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Novel Mean-Type Inequalities via Generalized Riemann-Type Fractional Integral for Composite Convex Functions: Some Special Examples

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Abstract: This study deals with a novel class of mean-type inequalities by employing fractional calculus and convexity theory. The high correlation between symmetry and convexity increases its significance. In this paper, we first establish an identity that is crucial in investigating fractional mean inequalities. Then, we establish the main results involving the error estimation of the Hermite–Hadamard inequality for composite convex functions via a generalized Riemann-type fractional integral. Such results are verified by choosing certain composite functions. These results give well-known examples in special cases. The main consequences can generalize many known inequalities that exist in other studies.

Keywords: mean inequalities; fractional integral; Hölder's inequality; Minkowski inequality

MSC: 26A33; 35J05

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1. Introduction

Fractional calculus has wide application in mathematics as well as in many other fields of the modern sciences, such as bio-engineering [1–3], biological membranes [4], medicine [5–7], geophysics [8], demography [9], the economy [10], physics [11] and also in signal processing. Over the past few decades, scientists have paid attention to the fractional theory of calculus and investigated and modeled many physical real phenomena using fractional calculus theory; for instance, fractional applications in epidemiology [12], the Atangana-Baleanu version of operators in convex analysis [13], impulsive Langevin equations in fractional settings [14], the application of fractional operators in inclusion theory [15–17], quantum calculus [18], variable order fractional engineering models based on thermostat control [19], etc.

Mathematical inequalities provide boundedness and uniqueness of solutions of boundary value problems, so they have became the backbone of mathematical methods. Due to their vast use in the field of mathematics as well as in other modern fields of science, their need and importance have inspired mathematicians to turn to more generalized and advanced inequalities [20–22]. Additionally, this group of inequalities has been applied in most studies studying fractional models, fractional BVPs and IVPs, etc. At present, the list of inequalities is very long and still growing. Studies by Beckenbach [23] are a good resource to survey these inequalities. The inequalities with general kernels and measures can be studied in the [24,25]. AlNemer et al. [26] and Zakarya et al. [27] established some Hardy and Coposn inequalities, respectively. The HH-inequality [28] is considered the fundamental inequality in the study of convexity. It helps us understand the geometrical aspects of a convex function. It can be written as:

Theorem 1. *If $\Phi : [c, d] \to \mathbb{R}$ is a convex function, then*

$$\Phi(\frac{c+d}{2}) \leq \frac{1}{d-c}\int_c^d \Phi(x)dx \leq \frac{\Phi(c)+\Phi(d)}{2}$$

holds. For the concave function above, inequality holds in the other direction.

Taking advantage of fractional operators, Farid *et al.* utilized a Riemann–Liouville fractional integral to study the error estimation of one of the most basic and famous Hermite–Hadamard (HH) inequalities by using the concept of convexity for strictly monotone mappings [29].

In this paper, to obtain more advanced results, we used a generalized Riemann–Liouville fractional integral [30] on HH-inequality. We establish the generalized identities and estimate the error of HH-inequality, which is further used in estimating errors of mid-point and trapezoidal inequalities for strictly monotonic convex functions.

The inspiration behind this paper is the recent work conducted by Farid et al. in [29]. We develop a generalized identity for Rieman-type fractional integrals and use it to investigate trapezoid-type inequalities for a class of composite convex functions with respect to a strictly monotone function. The basic purpose of this research is to obtain more advanced and refined results than exist in the literature.

The organization of the paper is as follows: the preliminaries are stated in Section 2; the main results and special cases, in the form of several examples and applications, are given in Section 3; and conclusive remarks are provided in Section 4.

2. Preliminaries

We give some preliminaries that are necessary to deal with our main results.

Definition 1. *A real-valued function Φ defined on $[c, d]$ is called convex if it satisfies*

$$\Phi(\eta x + (1-\eta)y) \leq \eta\Phi(x) + (1-\eta)\Phi(y),$$

where $0 \leq \eta \leq 1$ and $x, y \in [c, d]$.

The HH-inequality and its generalizations have been studied by many authors in [31–33]. Due to advancement and enhancement of effectiveness operators, mathematicians are struggling to invent new efficient mechanisms and extend the existing studies.

The convexity of a function w.r.t. a strictly monotone mapping given in [34] is presented as follows:

Definition 2. *The function Φ is convex w.r.t. a strictly monotone mapping $\mho$ if the composite function $\Phi \circ \mho^{-1}$ is convex.*

The following theorem gives the description of HH-inequality under a convex function w.r.t. a strictly monotone mapping [35].

Theorem 2. *Suppose I_1 and I_2 are sub-intervals of $(-\infty, +\infty)$, $\mho : I_2 \supset [c,d] \to R$ is a mapping with strict monotonicity property and $\Phi : [c,d] \subset I_1 \to R$ is a convex function w.r.t. $\mho$. Then*

$$\Phi\left(\mho^{-1}\left(\frac{\mho(c)+\mho(d)}{2}\right)\right) \leq \frac{1}{\mho(d)-\mho(c)} \int_{\mho(c)}^{\mho(d)} \Phi(\mho^{-1}(\nu))d\nu \leq \frac{\Phi(c)+\Phi(d)}{2}.$$

The following definition is an extension of the classical Gamma function. For more details, see [36].

Definition 3. *The k-Gamma function denoted by Γ_k is formulated as*

$$\Gamma_k(z) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{z}{k}-1}}{(z)_{n,k}} \text{ where } k > 0 \text{ and } z \in \mathbb{C}\backslash\mathbb{Z}^-.$$

Another form is

$$\Gamma_k(z) = \int_0^\infty e^{-\frac{\nu^k}{k}} \nu^{z-1} d\nu, \quad z \in \mathbb{C} \text{ and } Re(z) > 0.$$

One can easily observe that $\nu\Gamma_k(\nu) = \Gamma_k(\nu + k)$.

Definition 4 ([37])**.** *The left and right sided fractional RL-integrals (Riemann-Liouville) of G with order w are given as*

$$I_{c+}^w G(x) = \frac{1}{\Gamma(w)} \int_c^x (x-\nu)^{w-1} G(\nu) \mathrm{d}\nu, \qquad x > c,$$

$$I_{d-}^w G(x) = \frac{1}{\Gamma(w)} \int_x^d (\nu-x)^{w-1} G(\nu) \mathrm{d}\nu, \qquad x < d.$$

The generalized RL-integrals introduced in [30] are as follows:

Definition 5. *The left and right generalized RL-integrals of G with order w are given as:*

$${}_kI_{c+}^w G(x) = \frac{1}{k\Gamma_k(w)} \int_c^x (x-\nu)^{\frac{w}{k}-1} G(\nu) \mathrm{d}\nu, \qquad x > c,$$

$${}_kI_{d-}^w G(x) = \frac{1}{k\Gamma_k(w)} \int_x^d (\nu-x)^{\frac{w}{k}-1} G(\nu) \mathrm{d}\nu, \qquad x < d.$$

Note that the obtained results of the current manuscript are connected with the findings of [38–40].

3. Main Results

This section consists of several novel mean-type inequalities involving the generalized Riemann–Liouville fractional integrals. The following lemma gives an integral identity that will be helpful to study the error estimation (lower and upper bounds estimation) of HH-inequality.

