Subgroups of direct products of limit groups over coherent right-angled Artin groups

by

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To the memory of my mum

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Declarations

The main results of this thesis are contained in Chapters 2, 3, 4, 5 and 6.

Chapter 2 covers the results in [40], Chapter 3, the ones in [67], Chapter 4 is based on [68] and Chapter 5, on [31]. The first paper [40] has been accepted for publication in Algebraic & Geometric Topology and the third one [68] in Journal of Algebra. The articles [31] and [67] have been submitted.

The articles [67] and [68] are joint work with Dessislava Hristova Kochloukova (Universidade Estadual de Campinas) and the article [31] is joint work with Montserrat Casals-Ruiz (University of the Basque Country UPV/EHU).

I declare that the material in this thesis is, to the best of my knowledge, my own, except where otherwise indicated or cited in the text. This material has not been submitted for any other degree of qualification.

Abstract

During the last 40 years, a large body of work has been directed to study the connection between finiteness properties of groups and their algebraic and algorithmic properties. One of the early results is due to Baumslag and Roseblade, who showed that while finitely generated subgroups of the direct product of two free groups are wild and untractable, the finitely presented ones have nice algebraic and algorithmic properties. This work was widely extended during the years to the class of finitely presented residually free groups viewed as subgroups of direct products of limit groups.

In this thesis we continue this study and show that the good behaviour of finitely presented subgroups extends to the class of finitely presented residually Droms RAAGs. More precisely, we give a complete characterisation of finitely presented residually Droms RAAGs and we obtain a number of consequences related to decision problems, growth of homology and the Bieri-Neumann-Strebel-Renz invariants. We also study the subgroup structure of direct products of limit groups over Droms RAAGs depending on their finiteness properties. Finally, we initiate the study of finitely presented subgroups of direct products of 2-dimensional coherent RAAGs.

Laburpena

Azken 40 urteotan, matematikari ugarik beraien lana zuzendu dute taldeen finitasun-propietateen eta haien propietate aljebraiko eta algoritmikoen arteko lotura aztertzera. Hasierako emaitzetako bat Baumslag eta Rosebladek eman zuten. Frogatu zuten bi talde askeren produktu zuzenaren azpitalde finituki sortuak basatiak diren arren, azpitalde finituki aurkeztuek propietate aljebraiko eta algoritmiko zurrunak dituztela. Lan hau asko hedatu da urteetan zehar limite taldeen produktu zuzenen azpitalde finituki aurkeztuetara, edo baliokidea dena, finituki aurkeztuak diren hondar askeetara.

Tesi honetan lan honekin jarraitzen dugu eta finituki aurkeztuak diren azpitaldeen jokabide ona klase zabalago batera hedatzen dela frogatzen dugu, hain zuzen ere, finituki aurkeztuak diren hondar Droms artindar angeluzuzenetara. Zehazki, guztiz karakterizatzen ditugu finituki aurkeztuak eta hondar Droms artindar angeluzuzenak diren taldeak, eta ondorioz, hainbat propietate lortzen ditugu problema algoritmikoekin, homologia taldeen hazkundearekin eta Bieri-Neumann-Strebel-Renz inbarianteekin lotuta. Droms erako talde artindar angeluzuzenen produktu zuzenen azpitaldeen egitura ere aztertzen dugu, haien finitasun propietateen arabera. Azkenik, 2-dimentsioko talde artindar angeluzuzen eta koherenteen produktu zuzenen azpitalde finituki aurkeztuak ere ikasten ditugu.

Resumen

Durante los últimos 40 años, varios matemáticos han dirigido su trabajo a estudiar la conexión entre las propiedades de finitud de los grupos y sus propiedades algebraicas y algorítmicas. Uno de los primeros resultados se debe a Baumslag y Roseblade, quienes demostraron que mientras los subgrupos finitamente generados del producto directo de dos grupos libres son caóticos, los finitamente presentados tienen buenas propiedades algebraicas y algorítmicas. Este trabajo se ha extendido ampliamente durante los años a la clase de grupos finitamente presentados que son también residualmente libres, vistos como subgrupos de productos directos de grupos límite.

En esta tesis continuamos este estudio y mostramos que el buen comportamiento de los subgrupos finitamente presentados se extiende a la clase de los grupos finitamente presentados que son residualmente grupos de Artin de ángulo recto de tipo Droms. Más concretamente, damos una caracterización completa de grupos finitamente presentados que son residualmente grupos de Artin de ángulo recto de tipo Droms y obtenemos consecuencias relacionadas con los problemas algorítmicos, el crecimiento de los grupos de homología y los invariantes de Bieri-Neumann-Strebel-Renz. También estudiamos la estructura de subgrupos de productos directos de grupos límite sobre grupos de Artin de ángulo recto de tipo Droms en función de sus propiedades de finitud. Finalmente, iniciamos el estudio de los subgrupos finitamente presentados de productos directos de grupos de Artin de ángulo recto coherentes y de dimensión 2.

Notation

Unless otherwise stated R denotes a commutative ring with unit. If G is a group, then we denote by $Z(G)$ the *center* of G, by G' the *commutator subgroup* of G, by $\gamma_i(G)$ the i^{th} *member of the lower central series*, and by G^n the *direct product* of n copies of G, i.e. $G^n = G \times \cdots \times G$. If S is a subset of G, $\langle\langle S \rangle\rangle_G$ (or just $\langle\langle S \rangle\rangle$ if the group is clear) will be the *normal closure* of S in G. The *centraliser* of an element g in G is denoted by $C_G(g)$ and $\sqrt{g}$ is a *root* of g in the group G (if it exists). If S is a generating set for G, we denote the *Cayley graph* of G with respect to S as $\Delta(G, S)$.

If H is a subgroup of G, the set of *left cosets* of H in G will be denoted by $H \backslash G$ and the set of *right cosets* by G/H. If H has finite index in G, we write it as $H <_{fi} G$. Suppose that N is a normal subgroup in G. We say that two elements $x, y \in G$ are equal *modulo* N if $xN = yN$ in the quotient group G/N.

If S and T are two simplicial complexes, $S * T$ denotes the *simplicial join* of S and T.

Suppose that $G_1, \ldots, G_n$ are groups and let S be a subgroup of $G_1 \times \cdots \times G_n$. For $i \in \{1, \ldots, n\}$ we denote the projection map $S \mapsto G_i$ by p_i. If $1 \leq j_1 < \cdots < j_s \leq n$, $p_{j_1,\ldots,j_s}$ is the projection map

$$S \mapsto G_{j_1} \times \cdots \times G_{j_s}.$$

The group L_i is defined to be $S \cap G_i$. We say that S is a *subdirect product* if each p_i is surjective and it is *full* if each L_i is non-trivial. An embedding $S \hookrightarrow G_1 \times \cdots \times G_n$ of a full subdirect product is *neat* if G_1 is abelian (possibly trivial), L_1 is of finite index in G_1 and G_i has trivial center for $i \in \{2, \ldots, n\}$.

Introduction

In the 1950's work of Mihailova [77] demonstrated that the subgroup structure of the direct product of two free groups is complicated and interesting since they may have undecidable algorithmic problems. This was the beginning of a long line of research developed by many mathematicians on subgroups of direct products of free groups, and, more generally, of limit groups. In this thesis we continue this study for subgroups of direct products of right-angled Artin groups, and, more broadly, of limit groups over right-angled Artin groups, which are a generalisation of limit groups.

From the work of Baumslag and Roseblade [7] and Bridson, Howie, Miller and Short [20], [21], [28], [27], [26], it follows that finitely presented subgroups of direct products of finitely generated free groups, or even of limit groups, have a rigid structure and that the main algorithmic problems are decidable in this class. For right-angled Artin groups, however, Bridson showed in [19] that this is not necessarily true. Specifically, there are finitely presented subgroups of direct products of right-angled Artin groups with undecidable decision problems. Thus, there is a big difference in the behaviour of finitely presented subgroups of direct products of right-angled Artin groups. This leads to the question of identifying the class of right-angled Artin groups for which finitely presented subgroups of the direct product of groups in this class have a rigid structure. In this work we give a partial answer to this question by studying specific classes of right-angled Artin groups.

In what follows, we give more details about the history of the subject, and subsequently, we provide an outline of the results we have obtained.

Finitely presented subgroups of the direct product of two free groups

One motivation for studying subgroups of direct products of free groups arises from their complicated algorithmic structure. The main group-theoretic algorithmic problems are the word problem, the conjugacy problem and the isomorphism problem. These were formulated by Dehn in 1911 ([41]) and their study is now an active area of mathematics lying at the intersection of algorithmic algebra and combinatorial group theory. Each of Dehn's three problems is known to be undecidable for finitely presented groups, but they are decidable in the class of finitely generated subgroups of finitely generated free groups. Even though the decidability of the word problem passes to subgroups, the other algorithmic problems do not for direct products of free groups. For example, in [77] Mihailova constructed a finitely generated subgroup of the direct product of two free groups of rank two with undecidable conjugacy and membership problems.

Mihailova's example is based on the so-called free corner pullback construction. Let

$$1 \longrightarrow R \longrightarrow F \xrightarrow{\phi} G \longrightarrow 1$$

be a finite presentation of a group G, that is F is a finitely generated free group and R is finitely generated as a normal subgroup of F. Let us denote by $F \times_\phi F$ the pullback object of the diagram

$$\begin{tikzcd} F \times_\phi F \arrow[r] \arrow[d] & F \arrow[d, "\phi"] \\ F \arrow[r, "\phi"'] & G \end{tikzcd}$$

in the category of groups, i.e. $F \times_\phi F = \{(x, y) \in F \times F \mid \phi(x) = \phi(y)\}$. It follows easily from the assumption that G is finitely presented that $F \times_\phi F$ is finitely generated. What Mihailova showed is that there are finitely presented groups G such that $F \times_\phi F$ has undecidable membership problem in $F \times F$. In [78], Miller used the same construction to show that there is a finitely generated subgroup of $F_2 \times F_2$ with undecidable conjugacy problem.

Later, Grunewald proved in [52] that the subgroups $F \times_\phi F$ with this complicated behaviour are not finitely presented. More precisely, he showed that if ϕ is not an isomorphism, then $F \times_\phi F$ is finitely presented if and only if G is finite, that is, if and only if $F \times_\phi F$ has finite index in $F \times F$.

Baumslag and Roseblade generalised Grunewald's result to all finitely presented subgroups of the direct product of two finitely generated free groups and they explored the difference between finitely generated and finitely presented subgroups. They first emphasised the complexity of the subgroups by proving that there are continuously many finitely generated non-isomorphic subgroups of the direct product of two free groups of rank two. Then they showed that finitely presented subgroups are considerably better behaved: if F and F' are two finitely generated free groups and S is a finitely presented subgroup of $F \times F'$, then S is either free or it is virtually the direct product of two finitely generated free groups. In particular, the classical algorithmic problems are decidable in the class of finitely presented subgroups of the direct product of two finitely generated free groups, so there is a big dichotomy between finitely generated and finitely presented subgroups.

In their paper, Baumslag and Roseblade mention that they could not extend the previous structure theorem to finitely presented subgroups of direct products of finitely many finitely generated free groups. Their proof has a homological flavour and uses the theory of spectral sequences. With the aim of shedding light on the result and providing new methods that could be generalised to direct products of finitely many free groups, alternative proofs were given. For instance, Bridson and Wise gave a proof using CAT(0) cube complexes ([25]), Short gave one using van Kampen diagrams ([90]) and Miller provided a simple algebraic proof ([79]).

Finiteness properties of subgroups of direct products of free groups

The spectrum of finiteness properties gets richer when we focus on subgroups of direct products with more than two factors, reflecting the more complex structure of their finitely presented subgroups.

The two most basic finiteness properties are those of being finitely generated and finitely presented, but there are more general properties. A group is finitely presented if it is the fundamental group of a cell complex with finitely many 1 and 2-cells, or equivalently, if it acts cellularly, properly, faithfully, freely and cocompactly on a simply-connected space X. Putting more constraints on X, we get new homotopical finiteness properties. A group G is of type F_n if it has an Eilenberg-MacLane space $K(G, 1)$ with finite n-skeleton, or equivalently, if it acts cellularly, properly, faithfully, freely and cocompactly on an $(n-1)$-connected cell complex Y. If instead of asking for $(n-1)$-connecteness we ask Y to be $(n-1)$-acyclic, we call G of type FH_n. A slightly weaker property is to be of type FP_n and it was introduced and developed by Bieri (see [11]). Finally, a group is of type w$FP_n(R)$ if the homology groups $H_k(G_0; R)$ are finitely generated as R-modules for every $G_0 < G$ of finite index and $k \leq n$. If $R = \mathbb{Z}$, w$FP_n(\mathbb{Z})$ is usually denoted just by wFP_n. A group is finitely generated if and only if it is of type F_1 if and only if it is of type FP_1, and it is finitely presented if and only if it is of type F_2. Moreover, F_n implies FH_n, FH_n implies FP_n and FP_n implies wFP_n.

In 1963 Stallings constructed the first example of a group which is finitely presented but not of type FP_3 (see [91]). This group is the kernel of the homomorphism $F_2 \times F_2 \times F_2 \mapsto \mathbb{Z}$ that sends all standard generators to 1. This was generalised by Bieri in [12], where he proved that if F_2^n is the direct product of n free groups of rank two and ϕ_n is the homomorphism $F_2^n \mapsto \mathbb{Z}$ sending all standard generators to 1, then $\ker \phi_n$ is of type F_{n-1} but not of type FP_n. These two examples indicate the great diversity that may be found amongst finitely presented subgroups of direct products of free groups. In the case of the direct product of two factors, by Baumslag and Roseblade's result we obtain that finitely presented subgroups are, in fact, of type F_∞. Nevertheless, in the direct product of more than two free groups finitely presented subgroups with different finiteness properties can be found.

The general study of subgroups of direct products of arbitrarily many finitely generated free groups was conducted by Bridson, Howie, Miller and Short. In [26] they described the homological finiteness properties of subgroups of direct products of free and surface groups. More precisely, they showed that if $\Gamma_1, \dots, \Gamma_n$ are finitely generated free or surface groups and S is a subgroup of type $\mathrm{w}FP_n(\mathbb{Q})$ of $\Gamma_1 \times \cdots \times \Gamma_n$, then S is virtually the direct product of free or surface groups. The ingredients in the proof are the surface group analogues of those used in [7] for free groups, together with an inductive argument enabled by the theory of spectral sequences. For instance, in the case of free groups they use a theorem of Marshall Hall which states that every non-trivial element of a finitely generated free group is primitive in a subgroup of finite index. In the case of surface groups, they use the fact that if Σ is the fundamental group of a closed surface and γ is a non-trivial element of Σ, then there is a subgroup $\Sigma_0 < \Sigma$ of finite index such that Σ_0 is an HNN extension $S*_C$ where C is infinite cyclic and the stable letter is γ (see [86]).

Limit groups

Bridson, Howie, Miller and Short continued the programme by studying subgroups of direct products of limit groups. Since specific attributes of surface groups were used in [26], the authors required several new ideas to complete this programme.

Limit groups arise naturally from several points of view. The name limit group was introduced by Sela to emphasise the fact that these are precisely the groups that appear when one takes limits of stable sequences of homomorphisms $\phi_n \colon G \mapsto F$, where G is an arbitrary finitely generated group and F is a free group (see [88], [89], [87]). Remeslennikov, however, studied them in the language of model theory as $\exists$-free groups ([83]). The existential theory of a group G is the set of first order sentences that in the prenex normal form only have existential quantifiers and are true in G, and non-abelian limit groups are precisely groups that have the same existential theory as a free group.

The structure of limit groups is well understood. One can give a hierarchical description in terms of their non-trivial cyclic JSJ-decomposition. More concretely, limit groups are finitely generated subgroups of ω-residually free tower (ω-rft) groups, where a ω-rft group is the fundamental group of a tower space assembled from graphs, tori and surfaces in a hierarchical manner. The number of stages in the construction is the height of the tower, and the height of a limit group Γ is precisely the minimal height of an ω-rft group that has a subgroup isomorphic to Γ. Examples of limit groups include all finitely generated free or free abelian groups, and fundamental groups of closed surfaces of Euler characteristic at most -2.

Limit groups and their subgroups play a key role in the theory of algebraic geometry over groups. In [8], Baumslag, Miasnikov and Remeslennikov lay down the foundations of algebraic geometry over groups, which bears a similarity with classical algebraic geometry. The interest for doing this comes from the desire to study equations over groups. The authors present group theoretic counterparts to algebraic sets, coordinate algebras and other concepts from algebraic geometry. The analog of the classical coordinate algebra is called a coordinate group.

In algebraic geometry, there is a one-to-one correspondence between algebraic sets, radical ideals and coordinate algebras. Therefore, studying an algebraic set S is equivalent to studying its coordinate algebra $R[x_1, \ldots, x_n]/Rad(S)$. Moreover, if R is Noetherian and an integral domain, then S can be decomposed as a finite union of irreducible algebraic sets,

$$S = V_1 \dot{\cup} \ldots \dot{\cup} V_m,$$

where V_i is an irreducible algebraic set for $i \in \{1, \ldots, m\}$. In this setting, the coordinate algebra associated to S is a finitely generated subalgebra of the direct product of the coordinate algebras of the irreducible algebraic sets $V_1, \ldots, V_m$.

In algebraic geometry over groups, under some conditions that substitute R being an integral domain and Noetherian, working with algebraic sets is also equivalent to working with coordinate groups. Also in this setting, coordinate groups are finitely generated subgroups of the direct product of the coordinate groups associated to irreducible algebraic sets. In this new world, a limit group over a group G is just a coordinate group of an irreducible algebraic set over G. Therefore, when we study equations over free groups, we have to understand subgroups of direct products of limit groups.

A more algebraic reason to study limit groups is related to residual properties. A simple way to obtain partial information about a group is to try to find quotient groups which are well known. In this spirit, if $\mathcal{P}$ is a class of groups, a group G is termed residually $\mathcal{P}$ if for each non-trivial $g \in G$ there is a group $H \in \mathcal{P}$ and a homomorphism $f\colon G \mapsto H$ such that $f(g) \neq 1$. The group G is fully residually $\mathcal{P}$ if for any finite subset $T \subseteq G$ there exists a homomorphism from G to a group in $\mathcal{P}$ which is injective on T. There is a large body of work devoted to finitely generated residually finite groups due to their interesting properties, for instance, they have decidable word problem. Residually free groups have also been studied for several decades but they have only recently acquired increased importance. It turns out that the class of limit groups is precisely the class of finitely generated fully residually free groups, and coordinate groups over free groups correspond to finitely generated residually free groups. As a result, the study of finitely generated residually free groups reduces to the study of finitely generated subgroups of direct products of limit groups.

Subgroups of direct products of limit groups and finitely presented residually free groups

Motivated by the rich and powerful interplay among group theory, topology and logic that limit groups provide, Bridson, Howie, Miller and Short embarked on a programme to generalise the results in their earlier paper [26] to this bigger class.

In [20] Bridson and Howie generalised the results in [26] to the class of groups having the same elementary theory as free groups, called elementarily free groups. As mentioned before, limit groups have the same existential theory as a free group. The elementary theory of a group G is the set of all first order sentences that are true in G. In particular, the class of elementarily free groups is an important subclass of the class of limit groups. The arguments in [20] used in a crucial way the fact that in the case of elementarily free groups, the top of the ω-residually free tower decomposition is a quadratic form. In the case of general limit groups, the top may be also an abelian block. Hence, their methods did not extend immediately to the whole class of limit groups.

After dealing with the general case for $n = 2$ in [21], Bridson, Howie, Miller and Short finally completed the programme in [27] by proving that if $\Gamma_1, \ldots, \Gamma_n$ are limit groups and S is a subgroup of type $\mathrm{w}FP_n(\mathbb{Q})$ of $\Gamma_1 \times \cdots \times \Gamma_n$, then S is virtually a direct product of limit groups. An unexpected aspect of the proof is that after some reductions that make use of geometric methods, the argument is mostly based on nilpotent groups, as the authors show that finitely presented subgroups virtually contain a term of the lower central series of the group. In the same article, the authors showed that if S is a full subdirect product of $\Gamma_1 \times \cdots \times \Gamma_n$ where $\Gamma_1, \ldots, \Gamma_n$ are non-abelian limit groups and S is finitely presented, then $p_{i,j}(S)$ has finite index in $\Gamma_i \times \Gamma_j$, where $p_{i,j}$ is the projection homomorphism $\Gamma_1 \times \cdots \times \Gamma_n \mapsto \Gamma_i \times \Gamma_j$ for $1 \leq i < j \leq n$. This is a significant generalisation of the result of Baumslag and Roseblade. However, it does not give a complete characterisation of finitely presented subgroups since the converse was not yet known to be true.

In [28] the authors succeded in proving the converse of this result, and as a consequence, in characterising finitely presented full subdirect products. The beauty of this result is that it holds in remarkable generality. To be precise, Bridson, Howie, Miller and Short proved what is known as The Virtually Surjective on Pairs Criterion. This criterion states that if S is a subgroup of a direct product $G_1 \times \cdots \times G_n$ of finitely presented groups $G_1, \ldots, G_n$ and if S is virtually surjective on pairs, then S is finitely presented.

Combining [27] and [28] we get that if $\Gamma_1, \ldots, \Gamma_n$ are non-abelian limit groups and S is a full subdirect product of $\Gamma_1 \times \cdots \times \Gamma_n$, then S is finitely presented if and only if $p_{i,j}(S)$ has finite index in $\Gamma_i \times \Gamma_j$ for $1 \leq i < j \leq n$. This result is enough to show that the conjugacy and the membership problems are decidable in this class of finitely presented subgroups. Recall that finitely generated residually free groups are just finitely generated subgroups of direct products of limit groups. In [28] it is shown that finitely presented residually free groups are finitely presented full subdirect products of $\Gamma_1 \times \cdots \times \Gamma_n$, where Γ_1 is free abelian, $S \cap \Gamma_1$ has finite index in Γ_1 and $\Gamma_2, \ldots, \Gamma_n$ are non-abelian limit groups. Thus, [28] gives a complete characterisation of finitely presented residually free groups.

After having seen the significance of projections onto pairs, Kuckuck suggested in [69] the following generalisation, named The Virtual Surjection Conjecture. Let $n \leq m$ be positive integers and let S be a subgroup of a direct product $G_1 \times \cdots \times G_m$, where G_i is of type F_n for $1 \leq i \leq m$. If S is virtually surjective on n-tuples, then S is of type F_n. The Virtual Surjection Conjecture is still an open problem but some cases and implications have been proved. For example, in [65] Kochloukova showed that if $\Gamma_1, \ldots, \Gamma_m$ are non-abelian limit groups and S is a finitely generated full subdirect product of type $\mathrm{w}FP_k(\mathbb{Q})$ for some fixed $k \in \{2, \ldots, m\}$, then S virtually surjects on k-tuples. In conjuction with the work of Kuckuck in [69], we get the following powerful result. Let $\Gamma_1, \ldots, \Gamma_m$ be non-abelian limit groups and let S be a full subdirect product of $\Gamma_1 \times \cdots \times \Gamma_m$. Then S is finitely presented and of type $\mathrm{w}FP_k$ (for $k \geq 2$) if and only if S virtually surjects on k-tuples.

Thanks to the understanding of limit groups and finitely presented residually free groups, other interesting properties have been proved. Of special interest to us are the Bieri-Neumann-Strebel-Renz invariants. We are interested in finiteness properties of subgroups such as being of type F_n or of type FP_n. We have seen that kernels of homomorphisms to $\mathbb{Z}$ (so these kernels contain the commutator subgroup of the group) are particular examples of groups having interesting finiteness properties. The Bieri-Neumann-Strebel-Renz invariants are specific open subsets in the character sphere $S(G)$ of a group G, and they control when a subgroup containing the commutator subgroup is of type FP_n or is of type F_n. The invariants are separated into two groups: the homotopical invariants $\{\Sigma^n(G)\}_n$ and the homological ones $\{\Sigma^n(G,\mathbb{Z})\}_n$. In general, they are difficult to compute, but they are known for some classes of groups, including limit groups. Kochloukova proved in [65] that for any non-abelian limit group the invariants are the empty set. In addition, together with Ferreira Lima, they calculated the invariant $\Sigma^1(G)$ for a finitely presented residually free group G (see [72]).

In [24] Bridson and Wilton, using the results in [27] and pulling back results about nilpotent groups, show that finitely presented subgroups of finitely generated residually free groups are separable and that the subgroups of type FP_∞ are virtual retracts. This can be seen also as an extension of the results in [94], where Wilton shows that finitely generated subgroups of finitely generated fully residually free groups are virtual retracts, and in particular, separable.

The hierarchical structure of limit groups and the characterisation of finitely presented residually free groups of [28] was used by Bridson and Kochloukova to calculate the L_2-Betti numbers, rank gradient and asymptotic deficiency of limit groups and of finitely presented residually free groups (see [22]). If G is a residually finite group, an exhausting normal chain is a filtration $(G_n)_{n\geq 1}$ of G such that G_n is normal in G, $G_{n+1} \subseteq G_n$ and $\bigcap_{n\geq 1} G_n = 1$. They proved that if G is a limit group, $(G_n)_{n\geq 1}$ is an exhausting normal chain of G and K is any field of characteristic 0, the L_2-Betti number of G $\beta_j(G)$, which may be computed as

$$\lim_{n\to\infty} \frac{\dim H_j(G_n;K)}{[G\colon G_n]},$$

is zero except for $j = 1$, where it equals $-\chi(G)$. They also computed the homotopical version of that result and in the case of finitely presented residually free groups, they found particular filtrations where those limits can be calculated.

In summary, finitely presented subgroups of direct products of limit groups have been deeply studied and thanks to the understanding of their structure we now know many of their interesting properties, for instance, that most of the classical algorithmic problems are decidable in this class.

Right-angled Artin groups

Right-angled Artin groups (RAAGs) were first introduced by Baudisch ([5]) in the 1970's and further developed by Droms ([44], [45], [46]). The class of RAAGs generalises the class of finitely generated free groups, by allowing relations saying that some or all of the generators commute. The spirit is to associate a group to a finite simplicial graph. In recent years, RAAGs have been studied extensively due to their actions on CAT(0) cube complexes and their rich subgroup structure. As a consequence of the work of Haglund and Wise ([53]), groups acting properly by combinatorial automorphisms on a CAT(0) cube complex with special quotient are subgroups of RAAGs. Many groups that are important in geometric group theory can be seen virtually as subgroups of RAAGs, including fundamental groups of closed, irreducible 3-manifolds (see [2] and [95]), all Coxeter groups (see [54]) and one-relator groups with torsion (see [2]).

Subgroups of RAAGs also play an important role in geometric group theory by providing examples of groups with interesting finiteness properties. We have seen previously that subgroups of direct products of free groups have fascinating finiteness properties. Recall that if F_2^n is the direct product of n free groups of rank two (which is a RAAG) and H_n is the kernel of the homomorphism $F_2^n \mapsto \mathbb{Z}$ sending all the standard generators to 1, then H_n is of type F_{n-1} but not of type FP_n. Bestvina and Brady in [9] gave a general construction which provides examples of subgroups of direct products of RAAGs having one type of finiteness property but not the other, extending the results of Stallings and Bieri in [91] and [12], respectively. In particular, they produced groups of type FP_∞ that are not finitely presented. In order to find these type of examples it is not sufficient to work with subgroups of direct products of free groups, since from the work of Bridson, Howie, Miller and Short it follows that for subgroups of F_2^n, being of type F_n is equivalent to being of type $FP_n(\mathbb{Q})$.

Main goal of the thesis

The richness of subgroups of RAAGs and the previous work on finitely presented subgroups of direct products of finitely generated free groups motivates the study of subgroups of direct products of RAAGs. Since finitely presented subgroups of direct products of finitely generated free groups have good algorithmic behaviour, one can wonder if this is the case for all finitely presented subgroups of direct products of RAAGs. In his work [19], Bridson shows that this is not the case as he shows that there is a RAAG A and a finitely presented subgroup S of $A \times A$ for which the conjugacy problem and the membership problem in $A \times A$ are undecidable.

Hence, on the one hand, by all the previous results, there is a subclass of the class of RAAGs containing free groups where finitely presented subgroups of the direct product have a nice structure, and on the other hand, by the result of Bridson, we know that this is not true in the entire class of RAAGs. As a consequence, one may wonder which is the class of RAAGs where finitely presented subgroups of the direct product of groups in this class have good algorithmic behaviour. Since we want finitely presented subgroups of direct products to have a controllable structure, in particular we need to have some control on subgroups of groups in this class. There are two main families of RAAGs where it is known that the conjugacy and the membership problems are decidable for finitely presented subgroups, namely the class of Droms RAAGs and the class of coherent RAAGs. In this work we focus on studying finitely presented subgroups of direct products of groups in these classes, and, more generally, of limit groups over groups in these classes, which is a concept extending the notion of limit group.

Droms RAAGs and coherent RAAGs

In general, not all subgroups of RAAGs are again RAAGs, and Droms RAAGs are precisely those with the property that all of their finitely generated subgroups are again RAAGs. In particular, finitely generated free groups are Droms RAAGs. They were characterised by Droms as the RAAGs where the defining graph does not contain full subgraphs that are squares or lines of length 3 (see [46]).

Droms RAAGs are a subclass of the class of coherent RAAGs. Recall that a group is coherent if every finitely generated subgroup is also finitely presented. Droms gave in [44] a characterisation of coherent RAAGs in terms of the associated graphs. He proved that a RAAG is coherent if and only if the defining graph does not contain induced cycles of length greater than 3. For instance, the RAAG associated to a tree, which is called a tree group, is coherent. Tree groups with trivial center are no longer Droms RAAGs, however, and the fact that they have finitely generated subgroups which are not RAAGs make them more difficult to work with.

Limit groups over RAAGs

Some of the characterisations of limit groups mentioned can be adapted to the setting of RAAGs. For instance, if GX is a RAAG and G is a finitely generated group, then G is fully residually GX if and only if G is the coordinate group of an irreducible algebraic set over GX if and only if the existential theory of GX is contained in that of G. Thus a limit group over a RAAG is defined to be a group that is a finitely generated fully residually RAAG. In this work we focus on limit groups over coherent RAAGs, and more particularly, on limit groups over Droms RAAGs. Both of these classes have been studied previously by Casals-Ruiz, Duncan and Kazachkov in [32], [33], [34] and [35].

Contents of thesis

In Chapter 2 we prove the analogs of the results of [28] and [27] of Bridson, Howie, Miller and Short in the case of limit groups over Droms RAAGs. Specifically, remember that they prove that subgroups of type w$FP_n(\mathbb{Q})$ of the direct product of n limit groups are virtually the direct product of limit groups, and they also characterise finitely presented subgroups in terms of their projections onto pairs. We generalise both results to limit groups over Droms RAAGs in this first chapter, and in addition, we obtain some consequences related to decision problems.

