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Some Novel Estimates of Hermite–Hadamard and Jensen Type Inequalities for (h_1, h_2)-Convex Functions Pertaining to Total Order Relation

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Abstract: There are different types of order relations that are associated with interval analysis for determining integral inequalities. The purpose of this paper is to connect the inequalities terms to total order relations, often called (CR)-order. In contrast to classical interval-order relations, total order relations are quite different and novel in the literature and are calculated as $\omega = \left\langle \omega_c, \omega_r \right\rangle = \left\langle \frac{\overline{\omega}+\underline{\omega}}{2}, \frac{\overline{\omega}-\underline{\omega}}{2} \right\rangle$. A major benefit of total order relations is that they produce more efficient results than other order relations. This study introduces the notion of CR-(h_1, h_2)-convex function using total order relations. Center and Radius order relations are a powerful tool for studying inequalities based on their properties and widespread application. Using this novel notion, we first developed some variants of Hermite–Hadamard inequality and then constructed Jensen inequality. Based on the results, this new concept is extremely useful in connection with a variety of inequalities. There are many new and well-known convex functions unified by this type of convexity. These results will stimulate further research on inequalities for fractional interval-valued functions and fuzzy interval-valued functions, as well as the optimization problems associated with them. For the purpose of verifying our main findings, we provide some nontrivial examples.

Keywords: Jensen inequality; (h_1, h_2)-convex function; Hermite–Hadamard inequality; Center-Raius-order relation

MSC: 05A30; 26D10; 26D15

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1. Introduction

There are many uncertainty problems in practical life that are disrupted if a specific number is used to describe them. Therefore, avoiding this kind of error and obtaining effective results are very important. In 1969, Moore [1] became the first to apply time-interval analysis to error analysis. This improved calculation accuracy and attracted the attention of many scholars. The goal of interval analysis is to use interval numbers as variables instead of numbers for analysis and to use interval operations instead of number operations to arrive at conclusions. Variables similar to the interval are widely used in real-life uncertain situations in various practical settings, such as graphics [2], decision making [3], automatic error analysis [4], etc. A wide range of excellent results have been achieved through interval analysis research, and readers are encouraged to consult the references [5–10].

Mathematicians have a great deal of experience with inequalities, especially those connected to Simpson, Ostrowski, Opial, Bullen, Jensen, and Hermite–Hadamard inequalities. Convexity and inequality are prominent concepts in many disciplines and applications, leading many scholars to study and apply generalized convexity to interval-valued functions over the past few decades. The importance of convexity has been recognized for years in fields such as economics, control theory, optimization theory, etc. A wide range of mathematical physics problems can be solved using generalized mappings convexity. Diverse convex mappings and inequalities can also be studied by using differential and integral equations. The areas in which they have made significant contributions include decision making, symmetry analysis, finance, electrical engineering, networks, operations research, numerical analysis, and equilibrium. Utilizing a number of fundamental integral inequalities, we explore how convexity might be encouraged subjectively. It has been reported recently that convex functions have been rigorously generalized, see [11–14]. There have recently been studies that extend some of these inequalities to functions with interval values, see [15–20]. Recently, these inequalities have been applied in several ways, see [21–24]. As a first step, Breckner introduces the concept of continuity among $\mathcal{IVFS}$, see [25]. Using the Hukuhara derivative, Chalco-Cano et al. [26] developed the Ostrowski inequality while Costa et al. [27] presented Opial-type inequality for $\mathcal{IVFS}$. Generally, the famous Hermite–Hadamard inequality has the following definition:

$$\vartheta\left(\frac{f+g}{2}\right) \leq \frac{1}{g-f}\int_f^g \vartheta(\sigma)d\sigma \leq \frac{\vartheta(f)+\vartheta(g)}{2}. \tag{1}$$

A great deal of attentionhas been paid to it because of the way in which it defines convex mappings. In the basic calculus, the first geometry-based interpretation was given by this inequality. Jensen inequality and Hermite–Hadamard inequality were first developed by these authors [28], for $\mathcal{IVFS}$. Various interval-based generalizations of these inequalities are presented here, see [29–32]. Initially, the following authors developed the concept of (h_1, h_2)-convex functions and presented the following results [33]. There are various authors who use the concept of (h_1, h_2)-convexity to prove the following results for diverse classes of convexity, see [34–37]. Bai et al. [38] and Afzal et al. [39] use the concept of (h_1, h_2)-convexity to prove the following results for Hermite–Hadamard inequality and Jensen-type inequality. The results presented by various old partial order relationships such as inclusion relation, pseudo relation, fuzzy order relation, etc., are not much more accurate than the ones provided by the CR-order relation. The validity of the claim can be demonstrated by comparing examples in the literature with those derived by using these old relations. Here are some recent developments using partial order relations for different types of convexity, see [40–45]. Hence, a CR-order relation is essential for studying inequalities and convexity which was presented by Bhunia's [46]. In 2022, several authors attempted to prove these inequalities using the notion of CR-order relation for different classes of convexity, see Refs. [47,48]. Shi et al. used cr-h-convex function and prove the following result [48].

Theorem 1 (See [48])**.** *Consider $\vartheta : [f, g] \to \mathbf{R_I}^+$. Define $h : (0, 1) \to \mathbb{R}^+$ and $h\left(\frac{1}{2}\right) \neq 0$. If $\vartheta \in SX(CR\text{-}h, [f, g], \mathbf{R_I}^+)$ and $\vartheta \in \mathbf{IR}_{[f,g]}$, then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\vartheta\left(\frac{f+g}{2}\right) \preceq_{CR} \frac{1}{g-f}\int_f^g \vartheta(\sigma)d\sigma \preceq_{CR} [\vartheta(f)+\vartheta(g)]\int_0^1 h(s)ds. \tag{2}$$

The set of all CR-h-convex functions over $[f, g]$ is denoted by $SX(CR\text{-}h, [f, g], \mathbf{R_I}^+)$. As well, Jensen-type inequality was established using the notion of CR-h-convex functions.

Theorem 2 (See [48]). *Let $f_i \in \mathbf{R}^+$ and $j_i \in [f,g]$. If h is a non-negative super multiplicative function and $\vartheta \in SX(CR\text{-}h,[f,g],\mathbf{R_I}^+)$, then*

$$\vartheta\left(\frac{1}{F_k}\sum_{i=1}^{k} f_i j_i\right) \preceq_{CR} \sum_{i=1}^{k} h\left(\frac{f_i}{F_k}\right)\vartheta(j_i). \tag{3}$$

Furthermore, it introduces a novel concept of interval (h_1,h_2)-convex functions based on CR-orders. In contrast to classical interval-order relations, total order relations are quite different and novel in the literature and are calculated as: $\omega_c = \frac{\underline{\omega}+\overline{\omega}}{2}$ and $\omega_r = \frac{\overline{\omega}-\underline{\omega}}{2}$, respectively, where $\omega = [\underline{\omega},\bar{\omega}]$. As an advantage of the present study, we introduce an entirely new notion of interval valued (h_1,h_2)-convex functions based on (CR)-order relations, a very novel notion in the literature. The beauty of this order relation is its ability to give more precise results when used with interval analysis.

We draw our research inspiration from the strong literature and specific articles, see Refs. [40,41,46–48], based on the CR-order relation, we introduced the idea of CR-(h_1,h_2)-convex functions. By using this new concept, we developed Hermite–Hadamard ($\mathcal{H}-\mathcal{H}$) and Jensen-type inequalities. Furthermore, for the sake of checking the validity of our main findings, some nontrivial examples are given.

