Applied Mathematics Letters 132 (2022) 108203

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Study of a homoclinic canard explosion from a degenerate center

Bo-Wei Qin [a,1], Kwok-Wai Chung [b,1], Antonio Algaba [c,1], Alejandro J. Rodríguez-Luis [d,*,1]

[a] *Research Institute of Intelligent Complex Systems, Fudan University, Shanghai 200433, China*
[b] *Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong*
[c] *Departamento de Ciencias Integradas, Centro de Estudios Avanzados en Física, Matemática y Computación, Universidad de Huelva, 21071 Huelva, Spain*
[d] *Departamento de Matemática Aplicada II, E.T.S. Ingeniería, Universidad de Sevilla, 41092 Sevilla, Spain*

ARTICLE INFO

Article history:
Received 16 February 2022
Received in revised form 15 May 2022
Accepted 15 May 2022
Available online 23 May 2022

Keywords:
Canard
Singularly perturbed system
Asymptotic expansion
Degenerate center
Homoclinic connection

ABSTRACT

Canard explosion is an appealing event occurring in singularly perturbed systems. In this phenomenon, upon variation of a parameter within an exponentially small range, the amplitude of a small limit cycle increases abruptly. In this letter we analyze the canard explosion in a limit cycle related to a degenerate center (with zero Jacobian matrix). We provide a second-order approximation of the critical value of the parameter for which the canard explosion occurs. Numerical results are compared with the analytical predictions and excellent agreements are found. As in this problem the canard explosion ends in a homoclinic connection, a very good approximation for the homoclinic curve in the parameter plane is also obtained.

1. Introduction

In dynamical systems with multiple time scales, canard explosion alludes to a sudden transition between a small amplitude limit cycle (canard cycle) and a relaxation oscillation (see, for instance, [1–8] and references therein). An important problem is the determination of the critical value of the parameter for which the canard explosion occurs. One efficient way for its computation is the asymptotic expansion method [9–11].

When the canard cycle emerges from a Hopf bifurcation, the problem is well studied and understood under generic conditions (in this case there is an algebraic solution) [1,5,9]. However, when some non-generic conditions occur, there is no explicit expression for the first integral and the computations become harder [10]. The situation is more complicated when the canard cycle appears from a nilpotent center [12]. In this case, the period of the emerging orbit becomes unbounded [2]. A new situation corresponds to the

* Corresponding author.
E-mail addresses: boweiqin@fudan.edu.cn (B.-W. Qin), makchung@cityu.edu.hk (K.-W. Chung), algaba@uhu.es (A. Algaba), ajrluis@us.es (A.J. Rodríguez-Luis).
[1] All the authors contributed equally to this work.