In Chapter 3 we follow the path that Kochloukova took in [65] generalising the results in [27]. She showed that limit groups are free-by-(torsion-free nilpotent) and by using this property, she proved that if S is a subgroup of type w$FP_s(\mathbb{Q})$ of the direct product of limit groups, then S virtually surjects onto s coordinates. In the first part of the chapter we generalise these results to the case of limit groups over Droms RAAGs. After that, following [22], we discuss the growth of homology groups and the volume gradients for limit groups over coherent RAAGs.

In Chapter 4 we compute the Bieri-Neumann-Strebel-Renz invariants for limit groups over Droms RAAGs and the first Σ-invariant for finitely presented residually Droms RAAGs. For the case of limit groups and finitely presented residually free groups these were computed by Kochloukova and Ferreira Lima in [65] and [72], respectively. Generalising their work, we are able to extend the results and apply them to our case.

Finally, Chapter 5 and 6 deal with the class of 2-dimensional coherent RAAGs. More precisely, in Chapter 5 we generalise Baumslag and Roseblade's result on subgroups of direct products of finitely generated free groups and describe the structure of finitely presented subgroups of the direct product of two 2-dimensional coherent RAAGs. Furthermore, we show that these finitely presented subgroups have good algorithmic properties. We also prove, by means of the Bieri-Neumann-Strebel-Renz invariants, that these finitely presented subgroups are actually of type F_∞. In Chapter 6 we show that for this class there is a new phenomena. Namely, recall that finitely presented subgroups of direct products of finitely generated free groups virtually contain a term of the lower central series of the group. For finitely presented subgroups of direct products of 2-dimensional coherent RAAGs, they are virtually an extension of a direct product by a polycyclic group.

In sum, thanks to the work of Baumslag and Roseblade in [7] we know that finitely presented subgroups of the direct product of two finitely generated free groups have a rigid structure and that the main algorithmic problems are decidable in this class. There is a counterexample given by Bridson in [19] which warns us that this good behaviour does not extend to all finitely presented subgroups of the direct product of two RAAGs. In this thesis we study the finiteness and algorithmic properties of subgroups of direct products of some coherent RAAGs, and as a corollary, we narrow the subclass of RAAGs where finitely presented subgroups of the direct product are complicated by showing that Droms RAAGs and 2-dimensional coherent RAAGs do not belong to it.

Chapter 1

Background

1.1 Background on Bass-Serre theory

We recall the basics of Bass-Serre theory and we refer the reader to [18] and [43] for further details.

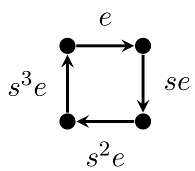
A *G-set* is a set X with a G-action on X. A function $\alpha \colon X_1 \mapsto X_2$ between G-sets is said to be a *G-map* if $\alpha(gx) = g\alpha(x)$ for all $g \in G, x \in X_1$. By the *quotient set* for the G-set X we mean $X/G = \{Gx \mid x \in X\}$, the set of G-orbits. By a *G-transversal* in X we mean a subset S of X which meets each orbit exactly once, so the composite $S \subseteq X \mapsto X/G$ is bijective. A *G-graph* (X, VX, EX, ι, τ) is a non-empty G-set X with specified non-empty G-subset VX, its complement $EX = X \setminus VX$, and two G-maps $\iota, \tau \colon EX \mapsto VX$. These are called the *incidence functions* of X. For $v \in VX$, we define $st(v) = \iota^{-1}(v) \cup \tau^{-1}(v)$. The number of elements in $st(v)$ is called the *valency* of v. If every vertex of X has finite valency, then X is said to be *locally finite*. By the *quotient graph* X/G we mean the graph

$$(X/G, VX/G, EX/G, \bar{\iota}, \bar{\tau}),$$

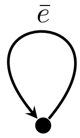
where

$$\bar{\iota}(Ge) = G\iota e, \quad \bar{\tau}(Ge) = G\tau e \quad \text{for all} \quad Ge \in EX/G.$$

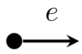
Example 1.1.1. If $G = C_4 = \langle s \mid s^4 = 1 \rangle$ and $S = \{s\}$, then the Cayley graph $\Delta(G, S)$ is



The quotient graph is



which lifts back to a G-transversal



We may define the incidence functions ι, τ on $EX^{\pm 1}$ by setting $\iota e^1 = \iota e, \tau e^1 = \tau e, \iota e^{-1} = \tau e, \tau e^{-1} = \iota e$ for all $e \in EX$. A *path* p in X is a finite sequence

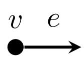
$$v_0, e_1^{\epsilon_1}, v_1, \ldots, v_{n-1}, e_n^{\epsilon_n}, v_n,$$

where $n \geq 0$, $v_i \in VX$, $e_i^{\epsilon_i} \in EX^{\pm 1}$, $\iota e_i^{\epsilon_i} = v_{i-1}, \tau e_i^{\epsilon_i} = v_i$. If for each i in $\{1, \ldots, n-1\}$ it happens that $e_{i+1}^{\epsilon_{i+1}} \neq e_i^{-\epsilon_i}$, then p is said to be *reduced.* If $v_0 = v_n$, p is said to be *closed.*

We say that X is a *forest* if the only reduced closed paths are the empty ones, and X is a *tree* if it is a connected forest. In a tree X, for any two vertices v, w of X there is a unique reduced path from v to w and it is called the *geodesic* from v to w.

Proposition 1.1.2. *[43, 2.6 Proposition] If X is a G-graph and X/G is connected, then there exist subsets $Y_0 \subseteq Y \subseteq X$ such that Y is a G-transversal in X, Y_0 is a subtree of X, $VY = VY_0$ and for each $e \in EY$, $\iota e \in VY = VY_0$.*

Example 1.1.3. In Example 1.1.1 Y is



and Y_0 is the vertex v.

We now introduce the main object of study for Bass-Serre theory.

Definition 1.1.4. By a *graph of groups* $(G(-), Y)$ we mean a connected graph (Y, VY, EY, ι, τ) together with a function $G(-)$ which assigns to each $v \in VY$ a group $G_v = G(v)$ and to each edge $e \in EY$ a distinguished subgroup $G_e = G(e)$ of $G(\iota e)$ and an injective homomorphism $t_e \colon G_e \mapsto G_{\tau e}$. For $g \in G_e$ we denote $t_e(g)$ by g^{t_e}.

We call the G_v, $v \in VY$, the *vertex groups* and the G_e, $e \in EY$, the *edge groups*. If the vertex and edge groups are the obvious ones, we simply say that Y is a graph of groups.

A *finite graph of groups* is a graph of groups where the underlying graph is finite.

Example 1.1.5. General construction. Suppose that (X, VX, EX, ι, τ) is a G-graph such that X/G is connected. Let us choose a fundamental G-transversal Y in X with subtree Y_0. A graph of groups can be defined as follows.

For each $e \in EY$ there are unique elements $\bar{\iota}e, \bar{\tau}e \in VY$ which lie in the same G-orbits as ιe and τe, respectively. In fact, $\bar{\iota}e = \iota e$. Then Y can be made into a graph with incidence functions $\bar{\iota}$ and $\bar{\tau}$. Since τe and $\bar{\tau}e$ lie in the same G-orbit, there is an element t_e in G such that $t_e\bar{\tau}e = \tau e$. Moreover, if $e \in EY_0$, then $\bar{\tau}e$ equals τe, so $t_e = 1$.

Now the stabiliser of an edge e, G_e, is a subgroup of the stabilisers of ιe and τe, $G_{\iota e}$ and $G_{\tau e}$. In addition, since $\bar{\iota}e = \iota e$, then $G_{\iota e} = G_{\bar{\iota}e}$ and $G_{\tau e} = t_e G_{\bar{\tau}} e t_e^{-1}$, so there is a monomorphism $t_e \colon G_e \mapsto G_{\bar{\tau}e}$ sending an element g to $t_e^{-1} g t_e$.

Definition 1.1.6. Let $(G(-), Y)$ be a graph of groups. Choose a maximal subtree Y_0 of Y, so $VY_0 = VY$. We define the associated *fundamental group* $\pi(G(-), Y, Y_0)$ to be the group presented as follows:

(1) The generating set is $\{t_e \mid e \in EY\} \cup \bigcup_{v \in VY} G_v$.

(2) The relations:

- the relations for G_v, for each $v \in VY$.
- $t_e^{-1} g t_e = g^{t_e}$ for all $e \in EY$, $g \in G_e \subseteq G_{\iota e}$, so $g^{t_e} \in G_{\tau e}$.
- $t_e = 1$ for all $e \in EY_0$.

The generators $\{t_e \mid e \in EY \setminus EY_0\}$ are called the *stable letters*.

Remark 1.1.7. If Y_0 and Y_1 are two different maximal subtrees of Y, then the fundamental groups $\pi(G(-), Y, Y_0)$ and $\pi(G(-), Y, Y_1)$ are isomorphic.

Example 1.1.8. Let $(G(-), Y)$ be a graph of groups and let Y_0 be a maximal subtree of Y.

(1) Suppose that $G_v = 1$ for all $v \in VY$. Then $\pi(G(-), Y, Y_0)$ is a free group of rank $|EY \setminus EY_0|$.

(2) Suppose that Y has one edge and two vertices $\iota e, \tau e$. Let $A = G_{\iota e}$, $B = G_{\tau e}$, $C = G_e$, so that C is a subgroup of A and there is a specified embedding $t\colon C \mapsto B$. Here $Y_0 = Y$ and the fundamental group is *the free product of A and B amalgamating C*, denoted $A *_C B$. It has presentation

$$\langle A, B \mid c = c^t \text{ for all } c \in C\rangle.$$

(3) Suppose that Y has one edge e and one vertex $v = \iota e = \tau e$. Let $A = G_v$, $C = G_e$, so that C is a subgroup of A and there is specified an embedding $t\colon C \mapsto A$. Here Y_0 consists of the single vertex and the fundamental group is the *HNN extension of A over C with stable letter t*, denoted $A*_C$. It has presentation

$$\langle A, t \mid t^{-1}ct = c^t \text{ for all } c \in C\rangle.$$

A graph of groups is said to be *reduced* if, given an edge with distinct endpoints v_1 and v_2, the subgroup $G_e < G_{v_1}$ is proper and the inclusion $G_e \mapsto G_{v_2}$ is proper. In particular, an HNN extension is always reduced. An amalgaman $A *_C B$ is reduced if $C \neq A$ and $C \neq B$. If a finite graph of groups is not reduced, it can be made reduced by iteratively collapsing edges that make it non-reduced. Thus, in this work we assume that finite graphs of groups are always reduced.

An HNN extension $A*_C$ with stable letter t is *ascending* if C or C^t is not a proper subgroup of A.

Theorem 1.1.9. *[18, 18.2 Theorem] Let $G = \pi(G(-), Y, Y_0)$ be the fundamental group of a graph of groups $(G(-), Y)$ with respect to a maximal subtree Y_0 of Y. Then the group G acts without inversion of edges on a tree T such that the underlying graph of the quotient graph T/G is isomorphic to Y and the stabilisers of the vertices and edges of the tree T are conjugate to the groups G_v, $v \in VY$, and G_e, $e \in EY$, respectively.*

Sketch of the proof. Define

$$VT = \dot{\bigcup}_{v\in VY} G/G_v, \quad ET = \dot{\bigcup}_{e\in EY} G/G_e,$$

$$\iota(gG_e) = gG_{\iota e} \quad \text{and} \quad \tau(gG_e) = gt_eG_{\tau e},$$

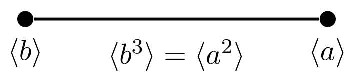
for $g \in G$, $e \in EY$. The group G acts on T by left multiplication. $\square$

The tree T in the previous theorem is named the associated *Bass-Serre tree.*

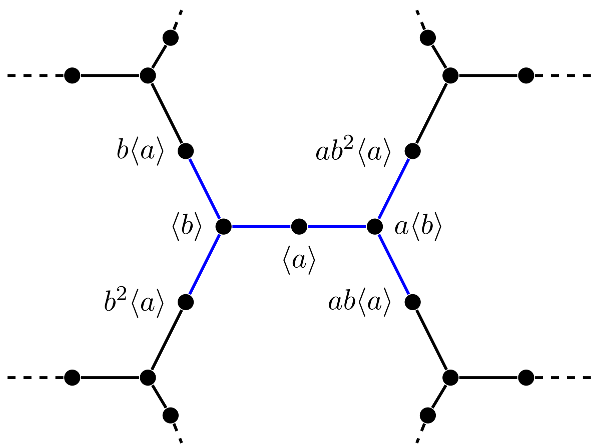
The next result gives the reverse implication of the above theorem, by showing that if a group acts on a tree, then such a group is the fundamental group of a graph of groups.

Theorem 1.1.10. *[43, 4.1 Theorem] Let T be a G-tree. Choose a fundamental G-transversal Y with subtree Y_0 and denote the incidence functions by $\bar{\iota}$, $\bar{\tau}$; choose, for each $e \in EY$, $t_e \in G$ such that $t_e\bar{\tau}e = \tau e$, with $t_e = 1$ if $e \in EY_0$, and form the resulting graph of groups $(G(-), Y)$ as in Example 1.1.5. Then G is isomorphic to $\pi(G(-), Y, Y_0)$.*

Example 1.1.11. [18, 18.7 Example] Let $G = \langle a, b \mid a^2 = b^3\rangle$, let $\varphi\colon G \mapsto S_3$ such that $\varphi(a) = (1\ 2)$ and $\varphi(b) = (1\ 2\ 3)$, and let H be the kernel of φ. Let us find a presentation of H in the form of a fundamental group of a graph of groups. The group G is the fundamental group of the graph of groups



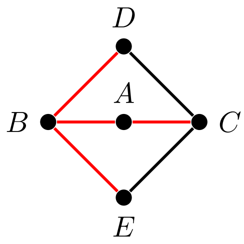
Then G acts on a tree T, where the vertices of this tree are the left cosets of subgroups $\langle a\rangle$ and $\langle b\rangle$ in G, and the edges are the left cosets of the subgroup $\langle a^2\rangle = \langle b^3\rangle$ in G. The vertices $g\langle b\rangle$ and $g\langle a\rangle$ are connected by the positively oriented edge $g\langle a^2\rangle$.



The group H acts on T by left multiplication. The set $\{1, b, b^2, a, ba, b^2a\}$ is a system of representatives of left cosets of H in G, so any vertex of the form $g\langle a\rangle$ is H-equivalent to either $\langle a\rangle$, $b\langle a\rangle$ or $b^2\langle a\rangle$. The set $\{1, b, b^2, a, ab, ab^2\}$ is also a system of representatives of left cosets of H in G, so any vertex of the form $g\langle b\rangle$ is H-equivalent to $\langle b\rangle$ or $a\langle b\rangle$. Therefore, there are 5 vertices in the graph T/H,

A, D, E, B, C and their representatives are the vertices $\langle a\rangle$, $b\langle a\rangle$, $b^2\langle a\rangle$, $\langle b\rangle$, $a\langle b\rangle$. Similarly it can be shown that there are exactly 6 equivalence classes of edges of the tree T (drawn in blue).

The vertices $b\langle a\rangle$ and $ab^2\langle a\rangle$ are H-equivalent, since $(ab^2a^{-1}b^{-1})b\langle a\rangle$ equals $ab^2\langle a\rangle$ and $t_1 = ab^2a^{-1}b^{-1} \in H$, so they are projected to the same vertex D. The vertices $b^2\langle a\rangle$ and $ab\langle a\rangle$ are also projected to the same vertex E. Thus, the graph $Y = T/H$ is



Let Y_0 be the maximal subtree of Y drawn in red. The stabilisers of all vertices and all edges in the group H are $\langle a^2\rangle$. In addition, all embeddings of edge groups into the corresponding vertex groups are identities, since $\langle a^2\rangle$ is the center of the group G. From this we deduce that H has presentation

$$\langle x, t_1, t_2 \mid t_1^{-1}xt_1 = x, t_2^{-1}xt_2 = x\rangle,$$

in which the letters x, t_1, t_2 correspond to $a^2, ab^2a^{-1}b^{-1}$, $aba^{-1}b^{-2}$.

In this thesis we will mainly work with cocompact, minimal, acylindrical actions of a group G on a tree T. The action is *cocompact* if the quotient graph T/G is compact. It is *minimal* if T is the only G-invariant subtree of T. It is k-*acylindrical* if the stabiliser of any geodesic path of length greater than k is trivial. If the value of k is not important, one says simply that the action is *acylindrical*. Analogously, a graph of groups decomposition is termed k-acylindrical if the action on the associated Bass-Serre tree is k-acylindrical. An element $g \in G$ is called *elliptic* if it fixes a point of T. Otherwise, it is called *hyperbolic* and in this case there is a unique embedded line in T, called the *axis*, on which g acts as translation.

1.2 Background on homology of groups

The homology theory of groups arose from both topological and algebraic sources and it offers a great deal of interactions between them. We here give a very brief summary of this fascinating subject just stating the definitions and results that will be used in the thesis. Brown's book [29] is a good reference for further details.

1.2.1 Definition of $H_*(G; M)$

Let G be a group and let $\mathbb{Z}G$ be the free $\mathbb{Z}$-module generated by the elements of G. Thus an element of $\mathbb{Z}G$ is uniquely expressible in the form $\sum_{g\in G} a(g)g$, where $a(g) \in \mathbb{Z}$ and $a(g) = 0$ for almost all g. The multiplication in G extends to a $\mathbb{Z}$-bilinear product $\mathbb{Z}G \times \mathbb{Z}G \mapsto \mathbb{Z}G$ and this makes $\mathbb{Z}G$ a ring, called the *integral group ring* of G.

A (left) $\mathbb{Z}G$-module, also called a *G-module*, consists of an abelian group A together with an action of G on A. For example, one has for any abelian group A the *trivial* module structure, with $g \cdot a = a$ for all $g \in G$, $a \in A$.

If R is a ring and M is an R-module with n generators, then we have an exact sequence $R^n \to M \to 0$. The surjection $R^n \mapsto M$ has a kernel K. If K admits m generators as an R-module, then we obtain a continuation of the previous sequence

$$\begin{tikzcd} R^m \arrow[r] \arrow[dr, two heads] & R^n \arrow[r] & M \arrow[r] & 0 \\ & K \arrow[u, hook] & & \end{tikzcd}$$

with the top row exact. The surjection $R^m \mapsto K$ has a kernel L. Choosing a free module that maps onto L and continuing the procedure, we obtain an exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0,$$

where each F_i is a free R-module. Such an exact sequence is called a *free resolution* of M. The *fundamental lemma of homological algebra* ensures that given any module M, free resolutions of M exist and are unique up to chain homotopy equivalence. A milder property for a module is to be projective. An R-module P is *projective* if it is a direct summand of a free module, i.e. there is a module Q such that $P \oplus Q$ is free. The fundamental lemma of homological algebra is applicable also to *projective resolutions*, not just to free ones.

The tensor product $M \otimes_R N$ is defined whenever M is a right R-module and N is a left R-module. It is the quotient of $M \otimes_{\mathbb{Z}} N$ obtained by introducing the relations $mr \otimes n = m \otimes rn$ for $m \in M$, $r \in R$, $n \in N$. In case R is a group ring $\mathbb{Z}G$, we can avoid having to consider both left and right modules. We can regard any left G-module M as a right G-module by setting $mg = g^{-1}m$ and in this way we can make sense out of the tensor product $M \otimes_{\mathbb{Z}G} N$ of two left G-modules. Note that $M \otimes_{\mathbb{Z}G} N$ is obtained from $M \otimes_{\mathbb{Z}} N$ by introducing the relations $g^{-1}m \otimes n = m \otimes gn$.

Let F be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ and let M be a G-module. We define the *homology of G with coefficients in M* by

$$H_*(G; M) = H_*(F \otimes_{\mathbb{Z}G} M).$$

If M is the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, then $H_*(G;\mathbb{Z})$ is denoted by $H_*(G)$ and called the *homology of G*.

Remark 1.2.1. $H_*(G; M)$ is independent of the projective resolution that we choose.

1.2.2 Topological interpretation

Suppose X is a *G-CW-complex*, by which we mean a CW-complex with a G-action that permutes the cells. Then G acts on the cellular chain complex $C_*(X)$, which therefore becomes a chain complex of $\mathbb{Z}G$-modules, so we have a sequence

$$\cdots \to C_1(X) \to C_0(X) \to \mathbb{Z} \to 0$$

of $\mathbb{Z}G$-modules. If G acts freely on X, then each module $C_n(X)$ is a direct sum of copies of $\mathbb{Z}G$, with one copy for each G-orbit of n-cells. In particular, $C_n(X)$ is a free $\mathbb{Z}G$-module. If, in addition, X is contractible, then the sequence is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

Definition 1.2.2. Let Y be a CW-complex with fundamental group G. We say that Y is an *Eilenberg-MacLane complex of type* $K(G,1)$ if its universal cover is contractible.

It is a fact that every group G admits a $K(G,1)$-complex Y and that Y is unique up to homotopy equivalence. In this case, the universal cover X of Y is a contractible, free G-CW-complex, so its cellular chain complex gives rise to a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

Theorem 1.2.3. *[29, (4.1) Proposition] Let Y be a $K(G,1)$-complex. Then*

$$H_*(G) = H_*(Y).$$

1.2.3 Finiteness properties

There are two natural definitions of the dimension of a group, depending on whether we think topologically or algebraically.

Definition 1.2.4. The *geometric dimension* of G, denoted gd G, is the smallest non-negative integer n such that there exists an n-dimensional $K(G,1)$-complex.

If no such n exists, we say that gd $G = \infty$.

Definition 1.2.5. The *cohomological dimension* of G, denoted cd G, is the smallest non-negative integer n such that there exists a projective resolution $P = (P_i)_{i \geq 0}$ of $\mathbb{Z}$ over $\mathbb{Z}G$ satisfying $P_i = 0$ for $i > n$.

If no such n exists, we say that cd $G = \infty$.

Since a $K(G, 1)$-complex yields a free resolution of length equal to its dimension, cd $G \leq$ gd G.

Definition 1.2.6. We say that a group G is of *type* F_n $(0 \leq n < \infty)$ if there is a $K(G, 1)$-complex with a finite n-skeleton, i.e. with only finitely many cells in dimensions $\leq n$. We say that G is of *type* F_∞ if there is a $K(G, 1)$ with all of its skeleta finite, and that G is of *type* F if there is a finite $K(G, 1)$-complex.

Definition 1.2.7. We say that a group G is of *type* FP_n $(0 \leq n < \infty)$ if there is a projective resolution P of $\mathbb{Z}$ over $\mathbb{Z}G$ such that P_i is finitely generated for $i \leq n$. We say that G is of *type* FP_∞ if there is a projective resolution P of $\mathbb{Z}$ over $\mathbb{Z}G$ with P_i finitely generated for all i, and that G is of *type* FP if there is a projective resolution such that P_i is finitely generated for all i and is 0 for sufficiently large i.

Definition 1.2.8. We say that a group is *weakly of type* $FP_n(\mathbb{Q})$ $(0 \leq n < \infty)$, and we denote it as w$FP_n(\mathbb{Q})$, if $H_i(G_0; \mathbb{Q})$ is finite dimensional for all $G_0 < G$ of finite index and $0 \leq i \leq n$.

Proposition 1.2.9.

(1) $F_n \implies FP_n \implies wFP_n(\mathbb{Q})$.

(2) Every group is of type F_0.

(3) G *is of type* F_1 *if and only if it is finitely generated.*

(4) G *is of type* F_2 *if and only if it is finitely presented.*

(5) G *is of type* FP_1 *if and only if it is finitely generated.*

(6) If G *is of type* FP_n *and* M *is a* G*-module which is finitely generated as an abelian group, then* $H_i(G; M)$ *is a finitely generated abelian group for* $0 \leq i \leq n$.

1.2.4 Euler characteristic of a group

Even though the Euler characteristic of a group may be defined in wider generality, here we define it for FP groups. If G is of type FP, the *Euler characteristic* $\chi(G)$ is defined by

$$\chi(G) = \sum (-1)^i rk_{\mathbb{Z}}(H_i(G)).$$

Proposition 1.2.10.

(1) If G is of type FP and H is a finite index subgroup of G, then $\chi(H) = [G\colon H]\chi(G)$.

(2) Let $1 \to N \to G \to Q \to 1$ be a short exact sequence of groups of type FP. Then $\chi(G) = \chi(N)\chi(Q)$.

(3) Let $G = A *_C B$ where A, B, C and G are of type FP. Then $\chi(G) = \chi(A) + \chi(B) - \chi(C)$.

1.2.5 Some tools to compute the homology of groups

In this last section we mention some of the tools that will be used in the thesis to compute the homology of groups.

If G is a group and H is a subgroup of G, Shapiro's Lemma relates the homology of H and G as follows.

Proposition 1.2.11 (Shapiro's Lemma)**.** *[29, (6.2) Proposition] If $H < G$ and M is an H-module, then*

$$H_*(H; M) \cong H_*(G; \mathbb{Z}G \otimes_{\mathbb{Z}H} M).$$

We will also make use of a special case of the Künneth formula for group homology.

Proposition 1.2.12. *[29, Chapter V, Section 2] Let G and H be groups and let K be a field. Then*

$$H_n(G \times H; K) \cong \bigoplus_{0 \leq p \leq n} H_p(G; K) \otimes_K H_{n-p}(H; K).$$

Since we will be often working with fundamental groups of graphs of groups, and more particularly with HNN extensions, the associated Mayer-Vietoris sequence will be useful for us. Details can be found in [29, Chapter VII, Section 9]. Let $G = A*_C$ with stable letter t. Then there is a long exact sequence

$$\cdots \to H_n(C; M) \xrightarrow{\alpha} H_n(A; M) \xrightarrow{\beta} H_n(G; M) \to H_{n-1}(C; M) \to \cdots,$$

where β is induced by the inclusion $A \mapsto G$ and α is the difference between the map induced by the inclusion twisted by the action of t under conjugation.

Finally, we review the Lyndon-Hochschild-Serre spectral sequence for homology. The theory of spectral sequences is rather technical, but the lecture notes [82] written by Antonio Díaz Ramos illustrate the basic notions via examples arising from Algebraic Topology and Group Theory. They are, therefore, a practical source to familiarise with spectral sequences. For full details the reader could consult [29, Chapter VII].

Theorem 1.2.13 (Lyndon-Hochschild-Serre spectral sequence for homology)**.** *Let $1 \to N \to G \to Q \to 1$ be a short exact sequence of groups and let M be a G-module. Then there is a first quadrant homological spectral sequence*

$$E^2_{p,q} = H_p(Q; H_p(N; M)) \implies H_{p+q}(G; M).$$

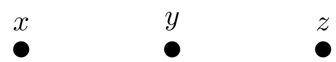
1.3 Background on right-angled Artin groups

Right-angled Artin groups were introduced by Baudisch in the 1970's [5] and further developed by Droms in the 1980's (see [44], [45], [46]). They have been studied extensively since that time in different branches of mathematics under different names: graph groups, partially commutative groups, semifree groups, etc. The notes [38] and [64] are a good source to learn about this fascinating class of groups.

Given a finite simplicial graph X, the corresponding *right-angled Artin group (RAAG)*, denoted by GX, is given by the following presentation. Let $V(X)$ denote the vertex set of X. Then

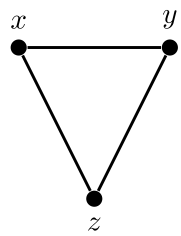
$$GX = \langle V(X) \mid xy = yx \iff x \text{ and } y \text{ are adjacent}\rangle.$$

For instance, if X is the following graph



then $GX = \langle x, y, z \mid \varnothing\rangle \cong F_3$. More generally, if X is a totally disconnected graph with n vertices, then the corresponding RAAG is the free group of rank n.

On the opposite side, if X is a complete subgraph, for example if X is



then $GX = \langle x, y, z \mid xy = yx, xz = zx, yz = zy \rangle \cong \mathbb{Z}^3$. If X is a complete graph with n vertices, then the corresponding RAAG is the free abelian group of rank n.

Alternatively, RAAGs are also defined by means of flag complexes. Recall that a *flag complex* is a simplicial complex such that any k-complete graph in the 1-skeleton of the complex spans a k-complex, so that a flag complex L is uniquely determined by its 1-skeleton $L^{(1)}$. The RAAG associated to L is the RAAG associated to $L^{(1)}$. Conversely, if X is a simplicial graph we can define the induced flag complex $\widehat{X}$ based on X as the flag complex with $\widehat{X}^{(1)} = X$.

1.3.1 The Salvetti complex

The goal of this section is to define a $K(G, 1)$-complex for a RAAG G, which will be called the *Salvetti complex* associated to G.

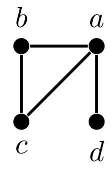
If G is a finitely presented group with finite generating set $S = \{s_1, \ldots, s_n\}$ and a finite set of relators R, then there is a way of obtaining a 2-dimensional cell complex, called the *presentation complex* of G, with fundamental group G. The complex has a single vertex, one loop at the vertex for each element of S and one 2-cell for each element of R with the boundary of the 2-cell attached along the word giving the relation.

If G is the RAAG associated to X, there is another associated cell complex, called the *Salvetti complex*, constructed as follows. We begin with a wedge of circles attached to a vertex and labeled by the generators. For each edge, say from s_i to s_j in X, we attach a 2-cell with boundary labeled by $s_i s_j {s_i}^{-1} {s_j}^{-1}$, giving a 2-torus. For each triangle in X connecting three vertices s_i, s_j, s_k, we attach a 3-torus with faces corresponding to the tori for the three edges of the triangle. This process is continued by attaching a k-torus for each complete subgraph of X with k-vertices. We denote by S_G the resulting cube complex.

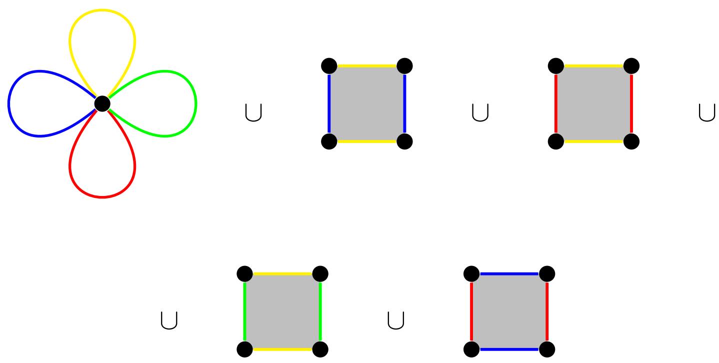
Theorem 1.3.1. *[38] Let G be a RAAG and let S_G be the associated Salvetti complex. Then S_G is a $K(G, 1)$-complex.*

Since S_G is a finite dimensional $K(G,1)$-complex, RAAGs are not only finitely presented, but they are also of type F.

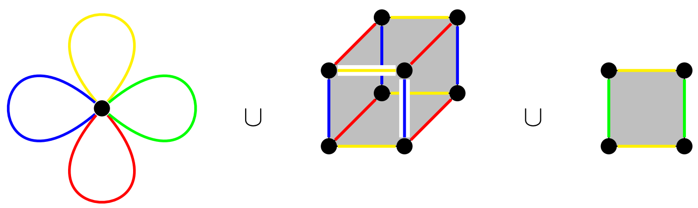
Example 1.3.2. Suppose that X is the following graph.