Lastly the article is designed as follows: Some basic background is provided in Section 2. The main findings are described in the following Sections 3 and 4. Section 5 explores a brief conclusion.

2. Preliminaries

A number of terms in the paper are used without being defined, see Refs. [47–49]. We will benefit greatly from having some basic arithmetic knowledge related to interval analysis for the rest of the paper.

$$\begin{aligned}
[\omega] &= [\underline{\omega},\overline{\omega}] \qquad (\underline{\omega} \leqq s \leqq \overline{\omega};\ s \in \mathbb{R}),\\
[\Omega] &= [\underline{\Omega},\overline{\Omega}] \qquad (\underline{\Omega} \leqq s \leqq \overline{\Omega};\ s \in \mathbb{R}),\\
[\omega]+[\Omega] &= [\underline{\omega},\overline{\omega}]+[\underline{\Omega},\overline{\Omega}] = [\underline{\omega}+\underline{\Omega},\overline{\omega}+\overline{\Omega}]
\end{aligned}$$

and

$$\sigma\omega = \sigma[\underline{\omega},\overline{\omega}] = \begin{cases} [\sigma\underline{\omega},\sigma\overline{\omega}], & (\sigma > 0);\\ \{0\}, & (\sigma = 0);\\ [\sigma\overline{\omega},\sigma\underline{\omega}], & (\sigma < 0), \end{cases}$$

where $\sigma \in \mathbb{R}$.

Consider $\mathbf{R}_I^+$ and $\mathbf{R}_I$, which represent the positive and bundle of all intervals of $\mathbf{R}$, respectively. Several algebraic properties of interval arithmetic will now be discussed.

Let $\omega = [\underline{\omega},\bar{\omega}] \in \mathbf{R}_I$, then $\omega_c = \frac{\overline{\omega}+\underline{\omega}}{2}$ and $\omega_r = \frac{\overline{\omega}-\underline{\omega}}{2}$ are the CR form of interval ω. It further can be expressed as:

$$\omega = \langle \omega_c,\omega_r\rangle = \left\langle \frac{\overline{\omega}+\underline{\omega}}{2},\frac{\overline{\omega}-\underline{\omega}}{2}\right\rangle.$$

In order to determinethe radius and center of an interval, we use the following relations:

Definition 1. *The CR-order relation for $\omega = [\underline{\omega},\overline{\omega}] = \langle\omega_c,\omega_r\rangle$, $\Omega = [\underline{\Omega},\overline{\Omega}] = \langle\Omega_c,\Omega_r\rangle \in \mathbf{R}_I$ represented as:*

$$\omega \preceq_{CR} \Omega \Longleftrightarrow \begin{cases} \omega_c < \Omega_c, & \text{if } \omega_c \neq \Omega_c;\\ \omega_r \leq \Omega_r, & \text{if } \omega_c = \Omega_c. \end{cases}$$

Definition 2 (see [48]). *Let $\mathcal{L} : [f, g]$ be an $\mathcal{IVF}$ such that $\mathcal{L} = [\underline{\mathcal{L}}, \overline{\mathcal{L}}]$. Then $\mathcal{L}$ is Riemann integrable (**IR**) on $[f, g]$ if and only if $\underline{\mathcal{L}}$ and $\overline{\mathcal{L}}$ are Riemann integrable on $[f, g]$, that is,*

$$(\mathbf{IR})\int_f^g \mathcal{L}(\mathrm{s})d\mathrm{s} = \left[(\mathbf{R})\int_f^g \underline{\mathcal{L}}(\mathrm{s})d\mathrm{s}, (\mathbf{R})\int_f^g \overline{\mathcal{L}}(\mathrm{s})d\mathrm{s}\right].$$

*The bundle of all (**IR**) $\mathcal{IVFS}$ on $[f, g]$ is represented by $\mathbf{IR}_{([f,g])}$.*

For CR-order relation, Shi et al. [48] verifies that integral hold order.

Theorem 3. *Let $\mathcal{L}, \mathcal{K} : [f, g]$ be $\mathcal{IVFS}$ given by $\mathcal{L} = [\underline{\mathcal{L}}, \overline{\mathcal{L}}]$ and $\mathcal{K} = [\underline{\mathcal{K}}, \overline{\mathcal{K}}]$. If $\mathcal{L}(\mathrm{s}) \preceq_{CR} \mathcal{K}(\mathrm{s})$, for all $\mathrm{s} \in [f, g]$, then*

$$\int_f^g \mathcal{L}(\mathrm{s})d\mathrm{s} \preceq_{CR} \int_f^g \mathcal{K}(\mathrm{s})d\mathrm{s}.$$

We will now provide an illustration to support the aforementioned Theorem.

Example 1. *Let $\mathcal{L} = [\mathrm{s}, 2\mathrm{s}]$ and $\mathcal{K} = [\mathrm{s}^2, \mathrm{s}^2 + 2]$, then for $\mathrm{s} \in [0, 1]$*

$$\mathcal{L}_{\mathcal{C}} = \frac{3\mathrm{s}}{2}, \mathcal{L}_{\mathcal{R}} = \frac{\mathrm{s}}{2}, \mathcal{K}_{\mathcal{C}} = \mathrm{s}^2 + 1 \ \ \textit{and} \ \ \mathcal{K}_{\mathcal{R}} = 1.$$

From Definition 1, we have $\mathcal{L}(\mathrm{s}) \preceq_{CR} \mathcal{K}(\mathrm{s})$ for $\mathrm{s} \in [0, 1]$ (see Figures 1–3).

Since,

$$\int_0^1 [\mathrm{s}, 2\mathrm{s}]d\mathrm{s} = \left[\frac{1}{2}, 1\right]$$

and

$$\int_0^1 [\mathrm{s}^2, \mathrm{s}^2 + 2]d\mathrm{s} = \left[\frac{1}{3}, \frac{7}{3}\right].$$

Also, from above Definition 1, we have

$$\int_0^1 \mathcal{L}(\mathrm{s})d\mathrm{s} \preceq_{CR} \int_0^1 \mathcal{K}(\mathrm{s})d\mathrm{s}.$$

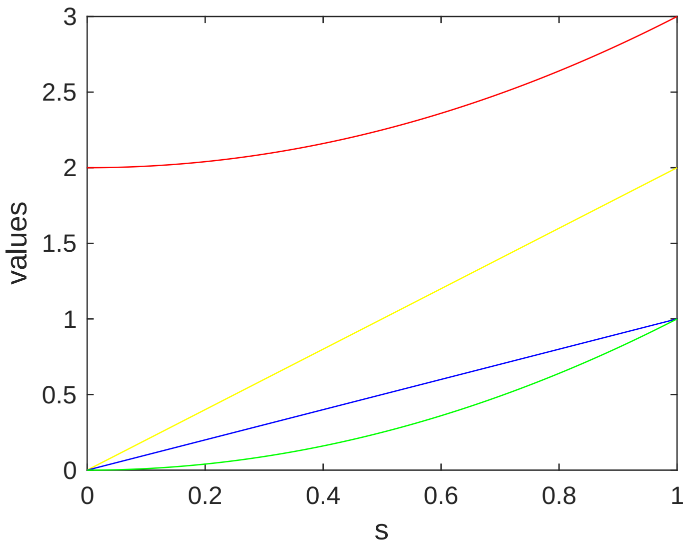

Figure 1. As shown in the above figure, $\mathrm{s}^2 + 2$ is shown as a red, 2s is shown as a yellow, s is shown as blue and s^2 as a green line, respectively. A clear indication of the validity of the CR-order relationship can be seen in the graph.