https://doi.org/10.1016/j.aml.2022.108203

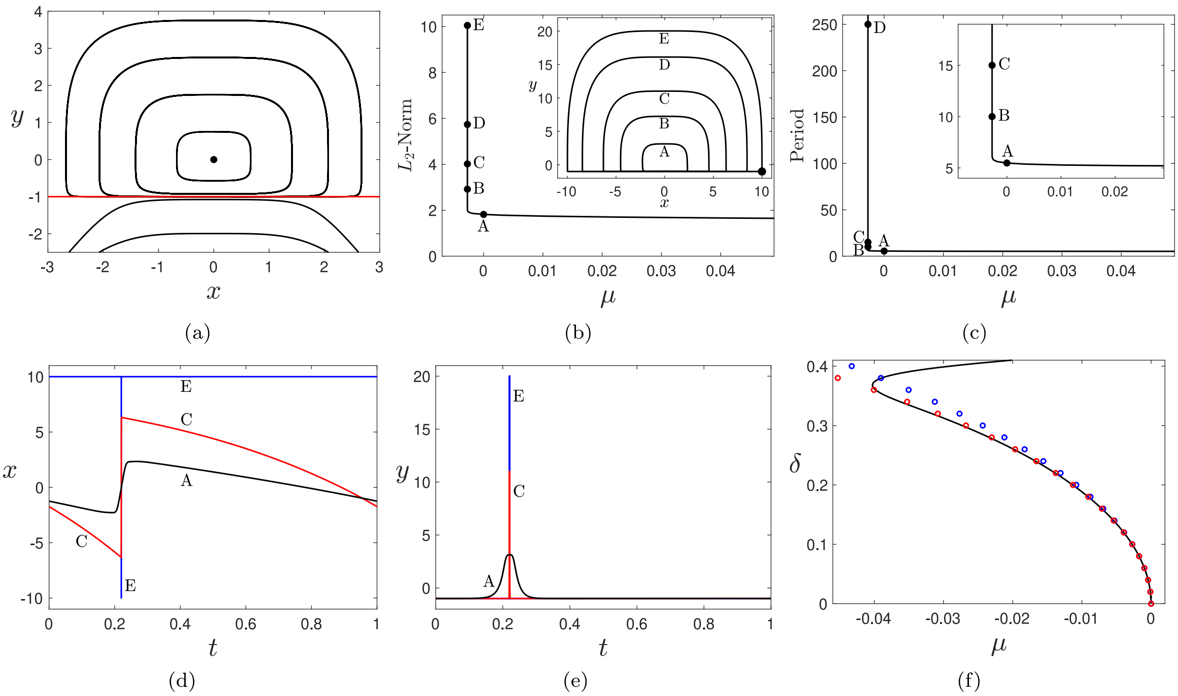

Fig. 1. (a) Phase portrait of the unperturbed system (1). The straight line $y = -1$ (in red) corresponds to a heteroclinic connection at infinity. For $\delta = 0.1$ and $a = 1$ in system (2): (b) bifurcation diagram L_2-Norm versus μ near the canard explosion which occurs when $\mu \approx -2.73441 \cdot 10^{-3}$. Four limit cycles (labeled A-D) illustrate the very rapid growth of the amplitude which is limited by a homoclinic connection (E) to the equilibrium marked with a bullet; (c) bifurcation diagram Period versus μ. In this case, the period tends to infinity since a homoclinic orbit, E, exists; (d) and (e) temporal profiles of orbits A (black), C (red) and E (blue), where period is normalized to 1. (f) Curve of the homoclinic connections in the (μ, δ)-parameter plane, when $a = 1$. The circles correspond to the analytical approximations (first-order in blue, second-order in red) provided in Theorem 1.

presence of a degenerate center (with zero Jacobian matrix). The aim of this letter is to provide a first example in the literature where the unperturbed system has a degenerate center. Specifically, we will consider

$$\dot{x} = y^3, \qquad \dot{y} = -x^3(1 + y). \tag{1}$$

This system has a degenerate center at the origin (surrounded by a continuum of periodic orbits which rotate clockwise) and the straight line $y = -1$ is a heteroclinic connection at infinity (see Fig. 1(a)).

We choose a simple perturbation of system (1) to analyze the canard explosion, namely

$$\dot{x} = y^3 + \delta x := P(x, y, \delta), \qquad \dot{y} = -x^3(1 + y) + \delta\mu + \delta^2 a x := Q(x, y, \delta, \mu). \tag{2}$$

Note that the scaling

$$X = \epsilon^{-1/4}x, \quad Y = y, \quad \tau = \epsilon^{3/4}t, \tag{3}$$

where $\delta = \epsilon^{3/4}$, $0 < \delta \ll 1$, transforms system (2) into the singularly perturbed system

$$\epsilon\frac{\mathrm{d}X}{\mathrm{d}\tau} = Y^3 + \epsilon X, \qquad \frac{\mathrm{d}Y}{\mathrm{d}\tau} = -X^3(1 + Y) + \mu + \epsilon a X. \tag{4}$$

The main result of this work is the following theorem that provides an approximation to the critical parameter value for which the canard explosion occurs in system (2) (and also in the singularly perturbed system (4)).

Theorem 1. *A canard explosion occurs in system* (2) *when* $\mu = \mu_2\delta^2 + \mu_4\delta^4 + \mathcal{O}(\delta^6)$, *where* μ_2 *and* μ_4 *are given in Eqs.* (11) and (25), *respectively. In system* (4) *it occurs for* $\mu = \mu_2\epsilon^{3/2} + \mu_4\epsilon^3 + \mathcal{O}(\epsilon^{9/2})$.

The rest of the letter is organized as follows. Section 2 is devoted to prove Theorem 1. Numerical results, which illustrate very good agreements with the theoretical predictions, are shown in Section 3. Finally, some conclusions are included.

