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New Results for Multivalued Mappings in Hausdorff Neutrosophic Metric Spaces

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Abstract: The fundamental goal of this paper is to derive common fixed-point results for a sequence of multivalued mappings contained in a closed ball over a complete neutrosophic metric space. A basic and distinctive procedure has been used to prove the proposed results.

Keywords: fixed point; Hausdorff neutrosophic metric space; multivalued mapping

MSC: 46S40; 47H10; 54H25

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1. Introduction

Multivalued functions are a particular type of relations rather than a generalization of single-valued functions. These functions assign more than one value to each input and often exist while reversing many-to-one functions. These functions rise with several results as extensions of single-valued functions over continuity, contraction mappings, fixed-point theorems, optimization, differentiation, integration, and topological degree theory. The following theorem is the first significant extension that has been done over Brouwer's work on fixed points.

Theorem 1 (Kakutani, 1941 [1]). *If $x \to \Phi(x)$ is an upper semi-continuous point-to-set mapping of an r-dimensional closed simplex S into $\mathfrak{R}(S)$, then there exists an $x_0 \in S$ such that $x_0 \in \Phi(x_0)$.*

In 1953, Strother [2] worked on an open question concerning fixed points; he asserted that a space with a fixed-point property for single-valued functions need not have the fixed-point property for multivalued functions, and this assertion has added credit to the multivalued functions. In the year 1969, Nadler extended the Banach Contraction Principle via multivalued contraction mappings.

Theorem 2 (Nadler, 1969 [3]). *Let (X, d) be a complete metric space and $CB(X)$ be the family of nonempty closed and bounded subsets of X. If $F : X \to CB(X)$ is a multivalued contraction mapping, then F has a fixed point.*

Since the establishment of such initiations over multivalued functions, many more fixed-point theorems for multivalued mappings have been demonstrated in various spaces. Beg et al. [4], Chaipunya et al. [5], Khan et al. [6], Mutlu et al. [7], Mustafa et al. [8], Mehmood et al. [9] and Arshad and Shoaib [10] are a few authors whose works have demonstrated, respectively, these developments in convex metric spaces, modular metric spaces, partial metric spaces, bipolar metric spaces, G-metric spaces, cone metric spaces and fuzzy metric spaces. In the interim, Rodriguez-Lopez and Romaguera [11] introduced the Hausdorff fuzzy metric for compact sets. By combining the ideas of fuzzy metrics and Hausdorff topology, Shoaib et al. [12] produced a fixed-point result for a family of multivalued mappings that are contractive on a sequence enclosed in a closed ball rather than the entire space.

Since the class of intuitionistic fuzzy metric spaces is more diverse than the class of fuzzy metric spaces, such a study is then applied to Hausdorff intuitionistic fuzzy metric spaces [13]. In light of these developments, this work aims to obtain a common fixed-point result for a family of multivalued mappings constructed over Hausdorff neutrosophic metric spaces. Additionally, an example is provided to demonstrate the applicability of the main result.

2. Preliminaries

Definition 1 ([14])**.** *Let $Y = [0,1]$. A binary operation $\diamond$ defined from $Y \times Y$ to Y is called:*

(i) *a continuous t-norm (shortly, ctn) if*

(n1) $\diamond$ *is associative, commutative and continuous;*

(n2) $\mathfrak{u} \diamond 1 = \mathfrak{u}$*, for all* $\mathfrak{u} \in Y$*;*

(n3) $\mathfrak{u} \diamond \beta \leq \mathfrak{v} \diamond \delta$ *whenever* $\mathfrak{u} \leq \mathfrak{v}$ *and* $\beta \leq \delta$ *for each* $\mathfrak{u}, \beta, \mathfrak{v}, \delta \in Y$.

(ii) *a continuous t-conorm [shortly, ctcn] if*

(cn1)$\diamond$ *is associative, commutative, continuous;*

(cn2)$\mathfrak{u} \diamond 0 = \mathfrak{u}$ *for all* $\mathfrak{u} \in Y$*;*

(cn3)$\mathfrak{u} \diamond \beta \leq \mathfrak{v} \diamond \delta$ *whenever* $\mathfrak{u} \leq \mathfrak{v}$ *and* $\beta \leq \delta$ *for each* $\mathfrak{u}, \beta, \mathfrak{v}, \delta \in Y$.

The neutrosophic set [15] is the basis for the space that served as the starting point for the suggested task. Three different types of values are given to each element in this set, measuring the degrees of membership, nonmembership, and indeterminacy. It is richer than the classical set, fuzzy set, and intuitionistic fuzzy set due to this characteristic. There are publications that define metrics over the neutrosophic sets, with [16–19] a few worth mentioning. The one selected for this study is found at [18].

Definition 2 ([18])**.** *A 6-tuple $(U, \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \diamond, \circledast)$ is said to be an Neutrosophic Metric Space (shortly, NMS) if U is an arbitrary nonempty set, $\diamond$ is a ctn, $\circledast$ is a ctcn, and $\mathfrak{A}, \mathfrak{B}$, and $\mathfrak{C}$ are neutrosophic sets on $U^2 \times \mathbb{R}^+$ satisfying the following conditions for all $\zeta, \nu, \delta, \omega \in U$, $\lambda \in \mathbb{R}^+$:*

1. $0 \leq \mathfrak{A}(\zeta,\nu,\lambda) \leq 1;\ 0 \leq \mathfrak{B}(\zeta,\nu,\lambda) \leq 1;\ 0 \leq \mathfrak{C}(\zeta,\nu,\lambda) \leq 1;$
2. $\mathfrak{A}(\zeta,\nu,\lambda) + \mathfrak{B}(\zeta,\nu,\lambda) + \mathfrak{C}(\zeta,\nu,\lambda) \leq 3;$
3. $\mathfrak{A}(\zeta,\nu,\lambda) = 1$ *if and only if* $\zeta = \nu$;
4. $\mathfrak{A}(\zeta,\nu,\lambda) = \mathfrak{A}(\nu,\zeta,\lambda);$
5. $\mathfrak{A}(\zeta,\nu,\lambda) \diamond \mathfrak{A}(\nu,\delta,\mu) \leq \mathfrak{A}(\zeta,\delta,\lambda+\mu)$*, for all* $\lambda, \mu > 0$;
6. $\mathfrak{A}(\zeta,\nu,\cdot) : [0,\infty) \to [0,1]$ *is neutrosophic continuous;*
7. $\lim_{\lambda\to\infty} \mathfrak{A}(\zeta,\nu,\lambda) = 1$ *for all* $\lambda > 0$;
8. $\mathfrak{B}(\zeta,\nu,\lambda) = 0$ *if and only if* $\zeta = \nu$;
9. $\mathfrak{B}(\zeta,\nu,\lambda) = \mathfrak{A}(\nu,\zeta,\lambda);$
10. $\mathfrak{B}(\zeta,\nu,\lambda) \circledast \mathfrak{B}(\nu,\delta,\mu) \geq \mathfrak{B}(\zeta,\delta,\lambda+\mu)$ *for all* $\lambda, \mu > 0$;
11. $\mathfrak{B}(\zeta,\nu,\cdot) : [0,\infty) \to [0,1]$ *is neutrosophic continuous;*
12. $\lim_{\lambda\to\infty} \mathfrak{B}(\zeta,\nu,\lambda) = 0$ *for all* $\lambda > 0$;
13. $\mathfrak{C}(\zeta,\nu,\lambda) = 0$ *if and only if* $\zeta = \nu$;
14. $\mathfrak{C}(\zeta,\nu,\lambda) = \mathfrak{C}(\nu,\zeta,\lambda);$
15. $\mathfrak{C}(\zeta,\nu,\lambda) \circledast \mathfrak{C}(\nu,\delta,\mu) \geq \mathfrak{C}(\zeta,\delta,\lambda+\mu)$ *for all* $\lambda, \mu > 0$;
16. $\mathfrak{C}(\zeta,\nu,\cdot) : [0,\infty) \to [0,1]$ *is neutrosophic continuous;*
17. $\lim_{\lambda\to\infty} \mathfrak{C}(\zeta,\nu,\lambda) = 0$ *for all* $\lambda > 0$.

Then, $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is called a Neutrosophic Metric on U. The functions $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ denote, respectively, the degrees of closedness, neturalness, and non-closedness, between ζ, ν, and δ with respect to λ.

The last condition of the aforementioned definition is omitted here since the domain of λ is $\mathbb{R}^+$.

Example 1 ([18])**.** *Let (U, d) be a metric space. Define $\omega \diamond \tau = \min\{\omega, \tau\}$ and $\omega \circledast \tau = \max\{\omega, \tau\}$ for all $\omega, \tau \in [0,1]$, and let $\mathfrak{A}_d, \mathfrak{B}_d, \mathfrak{C}_d : U^2 \times \mathbb{R}^+ \to [0,1]$ be defined by*

$$\mathfrak{A}(\zeta, \nu, \lambda) = \frac{\lambda}{\lambda + d(\zeta, \nu)}, \quad \mathfrak{B}(\zeta, \nu, \lambda) = \frac{d(\zeta, \nu)}{\lambda + d(\zeta, \nu)}, \quad \mathfrak{C}(\zeta, \nu, \lambda) = \frac{d(\zeta, \nu)}{\lambda},$$

for all $\zeta, \nu, \in U$ and $\lambda > 0$. Then, $(U, \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \diamond, \circledast)$ is an NMS.