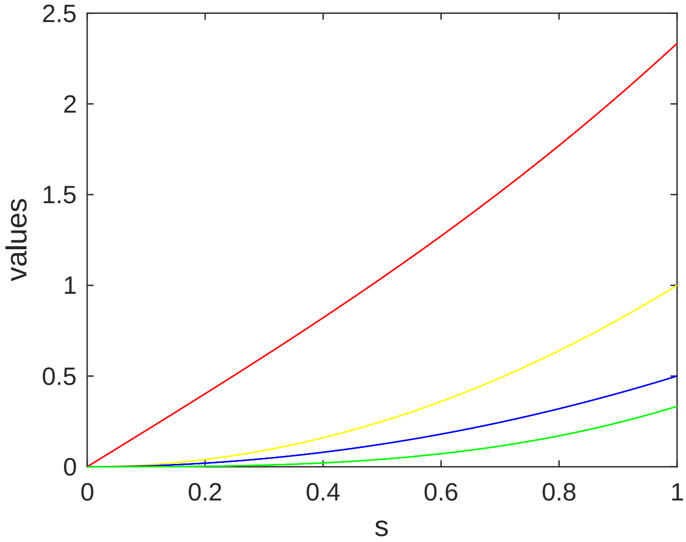

Figure 2. As shown in the above figure, $2s + \frac{s^3}{3}$ is shown as a red, s^2 is shown as a yellow, $\frac{s^2}{2}$ is shown as blue and $\frac{s^3}{3}$ as a green line, respectively. As can be seen from the graph, the Theorem 3 is valid.

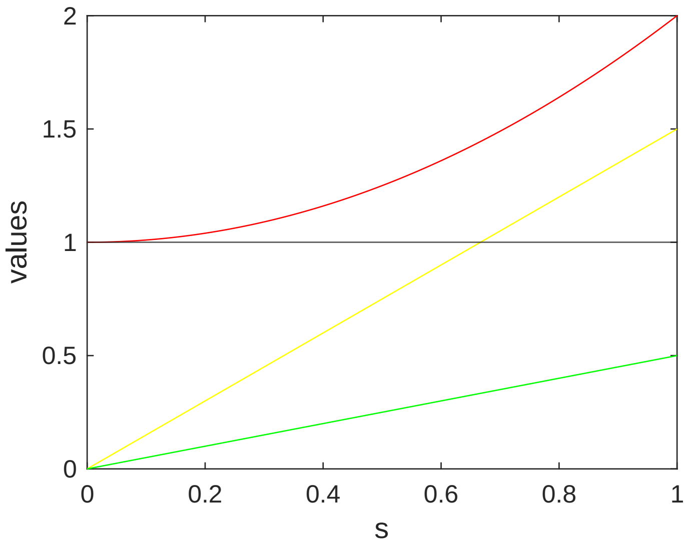

Figure 3. As shown in the above figure, $\mathcal{K}_{\mathcal{C}} = s^2 + 1$ is shown as a red, $\mathcal{L}_{\mathcal{C}} = \frac{3s}{2}$ is shown as a yellow, $\mathcal{L}_{\mathcal{R}} = 1$ is shown as black and $\mathcal{K}_{\mathcal{R}} = \frac{s}{2}$ as a green line, respectively.

Definition 3 (See [37]). *Consider $h : [0,1] \to \mathbf{R}^+$. Thus, we say $\vartheta : [f,g] \to \mathbf{R}^+$ is said to be h-convex function, or that $\vartheta \in SX(h,[f,g],\mathbf{R}^+)$, if for all $f_1, g_1 \in [f,g]$ and $\sigma \in [0,1]$, we have*

$$\vartheta(\sigma f_1 + (1-\sigma)g_1) \leq \; h(\sigma)\vartheta(f_1) + h(1-\sigma)\vartheta(g_1). \tag{4}$$

In (4), if "$\leq$" is replaced with "$\geq$", then it is called h-concave function or $\vartheta \in SV(h,[f,g],\mathbf{R}^+)$.

Definition 4 (See [37]). *Define $h_1, h_2 : [0,1] \to \mathbf{R}^+$. Thus, we say $\vartheta : [f,g] \to \mathbf{R}^+$ is known as (h_1,h_2)-convex function, or that $\vartheta \in SX((h_1,h_2),[f,g],\mathbf{R}^+)$, if for all $f_1, g_1 \in [f,g]$ and $\sigma \in [0,1]$, we have*

$$\vartheta(\sigma f_1 + (1-\sigma)g_1) \leq \; h_1(\sigma)h_2(1-\sigma)\vartheta(f_1) + h_1(1-\sigma)h_2(\sigma)\vartheta(g_1). \tag{5}$$

In (5), if "$\leq$" is replaced with "$\geq$", then it is called (h_1,h_2)-concave function or $\vartheta \in SV((h_1,h_2),[f,g],\mathbf{R}^+)$.

Let's discuss CR-order convexity now

Definition 5 (See [48]). *Consider $h : [0,1] \to \mathbf{R}^+$. Thus, we say $\vartheta = [\underline{\vartheta}, \overline{\vartheta}] : [f,g] \to \mathbf{R}_\mathbf{I}^+$ is called CR-h-convex function, or that $\vartheta \in SX(CR\text{-}h, [f,g], \mathbf{R}_\mathbf{I}^+)$, if $\forall\, f_1, g_1 \in [f,g]$ and $\sigma \in [0,1]$, we have*

$$\vartheta(\sigma f_1 + (1-\sigma)g_1) \preceq_{CR} \; h(\sigma)\vartheta(f_1) + h(1-\sigma)\vartheta(g_1). \tag{6}$$

In (6), if "$\preceq_{CR}$" is replaced with "$\succeq_{CR}$", then it is called CR-h-concave function or $\vartheta \in SV(CR\text{-}h, [f,g], \mathbf{R}_\mathbf{I}^+)$.

Definition 6. *Define $h_1, h_2 : [0,1] \to \mathbf{R}^+$. Thus, we say $\vartheta : [f,g] \to \mathbf{R}_\mathbf{I}^+$ is called CR-(h_1,h_2)-convex function, or that $\vartheta \in SX(CR\text{-}(h_1,h_2), [f,g], \mathbf{R}_\mathbf{I}^+)$, if for all $f_1, g_1 \in [f,g]$ and $\sigma \in [0,1]$, we have*

$$\vartheta(\sigma f_1 + (1-\sigma)g_1) \preceq_{CR} \; h_1(\sigma)h_2(1-\sigma)\vartheta(f_1) + h_1(1-\sigma)h_2(\sigma)\vartheta(g_1). \tag{7}$$

In (7), if "$\preceq_{CR}$" is replaced with "$\succeq_{CR}$", then it is called CR-(h_1,h_2)-concave function or $\vartheta \in SV(CR\text{-}(h_1,h_2), [f,g], \mathbf{R}_\mathbf{I}^+)$.

Remark 1.