2. Asymptotic expansion

The goal of this section is to demonstrate Theorem 1. First, eliminating the time variable, we obtain $Q(x,y,\delta,\mu) - y'(x)P(x,y,\delta) = 0$, where $' := \mathrm{d}/\mathrm{d}x$. We look for a solution in the following form

$$y(x) = \sum_{i=0}^{\infty} \delta^i y_i(x), \qquad \mu = \sum_{i=0}^{\infty} \delta^i \mu_i. \tag{5}$$

Then, for the zero-order solution, we have $y_0 = -1$, which is actually the critical manifold corresponding to the canard explosion.

In each order, we can obtain the corresponding linear equation as follows

$$y_i'(x) - x^3 y_i(x) + \mu_{i-1} = R_i(x), \tag{6}$$

of which $R_i(x)$ comprises all terms determined in preceding orders and $u(x) = e^{-x^4/4}$ is an integrating factor satisfying $u' = -ux^3$. Then, (6) can be rewritten as

$$(uy_i)' + u\mu_{i-1} = uR_i. \tag{7}$$

Thus, looking for solutions satisfying $\lim_{x\to\pm\infty} \frac{y_i(x)}{e^{|x|}} = 0$, the values of μ_{i-1} and $y_i(x)$ are uniquely found as

$$\mu_{i-1} = \frac{\int_{-\infty}^{\infty} u(x)R_i(x)\mathrm{d}x}{\int_{-\infty}^{\infty} u(x)\mathrm{d}x}, \qquad y_i(x) = \frac{1}{u(x)}\int_{-\infty}^{x} u(s)\left[R_i(s) - \mu_{i-1}\right]\mathrm{d}s. \tag{8}$$

Before finding the exact value of μ_{i-1}, we first define the following improper integral

$$I_n = \int_{-\infty}^{\infty} x^n u\,\mathrm{d}x,\ \ n \in \mathbb{N}, \tag{9}$$

that satisfies $I_n = [1 + (-1)^n]\, 2^{\frac{n-3}{2}}\, \Gamma\left(\frac{n+1}{4}\right)$. Note that, $I_n = 0$ if n is an odd number.

From the first-order equation, $y_1' - x^3 y_1 + \mu_0 = 0$, we obtain $\mu_0 = 0$ and $y_1(x) = 0$.

Considering the second-order equation, $y_2' - x^3 y_2 + \mu_1 = -ax$, we deduce that $\mu_1 = 0$ and $(uy_2)' = -aux$. In later calculations it will be useful to define $T_{m,n} = \int_{-\infty}^{\infty} x^m u y_n \mathrm{d}x$ for $m \geq 0$ and $n \geq 2$. For $n = 2$,

$$T_{m,2} = \frac{1}{m+1}\int_{-\infty}^{\infty} uy_2 \mathrm{d}x^{m+1} = -\frac{1}{m+1}\int_{-\infty}^{\infty} x^{m+1}(-aux)\,\mathrm{d}x \ \Rightarrow\ T_{m,2} = \frac{a}{m+1} I_{m+2}. \tag{10}$$

Therefore, $T_{m,2} = 0$ for odd m.

The analysis of the third-order equation allows to determine the first non-zero μ_i coefficient:

$$y_3'(x) - x^3 y_3(x) + \mu_2 = xy_2' \ \Rightarrow\ (uy_3)' = u\left(xy_2' - \mu_2\right) \ \Rightarrow\ \mu_2 I_0 = \int_{-\infty}^{\infty} xu\left(x^3 y_2 - ax\right)\mathrm{d}x$$

$$\Rightarrow \mu_2 = \frac{(T_{4,2} - aI_2)}{I_0} = -a\frac{2\sqrt{2}}{5\pi}\left[\Gamma\left(\frac{3}{4}\right)\right]^2 \approx -0.2703912958a. \tag{11}$$

Now, for $n = 3$,

$$T_{m,3} = -\frac{1}{m+1}\int_{-\infty}^{\infty} x^{m+1}(uy_3)'\,\mathrm{d}x = -\frac{1}{m+1}\int_{-\infty}^{\infty} x^{m+1}u\left[x\left(x^3y_2 - ax\right) - \mu_2\right]\mathrm{d}x$$
$$= \frac{1}{m+1}\left(aI_{m+3} + \mu_2 I_{m+1} - T_{m+5,2}\right) = \frac{1}{m+1}\left(aI_{m+3} + \mu_2 I_{m+1} - \frac{a}{m+6}I_{m+7}\right). \tag{12}$$

Considering the fourth-order equation, $y_4'(x) - x^3y_4(x) + \mu_3 = xy_3' + 3y_2y_2'$, we obtain $\mu_3 = 0$ and $(uy_4)' = u\left(xy_3' + 3y_2y_2'\right)$.