The presentation complex of GX is obtained in this way:



The Salvetti complex of GX is the following:

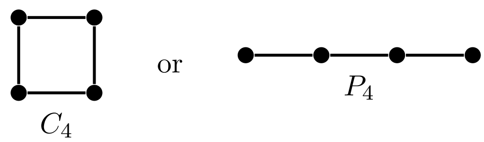


Recall that a *clique* in a simplicial graph is a complete subgraph. The complex S_G is an n-dimensional $K(G,1)$-complex, where n is equal to the number of vertices of the largest clique. The *dimension* of the group G is defined to be the dimension of the Salvetti complex S_G.

Note that every cell in S_G is a cycle because the cells are glued as a tori, so $H_j(G;\mathbb{Z}) \cong H_j(S_G;\mathbb{Z})$ is a free abelian group of rank d_j, where d_j is the number of j-cells in S_G, or equivalently, d_j is the number of j-vertex cliques in X. In particular, the abelianisation of G is $\mathbb{Z}^n$, where n is the number of vertices in X.

1.3.2 Droms RAAGs

As mentioned in the introduction, subgroups of RAAGs can be quite wild; in particular, not all finitely generated subgroups of RAAGs are themselves RAAGs. Droms provided a condition for a RAAG to have all its subgroups again of this type (see [46]). He showed that every finitely generated subgroup of GX is itself a RAAG if and only if no full subgraph of X has either of the forms C_4 or P_4 illustrated below:



If GX is a RAAG with the above properties, then GX is called a *Droms RAAG*. For instance, finitely generated free groups and free abelian groups are Droms RAAGs. The group $\mathbb{Z}^2 \times ((\mathbb{Z} \times F_3) * F_7)$ is also a Droms RAAG, but the RAAG associated to C_4, $F_2 \times F_2$, is not a Droms RAAG.

Alternatively, the class of Droms RAAGs has an algebraic description.

Definition 1.3.3. Let $\mathcal{C}$ be a class of groups. The *$Z*$-closure of $\mathcal{C}$*, denoted by $Z*(\mathcal{C})$, is the union of classes $(Z*(\mathcal{C}))_k$ defined inductively as follows. At level 0, the class $(Z*(\mathcal{C}))_0$ is $\mathcal{C}$. A group G lies in $(Z*(\mathcal{C}))_k$ if and only if

$$G \cong \mathbb{Z}^m \times (G_1 * \cdots * G_n),$$

where $m \in \mathbb{N} \cup \{0\}$ and the group G_i lies in $(Z*(\mathcal{C}))_{k-1}$ for all $i \in \{1, \dots, n\}$.

The *level of* G, denoted by $l(G)$, is the smallest k for which G belongs to $(Z*(\mathcal{C}))_k$.

Notice that if the class $\mathcal{C}$ contains only RAAGs, then so does its $Z*$-closure. In this terminology, Droms showed in [46] that the class of Droms RAAGs is the $Z*$-closure of the class of finitely generated free groups.

For example, if G is a Droms RAAG such that $l(G) = 0$, then G is a finite rank free group. If $l(G) = 1$, then

$$G \cong \mathbb{Z}^m \times F$$

for some $m \geq 1$, F free of finite rank and G is not $\mathbb{Z}$. If $l(G) = 2$, then

$$G \cong \mathbb{Z}^m \times \big((\mathbb{Z}^{n_1} \times F_{k_1}) * \cdots * (\mathbb{Z}^{n_l} \times F_{k_l})\big),$$

for some $m, n_i, k_i \in \mathbb{N} \cup \{0\}$, $\sum_i n_i \geq 1$, $l \geq 2$ and for $i \in \{1, \ldots, l\}$ F_{k_i} is free of finite rank k_i.

1.3.3 Coherent RAAGs

A group is called *coherent* if each of its finitely generated subgroups is finitely presented. Free groups and free abelian groups are, for example, coherent. More generally, Droms RAAGs are coherent. However, the RAAG $F_2 \times F_2$ is known not to be coherent (see the examples in the introduction of Bieri, [12]). Droms provided a characterisation of coherent RAAGs in terms of the associated graphs.

Theorem 1.3.4. *[44] The group GX is coherent if and only if X does not contain a full subgraph isomorphic to a cycle of length greater than* 3.

Theorem 1.3.5. *[44] If GX is a coherent RAAG, then GX splits as a finite graph of groups where the underlying graph is a tree and all the vertex groups are free abelian.*

The class of 2-dimensional coherent RAAGs will be very relevant to this work. Recall that GX is 2-dimensional if the corresponding Salvetti complex is 2-dimensional. Thus, 2-dimensional coherent RAAGs are precisely the RAAGs corresponding to forests, that is they are free products of tree groups, where by a *tree group* we mean a RAAG whose defining graph is a tree. For example, the RAAG corresponding to the graph P_4, which abusing notation we also denote by P_4, is a tree group but it is not a Droms RAAG.

1.4 Background on the Bieri-Neumann-Strebel-Renz invariants

In this section we summarise without proofs some of the main notions and results in the area of the Bieri-Neumann-Strebel-Renz invariants. We refer the reader to [14], [15] and [16] for a complete and extensive account of the theory. We mainly focus on the invariants for RAAGs computed by Meier, Meinert and VanWyk in [73].

1.4.1 Preliminaries on the Bieri-Neumann-Strebel-Renz invariants

Let G be a finitely generated group and let $Hom(G, \mathbb{R})$ be the vector space consisting of all homomorphisms $\chi \colon G \mapsto \mathbb{R}$ of G into the additive group of the field $\mathbb{R}$.

A *character* χ of G is an element of $Hom(G, \mathbb{R}) \setminus \{0\}$ and χ is *discrete* if $\chi(G) \cong \mathbb{Z}$.

We define an equivalence relation in $Hom(G,\mathbb{R})\setminus\{0\}$ where for two elements $\chi_1,\chi_2\in Hom(G,\mathbb{R})\setminus\{0\}$ we say that $\chi_1\sim\chi_2$ if there is $r\in\mathbb{R}$, $r>0$ such that $\chi_1=r\chi_2$. The equivalence class of χ_1 is denoted by $[\chi_1]$. Then the character sphere $S(G)$ is

$$S(G)=\frac{Hom(G,\mathbb{R})\setminus\{0\}}{\sim}.$$

The Bieri-Neumann-Strebel-Renz invariants are specific open subsets in $S(G)$. The invariants are separated in two groups: the homological invariants $\{\Sigma_D^n(G,M)\}_n$ and the homotopical invariants $\{\Sigma^n(G)\}_n$.

The homological invariants were studied by Bieri and Renz in [14]. Let $\chi\colon G\mapsto\mathbb{R}$ be a character and let G_χ be the monoid $\{g\in G\mid \chi(g)\geq 0\}$ of G. Let D be an integral domain and let M be a right DG-module. Then $\Sigma_D^n(G,M)$ is defined by

$$\Sigma_D^n(G,M)=\{[\chi]\in S(G)\mid M\text{ is of type }FP_n\text{ as a }DG_\chi\text{-module}\}.$$

When M is D, then $\Sigma_D^n(G,D)$ is usually denoted by $\Sigma^n(G,D)$ and we say that the invariant has coefficients in D. Also $\Sigma_{\mathbb{Z}}^n(G,M)$ is normally written as $\Sigma^n(G,M)$. Note that we have the inclusions

$$\cdots\subseteq\Sigma_D^n(G,M)\subseteq\Sigma_D^{n-1}(G,M)\subseteq\cdots\subseteq\Sigma_D^1(G,M)\subseteq S(G).$$

The homotopical invariant Σ^1 was defined at first for finitely generated metabelian groups and later, in [16], Bieri, Neumann and Strebel extended the definition to the class of all finitely generated groups. Suppose that X is a finite generating set for G and let $\Delta(G,X)$ be the Cayley graph of G with respect to X. Let $\Delta(G,X)_\chi$ be the subgraph of $\Delta(G,X)$ spanned by G_χ. Then, by definition,

$$\Sigma^1(G)=\{[\chi]\in S(G)\mid\Delta(G,X)_\chi\text{ is connected}\}.$$

This invariant coincides with $\Sigma^1(G,\mathbb{Z})$ by [14, (1.3)]. Renz extended this definition and introduced the invariant $\Sigma^n(G)$ for groups G of type F_n. If G is finitely presented and if $\langle X\mid R\rangle$ is a finite presentation of G, then $\Delta(G,X)$ can be extended to the Cayley complex $Y(X,R)$ by gluing 2-cells at every vertex attached at boundaries with labels that correspond to the elements of the set of relations R. If $Y(X,R)_\chi$ is the subcomplex of $Y(X,R)$ spanned by G_χ, then $\Sigma^2(G)$ is

$$\{[\chi]\in S(G)\mid\text{ there is a finite presentation }G=\langle X\mid R\rangle\text{ such that}$$

$$Y(X,R)_\chi\text{ is simply-connected}\}.$$

It is important to note that even though the definition of Σ^1 is independent from the choice of the finite generating set X, in the definition of $\Sigma^2(G)$ we have to check the condition for each finite presentation of G.

Recall that a complex is *n-connected* if all its reduced homotopy groups vanish up to an including dimension n. The definition of $\Sigma^n(G)$ generalises the definition of $\Sigma^2(G)$ by substituting the Cayley complex by the n-dimensional skeleton of a $K(G,1)$-complex and substituting 1-connectivity with $(n-1)$-connectivity. By [84], for $n \geq 2$ if G is of type F_n, then

$$\Sigma^n(G) = \Sigma^2(G) \cap \Sigma^n(G, \mathbb{Z}).$$

Hence, in order to calculate $\Sigma^n(G)$ it suffices to work with $\Sigma^2(G)$ and $\Sigma^n(G, \mathbb{Z})$.

The importance of the invariants $\Sigma^n(G)$ and $\Sigma^n(G, \mathbb{Z})$ lies in the fact that they control which subgroups of G containing the commutator subgroup are of type F_n and FP_n:

Theorem 1.4.1.

(1) [84] Let G be a group of type F_n and let N be a subgroup of G that contains the commutator subgroup G'. Then N is of type F_n if and only if

$$S(G, N) = \{[\chi] \in S(G) \mid \chi(N) = 0\} \subseteq \Sigma^n(G).$$

(2) [14] Let G be a group of type FP_n and let N be a subgroup of G that contains the commutator subgroup G'. Then N is of type FP_n if and only if

$$S(G, N) = \{[\chi] \in S(G) \mid \chi(N) = 0\} \subseteq \Sigma^n(G, \mathbb{Z}).$$

Lemma 1.4.2. *Let G be a group and let $\chi\colon G \mapsto \mathbb{R}$ be a character such that $\chi(c) \neq 0$ for some $c \in Z(G)$.*

(1) [14, Theorem C] If G is of type FP_n, then $[\chi] \in \Sigma^n(G, \mathbb{Z})$.

(2) [74, Lemma 2.1] If G is of type F_n, then $[\chi] \in \Sigma^n(G)$.

1.4.2 Some results on Σ-invariants: connections with actions on trees and Σ-invariants of direct products

In this subsection we collect results on Σ-invariants for fundamental groups of graphs of groups that we will require to show that finitely presented subgroups of the direct product of two 2-dimensional coherent RAAGs are of type F_∞. Since we will work with direct products of limit groups over RAAGs, we also state some results about Σ-invariants for direct products.

Theorem 1.4.3. *[85] Let G be the fundamental group of a finite graph of groups and let $\chi\colon G \mapsto \mathbb{R}$ be a character. Let χ_σ be the restriction of χ to a vertex group or an edge group G_σ and assume that χ_v is non-zero for a vertex group G_v. Let D be an integral domain and let M be a right DG-module.*

(1) If $n \geq 1$, if $[\chi_v] \in \Sigma^n(G_v, M)$ for all vertex groups G_v and if $[\chi_e] \in \Sigma^{n-1}(G_e, M)$ for all edge groups G_e, then $[\chi] \in \Sigma^n(G, M)$.

(2) If $n \geq 0$, if $[\chi] \in \Sigma^n(G, M)$ and if $[\chi_e] \in \Sigma^n(G_e, M)$ for all edge groups G_e, then $[\chi_v] \in \Sigma^n(G_v, M)$ for all vertex groups G_v.

(3) If $n \geq 1$, if $[\chi] \in \Sigma^n(G, M)$ and if $[\chi_v] \in \Sigma^{n-1}(G_v, M)$ for all vertex groups G_v, then $[\chi_e] \in \Sigma^{n-1}(G_e, M)$ for all edge groups G_e.

Theorem 1.4.4. *[75, Theorem B] Let a group G act on a 1-connected G-finite 2-complex X and let $\chi\colon G \mapsto \mathbb{R}$ be a character. Denote by χ_σ the restriction of χ to a cell σ of X. If $\chi_\sigma \neq 0$ and $[\chi_\sigma] \in \Sigma^{2-\dim(\sigma)}(G_\sigma)$ for all cells σ of X, then $[\chi] \in \Sigma^2(G)$.*

Theorem 1.4.5. *[36, Proposition 2.5] Let G be the fundamental group of a finite graph of groups Γ with G finitely generated. Assume that G is not an ascending HNN extension. If $[\chi] \in \Sigma^1(G)$, then χ is non-trivial on every edge group.*

We now collect the results on Σ-invariants for direct products. Let $\chi = (\chi_1, \chi_2)\colon G_1 \times G_2 \mapsto \mathbb{R}$ be a character, where by χ_i we denote the restriction of χ to G_i, $i \in \{1, 2\}$. Let π_i be the canonical projection $G_1 \times G_2 \mapsto G_i$. Then π_i induces

$$\pi_i^*\colon S(G_i) \mapsto S(G_1 \times G_2).$$

Lemma 1.4.6. *[51, Lemma 9.1] Let $\chi_1\colon G_1 \mapsto \mathbb{R}$ and $\chi_2\colon G_2 \mapsto \mathbb{R}$ be two characters and define G to be $G_1 \times G_2$ and χ to be $(\chi_1, \chi_2)\colon G \mapsto \mathbb{R}$.*

(1) Suppose that G_1 and G_2 are groups of type F_{n+m+1}. If $[\chi_1] \in \Sigma^n(G_1)$ and $[\chi_2] \in \Sigma^m(G_2)$, then $[\chi] \in \Sigma^{n+m+1}(G)$.

(2) Suppose that G_1 and G_2 are groups of type FP_{n+m+1}. If $[\chi_1] \in \Sigma^n(G_1, \mathbb{Z})$ and $[\chi_2] \in \Sigma^m(G_2, \mathbb{Z})$, then $[\chi] \in \Sigma^{n+m+1}(G, \mathbb{Z})$.

Lemma 1.4.7. *[51, Lemma 9.2] Let $\chi_i\colon G_i \mapsto \mathbb{R}$ be a character and let G be $G_1 \times G_2$.*

(1) Suppose that G is of type F_n. Then $[\chi_i] \in \Sigma^n(G_i)$ if and only if $\pi_i^*([\chi_i]) \in \Sigma^n(G_1 \times G_2)$.

(2) Suppose that G is of type FP_n. Then $[\chi_i] \in \Sigma^n(G_i, \mathbb{Z})$ if and only if $\pi_i^*([\chi_i]) \in \Sigma^n(G_1 \times G_2, \mathbb{Z})$.

Theorem 1.4.8. *[51, Corollary 2] Let $G_1, \dots, G_m$ be groups of type F_∞ and suppose that $\Sigma^1(G_i) = \Sigma^\infty(G_i)$ for each $i \in \{1, \dots, m\}$. Let*

$$\chi = (\chi_1, \dots, \chi_m) \colon G_1 \times \cdots \times G_m \mapsto \mathbb{R}$$

be a character with $\chi_{i_1}, \dots, \chi_{i_k}$ the only non-zero homomorphisms among $\chi_1, \dots, \chi_m$. Then

$$[\chi] \notin \Sigma^n(G_1 \times \cdots \times G_m) \iff \quad n \geq k \quad \text{and} \quad [\chi_{i_r}] \notin \Sigma^1(G_{i_r}) \text{ for } 1 \leq r \leq k.$$

1.4.3 The Bieri-Neumann-Strebel-Renz invariants for RAAGs

In this section we quote the results in [73] related to Σ-invariants for RAAGs relevant to this work.

Let X be a simplicial graph, let $\widehat{X}$ be the induced flag complex based on X and let GX be the associated RAAG. Let $\chi \colon GX \mapsto \mathbb{R}$ be a homomorphism. A vertex $v \in V(X)$ is *living* if $\chi(v) \neq 0$ and otherwise it is *dead*. We denote by $\mathcal{L}_\chi$ the full subgraph spanned by the living vertices, and let $\widehat{\mathcal{L}}_\chi$ be the induced flag complex based on $\mathcal{L}_\chi$.

Note that if for $i \in \{1, \dots, n\}$ $\widehat{X}_i$ is the flag complex corresponding to the RAAG GX_i, then the simplicial join of these flag complexes, $\widehat{X}_1 * \cdots * \widehat{X}_n$, is the flag complex corresponding to the RAAG $GX_1 \times \cdots \times GX_n$. Moreover, if

$$\chi \colon GX_1 \times \cdots \times GX_n \mapsto \mathbb{R}$$

is a homomorphism,

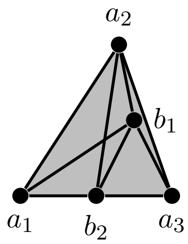
$$\widehat{\mathcal{L}}_\chi = \widehat{\mathcal{L}}_{\chi|GX_1} * \cdots * \widehat{\mathcal{L}}_{\chi|GX_n}.$$

Meier, Meinert and VanWyk characterise in [73] kernels of type FP_m or F_m of characters in terms of the topology of $\widehat{\mathcal{L}}_\chi \subseteq \widehat{X}$. We introduce the necessary terminology on simplicial complexes to state the results.

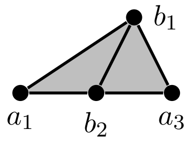
A complex is *n-acyclic over R* if its reduced homology groups over R, up to an including dimension n, are trivial. When the ring R may be deduced from the context, acyclic will always mean acyclic over R. We will be using a more general notion in the context of simplicial complexes.

Let K be a simplicial complex and let $V(K)$ be the set of vertices of K. If $\mathcal{V} \subseteq V(K)$, then $K - \mathcal{V}$ is the subcomplex of K formed by removing the vertices in $\mathcal{V}$ as well as the open stars of these vertices, where the *open star* of a vertex v consists of the interiors of all simplices that have v as a vertex.

For example, if K is



then $K - \{a_2\}$ is



Definition 1.4.9. The complex K is *m-n-acyclic over R* if for any k vertices $\{v_1, \ldots, v_k\}$ with $0 \leq k \leq m$ and $k < |V(K)|$, the complex $K - \{v_1, \ldots, v_k\}$ is n-acyclic over R.

In the above example K is 2-n-acyclic for any $n \in \mathbb{N}$ but it is not 3-0-acyclic.

For $v \in V(K)$ the *link* of v in K, $lk(v)$, is the flag subcomplex such that the vertex set is the set of vertices in $V(K)$ different to v and joined by an edge to v.

Definition 1.4.10. A subcomplex L of a simplicial complex K is (-1)-*acyclic-dominating* if it is non-empty, or equivalently, (-1)-acyclic. For $n \geq 0$, L is an *n-acyclic-dominating subcomplex of* K if for any vertex $v \in K - L$, the restricted link $lk_L(v) = lk(v) \cap L$ is $(n-1)$-acyclic and an $(n-1)$-acyclic-dominating subcomplex of the entire link $lk(v)$ of v in K.

There is an easy formula to compute the reduced homology groups of the join of two simplicial complexes.

Lemma 1.4.11. *Let X and Y be two simplicial complexes. Then*

$$\widetilde{H_{r+1}}(X * Y; R) = \bigoplus_{i+j=r} \widetilde{H_i}(X; R) \otimes_R \widetilde{H_j}(Y; R) \oplus \bigoplus_{i+j=r-1} Tor_1(\widetilde{H_i}(X; R); \widetilde{H_j}(Y; R)),$$

where $\widetilde{H_*}$ denotes the reduced homology group. Hence, the simplicial join of an r-acyclic complex and an s-acyclic complex is $(r+s+2)$-acyclic.

The results in [73] that will be used in this thesis are the following:

Corollary 1.4.12. *[73, Corollary A] Let X, $\widehat{X}$ and GX be as above. Let χ be a character $GX \mapsto \mathbb{Z}$. Then the kernel of χ is*

(1) F_n if and only if $\widehat{\mathcal{L}}_\chi \subseteq \widehat{X}$ is $(n-1)$-connected and $(n-1)$-$\mathbb{Z}$-acyclic-dominating,

(2) $FP_n(R)$ if and only if $\widehat{\mathcal{L}}_\chi \subseteq \widehat{X}$ is $(n-1)$-acyclic and $(n-1)$-acyclic-dominating.

Corollary 1.4.13. *[73, Corollary 7.1] One can determine if the kernel of a map $\chi\colon GX \mapsto \mathbb{Z}^n$ is F_n or $FP_n(R)$ by examining the induced maps $\phi \circ \chi\colon GX \mapsto \mathbb{Z}$ for $\phi \in Hom(\mathbb{Z}^n, \mathbb{Z}) \setminus \{0\}$.*

Corollary 1.4.14. *[73, Corollary B'] A RAAG GX has an abelian quotient of integral rank m with*

(1) F_n kernel if and only if $\widehat{X}$ is $(m-1)$-$(n-1)$-connected;

(2) $FP_n(R)$ kernel if and only if $\widehat{X}$ is $(m-1)$-$(n-1)$-acyclic.

1.5 Background on algebraic geometry over groups and limit groups

1.5.1 Elements of algebraic geometry over groups

In [8] Baumslag, Miasnikov and Remeslennikov lay down the foundations of algebraic geometry over groups and introduce group-theoretic counterparts of basic notions from algebraic geometry over fields. We recall the basics here and we refer to [8] for details.

The central notion of algebraic geometry over groups is the notion of a *G-group*, where G is a fixed group generated by a set A. We may imagine these G-groups as being algebras over a unitary commutative ring, where G plays the role of the coefficient ring. Therefore, instead of working in the category of groups, we consider the category of G-groups, that is groups which contain a designated subgroup isomorphic to the group G. If H and K are G-groups, then a homomorphism $\phi\colon H \mapsto K$ is a *G-homomorphism* if $\phi(g) = g$ for all $g \in G$. Thus, in the category of G-groups objects are G-groups and morphisms are G-homomorphisms.

By $Hom_G(H, K)$ we denote the set of all G-homomorphisms from H into K. A G-group H is termed *finitely generated G-group* if there exists a finite subset $C \subseteq H$ such that $C \cup G$ generates H. In the same way that it is shown that a free group is a free object in the category of groups, it is not hard to see that the free product $G * F(X)$ is a free object in the category of G-groups, where $F(X)$ is the free group with basis $X = \{x_1, \dots, x_n\}$. This group is called *the free G-group with basis X* and we denote it by $G[X]$.

For any element $s \in G[X]$, the formal equality $s = 1$ can be treated as an *equation over G*, and in general, for a subset $S \subseteq G[X]$, the formal equality $S = 1$ can be treated as a *system of equations over G* with coefficients in A. Elements from X are called *variables* and elements from $A^{\pm 1}$ are called *coefficients* or *constants*. To emphasise this we sometimes write $S(X, A) = 1$.

A *solution U* of the system $S(X) = 1$ over a group G is a tuple of elements $g_1, \dots, g_n \in G$ such that every equation from S vanishes at $(g_1, \dots, g_n)$. Equivalently, a solution U of the system $S = 1$ over G is a G-homomorphism $\pi_U \colon G[X] \mapsto G$ induced by the map $\pi_U \colon x_i \mapsto g_i$ such that $S \subseteq \ker(\pi_U)$.

Let us denote by $\langle\langle S \rangle\rangle$ the normal closure of S in $G[X]$. Then every solution $S(X) = 1$ in G gives rise to a G-homomorphism $G[X]/\langle\langle S \rangle\rangle \mapsto G$, and vice-versa. The set of all solutions over G of the system $S = 1$ is denoted by $V_G(S)$ and is called the *algebraic set defined by S*. For every system of equations S we set the *radical of the system S* to be the following subgroup of $G[X]$:

$$R(S) = \{T(X) \in G[X] \mid \forall g_1, \dots, g_n \quad (S(g_1, \dots, g_n) = 1 \implies T(g_1, \dots, g_n) = 1)\}.$$

It is easy to see that $R(S)$ is a normal subgroup of $G[X]$ that contains S, and by definition,

$$R(S) = \bigcap_{U \in V_G(S)} \ker(\pi_U).$$

For a subset $Y \subseteq G^n$, we define the *radical of Y* to be

$$R(Y) = \{T(X) \in G[X] \mid T(g_1, \dots, g_n) = 1 \quad \text{for all} \quad (g_1, \dots, g_n) \in Y\}.$$

In the lemma below we summarise the relations between radicals, systems of equations and algebraic sets.

Lemma 1.5.1.

(1) The radical of a system $S \subseteq G[X]$ contains the normal closure $\langle\langle S \rangle\rangle$ of S.

(2) Let Y_1 and Y_2 be subsets of G^n and let S_1, S_2 be subsets of $G[X]$. If $Y_1 \subseteq Y_2$, then $R(Y_2) \subseteq R(Y_1)$. If $S_1 \subseteq S_2$, then $R(S_1) \subseteq R(S_2)$.

(3) *For any family of sets* $\{Y_i \subseteq G^n \mid i \in I\}$, *we have*

$$R(\bigcup_{i\in I} Y_i) = \bigcap_{i\in I} R(Y_i).$$

(4) *A normal subgroup* H *of the group* $G[X]$ *is the radical of an algebraic set over* G *if and only if* $R(V_G(H)) = H$.

(5) *A set* $Y \subseteq G^n$ *is algebraic over* G *if and only if* $V_G(R(Y)) = Y$.

(6) *Let* $Y_1, Y_2 \subseteq G^n$ *be two algebraic sets. Then* $Y_1 = Y_2$ *if and only if* $R(Y_1) = R(Y_2)$. *Therefore, the radical of an algebraic set describes it uniquely.*

The quotient group

$$G_{R(S)} = G[X]/R(S)$$

is called the *coordinate group of the algebraic set* $V_G(S)$ *(or of the system* S*)*. There exists a one-to-one correspondence between algebraic sets and coordinate groups $G_{R(S)}$ (see [8, Theorem 4]).

If $V_G(S) \subseteq G^n$ and $V_G(S') \subseteq G^m$ are algebraic sets, then a map $\phi\colon V_G(S) \mapsto V_G(S')$ is a *morphism of algebraic sets* if there exist $f_1, \ldots, f_m \in G[x_1, \ldots, x_n]$ such that for any $(g_1, \ldots, g_n) \in V_G(S)$,

$$\phi(g_1, \ldots, g_n) = (f_1(g_1, \ldots, g_n), \ldots, f_m(g_1, \ldots, g_n)) \in V_G(S').$$

We say that $V_G(S)$ and $V_G(S')$ are *isomorphic* if there exist two morphisms $\psi\colon V_G(S) \mapsto V_G(S')$ and $\phi\colon V_G(S') \mapsto V_G(S)$ such that $\phi \circ \psi = 1$ and $\psi \circ \phi = 1$ and two systems of equations S and S' are called *equivalent* if the algebraic sets $V_G(S)$ and $V_G(S')$ are isomorphic.

Proposition 1.5.2. *Let* G *be a group and let* $V_G(S)$ *and* $V_G(S')$ *be two algebraic sets over* G. *Then the algebraic sets* $V_G(S)$ *and* $V_G(S')$ *are isomorphic if and only if the coordinate groups* $G_{R(S)}$ *and* $G_{R(S')}$ *are* G*-isomorphic.*

A G-group H is called G*-equationally Noetherian* if every system $S(X) = 1$ with coefficients from G is equivalent over G to a finite subsystem $S_0 = 1$ where $S_0 \subseteq S$, i.e. the system S and its subsystem S_0 define the same algebraic set. If G is G-equationally Noetherian, then we say that G is equationally Noetherian and in this case, every algebraic set V in G^n is a finite union of irreducible subsets, called *irreducible components of* V, and such a decomposition of V is unique.

1.5.2 Formulas in the languages $\mathcal{L}_A$ and $\mathcal{L}_G$

In this section we recall some basic notions of first-order logic and model theory. See [37] for further details.

Let G be a group generated by A. The standard first-order language of group theory, which we denote by $\mathcal{L}$, consists of a symbol for multiplication $\cdot$, a symbol for inversion $^{-1}$ and a symbol for the identity 1. To deal with G-groups, we have to enrich $\mathcal{L}$ by all the elements from G as constants. In fact, it suffices to enrich $\mathcal{L}$ by the constants that correspond to elements of A, so for every element $a \in A$ we introduce a new constant c_a. We denote the language $\mathcal{L}$ enriched by constants from A by $\mathcal{L}_A$, and by constants from G by $\mathcal{L}_G$.

A group word in variables X and constants A is a word $W(X, A)$ in the alphabet $(X \cup A)^{\pm 1}$. One may consider the word $W(X, A)$ as a term in the language $\mathcal{L}_A$. An *atomic formula* in the language $\mathcal{L}_A$ is a formula of the type $W(X, A) = 1$, where $W(X, A)$ is a group word in X and A. We interpret atomic formulas in $\mathcal{L}_A$ as equations over G, and vice-versa. A *Boolean combination* of atomic formulas in the language $\mathcal{L}_A$ is a disjunction of conjunctions of atomic formulas and their negations. Thus every Boolean combination Φ of atomic formulas in $\mathcal{L}_A$ can be written in the form $\Phi = \bigvee_{i=1}^{m} \Psi_i$, where each Ψ_i has one of the following forms:

$$\bigwedge_{j=1}^{n} (S_j(X, A) = 1), \quad \text{or} \quad \bigwedge_{j=1}^{n} (T_j(X, A) \neq 1) \quad \text{or}$$

$$\bigwedge_{j=1}^{n} (S_j(X, A) = 1) \wedge \bigwedge_{k=1}^{m} (T_k(X, A) \neq 1).$$

Every quantifier-free formula in the language $\mathcal{L}_A$ is logically equivalent, modulo the axioms of group theory, to a Boolean combination of atomic formulas. Moreover, every formula Φ in $\mathcal{L}_A$ with *free variables* $Z = \{z_1, \ldots, z_k\}$ is logically equivalent to a formula of the type

$$Q_1 x_1 Q_2 x_2 \ldots Q_l x_l \Psi(Z, X, A),$$

where $Q_i \in \{\forall, \exists\}$ and $\Psi(Z, X, A)$ is a Boolean combination of atomic formulas in variables from $X \cup Z$.

