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Combining Linear and Non-Linear Objectives in Spanning Tree Problems

by

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Abstract

A classical approach to multicriteria problems asks for the optimization of a suitable linear combination of the objectives. In this work we address such problems when one of the objectives is the linear function, the other is a non-linear one and we seek for a spanning tree of a given graph which optimizes the combination of the two functions. We consider both maximization and minimization problems and present the complexity status of 56 such problems, giving, whenever possible, polynomial solution algorithms.

1 Foreword

Combinatorial optimization problems with multiple objectives are often encountered in practice. One way of approaching such problems is provided by the linear (convex) combination of the different objectives into a single objective function, the different weights given to each objective corresponding to a kind of ranking of their relative importance in the opinion of the optimizer.

Obviously this does not change the nature of the problem when the objectives we combine are all linear, but this is not the case when some of them are non-linear. Let us observe that this linear combination is a classical way of approaching multicriteria optimization problems.

In this work we focus on one of the most basic combinatorial structures, namely the *spanning tree* of a given graph $G = (V, E)$. In each one of the problems considered, we assume that the objective function results from the convex combination of the usual linear objective, with respect to a given weighting of E, together with another objective which is non-linear in nature, with respect to a different weighting. Each problem is further characterized by the fact of being a maximization or a minimization one and by accepting weights in the set of all integers, $\mathtt{Z}$, or only in the set of non-negative integers, $\mathtt{N}$.

A companion paper addressing similar problems for directed instead of undirected graphs has been recently presented by the same authors [5]. Our starting point has been the set of NP-completeness results obtained long ago in [2, 3, 4] which considered most of the objectives we are interested to analyze, without combining them together.

Section 2 presents the general notation we use and the set of non-linear objectives we want to consider. In Sections 3 to 6 we study the problems, grouping them by "similar" objective functions. Finally an appendix is devoted to show that the optimization of cumulative function over a matroid is equivalent to the optimization of a particular linear function on a polymatroid.

2 Notation, problems and objectives

The recognition problems whose complexity status is addressed in this paper are formulated as follows.

Input:

- an undirected graph $G = (V, E)$;
- two weighings $w_i : E \to W, i = 1, 2, W \in \{\mathtt{N}, \mathtt{Z}\}$;
- a non-linear objective function z defined over the set $\mathcal{T}$ of spanning trees of G with E weighted by w_2;
- a positive rational $\lambda < 1$;
- a relation symbol $\Delta \in \{\leq, \geq\}$;
- a rational t.

Output: YES if there exists a spanning tree $T \in \mathcal{T}$ such that

$$f(T) := \lambda \sum_{e \in T} w_1(e) + (1 - \lambda)z \;\; \Delta \;\; t;$$

NO, otherwise.

In Table 1 we give the non-linear objective functions we are going to consider. For each function we report the mathematical formulation, the name and a short acronym. Note that we do not include in this table the four functions MAXROOTEDFLOW, MAXFLOW, SUMFLOW and SUMLEAF, discussed in [4], since finding an optimal spanning tree with only one of these objective function is already an NP-complete problem.

In the first column of this table $w(S)$ indicates the sum of the weights of the arcs in set $S \subseteq E$ (if S contains a single edge e we write for short $w(a)$ instead of

$w(\{e\})$; $\pi_\rho(i)$ denotes the set of arcs of T in the (unique) path from the root ρ to another vertex i of the graph, whereas $\pi(i,j)$ denotes the unique path of T between the two different vertices i and j; $f(e,T)$ ($f_\rho(e,T)$) indicates the *flow* (*rooted flow*) of edge e, i.e. the number of paths of T passing through edge e (having root ρ as terminal vertex); $\delta(e,T)$ is a binary variable set to 1 if edge e is incident to a leaf of T, and 0 otherwise; finally $\sigma(i)$ is the *cocycle* of $\{i\}$ that is the set of edges incident to vertex i in the given tree.

Table 1: The non-linear objective functions

Objective function	Name	Acronym
$\min_{e\in T}\ w(e)$	MINARC	ma
$\max_{e\in T}\ w(e)$	MAXARC	Ma
$\min_{e\in T}\ [w(e)\ \delta(e,T)]$	MINLEAF	ml
$\max_{e\in T}\ [w(e)\ \delta(e,T)]$	MAXLEAF	Ml
$\min_{e\in T}\ [w(e)f(e,T)]$	MINFLOW	mf
$\min_{e\in T}\ [w(e)\ f_\rho\ (e,T)]$	MINROOTEDFLOW	mrf
$\Sigma_{e\in T}\ [w(e)\ f_\rho(e,T)]$	SUMROOTEDFLOW	Σrf
$\min_{i,j\in N, i\neq j}\ w(\pi(i,j))$	MINPATH	mp
$\max_{i,j\in N, i\neq j}\ w(\pi(i,j))$	MAXPATH	Mp
$\min_{i\in N, i\neq \rho}\ w(\pi_\rho(i))$	MINROOTEDPATH	mrp
$\max_{i\in N}\ w(\pi_\rho(i))$	MAXROOTEDPATH	Mrp
$\min_{i\in N}\ w(\sigma(i))$	MINVALENCE	mv
$\max_{i\in N}\ w(\sigma(i))$	MAXVALENCE	Mv
$\sum_{e\in T}\ [w(e)p_{\phi(e)}]$	CUMULATIVE	Cum

The meaning of the objective functions is immediate, but for the CUMULATIVE function which requires an explanation.

The cumulative function has been introduced in [8] to formulate a lower bound for the Delivery Man Problem. A cumulative function is defined by the usual weighting of the edges plus a vector of n *penalties* $p_1, \ldots, p_n$. Then, a solution of a cumulative problem is given by a tree T and by a permutation ϕ which associates an integer in $\{1, \ldots, n\}$ to each edge of T. The value of the function is the product of the penalties vector times the weights of the edges of T, reordered accordingly to permutation ϕ. Without loss of generality we will assume in the following that the penalties are ordered so that

$$p_1 \geq p_2 \geq \ldots \geq p_n \tag{1}$$

Worth is noting that with this sorting, given a tree T, the permutation which minimizes the value of the cumulative function reorders the edges of T by nonincreasing weights. On the contrary the permutation which maximizes the value of the cumulative function reorders the edges by nondecreasing weights.