Lemma 1. *Consider a real function Φ and a strictly monotone real function $\mho$ defined on $[a_1, a_2]$ with $a_2 > a_1$ s.t. $(\Phi \circ \mho^{-1})$ is differentiable and $(\Phi \circ \mho^{-1})' \in L[a_1, a_2]$. In this case,*

$$\frac{\Phi(a_1)+\Phi(a_2)}{2} - \frac{\Gamma_k(u+k)}{2\left(\mho(a_2)-\mho(a_1)\right)^{\frac{u}{k}}} \left({}_kI_{\mho(a_1)^+}^u \Phi(a_2) + {}_kI_{\mho(a_2)^-}^u \Phi(a_1)\right)$$

$$= \frac{\mho(a_2)-\mho(a_1)}{2} \int_0^1 \left((1-\nu)^{\frac{u}{k}} - \nu^{\frac{u}{k}}\right) (\Phi \circ \mho^{-1})' \left(\nu\mho(a_1) + (1-\nu)\mho(a_2)\right) d\nu. \quad (1)$$

Proof. First we evaluate the integral

$$\int_0^1 (1-\nu)^{\frac{u}{k}} (\Phi \circ \mho^{-1})' \Big(\nu \mho(a_1) + (1-\nu)\mho(a_2) \Big) d\nu$$

$$= \frac{(1-\nu)^{\frac{u}{k}} (\Phi \circ \mho^{-1}) \Big(\nu \mho(a_1) + (1-\nu)\mho(a_2) \Big)}{\mho(a_1) - \mho(a_2)} \Bigg|_o^1$$

$$+ \frac{u}{k} \int_0^1 \frac{(1-\nu)^{\frac{u}{k}-1} (\Phi \circ \mho^{-1}) \Big(\nu \mho(a_1) + (1-\nu)\mho(a_2) \Big) d\nu}{\mho(a_1) - \mho(a_2)}.$$

$$= \frac{\Phi(a_2)}{\mho(a_2) - \mho(a_1)} - \frac{\frac{u}{k}}{\mho(a_2) - \mho(a_1)} \int_0^1 (1-\nu)^{\frac{u}{k}-1} (\Phi \circ \mho^{-1}) \Big(\nu \mho(a_1) + (1-\nu)\mho(a_2) \Big) d\nu.$$

$$= \frac{\Phi(a_2)}{\mho(a_2) - \mho(a_1)} - \frac{\frac{u}{k}}{\Big(\mho(a_2) - \mho(a_1)\Big)^{\frac{u}{k}+1}} \int_{\mho(a_1)}^{(\mho a_2)} (z - \mho(a))^{\frac{u}{k}-1} (\Phi \circ \mho^{-1})(z) dz.$$

$$= \frac{\Phi(a_2)}{\mho(a_2) - \mho(a_1)} - \frac{\Gamma_k(u+k)}{\Big(\mho(a_2) - \mho(a_1)\Big)^{\frac{u}{k}+1}} \Big({}_k I^u_{\mho(a_2)^-} \Phi(a_1) \Big). \tag{2}$$

Similarly, integrating by parts, we obtain

$$\int_0^1 \nu^{\frac{u}{k}} (\Phi \circ \mho^{-1})' \Big(\nu \mho(a_1) + (1-\nu)\mho(a_2) \Big) d\nu$$

$$= \frac{-\Phi(a_2)}{\mho(a_2) - \mho(a_1)} + \frac{\Gamma_k(u+k)}{\Big(\mho(a_2) - \mho(a_1)\Big)^{\frac{u}{k}+1}} \Big({}_k I^u_{\mho(a_1)^+} \Phi(a_2) \Big). \tag{3}$$

By substituting (2) and (3) in the (1), we can obtain the desired result. □

We derive the following error estimate of Theorem 2 with the help of Lemma 1.

Theorem 3. *Consider a real function Φ and a strictly monotone function $\mho$ defined on $[a_1, a_2]$ with $a_2 > a_1$ s.t. $\Phi \circ \mho^{-1}$ is differentiable and $(\Phi \circ \mho^{-1})' \in L[a_1, a_2]$. Then*

$$\left| \frac{\Phi(a_1) + \Phi(a_2)}{2} - \frac{\Gamma_k(u+k)}{2[\mho(a_2) - \mho(a_1)]^{\frac{u}{k}}} \left({}_k I^u_{\mho(a_1)^+} \Phi(a_2) +_k I^u_{\mho(a_2)^-} \Phi(a_1) \right) \right|$$

$$\leq \frac{|\mho(a_2) - \mho(a_1)|}{2(\frac{u}{k}+1)} \left(1 - \frac{1}{2^{\frac{u}{k}}} \right) \left(\left| (\Phi \circ \mho^{-1})'(\mho(a_1)) \right| + \left| (\Phi \circ \mho^{-1})'(\mho(a_2)) \right| \right), \tag{4}$$

holds whenever $|(\Phi \circ \mho^{-1})'|$ is convex.

Proof. From Lemma 1 with the properties of the absolute value function, the above inequality can be estimated by

$$\left|\frac{\Phi(a_1)+\Phi(a_2)}{2}-\frac{\Gamma_k(u+k)}{2\left(\mho(a_2)-\mho(a_1)\right)^{\frac{u}{k}}}\left({}_kI^u_{\mho(a_1)^+}\Phi(a_2)+{}_k I^u_{\mho(a_2)^-}\Phi(a_1)\right)\right|$$

$$\leq \frac{|\mho(a_2)-\mho(a_1)|}{2}\int_0^1\left|(1-\nu)^{\frac{u}{k}}-\nu^{\frac{u}{k}}\right|\left|(\Phi\circ\mho^{-1})'(\nu\mho(a_1)+(1-\nu)\mho(a_2))\right|d\nu. \quad (5)$$

Since $|(\Phi\circ\mho^{-1})'|$ is convex, therefore using this on the right-hand side of (5) will imply the following:

$$\left|\frac{\Phi(a_1)+\Phi(a_2)}{2}-\frac{\Gamma_k(u+k)}{2\left(\mho(a_2)-\mho(a_1)\right)^{\frac{u}{k}}}\left({}_kI^u_{\mho(a_1)^+}\Phi(a_2)+{}_k I^u_{\mho(a_2)^-}\Phi(a_1)\right)\right|$$

$$\leq \frac{|\mho(a_2)-\mho(a_1)|}{2}\int_0^1\left|(1-\nu)^{\frac{u}{k}}-\nu^{\frac{u}{k}}\right|\Big(\nu\left|(\Phi\circ\mho^{-1})'(\mho(a_1))\right|$$

$$+(1-\nu)\left|(\Phi\circ\mho^{-1})'(\mho(a_2))\right|\Big)d\nu$$

$$\leq \frac{|\mho(a_2)-\mho(a_1)|}{2}\Bigg(\int_0^{\frac{1}{2}}\left|(1-\nu)^{\frac{u}{k}}-\nu^{\frac{u}{k}}\right|\Big(\nu\left|(\Phi\circ\mho^{-1})'(\mho(a_1))\right|$$

$$+(1-\nu)\left|(\Phi\circ\mho^{-1})'(\mho(a_2))\right|\Big)d\nu$$

$$+\int_{\frac{1}{2}}^1\left|(1-\nu)^{\frac{u}{k}}-\nu^{\frac{u}{k}}\right|\Big(\nu\left|(\Phi\circ\mho^{-1})'(\mho(a_1))\right|+(1-\nu)\left|(\Phi\circ\mho^{-1})'(\mho(a_2))\right|\Big)d\nu\Bigg)$$

$$=\frac{|\mho(a_2)-\mho(a_1)|}{2}\Bigg(\left|(\Phi\circ\mho^{-1})'(\mho(a_1))\right|\int_0^{\frac{1}{2}}(\nu(1-\nu)^{\frac{u}{k}}-\nu^{\frac{u}{k}+1})d\nu$$

$$+\left|(\Phi\circ\mho^{-1})'(\mho(a_2))\right|\int_0^{\frac{1}{2}}((1-\nu)^{\frac{u}{k}+1}-\nu^{\frac{u}{k}}(1-\nu))d\nu$$

$$+\left|(\Phi\circ\mho^{-1})'(\mho(a_1))\right|\int_{\frac{1}{2}}^1(\nu^{\frac{u}{k}+1}-\nu(1-\nu)^{\frac{u}{k}})d\nu$$

$$+\left|(\Phi\circ\mho^{-1})'(\mho(a_2))\right|\int_{\frac{1}{2}}^1(\nu^{\frac{u}{k}}(1-\nu)-(1-\nu)^{\frac{u}{k}+1})d\nu\Bigg).$$

Next, some calculations will imply our desired result. □

Now, we present some special cases in the context of several examples, all of which have been proved in previous studies.