Remark 1 ([18])**.** *In an NMS $(U, \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \diamond, \circledast)$, $\mathfrak{A}(\zeta, \nu, \cdot) : [0, \infty) \to [0,1]$ is nondecreasing, $\mathfrak{B}(\zeta, \nu, \cdot) : [0, \infty) \to [0,1]$ is nonincreasing, and $\mathfrak{C}(\zeta, \nu, \cdot) : [0, \infty) \to [0,1]$ is decreasing for all $\zeta, \nu \in U$.*

Definition 3. *Let $(U, \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \diamond, \circledast)$ be an NMS.*

(a) *A sequence $\{\zeta_n\}$ converges to a point $\zeta \in U$ if for all $\lambda > 0$, $\lim_{\lambda \to \infty} \mathfrak{A}(\zeta_n, \zeta, \lambda) = 1$, $\lim_{\lambda \to \infty} \mathfrak{B}(\zeta_n, \zeta, \lambda) = 0$ and $\lim_{\lambda \to \infty} \mathfrak{C}(\zeta_n, \zeta, \lambda) = 0$. In this case, ζ is called the limit of the sequence ζ_n, and we write $\lim_{n \to \infty} \zeta_n = \zeta$, or $\zeta_n \to \zeta$.*

(b) *A sequence $\{\zeta_n\}$ in $(U, \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \diamond, \circledast)$ is said to be a Cauchy sequence if $\lim_{\lambda \to \infty} \mathfrak{A}(\zeta_n, \zeta_{n+p}, \lambda) = 1$, $\lim_{\lambda \to \infty} \mathfrak{B}(\zeta_n, \zeta_{n+p}, \lambda) = 0$ and $\lim_{\lambda \to \infty} \mathfrak{C}(\zeta_n, \zeta_{n+p}, \lambda) = 0$ for all $\lambda > 0$ and $p > 0$.*

(c) *The space U is said to be complete if and only if every Cauchy sequence in U is convergent. It is called compact if every sequence has a convergent subsequence.*

Definition 4. *Let $(U, \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \diamond, \circledast)$ be an NMS. Let $0 < \epsilon < 1$, $\lambda > 0$. An open ball $B(\zeta, \epsilon, \lambda)$ with center $\zeta \in U$ and radius ϵ is defined as*

$$B(\zeta, \epsilon, \lambda) = \left\{\nu \in U : \mathfrak{A}(\zeta, \nu, \lambda) > 1 - \epsilon,\ \mathfrak{B}(\zeta, \nu, \lambda) < \epsilon,\ \mathfrak{C}(\zeta, \nu, \lambda) < \epsilon\right\}.$$

Definition 5. *Let B be a nonempty subset of an NMS $(U, \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \diamond, \circledast)$. For $\omega \in U$ and $\lambda > 0$, we define that*

$$\begin{aligned}
\mathfrak{A}(\omega, B, \lambda) &= \sup\left\{\mathfrak{A}(\omega, \tau, \lambda) : \tau \in B\right\}, \\
\mathfrak{B}(\omega, B, \lambda) &= \inf\left\{\mathfrak{B}(\omega, \tau, \lambda) : \tau \in B\right\}, \\
\mathfrak{C}(\omega, B, \lambda) &= \inf\left\{\mathfrak{C}(\omega, \tau, \lambda) : \tau \in B\right\}.
\end{aligned}$$

Definition 6. *Let $(U, \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \diamond, \circledast)$ be an NMS. Let $C(U)$ be the collection of all nonempty compact subsets of U. Let $A, B \in C(U)$ and $\lambda > 0$. Define $H_{\mathfrak{A}}$, $H_{\mathfrak{B}}$, and $H_{\mathfrak{C}} : C(U) \times C(U) \times (0, \infty) \to \mathbb{R}^+$ by:*

$$\begin{aligned}
H_{\mathfrak{A}}(A, B, \lambda) &= \min\left\{\inf_{\omega \in A} \mathfrak{A}(\omega, B, \lambda),\ \inf_{\tau \in B} \mathfrak{A}(A, \tau, \lambda)\right\}, \\
H_{\mathfrak{B}}(A, B, \lambda) &= \max\left\{\sup_{\omega \in A} \mathfrak{B}(\omega, B, \lambda),\ \sup_{\tau \in B} \mathfrak{B}(A, \tau, \lambda)\right\}, \\
H_{\mathfrak{C}}(A, B, \lambda) &= \max\left\{\sup_{\omega \in A} \mathfrak{C}(\omega, B, \lambda),\ \sup_{\tau \in B} \mathfrak{C}(A, \tau, \lambda)\right\}.
\end{aligned}$$

The 5-tuple $(H_{\mathfrak{A}}, H_{\mathfrak{B}}, H_{\mathfrak{C}}, \diamond, \circledast)$ is called a Hausdorff NMS (shortly, HNMS).

Proposition 1. *Let $(U, \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \diamond, \circledast)$ be an NMS. Then $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ are continuous functions on $U \times U \times (0, \infty)$.*

Proof. Consider a sequence $(\zeta_n, \nu_n, \lambda_n)$ in $U \times U \times (0, \infty)$. For the sake of simplicity, let us denote it by $\{S_n\}$. Suppose the sequence $\{S_n\}$ converges to $S = (\zeta, \nu, \lambda)$, where $\zeta, \nu \in U$ and $\lambda > 0$.

Then, the sequences $\mathfrak{A}(S_n)$, $\mathfrak{B}(S_n)$ and $\mathfrak{C}(S_n)$ lie in $(0, 1]$. As $[0, 1]$ is compact, each of these sequences has converging subsequences, say, $\mathfrak{A}(S_{n_r})$, $\mathfrak{B}(S_{n_r})$, and $\mathfrak{C}(S_{n_r})$, to some points in $[0, 1]$.

Choose $\delta > 0$ such that $\delta < \frac{\lambda}{2}$. Then there is an $n_0 \in \mathbb{N}$ such that $|\lambda - \lambda_{n_r}| < \delta$ for all $n_r \geq n_0$. Hence,

$$\mathfrak{A}(S_{n_r}) \geq \mathfrak{A}\left(\zeta_{n_r}, \zeta, \frac{\delta}{2}\right) \diamond \mathfrak{A}(S - 2\delta) \diamond \mathfrak{A}\left(\nu, \nu_{n_r}, \frac{\delta}{2}\right),$$
$$\mathfrak{B}(S_{n_r}) \leq \mathfrak{B}\left(\zeta_{n_r}, \zeta, \frac{\delta}{2}\right) \circledast \mathfrak{B}(S - 2\delta) \circledast \mathfrak{B}\left(\nu, \nu_{n_r}, \frac{\delta}{2}\right),$$
$$\mathfrak{C}(S_{n_r}) \leq \mathfrak{C}\left(\zeta_{n_r}, \zeta, \frac{\delta}{2}\right) \circledast \mathfrak{C}(S - 2\delta) \circledast \mathfrak{C}\left(\nu, \nu_{n_r}, \frac{\delta}{2}\right),$$

for all $n_r \geq n_0$. We also have that

$$\mathfrak{A}(\zeta, \nu, \lambda + 2\delta) \geq \mathfrak{A}(\zeta, \zeta_{n_r}, \frac{\delta}{2}) \diamond \mathfrak{A}(S_{n_r}) \diamond \mathfrak{A}\left(\nu_{n_r}, \nu, \frac{\delta}{2}\right),$$
$$\mathfrak{B}(\zeta, \nu, \lambda + 2\delta) \leq \mathfrak{B}(\zeta, \zeta_{n_r}, \frac{\delta}{2}) \circledast \mathfrak{B}(S_{n_r}) \circledast \mathfrak{B}\left(\nu_{n_r}, \nu, \frac{\delta}{2}\right),$$
$$\mathfrak{C}(\zeta, \nu, \lambda + 2\delta) \leq \mathfrak{C}(\zeta, \zeta_{n_r}, \frac{\delta}{2}) \circledast \mathfrak{C}(S_{n_r}) \circledast \mathfrak{C}\left(\nu_{n_r}, \nu, \frac{\delta}{2}\right)$$

for all $n_r \geq n_0$.

