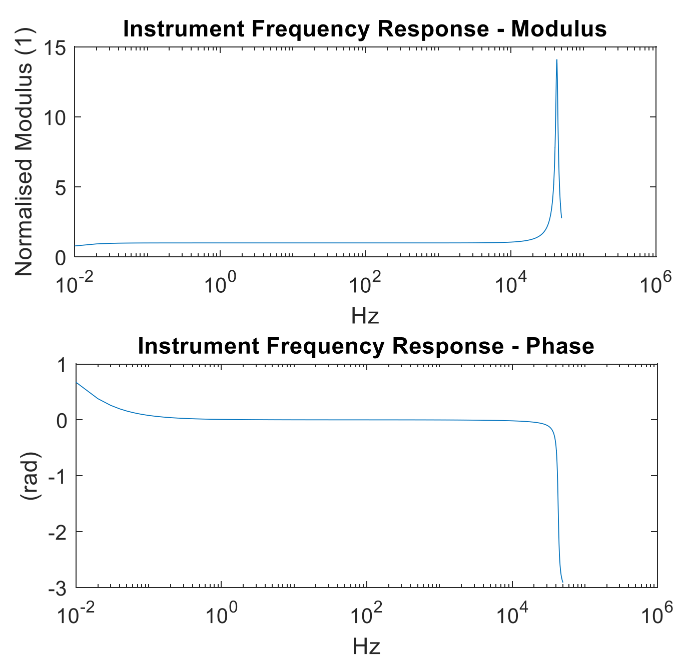

Fig. 1. Accelerometer frequency response – Normalised modulus and phase.

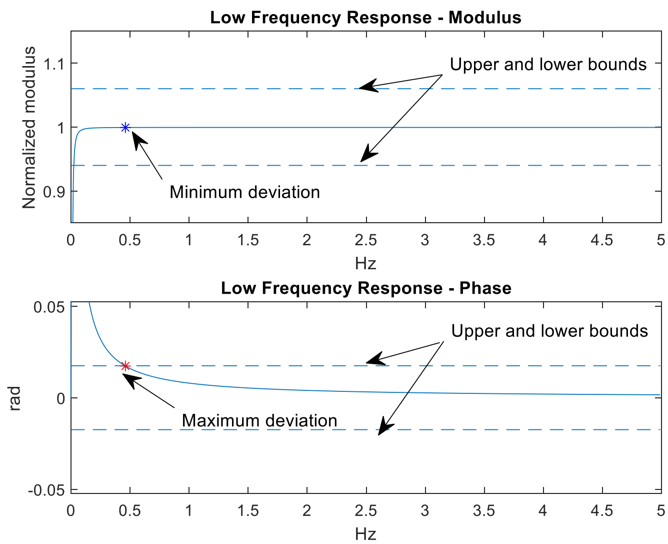

Fig. 2. Accelerometer low frequency response, bandwidth tolerances and limit deviations.

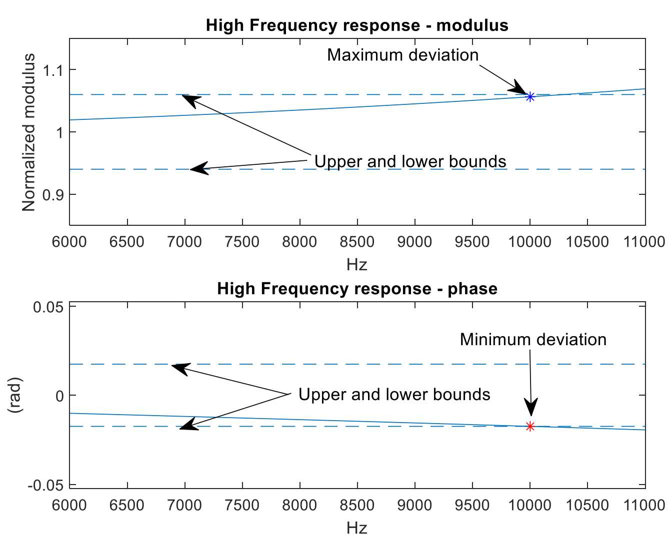

Fig. 3. Accelerometer high frequency response, bandwidth tolerances and limit deviations.

$$x_i(t) = \sum_{l=1}^{n} c_l \cos(2\pi f_l t + \varphi_l), \tag{55}$$

were simulated, with $n = 5$, f_l in the band B, $\varphi_l \in [-\pi, +\pi]$ and c_l in the range from 0.1 $\frac{m}{s^2}$ to 1 $\frac{m}{s^2}$. To cover the whole band properly, B was divided in 5 sub-bands and each f_l was selected from each sub-band. For each simulated signal, the error $e_i(t)$ and the corresponding ratio $e_{rms,i}/x_{rms,i}$ were calculated. An example result for a test is shown in Fig. 4.