Example 1. *By setting* $\mho(x) = 1/x$ *in (4), we obtain*

$$\left|\frac{\Phi(a_1)+\Phi(a_2)}{2} - \frac{\Gamma_k(u+k)}{2}\left(\frac{a_1 a_2}{a_2 - a_1}\right)^{\frac{u}{k}}\left({}_kI^u_{(\frac{1}{a_1})^-}\Phi\circ g(\frac{1}{a_2}) + {}_kI^u_{(\frac{1}{a_2})^+}\Phi\circ g(\frac{1}{a_1})\right)\right|$$

$$\leq \frac{|a_1 - a_2|}{2|a_1 a_2|(\frac{u}{k}+1)}\left(1 - \frac{1}{2^{\frac{u}{k}}}\right)\left(a_1^2\left|\Phi'(a_1)\right| + a_2^2\left|\Phi'(a_2)\right|\right),$$

where $g(\nu) = \dfrac{1}{\nu}$.

Example 2. *By setting* $\mho(x) = \dfrac{1}{x}$ *and* $\dfrac{u}{k} = 1$ *in (4), we obtain*

$$\left|\frac{\Phi(a_1)+\Phi(a_2)}{2} - \frac{k a_1 a_2}{a_2 - a_1}\int_{\frac{1}{a_2}}^{\frac{1}{a_1}}(\Phi\circ g)(\nu)d\nu\right| \leq \frac{|a_1 - a_2|}{8|a_1 a_2|}\left(a_1^2\left|\Phi'(a_1)\right| + a_2^2\left|\Phi'(a_2)\right|\right),$$

where $g(\nu) = \dfrac{1}{\nu}$.

Example 3. *By setting* $\mho(x) = x^r$ *where* $r \neq 0$ *in (4), we obtain*

$$\left|\frac{\Phi(a_1)+\Phi(a_2)}{2} - \frac{r^{\frac{u}{k}}\Gamma_k(u+k)}{2(a_2^r - a_1^r)^{\frac{u}{k}}}\left({}^r_kI^u_{a_1^+}\Phi(\nu) + {}^r_kI^u_{a_2^-}\Phi(\nu)\right)\right|$$

$$\leq \frac{|a_2^r - a_1^r|}{2|r|(\frac{u}{k}+1)}\left(1 - \frac{1}{2^{\frac{u}{k}}}\right)\left(a_1^{1-r}\left|\Phi'(a_1)\right| + a_2^{1-r}\left|\Phi'(a_2)\right|\right).$$

Example 4. *By setting* $\mho(x) = x^r$ *where* $r \neq 0$ *and* $\dfrac{u}{k} = 1$ *in (4), we obtain*

$$\left|\frac{\Phi(a_1)+\Phi(a_2)}{2} - \frac{kr}{2(a_2^r - a_1^r)}\int_{a_1}^{a_2}\nu^{r-1}f(\nu)d\nu\right| \leq \frac{|a_2^r - a_1^r|}{8|r|}\left(a_1^{1-r}\left|\Phi'(a_1)\right| + a_2^{1-r}\left|\Phi'(a_2)\right|\right).$$

Example 5. *By setting* $\mho(x) = \log_e x$ *in (4), we obtain*

$$\left|\frac{\Phi(a_1)+\Phi(a_2)}{2} - \frac{\Gamma_k(u+k)}{2(ln(a_2) - ln(a_1))^{\frac{u}{k}}}\left({}_kI^u_{ln(a_1)^+}\Phi(a_2) + {}_kI^u_{ln(a_2)^-}\Phi(a_1)\right)\right|$$

$$\leq \frac{|ln(a_2) - ln(a_1)|}{2(\frac{u}{k}+1)}\left(1 - \frac{1}{2^{\frac{u}{k}}}\right)\left(a_1\left|\Phi'(a_1)\right| + a_2\left|\Phi'(a_2)\right|\right).$$

Example 6. *By setting* $\mho(x) = \log_e x$ *with* $\dfrac{u}{k} = 1$ *in (4), we obtain*

$$\left|\frac{\Phi(a_1)+\Phi(a_2)}{2} - \frac{k}{ln(a_2) - ln(a_1)}\int_{a_1}^{a_2}\frac{\Phi(u)}{u}du\right| \leq \frac{ln(a_2) - ln(a_1)}{8}\left(a_1\left|\Phi'(a_1)\right| + a_2\left|\Phi'(a_2)\right|\right).$$

Next we present the following theorem.

Theorem 4. *Consider a real function Φ and a strictly monotone function $\mho$ defined on $[a_1, a_2]$ with $a_2 > a_1$ s.t. $\Phi \circ \mho^{-1}$ is differentiable and $(\Phi \circ \mho^{-1})' \in L[a_1, a_2]$. Then*

$$\left|\frac{\Phi(a_1)+\Phi(a_2)}{2} - \frac{\Gamma_k(u+k)}{2\Big(\mho(a_2)-\mho(a_1)\Big)^{\frac{u}{k}}}\left({}_kI^u_{\mho(a_1)^+}\Phi(a_2) + {}_kI^u_{\mho(a_2)^-}\Phi(a_1)\right)\right|$$

$$\leq \frac{|\mho(a_2)-\mho(a_1)|}{2^{\frac{1}{q}}(\frac{u}{k}+1)}\left(1-\frac{1}{2^{\frac{u}{k}}}\right)\left(\left|(\Phi\circ\mho^{-1})'(\mho(a_1))\right|^q + \left|(\Phi\circ\mho^{-1})'(\mho(a_2))\right|^q\right)^{\frac{1}{q}}, \quad (6)$$

whenever $|(\Phi \circ \mho^{-1})'|^q, q \geq 1$ is convex.

Proof. In two cases, the proof will be completed:
Case(i). For $q = 1$.
Via the convexity of $|(\Phi \circ \mho^{-1})'|$ and the properties of the absolute value function in Lemma 1, the above inequality can be obtained.
Case (ii): For $q > 1$.
We use the power mean inequality and the properties of the absolute value function to R.H.S of Lemma 1. We have

$$\left|\frac{\Phi(a_1)+\Phi(a_2)}{2} - \frac{\Gamma_k(u+k)}{2\Big(\mho(a_2)-\mho(a_1)\Big)^{\frac{u}{k}}}\left({}_kI^u_{\mho(a_1)^+}\Phi(a_2) + {}_kI^u_{\mho(a_2)^-}\Phi(a_1)\right)\right|$$

$$\leq \frac{|\mho(a_2)-\mho(a_1)|}{2}\left(\int_0^1\left|(1-\nu)^{\frac{u}{k}}-\nu^{\frac{u}{k}}\right|\right)^{1-\frac{1}{q}} \quad (7)$$

$$\times\left(\int_0^1\left|(1-\nu)^{\frac{u}{k}}-\nu^{\frac{u}{k}}\right|\left|(\Phi\circ\mho^{-1})'\Big(\nu\mho(a_1)+(1-\nu)\mho(a_2)\Big)\right|^q d\nu\right)^{\frac{1}{q}}.$$

This can be written as

$$\int_0^1\left|(1-\nu)^{\frac{u}{k}}-\nu^{\frac{u}{k}}\right|d\nu = \int_0^{\frac{1}{2}}\left((1-\nu)^{\frac{u}{k}}-\nu^{\frac{u}{k}}\right)d\nu + \int_{\frac{1}{2}}^1\left(\nu^{\frac{u}{k}}-(1-\nu)^{\frac{u}{k}}\right)d\nu$$

$$= \frac{2}{(\frac{u}{k}+1)}\left(1-\frac{1}{2^{\frac{u}{k}}}\right). \quad (8)$$