A first-order formula Φ is called a *sentence* if Φ does not contain free variables. A sentence Φ is called *universal* if and only if Φ is equivalent to a formula of the type

$$\forall x_1 \forall x_2 \ldots \forall x_l \Phi(X, A),$$

where $\Phi(X, A)$ is a Boolean combination of atomic formulas in variables from X. A *quasi identity* in the language $\mathcal{L}_A$ is a universal formula of the form

$$\forall x_1 \ldots x_l \big(\bigwedge_{i=1}^{m} (r_i(X,A) \neq 1 \vee s(X,A)) \big),$$

where $r_i(X,A)$ and $s(X,A)$ are terms.

1.5.3 First order logic and algebraic geometry

The connection between algebraic geometry over groups and logic has been shown to be very deep and fruitful and, in particular, led to a solution of the Tarski's problems on the elementary theory of free groups (see [61], [89]).

In [83] and [76] Miasnikov and Remeslennikov established relations between universal classes of groups, algebraic geometry and residual properties of groups. In order to state these theorems, we shall make use of the following notions. Let H and K be G-groups. We say that a family of G-homomorphisms $\mathcal{F} \subseteq Hom_G(H,K)$ *G-separates* (*G-discriminates*) H into K if for every non-trivial element $h \in H$ (every finite set of non-trivial elements $H_0 \subseteq H$) there exists $\phi \in \mathcal{F}$ such that $\phi(h) \neq 1$ ($\phi(h) \neq 1$ for every $h \in H_0$). In this case, we say that H is *G-separated by K* or that H is *G-residually K* (*G-discriminated by K* or that H is *G-fully residually K*). In the case when $G = 1$, we simply say that H is separated (discriminated) by K.

Theorem 1.5.3. *Let G be an equationally Noetherian ($G-$) group. Then the following classes coincide:*

(1) the class of all coordinate groups of algebraic sets over G (defined by systems of equations with coefficients in G);

(2) the class of all finitely generated ($G-$) groups that are ($G-$) separated by G;

(3) the class of all finitely generated ($G-$) groups that satisfy all the quasi-identities (in the language $\mathcal{L}_G$ or $\mathcal{L}_A$) that are satisfied by G.

Furthermore, a coordinate group of an algebraic set $V_G(S)$ is ($G-$) separated by G by homomorphisms π_U, $U \in V_G(S)$, corresponding to solutions.

Theorem 1.5.4. *Let G be an equationally Noetherian ($G-$) group. Then the following classes coincide:*

(1) the class of all coordinate groups of irreducible algebraic sets over G (defined by systems of equations with coefficients in G);

(2) the class of all finitely generated $(G-)$ groups that are $(G-)$ discriminated by G;

(3) the class of all finitely generated $(G-)$ groups that satisfy all universal sentences (in the language $\mathcal{L}_G$ or $\mathcal{L}_A$) that are satisfied by G.

Furthermore, a coordinate group of an irreducible algebraic set $V_G(S)$ is $(G-)$ discriminated by G by homomorphisms π_U, $U \in V_G(S)$, corresponding to solutions.

1.5.4 Limit groups over coherent RAAGs and limit groups over Droms RAAGs

In the previous context, we define a limit group over G to be a coordinate group of an irreducible algebraic set over G (or the equivalent definitions in Theorem 1.5.4).

However, if we work in the category of groups instead of in the category of G-groups, the previous relations between universal classes of groups, algebraic geometry and residual properties hold in a weaker way. More explicitly, a finitely generated group H is discriminated by G if and only if the universal theory of G is contained in the universal theory of H (but the inclusion may be proper). As in the previous case, this is equivalent to H being a coordinate group of an irreducible algebraic set over G. Similarly for finitely generated groups separated by G. In this setting, we define a *limit group over* G to be a group H that is finitely generated and *fully residually* G (or discriminated by G), that is for any finite set of non-trivial elements $S \subseteq H$ there exists a homomorphism from H to G that is non-trivial on S. A group H is *residually* G (or separated by G) if for every $1 \neq h \in H$ there is a homomorphism ϕ from H to G such that $1 \neq \phi(h) \in G$. The theory in [8] still ensures that finitely generated residually G groups are precisely finitely generated subgroups of direct products of finitely many limit groups over G ([8, Theorem F1]).

In this thesis we focus on the class of limit groups over coherent RAAGs, that is the union of the classes of limit groups over G, where G is a coherent RAAG.

The structure of limit groups over free groups (known as *limit groups*) is well understood. Sela characterised them as groups that admit a faithful action on a real tree induced by a sequence of homomorphisms to a free group. In addition, one can give a hierarchical description in terms of their non-trivial cyclic JSJ-decomposition. For further reference, see Sela's original papers [88], [89], [87] or the introductory notes of Bestvina and Feighn [10].

Casals-Ruiz and Kazachkov studied in [34] limit groups over RAAGs. They gave a dynamical characterisation of limit groups over RAAGs generalising the work of Sela. They showed that a finitely generated group is a limit group over a RAAG if and only if it acts nicely on a *real cubing*, which is a higher-dimensional generalisation of real trees. They also proved that any limit group over a RAAG G is a subgroup of a *graph tower over* G. In [35], together with Duncan, they focused on limit groups over coherent RAAGs and provided a neat hierarchical description of graph towers (see Proposition 1.5.6). We state here the results that we will use. If GX is a RAAG, the elements of X are termed the *canonical generators of* GX. A *non-exceptional surface* is a surface which is not a non-orientable surface of genus 1, 2 or 3.

Definition 1.5.5. Let H be a group and $Z \subseteq H$ the centraliser of an element. Then the group $G = H *_Z (Z \times \mathbb{Z}^n)$ is said to be obtained from H by an *extension of a centraliser*. An *ICE group over* H is a group obtained from H by applying finitely many times the extension of a centraliser construction.

Proposition 1.5.6. *[35, Lemma 7.3] Let G be a coherent right-angled Artin group. A graph tower over G of height 0 is a coherent RAAG H which is obtained from G by extending centralisers of canonical generators of G.*

A graph tower over G of height ≥ 1 can be obtained as a free product with amalgamation, where the edge group is a free abelian group, one of the vertex groups is a graph tower over G of lower height and the other vertex group is either free abelian or the direct product of a free abelian group and the fundamental group of a non-exceptional surface.

Theorem 1.5.7. *[34, Theorem 8.1] Let Γ be a limit group over a RAAG G. Then Γ is a subgroup of a graph tower over G.*

Furthermore, if we restrict to limit groups over Droms RAAGs, then they admit a nice hierarchical structure that comes from the fact that Droms RAAGs are the direct product of a free abelian group (possibly trivial) and a free product, so in this case we can use the work of Casals-Ruiz and Kazachkov (see [33]) or of Jaligot and Sela (see [59]) on limit groups over free products to obtain a hierarchy for limit groups over Droms RAAGs. If G is a Droms RAAG and Γ is a limit group over G, then Γ can be inductively defined depending on its *height*, $h(\Gamma)$, as follows:

Proposition 1.5.8. *[27, Proposition 2.1] Let G be a Droms RAAG such that $l(G) = 0$ (that is, a finitely generated free group) and let Γ be a limit group over G of height $h(\Gamma)$.*

If $h(\Gamma) = 0$, then Γ equals $M_1 * \cdots * M_j$ and $M_1, \ldots, M_j$ are free abelian groups or surface groups with Euler characteristic at most -2.

If $h(\Gamma) \geq 1$, then Γ is the fundamental group of a finite graph of groups that has infinite cyclic or trivial edge stabilisers and the vertex groups are limit groups *over G of height at most $h(\Gamma) - 1$. Moreover, at least one of the vertex groups is non-abelian. This decomposition may be chosen to be acylindrical.*

Proposition 1.5.9. *[33, Theorem 8.2] Let G be a Droms RAAG with $l(G) \geq 1$ so that G is of the form $\mathbb{Z}^m \times (G_1 * \cdots * G_n)$ and $l(G_i) \leq l(G) - 1$ for $i \in \{1, \ldots, n\}$ and let Γ be a limit group over G. Then Γ is $\mathbb{Z}^l \times \Lambda$ where Λ is a limit group over $G_1 * \cdots * G_n$ and if $m = 0$, then $l = 0$. If $h(\Lambda) = 0$, then*

$$\Lambda = A_1 * \cdots * A_j,$$

where for each $t \in \{1, \ldots, j\}$ A_t is a limit group over G_i for some $i \in \{1, \ldots, n\}$.

If $h(\Lambda) \geq 1$, then Λ is the fundamental group of a finite graph of groups that has infinite cyclic or trivial edge stabilisers and the vertex groups are limit groups over $G_1 * \cdots * G_n$ of height at most $h(\Lambda) - 1$. Moreover, at least one of the vertex groups has trivial center. This decomposition may be chosen to be acylindrical.

The last property of limit groups over Droms RAAGs that we will use is that they are precisely finitely generated subgroups of ICE groups over Droms RAAGs (recall Definition 1.5.5).

Theorem 1.5.10. *[35, Theorem 8.1] All ICE groups over Droms RAAGs are limit groups over Droms RAAGs. Moreover, a group is a limit group over a Droms RAAG if and only if it is a finitely generated subgroup of an ICE group over a Droms RAAG.*

The class of ICE groups over Droms RAAGs can be given a hierarchical structure that is obtained from understanding centralisers in Droms RAAGs.

If G is a Droms RAAG such that $l(G) = 0$, then the centraliser of any non-trivial element is an infinite cyclic group. Now let G be a Droms RAAG of the form $\mathbb{Z}^m \times (G_1 * \cdots * G_n)$ for $m, n \in \mathbb{N} \cup \{0\}$ such that $l(G_i) \leq l(G) - 1$ for $i \in \{1, \ldots, n\}$. Let g be an element in G.

Firstly, if $g \in \mathbb{Z}^m$, then $C_G(g) = G$, so the extension of a centraliser construction gives

$$G *_{C_G(g)} (C_G(g) \times \mathbb{Z}^k) \cong (\mathbb{Z}^m \times \mathbb{Z}^k) \times (G_1 * \cdots * G_n).$$

Secondly, assume that $g \in \mathbb{Z}^m g_0$, $g_0 \neq 1$ and g_0 is an elliptic element in

$$G_1 * \cdots * G_n.$$

Then g_0 lies in the conjugate of G_i for some $i \in \{1, \ldots, n\}$. Since $C_G(g) = C_G(g_0)$, we may assume that g equals g_0. Moreover, extensions of centralisers are conjugate invariant, so we may suppose that $g = g_0 \in G_i$ for some $i \in \{1, \ldots, n\}$. Thus,

$$C_G(g) = \mathbb{Z}^m \times C_{G_i}(g).$$

Therefore,

$$G *_{C_G(g)} (C_G(g) \times \mathbb{Z}^k) \cong \mathbb{Z}^m \times \Big(G_1 * \cdots * G_{i-1} * \big(G_i *_{C_{G_i}(g)} (\mathbb{Z}^k \times C_{G_i}(g))\big) * G_{i+1} * \cdots * G_n\Big).$$

That is, this extension is constructed by extending the centraliser of g in G_i and then doing the free product operation and adding the center $\mathbb{Z}^m$.

Thirdly, assume that $g \in \mathbb{Z}^m g_0$, $g_0 \neq 1$ and g_0 is a hyperbolic element in $G_1 * \cdots * G_n$. Then $C_G(g) = C_G(g_0)$ and we can assume without loss of generality that $g = g_0$. Then $C_G(g) = \mathbb{Z}^m \times \langle \sqrt{g} \rangle \cong \mathbb{Z}^m \times \langle g_1 \rangle \cong \mathbb{Z}^{m+1}$ and

$$G *_{C_G(g)} (C_G(g) \times \mathbb{Z}^k) \cong \Big(\mathbb{Z}^m \times (G_1 * \cdots * G_n)\Big) *_{\mathbb{Z}^m \times \langle g_1 \rangle} (\mathbb{Z}^m \times \langle g_1 \rangle \times \mathbb{Z}^k) \cong$$

$$\mathbb{Z}^m \times \Big((G_1 * \cdots * G_n) *_{\mathbb{Z}} \mathbb{Z}^{k+1}\Big).$$

In all the cases we obtain a group of the type $H = \mathbb{Z}^{m'} \times (H_1 * \cdots * H_n)$ for $m' \geq m$ or $H = \mathbb{Z}^{m'} \times (A *_B C)$ for $m' = m$. If we continue applying the extension of a centraliser construction for the centraliser of some element $h \in H$, we can consider as before the cases when $h \in \mathbb{Z}^{m'} \cup H_1 * \cdots * H_n \cup A *_B C$ and in the case when $h \in H_1 * \ldots * H_n$ or $h \in A *_B C$, then the element h can be either elliptic or hyperbolic with respect to that decomposition.

Moreover, in [35, Lemma 7.5] it is proved that any ICE group over a Droms RAAG can be obtained by first extending centralisers of central elements (which gives again a Droms RAAG), then extending centralisers of elliptic elements and finally considering extensions of centralisers of hyperbolic elements.

Thus, we get the following hierarchical structure depending on the *level* of the ICE group:

Proposition 1.5.11. *Assume that G is a Droms RAAG of level 0, that is G is a free group. An ICE group over G of level 0 is precisely G. An ICE group over G of level $k \geq 1$ is an amalgamated free product over $\mathbb{Z}^n$ of an ICE group over G of level $\leq k-1$ and a free abelian group $\mathbb{Z}^n \times \mathbb{Z}^m$, where $\mathbb{Z}^n$ embeds in $\mathbb{Z}^n \times \mathbb{Z}^m$ naturally.*

Assume that G is a Droms RAAG of level $l \geq 1$, that is

$$G = \mathbb{Z}^m \times (G_1 * \cdots * G_n)$$

where each G_i is a Droms RAAG of level less than l. An ICE group over G of level

0 *is of the form* $\mathbb{Z}^{m'} \times (H_1 * \cdots * H_n)$ *where* $m' \geq m$ *and* H_i *is an ICE group over* G_i *for* $i \in \{1, \ldots, n\}$.

An ICE group over G *of level* $k \geq 1$ *is an amalgamated free product over* $\mathbb{Z}^n$ *of an ICE group over* G *of level* $\leq k-1$ *and a free abelian group* $\mathbb{Z}^n \times \mathbb{Z}^m$, *where* $\mathbb{Z}^n$ *embeds in* $\mathbb{Z}^n \times \mathbb{Z}^m$ *naturally.*

Chapter 2

Subgroups of direct products of limit groups over Droms RAAGs

2.1 Introduction and outline

In 1984 Baumslag and Roseblade characterised finitely presented subgroups of the direct product of two finitely generated free groups, showing that up to finite index they are themselves a direct product of free groups. This result was generalised in a series of papers by Bridson, Howie, Miller and Short, culminating in a characterisation of subgroups of direct products of finitely many limit groups, assuming that the subgroups satisfy suitable finiteness properties.

In this chapter we aim to study the structure of limit groups over Droms RAAGs and residually Droms RAAGs. Droms RAAGs are precisely those RAAGs with the property that all of their finitely generated subgroups are again RAAGs, and they can also be described as the ones where the defining graph does not contain induced squares or straight line paths of length 3. Alternatively, the class of Droms RAAGs can be described as the $Z*$-closure of the class of finitely generated free groups (see Section 1.3.2). We prove the analogs of the results in [28] and [27] for subgroups of direct products of limit groups to the case of limit groups over Droms RAAGs.

Theorem 2.3.1. If $\Gamma_1, \dots, \Gamma_n$ are limit groups over Droms RAAGs and S is a subgroup of $\Gamma_1 \times \cdots \times \Gamma_n$ of type $\mathrm{w}FP_n(\mathbb{Q})$, then S is virtually a direct product of limit groups over Droms RAAGs.

Theorem 2.3.1 is the analogue of Theorem A proved by Bridson, Howie, Miller and Short in [27]. Following the spirit of [27], we deduce the theorem from the following result.

Theorem 2.8.1. Let $\Gamma_1, \dots, \Gamma_n$ be limit groups over Droms RAAGs such that each Γ_i has trivial center and let $S < \Gamma_1 \times \cdots \times \Gamma_n$ be a finitely generated full subdirect product. Then either:

(1) S is of finite index; or

(2) S is of infinite index and has a finite index subgroup $S_0 < S$ such that $H_j(S_0; \mathbb{Q})$ has infinite dimension for some $j \leq n$.

Remember that finitely generated residually Droms RAAGs are precisely finitely generated subgroups of direct products of limit groups over Droms RAAGs, so Theorem 2.3.1 gives us the following result.

Corollary. *For all $n \in \mathbb{N}$, a residually Droms RAAG is of type F_n if and only if it is of type $FP_n(\mathbb{Q})$.*

In Section 2.9 we study finitely presented residually Droms RAAGs and prove the following result.

Theorem 2.9.1. Let S be a finitely generated residually Droms RAAG. The following are equivalent:

(1) S is finitely presented;

(2) S is of type $FP_2(\mathbb{Q})$;

(3) S is of type $\mathrm{w}FP_2(\mathbb{Q})$;

(4) there exists a neat embedding $S \hookrightarrow \Gamma_0 \times \cdots \times \Gamma_n$ into a product of limit groups over Droms RAAGs such that the image of S under the projection to $\Gamma_i \times \Gamma_j$ has finite index for $0 \leq i < j \leq n$;

(5) for every neat embedding $S \hookrightarrow \Gamma_0 \times \cdots \times \Gamma_n$ into a product of limit groups over Droms RAAGs the image of S under the projection to $\Gamma_i \times \Gamma_j$ has finite index for $0 \leq i < j \leq n$.

For any commutative ring R, the property $FP_n(R)$ is inherited by finite index subgroups and persists in finite extensions. Nevertheless, if H is a subgroup in G and H is of type $FP_n(R)$, then it does not mean that G is of type $FP_n(R)$. For example, $\mathbb{Z}$ is of type $FP_1(\mathbb{Z})$ but an infinitely generated free group is not of type $FP_1(\mathbb{Z})$. However, when restricted to subgroups of direct products of limit groups over Droms RAAGs, it follows from Theorem 2.9.1 that any full subdirect product of limit groups over Droms RAAGs that contains a finitely presented full subdirect product is again finitely presented. We generalise this as follows:

Theorem 2.9.2. Let $\Gamma_1 \times \cdots \times \Gamma_k$ be the direct product of limit groups over Droms RAAGs where Γ_1 is free abelian and Γ_i is a limit group over a Droms RAAG with trivial center for $i \in \{2, \ldots, k\}$. Let $n \in \mathbb{N} \setminus \{1\}$, let $S < \Gamma_1 \times \cdots \times \Gamma_k$ be a full subdirect product and let $T < \Gamma_1 \times \cdots \times \Gamma_k$ be a subgroup that contains S. If S is of type $FP_n(\mathbb{Q})$, then so is T.

In [28, Section 7] Bridson, Howie, Miller and Short solve the multiple conjugacy problem for finitely presented residually free groups. Their argument can also be used to solve it for the case of finitely presented residually Droms RAAGs. Thus, in Section 2.9 we review their proof and show:

Theorem 2.9.6. The multiple conjugacy problem is decidable in every finitely presented residually Droms RAAG.

The same authors also solve in [28] the membership problem for finitely presented subgroups of finitely presented residually free groups. A key point in the proof of this result is that limit groups are subgroup separable (see [94]). This is unknown for limit groups over Droms RAAGs.

Question 2.1.1. Are limit groups over Droms RAAGs subgroup separable?

In the affirmative case, we would get, following the same reasoning as in [28, Section 7], that the membership problem for finitely presented subgroups of finitely presented residually Droms RAAGs is decidable. In addition, we could also use the arguments that Bridson and Wilton developed in [24] to get some residual results. More precisely, we would obtain that subgroups of type $FP_\infty(\mathbb{Q})$ of finitely generated residually Droms RAAGs are virtual retracts, and that the finitely presented subgroups are separable.

2.2 Limit groups over Droms RAAGs

In this section we state some of the results that Casals-Ruiz, Duncan and Kazachkov showed in [35] for limit groups over coherent RAAGs that we will use for the particular case of limit groups over Droms RAAGs.

Property 2.2.1. Limit groups over Droms RAAGs are finitely presented (see [35, Corollary 7.8]).

Property 2.2.2. Finitely generated subgroups of limit groups over Droms RAAGs are limit groups over Droms RAAGs (see [35, Theorem 8.1]).

Property 2.2.3. Limit groups over Droms RAAGs are of type F_∞. This follows from [35, Corollary 9.5].

There is another property proved recently that will be used later on:

Property 2.2.4. Limit groups over Droms RAAGs are cyclic subgroup separable (see [48]).

We now define a class of groups that contains all limit groups over Droms RAAGs.

Definition 2.2.5. The class of finitely presented groups $\mathcal{C}$ is defined in a hierarchical manner. It is the union of the classes $\mathcal{C}_n$ defined as follows.

At level 0 we have the class $\mathcal{C}_0$ consisting of groups of the form $\mathbb{Z}^n \times (A * B)$ where $n \in \mathbb{N} \cup \{0\}$ and A and B are non-trivial finitely presented groups where at least one of A and B has at least cardinality 3.

A group Γ lies in $\mathcal{C}_n$ if and only if $\Gamma \cong \mathbb{Z}^m \times \Gamma'$ where $m \in \mathbb{N} \cup \{0\}$ and Γ' is a finitely generated acylindrical graph of finitely presented groups, where all of the edge groups are cyclic, and at least one of the vertex groups lies in $\mathcal{C}_{n-1}$. Moreover, another property needs to be satisfied. Suppose that $\mathbb{Z}^{m'} \times \Gamma''$ is the vertex group of the decomposition of Γ' lying in $\mathcal{C}_{n-1}$. If $m = 0$, then $m' = 0$.

An immediate consequence of Proposition 1.5.8 and Proposition 1.5.9 is

Property 2.2.6. If Γ is a limit group over a Droms RAAG, then it lies in $\mathcal{C}$.

Finally, we prove another property that will be used in the proofs of the results.

Property 2.2.7. Let Γ be a limit group over a Droms RAAG such that Γ has trivial center and let S be a subgroup of Γ such that $H_1(S;\mathbb{Q})$ is finite dimensional. Then S is finitely generated (and hence a limit group over a Droms RAAG).

Proof. [23, Theorem 2] states that if Γ is a limit group and S is a subgroup of Γ such that $H_1(S;\mathbb{Q})$ is finite dimensional, then S is finitely generated. The proof of this result uses [27, Proposition 2.1] and [23, Lemma 4.1]. We have the analogous results to [27, Proposition 2.1] for limit groups over Droms RAAGs, namely, Proposition 1.5.8 and Proposition 1.5.9. Thus, if we check that [23, Lemma 4.1] holds also in the case of limit groups over Droms RAAGs, then we can use the same proof as in

[23, Theorem 2] in order to show our property. That is, it suffices to check that if Γ is residually a Droms RAAG and $S < \Gamma$ is a non-cyclic subgroup, then $H_1(S;\mathbb{Q})$ has dimension at least 2.

Indeed, if S is abelian the result is obvious. If not, there are two elements $s,t \in S$ such that $[s,t] \neq 1$ in S (they do not commute). Since Γ is residually a Droms RAAG, there is $\phi\colon \Gamma \mapsto G$ a homomorphism with G a Droms RAAG such that $[\phi(s),\phi(t)] \neq 1$. Then $\langle \phi(s), \phi(t)\rangle$ is a free group (see [5, Theorem 1.2]). Thus, S maps onto a non-abelian Droms RAAG and hence onto a free abelian group of rank at least 2. $\square$

2.3 From Theorem 2.8.1 to Theorem 2.3.1

The goal of this section is to show that Theorem 2.3.1 follows from Theorem 2.8.1. Let us recall both theorems:

Theorem 2.3.1. *If $\Gamma_1, \dots, \Gamma_n$ are limit groups over Droms RAAGs and S is a subgroup of $\Gamma_1 \times \cdots \times \Gamma_n$ of type $wFP_n(\mathbb{Q})$, then S is virtually a direct product of limit groups over Droms RAAGs.*

Theorem 2.8.1. Let $\Gamma_1, \dots, \Gamma_n$ be limit groups over Droms RAAGs such that each Γ_i has trivial center and let $S < \Gamma_1 \times \cdots \times \Gamma_n$ be a finitely generated full subdirect product. Then either:

(1) S is of finite index; or

(2) S is of infinite index and has a finite index subgroup $S_0 < S$ such that $H_j(S_0;\mathbb{Q})$ has infinite dimension for some $j \leq n$.

We first reduce Theorem 2.3.1 to a simpler case:

Proposition 2.3.2. *If Theorem 2.3.1 holds under the below assumptions (1)–(5), Theorem 2.3.1 holds in general:*

(1) Each Γ_i has trivial center.

(2) $n \geq 2$.

(3) Each projection map $p_i\colon S \mapsto \Gamma_i$ is surjective.

(4) Each intersection $L_i = S \cap \Gamma_i$ is non-trivial.

(5) Each Γ_i splits as an HNN extension over an infinite cyclic subgroup C_i or the trivial group with stable letter $t_i \in L_i$.

Proof. (1) Assume that $\Gamma_i \cong \mathbb{Z}^{m_i} \times \Gamma_i'$ for $m_i \in \mathbb{N} \cup \{0\}$ and Γ_i' is a limit group over a Droms RAAG with trivial center $i \in \{1, \ldots, n\}$. Then

$$\Gamma_1 \times \cdots \times \Gamma_n \cong \mathbb{Z}^{m_1 + \cdots + m_n} \times (\Gamma_1' \times \cdots \times \Gamma_n').$$

This reduces to the case where precisely one of the Γ_i is abelian, say Γ_1, and Γ_i is a limit group over a Droms RAAG such that Γ_i has trivial center for $i \geq 2$.

By the basis extension property for free abelian groups, there is a decomposition of Γ_1 as a direct sum $M_1 \oplus R_1$ where $L_1 = S \cap \Gamma_1$ has finite index in M_1. Thus, up to finite index we may assume that $\Gamma_1 = L_1 \oplus R_1$. Since $S \cap R_1$ is trivial, the projection homomorphism

$$f \colon L_1 \oplus R_1 \times \Gamma_2 \times \cdots \times \Gamma_n \mapsto L_1 \times \Gamma_2 \times \cdots \times \Gamma_n$$

descends to a monomorphism $f_{|S} \colon S \mapsto f(S)$. Thus, S is isomorphic to a subgroup

$$T < L_1 \times \Gamma_2 \times \cdots \times \Gamma_n.$$

But since $L_1 \subseteq T$, then $T = U \times L_1$ for some subgroup U of $\Gamma_2 \times \cdots \times \Gamma_n$.

In conclusion, $S \cong L_1 \times U$ with $U < \Gamma_2 \times \cdots \times \Gamma_n$ and L_1 is a free abelian group. Since S is of type w$FP_n(\mathbb{Q})$, so is U. Therefore, it suffices to check Theorem 2.3.1 for U.

(2) If $n = 1$, then $S < \Gamma_1$ has type w$FP_1(\mathbb{Q})$, so it is finitely generated by Property 2.2.7. Therefore, it is a limit group over a Droms RAAG.

(3) Since S has type w$FP_n(\mathbb{Q})$ for some $n \geq 2$, then S is in particular finitely generated. Hence, $p_i(S)$ is also finitely generated, so it is a limit group over a Droms RAAG. Therefore, we can replace each Γ_i by $p_i(S)$.

(4) Assume that there is a L_i, say L_n, which is trivial. Then the projection map $q_n \colon S \mapsto \Gamma_1 \times \cdots \times \Gamma_{n-1}$ is injective, so

$$S \cong q_n(S) < \Gamma_1 \times \cdots \times \Gamma_{n-1}.$$

After iterating this argument, we may assume that each L_i is non-trivial.

(5) The subgroup L_i of Γ_i is normal by (3) and non-trivial by (4). If we denote by T the Bass-Serre tree corresponding to the splitting of Γ_i described in Proposition 1.5.8 or 1.5.9, by [20, Corollary 2.2] L_i contains a hyperbolic isometry t_i. Then by [20, Theorem 3.1], there is Λ_i a finite index subgroup in Γ_i such that Λ_i is an HNN extension with stable letter t_i and with trivial or cyclic edge stabiliser.

We can replace each Γ_i by the subgroup of finite index Λ_i and S by $T = S \cap (\Lambda_1 \times \cdots \times \Lambda_n)$. Then T has finite index in S, so in order to prove Theorem 2.3.1 it suffices to show that T is virtually the direct product of limit groups over Droms RAAGs. $\square$

Finally, assume that Theorem 2.8.1 holds and let us prove Theorem 2.3.1. Assume that the group S is as in Theorem 2.3.1 with the additional assumptions of Proposition 2.3.2. By Theorem 2.8.1, S has finite index in $\Gamma_1 \times \cdots \times \Gamma_n$ or S has infinite index and there is a subgroup $S_0 < S$ of finite index in S such that $H_j(S_0; \mathbb{Q})$ has infinite dimension for some $j \in \{0, \ldots, n\}$. That is, in the latter case, S_0 is not of type w$FP_j(\mathbb{Q})$, and since S_0 has finite index in S, this implies that S is not of type w$FP_j(\mathbb{Q})$. Thus, S is not of type w$FP_n(\mathbb{Q})$.

2.4 Organisation of the proof of Theorem 2.8.1

We have shown in the previous section that Theorem 2.3.1 follows from Theorem 2.8.1. We now focus on proving Theorem 2.8.1 so the properties of Proposition 2.3.2 can be assumed. The proof extends from Section 2.5 to Section 2.8.

It is known that non-trivial, finitely generated normal subgroups in free products have finite index (see, for instance, the article [6]). In Section 2.5 we prove the following generalisation of that result. Recall the definition of the class $\mathcal{C}$ in Definition 2.2.5.

Theorem 2.5.1. Let Γ be a group in the class $\mathcal{C}$ with trivial center and suppose that Γ has subgroups $1 \neq N < G < \Gamma$ with N normal in Γ and G finitely generated. Then $|\Gamma \colon G| < \infty$.

In Section 2.6 we use this result and Proposition 2.3.2 (5) to deduce the following useful fact:

Theorem 2.6.4. Let $\Gamma_1, \ldots, \Gamma_n$ be limit groups over Droms RAAGs. Suppose that $S < \Gamma_1 \times \cdots \times \Gamma_n$ is a finitely generated subgroup of type w$FP_2(\mathbb{Q})$ with the properties of Proposition 2.3.2. Then Γ_j / L_j is virtually nilpotent.

We also state some properties of virtually nilpotent groups in Section 2.6 that will be used in the proof of Theorem 2.8.1.

In Section 2.7 we show that there is a subgroup N_0 in $\Gamma_1 \times \cdots \times \Gamma_n$ with infinite dimensional k-th homology for some $k \in \{0, \ldots, n\}$. More precisely, we prove the following:

Theorem 2.7.1. Let $\Gamma_1, \ldots, \Gamma_n$ be limit groups over Droms RAAGs such that Γ_i has trivial center and let N be the kernel of an epimorphism $\Gamma_1 \times \cdots \times \Gamma_n \mapsto \mathbb{Z}$. Then there is a subgroup of finite index $N_0 \subseteq N$ such that at least one of the homology groups $H_k(N_0;\mathbb{Q})$ has infinite dimension.