Letting $n_r \to \infty$ in the above inequalities, we obtain that

$$\lim_{n_r \to \infty} \mathfrak{A}(S_{n_r}) \geq 1 \diamond \mathfrak{A}(S - 2\delta) \diamond 1 = \mathfrak{A}(S - 2\delta)$$
$$\lim_{n_r \to \infty} \mathfrak{B}(S_{n_r}) \leq 0 \circledast \mathfrak{B}(S - 2\delta) \circledast 0 = \mathfrak{B}(S - 2\delta),$$
$$\lim_{n_r \to \infty} \mathfrak{C}(S_{n_r}) \leq 0 \circledast \mathfrak{C}(S - 2\delta) \circledast 0 = \mathfrak{C}(S - 2\delta);$$
$$\mathfrak{A}(\zeta, \nu, \lambda + 2\delta) \geq 1 \diamond \lim_{n_r \to \infty} \mathfrak{A}(S_{n_r}) \diamond 1 = \lim_{n_r \to \infty} \mathfrak{A}(S_{n_r}),$$
$$\mathfrak{B}(\zeta, \nu, \lambda + 2\delta) \leq 0 \circledast \lim_{n_r \to \infty} \mathfrak{B}(S_{n_r}) \circledast 0 = \lim_{n_r \to \infty} \mathfrak{B}(S_{n_r}),$$
$$\mathfrak{C}(\zeta, \nu, \lambda + 2\delta) \leq 0 \circledast \lim_{n_r \to \infty} \mathfrak{C}(S_{n_r}) \circledast 0 = \lim_{n_r \to \infty} \mathfrak{C}(S_{n_r}).$$

Since the functions $\lambda \mapsto \mathfrak{A}(S), \lambda \mapsto \mathfrak{B}(S)$ and $\lambda \mapsto \mathfrak{C}(S)$, we can deduce that

$$\lim_{n_r \to \infty} \mathfrak{A}(S_{n_r}) = \mathfrak{A}(S),$$
$$\lim_{n_r \to \infty} \mathfrak{B}(S_{n_r}) = \mathfrak{B}(S),$$
$$\lim_{n_r \to \infty} \mathfrak{C}(S_{n_r}) = \mathfrak{C}(S).$$

Therefore, $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ are continuous on $U \times U \times (0, \infty)$. □

3. The Main Result

Results that were essential to proving the main result are initially presented in this section.

Lemma 1. *Let $(U, \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \diamond, \circledast)$ be an NMS. Then for $\omega \in U$, $B \in C(U)$ and $\lambda > 0$, there is a $\tau_0 \in B$ such that*

$$\mathfrak{A}(\omega, B, \lambda) = \mathfrak{A}(\omega, \tau_0, \lambda),$$
$$\mathfrak{B}(\omega, B, \lambda) = \mathfrak{B}(\omega, \tau_0, \lambda),$$
$$\mathfrak{C}(\omega, B, \lambda) = \mathfrak{C}(\omega, \tau_0, \lambda).$$

Proof. By the continuity of the functions $\nu \to \mathfrak{A}(\omega,\nu,\lambda)$, $\nu \to \mathfrak{B}(\omega,\nu,\lambda)$, $\nu \to \mathfrak{C}(\omega,\nu,\lambda)$ and by the compactness of B, we can find a $\tau_0 \in B$ such that

$$\sup_{\tau\in B}\mathfrak{A}(\omega,\tau,\lambda)=\mathfrak{A}(\omega,\tau_0,\lambda),$$
$$\inf_{\tau\in B}\mathfrak{B}(\omega,\tau,\lambda)=\mathfrak{B}(\omega,\tau_0,\lambda),$$
$$\inf_{\tau\in B}\mathfrak{C}(\omega,\tau,\lambda)=\mathfrak{C}(\omega,\tau_0,\lambda).$$

Then, it is easy to conclude that

$$\mathfrak{A}(\omega,B,\lambda)=\mathfrak{A}(\omega,\tau_0,\lambda),$$
$$\mathfrak{B}(\omega,B,\lambda)=\mathfrak{B}(\omega,\tau_0,\lambda),$$
$$\mathfrak{C}(\omega,B,\lambda)=\mathfrak{C}(\omega,\tau_0,\lambda).$$

□

Lemma 2. *Let $(U,\mathfrak{A},\mathfrak{B},\mathfrak{C},\diamond,\circledast)$ be an NMS. Let $\big(C(U),H_{\mathfrak{A}},H_{\mathfrak{B}},H_{\mathfrak{C}},\diamond,\circledast\big)$ be an HNMS. Then for all $A,B\in C(U)$, for each $\omega\in A$ and for all $\lambda>0$ there exists $\tau_\omega\in B$ such that*

$$H_{\mathfrak{A}}(A,B,\lambda)\le\mathfrak{A}(\omega,\tau_\omega,\lambda),$$
$$H_{\mathfrak{B}}(A,B,\lambda)\ge\mathfrak{B}(\omega,\tau_\omega,\lambda),$$
$$H_{\mathfrak{C}}(A,B,\lambda)\ge\mathfrak{C}(\omega,\tau_\omega,\lambda).$$

Proof. First,

$$\mathfrak{A}(\omega,B,\lambda)\ge\inf_{\omega\in A}\mathfrak{A}(\omega,B,\lambda)\ge\min\Big\{\inf_{\omega\in A}\mathfrak{A}(\omega,B,\lambda),\ \inf_{\tau\in B}\mathfrak{A}(A,\tau,\lambda)\Big\},$$
$$\mathfrak{B}(\omega,B,\lambda)\le\sup_{\omega\in A}\mathfrak{B}(\omega,B,\lambda)\le\max\Big\{\sup_{\omega\in A}\mathfrak{B}(\omega,B,\lambda),\ \sup_{\tau\in B}\mathfrak{B}(A,\tau,\lambda)\Big\},$$
$$\mathfrak{C}(\omega,B,\lambda)\le\sup_{\omega\in A}\mathfrak{C}(\omega,B,\lambda)\le\max\Big\{\sup_{\omega\in A}\mathfrak{C}(\omega,B,\lambda),\ \sup_{\tau\in B}\mathfrak{C}(A,\tau,\lambda)\Big\}.$$

Using Lemma 1, one writes

$$\mathfrak{A}(\omega,\tau_\omega,\lambda)\ge H_{\mathfrak{A}}(A,B,\lambda),$$
$$\mathfrak{B}(\omega,\tau_\omega,\lambda)\le H_{\mathfrak{B}}(A,B,\lambda),$$
$$\mathfrak{C}(\omega,\tau_\omega,\lambda)\le H_{\mathfrak{C}}(A,B,\lambda).$$

□

To start with the main results, let us take some notes:
$(U,\mathfrak{A},\mathfrak{B},\mathfrak{C},\diamond,\circledast)$ is an NMS, Ω is an index set, and $\zeta_0\in U$.
Let $\{\mathscr{F}_\alpha\}_{\alpha\in\Omega}$ be a family of multivalued mappings from U to $C(U)$.
For some $\omega\in\Omega$, we can then find $\zeta_1\in\mathscr{F}_\omega(\zeta_0)$ such that for all $\lambda>0$,

$$\mathfrak{A}(\zeta_0,\mathscr{F}_\omega(\zeta_0),\lambda)=\mathfrak{A}(\zeta_0,\zeta_1,\lambda),$$
$$\mathfrak{B}(\zeta_0,\mathscr{F}_\omega(\zeta_0),\lambda)=\mathfrak{B}(\zeta_0,\zeta_1,\lambda),$$
$$\mathfrak{C}(\zeta_0,\mathscr{F}_\omega(\zeta_0),\lambda)=\mathfrak{C}(\zeta_0,\zeta_1,\lambda).$$

Choose $\zeta_2\in\mathscr{F}_\tau(\zeta_1)$ such that

$$\mathfrak{A}(\zeta_1,\mathscr{F}_\tau(\zeta_1),\lambda)=\mathfrak{A}(\zeta_1,\zeta_2,\lambda),$$
$$\mathfrak{B}(\zeta_1,\mathscr{F}_\tau(\zeta_1),\lambda)=\mathfrak{B}(\zeta_1,\zeta_2,\lambda),$$
$$\mathfrak{C}(\zeta_1,\mathscr{F}_\tau(\zeta_1),\lambda)=\mathfrak{C}(\zeta_1,\zeta_2,\lambda).$$

Continuing the process, we get a sequence $\{\zeta_n\}$ in U such that $\zeta_{n+1} \in \mathscr{F}_\beta(\zeta_n)$, and for all $\lambda > 0$,

$$\mathfrak{A}(\zeta_n, \mathscr{F}_\beta(\zeta_n), \lambda) = \mathfrak{A}(\zeta_n, \zeta_{n+1}, \lambda),$$
$$\mathfrak{B}(\zeta_n, \mathscr{F}_\beta(\zeta_n), \lambda) = \mathfrak{B}(\zeta_n, \zeta_{n+1}, \lambda)$$
$$\mathfrak{C}(\zeta_n, \mathscr{F}_\beta(\zeta_n), \lambda) = \mathfrak{C}(\zeta_n, \zeta_{n+1}, \lambda).$$

For the sake of clarity, let us denote the sequence $\{\zeta_n\}$ by $\{(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})(\mathscr{F}_\alpha(\zeta_n)\}_{\alpha\in\Omega}$.

We make the below-mentioned assumptions, which stand for the results proposed here:

(i) $(U, \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \diamond, \circledast)$ is a complete NMS;

(ii) The ctn $\diamond$ and the ctcn $\circledast$ are defined, respectively, by

$$\omega \diamond \omega \geq \omega \text{ or } \omega \diamond \tau = \min\{\omega, \tau\};$$
$$\omega \circledast \omega \leq \omega \text{ or } \omega \circledast \tau = \max\{\omega, \tau\};$$

(iii) $\big(C(U), H_\mathfrak{A}, H_\mathfrak{B}, H_\mathfrak{C}, \diamond, \circledast\big)$ is an HNMS;

(iv) $\big\{\mathscr{F}_\alpha\big\}_{\alpha\in\Omega}$ is a family of multivalued mappings from U to $C(U)$.