Since $|(\Phi \circ \mho^{-1})'|^q$ is convex, therefore

$$\int_0^1\left|(1-\nu)^{\frac{u}{k}}-\nu^{\frac{u}{k}}\right|\left|(\Phi\circ\mho^{-1})'\Big(\nu\mho(a_1)+(1-\nu)\mho(a_2)\Big)\right|^q d\nu$$

$$\leq \int_0^{\frac{1}{2}}\left((1-\nu)^{\frac{u}{k}}-\nu^{\frac{u}{k}}\right)\left(\nu\left|(\Phi\circ\mho^{-1})'(\mho(a_1))\right|^q + (1-\nu)\left|(\Phi\circ\mho^{-1})'(\mho(a_2))\right|^q\right)d\nu$$

$$+\int_{\frac{1}{2}}^1\left(\nu^{\frac{u}{k}}-(1-\nu)^{\frac{u}{k}}\right)\left(\nu\left|(\Phi\circ\mho^{-1})'(\mho(a_1))\right|^q + (1-\nu)\left|(\Phi\circ\mho^{-1})'(\mho(a_2))\right|^q\right)d\nu. \quad (9)$$

$$= \left|(\Phi\circ\mho^{-1})'(\mho(a_1))\right|^q\left(\int_o^{\frac{1}{2}}\nu\left((1-\nu)^{\frac{u}{k}}-\nu^{\frac{u}{k}+1}\right)d\nu + \int_{\frac{1}{2}}^1\nu\left(\nu^{\frac{u}{k}}-(1-\nu)^{\frac{u}{k}}\right)d\nu\right)$$

$$+ |(\Phi\circ\mho^{-1})'(\mho(a_2))|^q\left(\int_o^{\frac{1}{2}}\left((1-\nu)^{\frac{u}{k}+1}-\nu^{\frac{u}{k}}(1-\nu)\right)d\nu+\int_{\frac{1}{2}}^1\left(\nu^{\frac{u}{k}}(1-\nu)-(1-v)^{\frac{u}{k}+1}\right)d\nu\right)$$

$$= \left|(\Phi\circ\mho^{-1})'(\mho(a_1))\right|^q\frac{1}{(\frac{u}{k}+1)}\left(1-\frac{1}{2^{\frac{u}{k}}}\right)+\left|(\Phi\circ\mho^{-1})'(\mho(a_2))\right|^q\frac{1}{(\frac{u}{k}+1)}\left(1-\frac{1}{2^{\frac{u}{k}}}\right). \quad (10)$$

Next, some calculations with the use of (10), (9) and (8) in (7) will imply our desired result. □

Now, the following examples show the application of the conclusion of the above theorem.

Example 7. *By setting $\mho(x)=x$ in (6), we obtain*

$$\left|\frac{\Phi(a_1)+\Phi(a_2)}{2}-\frac{\Gamma_k(u+k)}{2(a_2-a_1)^{\frac{u}{k}}}\left({}_kI^u_{(a_1)^+}\Phi(a_2)+{}_kI^u_{(a_2)^-}\Phi(a_1)\right)\right|$$

$$\le\frac{|a_2-a_1|}{2^{\frac{1}{q}}(\frac{u}{k}+1)}\left(1-\frac{1}{2^{\frac{u}{k}}}\right)\left(\left|\Phi'(a_1)\right|^q+\left|\Phi'(a_2)\right|^q\right)^{\frac{1}{q}}.$$

Example 8. *By setting $\Phi(x)=\dfrac{1}{x}$ in (6), we obtain*

$$\left|\frac{\Phi(a_1)+\Phi(a_2)}{2}-\frac{\Gamma_k(u+k)}{2}\left(\frac{a_1a_2}{a_2-a_1}\right)^{\frac{u}{k}}\left({}_kI^{\frac{u}{k}}_{\frac{1}{a_1}^-}\Phi\circ g(\frac{1}{a_2})+{}_kI^{\frac{u}{k}}_{\frac{1}{a_2}^+}\Phi\circ g(\frac{1}{a_1})\right)\right|$$

$$\le\frac{|a_1-a_2|}{2^{\frac{1}{q}}|a_1a_2|(\frac{u}{k}+1)}\left(1-\frac{1}{2^{\frac{u}{k}}}\right)\left(a_1^{2q}\left|\Phi'\mho(a_1)\right|^q+a_2^{2q}\left|\Phi'\mho(a_2)\right|^q\right)^{\frac{1}{q}}.$$

Example 9. *By setting $\mho(x)=\ln(x)$ in (6), we obtain*

$$\left|\frac{\Phi(a_1)+\Phi(a_2)}{2}-\frac{\Gamma_k(u+k)}{2(lna_2-lna_1)^{\frac{u}{k}}}\left({}_kI^u_{lna_1^+}\Phi(a_2)+{}_kI^u_{lnb^-}\Phi(a_1)\right)\right|$$

$$\le\frac{|ln(a_2)-ln(a_1)|}{2^{\frac{1}{q}}(\frac{u}{k}+1)}\left(1-\frac{1}{2^{\frac{u}{k}}}\right)\left(a_1^q\left|\Phi'\mho(a_1)\right|^q+a_2^q\left|\Phi'\mho(a_2)\right|^q\right)^{\frac{1}{q}}.$$

Example 10. *By setting $\mho(x)=x^r$ where $r\neq0$ in (6), we obtain*

$$\left|\frac{\Phi(a_1)+\Phi(a_2)}{2}-\frac{r^{\frac{u}{k}}\Gamma_k(u+k)}{2}(a_2^r-a_1^r)^{\frac{u}{k}}\left({}_kI^u_{a_1^+}\Phi(a_2)+{}_kI^u_{a_2^-}\Phi(a_1)\right)\right|$$

$$\le\frac{|a_2^r-a_1^r|}{2^{\frac{1}{q}}|r|(\frac{u}{k}+1)}\left(1-\frac{1}{2^{\frac{u}{k}}}\right)\left(a_1^{(1-r)q}\left|\Phi'\mho(a_1)\right|^q+a_2^{(1-r)q}\left|\Phi'\mho(a_2)\right|^q\right)^{\frac{1}{q}}.$$

For the next theorem, the following Lemma will be helpful.

Lemma 2 ([41]). *For $y>x\ge0$ and $\alpha\in(0,1)$, we have*

$$\left|x^\alpha-y^\alpha\right|\le\left(y-x\right)^\alpha.$$

Theorem 5. *Consider a real function Φ and a strictly monotone real function $\mho$ defined on $[a_1, a_2]$ with $a_2 > a_1$ such that $\Phi \circ \mho^{-1}$ is differentiable and $(\Phi \circ \mho^{-1})' \in L[a_1, a_2]$ If $|(\Phi \circ \mho^{-1})'|^q$, $q \geq 1$ is convex, then*

$$\left|\frac{\Phi(a_1)+\Phi(a_2)}{2} - \frac{\Gamma_k(u+k)}{2\Big(\mho(a_2)-\mho(a_1)\Big)^{\frac{u}{k}}}\left({}_kI^u_{\mho(a_1)^+}\Phi(a_2) +_k I^u_{\mho(a_2)^-}\Phi(a_1)\right)\right|$$

$$\leq \frac{|\mho(a_2)-\mho(a_1)|}{2^{1+\frac{1}{q}}\left(\frac{up}{k}+1\right)^{\frac{1}{p}}}\left(\left|(\Phi\circ\mho^{-1})'(\mho(a_1))\right|^q + \left|(\Phi\circ\mho^{-1})'(\mho(a_2))\right|^q\right)^{\frac{1}{q}}, \tag{11}$$

s.t. $\frac{1}{q} + \frac{1}{p} = 1.$

Proof. The absolute value function along with the Holder's inequality on R.H.S of Lemma 1, give

$$\left|\frac{\Phi(a_1)+\Phi(a_2)}{2} - \frac{\Gamma_k(u+k)}{2\Big(\mho(a_2)-\mho(a_1)\Big)^{\frac{u}{k}}}\left({}_kI^u_{\mho(a_1)^+}\Phi(a_2) +_k I^u_{\mho(a_2)^-}\Phi(a_1)\right)\right|$$

$$\leq \frac{|\mho(a_2)-\mho(a_1)|}{2}\left(\int_0^1\left|(1-\nu)^{\frac{u}{k}} - \nu^{\frac{u}{k}}\right|^p\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\left|(\Phi\circ\mho^{-1})'\Big(\nu\mho(a_1)+(1-\nu)\mho(a_2)\Big)\right|^q d\nu\right)^{\frac{1}{q}},$$