Finally, Section 2.8 is devoted to the proof of Theorem 2.8.1 which uses an inductive argument and the lower central series obtained from the fact that Γ_j/L_j is virtually nilpotent.

2.5 Proof of Theorem 2.5.1

In this section we prove Theorem 2.5.1.

Theorem 2.5.1. *Let Γ be a group in the class $\mathcal{C}$ with trivial center and suppose that Γ has subgroups $1 \neq N < G < \Gamma$ with N normal in Γ and G finitely generated. Then $|\Gamma\colon G| < \infty$.*

Proof. Let $\Gamma \in \mathcal{C}$ with trivial center. Then Γ acts non-trivially, acylindrically, cocompactly and minimally on a tree T with trivial or cyclic edge stabilisers. By [27, Lemma 5.1], T/G is finite. Thus, for the stabiliser Γ_e of any edge e of T

$$|G\backslash\Gamma/\Gamma_e|$$

is finite. If there is an edge stabiliser which is trivial, this implies that $|\Gamma\colon G|$ is finite.

It remains to show that if all the edge stabilisers are infinite cyclic, then $|\Gamma\colon G|$ is still finite. We prove this in the following proposition. □

Proposition 2.5.2. *Let $\Gamma \in \mathcal{C}$ with trivial center and let D, G be subgroups of Γ with D cyclic and G finitely generated. If $|G\backslash\Gamma/D| < \infty$, then $|\Gamma\colon G| < \infty$.*

Proof. Let us prove it by induction on the level $l = l(\Gamma)$ in the hierarchy of $\mathcal{C} = \cup_n \mathcal{C}_n$ where Γ first appears.

If $l = 0$, then Γ is a finitely presented free product, so it has a non-trivial, acylindrical, cocompact action on a tree T with trivial edge stabilisers. If $l > 0$ Γ has a non-trivial, acylindrical, cocompact action on a tree T with infinite cyclic edge stabilisers and the stabiliser of some vertex w is in $\mathcal{C}_{l-1}$ and has trivial center.

Let c be a generator of the cyclic subgroup D. We distinguish two cases depending on whether c is elliptic or hyperbolic.

Suppose that c is elliptic and that it fixes a vertex t of T. The proof that $X = T/G$ is finite is the same proof as in [27, Proposition 5.2]:

First, we show that X has finite diameter. Since $|G\backslash\Gamma/D|$ is finite, there are $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that

$$\Gamma = G\gamma_1 D \dot{\cup} \cdots \dot{\cup} G\gamma_n D.$$

Since $c \cdot t = t$, the Γ-orbit of t consists of finitely many G-orbits, namely $G(\gamma_i t)$ $i \in \{1, \ldots, n\}$. Since the action of Γ on T is cocompact, there is $m > 0$ such that T is the m-neighborhood of Γt, so X is the m-neighborhood of the finitely many vertices $G(\gamma_i t)$. Therefore, X has finite diameter.

Second, $\pi_1(X)$ has finite rank. Indeed, since G is the fundamental group of the graph of groups X, there is $r\colon G \mapsto \pi_1(X)$ a retract and G is finitely generated, so $\pi_1(X)$ is also finitely generated.

Third, X has finitely many valency 1 vertices. Suppose that there are infinitely many valency 1 vertices. Then there is $\overline{u} = Gu$ a vertex with valency 1 where u is a vertex in T and $\overline{e} = Ge$ is the unique edge in X incident at $\overline{u}$ (e is an edge in T incident at u) such that $G_{\overline{u}} = G_{\overline{e}}$, where $G_{\overline{u}}$ and $G_{\overline{e}}$ denote the stabilisers of u and e in G, respectively. Otherwise, G would not be finitely generated. Since G acts by isometries in T and $\overline{u} = Gu$ has valency 1 in X, $G_{\overline{u}}$ acts transitively on the link $\mathrm{lk}(u)$ of u. Hence $|\mathrm{lk}(u)| = |G_{\overline{u}}\colon G_{\overline{e}}| = 1$, so u is a valency 1 vertex of T. This is a contradiction since T is minimal as a Γ-tree.

In conclusion, X has finite diameter, finite rank and finitely many valency 1 vertices, so X is a finite graph.

Suppose that $l = 0$. Then $|G\backslash\Gamma| = |\Gamma\colon G|$ is finite because the number of edges in X is an upper bound for that number. If $l > 0$ there is a vertex w such that its stabiliser, Γ_w, is in $\mathcal{C}_{l-1}$. Recall that by the definition of the class $\mathcal{C}$ Γ_w has trivial center. The number $|(G \cap \Gamma_w)\backslash\Gamma_w/\Gamma_e|$ is finite because it is bounded above by the number of edges in X incident at Gw that are images of edges $\gamma e \in \Gamma e$. So by inductive hypothesis $G \cap \Gamma_w$ has finite index in Γ_w. Define the action $\alpha\colon \Gamma_w \times G\backslash\Gamma \mapsto G\backslash\Gamma$ by

$$\alpha((x, G\gamma)) = G(\gamma x).$$

Then

$$G\backslash\Gamma = \dot{\bigcup}_{\gamma \in G\backslash\Gamma/\Gamma_w} \mathrm{Orbit}(G\gamma).$$

Note that $|G\backslash\Gamma/\Gamma_w| < \infty$ because it is bounded above by the number of edges in X. Moreover, by the Orbit-Stabiliser Theorem, $\mathrm{Orbit}(G\gamma)$ is in bijection with

$$\Gamma_w/(\gamma^{-1}G\gamma \cap \Gamma_w)$$

and this is finite. In conclusion, $G\backslash\Gamma$ is finite.

The argument for the case when c acts hyperbolically on T is the same as in [27, Proposition 5.2]. For sake of completeness, we include it here. Suppose that c acts hyperbolically on T with axis A. The double coset decomposition of Γ implies that the axes $\gamma(A)$ for $\gamma \in \Gamma$ belong to only finitely many G-orbits. On the other hand, the convex hull of $\bigcup_{\gamma\in\Gamma} \gamma(A)$ is a Γ-invariant subtree of T, but T is minimal as a Γ-tree, so that convex hull is the whole of T.

Let T_0 be the minimal G-invariant subtree of T. If $T_0 = T$, by [4, Proposition 7.9] $X = T/G$ is finite and the same argument as in the previous case may be used to conclude that $|\Gamma\colon G| < \infty$.

It remains to consider the case $T_0 \neq T$. For any subgraph Y of T and any $g \in G$, we have

$$d(g(Y), T_0) = d(g(Y), g(T_0))$$

because T_0 is G-invariant, and

$$d(g(Y), g(T_0)) = d(Y, T_0)$$

because the action is by isometries.

Since the axes $\gamma(A)$ for $\gamma \in \Gamma$ belong to only finitely many G-orbits, there is a global upper bound K on $d(\gamma(A), T_0)$ as γ varies over Γ. Since $T \neq T_0$ and T is spanned by the Γ-orbit of A, there is a translate $\gamma(A)$ of A not contained in T_0. Recall that the action of Γ on T is acylindrical, say k-acylindrical. Choose a vertex u on $\gamma(A)$ with $d(u, T_0) > K + k + 2$ and let Γ_u denote its stabiliser in Γ. Let p be the vertex at distance K from T_0 on the unique shortest path from T_0 to u. Since $d(\gamma(A), T_0) \leq K$, the geodesic $[p, u]$ is contained in $\gamma(A)$. Similarly $[p, u]$ is contained in any translate of A that passes through u. In particular, if $\delta \in \Gamma_u$, then $[p, u] \subseteq \gamma(A)$ and since δ fixes u we have $\delta(p) = p$ or $\delta(p) = p'$, where p' is the unique point of $\gamma(A)$ other than p with $d(u, p) = d(u, p')$.

If δ fixes the edge of $[p, u]$ incident at u, then $\delta(p) = p$ and δ fixes $[p, u]$ pointwise, which contradicts the k-acylindricality of the action unless $\delta = 1$. Thus, the stabiliser of this edge is trivial, which is a contradiction unless $l = 0$. If $l = 0$, then by the definition of $\mathcal{C}$, replacing u by an adjacent vertex if necessary, we may assume that $|\Gamma_u| > 2$. Choose distinct non-trivial elements δ_1, δ_2 in Γ_u. It cannot be that all three of δ_1, δ_2, $\delta_1\delta_2^{-1}$ send p' to p. Thus one of them fixes p, hence $[p, u]$, which again contradicts the k-acylindricality of the action. $\square$

2.6 Proof of Theorem 2.6.4 and nilpotent quotients

From now on we shall restrict to the conditions of Theorem 2.3.1 with the additional assumptions of Proposition 2.3.2. That is, each projection map $p_i \colon S \mapsto \Gamma_i$ is surjective, each Γ_i has trivial center and each Γ_i splits as an HNN extension over an infinite cyclic subgroup (or the trivial group) with stable letter $t_i \in L_i$.

The goal of this section is to prove that Γ_i/L_i is virtually nilpotent. The proofs are the analogues of the proofs from [27] by using the properties that we have shown for limit groups over Droms RAAGs.

We introduce the following notation. We write K_i for the kernel of the i-th projection map p_i and $N_{i,j}$ for the image of K_i under the j-th projection map p_j. That is,

$$K_i = \{(s_1, \dots, s_{i-1}, 1, s_{i+1}, \dots, s_n) \in S\},$$

$$N_{i,j} = \{s_j \in \Gamma_j \mid (*, \dots, *, 1, *, \dots, *, s_j, *, \dots, *) \in S\}.$$

Note that $N_{i,j}$ is a normal subgroup in Γ_j because p_j is surjective.

Lemma 2.6.1. *[27, Lemma 6.1] The iterated commutator*

$$[N_{1,j}, N_{2,j}, \dots, N_{j-1,j}, N_{j+1,j}, \dots, N_{n,j}]$$

is contained in L_j.

Let $\Gamma_i = B_i *_{C_i}$ be the HNN splitting of Γ_i with stable letter $t_i \in L_i$ and C_i an infinite cyclic subgroup (or the trivial group). Let us consider, for $i \neq j$, the group $A_{i,j} = p_j({p_i}^{-1}(C_i))$. Since $N_{i,j} = p_j({p_i}^{-1}(\{1\}))$, then

$$N_{i,j} < A_{i,j} < \Gamma_j.$$

The strategy is to show that $N_{i,j}$ has finite index in Γ_j.

Lemma 2.6.2. *Let $\Gamma_1, \dots, \Gamma_n$ be limit groups over Droms RAAGs such that Γ_i has trivial center. If $S < \Gamma_1 \times \cdots \times \Gamma_n$ is finitely generated, of type $wFP_2(\mathbb{Q})$ and if S satisfies conditions (1) to (5) of Proposition 2.3.2, then for all i, j*

(1) $|\Gamma_j \colon A_{i,j}| < \infty$, and

(2) $A_{i,j}/N_{i,j}$ is cyclic.

Proof. (1) Let us prove it for $i = 1$. The HNN decomposition $B_1 *_{C_1}$ of Γ_1 pulls back to an HNN decomposition of S of the form $\hat{B}_1 *_{\hat{C}_1}$ with stable letter $\hat{t}_1 =$

$(t_1, 1, \ldots, 1) \in S$, where $\hat{B}_1 = p_1^{-1}(B_1)$, $\hat{C}_1 = p_1^{-1}(C_1) = K_1 \rtimes \langle \hat{c}_1 \rangle$ and $\hat{c}_1$ is a choice of a lift of a generator of C_1. Consider the Mayer-Vietoris sequence for the HNN decomposition of S:

$$\cdots \to H_2(S; \mathbb{Q}) \xrightarrow{\alpha} H_1(\hat{C}_1; \mathbb{Q}) \xrightarrow{\phi} H_1(\hat{B}_1; \mathbb{Q}) \to H_1(S; \mathbb{Q}) \to \cdots$$

where ϕ is the difference between the map induced by the inclusion and the map induced by the inclusion twisted by the action of $\hat{t}_1$ under conjugation. By definition of $\hat{t}_1$ and K_1, one has that $\hat{t}_1$ commutes with K_1, so ϕ factors through the map $H_1(\hat{C}_1; \mathbb{Q}) \mapsto H_1(\langle \hat{c}_1 \rangle; \mathbb{Q})$. In particular, the image of ϕ has dimension at most 1. Since $H_2(S; \mathbb{Q})$ is finite dimensional by hypothesis, then its image under the map α is also finite dimensional. Thus, $H_1(\hat{C}_1; \mathbb{Q})$ is finite dimensional. For each $j \geq 2$ $A_{1,j}$ equals $p_j(\hat{C}_1)$ and since there is a surjection $H_1(\hat{C}_1; \mathbb{Q}) \mapsto H_1(A_{1,j}; \mathbb{Q})$, then $H_1(A_{1,j}; \mathbb{Q})$ is also finite dimensional. Hence, by Property 2.2.7, $A_{1,j}$ is finitely generated. Since it contains the non-trivial subgroup L_j, from Theorem 2.5.1 we get that $A_{1,j}$ has finite index in Γ_j.

(2) By definition,

$$A_{i,j}/N_{i,j} = p_j(\hat{C}_i)/p_j(K_i),$$

and p_j is surjective, so $A_{i,j}/N_{i,j}$ is a homomorphic image of $\hat{C}_i/K_i$. □

Proposition 2.6.3. *Let $\Gamma_1, \ldots, \Gamma_n$ be limit groups over Droms RAAGs such that Γ_i has trivial center. If $S < \Gamma_1 \times \cdots \times \Gamma_n$ is a finitely generated subgroup of type $wFP_2(\mathbb{Q})$ and S satisfies conditions (1) to (5) of Proposition 2.3.2, then $N_{i,j} \subseteq \Gamma_j$ is of finite index for all i, j.*

Proof. [27, Proof of Proposition 6.4] Let us prove it for the case $(i, j) = (2, 1)$. Let T be the projection of S to $\Gamma_1 \times \Gamma_2$ and define M_i to be $T \cap \Gamma_i$ for $i \in \{1, 2\}$. Notice that $M_1 = N_{2,1}$ and $M_2 = N_{1,2}$. Since the projection maps $S \mapsto \Gamma_1$ and $S \mapsto \Gamma_2$ are surjective,

$$\Gamma_1/M_1 \cong T/(M_1 \times M_2) \cong \Gamma_2/M_2.$$

We will assume that these groups are infinite and obtain a contradiction.

By Lemma 2.6.2, $T/(M_1 \times M_2)$ is virtually cyclic, so there is $T_0 < T$ a finite index subgroup containing $M_1 \times M_2$ such that $T_0/(M_1 \times M_2)$ is infinite cyclic. Hence, for $i \in \{1, 2\}$, $G_i = p_i(T_0)$ is a finite index subgroup containing M_i such that G_i/M_i is infinite cyclic. Choose $\tau \in T_0$ such that $\tau(M_1 \times M_2)$ generates $T_0/(M_1 \times M_2)$ and let $\tau_i = p_i(\tau) \in G_i$ for $i \in \{1, 2\}$.

Since G_i has finite index in Γ_i, the HNN decomposition of Γ_i from Proposition

2.3.2 induces an HNN decomposition $G_i = B'_i *_{C'_i}$ with stable letter $t'_i \in L_i$, where $C'_i = G_i \cap C_i$ and t'_i is an appropriate power of the stable letter t_i of Γ_i. Notice that $C'_i \cap M_i = 1$. For each $i \in \{1,2\}$ there is an index 2 subgroup Δ_i in G_i and an element $x_i \in M_i \cap \Delta_i \cap L_i$ such that $\bar{x}_i$ generates a free $\mathbb{Q}[\tau_i^{\pm 1}]$-submodule of $H_1(M_i \cap \Delta_i; \mathbb{Q})$ (see [27, Proposition 6.3]).

Now set $M'_i = M_i \cap \Delta_i$. Then $\bar{x}_1 \otimes \bar{x}_2$ generates a free $\mathbb{Q}[\tau_1^{\pm 1}, \tau_2^{\pm 1}]$ submodule of $H_1(M'_1; \mathbb{Q}) \otimes_{\mathbb{Q}} H_1(M'_2; \mathbb{Q}) \subseteq H_2(M'_1 \times M'_2; \mathbb{Q})$.

Let us define T_1 to be $(M'_1 \times M'_2) \rtimes \langle \tau \rangle$. Note that T_1 has finite index in T_0. Let $S_1 < S$ be the preimage of T_1 under the projection $S \mapsto T$. Then S_1 has also finite index in S. Considering the Lyndon—Hochschild—Serre spectral sequence for the short exact sequence

$$1 \to M'_1 \times M'_2 \to T_1 \to \langle \tau \rangle \to 1,$$

we see that

$$H_0(\langle \tau \rangle; H_2(M'_1 \times M'_2; \mathbb{Q})) \subseteq H_2(T_1; \mathbb{Q})$$

has an infinite dimensional subspace generated by

$$\{(\tau_1^m x_1 \tau_1^{-m}) \otimes (\tau_2^n x_2 \tau_2^{-n}) \mid m, n \in \mathbb{Z}\}.$$

In particular, the image of the map $H_2(L_1 \times L_2; \mathbb{Q}) \mapsto H_2(T_1; \mathbb{Q})$ induced by inclusion is infinite dimensional. But this contradicts the hypothesis that $H_2(S_1; \mathbb{Q})$ is finite dimensional, since the inclusion $L_1 \times L_2 \mapsto T_1$ factors through S_1. $\square$

Theorem 2.6.4 now follows easily.

Theorem 2.6.4. *Let $\Gamma_1, \ldots, \Gamma_n$ be limit groups over Droms RAAGs. Suppose that $S < \Gamma_1 \times \cdots \times \Gamma_n$ is a finitely generated subgroup of type $wFP_2(\mathbb{Q})$ with the properties of Proposition 2.3.2. Then Γ_j / L_j is virtually nilpotent.*

Proof. Let G be a group, let L be a normal subgroup in G and let $N_1, \ldots, N_n$ be normal subgroups of finite index in G such that $[N_1, \ldots, N_n] \subseteq L$. If we denote the group $N_1 \cap \cdots \cap N_n$ by N, then N has finite index in G and $[N, \ldots, N]$ is a subgroup of L. Now N is normal in G, and accordingly $[NL, \ldots, NL]$ is also a subgroup in L. Therefore, NL/L is nilpotent and it has finite index in G/L because N has finite index in G. $\square$

We finally state two results about finitely generated virtually nilpotent groups that will be used in the proof of Theorem 2.3.1.

Lemma 2.6.5. *[27, Lemma 8.1] Let G be a finitely generated virtually nilpotent group and let H be a subgroup of infinite index. Then there exists a subgroup K of finite index in G and an epimorphism $f\colon K \mapsto \mathbb{Z}$ such that $(H \cap K) \subseteq ker\ (f)$.*

Repeated applications of the previous lemma yield the following result.

Corollary 2.6.6. *[27, Corollary 8.2] Let G be a finitely generated virtually nilpotent group and let H be a subgroup of G. Then there is a subnormal chain*

$$H_0 < H_1 < \cdots < H_r = G,$$

where H_0 is a subgroup of finite index in H and for each i the group H_{i+1}/H_i is either finite or cyclic.

2.7 Proof of Theorem 2.7.1

The aim of this section is to prove Theorem 2.7.1:

Theorem 2.7.1. *Let $\Gamma_1, \ldots, \Gamma_n$ be limit groups over Droms RAAGs such that Γ_i has trivial center and let N be the kernel of an epimorphism $\Gamma_1 \times \cdots \times \Gamma_n \mapsto \mathbb{Z}$. Then there is a subgroup of finite index $N_0 \subseteq N$ such that at least one of the homology groups $H_k(N_0;\mathbb{Q})$ has infinite dimension.*

Recall that by Property 2.2.3 limit groups over Droms RAAGs are of type F_∞. Thus, the following result can be used in our case:

Proposition 2.7.2. *[27, Proposition 7.1] If $\Gamma_1, \ldots, \Gamma_n$ are groups of type $FP_n(\mathbb{Z})$ and $\phi\colon \Gamma_1 \times \cdots \times \Gamma_n \mapsto \mathbb{Z}$ has non-trivial restriction to each factor, then $H_j(ker\ \phi;\mathbb{Z})$ is finitely generated for $j \leq n-1$.*

By using that result, in [27, Theorem 7.2] they show that if $\Gamma_1, \ldots, \Gamma_n$ are non-abelian limit groups and S is the kernel of an epimorphism $\Gamma_1 \times \cdots \times \Gamma_n \mapsto \mathbb{Z}$ such that the restriction to each of the Γ_i is an epimorphism, then $H_n(S;\mathbb{Q})$ has infinite dimension.

The same statement holds for limit groups over Droms RAAGs with trivial center, namely,

Theorem 2.7.3. *Let $\Gamma_1, \ldots, \Gamma_n$ be limit groups over Droms RAAGs such that Γ_i has trivial center and let S be the kernel of an epimorphism $\phi\colon \Gamma_1 \times \cdots \times \Gamma_n \mapsto \mathbb{Z}$. If the restriction of ϕ to each of the Γ_i is an epimorphism, then $H_n(S;\mathbb{Q})$ has infinite dimension.*

The proof is an adaptation of the proof from [27] to the setting of limit groups over Droms RAAGs:

Proof. The proof is by induction on n. If $n = 1$, then the group $S = \ker\, \phi$ is a normal subgroup of a limit group over a Droms RAAG with trivial center. If $H_1(S;\mathbb{Q})$ were finite dimensional, then S would be finitely generated by Property 2.2.7. Then S would have finite index in Γ_1 (Theorem 2.5.1) and this is a contradiction because ϕ is an epimorphism.

If $n \geq 2$, let us consider the Lyndon—Hochschild—Serre spectral sequence for

$$1 \to S_{n-1} = \ker\,(p_n) \to S \xrightarrow{p_n} \Gamma_n \to 1,$$

where S_{n-1} is the kernel of the restriction of ϕ to $\Gamma_1 \times \cdots \times \Gamma_{n-1}$. By Proposition 2.7.2, there are only finite dimensional $\mathbb{Q}$-vector spaces in the region $0 \leq q \leq n-2$. Since Γ_i is of type FP_∞, the terms on the E^2 page involved in the calculation of $H_{n-1}(S;\mathbb{Q})$ are finite dimensional except for possibly $E^2_{0,n}$ and $E^2_{1,n-1}$. Thus, in order to prove that $H_n(S;\mathbb{Q})$ is infinite dimensional, it suffices to show that $H_1(\Gamma_n; H_{n-1}(S_{n-1};\mathbb{Q}))$ is infinite dimensional.

Since $\Gamma_1 \times \cdots \times \Gamma_{n-1}$ is a group of type FP_∞, we may consider a free resolution $\mathcal{F}$ of $\mathbb{Z}$ over $\mathbb{Z}[\Gamma_1 \times \cdots \times \Gamma_{n-1}]$ with each F_i finitely generated free. Then $M = H_{n-1}(S_{n-1};\mathbb{Q})$ is a homology group of a chain complex of free $R = \mathbb{Q}[t, t^{-1}]$ modules of finite rank. Since R is Noetherian, then M is finitely generated as an R-module. By inductive hypothesis, M has infinite dimension. Then by the classification of finitely generated modules over a PID, $M = M_0 \oplus R$.

Note that $\Gamma_n = L_n \rtimes \langle t_n \rangle$, where $L_n = S \cap \Gamma_n$ and $\phi(t_n)$ is a fixed generator of $\mathbb{Z}$. By the definitions of S_{n-1} and L_n, the Γ_n-action on M factors through $\langle t_n \rangle$. So the direct sum decomposition passes to M considered as a $\mathbb{Q}\Gamma_n$-module. Hence,

$$H_1(\Gamma_n; M) = H_1(\Gamma_n; M_0) \oplus H_1(\Gamma_n; R).$$

As a $\mathbb{Q}\Gamma_n$-module,

$$R = \mathbb{Q}\Gamma_n \otimes_{\mathbb{Q}L_n} \mathbb{Q},$$

so by Shapiro's Lemma $H_1(\Gamma_n; R) \cong H_1(L_n;\mathbb{Q})$. The group L_n is the kernel of the epimorphism $\phi_{|\Gamma_n}$, so it has infinite index in Γ_n. Hence, L_n is infinitely generated (Theorem 2.5.1). In conclusion, $H_1(L_n;\mathbb{Q})$ is infinite dimensional by Property 2.2.7. $\square$

Let us end the section by proving Theorem 2.7.1.

Proof of Theorem 2.7.1. Suppose that the restriction of ϕ to some Γ_i is trivial, for example to Γ_1. Then $\phi\colon \Gamma_2 \times \cdots \times \Gamma_n \mapsto \mathbb{Z}$ is an epimorphism. Thus, we may assume that the restriction of ϕ to each of the factors is non-trivial. Then $\phi(\Gamma_i) = m_i\mathbb{Z}$ for some non-zero integer m_i. We may replace Γ_i by $\Delta_i = \phi^{-1}(m_i\mathbb{Z})$. Note that Δ_i has finite index in Γ_i and that the restriction of ϕ to $\Delta_1 \times \cdots \times \Delta_n$ satisfies the conditions of Theorem 2.7.3. $\square$

2.8 Proof of Theorem 2.8.1

Let us recall the statement of Theorem 2.8.1:

Theorem 2.8.1. *Let $\Gamma_1, \ldots, \Gamma_n$ be limit groups over Droms RAAGs such that each Γ_i has trivial center and let $S < \Gamma_1 \times \cdots \times \Gamma_n$ be a finitely generated full subdirect product. Then either:*

(1) S is of finite index; or

(2) S is of infinite index and has a finite index subgroup $S_0 < S$ such that $H_j(S_0; \mathbb{Q})$ has infinite dimension for some $j \leq n$.

Proof. Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ and $L = L_1 \times \cdots \times L_n$. We only need to consider the case when S has infinite index in Γ. By contradiction, suppose that for all the subgroups S_0 of finite index in S and for all $0 \leq j \leq n$, $H_j(S_0; \mathbb{Q})$ has finite dimension.

By Theorem 2.6.4 the group Γ_i/L_i is virtually nilpotent for all $i \in \{1, \ldots, n\}$. Thus, Γ/L is virtually nilpotent. By applying Lemma 2.6.5 to Γ/L, there is a finite index subgroup Λ in Γ containing L and an epimorphism $f\colon \Lambda/L \mapsto \mathbb{Z}$ such that

$$\Lambda/L \cap S/L \subseteq \ker f.$$

Define g to be $f \circ \pi$, where $\pi\colon \Lambda \mapsto \Lambda/L$ is the projection map. By definition, g is an epimorphism and $\Lambda \cap S$ is contained in $\ker g$.

We replace S by $S \cap \Lambda$. Let $q_i\colon \Gamma \mapsto \Gamma_i$ be the i-th projection map. Then we also replace Γ_i by $q_i(\Lambda)$. By using this argument we ensure that $L \subseteq S \subseteq N$, where N is the kernel of an epimorphism $\Lambda \mapsto \mathbb{Z}$. By Theorem 2.7.1 there is a finite index subgroup N_0 in N and $j \leq n$ such that $H_j(N_0; \mathbb{Q})$ has infinite dimension. By Corollary 2.6.6, there is a subgroup S_0 contained in $S \cap N_0$ which has finite index in S and a series $S_0 \triangleleft S_1 \triangleleft \cdots \triangleleft S_k = N_0$ with S_{i+1}/S_i finite or cyclic for each i.

We now use the following lemma to reach a contradiction.

Lemma 2.8.2. *[27, Lemma 8.3] Let S_0 be a normal subgroup in S_1 with S_1/S_0 finite or cyclic. If $H_j(S_0;\mathbb{Q})$ is finite dimensional for $0 \le j \le n$, then $H_j(S_1;\mathbb{Q})$ is finite dimensional for $0 \le j \le n$.*

Coming back to the main proof, by repeatedly applying this lemma, we obtain that $H_j(N_0;\mathbb{Q})$ has finite dimension for all $j \le n$, contradicting Theorem 2.7.1. □

2.9 Finitely presented residually Droms RAAGs

The goal of this section is to understand finitely presented residually Droms RAAGs. Finitely generated residually Droms RAAGs are precisely coordinate groups over Droms RAAGs or, equivalently, finitely generated subgroups of direct products of limit groups over Droms RAAGs. Therefore, we view them as finitely presented subgroups of direct products of limit groups over Droms RAAGs. This section is based on the earlier work [28] of Bridson, Howie, Miller and Short.

Theorem 2.9.1. *Let S be a finitely generated residually Droms RAAG. The following are equivalent:*

(1) S is finitely presented;

(2) S is of type $FP_2(\mathbb{Q})$;

(3) S is of type $wFP_2(\mathbb{Q})$;

(4) there exists a neat embedding $S \hookrightarrow \Gamma_0 \times \cdots \times \Gamma_n$ into a product of limit groups over Droms RAAGs such that the image of S under the projection to $\Gamma_i \times \Gamma_j$ has finite index for $0 \le i < j \le n$;

(5) for every neat embedding $S \hookrightarrow \Gamma_0 \times \cdots \times \Gamma_n$ into a product of limit groups over Droms RAAGs the image of S under the projection to $\Gamma_i \times \Gamma_j$ has finite index for $0 \le i < j \le n$.

Proof. We first check that for every finitely generated residually Droms RAAG there is a neat embedding.

Clearly, there is an embedding $S \hookrightarrow \Gamma_0 \times \cdots \times \Gamma_n$ where Γ_i is a limit group over a Droms RAAG. Each group Γ_i is of the form $\mathbb{Z}^{m_i} \times \Gamma_i'$ with $m_i \in \mathbb{N} \cup \{0\}$ and Γ_i' is a limit group over a Droms such that Γ_i' has trivial center. Therefore, we may assume that S is a subgroup of the direct product $\Gamma_0 \times \cdots \times \Gamma_n$ where Γ_0 is a free abelian group and Γ_i is a limit group over a Droms RAAG with trivial center for $i > 0$.

By the basis extension property for free abelian groups, there is a decomposition of Γ_0 as a direct sum $M_0 \oplus R_0$ where $L_0 = S \cap \Gamma_0$ has finite index in M_0. Since the intersection $S \cap R_0$ is trivial, the projection homomorphism $f\colon M_0 \oplus R_0 \times \Gamma_1 \times \cdots \times \Gamma_n \mapsto M_0 \times \Gamma_1 \times \cdots \times \Gamma_n$ descends to a monomorphism $f_{|S}\colon S \mapsto f(S)$. Thus, S is isomorphic to a subgroup of $M_0 \times \Gamma_1 \times \cdots \times \Gamma_n$ and we may now assume that L_0 has finite index in Γ_0. Finally, we may also suppose that S is a full subdirect product as we did in Proposition 2.3.2.

Recall that each projection map $S \mapsto \Gamma_i \times \Gamma_j$ is denoted by $p_{i,j}$ for $0 \leq i < j \leq n$.

The implications (1) $\implies$ (2), (2) $\implies$ (3) are clear. Since neat embeddings exist, (5) $\implies$ (4). By [28, Theorem A], (4) implies (1). Thus it suffices to prove that (3) implies (5).