(i) *If $h_1 = h_2 = 1$, Definition 6 becomes a CR-P-function [48];*
(ii) *If $h_1(\sigma) = \frac{1}{h_1(\sigma)}$, $h_2 = 1$ Definition 6 becomes a CR-GL-convex function [48];*
(iii) *If $h_1(\sigma) = h_1(\sigma)$, $h_2 = 1$ Definition 6 becomes a CR-h-convex function [48];*
(iv) *If $h_1(\sigma) = \sigma^s$, $h_2 = 1$ Definition 6 becomes a CR-s-convex function [48].*

3. Main Results

Proposition 1. *Consider $\vartheta : [f,g] \to \mathbf{R}_\mathbf{I}^+$ given by $[\underline{\vartheta}, \overline{\vartheta}] = \langle \vartheta_c, \vartheta_r \rangle$. If ϑ_c and ϑ_r are (h_1,h_2)-convex over $[f,g]$, then ϑ is called CR-(h_1,h_2)-convex function over $[f,g]$.*

Proof. Since ϑ_c and ϑ_r are (h_1,h_2)-convex over $[f,g]$, then for each $\sigma \in (0,1)$ and for all $f_1, g_1 \in [f,g]$, we have

$$\vartheta_c(\sigma f_1 + (1-\sigma)g_1) \le h_1(\sigma)h_2(1-\sigma)\vartheta_c(f_1) + h_1(1-\sigma)h_2(\sigma)\vartheta_c(g_1),$$

and

$$\vartheta_r(\sigma f_1 + (1-\sigma)g_1) \le h_1(\sigma)h_2(1-\sigma)\vartheta_r(f_1) + h_1(1-\sigma)h_2(\sigma)\vartheta_R(g_1).$$

Now, if

$$\vartheta_c(\sigma f_1 + (1-\sigma)g_1) \ne h_1(\sigma)h_2(1-\sigma)\vartheta_c(f_1) + h_1(1-\sigma)h_2(\sigma)\vartheta_c(g_1),$$

then for each $\sigma \in (0,1)$ and for all $f_1, g_1 \in [f,g]$,

$$\vartheta_c(\sigma f_1 + (1-\sigma)g_1) < h_1(\sigma)h_2(1-\sigma)\vartheta_c(f_1) + h_1(1-\sigma)h_2(\sigma)\vartheta_c(g_1).$$

Accordingly,

$$\vartheta_c(\sigma f_1 + (1-\sigma)g_1) \le h_1(\sigma)h_2(1-\sigma)\vartheta_c(f_1) + h_1(1-\sigma)h_2(\sigma)\vartheta_c(g_1).$$

Otherwise, for each $\sigma \in (0,1)$ and for all $f_1, g_1 \in [f,g]$,

$$\vartheta_r(\sigma f_1 + (1-\sigma)g_1) \le h_1(\sigma)h_2(1-\sigma)\vartheta_r(f_1) + h_1(1-\sigma)h_2(\sigma)\vartheta_r(g_1)$$

implies

$$\vartheta(\sigma f_1 + (1-\sigma)g_1) \preceq_{CR} h_1(\sigma)h_2(1-\sigma)\vartheta(f_1) + h_1(1-\sigma)h_2(\sigma)\vartheta(g_1).$$

Based on the foregoing and the Definition 6, this can be stated as follows:

$$\vartheta(\sigma f_1 + (1-\sigma)g_1) \preceq_{CR} h_1(\sigma)h_2(1-\sigma)\vartheta(f_1) + h_1(1-\sigma)h_2(\sigma)\vartheta(g_1)$$

for each $\sigma \in (0,1)$ and for all $f_1, g_1 \in [f,g]$. This completes the proof. $\square$

Example 2. *Consider* $[f,g] = [0,1], h_1(s) = s$ *and* $h_2(s) = 1$ *for all* $s \in [0,1]$. *If* $\vartheta : [f,g] \to \mathbf{R_I}^+$ *is defined as*

$$\vartheta(\sigma) = [-2\sigma^2 + 3, 2\sigma^2 + 4], \quad \sigma \in [0,1].$$

Then,

$$\vartheta_C(\sigma) = \frac{7}{2}, \vartheta_R(\sigma) = 2\sigma^2 + \frac{1}{2}, \quad \sigma \in [0,1].$$

It is obvious that $\vartheta_C(\sigma), \vartheta_R(\sigma)$ *are* (h_1, h_2) *convex functions over* $[0,1]$ *(see Figure 4). This implies that from Proposition 1,* ϑ *is also CR-*(h_1, h_2) *convex function on* $[0,1]$.

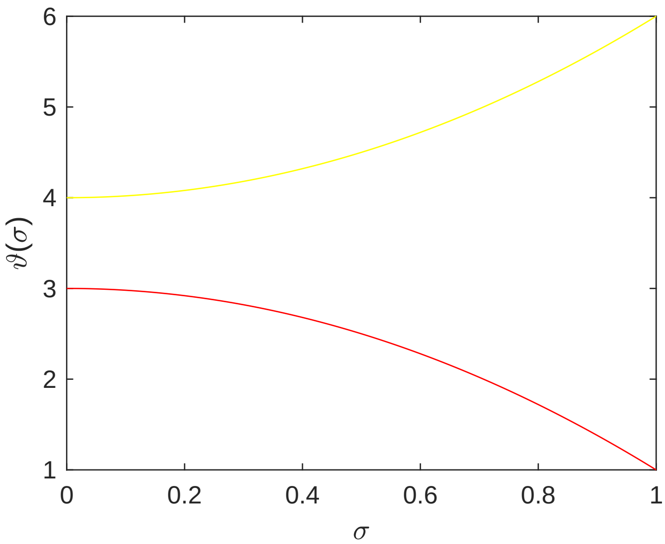

Figure 4. As shown in the example above, $\underline{\vartheta}$ is shown as a red and $\overline{\vartheta}$ as a yellow line, respectively.

Next, we will establish $\mathcal{H}-\mathcal{H}$ inequality for CR-(h_1, h_2)-convex function. In what follows, let $H(x,y) = h_1(x)h_2(y)$.

Theorem 4. *Consider* $h_1, h_2 : (0,1) \to \mathbf{R}^+$ *and* $h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \neq 0$. *Let* $\vartheta : [f,g] \to \mathbf{R_I}^+$, *if* $\vartheta \in SX(CR\text{-}(h_1,h_2), [f,g], \mathbf{R_I}^+)$ *and* $\vartheta \in \mathbf{IR}_{[\mathbf{f},\mathbf{g}]}$, *we have*

$$\frac{1}{2\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]}\vartheta\left(\frac{f+g}{2}\right) \preceq_{CR} \frac{1}{g-f}\int_f^g \vartheta(\sigma)d\sigma \preceq_{CR} [\vartheta(f) + \vartheta(g)]\int_0^1 H(s, 1-s)ds. \tag{8}$$

Proof. As $\vartheta \in SX(CR\text{-}(h_1,h_2), [f,g], \mathbf{R_I}^+)$, we have

$$\frac{1}{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]}\vartheta\left(\frac{f+g}{2}\right) \preceq_{CR} \vartheta(sf + (1-s)g) + \vartheta((1-s)f + sg).$$