Theorem 3. *Let $\{(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})(\mathscr{F}_\alpha(\zeta_n)\}_{\alpha\in\Omega}$ be a sequence generated by ζ_0 as above. Suppose that $\zeta, \nu \in \overline{B(\zeta_0, \epsilon, \lambda)} \cap \big\{(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})\ \mathscr{F}_\alpha(\zeta_n) : \alpha \in \Omega\big\}$ with $\zeta \neq \nu$, $0 < p_{\,i,j} \leq \kappa < 1$, $\zeta_0 \in U$ and $i, j \in \Omega$ with $i \neq j$.*

If, for all $\lambda > 0$,

$$\begin{aligned} H_\mathfrak{A}(\mathscr{F}_i(\zeta), \mathscr{F}_j(\nu), p_{\,i,j}\,\lambda) &\geq \mathfrak{A}(\zeta, \nu, \lambda), \\ H_\mathfrak{B}(\mathscr{F}_i(\zeta), \mathscr{F}_j(\nu), p_{\,i,j}\,\lambda) &\leq \mathfrak{B}(\zeta, \nu, \lambda), \\ H_\mathfrak{C}(\mathscr{F}_i(\zeta), \mathscr{F}_j(\nu), p_{\,i,j}\,\lambda) &\leq \mathfrak{C}(\zeta, \nu, \lambda), \end{aligned} \tag{1}$$

and, for some $\lambda > 0$,

$$\begin{aligned} \mathfrak{A}(\zeta_1, \zeta_2, (1-\kappa)\lambda) &\geq 1 - \epsilon, \\ \mathfrak{B}(\zeta_1, \zeta_2, (1-\kappa)\lambda) &\leq \epsilon, \\ \mathfrak{C}(\zeta_1, \zeta_2, (1-\kappa)\lambda) &\leq \epsilon, \end{aligned} \tag{2}$$

then

(1) $\{(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})(\mathscr{F}_\alpha(\zeta_n)\}_{\alpha\in\Omega}$ is a sequence in $\overline{B(\zeta_0, \epsilon, \lambda)}$;

(2) $\{(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})(\mathscr{F}_\alpha(\zeta_n)\}_{\alpha\in\Omega}$ converges to some δ in $\overline{B(\zeta_0, \epsilon, \lambda)}$;

(3) If (1) and (2) hold for δ, then the family of multivalued mappings $\big\{\mathscr{F}_\alpha\big\}_{\alpha\in\Omega}$ in $\overline{B(\zeta_0, \epsilon, \lambda)}$ has a common fixed point.

Proof. If $\zeta_0 = \zeta_1$, then ζ_0 is a common fixed point of $\mathscr{F}_\omega$ for all $\omega \in \Omega$.

Let $\zeta_0 \neq \zeta_1$. Then, by Lemma 2, we have

$$\mathfrak{A}(\zeta_1, \zeta_2, \lambda) \geq H_\mathfrak{A}(\mathscr{F}_\omega(\zeta_0), \mathscr{F}_\tau(\zeta_1), \lambda),$$
$$\mathfrak{B}(\zeta_1, \zeta_2, \lambda) \leq H_\mathfrak{B}(\mathscr{F}_\omega(\zeta_0), \mathscr{F}_\tau(\zeta_1), \lambda),$$
$$\mathfrak{C}(\zeta_1, \zeta_2, \lambda) \leq H_\mathfrak{C}(\mathscr{F}_\omega(\zeta_0), \mathscr{F}_\tau(\zeta_1), \lambda).$$

Then, it follows from induction that

$$\begin{aligned} \mathfrak{A}(\zeta_n, \zeta_{n+1}, \lambda) &\geq H_\mathfrak{A}(\mathscr{F}_i(\zeta_{n-1}), \mathscr{F}_\alpha(\zeta_n), \lambda), \\ \mathfrak{B}(\zeta_n, \zeta_{n+1}, \lambda) &\leq H_\mathfrak{B}(\mathscr{F}_i(\zeta_{n-1}), \mathscr{F}_\alpha(\zeta_n), \lambda), \\ \mathfrak{C}(\zeta_n, \zeta_{n+1}, \lambda) &\leq H_\mathfrak{C}(\mathscr{F}_i(\zeta_{n-1}), \mathscr{F}_\alpha(\zeta_n), \lambda). \end{aligned} \tag{3}$$

Let us first show that $\zeta_n \in \overline{B(\zeta_0, \epsilon, \lambda)}$.

From (2), we get

$$\begin{aligned}
\mathfrak{A}(\zeta_0,\zeta_1,\lambda) &= \mathfrak{A}(\zeta_0,\mathscr{F}_\omega(\zeta_0),\lambda) > \mathfrak{A}(\zeta_0,\zeta_1,(1-\kappa)\lambda) \geq 1-\epsilon,\\
\mathfrak{B}(\zeta_0,\zeta_1,\lambda) &= \mathfrak{B}(\zeta_0,\mathscr{F}_\omega(\zeta_0),\lambda) < \mathfrak{B}(\zeta_0,\zeta_1,(1-\kappa)\lambda) \leq \epsilon,\\
\mathfrak{C}(\zeta_0,\zeta_1,\lambda) &= \mathfrak{C}(\zeta_0,\mathscr{F}_\omega(\zeta_0),\lambda) < \mathfrak{C}(\zeta_0,\zeta_1,(1-\kappa)\lambda) \leq \epsilon.
\end{aligned}$$

This shows that $\zeta_1 \in \overline{B(\zeta_0,\epsilon,\lambda)}$.
Let $\zeta_2,\zeta_3,\cdots,\zeta_j \in \overline{B(\zeta_0,\epsilon,\lambda)}$. From (1), we have

$$\begin{aligned}
\mathfrak{A}(\zeta_j,\zeta_{j+1},\lambda) &\geq H_{\mathfrak{A}}(\mathscr{F}_\beta(\zeta_{j-1}),\mathscr{F}_\rho(\zeta_j),\lambda) \geq \mathfrak{A}\left(\zeta_{j-1},\zeta_j,\frac{\lambda}{p_{\beta,\rho}}\right)\\
&\geq H_{\mathfrak{A}}\left(\mathscr{F}_\mu(\zeta_{j-2}),\mathscr{F}_\beta(\zeta_{j-1}),\frac{\lambda}{p_{\beta,\rho}}\right) \geq \mathfrak{A}\left(\zeta_{j-2},\zeta_{j-1},\frac{\lambda}{p_{\mu,m}\,p_{\beta,\rho}}\right)\\
&\geq \mathfrak{A}\left(\zeta_{j-2},\zeta_{j-1},\frac{\lambda}{\kappa^2}\right) \geq \cdots \geq \mathfrak{A}\left(\zeta_0,\zeta_1,\frac{\lambda}{\kappa^j}\right)\\
&\geq \mathfrak{A}\left(\zeta_0,\zeta_1,\frac{\lambda}{\kappa^j}\right).
\end{aligned} \tag{4}$$

$$\begin{aligned}
\mathfrak{B}(\zeta_j,\zeta_{j+1},\lambda) &\leq H_{\mathfrak{B}}(\mathscr{F}_\beta(\zeta_{j-1}),\mathscr{F}_\rho(\zeta_j),\lambda) \leq \mathfrak{B}\left(\zeta_{j-1},\zeta_j,\frac{\lambda}{p_{\beta,\rho}}\right)\\
&\leq H_{\mathfrak{B}}\left(\mathscr{F}_\mu(\zeta_{j-2}),\mathscr{F}_\beta(\zeta_{j-1}),\frac{\lambda}{p_{\beta,\rho}}\right) \leq \mathfrak{B}\left(\zeta_{j-2},\zeta_{j-1},\frac{\lambda}{p_{\mu,m}\,p_{\beta,\rho}}\right)\\
&\leq \mathfrak{B}\left(\zeta_{j-2},\zeta_{j-1},\frac{\lambda}{\kappa^2}\right) \leq \cdots \leq \mathfrak{B}\left(\zeta_0,\zeta_1,\frac{\lambda}{\kappa^j}\right)\\
&\leq \mathfrak{B}\left(\zeta_0,\zeta_1,\frac{\lambda}{\kappa^j}\right).
\end{aligned} \tag{5}$$

$$\begin{aligned}
\mathfrak{C}(\zeta_j,\zeta_{j+1},\lambda) &\leq H_{\mathfrak{C}}(\mathscr{F}_\beta(\zeta_{j-1}),\mathscr{F}_\rho(\zeta_j),\lambda) \leq \mathfrak{C}\left(\zeta_{j-1},\zeta_j,\frac{\lambda}{p_{\beta,\rho}}\right)\\
&\leq H_{\mathfrak{C}}\left(\mathscr{F}_\mu(\zeta_{j-2}),\mathscr{F}_\beta(\zeta_{j-1}),\frac{\lambda}{p_{\beta,\rho}}\right) \leq \mathfrak{C}\left(\zeta_{j-2},\zeta_{j-1},\frac{\lambda}{p_{\mu,m}\,p_{\beta,\rho}}\right)\\
&\leq \mathfrak{C}\left(\zeta_{j-2},\zeta_{j-1},\frac{\lambda}{\kappa^2}\right) \leq \cdots \leq \mathfrak{C}\left(\zeta_0,\zeta_1,\frac{\lambda}{\kappa^j}\right)\\
&\leq \mathfrak{C}\left(\zeta_0,\zeta_1,\frac{\lambda}{\kappa^j}\right).
\end{aligned} \tag{6}$$