We apply Lemma 2, to obtain the following:

$$\int_0^1\left|(1-\nu)^{\frac{u}{k}} - \nu^{\frac{u}{k}}\right|^p d\nu \leq \int_0^1\left|1-2\nu\right|^{\frac{up}{k}} d\nu$$

$$= \int_0^{\frac{1}{2}}\Big(1-2\nu\Big)^{\frac{up}{k}} d\nu + \int_{\frac{1}{2}}^1\Big(2\nu-1\Big)^{\frac{up}{k}} d\nu$$

$$= \frac{1}{\frac{up}{k}+1}.$$

Now convexity of $|(\Phi\circ\mho{-}1)'|^q$ implies that

$$\int_0^1\left|(\Phi\circ\mho^{-1})'\Big(\nu\mho(a_1)+(1-\nu)\mho(a_2)\Big)\right|^q d\nu$$

$$\leq \int_0^1\left(\nu\left|(\Phi\circ\mho^{-1})'(\mho(a_1))\right|^q + (1-\nu)\left|(\Phi\circ\mho^{-1})'(\mho(a_2)\right|^q\right)d\nu$$

$$= \frac{\left|(\Phi\circ\mho^{-1})'(\mho(a_1))\right|^q + \left|(\Phi\circ\mho^{-1})'(\mho(a_2)\right|^q}{2}.$$

Hence by using the computations above, we can obtain the desired result (11). □

About the above theorem, we state some cases in the form of several examples.

Example 11. *By setting* $\mho(x) = \dfrac{1}{x}$ *in (11), we obtain*

$$\left|\frac{\Phi(a_1)+\Phi(a_2)}{2} - \frac{\Gamma_k(u+k)(a_1 a_2)^{\frac{u}{k}}}{2(a_2-a_1)^{\frac{u}{k}}}\left({}_kI^u_{\frac{1}{a_1}-}(\Phi\circ g)(\frac{1}{a_2}) + {}_kI^u_{\frac{1}{a_2}-}(\Phi\circ g)(\frac{1}{a_1})\right)\right|$$

$$\leq \frac{|a_1-a_2|}{2^{1+\frac{1}{q}}(\frac{up}{k}+1)^{\frac{1}{p}}|a_1a_2|}\left(a_1^{2q}\left|\Phi'(a_1)\right|^q + a_2^{2q}\left|\Phi'(a_2)\right|^q\right)^{\frac{1}{q}},$$

where $g(\nu) = \frac{1}{\nu}$.

Example 12. *By setting* $\mho(x) = x^r$ *where* $r \neq 0$ *in (11), we obtain*

$$\left|\frac{\Phi(a_1)+\Phi(a_2)}{2} - \frac{r^{\frac{u}{k}}\Gamma_k(u+k)}{2(a_2^r-a_1^r)}\left({}_kI^u_{a_1^+}\Phi(a_2) + {}_kI^u_{a_2^-}\Phi(a_1)\right)\right|$$

$$\leq \frac{|a_2^r-a_1^r|}{2^{1+\frac{1}{q}}|r|(\frac{up}{k}+1)^{\frac{1}{p}}}\left(a_1^{(1-r)q}\left|\Phi'(a_1)\right|^q + a_2^{(1-r)q}\left|\Phi'(a_2)\right|^q\right)^{\frac{1}{q}}.$$

Example 13. *By setting* $\mho(x) = ln(x)$ *in (11), we obtain*

$$\left|\frac{\Phi(a_1)+\Phi(a_2)}{2} - \frac{\Gamma_k(u+k)}{2(lna_2-lna_1)^{\frac{u}{k}}}\left({}_kI^u_{(lna_1)^+}\Phi(a_2) + {}_kI^u_{(lna_2)^-}\Phi(a_1)\right)\right|$$

$$\leq \frac{ln(a_2)-ln(a_1)}{2^{1+\frac{1}{q}}(\frac{up}{k}+1)^{\frac{1}{p}}}\left(a_1^{q}\left|\Phi'(a_1)\right|^q + a_2^{q}\left|\Phi'(a_2)\right|^q\right)^{\frac{1}{q}}.$$

The next identity will be useful to study the error estimation of the inequality (2).

Lemma 3. *We use Lemma 1 to establish the following:*

$$\frac{2^{\frac{u}{k}-1}\Gamma_k(u+k)}{\left(\mho(a_2)-\mho(a_1)\right)^{\frac{u}{k}}}\left({}_kI^u_{(\frac{\mho(a_2)+\mho(a_1)}{2})^+}\Phi(a_2) + {}_kI^u_{(\frac{\mho(a_1)+\mho(a_2)}{2})^-}\Phi(a_1)\right)$$

$$-\Phi\left(\mho^{-1}(\frac{\mho(a_1)+\mho(a_2)}{2})\right)$$

$$= \frac{\mho(a_2)-\mho(a_1)}{4}\left(\int_0^1 \nu^{\frac{u}{k}}(\Phi\circ\mho^{-1})'\left(\frac{\mho(a_1)\nu}{2} + (\frac{2-\nu}{2})\mho(a_2)\right)d\nu\right.$$

$$\left.-\int_0^1 \nu^{\frac{u}{k}}(\Phi\circ\mho^{-1})'\left(\frac{\mho(a_1)(2-\nu)}{2} + (\frac{\nu}{2})\mho(a_2)\right)d\nu\right).$$

Proof. Evaluating the integral by parts, we have

$$\int_0^1 \nu^{\frac{u}{k}}(\Phi\circ\mho^{-1})'\left(\frac{\mho(a_1)\nu}{2} + (\frac{2-\nu}{2})\mho(a_2)\right)d\nu$$

$$= \frac{\nu^{\frac{u}{k}}(\Phi \circ \mho^{-1})\left(\frac{\mho(a_1)\nu}{2} + (\frac{2-\nu}{2})\mho(a_2)\right)}{\frac{\mho(a_1)-\mho(a_2)}{2}}\Bigg|_0^1$$

$$- \frac{u}{k}\int_0^1 \nu^{\frac{u}{k}-1}\frac{(\Phi \circ \mho^{-1})\left(\frac{\mho(a_1)\nu}{2} + (\frac{2-\nu}{2})\mho(a_2)\right)}{\frac{\mho(a_1)-\mho(a_2)}{2}}d\nu. \tag{12}$$

$$= -\frac{2\Phi\left(\mho^{-1}(\frac{\mho(a_1)+\mho(a_2)}{2})\right)}{(\mho(a_2)-\mho(a_1))} + \frac{2\frac{u}{k}}{\mho(a_2)-\mho(a_1)}$$

$$\times \int_0^1 \nu^{\frac{u}{k}}(\Phi \circ \mho^{-1}\left(\frac{\mho(a_1)\nu}{2} + (\frac{2-\nu}{2})\mho(a_2)\right)d\nu.$$

By changing the variable, we obtain the following:

$$= -\frac{2\Phi(\mho^{-1}(\frac{\mho(a_1)+\mho(a_2)}{2}))}{(\mho(a_2)-\mho(a_1))} + \frac{2^{\frac{u}{k}+1}\Gamma_k(u+k)}{\mho(a_2)-\mho(a_1)^{\frac{u}{k}}}\left({}_kI^u_{(\frac{\mho(a_1)+\mho(a_2)}{2})^+}\Phi(a_2)\right).$$