The complexity status of the problems with only one non-linear objective is reported in Table 2. We use a "∗" to indicate that the problem is solvable in polynomial time, and a "!" to denote that it is NP-complete. In the first column we report the name of the non-linear objective considered, whereas in the remaining columns, for a given pair (relation,weighting), we give the complexity of the problem and the reference where the result has been proved. If a "t" appears, instead of a reference, then the result can be trivially obtained; if a "–" appears, then the result is proved below in this section. If the weighting is in Z and no symbol nor reference is given, then the result immediately descends from the analogous result with weighting in N. Finally the results for the CUMULATIVE function are discussed in Section 6.

Table 2: Single non-linear objective function

	Status/notes							
Name	$\le$, N		$\ge$, N		$\le$, Z		$\ge$, Z	
MINARC	∗	t	∗	t	∗	t	∗	t
MAXARC	∗	[1]	∗	t	∗	t	∗	t
MINLEAF	∗	t	!	[4]	∗	t	!	
MAXLEAF	!	[4]	∗	t	!	[4]	∗	t
MINFLOW	∗	[4]	!	[4]	!	[4]	!	
MINROOTEDFLOW	∗	[4]	!	[4]	!	[4]	!	
SUMROOTEDFLOW	∗	[7]	!	[3]	!	[2]	!	
MINPATH	∗	t	∗	t	!	[4]	∗	[2]
MAXPATH	∗	[7]	!	[3]	!	[2]	!	
MINROOTEDPATH	∗	t	∗	t	!	[4]	!	–
MAXROOTEDPATH	∗	[7]	!	[3]	!	[2]	!	
MINVALENCE	∗	t	!	[4]	∗	t	!	
MAXVALENCE	!	[3]	∗	t	!		∗	–
CUMULATIVE	∗	[8]	∗		∗		∗	

In Table 3 we report the status of the problems when the combination of the linear function, with a non-linear objective is considered. The symbol "(!)" indicates that the problem with the single non-linear objective is already NP-complete, hence implying the same result for the problem with the combined function. The complexity status of the 30 remaining problems has been proved in this work.

Table 3: Combining linear and non-linear objective functions

	Status/notes			
Name	$\leq$, N	$\geq$, N	$\leq$, Z	$\geq$, Z
MINARC	$*$	$*$	$*$	$*$
MAXARC	$*$	$*$	$*$	$*$
MINLEAF	$*$	(!)	$*$	(!)
MAXLEAF	(!)	$*$	(!)	$*$
MINFLOW	!	(!)	(!)	(!)
MINROOTEDFLOW	!	(!)	(!)	(!)
SUMROOTEDFLOW	!	(!)	(!)	(!)
MINPATH	$*$	$*$	(!)	!
MAXPATH	!	(!)	(!)	(!)
MINROOTEDPATH	$*$	$*$	(!)	(!)
MAXROOTEDPATH	!	(!)	(!)	(!)
MINVALENCE	$*$	(!)	$*$	(!)
MAXVALENCE	(!)	$*$	(!)	$*$
CUMULATIVE	$*$	$*$	$*$	$*$

The NP-completeness results of this paper have been obtained through a polynomial-time transformation from one of the two following well-known NP-complete problems (see .e.g. [9]).

EXACT COVER BY 3-SETS (X3C)

Input:

– a set $S := \{s_1, \ldots, s_{3q}\}$ of $3q$ elements;

– a collection $C := \{c_1, \ldots, c_s\}$ of subsets of S of three elements each ($s > q$)

Output: YES if there exists an "exact cover" of S, that is a subset $\tilde{C}$ of C such that $\mid \tilde{C} \mid = q$ and each element of S is contained in one of the subsets of $\tilde{C}$.

NO, otherwise.

HAMILTONIAN PATH BETWEEN TWO VERTICES (HP2)

Input: a graph $G = (V, E)$, two vertices $r, s \in V$.

Output: YES if there exists an Hamiltonian path starting with vertex r and ending with vertex s.

NO, otherwise.

We conclude this section proving the complexity status of the two problems with single objective function, that have not been considered before.

Theorem 2.1 *Given a graph with edges weighted in* $\mathbb{Z}$ *and a rational* t*, then the problem of finding a spanning tree with value of the MINROOTEDPATH greater or equal to* t *is NP-complete.*

Proof. We use a transformation from HAMILTONIAN PATH BETWEEN TWO VERTICES. Given an instance of HP2 we add to the graph G two vertices, ρ and ν and the two edges (ρ, r) and (ν, s). We give weight one to all edges, except (ν, s) which is given weight $-$M, where M is a large positive number. Then we set $t = n - M$. The MINROOTEDPATH goes from the root ρ to vertex ν and it has a value greater or equal to t if and only if there is an Hamiltonian path from r to s, in graph G. □

Theorem 2.2 *Given a graph* $G = (V, E)$ *with edges weighted in* $\mathbb{Z}$ *and a rational* t*, then the problem of finding a spanning tree with value of the MAXVALENCE greater or equal to* t *is solvable in polynomial time.*

Proof. The optimal tree can be found with the following algorithm. We consider, in turn, each vertex $v \in V$ as a candidate to determine the MAXVALENCE value. Then we construct a spanning tree maximizing the valence of v, and finally we select, among the n trees obtained with the n choices of v, the one having maximum MAXVALENCE value.

Given the vertex v we begin to construct the tree by selecting all the edges (v, j), for $j \in V, j \neq v$, with nonnegative value. Then we use a greedy algorithm to try to complete the tree, without using any other edge incident to v. If we succeed in finding a spanning tree, than we are done; otherwise there are components of G that can be connected to the remaining of the graph only through vertex v. In this case we complete the tree by adding, for each component C, an edge with maximum weight which connects v to C. □

3 Bottleneck functions

In this section we consider problems in which the value of the non-linear objective function is equal either to the minimum weight or to the maximum weight of an edge. If the minimization (resp. maximization) is over all the edges of T, then the function is MINARC (resp. MAXARC). If instead the minimization (resp. maximization) is on the set of edges incident to a leaf, then the function is MINLEAF (resp. MAXLEAF). The complexity of the spanning trees with objective function depending on the leaves of the tree has been systematically studied in [6].