Now we define $S_m = \int_{-\infty}^{\infty} x^m u y_2^2 \mathrm{d}x$ with $m \geq 0$ for convenience. Then,

$$\begin{aligned}
T_{m,4} &= -\frac{1}{m+1}\int_{-\infty}^{\infty} x^{m+1}(uy_4)'\,\mathrm{d}x \\
&= -\frac{1}{m+1}\int_{-\infty}^{\infty} x^{m+1}u\left[x\left(x^3y_3 - \mu_2 + x\left(x^3y_2 - ax\right)\right) + 3y_2\left(x^3y_2 - ax\right)\right]\mathrm{d}x \\
&= \frac{1}{m+1}\left(aI_{m+4} + \mu_2 I_{m+2} + 3aT_{m+2,2} - T_{m+6,2} - T_{m+5,3} - 3S_{m+4}\right) \\
&= \frac{1}{m+1}\left[\frac{a}{(m+6)(m+11)}I_{m+12} - \frac{(2m+13)a}{(m+6)(m+7)}I_{m+8} - \frac{\mu_2}{m+6}I_{m+6}\right. \\
&\quad \left. + \left(1 + \frac{3a}{m+3}\right)aI_{m+4} + \mu_2 I_{m+2} - 3S_{m+4}\right].
\end{aligned} \tag{13}$$

The analysis of the fifth-order equation, after some laborious calculations, leads to the value of μ_4.

$$\begin{aligned}
&y_5'(x) - x^3y_5(x) + \mu_4 = 3y_2y_3' + 3y_3y_2' + xy_4' \quad\Rightarrow\quad (uy_5)' = u\left(3y_2y_3' + 3y_3y_2' + xy_4' - \mu_4\right) \\
&\Rightarrow \mu_4 I_0 = 3\int_{-\infty}^{\infty} u\,\mathrm{d}(y_2y_3) + \int_{-\infty}^{\infty} xu\,\mathrm{d}y_4 = 3\int_{-\infty}^{\infty} x^3uy_2y_3\,\mathrm{d}x + \int_{-\infty}^{\infty}\left(x^4 - 1\right)uy_4\,\mathrm{d}x = 3\alpha + \beta.
\end{aligned} \tag{14}$$

Then, using (10), (11) and (12)–(14), the integrals α and β can be computed as

$$\begin{aligned}
\alpha &= \int_{-\infty}^{\infty} x^3uy_2y_3\,\mathrm{d}x = \int_{-\infty}^{\infty}(uy_2)(uy_3)\,\mathrm{d}\left(\frac{1}{u}\right) = \int_{-\infty}^{\infty} u\left[axy_3 - y_2\left(x\left(x^3y_2 - ax\right) - \mu_2\right)\right]\mathrm{d}x \\
&= aT_{1,3} + aT_{2,2} + \mu_2 T_{0,2} - S_4 \quad\Rightarrow\quad \alpha = \frac{3a\mu_2}{2}I_2 + \frac{5a^2}{6}I_4 - \frac{a^2}{14}I_8 - S_4, \\
\beta &= \int_{-\infty}^{\infty}\left(x^4 - 1\right)uy_4\,\mathrm{d}x = T_{4,4} - T_{0,4} = -\mu_2 I_2 - (1+a)aI_4 + \frac{11\mu_2}{30}I_6 \\
&\quad + \frac{(107 + 18a)a}{210}I_8 - \frac{\mu_2}{50}I_{10} - \frac{4a}{75}I_{12} + \frac{a}{750}I_{16} + 3S_4 - \frac{3}{5}S_8.
\end{aligned} \tag{15}$$

The following proposition is useful in finding S_4 and S_8.