Similarly,

$$\int_0^1 \nu^{\frac{u}{k}}(\Phi \circ \mho^{-1})'\left(\frac{\mho(a_1)(2-\nu)}{2} + (\frac{\nu}{2})\mho(a_2)\right)d\nu$$

$$= \frac{2\Phi(\mho^{-1}(\frac{\mho(a_1)+\mho(a_2)}{2}))}{(\mho(a_2)-\mho(a_1))}$$

$$- \frac{2\frac{u}{k}}{\mho(a_2)-\mho(a_1)}\int_0^1 \nu^{\frac{u}{k}-1}(\Phi \circ \mho^{-1})\left(\frac{\mho(a_1)\nu}{2}) + (\frac{2-\nu}{2})\mho(a_2)\right)dt. \tag{13}$$

$$= \frac{2\Phi(\mho^{-1}(\frac{\mho(a_1)+\mho(a_2)}{2}))}{(\mho(a_2)-\mho(a_1))} + \frac{2^{\frac{u}{k}+1}\Gamma_k(u+k)}{\left(\mho(a_2)-\mho(a_1)\right)^{\frac{u}{k}}}\left({}_kI^u_{(\frac{\mho(a_1)+\mho(a_2)}{2})^-}\Phi(a_1)\right).$$

Lemma 3 is obtained by using (12) and (13). □

With the help of Lemma 3, we establish the error estimate of the HH-inequality.

Theorem 6. *Using the assumption of Theorem 3,*

$$\Bigg|\frac{2^{\frac{u}{k}-1}\Gamma_k(u+k)}{\left(\mho(a_2)-\mho(a_1)\right)^{\frac{u}{k}}}\left({}_kI^u_{(\frac{\mho(a_2)+\mho(a_1)}{2})^+}\Phi(a_2) + {}_kI^u_{(\frac{\mho(a_1)+\mho(a_2)}{2})^-}\Phi(a_1)\right) \tag{14}$$

$$- \Phi\left(\mho^{-1}(\frac{\mho(a_1)+\mho(a_2)}{2})\right)\Bigg|$$

$$\leq \frac{|\mho(a_2)-\mho(a_1)|}{4(\frac{\mu}{k}+1)}\left(\left|(\Phi \circ \mho^{-1})'\mho(a_1))\right| + \left|(\Phi \circ \mho^{-1})'\mho(a_2))\right|\right), \tag{15}$$

holds.

Proof. By convexity of $|(\Phi \circ \mho^{-1})'|$ and applying the properties of the absolute value function in Lemma 3, we obtain

$$\Bigg|\frac{2^{\frac{u}{k}-1}\Gamma_k(u+k)}{(\mho(a_2)-\mho(a_1))^{\frac{u}{k}}}\left({}_kI^u_{(\frac{\mho(a_1)+\mho(a_2)}{2})^+}\Phi(a_2)+{}_kI^u_{(\frac{\mho(a_1)+\mho(a_2)}{2})^-}\Phi(a_1)\right)$$

$$-\Phi\left(\mho^{-1}\left(\frac{\mho(a_1)+\mho(a_2)}{2}\right)\right)\Bigg|$$

$$\leq \frac{|\mho(a_2)-\mho(a_1)|}{4}\left(\int_0^1\left|\nu^{\frac{u}{k}}(\Phi\circ\mho^{-1})'\left(\frac{\mho(a_1)\nu}{2}+(\frac{2-\nu}{2})\mho(a_2)\right)\right|d\nu\right.$$

$$\left.+\int_0^1\left|\nu^{\frac{u}{k}}(\Phi\circ\mho^{-1})'\left(\frac{\mho(a_1)(2-\nu)}{2}+(\frac{\nu}{2})\mho(a_2)\right)\right|d\nu\right)$$

$$\leq \frac{|\mho(a_2)-\mho(a_1)|}{4}\left(\left(\left|(\Phi\circ\mho^{-1})'\mho(a_1))\right|+\left|(\Phi\circ\mho^{-1})'\mho(a_2))\right|\right)\int_0^1\nu^{\frac{u}{k}}d\nu\right).$$

Next, the little calculations will imply our desired result. □

Example 14. *By setting $\mho(x)=\frac{1}{x}$ in (14), we obtain*

$$\left|\frac{2^{\frac{u}{k}-1}(\Gamma_k(u+k))(a_1a_2)^{\frac{u}{k}}}{(a_2-a_1)^{\frac{u}{k}}}\left({}_kI^u_{(\frac{a_1+a_2}{2a_1a_2})^-}\Phi\circ g\left(\frac{1}{a_2}\right)+{}_kI^u_{(\frac{a_1+a_2}{2a_1a_2})^+}\Phi\circ g\left(\frac{1}{a_1}\right)\right)-\Phi\left(\frac{2a_1a_2}{a_1+a_2}\right)\right|$$

$$\leq \frac{|a_1-a_2|}{4(\frac{u}{k}+1)|a_1a_2|}\left(a_1^2\left|\Phi'(a_1)\right|+a_2^2\left|\Phi'(a_2)\right|\right),$$

where $g(\nu)=\dfrac{1}{\nu}$.

Example 15. *By setting $\mho(x)=x^r$ where $r\neq 0$ in (14), we obtain*

$$\left|\frac{2^{\frac{u}{k}-1}(\Gamma_k(u+k))(r)^{\frac{u}{k}}}{(a_2^r-a_1^r)^{\frac{u}{k}}}\left({}^r_kI^u_{(\frac{a_1^r+a_2^r}{2})^{\frac{1}{r}+}}\Phi(a_2)+{}^r_kI^u_{(\frac{a_1^r+a_2^r}{2})^{\frac{1}{r}-}}\Phi(a_1)\right)-\Phi\left(\left(\frac{a_1^r+a_2^r}{2}\right)^{\frac{1}{r}}\right)\right|$$
$$\leq \frac{|a_2^r-a_1^r|}{4(\frac{u}{k}+1)|r|}\left(a_1^{(1-r)}\left|\Phi'(a_1)\right|+a_2^{(1-r)}\left|\Phi'(a_2)\right|\right).$$

Example 16. *By setting $\mho(x)=\log_e x(x)$ in (14), we obtain*

$$\Bigg|\frac{2^{\frac{u}{k}-1}\Gamma_k(u+k)}{ln(a_2)-ln(a_1)}\left({}_kI^u_{(\frac{ln(a_1)+ln(a_2)}{2})^+}\Phi(a_2)+{}_kI^u_{(\frac{ln(a_1)+ln(a_2)}{2})^-}\Phi(a_1)\right)$$

$$-\Phi\left(exp\left(\frac{ln(a_1)+ln(a_2)}{2}\right)\right)\Bigg|$$

$$\leq \frac{|ln(a_2)-ln(a_1)|}{4(\frac{u}{k}+1)}\left(a_1\left|\Phi'(a_1)\right|+a_2\left|\Phi'(a_2)\right|\right).$$

Next, we prove the theorem.