The problems in which the bottleneck edge is selected among all the edges of E are in P, as shown in the following.

Theorem 3.1 *Problems* $\langle \text{ma},\leq,\texttt{Z}\rangle$, $\langle \text{ma},\geq,\texttt{Z}\rangle$, $\langle \text{Ma},\leq,\texttt{Z}\rangle$ *and* $\langle \text{Ma},\geq,\texttt{Z}\rangle$ *are solvable in polynomial time.*

Proof. We first describe a polynomial algorithm for problem $\langle \text{ma},\leq,\texttt{Z}\rangle$, then we show how to modify the algorithm so that it can solve also the three remaining problems.

Let us suppose that we know the edge e having the minimum w_2 weight, in an optimum tree T^* (i.e. $w_2(e) = \min_{l\in T^*} w_2(l)$). Then we can determine T^* as follows. We remove from graph G all the edges $l \in E$ such that $w_2(l) < w_2(e)$, then we find a minimum cost tree T, with respect to the linear objective, with the additional constraint that $e \in T$. By construction edge e has the minimum w_2 weight, among the edges of T, and the linear objective is minimized, thus $T \equiv T^*$. (An immediate method for solving the above constrained tree problem is to initialize the set $T = \{e\}$, thus fixing e in the solution, then to complete the tree with the Greedy algorithm, (see e.g. [11]) applied with weighting function w_1.) Since we do not know 'a-priori' the edge which has the minimum w_2 weight in the optimal tree, then we apply the above procedure fixing, in turn, each edge of E. The overall time complexity of this naive algorithm is $O(|E|^2)$, so proving that $\langle \text{ma},\leq,\texttt{Z}\rangle$ is in P.

Problem $\langle \text{ma},\geq,\texttt{Z}\rangle$ can be solved with a similar algorithm, but constructing, in a greedy way, the tree which maximizes the linear objective function, and returning NO if no spanning tree has value larger or equal to t, and YES otherwise.

Problems $\langle \text{Ma},\leq,\texttt{Z}\rangle$ and $\langle \text{Ma},\geq,\texttt{Z}\rangle$ can be solved analogously, but we have to remove from G all the edges with weight w_2 larger than the weight of the fixed edge. □

From the above theorem it immediately descends the following.

Corollary 3.1 *Problems* $\langle \text{ma},\leq,\texttt{N}\rangle$, $\langle \text{ma},\geq,\texttt{N}\rangle$, $\langle \text{Ma},\leq,\texttt{N}\rangle$ *and* $\langle \text{Ma},\geq,\texttt{N}\rangle$ *are solvable in polynomial time.*

When we select the bottleneck edge in the set of the edges incident to a leaf, the complexity status changes with the problem. In order to establish the complexity of problems ⟨ml,≤,Z⟩ and ⟨Ml,≥,Z⟩ we need the following

Lemma 3.1 *Given an instance of* ⟨ml,≤,Z⟩ *(resp.* ⟨ML,≥,Z⟩*), if vertex i is a leaf of a tree T^* which minimizes (resp. maximizes) $f(T)$, and $e = (i, j)$ is the edge determining the MINLEAF (resp. MAXLEAF) value, then T^* can be determined by constructing the spanning tree of minimum (resp. maximum) value, with respect to the linear objective function, in the graph G' obtained from G removing all edges incident into i, but e.*

Proof. We give the proof only for problem ⟨ml,≤,Z⟩, but the proof for ⟨ML,≥,Z⟩ can be easily obtained with simple changes of our reasoning.

Let T be the tree determined by computing the optimal spanning tree of the modified graph G', with respect to the linear function. Since the edge e is imposed in the solution the value of the tree can be written as

$$f(T) = \lambda \min_{T'' \in G''} \sum_{l \in T''} w_1(l) + \lambda w_1(e) + (1-\lambda) \min_{l \in T} w_2(l)\delta(l, T) \tag{2}$$

where G'' is obtained from G' by removing edge e and vertex i. The value of the optimal tree is

$$f(T^*) = \lambda \sum_{l \in T^* \setminus \{e\}} w_1(l) + \lambda w_1(e) + (1-\lambda) w_2(e) \tag{3}$$

Comparing equations (2) and (3) we can see that the value of the first term in (2) is smaller or equal than the value of the first term in (3), whilst the values of the second terms are identical. If $\min_{l \in T} w_2(l)\delta(l, T) < w_2(e)$, then $f(T) < f(T^*)$ and T^* is not the optimal tree: a contradiction. On the other hand $\min_{l \in T} w_2(l)\delta(l, T)$ cannot be larger than $w_2(e)$, so the two values are equal and $f(T) \leq f(T^*)$, thus proving that T is an optimal tree for the complete objective function. □

Theorem 3.2 *Problems* ⟨ml,≤,Z⟩ *and* ⟨Ml,≥,Z⟩ *are solvable in polynomial time.*

Proof. Let us consider problem ⟨ml,≤,Z⟩ (resp. ⟨Ml,≥,Z⟩). Since we do not know the edge and the vertex determining the MINLEAF (resp. MAXLEAF) value, then we select, in turn, each edge $e = (h, k) \in E$. For a given edge we apply two times the algorithm of Lemma 3.1 imposing, respectively, that vertex h or k is a leaf. If no one of the spanning trees determined as above has value less or equal (resp. greater or equal) to t, then the answer is NO, otherwise the answer is YES. □

Theorem 3.3 *Problems* $\langle \text{ml},\geq,\texttt{N}\rangle$ *and* $\langle \text{Ml},\leq,\texttt{N}\rangle$ *are NP-complete.*

Proof. To show that the two problems are NP-complete we use a transformation from HAMILTONIAN PATH BETWEEN TWO VERTICES. Let graph $\hat{G} = (\hat{V}, \hat{E})$ and the two vertices $r \in \hat{V}$ and $s \in \hat{V}$, be the elements defining an instance of HP2. We construct an instance of $\langle \text{ml},\geq,\texttt{N}\rangle$ with graph G given by:

$$V := \hat{V} \cup \{r', s'\},$$
$$E := \hat{E} \cup \{(r, r'), (s, s')\},$$
$$w_1(e) := 1, \text{ for each } e \in E,$$
$$w_2(e) := \begin{cases} 2 & \text{if } e = (r, r') \text{ or } e = (s, s'), \\ 1 & \text{otherwise,} \end{cases}$$

then we set $\lambda := \frac{1}{2}$ and $t := \frac{n+3}{2}$.