Note that since L_0 has finite index in Γ_0, then $p_{0,i}(S) \cap \Gamma_0$ has finite index in Γ_0, so $p_{0,i}(S)$ has finite index in $\Gamma_0 \times \Gamma_i$ for each $i > 0$.

Second, let us show that for $1 \leq i < j \leq n$, $p_{i,j}(S)$ has finite index in $\Gamma_i \times \Gamma_j$. There is an exact sequence

$$1 \to \Gamma_0 \to \Gamma_0 \times \cdots \times \Gamma_n \xrightarrow{q} \Gamma_1 \times \cdots \times \Gamma_n \to 1,$$

and this exact sequence descends to another exact sequence for S:

$$1 \to L_0 \to S \to q(S) \to 1.$$

Thus, $S/L_0 \cong q(S) < \Gamma_1 \times \cdots \times \Gamma_n$.

Let us take $\overline{S_0}$ a finite index subgroup in S/L_0. Then $\overline{S_0}$ is of the form S_0L_0/L_0 with S_0L_0 of finite index in S. The group L_0 is finitely generated and by hypothesis, $\dim H_2(S_0L_0;\mathbb{Q})$ is finite.

Hence, $H_2(\overline{S_0};\mathbb{Q})$ is finite dimensional for all finite index subgroups $\overline{S_0}$ in the group S/L_0, so by Proposition 2.6.3, for $1 \leq i < j \leq n$ the group $p_{i,j}(q(S)) = p_{i,j}(S)$ has finite index in $\Gamma_i \times \Gamma_j$. $\square$

By Theorem 2.9.1 any subgroup of the direct product of limit groups over Droms RAAGs that contains a finitely presented full subdirect product is again finitely presented. Theorem 2.9.2 generalises this.

Theorem 2.9.2. *Let $\Gamma_1 \times \cdots \times \Gamma_k$ be the direct product of limit groups over Droms RAAGs where Γ_1 is free abelian and Γ_i is a limit group over a Droms RAAG with trivial center for $i \in \{2, \ldots, k\}$. Let $n \in \mathbb{N} \setminus \{1\}$, let $S < \Gamma_1 \times \cdots \times \Gamma_k$ be a full* *subdirect product and let $T < \Gamma_1 \times \cdots \times \Gamma_k$ be a subgroup that contains S. If S is of type $FP_n(\mathbb{Q})$, then so is T.*

Proof. The quotient group Γ_1/L_1 is abelian, so in particular, it is nilpotent.

Moreover, if S is of type $FP_n(\mathbb{Q})$ for $n \geq 2$, it is in particular of type $FP_2(\mathbb{Q})$. Then by Theorem 2.6.4, Γ_i/L_i is virtually nilpotent for $i \in \{2, \ldots, k\}$. Thus, D/L is virtually nilpotent, where $D = \Gamma_1 \times \cdots \times \Gamma_k$ and $L = (S \cap \Gamma_1) \times \cdots \times (S \cap \Gamma_k)$.

By Corollary 2.6.6 there is a finite index subgroup $S_0 < S$ and a subnormal chain

$$S_0 \lhd S_1 \lhd \cdots \lhd S_l = T$$

such that each quotient S_{i+1}/S_i is either finite or infinite cyclic.

Since S is of type $FP_n(\mathbb{Q})$ and S_0 has finite index in S, then S_0 is also of type $FP_n(\mathbb{Q})$. Note that there is a short exact sequence

$$1 \to S_0 \to S_1 \to S_1/S_0 \to 1.$$

Moreover, S_1/S_0 is of type $FP_n(\mathbb{Q})$ because it is infinite cyclic or a finite group. Therefore, S_1 is of type $FP_n(\mathbb{Q})$. By iterating this argument we obtain that T is of type $FP_n(\mathbb{Q})$. □

Finally, we focus on the multiple conjugacy problem for finitely presented residually Droms RAAGs.

The *multiple conjugacy problem* for a finitely generated group G (given by a finite generating set) asks if there is an algorithm that, given a natural number l and two l-tuples of elements in the generators of G, say $x = (x_1, \ldots, x_l)$ and $y = (y_1, \ldots, y_l)$, can determine if there exists $g \in G$ such that $g^{-1}x_ig = y_i$ in G for $i \in \{1, \ldots, l\}$.

When solving the multiple conjugacy problem for finitely presented residually free groups (see [28]), the authors first show a result for bicombable groups. Fundamental groups of compact non-positively curved spaces are bicombable groups, so in particular, limit groups over Droms RAAGs are bicombable (see [35, Corollary 9.5]).

Proposition 2.9.3. *[28, Proposition 7.1] Let Γ be a bicombable group, let $H < \Gamma$ be a subgroup, and suppose that there exists a subgroup $L < H$ normal in Γ such that Γ/L is nilpotent. Then H has a decidable multiple conjugacy problem.*

Second, they state a result that relates the decidability of the multiple conjugacy problem for a finite index subgroup and the whole group:

Lemma 2.9.4. *[28, Lemma 7.2] Suppose G is a group in which roots are unique and $H < G$ is a subgroup of finite index. If the multiple conjugacy problem for H is decidable, then the multiple conjugacy problem for G is decidable.*

In order to apply this lemma, we need to check that limit groups over Droms RAAGs have unique roots. Recall that a group G is said to have *unique roots* if for all $x, y \in G$ and $n \neq 0$, one has that $x = y \iff x^n = y^n$. RAAGs have unique roots (see, for instance, [80, Lemma 6.3]). Hence, it suffices to show that if H has unique roots and G is fully residually H, then G has unique roots:

Lemma 2.9.5. *Let H be a group in which roots are unique and let G be fully residually H. Then G has unique roots.*

Proof. Suppose that there are two elements $x \neq y \in G$ such that $x^n = y^n$. Since G is fully residually H, then there is a homomorphism $\phi\colon G \mapsto H$ such that $\phi(x) \neq \phi(y)$. However, $\phi(x)^n = \phi(y)^n$ and this contradicts the fact that H has unique roots. □

Building on the previous results and Theorem 2.6.4, we now prove Theorem 2.9.6:

Theorem 2.9.6. *The multiple conjugacy problem is decidable in every finitely presented residually Droms RAAG.*

Proof. Let S be a finitely presented residually Droms RAAG. Then S can be viewed as a full subdirect product of $D = \Gamma_1 \times \cdots \times \Gamma_n$ where Γ_i is a limit group over a Droms RAAG, $L = L_1 \times \cdots \times L_n$ is normal in D where $L_i = \Gamma_i \cap S$ and D/L is virtually nilpotent. Let N be a finite index subgroup in D such that $L < N$ and N/L is nilpotent and let S_0 be $N \cap S$. By Proposition 2.9.3, S_0 has decidable multiple conjugacy problem, and applying Lemma 2.9.4 we get that S has decidable multiple conjugacy problem. □

Chapter 3

On subdirect products of type FP_n of limit groups over Droms RAAGs

3.1 Introduction and outline

One of the main results of Chapter 2 is that if $\Gamma_1, \dots, \Gamma_n$ are limit groups over Droms RAAGs and S is a full subdirect product of $\Gamma_1 \times \cdots \times \Gamma_n$, then S is finitely presented if and only if $p_{i,j}(S)$ is of finite index in $\Gamma_i \times \Gamma_j$ for all $1 \leq i < j \leq n$, where $p_{i,j}$ denotes the projection map $S \mapsto \Gamma_i \times \Gamma_j$. After having seen the importance of projections onto pairs, Kuckuck suggested in [69] the following generalisation.

The Virtual Surjection Conjecture *Let $n \leq m$ be positive integers and let H be a subgroup of the direct product $G_1 \times \cdots \times G_m$, where G_i is of type F_n for $1 \leq i \leq m$. If H is virtually surjective on n-tuples, then H is of type F_n.*

The Virtual Surjection Conjecture is still an open problem, but Kochloukova showed in [65] that if $\Gamma_1, \dots, \Gamma_m$ are non-abelian limit groups and S is a finitely generated full subdirect product of type $FP_k(\mathbb{Q})$ for some fixed $k \in \{2, \dots, m\}$, then S virtually surjects on k-tuples. The converse is, however, an open problem, so there is no characterisation of residually free groups of type $FP_s(\mathbb{Q})$ for $s \geq 3$. One of the main goals of this chapter is to extend this result to the class of limit groups over Droms RAAGs.

One of the key properties of limit groups used by Kochloukova in the proof is that they are free-by-(torsion-free nilpotent). Our first result is a generalisation of this fact for limit groups over Droms RAAGs.

Proposition 3.2.12. Limit groups over Droms RAAGs are free-by-(torsion-free nilpotent).

In Section 3.3 we extend the result about full subdirect products of type $FP_s(\mathbb{Q})$ of direct products of limit groups to the class of limit groups over Droms RAAGs.

Theorem 3.3.8. Let $\Gamma_1, \ldots, \Gamma_m$ be limit groups over Droms RAAGs such that each Γ_i has trivial center and let $S < \Gamma_1 \times \cdots \times \Gamma_m$ be a finitely generated full subdirect product of type w$FP_s(\mathbb{Q})$ for some $s \in \{2, \ldots, m\}$. Then for every canonical projection

$$p_{j_1,\ldots,j_s} \colon S \mapsto \Gamma_{j_1} \times \cdots \times \Gamma_{j_s},$$

the index of $p_{j_1,\ldots,j_s}(S)$ in $\Gamma_{j_1} \times \cdots \times \Gamma_{j_s}$ is finite.

In particular, if S is of type $FP_s(\mathbb{Q})$, then for every canonical projection $p_{j_1,\ldots,j_s}$, the index of $p_{j_1,\ldots,j_s}(S)$ in $\Gamma_{j_1} \times \cdots \times \Gamma_{j_s}$ is finite.

In Section 3.4 we discuss the growth of homology groups and the volume gradients of limit groups over coherent RAAGs and of finitely presented residually Droms RAAGs. If G is a group, (B_n) is an *exhausting chain* of G if (B_n) is a chain of subgroups of finite index such that $B_{n+1} \subseteq B_n$ and $\bigcap_n B_n = 1$, and it is an *exhausting normal chain* if it is an exhausting chain and each B_n is normal in G. As every residually finite group has an exhausting normal chain, limit groups over coherent RAAGs have exhausting normal chains. The *growth of homology groups with respect to the exhausting normal chain* (B_n) *and field* K is measured by the limit

$$\lim_{n\to\infty} \dim_K H_i(B_n; K)/[G \colon B_n],$$

whenever this limit exits. A natural question to ask is whether this limit (when it exists) is an invariant of G or it is an artefact of the chain that we choose. Thanks to the *approximation theorem* of Lück we know that if K has characteristic 0, G is residually finite, finitely presented and of type FP_m, then the limit exists for $i < m$ and equals the L^2-Betti number $\beta_i(G)$. In particular, in this case the limit is independent of the exhausting normal chain.

For right-angled Artin groups, Avramidi, Okun and Schreve computed in [3] the growth of homology groups over an arbitrary field. They showed that the limit exists, it is independent of the chain but it does depend on the field.

In this chapter we compute the growth of homology groups for limit groups over coherent RAAGs and we show that the limit is independent of the field. As a consequence, we get the L^2-Betti numbers of limit groups over coherent RAAGs. L^2-homology is a homology theory deeply connected to 3-manifold theory. The L^2-Euler characteristic is used to compute the Thurston norm of a 3-manifold and the L^2-Betti numbers also have a strong connection with group fibring. They are a generalisation of the Betti numbers with the additional powerful property that they behave nicely when taking finite index subgroups.

We also consider the following homotopical analogues. Given a group G of type F_m, the *m-volume* $\mathrm{vol}_m(G)$ was defined in [22] as the least number of m-cells among all classifying spaces $K(G,1)$ with finite m-skeleton. For instance, $\mathrm{vol}_1(G)$ is the minimal number of generators of G, $d(G)$. The 2-volume, $\mathrm{vol}_2(G)$, bounds the *deficiency* def (G) of G, which is the infimum of $|X|-|R|$ over all finite presentations $\langle X \mid R\rangle$ of G. The *m-dimensional volume gradient of G with respect to* (B_n) is

$$lim_{n\to\infty}\mathrm{vol}_m(B_n)/[G\colon B_n].$$

The most studied volume gradient is the 1-dimensional one, that is the *rank gradient of G with respect to* (B_n), which is defined to be $\lim_n d(B_n)/[G\colon B_n]$ and it is denoted by $RG(G,(B_n))$. Its study was initiated by Lackenby in [70] in connection with the study of largeness of 3-manifolds and a combinatorial approach to the rank gradient was developed by Abert, Nikolov and Jaikin-Zapirain in [1].

In [22] Bridson and Kochloukova calculated the above volume and homology type gradients for limit groups and in the case of residually free groups they found particular filtrations where the homology growth can be calculated.

We compute the homology growth and the volume gradients for limit groups over coherent RAAGs. We use the approach from [22] and we show that limit groups over coherent RAAGs are slow above dimension 1. A group G of type F is *slow above dimension 1* if it is residually finite and for every exhausting normal chain (B_n) there exists a finite $K(B_n,1)$ with $r_k(B_n)$ k-cells such that

$$\lim_{n\to\infty}\frac{r_k(B_n)}{[G\colon B_n]}=0$$

for all $k\geq 2$. We say that G is *slow* if it satisfies the additional requirement that the limit exists and is zero for $k=1$ as well. This implies the following results.

Theorem 3.4.14. Let G be a limit group over a coherent RAAG and let (B_n) be an exhausting normal chain in G. Then

(1) The rank gradient $RG(G,(B_n)) = \lim_{n\to\infty}\frac{d(B_n)}{[G\colon B_n]} = -\chi(G)$.

(2) The deficiency gradient $DG(G,(B_n)) = \lim_{n\to\infty}\frac{\mathrm{def}(B_n)}{[G\colon B_n]} = \chi(G)$.

(3) The k-dimensional volume gradient $\lim_{n\to\infty}\frac{\mathrm{vol}_k(B_n)}{[G\colon B_n]} = 0$ for $k\geq 2$.

Theorem 3.4.15. Let K be a field, G a limit group over a coherent RAAG and (B_n) an exhausting normal chain in G. Then

(1) $\lim_{n\to\infty} \frac{\dim_K H_1(B_n;K)}{[G\colon B_n]} = -\chi(G)$.

(2) $\lim_{n\to\infty} \frac{\dim_K H_j(B_n;K)}{[G\colon B_n]} = 0$ for all $j \geq 2$.

Using results from [22] we also compute the homology growth up to dimension m of residually Droms RAAGs of type FP_m for $m \geq 2$. We cannot apply the same method to the class of residually coherent RAAGs since Theorem 3.3.8 is a key result in this method and in Chapter 5 it is proved that this no longer holds for coherent RAAGs.

Theorem 3.4.16. Let $m \geq 2$, let G be a residually Droms RAAG of type FP_m and let ρ be the largest integer such that G contains a direct product of ρ non-abelian free groups. Then there exists an exhausting sequence (B_n) in G so that for all fields K,

(1) if G is not of type FP_∞, then

$$\lim_{n\to\infty} \frac{\dim H_i(B_n;K)}{[G\colon B_n]} = 0 \text{ for all } 0 \leq i \leq m;$$

(2) if G is of type FP_∞, then for all $j \geq 1$

$$\lim_{n\to\infty} \frac{\dim H_\rho(B_n;K)}{[G\colon B_n]} = (-1)^\rho \chi(G), \quad \lim_{n\to\infty} \frac{\dim H_j(B_n;K)}{[G\colon B_n]} = 0 \quad \text{for all} \quad j \neq \rho.$$

3.2 Limit groups over Droms RAAGs are free-by-(torsion-free nilpotent)

Definition 3.2.1. A graph is called *triangulated* if it contains no induced copy of C_n for $n \geq 4$, where C_n is the circle with n vertices.

Theorem 3.2.2. *[47, Theorem 2] If G is the RAAG corresponding to the graph X, the commutator subgroup G' is free if and only if X is triangulated.*

Recall that Droms RAAGs can be described as the RAAGs where the defining graph does not contain induced squares or straight line paths with 3 edges (see [46]). In particular, if G is a Droms RAAG, then G' is free.

Lemma 3.2.3. *Droms RAAGs are free-by-(free abelian).*

Proof. Let G be a Droms RAAG corresponding to the graph X. Then G' is free, the abelianisation is $\mathbb{Z}^n$ where n is the number of vertices in the graph X and there is a short exact sequence

$$1 \longrightarrow G' \longrightarrow G \longrightarrow \mathbb{Z}^n \longrightarrow 1.$$

$\square$

Definition 3.2.4. A filtration $\{G_i\}_{i\geq 1}$ of normal subgroups of a group G is a *torsion-free exhausting normal chain* if G/G_i is torsion-free and $\bigcap_{i\geq 1} G_i = 1$, and it is a *non-abelian exhausting normal chain* if $\bigcap_{i\geq 1} G_i = 1$ and for each finitely generated abelian subgroup M of G there is $i = i(M)$ such that $G_i \cap M = 1$.

Lemma 3.2.5. *Let G be a group and let $\{G_i\}_{i\geq 1}$ be a filtration of normal subgroups of G. If $\{G_i\}_{i\geq 1}$ is a torsion-free exhausting normal chain, it is also a non-abelian exhausting normal chain.*

Proof. Let M be a finitely generated abelian subgroup of G. Since $\bigcap_{i\geq 1} G_i = 1$, there is i such that $G_i \cap M$ is not M. The group $M/(M \cap G_i)$ embeds in G/G_i, so it is non-trivial and torsion-free, that is a finite rank free abelian group. As M has finite Hirsch length, we deduce that $M \cap G_j$ is trivial for sufficiently large j. $\square$

Lemma 3.2.6. *Let H be a group, let G be $\mathbb{Z}^m \times H$ for some $m \in \mathbb{N}$ and let us denote the natural projection map $G \mapsto H$ by p. Suppose that $\{G_i\}_{i\geq 1}$ is a non-abelian exhausting normal chain of G. Then $\{p(G_i)\}_{i\geq 1}$ is a non-abelian exhausting normal chain of H.*

Proof. Let us denote $\mathbb{Z}^m$ by A and let us define H_i to be $p(G_i)$. Thus, $AG_i = AH_i$. Let C be a finitely generated abelian subgroup of H. Note that the group $A(H_i \cap C)$ is contained in

$$AH_i \cap AC = AG_i \cap AC = A(G_i \cap AC).$$

Since the filtration $\{G_i\}$ is a non-abelian exhausting normal chain, there is $i_1 = i(AC)$ such that $G_{i_1} \cap AC = 1$. To sum up, since $A \cap C \subseteq A \cap H = 1$, then $A(H_i \cap C)$ being equal to A implies that $H_i \cap C = 1$ for $i \geq i_1$. $\square$

Proposition 3.2.7. *Let G be a Droms RAAG and let $\{G_i\}_{i\geq 1}$ be a non-abelian exhausting normal chain of G. Then for sufficiently large i_0 the group G_{i_0} is free.*

Proof. Let G be a Droms RAAG of level $l(G)$. If $l(G) = 0$, then G is a free group, so the statement holds. If $l(G) \geq 1$, then G is $\mathbb{Z}^m \times (K_1 * \cdots * K_k)$ for some $m \in \mathbb{N} \cup \{0\}$ and $K_1, \ldots, K_k$ are Droms RAAGs of level less than $l(G)$.

Let us denote $\mathbb{Z}^m$ and $K_1 * \cdots * K_k$ by A and H, respectively, and the projection map $G \mapsto H$ by p. By hypothesis there is $l = l(A)$ such that $A \cap G_l = 1$.

Let us define H_i to be $p(G_i)$. From Lemma 3.2.6 we get that $\{H_i\}_{i\geq 1}$ is a non-abelian exhausting normal chain of H, so $\{H_i\}_{i\geq l}$ is a non-abelian exhausting normal chain and $G_i \cong H_i$.

The group H_i is a free product of conjugates of $H_i \cap K_j$ for $j \in \{1, \ldots, k\}$ and a free group. For $j \in \{1, \ldots, k\}$ $\{H_i \cap K_j\}_{i\geq l}$ is a non-abelian exhausting normal chain of K_j, so by inductive hypothesis, there is i_j such that $H_{i_j} \cap K_j$ is free. In conclusion, by taking i_0 to be $\max\{l, i_1, \ldots, i_k\}$ we have that H_{i_0} is a free group. □

Theorem 3.2.8. *Let G be a Droms RAAG and let $\{G_i\}_{i\geq 1}$ be a torsion-free exhausting normal chain. Then for sufficiently large i_0 the group G_{i_0} is free.*

Proof. It follows from Lemma 3.2.5 and Proposition 3.2.7. □

In order to show that limit groups over Droms RAAGs are free-by-(torsion-free nilpotent), we will show that ICE groups over Droms RAAGs are free-by-(torsion-free nilpotent). A limit group over a Droms RAAG is a finitely generated subgroup of an ICE group over a Droms RAAG, so it will also be free-by-(torsion-free nilpotent) (see Section 1.5.4).

Theorem 3.2.9. *Let G be a Droms RAAG and let K be an ICE group over G. Let $\{K_i\}_{i\geq 1}$ be a non-abelian exhausting normal chain of K. Then for sufficiently large i_0 the group K_{i_0} is free.*

Proof. We prove it by induction on the level $l(G)$ of G as a Droms RAAG. Suppose that $l(G) = 0$. If K has level 0, then K is precisely G (which is a free group) and the result holds. If K has level ≥ 1, then K is of the form

$$H *_{\mathbb{Z}^n} (\mathbb{Z}^m \times \mathbb{Z}^n),$$

and H is an ICE group over G of smaller level than K. Since K_i is a normal subgroup of K, then K_i inherits a graph of groups decomposition where the vertex groups are of the form

$$(K_i \cap H)^{g_{\alpha_j}} \quad \text{or} \quad (K_i \cap (\mathbb{Z}^m \times \mathbb{Z}^n))^{g_{\alpha_j}}, \quad \text{for} \quad g_{\alpha_j} \in K,$$

and the edge groups are of the form

$$(K_i \cap \mathbb{Z}^n)^{g_{\beta_j}} \quad \text{for} \quad g_{\beta_j} \in K.$$

Note that $\{K_i \cap H\}_{i\geq 1}$ is a non-abelian exhausting normal chain of H, so by inductive hypothesis there is i_1 such that $K_{i_1} \cap H$ is free. In addition, the filtration $\{K_i\}_{i\geq 1}$ is a non-abelian exhausting normal chain, so there is $i_2 = i(\mathbb{Z}^m \times \mathbb{Z}^n)$ such that $K_{i_2} \cap (\mathbb{Z}^m \times \mathbb{Z}^n)$ is trivial. In particular, $K_{i_2} \cap \mathbb{Z}^n$ is also trivial. Therefore, taking i_0 to be $\max\{i_1, i_2\}$ we get that K_{i_0} is free.

Now suppose that $l(G) \geq 1$. Then G is of the form

$$\mathbb{Z}^m \times (G_1 * \cdots * G_n)$$

for some $m \in \mathbb{N} \cup \{0\}$ and G_i is a Droms RAAG with $l(G_i) \leq l(G) - 1$ for each $i \in \{1, \ldots, n\}$. If K has level 0 as an ICE group over G, then

$$K = \mathbb{Z}^{m'} \times (H_1 * \cdots * H_n),$$

where $m' \geq m$ and H_i is an ICE group over G_i for $i \in \{1, \ldots, n\}$. Let us denote the projection map $K \mapsto H_1 * \cdots * H_n$ by p.

By hypothesis there is $i_1 = i(\mathbb{Z}^{m'})$ such that $\mathbb{Z}^{m'} \cap K_{i_1} = 1$. In addition, by Lemma 3.2.6, $\{N_i\}_{i\geq 1}$ is a non-abelian exhausting normal chain of $H_1 * \cdots * H_n$, where $N_i = p(K_i)$ and $N_i \cong K_i$ for $i \geq i_1$.

The group N_i is a free product of conjugates of $N_i \cap H_j$ for $j \in \{1, \ldots, n\}$ and a free group. For $j \in \{1, \ldots, n\}$, $\{N_i \cap H_j\}_{i\geq 1}$ is a non-abelian exhausting normal chain of H_j, so by inductive hypothesis there is r_j such that $N_{r_j} \cap H_j$ is free. In conclusion, taking i_0 to be $\max\{i_1, r_1, \ldots, r_n\}$ we have that N_{i_0} is a free group.

Finally, suppose that K is an ICE group over G of level $k \geq 1$. Then K is an amalgamated free product over $\mathbb{Z}^n$ of an ICE group over G of level $\leq k-1$ and a free abelian group $\mathbb{Z}^n \times \mathbb{Z}^m$. This case may be treated as the case when K is an ICE group of level greater than 0 over a free group. $\square$

We denote by $tor(G)$ the set of torsion elements of G. In the case when G is finitely generated nilpotent, then $\langle tor(G)\rangle$ is the maximal finite subgroup of G.

Theorem 3.2.10. *Let G be a Droms RAAG, let K be an ICE group over G and define K_i to be $K_i/\gamma_i(K) = \langle tor(K/\gamma_i(K))\rangle$. Then $K_{i+1} < K_i$, K_i is normal in K, K/K_i is torsion-free nilpotent and $\bigcap_{i\geq 1} K_i = 1$.*

Proof. By construction $K_i/\gamma_i(K)$ is a characteristic subgroup of $K/\gamma_i(K)$ and $K_{i+1} < K_i$. It remains to show that $\bigcap_i K_i = 1$. Suppose that $k \in (\bigcap_i K_i) \setminus \{1\}$. Since K is a limit group over G, there is a homomorphism $\varphi\colon K \mapsto G$ such that $\varphi(k) \neq 1$. Let $G_0 = \mathrm{im}(\varphi)$, so G_0 is a Droms RAAG. By [92, Theorem 6.4] we have that $G_0/\gamma_i(G_0)$

is torsion-free, hence $\varphi(K_i) = \varphi(\gamma_i(K))$. Then $\varphi(k) \in \bigcap_i \varphi(K_i) = \bigcap_i \varphi(\gamma_i(K)) \subseteq \bigcap_i \gamma_i(G) = 1$, a contradiction. $\square$

Corollary 3.2.11. *Every ICE group over a Droms RAAG is free-by-(torsion-free nilpotent).*

Proof. It follows from Lemma 3.2.5, Theorem 3.2.9 and Theorem 3.2.10. $\square$

Proposition 3.2.12. *Limit groups over Droms RAAGs are free-by-(torsion-free nilpotent).*

Proof. It follows from Theorem 1.5.10 and Corollary 3.2.11. $\square$

3.3 Corollaries on subdirect products of limit groups over Droms RAAGs

In this section we aim to prove Theorem 3.3.8, which is a generalisation of [65, Theorem 11].

Theorem 3.3.1. *[65, Theorem 11] Let $\Gamma_1, \dots, \Gamma_m$ be non-abelian limit groups and let $S < \Gamma_1 \times \cdots \times \Gamma_m$ be a finitely generated full subdirect product of type $wFP_s(\mathbb{Q})$ for some $s \in \{2, \dots, m\}$. Then for every canonical projection*

$$p_{j_1,\dots,j_s} : S \mapsto \Gamma_{j_1} \times \cdots \times \Gamma_{j_s},$$

the index of $p_{j_1,\dots,j_s}(S)$ in $\Gamma_{j_1} \times \cdots \times \Gamma_{j_s}$ is finite.

In particular, if S is of type $FP_s(\mathbb{Q})$, then for every canonical projection $p_{j_1,\dots,j_s}$, the index of $p_{j_1,\dots,j_s}(S)$ in $\Gamma_{j_1} \times \cdots \times \Gamma_{j_s}$ is finite.

Theorem 3.3.1 applies in bigger generality. We explain this in the following result.

Theorem 3.3.2. *Let $2 \le s \le m$ and let $S < G_1 \times \cdots \times G_m$ be a finitely generated subdirect product such that*

(1) there exist normal free subgroups L_i in G_i with $Q_i = G_i/L_i$ nilpotent;

(2) each G_i is finitely presented;

(3) for every $1 \le j_1 < \cdots < j_s \le m$ if $M_{j_1,\dots,j_s}$ is a subgroup of infinite index in $G_{j_1} \times \cdots \times G_{j_s}$, then there exists $i \le s$ such that $H_i(M_{j_1,\dots,j_s}; \mathbb{Q})$ is infinite dimensional;

(4) for every $1 \leq j_1 < \cdots < j_s \leq m$ and every subgroup H_{j_i} of finite index in G_{j_i} we have that if a subdirect product $M_{j_1,\ldots,j_s} < H_{j_1} \times \cdots \times H_{j_s}$ is finitely presented, then there is a subgroup K_{j_i} of finite index in H_{j_i} and N_{j_i} a normal subgroup of H_{j_i} such that K_{j_i}/N_{j_i} is nilpotent and $N_{j_1} \times \cdots \times N_{j_s} \subseteq M_{j_1,\ldots,j_s}$;

(5) S virtually surjects onto pairs;

(6) $L = L_1 \times \cdots \times L_m \subseteq S$;

(7) S is of type $wFP_n(\mathbb{Q})$.

Then for every canonical projection

$$p_{j_1,\ldots,j_s} \colon S \mapsto G_{j_1} \times \cdots \times G_{j_s},$$

the index of $p_{j_1,\ldots,j_s}(S)$ in $G_{j_1} \times \cdots \times G_{j_s}$ is finite.

Proof. We prove the result by induction on s. For $s = 2$ this is condition (5) from the statement.

Now suppose that $s \geq 3$ and that the statement holds for $k \leq s - 1$. We divide the proof in several steps.

1) Set Q to be $S/L < Q_1 \times \cdots \times Q_m$. Consider the Lyndon-Hochschild-Serre spectral sequence

$$E^2_{i,j} = H_i(Q; H_j(L; \mathbb{Q}))$$

that converges to $H_{i+j}(S; \mathbb{Q})$.