Theorem 7. *Using Theorem 4, we can write*

$$\Bigg|\frac{2^{\frac{u}{k}-1}\Gamma_k(u+k)}{(\mho(a_2)-\mho(a_1))^{\frac{u}{k}}}\left({}_kI^u_{(\frac{\mho(a_1)+\mho(a_2)}{2})^+}\Phi(a_2)+{}_kI^u_{(\frac{\mho(a_1)+\mho(a_2)}{2})^-}\Phi(a_1)\right)$$

$$-\Phi\left(\mho^{-1}\left(\frac{\mho(a_1)+\mho(a_2)}{2}\right)\right)\Bigg|$$

$$\leq\frac{\Big|\mho(a_2)-\mho(a_1)\Big|}{2^{2+\frac{1}{q}}(\frac{u}{k}+1)(\frac{u}{k}+2)^{\frac{1}{q}}}\left(\left(\Big|(\Phi\circ\mho^{-1})'\mho(a_1))\Big|^q(\frac{u}{k}+1)+\Big|(\Phi\circ\mho^{-1})'\mho(a_2))\Big|^q(\frac{u}{k}+3)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\Big|(\Phi\circ\mho^{-1})'\mho(a_1))\Big|^q(\frac{u}{k}+3)+\Big|(\Phi\circ\mho^{-1})'\mho(a_2))\Big|^q(\frac{u}{k}+1)\right)^{\frac{1}{q}}\right). \tag{16}$$

Proof. In two cases, the proof will be completed:
Case (i). For $q=1$.
Via the convexity of $|(\Phi\circ\mho^{-1})'|$ and utilizing the properties of the absolute value function (Lemma 3), the desired result can be obtained easily.
Case (ii). For $q>1$.
With the use of Power mean inequality and properties of the absolute value function in Lemma 3, we have

$$\Bigg|\frac{2^{\frac{u}{k}-1}\Gamma_k(u+k)}{\mho(a_2)-\mho(a_1)}^{\frac{u}{k}}\left({}_kI^u_{(\frac{\mho(a_1)+\mho(a_2)}{2})^+}\Phi(a_2)+{}_kI^u_{(\frac{\mho(a_1)+\mho(a_2)}{2})^-}\Phi(a_1)\right)$$

$$-\Phi\left(\mho^{-1}(\frac{\mho(a_1)+\mho(a_2)}{2})\right)\Bigg|$$

$$\leq\frac{|\mho(a_2)-\mho(a_1)|}{4}\left(\int_0^1\nu^{\frac{u}{k}}d\nu\right)^{1-\frac{1}{q}}\left(\left(\int_0^1\Big|\nu^{\frac{u}{k}}(\Phi\circ\mho^{-1})'\left(\frac{\mho(a_1)\nu}{2}+(\frac{2-\nu}{2})\mho(a_2)\right)\Big|^q d\nu\right.\right.$$

$$\left.\left.+\int_0^1\Big|\nu^{\frac{u}{k}}(\Phi\circ\mho^{-1})'(\frac{\mho(a_1)(2-\nu)}{2}+(\frac{\nu}{2})\mho(a_2))\Big|^q d\nu\right)^{\frac{1}{q}}\right)$$

$$\leq\frac{|\mho(a_2)-\mho(a_1)|}{4(\frac{u}{k}+1)^{1-\frac{1}{q}}}$$

$$\times\left(\left(\Big|(\Phi\circ\mho^{-1})'(\mho(a_1))\Big|^q\int_0^1\frac{\nu^{\frac{u}{k}+1}}{2}d\nu+\Big|(\Phi\circ\mho^{-1})'(\mho(a_2))\Big|^q\int_0^1\frac{(2-\nu)\nu^{\frac{u}{k}}}{2}d\nu\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\Big|(\Phi\circ\mho^{-1})'(\mho(a_1))\Big|^q\int_0^1\frac{(2-\nu)\nu^{\frac{u}{k}}}{2}d\nu\Big|(\Phi\circ\mho^{-1})'(\mho(a_2))\Big|^q\int_0^1\frac{\nu^{\frac{u}{k}+1}}{2}d\nu\right)^{\frac{1}{q}}\right)$$

$$=\frac{|\mho(a_2)-\mho(a_1)|}{2^{2+\frac{1}{q}}(\frac{u}{k}+1)(\frac{u}{k}+2)^{\frac{1}{q}}}\left(\left(\Big|(\Phi\circ\mho^{-1})'\mho(a_1))\Big|^q(\frac{u}{k}+1)+\Big|(\Phi\circ\mho^{-1})'\mho(a_2))\Big|^q(\frac{u}{k}+3)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\Big|(\Phi\circ\mho^{-1})'\mho(a_1))\Big|^q(\frac{u}{k}+3)+\Big|(\Phi\circ\mho^{-1})'\mho(a_2))\Big|^q(\frac{u}{k}+1)\right)^{\frac{1}{q}}\right).$$

□

The following examples show the applicability of the above result in special cases:

Example 17. *By setting* $\mho(x) = \frac{1}{x}$ *in (16), we obtain*

$$\left|\frac{2^{\frac{u}{k}-1}(\Gamma_k(u+k))(a_1a_2)^{\frac{u}{k}}}{(a_2-a_1)^{\frac{u}{k}}}\left({}_kI^u_{(\frac{a_1+a_2}{2a_1a_2})^-}\Phi\circ g(\frac{1}{a_2})+{}_kI^u_{(\frac{a_1+a_2}{2a_1a_2})^+}\Phi\circ g(\frac{1}{a_1})\right)-\Phi\left(\frac{2a_1a_2}{a_1+a_2}\right)\right|$$

$$\leq \frac{|a_1-a_2|}{2^{2+\frac{1}{q}}(\frac{u}{k}+1)(\frac{u}{k}+2)^{\frac{1}{q}}|a_1a_2|}\left(\left(a_1^{2q}\left|\Phi'(a_1)\right|^q(\frac{u}{k}+1)+a_2^{2q}\left|\Phi'(a_2)\right|^q(\frac{u}{k}+3)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(a_1^{2q}\left|\Phi'(a_1)\right|^q(\frac{u}{k}+3)+a_2^{2q}\left|\Phi'(a_2)\right|^q(\frac{u}{k}+1)\right)^{\frac{1}{q}}\right),$$

where $g(v) = \frac{1}{v}$.

Example 18. *By setting* $\mho(x) = x^r$ *where* $r \neq 0$ *in (16), we obtain*

$$\left|\frac{2^{\frac{u}{k}-1}(\Gamma_k(u+k))(r)^{\frac{u}{k}}}{(a_2^r-a_1^r)^{\frac{u}{k}}}\left({}^r_kI^u_{(\frac{a_1^r+a_2^r}{2})^{\frac{1}{r}}+}\Phi(a_2)+{}_kI^u_{(\frac{a_1^r+a_2^r}{2})^{\frac{1}{r}}-}\Phi(a_1)\right)-\Phi\left((\frac{a_1^r+a_2^r}{2})^{\frac{1}{r}}\right)\right|$$

$$\leq \frac{|a_2^r-a_1^r|}{2^{2+\frac{1}{q}}(\frac{u}{k}+1)(\frac{u}{k}+2)^{\frac{1}{q}}|r|}\left(\left(a_1^{q(1-r)}\left|\Phi'(a_1)\right|^q(\frac{u}{k}+1)+a_2^{q(1-r)}\left|\Phi'(a_2)\right|^q(\frac{u}{k}+3)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(a_1^{q(1-r)}\left|\Phi'(a_1)\right|^q(\frac{u}{k}+3)+a_2^{q(1-r)}\left|\Phi'(a_2)\right|^q(\frac{u}{k}+1)\right)^{\frac{1}{q}}\right).$$

Example 19. *By setting* $\mho(x) = ln(x)$ *in (16), we obtain*

$$\left|\frac{2^{\frac{u}{k}-1}\Gamma_k(u+k)}{ln(a_2)-ln(a_1)}\left({}_kI^u_{(\frac{ln(a_1)+ln(a_2)}{2})^+}\Phi(a_2)+{}_kI^u_{(\frac{ln(a_1)+ln(a_2)}{2})^-}\Phi(a_1)\right)\right.$$

$$\left.-\Phi\left(exp(\frac{ln(a_1)+ln(a_2)}{2})\right)\right|$$

$$\leq \frac{|ln(a_2)-ln(a_1)|}{2^{2+\frac{1}{q}}(\frac{u}{k}+1)(\frac{u}{k}+2)^{\frac{1}{q}}}\left(\left(a_1^q\left|\Phi'(a_1)\right|^q(\frac{u}{k}+1)+a_2^q\left|\Phi'(a_2)\right|^q(\frac{u}{k}+3)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(a_1^q\left|\Phi'(a_1)\right|^q(\frac{u}{k}+3)+a_2^q\left|\Phi'(a_2)\right|^q(\frac{u}{k}+1)\right)^{\frac{1}{q}}\right)^{\frac{1}{q}}.$$