Note that with this instance the contribution of the linear objective function is $\frac{n+1}{2}$ for any spanning tree of G, so only the MINLEAF objective can differentiate the value of the trees.

If there exists an Hamiltonian path from r to s, in $\hat{G}$, then there exists an Hamiltonian path from r' to s', in G. A path is a special tree with only two leaves (r' and s', in this case), so the MINLEAF value on this tree is 2 and the value of the objective function is $\frac{n+3}{2}$. Thus if $\hat{G}$ has the required path, then the answer to $\langle \text{ml},\geq,\texttt{N}\rangle$ is YES.

On the other hand note that any spanning tree T of G which is not Hamiltonian must have at least a leaf in $\hat{V}$. The MINLEAF value of tree T is 1 and $f(T) = \frac{n+2}{2}$. Therefore if $\hat{G}$ has no Hamiltonian path, from r to s, then no spanning tree of G has value at least equal to t and the answer is NO.

Problem $\langle \text{Ml},\leq,\texttt{N}\rangle$ can be proved to be NP-complete with a similar transformation. We use the same construction for sets V and E, and the same weights w_1. Instead, the weighting of the MAXLEAF objective is

$$w_2(e) := \begin{cases} 1 & \text{if } e = (r, r') \text{ or } e = (s, s'), \\ 2 & \text{otherwise,} \end{cases}$$

whilst the value of λ is set to $\frac{1}{2}$ and the target t is set to $\frac{n+2}{2}$. Following the above reasoning it is not difficult to see that if there is an Hamiltonian path form r to s, in $\hat{G}$, then there exist a spanning tree of G (namely a Hamiltonian path) with objective function value equal to t, so the answer is YES. If instead the required path of $\hat{G}$ does not exists, then no Hamiltonian path exists in G and any spanning tree must have at least a leaf in $\hat{V}$, thus the MAXLEAF value is 2 and the objective function value is strictly greater than t. It follows that the correct answer is NO. □

If a problem is NP-complete with weights restricted to be nonnegative integers, it obviously remains NP-complete if we assume that the weights are nonrestricted integers.

Corollary 3.2 *Problems* $\langle$ml,$\geq$,Z$\rangle$ *and* $\langle$Ml,$\leq$,Z$\rangle$ *are NP-complete.*

4 Flows and paths

Among the 28 problems with objective function depending on the flows and paths of the tree, 18 are already known to be NP-complete when the only non-linear objective is adopted. In this section we prove that when we combine these functions with the linear objective, then only four of the ten easy problems remain solvable in polynomial time.

Theorem 4.1 *Problem* $\langle$mrf,$\leq$,N$\rangle$ *is NP-complete.*

Proof. Given an instance of X3C we define a graph with the following rules (see Figure 1). The vertex set is

$$V := \{\rho, \alpha\} \cup V_S \cup V_C$$

where $V_S := \{v(s_j) : s_j \in S\}$ and $V_C := \{v(c_i) : c_i \in C\}$. The edge set is

$$E \ := \ \{(\rho, \alpha)\} \cup E_{CS} \cup E_{\rho C} \cup E_{\alpha C}$$

where $E_{CS} := \{(v(c_i), v(s_j)) : c_i \in C, s_j \in S, s_j \in c_i\}$, $E_{\rho C} := \{(\rho, v(c_i)) : c_i \in C\}$ and $E_{\alpha C} := \{(\alpha, v(c_i)) : c_i \in C\}$. The edges are given the weights

$$w_1(e) \ := \ \begin{cases} M & e \in E_{CS}, \\ 2 & e \in E_{\rho C}, \\ 0 & \text{otherwise}; \end{cases}$$
$$w_2(e) \ := \ \begin{cases} 1 & e = (\rho, \alpha), \\ M & \text{otherwise}; \end{cases}$$

where M is a large positive number. The rational λ is set to $\frac{1}{2}$ and the target is $t := (3qM + q + s + 1)/2$.

Observe that the big weight $w_1(e) = M$ for edges $e \in E_{CS}$ implies that only the minimum number of such edges is utilized in T thus imposing to all vertices of V_S to be leaves of T.

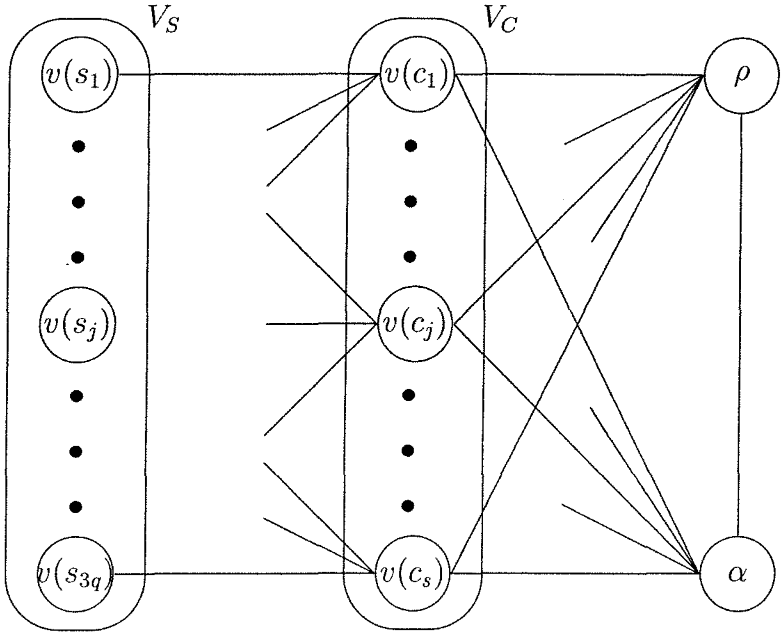

Figure 1: Reduction of X3C to ⟨mrf,≤,N⟩

Moreover observe that the particular weighting w_2 imposes that edge (ρ, α) belongs to the optimal tree, and the MINROOTEDFLOW value is achieved on this edge.