Now,

$$\begin{aligned}
\mathfrak{A}(\zeta_0,\nu_{j+1},\lambda) &\geq \mathfrak{A}\left(\zeta_0,\zeta_{j+1},\left(1-\kappa^{j+1}\right)\lambda\right)\\
&\geq \mathfrak{A}\left(\zeta_0,\zeta_1,(1-\kappa)\lambda\right) \diamond \mathfrak{A}\left(\zeta_1,\zeta_2,(1-\kappa)\kappa\,\lambda\right) \diamond \cdots \diamond \mathfrak{A}\left(\zeta_j,\zeta_{j+1},(1-\kappa)\kappa^j\,\lambda\right)\\
&\geq \mathfrak{A}\left(\zeta_0,\zeta_1,(1-\kappa)\lambda\right) \diamond \mathfrak{A}\left(\zeta_0,\zeta_1,(1-\kappa)\lambda\right) \diamond \cdots \diamond \mathfrak{A}\left(\zeta_0,\zeta_1,(1-\kappa)\lambda\right)\\
&\geq (1-\epsilon) \diamond (1-\epsilon) \diamond \cdots \diamond (1-\epsilon)\\
&\geq 1-\epsilon,
\end{aligned}$$

$$
\begin{aligned}
\mathfrak{B}(\zeta_0, \nu_{j+1}, \lambda) &\leq \mathfrak{B}\Big(\zeta_0, \zeta_{j+1}, \big(1-\kappa^{j+1}\big)\lambda\Big) \\
&\leq \mathfrak{B}\Big(\zeta_0, \zeta_1, (1-\kappa)\lambda\Big) \diamondsuit \mathfrak{B}\Big(\zeta_1, \zeta_2, (1-\kappa)\kappa\,\lambda\Big) \diamondsuit \cdots \diamondsuit \mathfrak{B}\Big(\zeta_j, \zeta_{j+1}, (1-\kappa)\kappa^j\,\lambda\Big) \\
&\leq \mathfrak{B}\Big(\zeta_0, \zeta_1, (1-\kappa)\lambda\Big) \diamondsuit \mathfrak{B}\Big(\zeta_0, \zeta_1, (1-\kappa)\lambda\Big) \diamondsuit \cdots \diamondsuit \mathfrak{B}\Big(\zeta_0, \zeta_1, (1-\kappa)\lambda\Big) \\
&\leq \epsilon \diamondsuit \epsilon \diamondsuit \cdots \diamondsuit \epsilon \\
&\leq \epsilon,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{C}(\zeta_0, \nu_{j+1}, \lambda) &\leq \mathfrak{C}\Big(\zeta_0, \zeta_{j+1}, \big(1-\kappa^{j+1}\big)\lambda\Big) \\
&\leq \mathfrak{C}\Big(\zeta_0, \zeta_1, (1-\kappa)\lambda\Big) \diamondsuit \mathfrak{C}\Big(\zeta_1, \zeta_2, (1-\kappa)\kappa\,\lambda\Big) \diamondsuit \cdots \diamondsuit \mathfrak{C}\Big(\zeta_j, \zeta_{j+1}, (1-\kappa)\kappa^j\,\lambda\Big) \\
&\leq \mathfrak{C}\Big(\zeta_0, \zeta_1, (1-\kappa)\lambda\Big) \diamondsuit \mathfrak{C}\Big(\zeta_0, \zeta_1, (1-\kappa)\lambda\Big) \diamondsuit \cdots \diamondsuit \mathfrak{C}\Big(\zeta_0, \zeta_1, (1-\kappa)\lambda\Big) \\
&\leq \epsilon \diamondsuit \epsilon \diamondsuit \cdots \diamondsuit \epsilon \\
&\leq \epsilon.
\end{aligned}
$$

Hence, we have that $\zeta_{j+1} \in \overline{B(\zeta_0, \epsilon, \lambda)}$.
Now, for all $n \in \mathbb{N}$ and $\lambda > 0$, the inequalities (4), (5), and (6) can be written as

$$
\begin{aligned}
\mathfrak{A}(\zeta_n, \zeta_{n+1}, \lambda) &\geq \mathfrak{A}\Big(\zeta_0, \zeta_1, \frac{\lambda}{\kappa^n}\Big), \\
\mathfrak{B}(\zeta_n, \zeta_{n+1}, \lambda) &\leq \mathfrak{B}\Big(\zeta_0, \zeta_1, \frac{\lambda}{\kappa^n}\Big) \\
\mathfrak{C}(\zeta_n, \zeta_{n+1}, \lambda) &\leq \mathfrak{C}\Big(\zeta_0, \zeta_1, \frac{\lambda}{\kappa^n}\Big).
\end{aligned}
\tag{7}
$$

For each $n, m \in \mathbb{N},;\ m > n$, we have

$$
\begin{aligned}
\mathfrak{A}(\zeta_n, \zeta_m, \lambda) &> \mathfrak{A}\Big(\zeta_n, \zeta_m, \big(1-\kappa^{m-n}\big)\lambda\Big) \\
&\geq \mathfrak{A}\Big(\zeta_n, \zeta_{n+1}, \big(1-\kappa\big)\lambda\Big) \diamond \mathfrak{A}\Big(\zeta_{n+1}, \zeta_{n+2}, \big(1-\kappa\big)\kappa\,\lambda\Big) \\
&\qquad \diamond \cdots \diamond \mathfrak{A}\Big(\zeta_{m-1}, \zeta_m, \big(1-\kappa\big)\kappa^{m-n-1}\,\lambda\Big) \\
&\geq \mathfrak{A}\left(\zeta_0, \zeta_1, \frac{(1-\kappa)\lambda}{\kappa^n}\right) \diamond \mathfrak{A}\left(\zeta_0, \zeta_1, \frac{(1-\kappa)\kappa\,\lambda}{\kappa^{n+1}}\right) \\
&\qquad \diamond \cdots \diamond \mathfrak{A}\left(\zeta_0, \zeta_1, \frac{(1-\kappa)\kappa^{m-n-1}\,\lambda}{\kappa^{m-1}}\right) \\
&\geq \mathfrak{A}\left(\zeta_0, \zeta_1, \frac{(1-\kappa)\lambda}{\kappa^n}\right) \diamond \mathfrak{A}\left(\zeta_0, \zeta_1, \frac{(1-\kappa)\lambda}{\kappa^n}\right) \diamond \cdots \diamond \mathfrak{A}\left(\zeta_0, \zeta_1, \frac{(1-\kappa)\lambda}{\kappa^n}\right) \\
&\geq \mathfrak{A}\left(\zeta_0, \zeta_1, \frac{(1-\kappa)\lambda}{\kappa^n}\right).
\end{aligned}
$$

As $\lim_{\lambda \to \infty} \mathfrak{A}(\zeta, \nu, \lambda) = 1$ for all $\zeta, \nu \in U$, we have $\mathfrak{A}\Big(\zeta_0, \zeta_1, \frac{(1-\kappa)\lambda}{\kappa^n}\Big) = 1$ as $n \to \infty$.
Hence, $\mathfrak{A}(\zeta_n, \zeta_m, \lambda) = 1$ as $n \to \infty$.

$$
\begin{aligned}
\mathfrak{B}(\zeta_n,\zeta_m,\lambda) &< \mathfrak{B}\Big(\zeta_n,\zeta_m,\big(1-\kappa^{m-n}\big)\lambda\Big) \\
&\leq \mathfrak{B}\Big(\zeta_n,\zeta_{n+1},\big(1-\kappa\big)\lambda\Big) \diamondsuit \mathfrak{B}\Big(\zeta_{n+1},\zeta_{n+2},\big(1-\kappa\big)\kappa\,\lambda\Big) \\
&\qquad \diamondsuit \cdots \diamondsuit \mathfrak{B}\Big(\zeta_{m-1},\zeta_m,\big(1-\kappa\big)\kappa^{m-n-1}\,\lambda\Big) \\
&\leq \mathfrak{B}\left(\zeta_0,\zeta_1,\frac{(1-\kappa)\lambda}{\kappa^n}\right) \diamondsuit \mathfrak{B}\left(\zeta_0,\zeta_1,\frac{(1-\kappa)\kappa\,\lambda}{\kappa^{n+1}}\right) \\
&\qquad \diamondsuit \cdots \diamondsuit \mathfrak{B}\left(\zeta_0,\zeta_1,\frac{(1-\kappa)\kappa^{m-n-1}\,\lambda}{\kappa^{m-1}}\right) \\
&\leq \mathfrak{B}\left(\zeta_0,\zeta_1,\frac{(1-\kappa)\lambda}{\kappa^n}\right) \diamondsuit \mathfrak{B}\left(\zeta_0,\zeta_1,\frac{(1-\kappa)\lambda}{\kappa^n}\right) \diamondsuit \cdots \diamondsuit \mathfrak{B}\left(\zeta_0,\zeta_1,\frac{(1-\kappa)\lambda}{\kappa^n}\right) \\
&\leq \mathfrak{B}\left(\zeta_0,\zeta_1,\frac{(1-\kappa)\lambda}{\kappa^n}\right).
\end{aligned}
$$