Proposition 1. *For $m \geq 0$,*

$$S_m = -\frac{1}{m+1}S_{m+4} + \frac{2a^2}{(m+1)(m+3)}I_{m+4}. \tag{16}$$

Proof. From the definition of S_m, we have

$$\begin{aligned}
S_m &= -\int_{-\infty}^{\infty} x^{m-3}y_2^2\,\mathrm{d}u = \int_{-\infty}^{\infty} u\left[(m-3)x^{m-4}y_2^2 + 2x^{m-3}y_2\left(x^3y_2 - ax\right)\right]\mathrm{d}x \\
&= (m-3)S_{m-4} + 2\left(S_m - aT_{m-2,2}\right) \quad\Rightarrow\quad S_m = -(m-3)S_{m-4} + \frac{2a^2}{m-1}I_m,
\end{aligned} \tag{17}$$

from which (16) follows if m is replaced by $m+4$ in (17). This completes the proof. □

From (17), we have

$$S_8 = -5S_4 + \frac{2a^2}{7} I_8 \quad \text{and} \quad S_4 = -S_0 + \frac{2a^2}{3} I_4. \tag{18}$$

It follows from (16) that

$$S_0 = -\lim_{n\to\infty} \frac{(-1)^n S_{4n+4}}{\prod_{i=0}^{n}(4i+1)} + 2a^2 \sum_{n=0}^{\infty} \frac{(-1)^n I_{4n+4}}{(4n+3)\prod_{i=0}^{n}(4i+1)}. \tag{19}$$

The following proposition shows that the limit in (19) vanishes.

Proposition 2. $\lim_{n\to\infty} \frac{S_{4n+4}}{\prod_{i=0}^{n}(4i+1)} = 0.$

Proof. We first show that

$$x^2 \left[u(x)y_2(x)\right]^2 \le a^2 [u(x)]^2. \tag{20}$$

We assume that $a < 0$ and define

$$f(x) = \frac{-au(x)}{x} - u(x)y_2(x), \qquad x \in (-\infty, 0). \tag{21}$$

Differentiating (21) with respect to x, we have $f'(x) = a\left(x^4 + x^3 + 1\right)u(x)/x^2 < 0$, for all $x \in (-\infty, 0)$. Therefore, $f(x)$ is strictly decreasing. Moreover, $\lim_{x\to-\infty} f(x) = 0$. Hence, for $x \in (-\infty, 0)$, we have $f(x) < 0$, that is, $-u(x)y_2(x) < au(x)/x$. Furthermore, both sides are positive since $y_2(x) = -\frac{a}{u(x)} \int_{-\infty}^{x} su(s)\,\mathrm{d}s$ and then $y_2(x) < 0$ for $x \in (-\infty, 0)$.

Therefore, we have proved (20) for $a < 0$ and $x \in (-\infty, 0)$. Taking into account that $y_2(-x) = y_2(x)$ and $u(-x) = u(x)$ for all $x \in (-\infty, \infty)$, the inequality (20) is verified for all $a \in \mathbb{R}$ and $x \in (-\infty, \infty)$.

Thus, using (20) we have $S_{4n+4} = \int_{-\infty}^{\infty} \frac{x^{4n+4}}{u(x)} \left[u(x)y_2(x)\right]^2 \mathrm{d}x \le a^2 \int_{-\infty}^{\infty} x^{4n+2} u \,\mathrm{d}x = a^2 I_{4n+2}$. Since $I_{4n+2} = 4^n\sqrt{2}\Gamma\left(n+\frac{3}{4}\right) = \sqrt{2}\Gamma\left(\frac{3}{4}\right)\prod_{i=0}^{n-1}(4i+3)$, we have

$$0 \le \lim_{n\to\infty} \frac{S_{4n+4}}{\prod_{i=0}^{n}(4i+1)} \le \sqrt{2}a^2\Gamma\left(\frac{3}{4}\right) \lim_{n\to\infty}\left[\frac{1}{4n+1}\prod_{i=0}^{n-1}\left(\frac{4i+3}{4i+1}\right)\right] = 0. \tag{22}$$

This completes the proof. $\square$

Furthermore,

$$I_{4n+4} = \frac{1}{\sqrt{2}}\Gamma\left(\frac{1}{4}\right)\prod_{i=0}^{n}(4i+1) \quad \text{and} \quad \sum_{n=0}^{\infty}\frac{(-1)^n}{(4n+3)} = \frac{\sqrt{2}}{4}\left[\frac{\pi}{2} - \ln(1+\sqrt{2})\right]. \tag{23}$$

From (19), (22) and (23), we obtain

$$S_0 = \frac{a^2}{2}\Gamma\left(\frac{1}{4}\right)\left[\frac{\pi}{2} - \ln(1+\sqrt{2})\right]. \tag{24}$$