Theorem 8. *By use of Theorem 5, we obtain the following inequality:*

$$\left|\frac{2^{\frac{u}{k}-1}\Gamma_k(u+k)}{\left(\mho(b)-\mho(a)\right)^{\frac{u}{k}}}\left({}_kI^u_{(\frac{\mho(a)+\mho(b)}{2})^+}\Phi(b)+{}_kI^u_{(\frac{\mho(a)+\mho(b)}{2})^-}\Phi(a)\right)-\Phi\left(\mho^{-1}(\frac{\mho(a)+\mho(b)}{2})\right)\right|$$

$$\leq \frac{|\mho(b)-\mho(a)|}{4^{1-\frac{1}{p}}(\frac{up}{k}+1)^{\frac{1}{p}}}\left(\left|(\Phi\circ\mho^{-1})'\mho(a)\right|+\left|(\Phi\circ\mho^{-1})'\mho(b))\right|\right). \tag{17}$$

Proof. Making use of Hölder's inequality and the properties of the absolute value function in Lemma 3, we obtain

$$\Bigg|\frac{2^{\frac{u}{k}-1}\Gamma_k(u+k)}{\left(\mho(a_2)-\mho(a_1)\right)^{\frac{u}{k}}}\left({}_kI^u_{(\frac{\mho(a_1)+\mho(a_2)}{2})^+}\Phi(a_2)+{}_kI^u_{(\frac{\mho(a_1)+\mho(a_2)}{2})^-}\Phi(a_1)\right)$$

$$-\Phi\left(\mho^{-1}(\frac{\mho(a_1)+\mho(a_2)}{2})\right)\Bigg|$$

$$\leq \frac{|\mho(a_2)-\mho(a_1)|}{4}\left(\int_0^1 \nu^{\frac{up}{k}}d\nu\right)^{\frac{1}{p}}\left(\int_0^1\left|(\Phi\circ\mho^{-1})'\left(\frac{\mho(a_1)\nu}{2}+(\frac{2-\nu}{2})\mho(a_2)\right)\right|d\nu\right.$$

$$\left.+\int_0^1\left|\nu^{\frac{u}{k}}(\Phi\circ\mho^{-1})'\left(\frac{\mho(a_1)(2-\nu)}{2}+(\frac{\nu}{2})\mho(a_2)\right)\right|d\nu\right). \tag{18}$$

As $|(\Phi\circ\mho^{-1})'|$ is convex, therefore R.H.S. of the inequality (18) will take the following form

$$\leq \frac{|\mho(a_2)-\mho(a_1)|}{4(\frac{up}{k}+1)^{\frac{1}{p}}}\left(\left|(\Phi\circ\mho^{-1})'\mho(a_1))\right|^q\right.$$

$$\left.\times\int_0^1\frac{\nu}{2}d\nu+\left|(\Phi\circ\mho^{-1})'\mho(a_2))\right|^q\int_0^1(\frac{2-\nu}{2})d\nu\right)^{\frac{1}{q}}$$

$$+\left|(\Phi\circ\mho^{-1})'\mho(a_1))\right|^q\int_0^1(\frac{2-\nu}{2})d\nu+\left(\left|(\Phi\circ\mho^{-1})'\mho(a_2))\right|^q\int_0^1\frac{\nu}{2}d\nu\right)^{\frac{1}{q}}$$

$$\leq \frac{|\mho(a_2)-\mho(a_1)|}{4(\frac{up}{k}+1)^{\frac{1}{p}}}\left(\left(\frac{\left|(\Phi\circ\mho^{-1})'\mho(a_1))\right|^q+3\left|(\Phi\circ\mho^{-1})'\mho(a_2))\right|^q}{4}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{3\left|(\Phi\circ\mho^{-1})'\mho(a_1))\right|^q+\left|(\Phi\circ\mho^{-1})'\mho(a_2))\right|^q}{4}\right)^{\frac{1}{q}}\right)$$

$$\leq \frac{|\mho(a_2)-\mho(a_1)|}{4^{1-\frac{1}{p}}(\frac{up}{k}+1)^{\frac{1}{p}}}\left(\left|(\Phi\circ\mho^{-1})'\mho(a_1)\right|+\left|(\Phi\circ\mho^{-1})'\mho(a_2)\right|\right).$$

In above computations, we use $a_1^p+a_2^q\leq(a_1+a_2)^q$ where $q>1$ and $a_1,a_2\geq 0$ and obtain our desired result. □

The following examples show the applicability of the latter result (Theorem 8) in special cses.

Example 20. *By setting* $\mho(x) = \frac{1}{x}$ *in (17), we obtain*

$$\left| \frac{2^{\frac{u}{k}-1}(\Gamma_k(u+k))(a_1a_2)^{\frac{u}{k}}}{(a_2-a_1)^{\frac{u}{k}}} \left({}_kI^u_{(\frac{a_1+a_2}{2a_1a_2})^-} \Phi \circ g(\frac{1}{a_2}) + {}_kI^u_{(\frac{a_1+a_2}{2a_1a_2})^+} \Phi \circ g(\frac{1}{a_1}) \right) - \Phi\left(\frac{2a_1a_2}{a_1+a_2}\right) \right|$$

$$\leq \frac{|a_1-a_2|}{4^{1-\frac{1}{p}}(\frac{up}{k}+1)^{\frac{1}{p}}|a_1a_2|} \left(a_1^2 \left|\Phi'(a_1)\right| + a_2^2 \left|\Phi'(a_2)\right| \right),$$

where $g(\nu) = \frac{1}{\nu}$.

Example 21. *By setting* $\mho(x) = x^r$ *in (17), we obtain*

$$\left| \frac{2^{\frac{u}{k}-1}(\Gamma_k(u+k))(r)^{\frac{u}{k}}}{(a_2^r-a_1^r)^{\frac{u}{k}}} \left({}_kI^u_{(\frac{a_1^r+a_2^r}{2})^{\frac{1}{r}}+} \Phi(a_2) + {}_kI^u_{(\frac{a_1^r+a_2^r}{2})^{\frac{1}{r}}-} \Phi(a_1) \right) - \Phi\left(\left(\frac{a_1^r+a_2^r}{2} \right)^{\frac{1}{r}} \right) \right|$$

$$\leq \frac{|a_2^r-a_1^r|}{4^{1-\frac{1}{p}}|r|(\frac{up}{k}+1)^{\frac{1}{p}}} \left(a_1^{1-r} \left|\Phi'(a_1)\right| + a_2^{1-r} \left|\Phi'(a_2)\right| \right).$$

Example 22. *By setting* $\mho(x) = log_e x$ *in (17), we obtain*

$$\left| \frac{2^{\frac{u}{k}-1}\Gamma_k(u+k)}{ln(a_2)-ln(a_1)} \left({}_kI^u_{(\frac{ln(a_1)+ln(a_2)}{2})^+} \Phi(a_2) + {}_kI^u_{(\frac{ln(a_1)+ln(a_2)}{2})^-} \Phi(a_1) \right) - \Phi\left(exp\left(\frac{ln(a_1)+ln(a_2)}{2} \right) \right) \right|$$

$$\leq \frac{|ln(a_2)-ln(a_1)|}{4^{1-\frac{1}{p}}(\frac{up}{k}+1)^{\frac{1}{p}}} \left(a_1 \left|\Phi'(a_1)\right| + a_2 \left|\Phi'(a_2)\right| \right).$$

4. Conclusions

The fractional calculus theory and integral operators have been used to yield more generalized inequalities. In this article, we utilized the generalized form of the Riemann-type fractional integral to obtain mean-type inequalities. The main results were based on identity. The consequences were verified to correspond to different choices for certain functions. The findings of this research reduced to the findings of [29] just by replacing $k = 1$. Similarly, some other results that exist in the literature were recreated. The proven results in this research are hopefully helpful in the field of modified scientific. In the future, we are committed to obtaining more generalized and refined inequalities for fractional operators.

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