Let us define $X(T)$ as the set of edges of T incidents to ρ, minus edge (ρ, α) (i.e. $X(T) = \sigma(\rho) \cap T \setminus \{(\rho, \alpha)\}$), and define $Y(T)$ as the set of edges of $E_{CS} \cap T$ which belongs to a rooted path of length two (i.e. $(v(c_i), v(s_j)) \in Y(T) \Rightarrow (\rho, v(c_i)) \in T$). Note that the above definition of $Y(T)$ implies that any edge $E_{CS} \cup T \setminus Y(T)$ belongs to a rooted path of length three, which includes vertex α. Finally let us define $x = |X(T)|$ and $y = |Y(T)|$.

Given any optimal spanning tree T^*, the contribution to the objective function value due to the linear objective is $3qM + 2x$, whereas the contribution due to the non-linear objective is equal to the number of vertices in the subtree rooted at α, i.e. $1 + (s - x) + (3q - y)$. It follows that the objective function value is $f(T^*) = \frac{1}{2}(3qM + 3q + s + 1 + x - y)$. From the structure of the graph we know that $0 \leq x \leq s$, $0 \leq y \leq 3q$ and $y \leq 3x$, so relaxing the integrality constraints on the values of x and y, the pair (x, y) which minimizes $f(T^*)$ can be obtained by solving the continuous linear program

$$\min\{x - y : x, y \in \Re, x \leq s, y \leq 3q, y \leq 3x, x, y \geq 0\}, \tag{4}$$

It is not difficult to see that the unique optimal solution of (4) is $x^* = q, y^* = 3q$. It follows that if X3C has answer YES, then there exists a corresponding spanning tree T^* of G such that $(\rho, v(c_i)) \in T^*$ for each of the q subsets c_i in the solution of $X3C$. Moreover for each of these subsets the three edges $\{(v(c_i), v(s_j)) : s_j \in c_i\}$ belongs to the tree. The remaining vertices of V_C are connected to the root through paths of length two using vertex α. The value of this tree is exactly t and also the answer to ⟨mrf,≤,N⟩ is YES. If instead the answer to X3C is NO, then, from the analysis of (4), we know that any spanning tree has value strictly larger than t and also the answer to ⟨mrf,≤,N⟩ is NO. □

Theorem 4.2 *Problem ⟨mf,≤,N⟩ is NP-complete.*

Proof. We use a transformation from X3C similar to the previous one. Given an instance of X3C we define a graph obtained from that used in the proof of Theorem 4.1, by adding k new vertices and k edges each of which connects one of the new vertices to ρ. The weighings are:

$$w_1(e) \; := \; \begin{cases} M & e \in E_{CS}, \\ K & e \in E_{\rho C}, \\ 0 & \text{otherwise}; \end{cases}$$

$$w_2(e) \; := \; \begin{cases} 1 & e = (\rho, \alpha), \\ M & \text{otherwise} \; ; \end{cases}$$

where K is a large integer number such that $(K \bmod 4) = 0$, and $K << M$. This weighting imposes that each vertex of V_C is a leaf in any optimal tree an that the MINFLOW is achieved on edge (ρ, α) (which is the only one having a "small" w_2 weight). Using the notation introduced in the proof of Theorem 4.1 the objective function value associated to a tree T is

$$f(T) = f(x, y) = Kx + 3qM + (x + y + k + 1)(1 + s - x + 3q - y)$$

Considering the optimization version of ⟨mf,≤,N⟩, we need to find the minimum value of the quadratic concave function $f(T)$ for $T \in \mathcal{T}$. Reminding the bounds on the values of x and y (see again Theorem 4.1), the continuous relaxation of the problem is $\min\{f(x, y) : x, y \in P\}$ where

$$P = \{(x, y) \in \Re^2 : x \leq s, y \leq 3q, y \leq 3x, x, y \geq 0\}$$

Since f is concave the optimal solution is achieved at one of the four vertices of P (namely (0,0), (s,0), (q,3q) and (s,3q)).

If we define the number of new vertices $k = K/4 + s - q + 1$, with algebraic manipulations one can see that the vertex $x = q, y = 3q$ determines an objective function value strictly smaller than that of the other three vertices. Therefore if we let $\lambda := 1/2$ and $t := f(q, 3q)/2$, then the answer to $\langle$mf,$\leq$,N$\rangle$ is YES if and only if the answer to X3C is YES too. □

Theorem 4.3 *Problem* $\langle\Sigma$rf,$\leq$,N$\rangle$ *is NP-complete.*

Proof. We use the same transformation from X3C adopted for $\langle$mrf,$\leq$,N$\rangle$, but with weighting

$$w_1(e) := \begin{cases} 2 & e \in E_{\rho C}, \\ 0 & \text{otherwise}; \end{cases}$$
$$w_2(e) := \begin{cases} 1 & e \in E_{\alpha C}, \\ M & e \in E_{CS}, \\ 0 & \text{otherwise}\ ; \end{cases}$$

where M is a large positive number. The rational λ is set to $1/2$ and the target is $t := (3qM + q + s)/2$. Again the large value M given to the w_2 weights for the edges in E_{CS} imposes that in any optimal solution the vertices of V_S are leaves. Using the same definition of x and y introduced in the proof of Theorem 4.1 we see that the contribution to the objective function value due to the linear objective is $2x$, whereas the contribution due to the SUMROOTEDFLOW objective is $3qM + s - x + 3q - y$, thus the value of an optimal spanning tree T^* is $f(T^*) = \frac{1}{2}(3qM + 3q + s + x - y)$. Relaxing the integrality constraints on x and y we can see that the minimum feasible value of $f(T^*)$ is equal to t and it is obtained at $x^* = q, y^* = 3q$. It follows that the X3C instance and the instance of $\langle\Sigma$rf,$\leq$,N$\rangle$ have always the same solution. □

The following results consider the combination of the linear objective with the maximum rooted path or with the maximum path of T.