As $\lim_{\lambda\to\infty} \mathfrak{B}(\zeta,\nu,\lambda) = 0$ for all $\zeta,\nu \in U$, we have $\mathfrak{B}\Big(\zeta_0,\zeta_1,\frac{(1-\kappa)\lambda}{\kappa^n}\Big) = 0$ as $n \to \infty$. Hence, $\mathfrak{B}(\zeta_n,\zeta_m,\lambda) = 0$ as $n \to \infty$. Further,

$$
\begin{aligned}
\mathfrak{C}(\zeta_n,\zeta_m,\lambda) &< \mathfrak{C}\Big(\zeta_n,\zeta_m,\big(1-\kappa^{m-n}\big)\lambda\Big) \\
&\leq \mathfrak{C}\Big(\zeta_n,\zeta_{n+1},\big(1-\kappa\big)\lambda\Big) \diamondsuit \mathfrak{C}\Big(\zeta_{n+1},\zeta_{n+2},\big(1-\kappa\big)\kappa\,\lambda\Big) \\
&\qquad \diamondsuit \cdots \diamondsuit \mathfrak{C}\Big(\zeta_{m-1},\zeta_m,\big(1-\kappa\big)\kappa^{m-n-1}\,\lambda\Big) \\
&\leq \mathfrak{C}\left(\zeta_0,\zeta_1,\frac{(1-\kappa)\lambda}{\kappa^n}\right) \diamondsuit \mathfrak{C}\left(\zeta_0,\zeta_1,\frac{(1-\kappa)\kappa\,\lambda}{\kappa^{n+1}}\right) \\
&\qquad \diamondsuit \cdots \diamondsuit \mathfrak{C}\left(\zeta_0,\zeta_1,\frac{(1-\kappa)\kappa^{m-n-1}\,\lambda}{\kappa^{m-1}}\right) \\
&\leq \mathfrak{C}\left(\zeta_0,\zeta_1,\frac{(1-\kappa)\lambda}{\kappa^n}\right) \diamondsuit \mathfrak{C}\left(\zeta_0,\zeta_1,\frac{(1-\kappa)\lambda}{\kappa^n}\right) \diamondsuit \cdots \diamondsuit \mathfrak{C}\left(\zeta_0,\zeta_1,\frac{(1-\kappa)\lambda}{\kappa^n}\right) \\
&\leq \mathfrak{C}\left(\zeta_0,\zeta_1,\frac{(1-\kappa)\lambda}{\kappa^n}\right).
\end{aligned}
$$

As $\lim_{\lambda\to\infty} \mathfrak{C}(\zeta,\nu,\lambda) = 0$ for all $\zeta,\nu \in U$, we have $\mathfrak{C}\Big(\zeta_0,\zeta_1,\frac{(1-\kappa)\lambda}{\kappa^n}\Big) = 0$ as $n \to \infty$.
Hence, $\mathfrak{C}(\zeta_n,\zeta_m,\lambda) = 0$ as $n \to \infty$.
That is, $\Big\{(\mathfrak{A},\mathfrak{B},\mathfrak{C})\ \mathscr{F}_\alpha(\zeta_n)\Big\}$ is a Cauchy sequence in $\overline{B(\zeta_0,\epsilon,\lambda)}$.

As every closed ball in a complete NMS is complete, $\overline{B(\zeta_0,\epsilon,\lambda)}$ is complete. Therefore there exists a point ζ in $\overline{B(\zeta_0,\epsilon,\lambda)}$ such that $\lim_{\lambda\to\infty} \zeta_n = \zeta$.

We can now choose some $\alpha_0 \in \Omega$ such that

$$
\mathfrak{A}(\delta,\mathscr{F}_{\alpha_0}(\delta),\lambda) \geq \mathfrak{A}(\delta,\zeta_n,(1-\kappa)\lambda) \diamond \mathfrak{A}(\zeta_n,\mathscr{F}_{\alpha_0}(\delta),\kappa\,\lambda).
$$

By Lemma 2, we have

$$
\begin{aligned}
\mathfrak{A}(\delta,\mathscr{F}_{\alpha_0}(\delta),\lambda) &\geq \mathfrak{A}(\delta,\zeta_n,(1-\kappa)\lambda) \diamond H_{\mathfrak{A}}(\mathscr{F}_\epsilon(\zeta_{n-1}),\mathscr{F}_{\alpha_0}(\delta),\kappa\lambda) \\
&\geq \mathfrak{A}(\delta,\zeta_n,(1-\kappa)\lambda) \diamond \mathfrak{A}\left(\zeta_{n-1},\delta,\frac{\kappa\lambda}{p_{\,\epsilon,\alpha_0}}\right) \\
&\geq \mathfrak{A}(\delta,\zeta_n,(1-\kappa)\lambda) \diamond \mathfrak{A}(\zeta_{n-1},\delta,\lambda).
\end{aligned}
$$

Letting $n \to \infty$, we have $\mathfrak{A}(\delta, \mathscr{F}_{\alpha_0}(\delta), \lambda) \geq 1 \diamond 1 = 1$.

$$\begin{aligned}\mathfrak{B}(\delta, \mathscr{F}_{\alpha_0}(\delta), \lambda) &\leq \mathfrak{B}(\delta, \zeta_n, (1-\kappa)\lambda) \ \circledast\ H_{\mathfrak{B}}(\mathscr{F}_{\epsilon}(\zeta_{n-1}), \mathscr{F}_{\alpha_0}(\delta), \kappa\lambda) \\ &\leq \mathfrak{B}(\delta, \zeta_n, (1-\kappa)\lambda) \ \circledast\ \mathfrak{B}\left(\zeta_{n-1}, \delta, \frac{\kappa\lambda}{p_{\ \epsilon,\alpha_0}}\right) \\ &\leq \mathfrak{B}(\delta, \zeta_n, (1-\kappa)\lambda) \ \circledast\ \mathfrak{B}(\zeta_{n-1}, \delta, \lambda).\end{aligned}$$

Letting $n \to \infty$, we have $\mathfrak{B}(\delta, \mathscr{F}_{\alpha_0}(\delta), \lambda) \leq 0 \ \circledast\ 0 = 0$.

$$\begin{aligned}\mathfrak{C}(\delta, \mathscr{F}_{\alpha_0}(\delta), \lambda) &\leq \mathfrak{C}(\delta, \zeta_n, (1-\kappa)\lambda) \ \circledast\ H_{\mathfrak{C}}(\mathscr{F}_{\epsilon}(\zeta_{n-1}), \mathscr{F}_{\alpha_0}(\delta), \kappa\lambda) \\ &\leq \mathfrak{C}(\delta, \zeta_n, (1-\kappa)\lambda) \ \circledast\ \mathfrak{C}\left(\zeta_{n-1}, \delta, \frac{\kappa\lambda}{p_{\ \epsilon,\alpha_0}}\right) \\ &\leq \mathfrak{C}(\delta, \zeta_n, (1-\kappa)\lambda) \ \circledast\ \mathfrak{C}(\zeta_{n-1}, \delta, \lambda).\end{aligned}$$

Letting $n \to \infty$, we have $\mathfrak{C}(\delta, \mathscr{F}_{\alpha_0}(\delta), \lambda) \leq 0 \ \circledast\ 0 = 0$.
These deductions imply that $\delta \in \mathscr{F}_{\alpha_0}(\delta)$.
Hence, $\delta \in \cap\{\mathscr{F}_{\alpha_0}(\delta)\}_{\alpha_0 \in \Omega}$.
This completes the proof. □

Let us bring here another notation for a sequence as before:
Let $\mathscr{F}$ be a multivalued mapping from U to $C(U)$. Then, for all $\lambda > 0$, there exists $\zeta_1 \in \mathscr{F}(\zeta_0)$ such that

$$\begin{aligned}\mathfrak{A}(\zeta_0, \mathscr{F}(\zeta_0), \lambda) &= \mathfrak{A}(\zeta_0, \zeta_1, \lambda), \\ \mathfrak{B}(\zeta_0, \mathscr{F}(\zeta_0), \lambda) &= \mathfrak{B}(\zeta_0, \zeta_1, \lambda), \\ \mathfrak{C}(\zeta_0, \mathscr{F}(\zeta_0), \lambda) &= \mathfrak{C}(\zeta_0, \zeta_1, \lambda).\end{aligned}$$

Let $\zeta_2 \in \mathscr{F}(\zeta_1)$, such that

$$\begin{aligned}\mathfrak{A}(\zeta_1, \mathscr{F}(\zeta_1), \lambda) &= \mathfrak{A}(\zeta_1, \zeta_2, \lambda), \\ \mathfrak{B}(\zeta_1, \mathscr{F}(\zeta_1), \lambda) &= \mathfrak{B}(\zeta_1, \zeta_2, \lambda), \\ \mathfrak{C}(\zeta_1, \mathscr{F}(\zeta_1), \lambda) &= \mathfrak{C}(\zeta_1, \zeta_2, \lambda).\end{aligned}$$

Thus, we can construct a sequence $\{\zeta_n\}$ of points in U such that $\zeta_{n+1} \in \mathscr{F}(\zeta_n)$, and

$$\begin{aligned}\mathfrak{A}(\zeta_n, \mathscr{F}(\zeta_n), \lambda) &= \mathfrak{A}(\zeta_n, \zeta_{n+1}, \lambda), \\ \mathfrak{B}(\zeta_n, \mathscr{F}(\zeta_n), \lambda) &= \mathfrak{B}(\zeta_n, \zeta_{n+1}, \lambda), \\ \mathfrak{C}(\zeta_n, \mathscr{F}(\zeta_n), \lambda) &= \mathfrak{C}(\zeta_n, \zeta_{n+1}, \lambda).\end{aligned}$$

for all $\lambda > 0$. We denote this sequence by $\{(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})(\mathscr{F}(\zeta_n)\}$.