Integrating (change of variables), we have

$$
\begin{aligned}
\frac{1}{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]}\vartheta\left(\frac{f+g}{2}\right) &\preceq_{CR} \left[\int_0^1 \vartheta(sf+(1-s)g)ds + \int_0^1 \vartheta((1-s)f+sg)ds\right] \\
&= \left[\int_0^1 \underline{\vartheta}(sf+(1-s)g)ds + \int_0^1 \underline{\vartheta}((1-s)f+sg)ds,\right. \\
&\quad \left.\int_0^1 \overline{\vartheta}(sf+(1-s)g)ds + \int_0^1 \overline{\vartheta}((1-s)f+sg)ds\right] \\
&= \left[\frac{2}{g-f}\int_f^g \underline{\vartheta}(\sigma)d\sigma, \frac{2}{g-f}\int_f^g \overline{\vartheta}(\sigma)d\sigma\right] \\
&= \frac{2}{g-f}\int_f^g \vartheta(\sigma)d\sigma. \qquad (9)
\end{aligned}
$$

By Definition 6, we have

$$\vartheta(sf+(1-s)g) \preceq_{CR} h_1(s)h_2(1-s)\vartheta(f) + h_1(1-s)h_2(s)\vartheta(g).$$

Integrating (change of variables), we have

$$\int_0^1 \vartheta(sf+(1-s)g)ds \preceq_{CR} \vartheta(f)\int_0^1 h_1(s)h_2(1-s)ds + \vartheta(g)\int_0^1 h_1(1-s)h_2(s)ds.$$

Accordingly,

$$\frac{1}{g-f}\int_f^g \vartheta(\sigma)d\sigma \preceq_{CR} [\vartheta(f)+\vartheta(g)]\int_0^1 H(s,1-s)ds. \qquad (10)$$

Now, unite (9) and (10), we obtain desired outcome

$$\frac{1}{2\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]}\vartheta\left(\frac{t+u}{2}\right) \preceq_{CR} \frac{1}{g-f}\int_f^g \vartheta(\sigma)d\sigma \preceq_{CR} [\vartheta(f)+\vartheta(g)]\int_0^1 H(s,1-s)ds.$$

□

Remark 2.

(i) *If $h_1(s) = h_2(s) = 1$, Theorem 4 incorporates output for CR-P-function:*

$$\frac{1}{2}\vartheta\left(\frac{f+g}{2}\right) \preceq_{CR} \frac{1}{g-f}\int_f^g \vartheta(\sigma)d\sigma \preceq_{CR} [\vartheta(f)+\vartheta(g)];$$

(ii) *If $h_1(s) = \frac{1}{h(s)}$ and $h_2(s) = 1$, Theorem 4 incorporates output for CR-h-GL-function:*

$$\frac{h\left(\frac{1}{2}\right)}{2}\vartheta\left(\frac{f+g}{2}\right) \preceq_{CR} \frac{1}{g-f}\int_f^g \vartheta(\sigma)d\sigma \preceq_{CR} \int_0^1 \frac{ds}{h(s)};$$

(iii) *If $h_1(s) = h(s)$ and $h_2(s) = 1$, Theorem 4 incorporates output for CR-h-convex function:*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\vartheta\left(\frac{f+g}{2}\right) \preceq_{CR} \frac{1}{g-f}\int_f^g \vartheta(\sigma)d\sigma \preceq_{CR} \int_0^1 h(s)ds;$$

(iv) *If $h_1(s) = \frac{1}{h_1(s)}$ and $h_2(s) = \frac{1}{h_2(s)}$, Theorem 4 incorporates output for CR-(h_1,h_2)-convex function:*

$$\frac{1}{2\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]}\vartheta\left(\frac{f+g}{2}\right) \preceq_{CR} \frac{1}{g-f}\int_f^g \vartheta(\sigma)d\sigma \preceq_{CR} \int_0^1 H(s,1-s)ds.$$

Example 3. *Looking back at Example 2, one has*

$$\frac{1}{2\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]}\vartheta\left(\frac{f+g}{2}\right)=\vartheta\left(\frac{1}{2}\right)=\left[\frac{5}{2},\frac{9}{2}\right],$$

$$\frac{1}{g-f}\int_f^g \vartheta(\sigma)d\sigma=\left[\int_0^1(-2\sigma^2+3)d\sigma,\int_0^1(2\sigma^2+4)d\sigma\right]=\left[\frac{7}{3},\frac{14}{3}\right],$$

$$[\vartheta(f)+\vartheta(g)]\int_0^1 H(s,1-s)ds=[2,5].$$

So, we have

$$\left[\frac{5}{2},\frac{9}{2}\right]\preceq_{CR}\left[\frac{7}{3},\frac{14}{3}\right]\preceq_{CR}[2,5].$$

This verify the above theorem.

Theorem 5. *Define* $h_1,h_2:(0,1)\to\mathbf{R}^+$ *and* $h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\neq 0$. *Let* $\vartheta:[f,g]\to\mathbf{R_I}^+$, *if* $\vartheta\in SX(CR\text{-}(h_1,h_2),[f,g],\mathbf{R_I}^+)$ *and* $\vartheta\in\mathbf{IR}_{[\mathbf{f},\mathbf{g}]}$, *we have*

$$\frac{1}{4\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\vartheta\left(\frac{f+g}{2}\right)\preceq_{CR}\triangle_1\preceq_{CR}\frac{1}{g-f}\int_f^g\vartheta(\sigma)d\sigma\preceq_{CR}\triangle_2$$
$$\preceq_{CR}\left\{[\vartheta(f)+\vartheta(g)]\left[\frac{1}{2}+H\left(\frac{1}{2},\frac{1}{2}\right)\right]\right\}\int_0^1 H(s,1-s)ds,$$

where

$$\triangle_1=\frac{1}{4H\left(\frac{1}{2},\frac{1}{2}\right)}\left[\vartheta\left(\frac{3f+g}{4}\right)+\vartheta\left(\frac{3g+f}{4}\right)\right]$$

and

$$\triangle_2=\left[\vartheta\left(\frac{f+g}{2}\right)+\frac{\vartheta(f)+\vartheta(g)}{2}\right]\int_0^1 H(s,1-s)ds.$$

Proof. Take $\left[f,\frac{f+g}{2}\right]$, we have

$$\vartheta\left(\frac{3f+g}{4}\right)\preceq_{CR}H\left(\frac{1}{2},\frac{1}{2}\right)\vartheta\left(sf+(1-s)\frac{f+g}{2}\right)+H\left(\frac{1}{2},\frac{1}{2}\right)\vartheta\left((1-s)f+s\frac{f+g}{2}\right).$$

Integrating (change of variables), we have

$$\begin{aligned}\vartheta\left(\frac{3f+g}{2}\right)&\preceq_{CR}H\left(\frac{1}{2},\frac{1}{2}\right)\left[\int_0^1\vartheta\left(sf+(1-s)\frac{f+g}{2}\right)ds+\int_0^1\vartheta\left(s\frac{f+g}{2}+(1-s)g\right)ds\right]\\&=H\left(\frac{1}{2},\frac{1}{2}\right)\left[\frac{2}{g-f}\int_f^{\frac{f+g}{2}}\vartheta(\sigma)d\sigma+\frac{2}{g-f}\int_f^{\frac{f+g}{2}}\vartheta(\sigma)d\sigma\right]\\&=H\left(\frac{1}{2},\frac{1}{2}\right)\left[\frac{4}{g-f}\int_f^{\frac{f+g}{2}}\vartheta(\sigma)d\sigma\right].\end{aligned}\tag{11}$$

Accordingly,

$$\frac{1}{4H\left(\frac{1}{2},\frac{1}{2}\right)}\vartheta\left(\frac{3f+g}{2}\right)\preceq_{CR}\frac{1}{g-f}\int_f^{\frac{f+g}{2}}\vartheta(\sigma)d\sigma.\tag{12}$$