From the probability distribution, the expected value for the ratio e_{rms}/x_{rms} is 0.024. Result stability is confirmed by repeated tests in the same conditions giving a standard deviation for the expected e_{rms}/x_{rms} of 0.13%.

Therefore, it appears that the proposed formula yields a conservative evaluation of relative standard uncertainty, which is appropriate in view of the *limited information* typically available.

6. Hints for practical uncertainty evaluation

Let us now discuss the application of the method above to typical

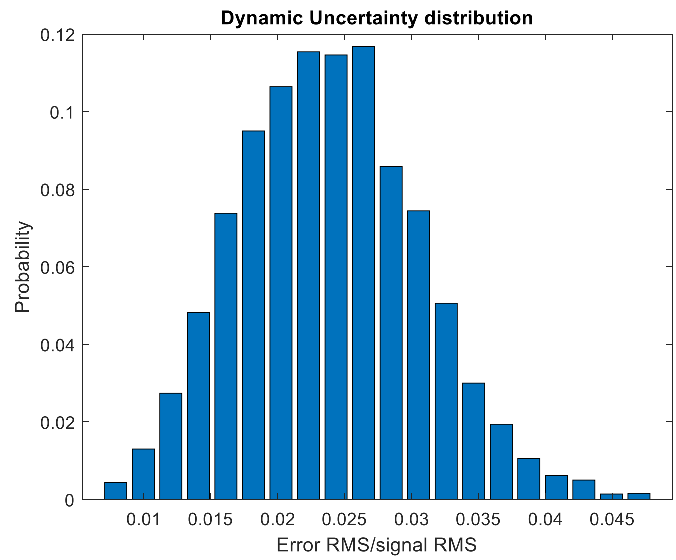

Fig. 4. Probability distribution for the ratio e_{rms}/x_{rms} with N = 5000 samples.

dynamic measurements, such as vibration measurement. Typical transducers for such measurements are either piezo-electric accelerometers, for absolute motion monitoring, or eddy-current proximity probes, for relative motion. Let us then assume a more general model instrument indication, capable of accounting for other typical uncertainty sources, that is:

$$y(t) = (k + \delta k) \sum_{i=1}^{n} \alpha(f_i) c_i \cos(2\pi f_i t + \varphi_i + \varphi(f_i)) + (h + \delta h) + v(t), \tag{56}$$

where h is an additive term which accounts for a possible non-zero output, in correspondence with a zero input, which is typically the case with eddy-current proximity probes, δk and δh are multiplicative and additive deviations, and $v(t)$ is additive noise that includes noise in the process, due to a non-perfectly harmonic phenomenon, and measurement noise.

The corresponding expression for the error will then be:

$$e(t) = \left(\frac{\delta k}{k} + \delta\alpha\right) \sum_{i=1}^{n} c_i \cos(2\pi f_i t + \varphi_i)$$

$$-\delta\varphi \sum_{i=1}^{n} c_i \sin(2\pi f_i t + \varphi_i) + k^{-1}(h + v(t)) \tag{57}$$

It should be noted that the multiplicative systematic effect due to sensitivity (normalised) deviation $\delta k/k$ behaves in a way very similar to $\delta\alpha$ therefore it is again convenient to average over time. Therefore, the final expression for relative standard uncertainty evaluation, accounting for all the uncertainty sources considered above, will be:

$$\frac{u}{x_{rms}} = \sqrt{u_\alpha^2 + u_\varphi^2 + k^{-2}\left(u_k^2 + x_{rms}^{-2}\left(u_h^2 + v_{rms}^2\right)\right)} \tag{58}$$

The relationship between relative standard uncertainty and signal-to-noise ratio (SNR), a common feature in dynamic measurement, can be noted. Indeed:

$$\mathrm{SNR} = 10\log_{10}\left(\frac{x_{rms}}{u}\right)^2 = 20\log_{10}\left(\frac{x_{rms}}{u}\right) \tag{59}$$

Let us then briefly discuss its practical application, leaving aside, for now, the evaluation of noise, to be treated at the end of this section.