Theorem 4.4 *Problems* $\langle$Mrp,$\leq$,N$\rangle$ *and* $\langle$Mp,$\leq$,N$\rangle$ *are NP-complete.*

Proof. The same graph utilized in Theorem 4.1 reduces X3C to $\langle$Mrp,$\leq$,N$\rangle$ if one uses the following weights.

$$w_1(e) := \begin{cases} 1 & e \in E_{\rho C}, \\ 0 & \text{otherwise}; \end{cases}$$
$$w_2(a) := \begin{cases} 0 & e \in \sigma(\rho), \\ M & \text{otherwise}. \end{cases}$$

where M is a large positive number. The rational λ is set to $\frac{1}{2}$ and the target is $t := (M+q)/2$. The objective function results from the sum of a term equal to x, the number of vertices of V_C connected to ρ, plus either $2M$ or M depending from the fact that there is in T a path from ρ to a vertex of V_S going through α or not. Obviously the minimum is obtain with the second choice (MAXROOTEDPATH equal M) and with the smallest possible x, i.e. if $x = q$, thus identifying the cover.

To see that also ⟨Mp,≤,N⟩ is NP-complete one must only reduce to it the previous problem, by the addition of another vertex β connected only to ρ and having $w_1 = 0$ and $w_2 = M$. □

By changing the sign of the weights in the construction utilized for Theorem 4.4 one can easily prove the result of the following corollary.

Corollary 4.1 *Problem ⟨mp,≥,Z⟩ is NP-complete.*

We now turn our attention to problems which are solvable in polynomial time.

Theorem 4.5 *Problems ⟨mp,≤,N⟩ and ⟨mp,≥,N⟩ can be solved in polynomial time.*

Proof. If the weighting is in N no edge has a negative weight, so the path with minimum value is a single arc and MINPATH equal MINARC. Since we already know (Theorem 3.1) that ⟨ma,≤,N⟩ and ⟨ma,≥,N⟩ are polynomially solvable, then the thesis holds. □

Theorem 4.6 *Problems ⟨mrp,≤,N⟩ and ⟨mrp,≥,N⟩ can be solved in polynomial time.*

Proof. Due to the weighting in N the MINROOTEDPATH is a single edge belonging to $\sigma(\rho)$. Therefore we can solve the two problems by fixing, in turn, an edge $e \in \sigma(\rho)$ in the solution, removing all other edges $l \in \sigma(\rho)$ with $w_2(l) < w_2(e)$ and completing the tree with a Greedy algorithm. Among the $|\sigma(\rho)|$ trees generated we select that having minimum (resp. maximum) objective function value, thus solving the optimization version of ⟨mrp,≤,N⟩ (resp. ⟨mrp,≥,N⟩). □

5 Valence

The complexity status of the problems with non-linear objective function depending on the valence of the vertices (MINVALENCE and MAXVALENCE) is similar to that of the problems with bottleneck function depending on the leaves (MINLEAF and MAXLEAF).

Theorem 5.1 *Problems* $\langle$mv,$\leq$,Z$\rangle$, $\langle$Mv,$\geq$,Z$\rangle$ *are solvable in polynomial time.*

Proof. Consider the optimization version of problem $\langle$mv,$\leq$,Z$\rangle$: our goal is to find the spanning tree which minimizes $f(T)$. More precisely we want to find

$$f(T^*) = \min_{T \in \mathcal{T}}\{f(T)\} = \min_{T \in \mathcal{T}}\{\lambda \sum_{e \in T} w_1(e) + (1-\lambda) \min_{v \in V} w_2(\sigma_v)\} \tag{5}$$

Equation (5) can be rewritten as

$$\begin{aligned} f(T^*) &= \min_{v \in V} \min_{T \in \mathcal{T}}\{\lambda \sum_{e \in T} w_1(e) + (1-\lambda) w_2(\sigma_v)\} \\ &= \min_{v \in V} \min_{T \in \mathcal{T}}\{\lambda \sum_{e \in T \setminus \sigma_v} w_1(e) + \lambda w_1(\sigma_v) + (1-\lambda) w_2(\sigma_v)\} \end{aligned} \tag{6}$$

Given a vertex $v \in V$ and a spanning tree T of G, if we define

$$w(e) = \begin{cases} \lambda w_1(e) & e \in E \setminus \sigma_v \\ \lambda w_1(e) + (1-\lambda) w_2(e) & e \in \sigma_v \end{cases}$$

then the value computed in (6), for a given vertex v and tree T, is equal to the sum of the edge weights, when the weighting w is adopted. It follows that we can find an optimum tree T^* by considering, in turn, a vertex $v \in V$, defining the corresponding weights w, determining a spanning tree which minimizes the sum of the edge weights, and choosing, among the $|V|$ trees determined, the one with minimum objective function value. We have thus solved in polynomial time the optimization version of $\langle$mv,$\leq$,Z$\rangle$, so also the decision problem is in P.

The second problem $\langle$Mv,$\geq$,Z$\rangle$ can be solved with a similar algorithm which maximizes the objective function instead of minimizing. □

Corollary 5.1 *Problems* $\langle$mv,$\leq$,N$\rangle$ *and* $\langle$Mv,$\geq$,N$\rangle$ *are solvable in polynomial time.*

The complexity status changes for the remaining problems with valence dependent objective function.