Note that since each L_i is a free group, for every $t \leq m$ we have that

$$H_t(L; \mathbb{Q}) \cong \bigoplus_{1 \leq j_1 < j_2 < \cdots < j_t \leq m} H_1(L_{j_1}; \mathbb{Q}) \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} H_1(L_{j_t}; \mathbb{Q}), \tag{3.1}$$

where each summand is Q-invariant and the action of Q on the group

$$H_1(L_{j_1}; \mathbb{Q}) \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} H_1(L_{j_i}; \mathbb{Q})$$

factors through the canonical map

$$h_{j_1,\ldots,j_i} \colon Q_1 \times \cdots \times Q_n \mapsto Q_{j_1} \times \cdots \times Q_{j_i}.$$

Thus,

$$E^2_{0,s} = H_0(Q; H_s(L; \mathbb{Q})) \cong \bigoplus_{1 \leq j_1 < j_2 < \ldots < j_s \leq m} \Big(H_1(L_{j_1}; \mathbb{Q}) \otimes \cdots \otimes H_1(L_{j_s}; \mathbb{Q}) \otimes_{\mathbb{Q}Q} \mathbb{Q} \Big).$$

The group L_i is finitely generated as a normal subgroup of G_i, so $H_i(L_i;\mathbb{Q})$ is finitely generated as a $\mathbb{Q}Q_i$-module. Hence,

$$H_1(L_{j_1};\mathbb{Q}) \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} H_1(L_{j_i};\mathbb{Q})$$

is finitely generated as a $\mathbb{Q}[Q_{j_1} \times \cdots \times Q_{j_i}]$-module. From the inductive hypothesis, $p_{j_1,\dots,j_i}(S)$ has finite index in $G_{j_1} \times \cdots \times G_{j_i}$ for $i \leq s-1$, so $h_{j_1,\dots,j_i}(Q)$ has finite index in $Q_{j_1} \times \cdots \times Q_{j_i}$ for $i \leq s-1$. In conclusion, $H_j(L;\mathbb{Q})$ is a finitely generated $\mathbb{Z}Q$-module for $j \leq s-1$. The group Q is nilpotent, so the group ring $\mathbb{Q}Q$ is Noetherian (see [56, Theorem 1]). Hence,

$$E^2_{i,j} \text{ is finite dimensional over } \mathbb{Q} \text{ for every } j \leq s-1. \tag{3.2}$$

For $i \geq 2$ we have that $s+1-i \leq s-1$, so by (3.2) $E^i_{i,s+1-i}$ is finite dimensional. Thus, for all differential maps $d^i_{i,j} \colon E^i_{i,s+1-i} \mapsto E^i_{0,s}$, $\mathrm{im}(d^i_{i,j})$ is finite dimensional. Hence,

$$E^{i+1}_{0,s} = \ker(d^i_{0,s})/\mathrm{im}(d^i_{i,j}) = E^i_{0,s}/\mathrm{im}(d^i_{i,j})$$

is finite dimensional if and only if $E^i_{0,s}$ is finite dimensional. This implies that $E^2_{0,s}$ is finite dimensional if and only if $E^\infty_{0,s}$ is finite dimensional. Combining this with the convergence of the spectral sequence and (3.2) we deduce that

$$H_s(S;\mathbb{Q}) \text{ is finite dimensional if and only if}$$

$$E^2_{0,s} = H_0(Q; H_s(L;\mathbb{Q})) \text{ is finite dimensional.}$$

The condition that $H_s(S;\mathbb{Q})$ is finite dimensional implies that for each $1 \leq j_1 < j_2 < \cdots < j_s \leq m$,

$$\dim_{\mathbb{Q}} \big(H_1(L_{j_1},\mathbb{Q}) \otimes \cdots \otimes H_1(L_{j_s},\mathbb{Q}) \otimes_{\mathbb{Q}Q} \mathbb{Q}\big) = \tag{3.3}$$

$$\dim_{\mathbb{Q}} \big(H_1(L_{j_1},\mathbb{Q}) \otimes \cdots \otimes H_1(L_{j_s},\mathbb{Q}) \otimes_{\mathbb{Q}[h_{j_1,\dots,j_s}(Q)]} \mathbb{Q}\big) < \infty.$$

2) Let $\widetilde{S}$ be a subgroup of finite index in $p_{j_1,\dots,j_s}(S)$ that contains the group $\widetilde{L} = L_{j_1} \times \cdots \times L_{j_s}$ and set $\widetilde{Q}$ to be $\widetilde{S}/\widetilde{L}$. Then $\widetilde{S}$ is an extension of $\widetilde{L}$ by a subgroup of $\widetilde{Q}$ of finite index in $h_{j_1,\dots,j_s}(Q)$.

By the same argument as above, for $r \leq s-1$, $H_r(\widetilde{L};\mathbb{Q})$ is finitely generated as a $\mathbb{Q}\widetilde{Q}$-module. We consider the Lyndon-Hochschild-Serre spectral sequence

$$\widetilde{E}^2_{i,j} = H_i(\widetilde{Q}; H_j(\widetilde{L}; \mathbb{Q}))$$

that converges to $H_{i+j}(\widetilde{S}; \mathbb{Q})$. We again get that $\widetilde{E}^\infty_{i,j}$ is finite dimensional for all $j \leq s-1$. Then $H_s(\widetilde{S}; \mathbb{Q})$ is finite dimensional if and only if $\widetilde{E}^\infty_{0,s}$ is finite dimensional. From (3.3) we deduce that $\widetilde{E}^2_{0,s}$ is finite dimensional, so both $\widetilde{E}^\infty_{0,s}$ and $H_s(\widetilde{S}; \mathbb{Q})$ are finite dimensional.

3) Now we consider $\widetilde{S}_0$ an arbitrary subgroup of finite index in $p_{j_1,\ldots,j_s}(S)$. We view $p_{j_1,\ldots,j_s}(S)$ as a subgroup of $G_{j_1} \times \cdots \times G_{j_s}$. As S virtually surjects onto pairs we have that $p_{j_1,\ldots,j_s}(S)$ virtually surjects onto pairs, so by [28, Theorem A] $p_{j_1,\ldots,j_s}(S)$ is finitely presented. Hence $\widetilde{S}_0$ is finitely presented. Then, by condition (4) applied to $\widetilde{S}_0$ considered as a subdirect product of $p_{j_1}(\widetilde{S}_0) \times \cdots \times p_{j_s}(\widetilde{S}_0)$, there is a subgroup of finite index K_{j_i} in $p_{j_i}(\widetilde{S}_0)$ such that for sufficiently big t we have that $\gamma_t(K_{j_i}) \subseteq \widetilde{S}_0$. By condition (1) we can choose t sufficiently big so that $\gamma_t(G_k) \subseteq L_k$ is free for every $1 \leq k \leq m$. In particular, $\gamma_t(K_{j_i})$ is free.

For every $j \in \{1, \ldots, m\} \setminus \{j_1, \ldots, j_s\}$ we set K_j to be G_j. Then

$$S_0 = S \cap (K_1 \times \cdots \times K_m)$$

is a subgroup of finite index in S such that $\gamma_t(K_1) \times \cdots \times \gamma_t(K_m) \subseteq S_0$. By condition (7), $H_i(S_0; \mathbb{Q})$ is finite dimensional for all $i \leq s$. Applying step 2) for S_0 instead of S, K_j instead of G_j and $\gamma_t(K_j)$ instead of L_j we deduce that

$$\dim_{\mathbb{Q}} H_i(p_{j_1,\ldots,j_s}(S_0); \mathbb{Q}) < \infty \quad \text{for} \quad i \leq s.$$

But this combined with condition (3) implies that $p_{j_1,\ldots,j_s}(S_0)$ has finite index in $K_{j_1} \times \cdots \times K_{j_s}$. Since $S_0 \subseteq S$ and each K_{j_i} has finite index in G_{j_i}, we deduce that $p_{j_1,\ldots,j_s}(S)$ has finite index in $G_{j_1} \times \cdots \times G_{j_s}$. □

Lemma 3.3.3. *[65, Lemma 6] Let $Q_1, \ldots, Q_i$ be finitely generated nilpotent groups and let V_j be a finitely generated $\mathbb{Q}Q_j$-module such that V_j contains a cyclic non-zero free $\mathbb{Q}Q_j$-submodule W_j for $1 \leq j \leq i$. Suppose that $\widetilde{Q}$ is a subgroup of $Q_1 \times \cdots \times Q_i$ such that $V_1 \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} V_i$ is finitely generated as a $\mathbb{Q}\widetilde{Q}$-module. Then $\widetilde{Q}$ has finite index in $Q_1 \times \cdots \times Q_i$.*

Proposition 3.3.4. *[65, Proposition 7] Let G be a group of negative Euler characteristic such that the trivial $\mathbb{Q}G$-module $\mathbb{Q}$ has a free resolution with finitely generated modules and finite length. Then for any normal subgroup M of G such that the* *group $Q = G/M$ is torsion-free nilpotent and M is free, the $\mathbb{Q}Q$-module $V = M/[M,M] \otimes_{\mathbb{Z}} \mathbb{Q}$ has a non-zero free $\mathbb{Q}Q$-submodule, where Q acts on $M/[M,M]$ via conjugation.*

Theorem 3.3.5. *Let $2 \leq s \leq m$ and let $S < G_1 \times \cdots \times G_m$ be a finitely generated subdirect product such that*

(1) for each $1 \leq i \leq m$ the trivial $\mathbb{Q}G_i$-module $\mathbb{Q}$ has a free resolution of finite length with finitely generated modules;

(2) for each $1 \leq i \leq m$ there is a normal free subgroup L_i of G_i such that G_i/L_i is torsion-free nilpotent;

(3) $L_1 \times \cdots \times L_m \subseteq S$;

(4) S is of type $FP_s(\mathbb{Q})$;

(5) $\chi(G_i) < 0$.

Then for every canonical projection $p_{j_1,\dots,j_s} \colon S \mapsto G_{j_1} \times \cdots \times G_{j_s}$, the index of $p_{j_1,\dots,j_s}(S)$ in $G_{j_1} \times \cdots \times G_{j_s}$ is finite.

Proof. Note that since G_i is of type $FP_\infty(\mathbb{Q})$, it is of type $FP_1(\mathbb{Q})$ and so it is finitely generated. By Proposition 3.3.4, $V_i = (L_i/[L_i, L_i]) \otimes_{\mathbb{Z}} \mathbb{Q}$ has a non-zero free $\mathbb{Q}Q_i$-submodule, where $Q_i = G_i/L_i$ acts via conjugation.

Let

$$\mathcal{F} \colon \ \cdots \to F_i \to F_{i-1} \to \cdots \to F_0 \to \mathbb{Q} \to 0$$

be a free resolution of the trivial $\mathbb{Q}S$-module $\mathbb{Q}$ with F_i finitely generated for $i \leq s$. We define L to be $L_1 \times \cdots \times L_m$ and since each L_i is free, by the Künneth formula

$$H_s(L; \mathbb{Q}) \cong \bigoplus_{1 \leq j_1 < \cdots < j_s \leq m} V_{j_1} \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} V_{j_s}.$$

Note that

$$H_s(L; \mathbb{Q}) \cong H_s(\mathcal{F} \otimes_{\mathbb{Q}L} \mathbb{Q}) = \ker(d_s)/\mathrm{im}(d_{s+1}),$$

where $d_j \colon F_j \otimes_{\mathbb{Q}L} \mathbb{Q} \mapsto F_{j-1} \otimes_{\mathbb{Q}L} \mathbb{Q}$ is the differential map of $\mathcal{F} \otimes_{\mathbb{Q}L} \mathbb{Q}$. If we denote by $\widehat{Q}$ the quotient group S/L, then $\widehat{Q}$ is a finitely generated nilpotent group, so $\mathbb{Q}\widehat{Q}$ is a Noetherian ring. Since $F_s \otimes_{\mathbb{Q}L} \mathbb{Q}$ is a finitely generated $\mathbb{Q}\widehat{Q}$-module, we deduce that $\ker(d_s)$ is a finitely generated $\mathbb{Q}\widehat{Q}$-module. In particular, $H_s(L; \mathbb{Q})$ and $V_{j_1} \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} V_{j_s}$ are finitely generated $\mathbb{Q}\widehat{Q}$-modules. Note that the action of $\widehat{Q}$ on $V_{j_1} \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} V_{j_s}$ factors through $p_{j_1,\dots,j_s}(S)/L_{j_1} \times \cdots \times L_{j_s}$. Then, by Lemma 3.3.3,

$p_{j_1,\ldots,j_s}(S)/L_{j_1} \times \cdots \times L_{j_s}$ has finite index in $Q_{j_1} \times \cdots \times Q_{j_s}$. This is equivalent to $p_{j_1,\ldots,j_s}(S)$ having finite index in $G_{j_1} \times \cdots \times G_{j_s}$. $\square$

The next step is to prove that limit groups over Droms RAAGs satisfy the conditions of Theorem 3.3.5.

Lemma 3.3.6. *Let G be a Droms RAAG and let Γ be a limit group over G. Then $\chi(\Gamma) \leq 0$. Furthermore, $\chi(\Gamma) = 0$ if and only if Γ has non-trivial center. The latter happens precisely when $\Gamma = \mathbb{Z}^l \times \Lambda$ for some $l \geq 1$.*

Proof. Let us prove it by induction on the level $l(G)$ of G. If $l(G) = 0$, then G is a free group, so the result follows from [65, Lemma 5].

Now assume that $l(G) \geq 1$. Then G is

$$\mathbb{Z}^m \times (G_1 * \cdots * G_n),$$

where $m \in \mathbb{N} \cup \{0\}$ and G_i is a Droms RAAG such that $l(G_i) \leq l(G) - 1$ for $i \in \{1, \ldots, n\}$. From Proposition 1.5.9 we get that Γ is of the form $\mathbb{Z}^l \times \Lambda$ where Λ is a limit group over $G_1 * \cdots * G_n$ and if $m = 0$, then $l = 0$. Therefore,

$$\chi(\Gamma) = \chi(\mathbb{Z}^l)\chi(\Lambda),$$

so if $l \geq 1$, then $\chi(\Gamma) = 0$. Let us compute $\chi(\Lambda)$. If the height of Λ is 0, i.e. $h(\Lambda) = 0$, then

$$\Lambda = A_1 * \cdots * A_j,$$

where for each $t \in \{1, \ldots, j\}$ A_t is a limit group over G_i for some $i \in \{1, \ldots, n\}$. Hence,

$$\chi(\Lambda) = \sum_{t \in \{1,\ldots,j\}} \chi(A_t) - (j-1),$$

so applying the inductive hypothesis, we get that $\chi(\Lambda) \leq 1 - j$. If $j \geq 2$, then $\chi(\Lambda) < 0$. If $j = 1$, then $\chi(\Lambda) = \chi(A_1)$ and A_1 is a limit group over G_i. Thus, by induction $\chi(A_1) \leq 0$ and $\chi(A_1) = 0$ if and only if A_1 has non-trivial center.

If $h(\Lambda) \geq 1$, then Λ acts cocompactly on a tree T where the edge stabilisers are cyclic and the vertex groups are limit groups over $G_1 * \cdots * G_n$ of height at most $h(\Lambda) - 1$. Moreover, at least one vertex group H_{v_0} has trivial center and so by inductive hypothesis $\chi(H_{v_0}) < 0$. If X is the quotient graph T/Λ,

$$\chi(\Lambda) = \sum_{v \in V(X)} \chi(H_v) - \sum_{e \in E(X)} \chi(H_e) \leq \sum_{v \in V(X)} \chi(H_v) \leq \chi(H_{v_0}) < 0.$$

$\square$

Lemma 3.3.7. *Let Γ be a limit group over a Droms RAAG. Then the trivial $\mathbb{Q}\Gamma$-module $\mathbb{Q}$ has a free resolution with finitely generated modules and of finite length.*

Proof. A limit group over a Droms RAAG is of type F. This follows from the hierarchies in Proposition 1.5.8 and Proposition 1.5.9. $\square$

Let us finish this section by proving Theorem 3.3.8.

Theorem 3.3.8. *Let $\Gamma_1, \ldots, \Gamma_m$ be limit groups over Droms RAAGs such that each Γ_i has trivial center and let $S < \Gamma_1 \times \cdots \times \Gamma_m$ be a finitely generated full subdirect product of type $wFP_s(\mathbb{Q})$ for some $s \in \{2, \ldots, m\}$. Then for every canonical projection*

$$p_{j_1,\ldots,j_s} \colon S \mapsto \Gamma_{j_1} \times \cdots \times \Gamma_{j_s},$$

the index of $p_{j_1,\ldots,j_s}(S)$ in $\Gamma_{j_1} \times \cdots \times \Gamma_{j_s}$ is finite.

In particular, if S is of type $FP_s(\mathbb{Q})$, then for every canonical projection $p_{j_1,\ldots,j_s}$, the index of $p_{j_1,\ldots,j_s}(S)$ in $\Gamma_{j_1} \times \cdots \times \Gamma_{j_s}$ is finite.

Proof. By Theorem 2.6.4 the quotient group $\Gamma_i/(S \cap \Gamma_i)$ is virtually nilpotent for every $i \in \{1, \ldots, m\}$. By substituting Γ_i and S with subgroups of finite index if necessary, we can assume that $\gamma_{m_i}(\Gamma_i) \subseteq S$ for some m_i. By Proposition 3.2.12, for some $n_i \geq m_i$ we have that $L_i = \gamma_{n_i}(\Gamma_i)$ is free, Γ_i/L_i is nilpotent and $L_i \subseteq S$. Now by substituting again Γ_i and S with subgroups of finite index if necessary but without changing L_i, we can assume that L_i is a normal subgroup of Γ_i such that L_i is free, Γ_i/L_i is torsion-free nilpotent and $L_1 \times \cdots \times L_m \subseteq S$, so conditions (1) and (6) from Theorem 3.3.2 hold. The other conditions from Theorem 3.3.2 hold when each Γ_i is a limit group over a Droms RAAG such that Γ_i has trivial center: condition (2) is [35, Corollary 7.8], condition (3) is Theorem 2.8.1, condition (5) is Theorem 2.9.1. Finitely generated subgroups of limit groups over Droms RAAGs are again limit groups over Droms RAAGs, so condition (4) follows from Theorem 2.6.4. $\square$

3.4 On the L^2-Betti numbers and volume gradients of limit groups over Droms RAAGs and their subdirect products

The aim of this section is to study the growth of homology groups and the volume gradients for limit groups over Droms RAAGs and for finitely presented residually

Droms RAAGs. Some of the results concerning limit groups over Droms RAAGs hold in a more general setting, more precisely, they also hold for limit groups over coherent RAAGs. Thus, these results (see Theorem 3.4.14 and Theorem 3.4.15) will be stated for limit groups over coherent RAAGs. However, the result for finitely presented residually Droms RAAGs makes use of Theorem 2.9.1 and in Chapter 5 it will be shown that this no longer holds for coherent RAAGs. Thus, Theorem 3.4.16 is just stated for residually Droms RAAGs.

In order to study the homology growth and volume gradients, we work with *exhausting normal chains*: a chain (B_n) of normal subgroups of finite index such that $B_{n+1} \subseteq B_n$ and $\bigcap_n B_n = 1$. Note that if a group is residually finite, then it has an exhausting normal chain. In particular, limit groups over coherent RAAGs have exhausting normal chains.

Given a group G of homotopical type F_m, the *m-volume* $\mathrm{vol}_m(G)$ is defined to be the least number of m-cells among all classifying spaces $K(G,1)$ with finite m-skeleton. For instance, $\mathrm{vol}_1(G)$ is the minimal number of generators of G, $d(G)$.

One of the aims of this section is to prove Theorem 3.4.14 and Theorem 3.4.15. These two results are proved in [22] for limit groups via more technical results that make use of *slowness* of limit groups (see Section 3.4.1 for the definition). In [22] it is shown that limit groups are slow above dimension 1 and hence are K-slow above dimension 1. Thus, the key point is to show that limit groups over coherent RAAGs are also slow above dimension 1 (see Section 3.4.2) as then, the results follow from the following theorems.

Theorem 3.4.1. *[22, Theorem D] If a residually finite group G of type F is slow above dimension 1, then with respect to every exhausting normal chain (B_n),*

(1) Rank gradient:

$$RG(G,(B_n)) = \lim_{n\to\infty} \frac{d(B_n)}{[G\colon B_n]} = -\chi(G).$$

(2) Deficiency gradient:

$$DG(G,(B_n)) = \lim_{n\to\infty} \frac{\mathit{def}(B_n)}{[G\colon B_n]} = \chi(G).$$

Lemma 3.4.2. *[22, Lemma 5.2] Let K be a field and let G be a residually finite group of type F with an exhausting normal chain (B_n). If G is K-slow above dimension 1, then*

$$\lim_{n\to\infty} \frac{\dim H_1(B_n;K)}{[G\colon B_n]} = -\chi(G).$$

3.4.1 Preliminaries on groups that are K-slow above dimension 1 and slow above dimension 1

Let us start defining slowness and K-slowness.

Definition 3.4.3. Let G be a group. A sequence of non-negative integers $(r_j)_{j\geq 0}$ is a *volume vector* for G if there is a classifying space $K(G,1)$ that, for all $j \in \mathbb{N}$, has exactly r_j j-cells.

Definition 3.4.4. A group G of homotopical type F is *slow above dimension* 1 if it is residually finite and for every exhausting normal chain (B_n) there exist volume vectors $(r_j(B_n))_j$ for B_n with finitely many non-zero entries, so that

$$\lim_{n\to\infty} \frac{r_j(B_n)}{[G\colon B_n]} = 0$$

for all $j \geq 2$. The group G is *slow* if it satisfies the additional requirement that the limit exists and is zero for $j = 1$ as well.

Example 3.4.5. [22, Examples 4.4] Finitely generated torsion-free nilpotent groups are slow. The trivial group is slow. Free groups are slow above dimension 1. Surface groups are slow above dimension 1.

The next proposition shows how to obtain slow groups above dimension 1 using graphs of groups.

Proposition 3.4.6. *[22, Proposition 4.5] If a residually finite group G is the fundamental group of a finite graph of groups where all of the edge groups are slow and all of the vertex groups are slow above dimension* 1*, then G is slow above dimension* 1.

Definition 3.4.7. Let K be a field and let G be a residually finite group. Then G is *K-slow above dimension 1* if for every exhausting normal chain (B_n) we have

$$\lim_{n\to\infty} \frac{\dim_K H_j(B_n;K)}{[G\colon B_n]} = 0$$

for all $j \geq 2$.

The group G is *K-slow* if it satisfies the additional requirement that the limit exists and is zero for $j = 1$ as well.

It follows directly from the definitions that if a group G is slow above dimension 1 (respectively, slow), then it is K-slow above dimension 1 (respectively, K-slow).

Proposition 3.4.8. *[22, Proposition 5.3] Let K be a field. If a residually finite group G is the fundamental group of a finite graph of groups where all of the edge groups are K-slow and all of the vertex groups are K-slow above dimension 1, then G is K-slow above dimension 1.*

3.4.2 Limit groups over coherent RAAGs are slow above dimension 1

Lemma 3.4.9. *Coherent RAAGs are slow above dimension 1. In particular, Droms RAAGs are slow above dimension 1.*

Proof. Recall that coherent RAAGs split as fundamental groups of finite graphs of groups where all the vertex groups are free abelian. Thus, by Proposition 3.4.6 coherent RAAGs are slow above dimension 1. □

Not all RAAGs are, however, slow above dimension 1. See, for instance, [3, Corollary 2].

We now prove some results that will be used in order to show that limit groups over coherent RAAGs are slow above dimension 1.

Lemma 3.4.10. *[93] Let $1 \to C \to D \to E \to 1$ be a short exact sequence of groups of type F. Suppose that there are classifying spaces $K(C, 1)$ and $K(E, 1)$ with $\alpha_t(C)$ and $\alpha_t(E)$ t-cells, respectively. Then there is a $K(D, 1)$-complex with $\alpha_i(D)$ i-cells such that*

$$\alpha_i(D) = \sum\nolimits_{0 \leq t \leq i} \alpha_t(C)\alpha_{i-t}(E).$$

Lemma 3.4.11. *Suppose that G is a group of type F, H is a normal subgroup of finite index in G and $(r_j(G))_j$ is a volume vector for G. Then $([G\colon H]r_j(G))_j$ is a volume vector for H.*

Proof. Let Y be a classifying space $K(G, 1)$ having exactly $r_j(G)$ j-cells for $j \in \mathbb{N} \cup \{0\}$. Then if we denote by $\widetilde{Y}$ the universal cover of Y, $\widetilde{Y}$ is contractible and $Y = \widetilde{Y}/G$. Therefore, $\widetilde{Y}/H$ is a classifying space for H with exactly $[G\colon H]r_j(G)$ j-cells. □

Lemma 3.4.12. *Let G be a residually finite group of type F and let H be a residually finite group where there is a short exact sequence*

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow H \xrightarrow{p} G \longrightarrow 1.$$

(1) If $n \geq 1$, then H is slow.

(2) If $n \geq 0$ and G is slow above dimension 1, then H is slow above dimension 1.

Proof. Let us denote $\mathbb{Z}^n$ by A.

(1) Let (B_i) be an exhausting normal chain in H. We need to show that for each i there is a $K(B_i, 1)$-complex with $r_j(B_i)$ j-cells such that for each $j \geq 1$

$$\lim_{i\to\infty} \frac{r_j(B_i)}{[H\colon B_i]} = 0.$$

For each i the short exact sequence from the statement induces a short exact sequence

$$1 \longrightarrow A \cap B_i \longrightarrow B_i \longrightarrow p(B_i) \longrightarrow 1.$$

Let us show that

$$[H\colon B_i] = [A\colon A\cap B_i][G\colon p(B_i)].$$

Indeed, note that $[H\colon B_i] = [H\colon AB_i][AB_i\colon B_i]$. Firstly,

$$[AB_i\colon B_i] = [A\colon A \cap B_i].$$

Secondly, $[H\colon AB_i]$ equals $[H/A\colon AB_i/A]$, and $H/A \cong G$ and $AB_i/A \cong p(B_i)$. Therefore, $[H\colon AB_i] = [G\colon p(B_i)]$.

Let $\alpha_j(G)$ be the number of j-cells in a fixed $K(G, 1)$-complex. By Lemma 3.4.11 there is a $K(p(B_i), 1)$-complex with $\alpha_j(p(B_i))$ j-cells such that

$$\alpha_j(p(B_i)) = [G\colon p(B_i)]\alpha_j(G).$$

Since $A \cap B_i$ has finite index in A, there is a $K(A\cap B_i, 1)$-complex with $\binom{n}{j}$ j-cells. By Lemma 3.4.10 there is a $K(B_i, 1)$-complex with $\alpha_j(B_i)$ j-cells such that

$$\alpha_j(B_i) = \sum\nolimits_{0\leq a\leq j} \binom{n}{j-a}\alpha_a(p(B_i)) = \sum\nolimits_{0\leq a\leq j} \binom{n}{j-a}[G\colon p(B_i)]\alpha_a(G),$$

and we set $r_j(B_i)$ to be $\alpha_j(B_i)$. Then

$$\lim_{i\to\infty} \frac{r_j(B_i)}{[H\colon B_i]} = \lim_{i\to\infty} \frac{[G\colon p(B_i)]\sum\nolimits_{0\leq a\leq j}\binom{n}{j-a}\alpha_a(G)}{[G\colon p(B_i)][A\colon B_i\cap A]} =$$

$$\sum\nolimits_{0\leq a\leq j} \binom{n}{j-a}\alpha_a(G) \lim_{i\to\infty} \frac{1}{[A\colon B_i \cap A]} = 0.$$

(2) If $n \geq 1$, then we apply (1). If $n = 0$, then by assumption G is slow above dimension 1. $\square$

Theorem 3.4.13. *Limit groups over coherent RAAGs are slow above dimension* 1.

Proof. Let G be a coherent RAAG and let Γ be a limit group over G. Then, by Proposition 1.5.6, Γ is a subgroup of a graph tower over G (see Proposition 1.5.6 and Theorem 1.5.7), say L. Let us prove by induction on the height of L that Γ is slow above dimension 1.

If L has height 0, then L is a coherent RAAG. Therefore, L is the fundamental group of a graph of groups where the vertex groups are free abelian. Thus Γ also admits a decomposition as a graph of groups where the vertex groups are free abelian, so by Proposition 3.4.6 we get that Γ is slow above dimension 1.

Now suppose that the height of L is greater than 0. Then L is a free product with amalgamation, where the edge group is a free abelian group, one of the vertex groups is a graph tower over G of lower height and the other vertex group is either free abelian or the direct product of a free abelian group and the fundamental group of a non-exceptional surface. Then Γ admits a decomposition as a graph of groups where the edge groups are free abelian, and the vertex groups are either subgroups of graph towers over G of lower height (and by induction, those vertex groups are slow above dimension 1) or free abelian groups or subgroups of the direct product of a free abelian group and the fundamental group of a non-exceptional surface. It suffices to show that finitely generated subgroups of the direct product of a free abelian group and the fundamental group of a non-exceptional surface are slow above dimension 1. As a consequence, by Proposition 3.4.6 we obtain that Γ is slow above dimension 1.

If H is a finitely generated subgroup of $\mathbb{Z}^m \times G_0$ where $m \in \mathbb{N} \cup \{0\}$ and G_0 is the fundamental group of a non-exceptional surface, then there is a short exact sequence

$$1 \longrightarrow A \longrightarrow H \longrightarrow N \longrightarrow 1,$$

where A is a free abelian group and N is a finitely generated subgroup of G_0. In particular, N is either a surface group or a free group, so N is slow above dimension 1. Then, by Lemma 3.4.12, H is slow above dimension 1. $\square$

We now have the necessary tools to prove Theorem 3.4.14, Theorem 3.4.15 and Theorem 3.4.16.

Theorem 3.4.14. *Let G be a limit group over a coherent RAAG and let (B_n) be an exhausting normal chain in G. Then*

(1) The rank gradient $RG(G,(B_n)) = \lim_{n\to\infty} \frac{d(B_n)}{[G\colon B_n]} = -\chi(G)$.

(2) The deficiency gradient $DG(G,(B_n)) = \lim_{n\to\infty} \frac{def(B_n)}{[G\colon B_n]} = \chi(G)$.

(3) The k*-dimensional volume gradient* $\lim_{n\to\infty} \frac{vol_k(B_n)}{[G\colon B_n]} = 0$ *for* $k \geq 2$.

Proof. Parts (1) and (2) follow from Theorem 3.4.1 and Theorem 3.4.13. Part (3) follows from Theorem 3.4.13. □

Theorem 3.4.15. *Let* K *be a field,* G *a limit group over a coherent RAAG and* (B_n) *an exhausting normal chain in* G*. Then*

(1) $\lim_{n\to\infty} \frac{\dim_K H_1(B_n;K)}{[G\colon B_n]} = -\chi(G)$.

(2) $\lim_{n\to\infty} \frac{\dim_K H_j(B_n;K)}{[G\colon B_n]} = 0$ *for all* $j \geq 2$.

Proof. It follows from Lemma 3.4.2 and the fact that Theorem 3.4.13 implies that every limit group over a coherent RAAG is K-slow above dimension 1. □

Theorem 3.4.16. *Let* $m \geq 2$*, let* G *be a residually Droms RAAG of type* FP_m *and let* ρ *be the largest integer such that* G *contains a direct product of* ρ *non-abelian free groups. Then there exists an exhausting sequence* (B_n) *in* G *so that for all fields* K*,*

(1) if G *is not of type* FP_∞*, then*

$$\lim_{n\to\infty} \frac{\dim H_i(B_n;K)}{[G\colon B_n]} = 0 \text{ for all } 0 \leq i \leq m;$$

(2) if G *is of type* FP_∞*, then for all* $j \geq 1$*,*

$$\lim_{n\to\infty} \frac{\dim H_\rho(B_n;K)}{[G\colon B_n]} = (-1)^\rho \chi(G), \quad \lim_{n\to\infty} \frac{\dim H_j(B_n;K)}{[G\colon B_n]} = 0 \quad \text{for all} \quad j \neq \rho.$$

Proof. From Theorem 2.9.1 we get that G is a full subdirect product of limit groups over Droms RAAGs $\Gamma_0 \times \Gamma_1 \times \cdots \times \Gamma_r$ such that Γ_0 is free abelian (possibly trivial), $G \cap \Gamma_0$ has finite index in Γ_0 and Γ_i has trivial center for $i \in \{1, 2, \ldots, r\}$.