In the case of high quality piezo-electric accelerometers, explicit statements on the uncertainties of the modulus and of the phase, for a selected frequency range, in the form:

$$\delta\alpha = \pm\Delta\alpha,\ \delta\varphi = \pm\Delta\varphi. \tag{60}$$

The uncertainty on k is typically expressed as a percentage value, i.e., in the form $\pm\Delta k/k$ and the uncertainty on h are often not mentioned, which may be interpreted as that they are considered negligible as compared to the dynamic effects. Let us also assume that uncertainty related to environmental conditions is negligible as well. Then, apart from noise, relative standard uncertainty can be evaluated from:

$$\frac{u}{x_{rms}} = \sqrt{\frac{\Delta\alpha^2 + \Delta\varphi^2 + (\Delta k/k)^2}{3}}. \tag{61}$$

where uniform distributions have been assumed for the variable involved. For example, if, in the frequency range of the device, the uncertainty on the sensitivity is rated within ±10 %, that on the module also within ±10 %, and that on the phase within ±1°, we obtain:

$$\frac{u}{x_{rms}} = \sqrt{\frac{(0.1)^2 + (0.0175)^2 + (0.1)^2}{3}} = 0.082, \tag{62}$$

which corresponds to SNR = 22 dB. It may be noted that the phase effect, in this case is negligible as compared to the modulus effect.

In the case of eddy-current proximity probes, the dynamic behaviour may be documented by presenting typical frequency response curves for both modulus and phase. Such curves usually have a low-pass behaviour, with deviation from the ideal behaviour asymmetrical with respect to zero, typically in a range $(-\Delta\alpha, 0)$ and $(-\Delta\varphi, 0)$, respectively. Yet an asymmetrical distribution would imply some correction of the result, which, in this case, would mean performing dynamic compensation. Yet this is usually avoided, in practical applications. Hence, symmetrical ranges can be assumed, i.e., $(-\Delta\alpha, +\Delta\alpha)$ and $(-\Delta\varphi, +\Delta\varphi)$. Concerning the other uncertainty sources, here both $\delta k/k$ ad δh are present, usually denoted as (uncertainty on the) incremental scale factor (ISF) and deviation from (best fit) straight line (DSL). Therefore, apart from noise, eq. (24) can be used. For example, in a given frequency range, viz. up to 1 kHz, the maximum deviation of the modulus is − 0.25 dB, of the phase − 10°, the ISF is rated within ±5 %, the DSL is ±0.025 mm, for x_{rms} = 1.0 mm, we obtain:

$$\frac{u}{x_{rms}} = \sqrt{\frac{(0.03)^2 + (0.175)^2 + (0.05)^2 + (0.025/1.0)^2}{3}} = 0.11, \tag{63}$$

with SNR = 19. Here the uncertainty on the phase is the most important effect.

Lastly, let us briefly discuss the evaluation of the rms value of the noise that directly affects the result as an additional uncertainty source. This can hardly be obtained from the data sheets of the devices, since it is strongly related to the experimental and environmental conditions.

One possibility, when applicable, is to record the output of the measuring system, for a zero measurand input, and to compute the corresponding rms value. But zeroing the input is often impossible, especially in the field.

Then, if the maximum frequency of interest for the phenomenon, call it f_{max}, is significantly smaller than $f_s/2$, where f_s is the sampling frequency, the noise in the band $(f_{max}, f_s/2)$ can be estimated as the difference between the original signal and the signal low-pass filtered up to f_{max}. If v'_{rms} is its rms value, the rms value of the noise can be estimated as:

$$v_{rms} = v'_{rms}\sqrt{\frac{f_s/2}{f_s/2 - f_{max}}}. \tag{64}$$

Lastly, if that is not even possible, at least in the case of harmonic processes, the spectrum of the signal can be estimated and the noise can be estimated as the difference between the original signal and the signal reconstructed through eq. (20) or eq. (21), where only the significant spectral components are included.

7. Conclusions

Systematic dynamic effects in linear measuring devices have been considered and simple formulas have been obtained, applicable to many classes of dynamic phenomena, including periodic, harmonic, transitory impulsive and stochastic stationary ones. Their practical application, in the case of absolute or relative vibration, has been discussed and hints for practical uncertainty evaluation have been provided.

CRediT authorship contribution statement

Giovanni Battista Rossi: Conceptualization, Supervision, Methodology, Writing – review & editing. **Francesco Crenna:** Methodology, Software, Validation, Writing – review & editing. **Marta Berardengo:** Methodology, Validation, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

This research was partially funded with grant FRA 2019, from the University of Genoa.