According to (14), $\mu_4 = (3\alpha + \beta)/I_0$. Thus, using (15), (18) and (24) we obtain

$$\begin{aligned} \mu_4 &= \left(\frac{16}{25} - 9a\right)\frac{2a}{5\pi^2}\left[\Gamma\left(\frac{3}{4}\right)\right]^4 + a^2\left(2 - \frac{3\sqrt{2}\pi}{4} + \frac{3\sqrt{2}\ln(1+\sqrt{2})}{2}\right) - \frac{38a}{525} \\ &\approx -a\,(0.01389179008 + 0.2849903285a)\,. \end{aligned} \tag{25}$$

In this way we have proved the result stated in Theorem 1.

Table 1
Comparison between analytical predictions and numerical results for $a = 1$ and for $a = -1$.

	$a = 1$			$a = -1$		
δ	μ(num.)	μ(1st)	μ(2nd)	μ(num.)	μ(1st)	μ(2nd)
1E−03	−2.70393E−7	−2.703913E−7	−2.703916E−7	2.70392E−7	2.703913E−7	2.703910E−7
1E−02	−2.70421E−5	−2.703913E−5	−2.704212E−5	2.70364E−5	2.703913E−5	2.703642E−5
1E−01	−2.73441E−3	−2.703913E−3	−2.733801E−3	2.67717E−3	2.703913E−3	2.676803E−3
2E−01	−1.13361E−2	−1.081565E−2	−1.129386E−2	1.04046E−2	1.081565E−2	1.038189E−2
3E−01	−2.73186E−2	−2.433522E−2	−2.675616E−2	2.23815E−2	2.433522E−2	2.213932E−2

3. Numerical results

To validate the analytical approximations of the previous section we present some numerical results obtained with AUTO [13] for system (2).

In Figs. 1(b)-1(c), for $\delta = 0.1$ and $a = 1$, we show the bifurcation diagrams L_2-Norm versus μ and Period versus μ, respectively, near the canard explosion. The sudden increase of amplitude and period occurs when $\mu \approx -2.73441 \cdot 10^{-3}$. To see the evolution of the limit cycles we draw several of them in Fig. 1(b). The smallest one (A) exists for $\mu = 0$ and three of them (B-D) are present along the explosion. Although it is usually the most frequent [1], in this system there are no relaxation oscillations but the explosion ends with a homoclinic orbit (E) which connects the equilibrium marked with a bullet, situated at $(x, y) \approx (9.99701, -0.99990)$ (the period tends to infinity since a homoclinic orbit exists).

The temporal profiles of orbits A, C and E (see Figs. 1(d)-1(e), where the period is normalized to 1) show that slow–fast motions are present: the slow motion appears when the orbit is close to $y = -1$ (where the unperturbed system (1) has a heteroclinic connection at infinity) and the fast motion in the semielliptical-like part. Indeed, the explosion occurs when the orbits are close enough to $y = -1$ (note that orbit A is close, but still outside the explosion zone).

In Fig. 1(f), for $a = 1$, we compare the curve of homoclinic connections (solid line) in the (μ, δ)-parameter plane with the analytical approximations for the canard explosion (circles; first-order in blue, second-order in red) stated in Theorem 1. We observe that the agreement of the second-order approximation is very good for δ values up to about 0.3.

Finally, in Table 1 for five values of δ, in the cases $a = 1$ and $a = -1$, we compare analytical predictions and numerical results. We observe that the first-order approximation only provides accurate results when δ is small enough. As expected, when δ is relatively large, the second-order approximation clearly improves the results.

Remark that, using scaling (3), all the numerical results can be easily translated to system (4). In this case, $\epsilon = 0.1^{4/3} \approx 0.0464$.

In summary, we are dealing with periodic orbits which experience a rapid growth in amplitude and period (in an extremely narrow range of the parameter μ) and have a slow–fast behavior (for δ or ϵ small enough). Due to its similarity with the situation that appears, for example, in the van der Pol system (see [11] and references therein), we call it "canard explosion" (with a certain abuse of language, since the standard conditions for a critical manifold with unstable and stable parts are not fulfilled in system (4), see [1]).