Now, consider $\left[\frac{f+g}{2}, g\right]$, one has

$$\frac{1}{4H\left(\frac{1}{2},\frac{1}{2}\right)}\vartheta\left(\frac{3g+f}{2}\right) \preceq_{CR} \frac{1}{g-f}\int_{\frac{f+g}{2}}^{g}\vartheta(\sigma)d\sigma. \tag{13}$$

Adding Equations (12) and (13), we have

$$\triangle_1 = \frac{1}{4H\left(\frac{1}{2},\frac{1}{2}\right)}\left[\vartheta\left(\frac{3f+g}{4}\right)+\vartheta\left(\frac{3g+f}{4}\right)\right] \preceq_{CR} \left[\frac{1}{g-f}\int_{f}^{g}\vartheta(\sigma)d\sigma\right].$$

Now, we have

$$\begin{aligned}
&\frac{1}{4\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\vartheta\left(\frac{f+g}{2}\right)\\
&= \frac{1}{4\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\vartheta\left(\frac{1}{2}\left(\frac{3f+g}{4}\right)+\frac{1}{2}\left(\frac{3g+f}{4}\right)\right)\\
&\preceq_{CR} \frac{1}{4\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\left[H\left(\frac{1}{2},\frac{1}{2}\right)\vartheta\left(\frac{3f+g}{4}\right)+H\left(\frac{1}{2},\frac{1}{2}\right)\vartheta\left(\frac{3g+f}{4}\right)\right]\\
&= \frac{1}{4H\left(\frac{1}{2},\frac{1}{2}\right)}\left[\vartheta\left(\frac{3f+g}{4}\right)+\vartheta\left(\frac{3g+f}{4}\right)\right] = \triangle_1\\
&\preceq_{CR} \frac{1}{4H\left(\frac{1}{2},\frac{1}{2}\right)}\left\{H\left(\frac{1}{2},\frac{1}{2}\right)\left[\vartheta(f)+\vartheta\left(\frac{f+g}{2}\right)\right]+H\left(\frac{1}{2},\frac{1}{2}\right)\left[\vartheta(g)+\vartheta\left(\frac{f+g}{2}\right)\right]\right\}\\
&= \frac{1}{2}\left[\frac{\vartheta(f)+\vartheta(g)}{2}+\vartheta\left(\frac{f+g}{2}\right)\right]\\
&\preceq_{CR} \left[\frac{\vartheta(f)+\vartheta(g)}{2}+\vartheta\left(\frac{f+g}{2}\right)\right]\int_0^1 H(s,1-s)ds = \triangle_2\\
&\preceq_{CR} \left[\frac{\vartheta(f)+\vartheta(g)}{2}+H\left(\frac{1}{2},\frac{1}{2}\right)\vartheta(f)+H\left(\frac{1}{2},\frac{1}{2}\right)\vartheta(g)\right]\int_0^1 H(s,1-s)ds\\
&\preceq_{CR} \left[\frac{\vartheta(f)+\vartheta(g)}{2}+H\left(\frac{1}{2},\frac{1}{2}\right)[\vartheta(f)+\vartheta(g)]\right]\int_0^1 H(s,1-s)ds\\
&\preceq_{CR} \left\{[\vartheta(f)+\vartheta(g)]\left[\frac{1}{2}+H\left(\frac{1}{2},\frac{1}{2}\right)\right]\right\}\int_0^1 H(s,1-s)ds.
\end{aligned}$$

This completes the proof. □

Example 4. *Looking back at Example 3, one has*

$$\frac{1}{4\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\vartheta\left(\frac{f+g}{2}\right) = \vartheta\left(\frac{1}{2}\right) = \left[\frac{5}{2},\frac{9}{2}\right],$$

$$\begin{aligned}
\triangle_1 &= \frac{1}{2}\left[\vartheta\left(\frac{1}{4}\right)+\vartheta\left(\frac{3}{4}\right)\right] = \left[\frac{19}{8},\frac{37}{8}\right],\\
\triangle_2 &= \left[\frac{\vartheta(0)+\vartheta(1)}{2}+\vartheta\left(\frac{1}{2}\right)\right]\int_0^1 H(s,1-s)ds,\\
&= \frac{1}{2}\left([2,5]+\left[\frac{5}{2},\frac{9}{2}\right]\right)\\
&= \left[\frac{9}{4},\frac{19}{4}\right]
\end{aligned}$$

and

$$\left\{[\vartheta(f)+\vartheta(g)]\left[\frac{1}{2}+H\left(\frac{1}{2},\frac{1}{2}\right)\right]\right\}\int_0^1 H(s,1-s)ds=[2,5].$$

As a result, we have

$$\left[\frac{5}{2},\frac{9}{2}\right]\preceq_{CR}\left[\frac{19}{8},\frac{37}{8}\right]\preceq_{CR}\left[\frac{7}{3},\frac{14}{3}\right]\preceq_{CR}\left[\frac{9}{4},\frac{19}{4}\right]\preceq_{CR}[2,5].$$

This verifies Theorem 5.

Theorem 6. *Let $\vartheta,\varphi:[f,g]\to\mathbf{R_I}^+$ and $h_1,h_2:(0,1)\to\mathbf{R}^+$ such that $h_1,h_2\neq 0$. If $\vartheta\in SX(CR\text{-}(h_1,h_2),[f,g],\mathbf{R_I}^+)$, $\varphi\in SX(CR\text{-}(h_1,h_2),[f,g],\mathbf{R_I}^+)$ and $\vartheta,\varphi\in\mathbf{IR}_{[\mathbf{f},\mathbf{g}]}$, then*

$$\frac{1}{g-f}\int_f^g\vartheta(\sigma)\varphi(\sigma)d\sigma\preceq_{CR}M(f,g)\int_0^1 H^2(s,1-s)ds+N(f,g)\int_0^1 H(s,s)H(1-s,1-s)ds,$$

where

$$M(f,g)=\vartheta(f)\varphi(f)+\vartheta(g)\varphi(g)$$

and

$$N(f,g)=\vartheta(f)\varphi(g)+\vartheta(g)\varphi(f).$$

Proof. Consider $\vartheta\in SX(CR\text{-}(h_1,h_2),[f,g],\mathbf{R_I}^+)$ and $\varphi\in SX(CR\text{-}(h_1,h_2),[f,g],\mathbf{R_I}^+)$, then

$$\vartheta(fs+(1-s)g)\preceq_{CR}h_1(s)h_2(1-s)\vartheta(f)+h_1(1-s)h_2(s)\vartheta(g)$$

and

$$\varphi(fs+(1-s)g)\preceq_{CR}h_1(s)h_2(1-s)\varphi(f)+h_1(1-s)h_2(s)\varphi(g).$$

Then, we obtain

$$\begin{aligned}&\vartheta(fs+(1-s)g)\varphi(fs+(1-s)g)\\&\preceq_{CR}H^2(s,1-s)\vartheta(f)\varphi(f)+H^2(1-s,s)[\vartheta(f)\varphi(g)+\vartheta(g)\varphi(f)]\\&\quad+H(s,s)H(1-s,1-s)\vartheta(g)\varphi(g).\end{aligned}$$