Corollary 1. *Let $\{(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})(\mathscr{F}(\zeta_n)\}$ be a sequence generated by ζ_0, as above. Assume that $0 < \kappa < 1$, $\zeta_0 \in U$, $\zeta, \nu \in \overline{B(\zeta_0, \epsilon, \lambda)} \cap \{U\ \mathscr{F}(\zeta_n)\}$ with $\zeta \neq \nu$.*
If, for all $\lambda > 0$,

$$\begin{aligned}H_{\mathfrak{A}}(\mathscr{F}(\zeta), \mathscr{F}(\nu), \kappa\lambda) &\geq \mathfrak{A}(\zeta, \nu, \lambda), \\ H_{\mathfrak{B}}(\mathscr{F}(\zeta), \mathscr{F}(\nu), \kappa\lambda) &\leq \mathfrak{B}(\zeta, \nu, \lambda), \\ H_{\mathfrak{C}}(\mathscr{F}(\zeta), \mathscr{F}(\nu), \kappa\lambda) &\leq \mathfrak{C}(\zeta, \nu, \lambda),\end{aligned} \tag{8}$$

and if for some $\lambda > 0$,

$$\begin{aligned}\mathfrak{A}(\zeta_0, \mathscr{F}(\zeta_0), (1-\kappa)\lambda) &\geq 1-\epsilon, \\ \mathfrak{B}(\zeta_0, \mathscr{F}(\zeta_0), (1-\kappa)\lambda) &\leq \epsilon, \\ \mathfrak{C}(\zeta_0, \mathscr{F}(\zeta_0), (1-\kappa)\lambda) &\leq \epsilon,\end{aligned} \tag{9}$$

then

(i) $\{(\mathfrak{A},\mathfrak{B},\mathfrak{C})(\mathscr{F}(\zeta_n)\}$ *is a sequence in* $\overline{B(\zeta_0,\epsilon,\lambda)}$*;*

(ii) $\{(\mathfrak{A},\mathfrak{B},\mathfrak{C})(\mathscr{F}(\zeta_n)\}$ *converges to* δ *for some* $\delta\in\overline{B(\zeta_0,\epsilon,\lambda)}$*;*

(iii) *If (8) and (9) hold for* δ*, then* $\mathscr{F}$ *has a fixed point in* $\overline{B(\zeta_0,\epsilon,\lambda)}$*.*

Corollary 2. *Let* $\{(\mathfrak{A},\mathfrak{B},\mathfrak{C})(\mathscr{F}(\zeta_n)\}$ *be a sequence generated by* ζ_0*, as in the previous corollary. Assume that for some* $0<\kappa<1$*,* $\zeta_0\in U$*,* $\zeta,\nu\in\overline{B(\zeta_0,\epsilon,\lambda)}$ *with* $\zeta\neq\nu$*. If for all* $\lambda>0$*,*

$$\begin{aligned}\mathfrak{A}(\mathscr{F}(\zeta),\mathscr{F}(\nu),\kappa\lambda)&\geq\mathfrak{A}(\zeta,\nu,\lambda),\\ \mathfrak{B}(\mathscr{F}(\zeta),\mathscr{F}(\nu),\kappa\lambda)&\leq\mathfrak{B}(\zeta,\nu,\lambda),\\ \mathfrak{C}(\mathscr{F}(\zeta),\mathscr{F}(\nu),\kappa\lambda)&\leq\mathfrak{C}(\zeta,\nu,\lambda),\end{aligned}$$

and if for some $\lambda>0$*,*

$$\begin{aligned}\mathfrak{A}(\zeta_0,\mathscr{F}(\zeta_0),(1-\kappa)\lambda)&\geq 1-\epsilon,\\ \mathfrak{B}(\zeta_0,\mathscr{F}(\zeta_0),(1-\kappa)\lambda)&\leq\epsilon\\ \mathfrak{C}(\zeta_0,\mathscr{F}(\zeta_0),(1-\kappa)\lambda)&\leq\epsilon,\end{aligned}$$

then $\mathscr{F}$ *has a fixed point in* $\overline{B(\zeta_0,\epsilon,\lambda)}$*.*

Example 2. *Let* $U=[0,2]$ *and* d *be a Euclidean metric on* U*. Define that* $\omega\diamond\tau=\min\{\omega,\tau\}$ *and* $\omega\circledast\tau=\max\{\omega,\tau\}$ *for all* $\omega,\tau\in[0,1]$*.* $\mathfrak{A}$*,* $\mathfrak{B}$*, and* $\mathfrak{C}$ *are defined by*

$$\begin{aligned}\mathfrak{A}(\zeta,\nu,\lambda)&=\frac{\lambda}{\lambda+d(\zeta,\nu)},\\ \mathfrak{B}(\zeta,\nu,\lambda)&=\frac{d(\zeta,\nu)}{\lambda+d(\zeta,\nu)},\\ \mathfrak{C}(\zeta,\nu,\lambda)&=\frac{d(\zeta,\nu)}{\lambda},\end{aligned}$$

for all $\zeta,\nu\in U$ *and* $\lambda>0$*. Then,* $(U,\mathfrak{A},\mathfrak{B},\mathfrak{C},\diamond,\circledast)$ *is an NMS. Consider the multivalued mapping* $\mathscr{F}_n:U\to C(U)$*,* $n=\omega,1,2,\cdots$ *defined by*

$$\mathscr{F}_\omega(\zeta)=\begin{cases}\left[\frac{\zeta}{4},\frac{5\zeta}{18}\right], & \text{if } \zeta\in[0,\frac{3}{2}],\\ \left[3\zeta,4\zeta\right], & \text{if } \zeta\in[\frac{3}{2},2],\end{cases}.$$

$$\mathscr{F}_n(\zeta)=\begin{cases}\left[\frac{\zeta}{4n},\frac{\zeta}{3n}\right], & \text{if } \zeta\in[0,\frac{3}{2}],\\ \left[3n\zeta,4n\zeta\right], & \text{if } \zeta\in[\frac{3}{2},2],\end{cases}$$

where $n=1,2,\cdots$*. Consider* $\zeta_0=\frac{1}{2}$ *and* $\epsilon=\frac{1}{2}$*; then,* $\overline{B(\zeta_0,\epsilon,\lambda)}=[0,\frac{3}{2}]$*. Now,*

$$\begin{aligned}\mathfrak{A}(\zeta_0,\mathscr{F}_\omega(\zeta_0),\lambda)&=\mathfrak{A}\Big(\frac{1}{2},\mathscr{F}_\omega(\frac{1}{2}),\lambda\Big)=\mathfrak{A}\Big(\frac{1}{2},\frac{5}{36},\lambda\Big),\\ \mathfrak{A}(\zeta_1,\mathscr{F}_1(\zeta_1),\lambda)&=\mathfrak{A}\Big(\frac{5}{36},\mathscr{F}_1(\frac{5}{36}),\lambda\Big)=\mathfrak{A}\Big(\frac{5}{36},\frac{5}{108},\lambda\Big),\\ \mathfrak{A}(\zeta_2,\mathscr{F}_2(\zeta_2),\lambda)&=\mathfrak{A}\Big(\frac{5}{108},\mathscr{F}_1(\frac{5}{108}),\lambda\Big)=\mathfrak{A}\Big(\frac{5}{108},\frac{5}{648},\lambda\Big).\end{aligned}$$

Thus, we obtain a sequence $\{\mathscr{F}_\alpha(\zeta_n)\} = \left\{\frac{1}{2}, \frac{5}{36}, \frac{5}{108}, \frac{5}{648}, \cdots\right\}$, *which is generated by* ζ_0. *For* $\zeta = \frac{8}{5}, \nu = \frac{9}{5}, \kappa = p_{1,\omega} = \frac{1}{4}$ *and* $\lambda = 1$, *we have*

$$H_{\mathfrak{A}}\left(\mathscr{F}_1\left(\frac{8}{5}\right), \mathscr{F}_\omega\left(\frac{9}{5}\right), \frac{1}{4}\right) = \min\left\{\inf_{\tau\in\mathscr{F}_1\left(\frac{8}{5}\right)} \mathfrak{A}\left(\tau, \mathscr{F}_\omega\left(\frac{9}{5}\right), \frac{1}{4}\right), \inf_{\partial\in\mathscr{F}_\omega\left(\frac{9}{5}\right)} \mathfrak{A}\left(\mathscr{F}_1\left(\frac{8}{5}\right), \partial, \frac{1}{4}\right)\right\}$$
$$= 0.238,$$

We also have that $\mathfrak{A}\left(\frac{8}{5}, \frac{9}{5}, \lambda\right) = \frac{1}{1 + |\frac{8}{5} - \frac{9}{5}|} = \frac{5}{6} = 0.833.$