4. Conclusions

The principal goal of this letter is to study a system, as simple as possible, exhibiting a canard explosion related to a degenerate center (with zero Jacobian matrix). As far as we know, it is the first time in the literature that an asymptotic expansion is found for this kind of problem. Specifically, we consider an unperturbed system which has a heteroclinic connection at infinity. The perturbation of this curve allows to find the corresponding asymptotic expansions, namely we are able to find the exact values of the first two terms of the critical value of the parameter μ. This analytical approximation agrees very well with the numerical results, even if the parameter δ is relatively large. Finally, it is worth noting that the canard explosion of the studied system ends in a homoclinic connection. In this way, the approximation obtained for the explosion is also valid for the corresponding curve of global connections.

Acknowledgments

This work has been partially supported by the *Ministerio de Economía y Competitividad*, Spain (MTM2017-87915-C2-1-P), by the *Ministerio de Ciencia, Innovación y Universidades*, Spain (PGC2018-096265-B-I00) and by the *Consejería de Economía, Innovación, Ciencia y Empleo de la Junta de Andalucía*, Spain (projects FQM-276, TIC-0130 and UHU-1260150). B.-W.Q. is also supported by the National Natural Science Foundation of China (No. 12001110).

References

[1] M. Krupa, P. Szmolyan, Relaxation oscillation and canard explosion, J. Differ. Equ. 174 (2001) 312–368, http://dx.doi.org/10.1006/jdeq.2000.3929.

[2] P. De Maesschalck, F. Dumortier, Canard solutions at non-generic turning points, Trans. Amer. Math. Soc. 358 (2006) 2291–2334, http://dx.doi.org/10.1090/S0002-9947-05-03839-0.

[3] C. Kuehn, Multiple Time Scale Dynamical Systems, Springer, Berlin, 2015, http://dx.doi.org/10.1007/978-3-319-12316-5.

[4] G.N. Gorelov, V.A. Sobolev, Duck-trajectories in a thermal explosion problem, Appl. Math. Lett. 5 (1992) 3–6, http://dx.doi.org/10.1016/0893-9659(92)90002-Q.

[5] E. Freire, E. Gamero, A.J. Rodríguez-Luis, First-order approximation for canard periodic orbits in a van der Pol electronic oscillator, Appl. Math. Lett. 12 (1999) 73–78, http://dx.doi.org/10.1016/S0893-9659(98)00152-9.

[6] E.K. Ersöz, M. Desroches, C.R. Mirasso, S. Rodrigues, Anticipation via canards in excitable systems, Chaos 29 (2019) 013111, http://dx.doi.org/10.1063/1.5050018.

[7] K.H.M. Nyman, P. Ashwin, P.D. Ditlevsen, Bifurcation of critical sets and relaxation oscillations in singular fast-slow systems, Nonlinearity 33 (2020) 2853–2904, http://dx.doi.org/10.1080/14689367.2019.1575337.

[8] E. Shchepakina, O. Korotkova, Canard explosion in chemical and optical systems, Discrete Cont. Dyn. Syst.-B 18 (2013) 495–512, http://dx.doi.org/10.3934/dcdsb.2013.18.495.

[9] M. Brøns, K. Uldall Kristiansen, On the approximation of the canard explosion point in singularly perturbed systems without an explicit small parameter, Dyn. Syst. 33 (2018) 136–158, http://dx.doi.org/10.1080/14689367.2017.1313390.

[10] B.W. Qin, K.W. Chung, A. Algaba, A.J. Rodríguez-Luis, Asymptotic expansions for a family of non-generic canards using parametric representation, Appl. Math. Lett. 106 (2020) 106355, http://dx.doi.org/10.1016/j.aml.2020.106355.

[11] A. Algaba, K.W. Chung, B.W. Qin, A.J. Rodríguez-Luis, Analytical approximation of the canard explosion in a van der Pol system with the nonlinear time transformation method, Physica D 406 (2020) 132384, http://dx.doi.org/10.1016/j.physd.2020.132384.

[12] B.W. Qin, K.W. Chung, A. Algaba, A.J. Rodríguez-Luis, Asymptotic expansions for a degenerate canard explosion, Physica D 418 (2021) 132841, http://dx.doi.org/10.1016/j.physd.2020.132841.

[13] E.J. Doedel, A.R. Champneys, F. Dercole, T. Fairgrieve, Y. Kuznetsov, B. Oldeman, R. Paffenroth, B. Sandstede, X. Wang, C. Zhang, AUTO-07P: Continuation and Bifurcation Software for Ordinary Differential Equations (with HomCont), Technical report, Concordia University, 2012.