Integrating (change of variables), we have

$$\begin{aligned}&\int_0^1\vartheta(fs+(1-s)g)\varphi(fs+(1-s)g)ds\\&=\left[\int_0^1\underline{\vartheta}(fs+(1-s)g)\underline{\varphi}(fs+(1-s)g)ds,\int_0^1\overline{\vartheta}(fs+(1-s)g)\overline{\varphi}(fs+(1-s)g)ds\right]\\&=\left[\frac{1}{g-f}\int_f^g\underline{\vartheta}(\sigma)\underline{\varphi}(\sigma)d\sigma,\frac{1}{g-f}\int_f^g\overline{\vartheta}(\sigma)\overline{\varphi}(\sigma)d\sigma\right]\\&=\frac{1}{g-f}\int_f^g\vartheta(\sigma)\varphi(\sigma)d\sigma\\&\preceq_{CR}\int_0^1[\vartheta(f)\varphi(f)+\vartheta(g)\varphi(g)]H^2(s,1-s)ds\\&\quad+\int_0^1[\vartheta(f)\varphi(g)+\vartheta(g)\varphi(f)]H(s,s)H(1-s,1-s)ds.\end{aligned}$$

As a result of this,

$$\frac{1}{g-f}\int_f^g \vartheta(\sigma)\varphi(\sigma)d\sigma \preceq_{CR} M(f,g)\int_0^1 H^2(s,1-s)ds + N(f,g)\int_0^1 H(s,s)H(1-s,1-s)ds.$$

This completes the proof. □

Example 5. *Let* $[f,g]=[1,2]$, $h_1(s)=s$ *and* $h_2(s)=1$, *for all* $s\in(0,1)$. *If* $\vartheta,\varphi:[f,g]\to \mathbf{R_I}^+$ *are defined as*

$$\vartheta(\sigma)=[-\sigma^2+2,\sigma^2+3] \ \text{ and } \ \varphi(\sigma)=[-\sigma+1,\sigma+2].$$

So, we have

$$\frac{1}{g-f}\int_f^g \vartheta(\sigma)\varphi(\sigma)d\sigma = \left[\frac{5}{12},\frac{227}{12}\right],$$

$$M(f,g)\int_0^1 H^2(s,1-s)ds = M(1,2)\int_0^1 s^2ds = \left[\frac{-8}{3},\frac{40}{3}\right]$$

and

$$N(f,g)\int_0^1 H(s,s)H(1-s,1-s)ds = N(1,2)\int_0^1 s(1-s)ds = \left[\frac{-10}{6},\frac{29}{6}\right].$$

Consequently,

$$\left[\frac{5}{12},\frac{227}{12}\right] \preceq_{CR} \left[\frac{-8}{3},\frac{40}{3}\right]+\left[\frac{-10}{6},\frac{29}{6}\right]=\left[\frac{-13}{3},\frac{109}{6}\right].$$

Therefore, the theorem above holds.

Theorem 7. *Let* $\varphi,\vartheta:[f,g]\to\mathbf{R_I}^+$ *and* $h_1,h_2:(0,1)\to\mathbf{R}^+$ *such that* $h_1,h_2\neq 0$. *If* $\vartheta\in SX(CR\text{-}(h_1,h_2),[f,g],\mathbf{R_I}^+)$, $\varphi\in SX(CR\text{-}(h_1,h_2),[f,g],\mathbf{R_I}^+)$ *and* $\vartheta,\varphi\in \mathbf{IR}_{[\mathbf{f},\mathbf{g}]}$, *then*

$$\frac{1}{2\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\vartheta\left(\frac{f+g}{2}\right)\varphi\left(\frac{f+g}{2}\right) \preceq_{CR} \frac{1}{g-f}\int_f^g \vartheta(\sigma)\varphi(\sigma)d\sigma + M(f,g)\int_0^1 H(s,s)H(1-s,1-s)ds + N(f,g)\int_0^1 H^2(s,1-s)ds.$$

Proof. Since $\vartheta\in SX(CR\text{-}(h_1,h_2),[f,g],\mathbf{R_I}^+)$ and $\varphi\in SX(CR\text{-}(h_1,h_2),[f,g],\mathbf{R_I}^+)$, one has

$$\vartheta\left(\frac{f+g}{2}\right) \preceq_{CR} H\left(\frac{1}{2},\frac{1}{2}\right)\vartheta(fs+(1-s)g)+H\left(\frac{1}{2},\frac{1}{2}\right)\vartheta(f(1-s)+sg)$$

and

$$\varphi\left(\frac{f+g}{2}\right) \preceq_{CR} H\left(\frac{1}{2},\frac{1}{2}\right)\varphi(fs+(1-s)g)+H\left(\frac{1}{2},\frac{1}{2}\right)\varphi(f(1-s)+sg).$$

Then, we have

$$
\begin{aligned}
&\vartheta\left(\frac{f+g}{2}\right)\varphi\left(\frac{f+g}{2}\right) \\
&\preceq_{CR} \left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2[\vartheta(fs+(1-s)g)\varphi(fs+(1-s)g)+\vartheta(f(1-s)+sg)\varphi(f(1-s)+sg)] \\
&+\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2[\vartheta(fs+(1-s)g)\varphi(f(1-s)+sg)+\vartheta(f(1-s)+sg)\varphi(fs+(1-s)g)] \\
&\preceq_{CR} \left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2[\vartheta(fs+(1-s)g)\varphi(fs+(1-s)g)+\vartheta(f(1-s)+sg)\varphi(f(1-s)+sg)] \\
&+\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2[H(s,1-s)\vartheta(f)+H(1-s,s)\vartheta(g))(H(1-s,s)\varphi(f)+H(s,1-s)\varphi(g)) \\
&+(H(1-s,s)\varphi(f)+H(s,1-s)\varphi(g))(H(s,1-s)\varphi(f)+H(1-s,s)\varphi(g))] \\
&\preceq_{CR} \left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2[\vartheta(fs+(1-s)g)\varphi(fs+(1-s)g)+\vartheta(f(1-s)+sg)\varphi(f(1-s)+sg)] \\
&+\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2\left[(2H(s,s)H(1-s,1-s))M(f,g)+\left(H^2(s,1-s)+H^2(1-s,s)\right)N(f,g)\right]
\end{aligned}
$$

Integrating (change of variables), we have

$$
\begin{aligned}
\int_0^1 \vartheta\left(\frac{f+g}{2}\right)\varphi\left(\frac{f+g}{2}\right)ds &= \left[\int_0^1 \underline{\vartheta}\left(\frac{f+g}{2}\right)\underline{\varphi}\left(\frac{f+g}{2}\right)ds, \int_0^1 \overline{\vartheta}\left(\frac{f+g}{2}\right)\overline{\varphi}\left(\frac{f+g}{2}\right)ds\right] \\
&\preceq_{CR} 2\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2\left[\frac{1}{g-f}\int_f^g \vartheta(\sigma)\varphi(\sigma)d\sigma\right] \\
&\quad +2\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2\left[M(f,g)\int_0^1 H(s,s)H(1-s,1-s)ds\right. \\
&\quad \left. +N(f,g)\int_0^1 H^2(s,1-s)ds\right].
\end{aligned}
$$

Divide both sides by $\frac{1}{2\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}$ above equation, we get

$$
\begin{aligned}
&\frac{1}{2\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\vartheta\left(\frac{f+g}{2}\right)\varphi\left(\frac{f+g}{2}\right) \preceq_{CR} \frac{1}{g-f}\int_f^g \vartheta(\sigma)\varphi(\sigma)d\sigma \\
&+M(f,g)\int_0^1 H(s,s)H(1-s,1-s)ds+N(f,g)\int_0^1 H^2(s,1-s)ds.
\end{aligned}
$$