It is clear that

$$H_{\mathfrak{A}}\left(\mathscr{F}_1\left(\frac{8}{5}\right), \mathscr{F}_\omega\left(\frac{9}{5}\right), \frac{1}{4}\right) < \mathfrak{A}\left(\frac{8}{5}, \frac{9}{5}, 1\right).$$

For all $\zeta, \nu \in \overline{B(\zeta_0, \epsilon, \lambda)} \cap \{\mathscr{F}_\alpha(\zeta_n)\}$, *we have*

$$\begin{aligned}
H_{\mathfrak{A}}(\mathscr{F}_n(\zeta), \mathscr{F}_\omega(\nu), \kappa\lambda) &= \min\left\{\inf_{\tau\in\mathscr{F}_n(\zeta)} \mathfrak{A}(\tau, \mathscr{F}_\omega(\nu), \kappa\lambda), \inf_{\partial\in\mathscr{F}_\omega(\nu)} \mathfrak{A}(\mathscr{F}_n(\zeta), \partial, \kappa\lambda)\right\} \\
&= \min\left\{\inf_{\tau\in\mathscr{F}_n(\zeta)} \mathfrak{A}\left(\tau, \left[\frac{\nu}{4}, \frac{5\nu}{18}\right], \frac{1}{4}\lambda\right), \inf_{\partial\in\mathscr{F}_\omega(\nu)} \mathfrak{A}\left(\left[\frac{\zeta}{4n}, \frac{\zeta}{3n}\right], \partial, \frac{1}{4}\lambda\right)\right\} \\
&= \min\left\{\mathfrak{A}\left(\frac{\zeta}{3n}, \frac{5\nu}{18}, \frac{1}{4}\lambda\right), \mathfrak{A}\left(\frac{\zeta}{4n}, \frac{\nu}{4}, \frac{1}{4}\lambda\right)\right\} \\
&= \min\left\{\frac{\frac{1}{4}\lambda}{\frac{1}{4}\lambda + |\frac{\zeta}{3n} - \frac{5\nu}{18}|}, \frac{\frac{1}{4}\lambda}{\frac{1}{4}\lambda + |\frac{\zeta}{4n} - \frac{\nu}{4}|}\right\}, \\
H_{\mathfrak{A}}(\mathscr{F}(\zeta), \mathscr{F}(\nu), \kappa\lambda) &= \frac{\frac{1}{4}\lambda}{\frac{1}{4}\lambda + |\frac{\zeta}{4} - \frac{\nu}{4}|} \geq \frac{\lambda}{\lambda + |\zeta - \nu|} = \mathfrak{A}(\zeta, \nu, \lambda).
\end{aligned}$$

We have

$$\begin{aligned}
H_{\mathfrak{B}}(\mathscr{F}_n(\zeta), \mathscr{F}_\omega(\nu), \kappa\lambda) &= \max\left\{\sup_{\tau\in\mathscr{F}_n(\zeta)} \mathfrak{B}(\tau, \mathscr{F}_\omega(\nu), \kappa\lambda), \sup_{\partial\in\mathscr{F}_\omega(\nu)} \mathfrak{B}(\mathscr{F}_n(\zeta), \partial, \kappa\lambda)\right\} \\
&= \max\left\{\sup_{\tau\in\mathscr{F}_n(\zeta)} \mathfrak{B}\left(\tau, \left[\frac{\nu}{4}, \frac{5\nu}{18}\right], \frac{1}{4}\lambda\right), \sup_{\partial\in\mathscr{F}_\omega(\nu)} \mathfrak{B}\left(\left[\frac{\zeta}{4n}, \frac{\zeta}{3n}\right], \partial, \frac{1}{4}\lambda\right)\right\} \\
&= \max\left\{\mathfrak{B}\left(\frac{\zeta}{3n}, \frac{5\nu}{18}, \frac{1}{4}\lambda\right), \mathfrak{B}\left(\frac{\zeta}{4n}, \frac{\nu}{4}, \frac{1}{4}\lambda\right)\right\} \\
&= \max\left\{\frac{|\frac{\zeta}{3n} - \frac{5\nu}{18}|}{\frac{1}{4}\lambda + |\frac{\zeta}{3n} - \frac{5\nu}{18}|}, \frac{|\frac{\zeta}{4n} - \frac{\nu}{4}|}{\frac{1}{4}\lambda + |\frac{\zeta}{4n} - \frac{\nu}{4}|}\right\}, \\
H_{\mathfrak{B}}(\mathscr{F}(\zeta), \mathscr{F}(\nu), \kappa\lambda) &= \frac{|\frac{\zeta}{4} - \frac{\nu}{4}|}{\frac{1}{4}\lambda + |\frac{\zeta}{4} - \frac{\nu}{4}|} \leq \frac{|\zeta - \nu|}{\lambda + |\zeta - \nu|} = \mathfrak{B}(\zeta, \nu, \lambda).
\end{aligned}$$

That is,

$$
\begin{aligned}
H_{\mathfrak{C}}(\mathscr{F}_n(\zeta),\mathscr{F}_\omega(\nu),\kappa\lambda) &= \max\left\{\sup_{\tau\in\mathscr{F}_n(\zeta)}\mathfrak{C}(\tau,\mathscr{F}_\omega(\nu),\kappa\lambda),\ \sup_{\partial\in\mathscr{F}_\omega(\nu)}\mathfrak{C}(\mathscr{F}_n(\zeta),\partial,\kappa\lambda)\right\}\\
&= \max\left\{\sup_{\tau\in\mathscr{F}_n(\zeta)}\mathfrak{C}\Big(\tau,\Big[\frac{\nu}{4},\frac{5\nu}{18}\Big],\frac{1}{4}\lambda\Big),\ \sup_{\partial\in\mathscr{F}_\omega(\nu)}\mathfrak{C}\Big(\Big[\frac{\zeta}{4n},\frac{\zeta}{3n}\Big],\partial,\frac{1}{4}\lambda\Big)\right\}\\
&= \max\left\{\mathfrak{C}\Big(\frac{\zeta}{3n},\frac{5\nu}{18},\frac{1}{4}\lambda\Big),\mathfrak{C}\Big(\frac{\zeta}{4n},\frac{\nu}{4},\frac{1}{4}\lambda\Big)\right\}\\
&= \max\left\{\frac{|\frac{\zeta}{3n}-\frac{5\nu}{18}|}{\frac{1}{4}\lambda},\frac{|\frac{\zeta}{4n}-\frac{\nu}{4}|}{\frac{1}{4}\lambda}\right\},\\
H_{\mathfrak{C}}(\mathscr{F}(\zeta),\mathscr{F}(\nu),\kappa\lambda) &= \frac{|\frac{\zeta}{4}-\frac{\nu}{4}|}{\frac{1}{4}\lambda}\le\frac{|\zeta-\nu|}{\lambda}=\mathfrak{C}(\zeta,\nu,\lambda).
\end{aligned}
$$

Hence, the contractive conditions hold over $\overline{B(\zeta_0,\epsilon,\lambda)}\cap\{\mathscr{F}_\alpha(\zeta_n)\}$.
For $\lambda=1$,

$$
\begin{aligned}
\mathfrak{A}(\zeta_0,\zeta_1,(1-\kappa)\lambda) &= \mathfrak{A}\Big(\frac{1}{2},\frac{5}{36},\frac{3}{4}\Big)=\frac{27}{40}>\frac{1}{2}=1-\epsilon,\\
\mathfrak{B}(\zeta_0,\zeta_1,(1-\kappa)\lambda) &= \mathfrak{B}\Big(\frac{1}{2},\frac{5}{36},\frac{3}{4}\Big)=1-\mathfrak{B}\Big(\frac{1}{2},\frac{5}{36},\frac{3}{4}\Big)=1-\frac{27}{40}=\frac{13}{40}<\frac{1}{2}=\epsilon,\\
\mathfrak{C}(\zeta_0,\zeta_1,(1-\kappa)\lambda) &= \mathfrak{C}\Big(\frac{1}{2},\frac{5}{36},\frac{3}{4}\Big)=\frac{1}{\mathfrak{C}\big(\frac{1}{2},\frac{5}{36},\frac{3}{4}\big)}-1=\frac{40}{27}-1=\frac{13}{27}<\frac{1}{2}=\epsilon.
\end{aligned}
$$

Hence, all the conditions of Theorem 3 are satisfied. Therefore, $\{\mathscr{F}_\alpha(\zeta_n)\}$ *is a sequence in* $\overline{B(\zeta_0,\epsilon,\lambda)}$ *and* $\{\mathscr{F}_\alpha(\zeta_n)\}\to 0\in\overline{B(\zeta_0,\epsilon,\lambda)}$. *Moreover,* $\{\mathscr{F}_\alpha:\alpha=0,1,2,\cdots\}$ *has a common fixed point* 0.

4. Conclusions

A common fixed-point theorem for multivalued mappings in the closed ball $\overline{B(\zeta_0,\epsilon,\lambda)}$ was developed and demonstrated in this manuscript. Over a complete HNMS, this task is completed. It guarantees that multivalued mappings have fixed points. In order to make the primary results effective, this paper additionally presents a specific example. The outcome presented here could be improved upon to match with generalized NMS.

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