Theorem 5.2 *Problems* $\langle$mv,$\geq$,N$\rangle$ *and* $\langle$Mv,$\leq$,N$\rangle$ *are NP-complete.*

Proof. To prove that $\langle$mv,$\geq$,N$\rangle$ is NP-complete we use the same transformation of Theorem 3.3. Given an instance of HP2 we construct the graph G as in the proof of Theorem 3.3. If HP2 has answer YES, then there exists an Hamiltonian path in G, from r' to s' (which is also a spanning tree of G). On this tree the valence is 2 for all vertices, but for r and s, at which it is associated the value 3, so

MINVALENCE $= 2$. The objective function value of the tree is $\frac{n+3}{2}$ and the target value is reached, thus also $\langle$mv,$\geq$,N$\rangle$ has answer YES. If HP2 has answer NO, then any spanning tree of G must have at least a leaf in $\hat{V}$. For this leaf the valence has value 1, thus the objective function value is $\frac{n+2}{2}$ and also the answer to the tree problem is NO.

The transformation given in the proof of Theorem 3.3 for problem $\langle$Ml,$\leq$,N$\rangle$ also applies to problem $\langle$Mv,$\leq$,N$\rangle$, but with target $t = \frac{n+5}{2}$. When an Hamiltonian path exists in G (so the answer to HP2 is YES), then the MAXVALENCE is achieved at a vertex $v \in V \backslash \{r, r', s, s'\}$ and its value is 4, thus the objective function value is $\frac{n+5}{2}$. If instead G is not Hamiltonian (so the answer to the HP2 is NO), then at least a vertex in $V \backslash \{r', s'\}$ must have more than two edges of the tree incident in it, so the MAXVALENCE is at least 5. It follows that the objective function value is strictly greater than t and both problems have answer NO. □

Corollary 5.2 *Problems* $\langle$mv,$\geq$,Z$\rangle$ *and* $\langle$Mv,$\leq$,Z$\rangle$ *are NP-complete.*

6 Cumulative function

The optimization version of problem $\langle$Cum,$\leq$,N$\rangle$ was first introduced in [8] to obtain a lower bound adopted, among others, in an effective branch-and-bound algorithm for the Delivery Man Problem. In [8] it is also shown that the greedy algorithm solves the problem. More specifically, it is well known that the spanning trees of a given graph G are the bases of a graphic matroid $M = (E, \mathcal{F})$ in which the ground set coincides with the edge set of G, and a subset $S \subseteq E$ is an independent set (i.e. $S \in \mathcal{F}$), iff it does not contain any circuit. Given a cost function $w : E \to \Re$, finding the minimum cost spanning tree of G is equivalent to find the minimum cost basis of M. This can be done with the following algorithm.

Greedy algorithm

(i) Choose an element $e_1 \in E$ such that $w(e_1)$ is a minimum;

(ii) assuming that $\{e_1, \ldots, e_i\}$ are chosen, find e_{i+1} such that $\{e_1, \ldots, e_{i+1}\} \in \mathcal{F}$ and $w(e_{i+1})$ is a minimum;

(iii) repeat (ii) until no such e_{i+1} exists.

The authors of [8] prove that the solution obtained with the greedy algorithm, considering only the linear weighting w, is also optimal for the cumulative case if we associate to the i-th element chosen the i-th penalty.

It is not difficult to see that ⟨Cum,≥,N⟩ can be solved with a greedy algorithm which, at each iteration, looks for the the maximum weight edge.

The same algorithms finds the optimal solution if the edge weights are unrestricted integers or real numbers. In the appendix we generalize the problem considering a generic matroid, instead of a graphic matroid, and we show that the above algorithm still works and that it is an implementation of the standard greedy algorithm for a particular polymatroid.

Unfortunately when the cumulative function is combined with the linear function the greedy fails in finding the optimal solution. In order to determine the complexity status of the problems with the CUMULATIVE objective we need to introduce a new model for the problem.

Given an instance of ⟨Cum,≤,Z⟩ (i.e. the graph $G = (V, E)$, the two edge weighings w_1 and w_2, the vector p of the cumulative penalties and the rational λ and t) we associated to this instance a *multigraph* $\tilde{G} = (\tilde{V}, \tilde{E})$ where $\tilde{V} = V$, and $\tilde{E}$ contains the n edges $(i,j)_1, \ldots, (i,j)_n$, for each $(i,j) \in E$, with weights

$$\tilde{w}(i,j,k) = \lambda w_1(i,j) + (1-\lambda)p_k w_2(i,j) \quad (i,j) \in E, k = 1, \ldots, n \tag{7}$$

Roughly speaking we can say that $\tilde{G}$ is obtained from G duplicating each edge n times and associating to the k-th copy the weight that it will have if it would be the k-th smaller edge of a spanning tree.

Note that the edge set of the multigraph is partitioned into n sets $\tilde{E}_1, \ldots, \tilde{E}_n$ each of which is identical to E, but has associated a weighting depending on the same cumulative penalty, i.e. $\tilde{E}_k = \{(i,j)_k \in \tilde{E}\}$ for $k = 1, \ldots, n$.

Problem ⟨Cum,≤,Z⟩ can be reformulated as follows:

(P') find a minimum weight tree T^* of $\tilde{G}$, such that not two edges of T^* belong to the same edge set $\tilde{E}_k$, for $k = 1, \ldots, n$.

If the solution to P' contains an edge $(i,j)_k \in \tilde{E}$, then the corresponding tree of G contains edge $(i,j) \in E$. The fact that we have chosen the k-th of the edges of $\tilde{E}$ having end vertices i and j, implies that we want $(i,j) \in E$ to be assigned the k-th penalty. Thus solving P' is equivalent to find a spanning tree T of G and to give a ranking to the edges of this tree.

Each feasible solution T to P' must satisfy two conditions which correspond to two different matroids defined on the same ground set:

(a) T must be acyclic, so we define a graphic matroid $M_1 = (E, \mathcal{F}_1)$ where $S \subseteq E$ is in $\mathcal{F}_1$ iff it contains no cycle;

(b) T contains one edge for each set $\tilde{E}_k$, for $k = 1, \ldots, n$, so we define a partition matroid $M_2 = (E, \mathcal{F}_2)$ where $S \subseteq E$ is in $\mathcal{F}_2$ iff for each pair of elements

$e, l \in S$ with $e \in \tilde{E}_a$, $l \in \tilde{E}_b$, it is $a \neq b$.