The authors want to thank anonymous Reviewers, whose comments were useful to improve the paper.

References

[1] Bartoli, et al., Traceable Dynamic Measurement of Mechanical Quantities: Objectives and First Results of This European Project", *XX IMEKO World Congress*, 2012, pp. 628–636.
[2] G.B. Rossi, Toward an interdisciplinary probabilistic theory of measurement, IEEE Trans. IM 61 (2012) 2097–2106.
[3] G.B. Rossi, F. Crenna, V. Piscopo, A. Scamardella, Comparison of spectrum estimation methods for the accurate evaluation of sea state parameters, Sensors 20 (2020) 1416, https://doi.org/10.3390/s20051416.
[4] L. Narens, R.D. Luce, Measurement: the theory of numerical assignments, Psychol. Bull. 99 (2) (1986) 166–180.
[5] B.D. Wright, A history of social science measurement, Educ. Meas. 16 (4) (1997) 33–45.
[6] K.D. Sommer, Modelling of measurement, system theory, and uncertainty evaluation, in: F. Pavese, A. Forbes (Eds.), Data Modeling for Metrology and Testing in Measurement Science, Birkhauser-Springer, Boston, 2009, pp. 275–298.
[7] G.B. Rossi, Measurement and Probability, Springer, Dordrecht, 2014.
[8] K.H. Ruhm, Dynamics and stability – a proposal for related terms in Metrology, from a mathematical point of view, Measurement 79 (2016) 311–320.
[9] K.H. Ruhm, Measurement plus observation – a new structure in metrology, Measurement 126 (2017) 421–432.
[10] L. Pendrill, Quality Assured Measurement Unification across Social and Physical Sciences, Springer, 2019.
[11] L. Mari, M. Wilson, A. Maul, Measurement across the Sciences, Springer, 2021.
[12] J.P. Hessling, Dynamic metrology, Meas. Sci. Technol. 19 (7p) (2008), 084008.
[13] M. Kobusch, S. Eichstädt, L. Klaus, T. Bruns, Investigations for the model-based dynamic calibration of force transducers by using shock excitation, Acta Imeko 4 (2) (2015) 45–51.
[14] S. Eichstaedt, C. Elster, T.J. Esward, J.P. Hessling, Deconvolution filters for the analysis of measurement processes, Metrologia 47 (2010) 522–533.
[15] G.B. Rossi, F. Crenna, A. Palazzo, Probabilistic framework for dynamic measurements for Markovian models with a uniform prior for the measurand, IEEE Trans. IM 68 (2019) 4357–4367.
[16] G.B. Rossi, F. Crenna, Dynamic Compensation for a Second Order Measuring Device, IEEE International Instrumentation and Measurement Technology Conference (I2MTC), Dubrovnik, Croatia, 2020, pp. 1–5, 2020.
[17] C. Elster, A. Link, Uncertainty evaluation for dynamic measurements modelled by a linear time-invariant system, Metrologia 45 (2008) 464–473.
[18] C. Elster, A. Link, Bruns, Analysis of dynamic measurements and determination of time-dependent measurement uncertainty using a second order model, Meas. Sci. Technol. 18 (2007) 3682–3687.
[19] E. Doebelin, Measurement Systems, Mc Graw Hill, 1990.
[20] G.B. Rossi, F. Crenna, M. Berardengo, Evaluation of the uncertainty due to dynamic effects in linear measuring devices - Preliminary results, Measurement: Sensors 18 (2021) 100086, https://doi.org/10.1016/j.measen.2021.100086.
[21] A. Papoulis, S.U. Pillai, Probability, Random Variables and Stochastic Processes, Mc Graw Hill, Boston, 2002.

[22] D.B. Percival, A.T. Walden, Spectral Analysis for Univariate Time Series, Cambridge University Press, Cambridge, 2020.
[23] J.S. Bendat, A.G. Piersol, Random Data: Analysis and Measurement Procedures, Wiley, New York, 1986.
[24] M.B. Priestly, Spectral Analysis and Time Series, Academic Press, London, 1981.
[25] A. Papoulis, The Fourier Integral and its Applications, Mc Graw Hill, New York, 1962.
[26] M.B. Priestley, Evolutionary spectra and non-stationary processes, J. Roy. Stat. Soc. B 27 (1965) 204–237.
[27] M.B. Priestley, Power spectral analysis of non-stationary random processes, J. Sound Vib. 6 (1967) 86–97.
[28] B. Boashash, Time-frequency Signal Analysis and Processing, Elsevier, Amsterdam, 2016.