Therefore, the proof is completed. □

Example 6. *Looking back at Example 5, we have*

$$
\begin{gathered}
\frac{1}{2\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\vartheta\left(\frac{f+g}{2}\right)\varphi\left(\frac{f+g}{2}\right) = \frac{1}{2}\vartheta\left(\frac{3}{2}\right)\varphi\left(\frac{3}{2}\right) = \left[\frac{-21}{16},\frac{147}{16}\right], \\
\frac{1}{g-f}\int_f^g \vartheta(\sigma)\varphi(\sigma)d\sigma = \left[\frac{5}{12},\frac{227}{12}\right], \\
M(f,g)\int_0^1 H(s,s)H(1-s,1-s)ds = M(1,2)\int_0^1 s(1-s)ds = \left[\frac{-8}{6},\frac{40}{6}\right]
\end{gathered}
$$

and

$$
N(f,g)\int_0^1 H^2(s,1-s)ds = N(1,2)\int_0^1 s^2ds = \left[\frac{-10}{3},\frac{29}{3}\right].
$$

It follows that

$$\left[\frac{-21}{16}, \frac{147}{16}\right] \preceq_{CR} \left[\frac{5}{12}, \frac{227}{12}\right] + \left[\frac{-8}{6}, \frac{40}{6}\right] + \left[\frac{-10}{3}, \frac{29}{3}\right] = \left[\frac{-17}{4}, \frac{141}{4}\right].$$

Therefore, the theorem above holds.

4. Jensen Type Inequality for CR-(h_1, h_2)-Convex Mapping

Following that, we will prove a Jensen-type inequality for the CR-(h_1, h_2)-convex function.

Theorem 8. *Let $f_i \in \mathbf{R}^+$ and $j_i \in [f, g]$. If h_1, h_2 are super multiplicative non-negative functions and $\vartheta \in SX(CR\text{-}(h_1, h_2), [f, g], \mathbf{R_I}^+)$. Then*

$$\vartheta\left(\frac{1}{F_k}\sum_{i=1}^{k} f_i j_i\right) \preceq_{CR} \sum_{i=1}^{k} H\left(\frac{f_i}{F_k}, \frac{F_{k-1}}{F_k}\right)\vartheta(j_i), \tag{14}$$

where $F_k = \sum_{i=1}^{k} f_i$.

Proof. If $k = 2$, then (14) holds. Suppose that (14) is also valid for $k-1$, then

$$\begin{aligned}
&\vartheta\left(\frac{1}{F_k}\sum_{i=1}^{k} f_i j_i\right) = \vartheta\left(\frac{f_k}{F_k}v_k + \sum_{i=1}^{k-1}\frac{f_i}{F_k}j_i\right) \\
&\preceq_{CR} h_1\left(\frac{f_k}{F_k}\right)h_2\left(\frac{F_{k-1}}{F_k}\right)\vartheta(j_k) + h_1\left(\frac{F_{k-1}}{F_k}\right)h_2\left(\frac{f_k}{F_k}\right)\vartheta\left(\sum_{i=1}^{k-1}\frac{f_i}{F_k}j_i\right) \\
&\preceq_{CR} h_1\left(\frac{f_k}{F_k}\right)h_2\left(\frac{F_{k-1}}{F_k}\right)\vartheta(j_k) + h_1\left(\frac{F_{k-1}}{F_k}\right)h_2\left(\frac{f_k}{F_k}\right)\sum_{i=1}^{k-1}\left[H\left(\frac{f_i}{F_k}, \frac{F_{k-2}}{F_{k-1}}\right)\vartheta(j_i)\right] \\
&\preceq_{CR} h_1\left(\frac{f_k}{F_k}\right)h_2\left(\frac{F_{k-1}}{F_k}\right)\vartheta(j_k) + \sum_{i=1}^{k-1} H\left(\frac{f_i}{F_k}, \frac{F_{k-2}}{F_{k-1}}\right)\vartheta(j_i) \\
&\preceq_{CR} \sum_{i=1}^{k} H\left(\frac{f_i}{F_k}, \frac{F_{k-1}}{F_k}\right)\vartheta(j_i).
\end{aligned}$$

It follows from Mathematical induction that the conclusion is correct. □

Remark 3.

(i) *If $h_1(s) = h_2(s) = 1$, Theorem 8 incorporates output for CR- P-function:*

$$\vartheta\left(\frac{1}{F_k}\sum_{i=1}^{k} f_i j_i\right) \preceq_{CR} \sum_{i=1}^{k}\vartheta(j_i);$$

(ii) *If $h_1(s) = \frac{1}{h_1(s)}$ and $h_2(s) = \frac{1}{h_2(s)}$, Theorem 8 incorporates output for CR-(h_1, h_2)-GL function:*

$$\vartheta\left(\frac{1}{F_k}\sum_{i=1}^{k} f_i j_i\right) \preceq_{CR} \sum_{i=1}^{k}\left[\frac{\vartheta(j_i)}{H\left(\frac{f_i}{F_k}, \frac{F_{k-1}}{F_k}\right)}\right];$$

(iii) *If $h_1(s) = s$ and $h_2(s) = 1$, Theorem 8 incorporates output for CR-convex function:*

$$\vartheta\left(\frac{1}{F_k}\sum_{i=1}^{k} f_i j_i\right) \preceq_{CR} \sum_{i=1}^{k}\frac{f_i}{F_k}\vartheta(j_i);$$

(iv) If $h_1(s) = h(s)$ and $h_2(s) = 1$, Theorem 8 incorporates output for CR-h-convex function:

$$\vartheta\left(\frac{1}{F_k}\sum_{i=1}^{k} f_i j_i\right) \preceq_{CR} \sum_{i=1}^{k} h\left(\frac{f_i}{F_k}\right)\vartheta(j_i);$$

(v) If $h_1(s) = \frac{1}{h(s)}$ and $h_2(s) = 1$, Theorem 8 incorporates output for CR-h-GL-function:

$$\vartheta\left(\frac{1}{F_k}\sum_{i=1}^{k} f_i j_i\right) \preceq_{CR} \sum_{i=1}^{k} \left[\frac{\vartheta(j_i)}{h\left(\frac{f_i}{F_k}\right)}\right];$$

(vi) If $h_1(s) = \frac{1}{(s)^s}$ and $h_2(s) = 1$, Theorem 8 incorporates output for CR-s-convex function:

$$\sigma\left(\frac{1}{F_k}\sum_{i=1}^{k} f_i j_i\right) \preceq_{CR} \sum_{i=1}^{k} \left(\frac{f_i}{F_k}\right)^{s}\vartheta(j_i).$$

5. Conclusions

In the present study, we developed the concept of (h_1, h_2)-Convex functions pertaining to CR-order relation for $\mathcal{IVFS}$. This new concept allows us to achieve much more precise results than other partial order relations since the interval difference between endpoints is much smaller in examples based on this new concept. Generalizations can be drawn from the recent findings described in [39,47,48]. Furthermore, some nontrivial examples are provided to test the validity of our main findings. Considering the widespread use of integral operators in engineering and other applied sciences, such as different types of mathematical modeling, and the fact that different integral operators are appropriate for various practical problems, our study of interval integral operator-type integral inequalities will expand their potential applications in practice. It might be interesting to determine equivalent inequalities for different types of convexity in the future. This concept is expected to be useful to other researchers in a variety of scientific fields.

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