Problem P' is then a *weighted matroid intersection* problem which can be solved in polynomial time (see e.g. [11]). We have thus proved the following.

Theorem 6.1 *Problems* $\langle$Cum,$\leq$,N$\rangle$, $\langle$Cum,$\geq$,N$\rangle$, $\langle$Cum,$\leq$,Z$\rangle$ *and* $\langle$Cum,$\geq$,Z$\rangle$ *are solvable in polynomial time.*

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Appendix

In this section we generalize the result presented at the beginning of Section 6, showing that the optimization of a cumulative function on a matroid is equivalent to the optimization of a linear function on a particular polymatroid.

Let $M = (E, \mathcal{F})$ be a given matroid in which E is the *ground set* and $\mathcal{F}$ is the family of the *independent sets*. Moreover let $r : 2^E \rightarrow \mathrm{Z}$ be the rank function of M and let $w : E \rightarrow \mathrm{Z}$ be a weighting function. Finally assume that the elements of E are ordered in such a way that $w(e_1) \geq w(e_2) \geq \ldots, \geq w(e_m)$.

Let us define the following 'cumulative' optimization problem

$$(P1) \qquad \max\{\sum_{e \in B} w(e) p_{\phi(e)} : B \in \mathcal{F}, |B| = r(E)\}$$

where ϕ is a relation which associates the index 1 to the element of B with largest weight, the index 2 to the element with second largest weight etc.

In order to associate a polymatroid to problem P1, let us introduce the function

$$f(S) = \sum_{i=1}^{r(S)} p_i, \quad S \subseteq E \tag{8}$$

The following theorem holds.

Theorem 6.2 *Function f is submodular.*

Proof. By definition a function is submodular if $f(S \cap T) + f(S \cup T) \leq f(S) + f(T)$ for all $S, T \subseteq E$. Thus we have to prove that

$$\sum_{i=1}^{r(S \cap T)} p_i + \sum_{i=1}^{r(S \cup T)} p_i \leq \sum_{i=1}^{r(S)} p_i + \sum_{i=1}^{r(T)} p_i \text{ for all } S, T \subseteq E \tag{9}$$

Function r is a rank function, so we know that $r(S \cap T) \leq \min(r(S), r(T))$ and $r(S \cup T) \geq \max(r(S), r(T))$. Thus if we subtract $\sum_{i=1}^{r(S \cap T)} p_i$ and $\sum_{i=1}^{r(T)} p_i$ from (9) we obtain

$$\sum_{i=r(T)+1}^{r(S \cup T)} p_i \leq \sum_{i=r(S \cap T)+1}^{r(S)} p_i \tag{10}$$

Since r is submodular, then the number of elements considered in the left-hand-side of (9) is smaller or equal than the number of elements considered in the right-hand-side, and the same holds for (10). Moreover we know that $r(T) \geq r(S \cap T)$, so remembering that the penalties p_i are ordered by non-increasing values we obtain that the value of the left-hand-side of (10) is smaller or equal than the value of the right-hand-side, and the thesis follows. □

Since function f is submodular it can be used to define the *cumulative polymatroid*

$$P_{cum} = \{x \in \Re^E : x(S) \leq \sum_{i=1}^{r(S)} p_i \ \text{ for all } \ S \subseteq E, x \geq 0\}, \tag{11}$$

and the associated linear problem

$$(P2) \qquad \max\{wx : x \in P_{cum}\} \tag{12}$$

It is not difficult to see that the optimal solution value of problem P2 is an upper bound on the optimal solution value of problem P1.

Note that since f assumes only integer values, then any vertex of P_{cum} is integral (see e.g. chapter 10 in [10]). Function f has another nice property: it is monotone (i.e. $S \subseteq T \Rightarrow f(S) \leq f(T)$), therefore the optimal solution can be determined by reordering the elements of E in such a way that $w(e_1) \geq \ldots \geq w(e_m)$ and defining the optimal vector with the recursion (see again [10]):

$$x^*(e_1) := f(e_1); \quad x^*(e_j) := f(\{e_1, \ldots, e_j\}) - f(\{e_1, \ldots, e_{j-1}\}), \quad \text{for } j = 2, \ldots, m.$$

Let $E_j = \{e_1, \ldots, e_j\}$, for $j = 1, \ldots, m$, then using the definition of f we can rewrite the recursion as:

$$x^*(e_1) := p_i; \quad x^*(e_j) := \sum_{i=1}^{r(E_j)} p_i - \sum_{i=1}^{r(E_{j-1})} p_i, \quad \text{for } j = 2, \ldots, m \tag{13}$$

The difference between $r(E_j)$ and $r(E_{j-1})$ is either equal to zero or one. It has value zero if the bases of E_{j-1} and those of E_j have the same cardinality; it has value one if the bases of E_j have exactly one more element that those of E_{j-1}. Therefore given an element $e \in E$ then $x^*(e)$ is either equal to zero or to a certain penalty p_i.

Further note that if $r(E_{j-1}) \neq r(E_j)$ then: (a) the element e_j must belong to any basis of E_j; and (b) given any basis B of E_{j-1} the set $B \cup \{e_j\}$ is a basis of E_j. From this observations we can derive an immediate greedy-like procedure which implements the above recursion.

Algorithm *greedy2*
$B := \{e_1\};\ i := 1;\ x^*(e_1) := p_1;$
for $j := 2$ **to** m **do**
  if $B \cup \{e_j\} \in \mathcal{F}$ **then**
    $B := B \cup \{e_j\};\ i := i + 1;\ x^*(e_j) := p_i$
  else $x^*(e_j) := 0;$
endfor

Given the optimal solution x^* of P2, we know that wx^* is an upper bound on the optimal solution value of P1, but due to the particular values of x^* it is $wx^* = \sum_{e \in B} w(e) p_{\phi(e)}$, where B is the basis of M identified by algorithm *greedy*2 and ϕ is the relation introduced in the definition of (P1). It follows that wx^* is also the optimal solution value of problem P1 and that B is the optimal basis. We have thus proved the following.

Theorem 6.3 *Algorithm greedy2 determines an optimal solution both for problem P1 and P2, and the two solution values coincide.*

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