Suppose that Γ_0 is trivial. By Theorem 2.6.4, for each $i \in \{1, \ldots, r\}$ the group $\Gamma_i/(G \cap \Gamma_i)$ is virtually nilpotent, so there is a subgroup of finite index $\widetilde{\Gamma}_i$ in Γ_i and a free normal subgroup F_i of Γ_i such that $\widetilde{\Gamma}_i/F_i$ is torsion-free nilpotent and $F_i \subseteq G$. Then, by Theorem 3.3.8 and [22, Theorem F], there is an exhausting normal chain (B_n) of $G \cap (\widetilde{\Gamma}_1 \times \cdots \times \widetilde{\Gamma}_r)$ with the desired properties. Nevertheless, the chain may not be normal in G.

If Γ_0 is non-trivial, then $G \cap \Gamma_0$ is non-trivial, free abelian and central, so part (1) follows from [22, Lemma 7.2].

It remains to consider part (2). From Theorem 2.3.1, G has a subgroup of finite index $H = H_1 \times \cdots \times H_r$ where each H_i is a limit group over a Droms RAAG. More specifically, H_i is a finite index normal subgroup of Γ_i, $i \in \{1, \ldots, r\}$. Let (B_i) be an exhausting normal chain in G such that each B_i is contained in H and decompose B_i as $(B_i \cap H_1) \times \cdots \times (B_i \cap H_r)$. Then, by the Künneth formula and Theorem 3.4.14 applied for each H_i, we can deduce that for $j \geq 1$

$$\lim_{n\to\infty} \frac{\dim H_j(B_n; K)}{[G\colon B_n]} =$$

$$\frac{1}{[G\colon H]} \sum_{j_1+\cdots+j_r=j} \prod_{1\leq s\leq r} \lim_{n\to\infty} \frac{\dim H_{j_s}(B_n \cap H_s; K)}{[H_s\colon B_n \cap H_s]} =$$

$$\frac{1}{[G\colon H]} \sum_{j_1+\cdots+j_r=j} \prod_{1\leq s\leq r} - \ \delta_{1,j_s} \chi(H_s) =$$

$$\frac{1}{[G\colon H]} (-1)^r \delta_{j,r} \chi(H) = (-1)^r \delta_{j,r} \chi(G),$$

where $\delta_{j,r}$ is the Kronecker symbol.

A limit group over a Droms RAAG does not contain a direct product of two or more non-abelian free groups since limit groups over Droms RAAGs are coherent (see [35, Corollary 7.8]) and the direct product of two non-abelian free groups is not coherent. In addition, every limit group over a Droms RAAG that has trivial center contains a non-abelian free group (see, for instance, Property 2.2.7), so $r = \rho$ unless one or more H_i has non-trivial center. If some H_i has non-trivial center, $\chi(H_i) = 0$ and so $\chi(H) = 0$ and $\chi(G) = 0$. $\square$

Chapter 4

On the Bieri-Neumann-Strebel-Renz invariants and limit groups over Droms RAAGs

4.1 Introduction and outline

The Bieri-Neumann-Strebel-Renz invariants are specific open subsets in the character sphere $S(G)$ of a group G and they are a tool to control when a subgroup containing the commutator subgroup is of type FP_n or of type F_n. These invariants are separated in two groups: the homotopical invariants $\{\Sigma^n(G)\}_n$ and the homological ones $\{\Sigma^n(G, \mathbb{Z})\}_n$. In Section 1.4.1 a brief introduction to the topic was given and the main notions were defined. In general, the invariants are difficult to compute, but they are known for some classes of groups. For example, the case of the Thompson group F and the generalised Thompson groups $F_{n,\infty}$ were treated by Bieri-Geoghegan-Kochloukova ([17]), Witzel-Zaremsky ([96]) and Zaremsky ([97]). For a finitely generated metabelian group G, the structure of the complement of $\Sigma^1(G)$ in the character sphere $S(G)$ as a rationally defined polyhedron was proved by Bieri-Groves in [13] and it has numerous applications in tropical geometry. The case when G is the fundamental group of a compact Kähler manifold was studied by Delzant in [42] and the case of a free-by-cyclic group was considered by Funke-Kielak and Kielak in [49] and [62], respectively. The case when G is a right-angled Artin group was settled by Meier-Meinert-VanWyk ([73]) but for general Artin groups just some particular cases are known; for instance, some even Artin groups were considered by Blasco García-Cogolludo Agustín-Martínez Pérez in [50] and by Kochloukova in [66]. The Bieri-Neumann-Strebel-Renz invariants are also known for limit groups (see [65]) and Ferreira Lima-Kochloukova partially computed them for finitely presented residually free groups in [72].

The goal of this chapter is to compute the Σ-invariants for limit groups over Droms RAAGs and the first invariant for finitely presented residually Droms RAAGs. For this aim, we generalise the results in [65] and [72]. More precisely, in this chapter we prove the following results.

Proposition 4.2.1. Let Γ be a limit group over a Droms RAAG.

(1) If Γ has trivial center, then $\Sigma^n(\Gamma) = \Sigma^n(\Gamma, \mathbb{Z}) = \Sigma^n(\Gamma, \mathbb{Q}) = \varnothing$ for every $n \geq 1$.

(2) In general, $\Sigma^n(\Gamma) = \Sigma^n(\Gamma, \mathbb{Z}) = \Sigma^n(\Gamma, \mathbb{Q}) = \{[\chi] \in S(\Gamma) \mid \chi(Z(\Gamma)) \neq 0\}$ for every $n \geq 1$.

The next result gives a necessary condition for a point from the character sphere $S(H)$ to belong to $\Sigma^m(H, \mathbb{Q})$ for a finitely presented residually Droms RAAG H.

Theorem 4.2.4. Let $m \geq 2$, $1 \leq n \leq m$ and let $H < \Gamma_1 \times \cdots \times \Gamma_m$ be a finitely presented full subdirect product of limit groups over Droms RAAGs $\Gamma_1, \ldots, \Gamma_m$ where each Γ_i has trivial center. Suppose that $[\chi] \in \Sigma^n(H, \mathbb{Q})$. Then

$$p_{j_1,\ldots,j_n}(H_\chi) = p_{j_1,\ldots,j_n}(H) \quad \text{for all} \quad 1 \leq j_1 < \cdots < j_n \leq m, \tag{4.1}$$

where $p_{j_1,\ldots,j_n} \colon H \mapsto \Gamma_{j_1} \times \cdots \times \Gamma_{j_n}$ is the canonical projection.

Note that whenever $\Sigma^n(H, \mathbb{Q}) \neq \varnothing$, then the group H is of type $FP_n(\mathbb{Q})$, so by Theorem 3.3.8 this implies that $[\Gamma_{j_1} \times \cdots \times \Gamma_{j_n} \colon p_{j_1,\ldots,j_n}(H)] < \infty$. As a consequence, Theorem 4.2.4 gives us a monoidal version of the Virtual Surjection Conjecture applied to subdirect products of limit groups over Droms RAAGs. Recall from Chapter 3 the Virtual Surjection Conjecture:

The Virtual Surjection Conjecture *Let $n \leq m$ be positive integers and let H be a subgroup of the direct product $G_1 \times \cdots \times G_m$, where G_i is of type F_n for $1 \leq i \leq m$. If H is virtually surjective on n-tuples, then H is of type F_n.*

The Monoidal Virtual Surjection Conjecture was named by Ferreira Lima and Kochloukova in [72] and it states that the reverse implication of Theorem 4.2.4 should also hold.

The Monoidal Virtual Surjection Conjecture *Let $n \geq m$ be positive integers and let $H < \Gamma_1 \times \cdots \times \Gamma_m$ be a full subdirect product of limit groups over*

Droms RAAGs where each Γ_i has trivial center and assume that H is of type FP_n and finitely presented. Then

$$[\chi] \in \Sigma^n(H, \mathbb{Q}) = \Sigma^n(H, \mathbb{Z}) = \Sigma^n(H)$$

if and only if

$$p_{j_1,\dots,j_n}(H_\chi) = p_{j_1,\dots,j_n}(H) \quad \text{for all} \quad 1 \leq j_1 < \dots < j_n \leq m, \tag{4.2}$$

where $p_{j_1,\dots,j_n} \colon H \mapsto \Gamma_{j_1} \times \dots \times \Gamma_{j_n}$ is the canonical projection.

Note that one of the key assumptions is that the limit groups over Droms RAAGs need to have trivial center. This hypothesis ensures that the Σ-invariants are the empty set as we see in Proposition 4.2.1, and this property will be used constantly in the arguments. In the subsequent results, however, we also mention the general case where we do not add any restrictions on the centers of the limit groups over Droms RAAGs.

The next result gives the converse of Theorem 4.2.4 for $n = 1$, so we indeed see that the Monoidal Virtual Surjection Conjecture holds for $n = 1$.

Corollary 4.2.9. (1) Let $H < \Gamma_1 \times \dots \times \Gamma_m$ be a finitely presented full subdirect product of limit groups over Droms RAAGs $\Gamma_1, \dots, \Gamma_m$ where each Γ_i has trivial center. Then where $p_i \colon H \mapsto \Gamma_i$ is the canonical projection.

(2) If H is a finitely presented residually Droms RAAG, then there exist finitely many subgroups $H_1, \dots, H_m$ of H such that

$$S(H) \setminus \Sigma^1(H) = \bigcup_{1 \leq i \leq m} S(H, H_i).$$

This result also shows that for a finitely presented residually Droms RAAG H the complement of the first invariant is a finite union of sub-spheres in the character sphere. Every sub-sphere $S(G, M)$, where M is a subgroup of an arbitrary finitely generated group G, is a finite intersection of closed rational semi-spheres of $S(G)$. As a consequence, from Corollary 4.2.9 we get that $S(H) \setminus \Sigma^1(H)$ is a finite union of finite intersections of closed, rationally defined semi-spheres of $S(G)$. In addition, we also see that the antipodality condition

$$S(H) \setminus \Sigma^1(H) = S(H) \setminus -\Sigma^1(H)$$

that naturally holds in the class of RAAGs also appears for finitely presented residually Droms RAAGs.

The last result of this chapter states that the Virtual Surjection Conjecture implies the discrete case of the Monoidal Virtual Surjection Conjecture. In particular, since the Virtual Surjection Conjecture holds in dimension 2 as Bridson, Howie, Miller and Short proved in [28], then we get that the discrete case of the Monoidal Virtual Surjection Conjecture is true in dimension 2.

Theorem 4.2.11. (1) Let $m \geq 2$, $1 \leq n \leq m$ and let $H < \Gamma_1 \times \cdots \times \Gamma_m$ be a full subdirect product of limit groups over Droms RAAGs $\Gamma_1, \ldots, \Gamma_m$ where each Γ_i has trivial center. Suppose that H is of type FP_n, finitely presented and that the Virtual Surjection Conjecture holds in dimension n. Then

$$[\chi] \in \Sigma^n(H, \mathbb{Q})_{dis} = \Sigma^n(H, \mathbb{Z})_{dis} = \Sigma^n(H)_{dis}$$

if and only if

$$p_{j_1,\ldots,j_n}(H_\chi) = p_{j_1,\ldots,j_n}(H) \quad \text{for all} \quad 1 \leq j_1 < \cdots < j_n \leq m, \tag{4.3}$$

where $p_{j_1,\ldots,j_n} : H \mapsto \Gamma_{j_1} \times \cdots \times \Gamma_{j_n}$ is the canonical projection. In particular, since the Virtual Surjection Conjecture holds in dimension 2, the result holds for $n = 2$.

(2) If H is a finitely presented residually Droms RAAG, then there exist finitely many subgroups $H_{i,j}$ of H, where $1 \leq i < j \leq n$, such that

$$S(H)_{dis} \setminus \Sigma^2(H)_{dis} = \bigcup_{1 \leq i < j \leq m} S(H, H_{i,j})_{dis}.$$

4.2 The Bieri-Neumann-Strebel-Renz invariants of limit groups over Droms RAAGs and their subdirect products

Let us start computing the Σ-invariants for limit groups over Droms RAAGs with trivial center.

Proposition 4.2.1. *Let Γ be a limit group over a Droms RAAG.*

(1) If Γ has trivial center, then $\Sigma^n(\Gamma) = \Sigma^n(\Gamma, \mathbb{Z}) = \Sigma^n(\Gamma, \mathbb{Q}) = \varnothing$ for every $n \geq 1$.

(2) In general, $\Sigma^n(\Gamma) = \Sigma^n(\Gamma, \mathbb{Z}) = \Sigma^n(\Gamma, \mathbb{Q}) = \{[\chi] \in S(\Gamma) \mid \chi(Z(\Gamma)) \neq 0\}$ for every $n \geq 1$.

Proof. (1) By [65, Lemma 29] if G is a free-by-(torsion-free nilpotent) group such that $\chi(G) < 0$ and the trivial $\mathbb{Q}G$-module $\mathbb{Q}$ has a free resolution of finite length and with finitely generated modules, then we have that $\Sigma^n(G, \mathbb{Q})$ and $\Sigma^n(G)$ are the empty set for every $n \geq 1$. Then it is enough to apply Lemma 3.3.6, Lemma 3.3.7 and the fact that for any group H of type F_n we have that

$$\Sigma^n(H) \subseteq \Sigma^n(H, \mathbb{Z}) \subseteq \Sigma^n(H, \mathbb{Q}).$$

(2) By the structure theory of limit groups over Droms RAAGs,

$$G \cong Z(G) \times G_0$$

where G_0 is a limit group over a Droms RAAG and G_0 has trivial center. By Lemma 1.4.2,

$$\{[\chi] \in S(G) \mid \chi(Z(G)) \neq 0\} \subseteq \Sigma^n(G) \subseteq \Sigma^n(G, \mathbb{Z}) \subseteq \Sigma^n(G, \mathbb{Q}).$$

The proof is completed by Lemma 4.2.8 and part (1). $\square$

We now prove the following result that implies that the forward direction of the Monoidal Virtual Surjection Conjecture holds for limit groups over Droms RAAGs with trivial center.

Proposition 4.2.2. *Let $m \geq 2$, $1 \leq n \leq m$ and let $H < \Gamma_1 \times \cdots \times \Gamma_m$ be a finitely generated full subdirect product. Assume further that*

(1) Γ_i is finitely generated and there is a free normal subgroup L_i of Γ_i such that Γ_i / L_i is polycyclic-by-finite for each $1 \leq i \leq m$;

(2) $N = L_1 \times \cdots \times L_m \subseteq H'$;

(3) for each i the Euler characteristic $\chi(\Gamma_i) < 0$ and there is a finite length free resolution of the trivial $\mathbb{Q}\Gamma_i$-module $\mathbb{Q}$ with all modules finitely generated;

(4) $[\chi] \in \Sigma^n(H, \mathbb{Q})$.

Then for any $1 \leq j_1 < \cdots < j_n \leq m$ we have that

$$\psi(H_\chi / N) = \psi(H/N),$$

where $\psi \colon H/N \mapsto \Gamma_{j_1}/L_{j_1} \times \cdots \times \Gamma_{j_n}/L_{j_n}$ is induced by $p_{j_1,\ldots,j_n}$.

Proof. By definition $[\chi] \in \Sigma^n(H, \mathbb{Q})$ is equivalent to $\mathbb{Q}$ being of type FP_n as a $\mathbb{Q}H_\chi$-module. Let

$$\mathcal{F}\colon \ \cdots \mapsto F_i \mapsto F_{i-1} \mapsto \cdots \mapsto F_0 \mapsto \mathbb{Q} \mapsto 0$$

be a free resolution of $\mathbb{Q}$ as a $\mathbb{Q}H_\chi$-module with F_i finitely generated for $i \leq n$. Then $\mathcal{F} \otimes_{\mathbb{Q}N} \mathbb{Q}$ is a complex whose modules $M_i = F_i \otimes_{\mathbb{Q}N} \mathbb{Q}$ are free $\mathbb{Q}[H_\chi/N]$-modules and M_i is finitely generated for each $i \leq n$.

Suppose first that χ is a discrete character. In [72, Lemma 5.2] the authors show that if H/N is polycyclic-by-finite and χ is a discrete character, then $\mathbb{Q}[H_\chi/N]$ is a Noetherian ring. Hence, for $i \leq n$ we have that M_i is a Noetherian $\mathbb{Q}[H_\chi/N]$-module. Thus, if $d_n\colon M_n \mapsto M_{n-1}$ is the differential of the complex $\mathcal{F} \otimes_{\mathbb{Q}N} \mathbb{Q}$, then we deduce that $\ker(d_n)$ is a finitely generated $\mathbb{Q}[H_\chi/N]$-module. Therefore,

$$H_n(N;\mathbb{Q}) \cong H_n(\mathcal{F} \otimes_{\mathbb{Q}N} \mathbb{Q}) = \ker(d_n)/\mathrm{im}(d_{n+1})$$

is finitely generated as a $\mathbb{Q}[H_\chi/N]$-module, so it is a Noetherian $\mathbb{Q}[H_\chi/N]$-module. On the other hand, by the Künneth formula

$$H_n(N;\mathbb{Q}) \cong \bigoplus_{1 \leq j_1 < \cdots < j_n \leq m} ((L_{j_1}/[L_{j_1}, L_{j_1}]) \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} ((L_{j_n}/[L_{j_n}, L_{j_n}]) \otimes_{\mathbb{Z}} \mathbb{Q})$$

and each direct summand is a $\mathbb{Q}[H/N]$-submodule, where the H/N action is induced by conjugation.

In particular, by Noetherianess,

$$W_{j_1,\ldots,j_n} = ((L_{j_1}/[L_{j_1}, L_{j_1}]) \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} ((L_{j_n}/[L_{j_n}, L_{j_n}]) \otimes_{\mathbb{Z}} \mathbb{Q})$$

is a finitely generated $\mathbb{Q}[H_\chi/N]$-submodule of $H_n(N;\mathbb{Q})$. Since the action of H_χ/N on $W_{j_1,\ldots,j_n}$ factors through $\psi(H_\chi/N)$, we deduce that

$$W_{j_1,\ldots,j_n} \text{ is a finitely generated } \mathbb{Q}[\psi(H_\chi/N)]\text{-module.} \tag{4.4}$$

Each Γ_i/L_i is polycyclic-by-finite, so there is a characteristic subgroup $\widehat{Q}_i$ of finite index in $Q_i = \Gamma_i/L_i$ that is torsion-free and polycyclic. By Proposition 3.3.4 condition (3) from the statement implies that each $L_{j_r}/[L_{j_r}, L_{j_r}] \otimes_{\mathbb{Z}} \mathbb{Q}$ contains a non-zero free cyclic $\mathbb{Q}\widehat{Q}_i$-submodule. Then

$$\mathbb{Q}[\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n}] \text{ embeds in } W_{j_1,\ldots,j_n}. \tag{4.5}$$

Let $\chi_0\colon H/N \mapsto \mathbb{R}$ be the discrete character induced by $\chi\colon H \mapsto \mathbb{R}$. Then there is $q_0 \in H/N$ such that $\chi(q_0) > 0$ and we have disjoint unions

$$H/N = \dot{\bigcup}_{\alpha\in\mathbb{Z}} q_0^\alpha \ker(\chi_0) \quad \text{and} \quad H_\chi/N = \dot{\bigcup}_{\alpha\geq 0} q_0^\alpha \ker(\chi_0).$$

Applying ψ,

$$\psi(H/N) = \bigcup_{\alpha\in\mathbb{Z}} \psi(q_0)^\alpha \psi(\ker(\chi_0)) \quad \text{and} \quad \psi(H_\chi/N) = \bigcup_{\alpha\geq 0} \psi(q_0)^\alpha \psi(\ker(\chi_0)),$$

but these last two unions are not necessarily disjoint.

Suppose that $\psi(H_\chi/N) \neq \psi(H/N)$. Then $\psi(q_0)^{-1} \notin \psi(H_\chi/N)$ and this implies that $\psi(H/N) = \psi(\ker(\chi_0)) \rtimes \langle\psi(q_0)\rangle$ and that there is a discrete character

$$\mu\colon K = \psi(H/N) \mapsto \mathbb{R}$$

with $\ker(\mu) = \psi(\ker(\chi_0))$ and $\mu(\psi(q_0)) = \chi_0(q_0)$.

Furthermore, note that $\psi(H_\chi/N) = K_\mu = \{k \in K \mid \mu(k) \geq 0\}$. Now (4.4) means that $W_{j_1,\ldots,j_n}$ is finitely generated as a $\mathbb{Q}K_\mu$-module. Let

$$K_1 = K \cap (\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n})$$

and $\mu_1 = \mu_{|K_1}\colon K_1 \mapsto \mathbb{R}$. Since $\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n}$ has finite index in $Q_{j_1} \times \cdots \times Q_{j_n}$, we have that $[K\colon K_1] < \infty$. Then, by [73, Theorem 9.3],

$$W_{j_1,\ldots,j_n} \text{ is finitely generated as a } \mathbb{Q}(K_1)_{\mu_1}\text{-module.} \tag{4.6}$$

Note that

$$\mathbb{Q}(K_1)_{\mu_1} = \mathbb{Q}[K_\mu \cap (\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n})],$$

and since μ_1 is a discrete character [72, Lemma 5.2] implies that $\mathbb{Q}(K_1)_{\mu_1}$ is a Noetherian ring. From the Noetherianess of $\mathbb{Q}(K_1)_{\mu_1}$, (4.5) and (4.6) we get that $\mathbb{Q}[\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n}]$ is finitely generated as a $\mathbb{Q}(K_1)_{\mu_1}$-module.

This easily leads to a contradiction. Indeed, by the fact that $\mathbb{Q}(K_1)_{\mu_1}$ is Noetherian and that $\mathbb{Q}K_1$ is a $\mathbb{Q}(K_1)_{\mu_1}$-submodule of $\mathbb{Q}[\widehat{Q}_{j_1} \times \cdots \times \widehat{Q}_{j_n}]$ we deduce that $\mathbb{Q}K_1$ is finitely generated as a $\mathbb{Q}(K_1)_{\mu_1}$-module, which is false for a character μ_1.

The fact that the case of a discrete character implies the general case follows from [72, Subsection 6.1]. $\square$

As we shall apply Proposition 4.2.2 to our case, the next step is to show that there is a quotient of Γ_i which is polycyclic-by-finite.

Proposition 4.2.3. *Let $m \geq 2$, $1 \leq n \leq m$ and let $H < \Gamma_1 \times \cdots \times \Gamma_m$ be a finitely presented full subdirect product of limit groups over Droms RAAGs $\Gamma_1, \ldots, \Gamma_m$ where each Γ_i has trivial center. Then, for each $1 \leq i \leq m$, there exists a free normal subgroup $F_i < \Gamma_i$ such that $F_i \subseteq H'$ and Γ_i/F_i is a polycyclic-by-finite group.*

Proof. Recall the notation from Chapter 2. If $p_i\colon H \mapsto \Gamma_i$ is the canonical projection map, then $N_{i,j}$ is defined to be $p_j(\ker(p_i))$ and N_j is

$$\bigcap_{i \neq j} N_{i,j}.$$

From Section 2.6 we get that

$$\gamma_{m-1}(N_j) \subseteq [N_{1,j}, \ldots, N_{j-1,j}, N_{j+1,j}, \ldots, N_{m,j}] \subseteq H \quad \text{and} \quad [\Gamma_j : N_j] < \infty,$$

so that $\Gamma_j/\gamma_{m-1}(N_j)$ is nilpotent-by-finite. Moreover, Proposition 3.2.12 states that limit groups over Droms RAAGs are free-by-(torsion-free nilpotent), so there is a free normal subgroup $\widetilde{L}_j < \Gamma_j$ such that $\Gamma_j/\widetilde{L}_j$ is torsion-free nilpotent. Let us define $\widehat{L}_j$ to be the group

$$\gamma_{m-1}(N_j) \cap \widetilde{L}_j \subseteq H.$$

Then $\Gamma_j/\widehat{L}_j$ is nilpotent-by-finite. Set $\widehat{N}$ to be $\widehat{L}_1 \times \cdots \times \widehat{L}_m \subseteq H$ and finally define L_j to be

$$\widehat{L}_j \cap (\widehat{N} \cap H') = \widehat{L}_j \cap H'.$$

Since $\widehat{L}_j$ is normal in Γ_j and $p_j(H) = \Gamma_j$, then L_j is also normal in Γ_j and

$$\widehat{L}_j/L_j \cong \widehat{L}_j/(\widehat{L}_j \cap H')$$

is isomorphic to a subgroup of H/H'. In particular, it is a finitely generated abelian group. In conclusion, Γ_j/L_j is polycyclic-by-finite. $\square$

Theorem 4.2.4. *Let $m \geq 2$, $1 \leq n \leq m$ and let $H < \Gamma_1 \times \cdots \times \Gamma_m$ be a finitely presented full subdirect product of limit groups over Droms RAAGs $\Gamma_1, \ldots, \Gamma_m$ where each Γ_i has trivial center. Suppose that $[\chi] \in \Sigma^n(H, \mathbb{Q})$. Then*

$$p_{j_1,\ldots,j_n}(H_\chi) = p_{j_1,\ldots,j_n}(H) \quad \textit{for all} \quad 1 \leq j_1 < \cdots < j_n \leq m, \tag{4.7}$$

where $p_{j_1,\ldots,j_n}\colon H \mapsto \Gamma_{j_1} \times \cdots \times \Gamma_{j_n}$ is the canonical projection.

Proof. It suffices to show that we can apply Proposition 4.2.2. Conditions (1) and (2) are Proposition 4.2.3 and condition (3) is Lemma 3.3.6 and Lemma 3.3.7. □

The next step is to prove Corollary 4.2.9. For that, we start with a technical result.

Lemma 4.2.5. *Let $H < \Gamma_1 \times \cdots \times \Gamma_m$ be a subdirect product. Assume further that*

(1) the group Γ_i is finitely generated and there is a free normal subgroup L_i of Γ_i such that Γ_i / L_i is polycyclic-by-finite for each $1 \leq i \leq m$;

(2) $N = L_1 \times \cdots \times L_m \subseteq H'$.

Then

$$\{[\chi] \in S(H) \mid p_i(H_\chi) = p_i(H) = \Gamma_i \text{ for every } 1 \leq i \leq m\} \subseteq \Sigma^1(H).$$

Proof. Let us check that under the above conditions the group H is finitely generated. Indeed, fix a finite subset $A_i \subseteq L_i$ such that $L_i = \langle\langle A_i \rangle\rangle_{\Gamma_i}$ and a finite subset $B_i \subseteq H$ such that $\Gamma_i = p_i(H) = \langle p_i(B_i) \rangle$. Then

$$L_i = \langle\langle A_i \rangle\rangle_{\Gamma_i} = \langle\langle A_i \rangle\rangle_{p_i(H)} = \langle\langle A_i \rangle\rangle_{\langle p_i(B_i) \rangle} = \langle\langle A_i \rangle\rangle_{\langle B_i \rangle} \subseteq \langle A_i \cup B_i \rangle$$

and this implies that

$$N = L_1 \times \cdots \times L_m \subseteq \Big\langle \bigcup_{1 \leq i \leq m} (A_i \cup B_i) \Big\rangle \subseteq H.$$

In addition, H/N is a subgroup of the polycyclic-by-finite group

$$\Gamma_1 / L_1 \times \cdots \times \Gamma_m / L_m,$$

hence H/N is finitely generated, so there is a finite subset $C \subseteq H$ such that $H = N\langle C \rangle$. Thus $C \cup \bigcup_{1 \leq i \leq m} (A_i \cup B_i)$ is a finite generating set for H.

By [16] we have that $[\chi] \in \Sigma^1(H) = \Sigma^1(H, \mathbb{Z})$ if there is a finitely generated submonoid M of H_χ such that H' is finitely generated as a M-group, where M acts via conjugation. Therefore, we just need to show that if $[\chi] \in S(H)$ and $p_i(H_\chi) = p_i(H) = \Gamma_i$ for every $1 \leq i \leq m$, then H' is finitely generated as a M-group for some finitely generated submonoid M of H_χ.

Since $p_i(H_\chi) = \Gamma_i$ and each Γ_i is finitely generated, there is a finitely generated monoid M such that $M \subseteq H_\chi$ and $p_i(M) = \Gamma_i$ for $1 \le i \le m$. Then,

$$L_i = \langle\langle A_i \rangle\rangle_{\Gamma_i} = \langle\langle A_i \rangle\rangle_{p_i(M)} \quad \text{and} \quad N = L_1 \times \cdots \times L_m \subseteq \big\langle\big\langle \bigcup_{1 \le i \le m} A_i \big\rangle\big\rangle_M \subseteq H'.$$

Finally, since H'/N is a subgroup of the polycyclic-by-finite group H/N, we deduce that it is finitely generated, so there is a finite subset D of H' such that $H' = N\langle D \rangle$. Hence,

$$H' = \big\langle\big\langle \big(\bigcup_{1 \le i \le m} A_i \big) \cup D \big\rangle\big\rangle_M.$$

□

In [72] the following result is proved.

Theorem 4.2.6. *[72, Theorem B] Let $H < \Gamma_1 \times \cdots \times \Gamma_m$ be a finitely presented full subdirect product of non-abelian limit groups $\Gamma_1, \ldots, \Gamma_m$ with $m \ge 1$. Then*

$$\Sigma^1(H) = \Sigma^1(H, \mathbb{Q}) = \{[\chi] \in S(H) \mid p_i(H_\chi) = p_i(H) = \Gamma_i \textit{ for every } 1 \le i \le m\}.$$

We show that the above result works in a more general setting.

Proposition 4.2.7. *Let $H < \Gamma_1 \times \cdots \times \Gamma_m$ be a finitely generated full subdirect product. Assume further that*

(1) the group Γ_i is finitely generated and there is a free normal subgroup L_i of Γ_i such that Γ_i/L_i is polycyclic-by-finite for each $1 \le i \le m$;

(2) $N = L_1 \times \cdots \times L_m \subseteq H'$;

(3) for each i the Euler characteristic $\chi(\Gamma_i) < 0$ and there is a finite length free resolution of the trivial $\mathbb{Q}\Gamma_i$-module $\mathbb{Q}$ with all modules finitely generated.

Then

$$\Sigma^1(H) = \Sigma^1(H, \mathbb{Q}) = \{[\chi] \in S(H) \mid p_i(H_\chi) = p_i(H) = \Gamma_i \textit{ for every } 1 \le i \le m\}.$$

Proof. For a general group G we have that $\Sigma^1(G) = \Sigma^1(G, \mathbb{Z}) \subseteq \Sigma^1(G, \mathbb{Q})$. Applying Proposition 4.2.2 for $n = 1$ we have that

$$\Sigma^1(H) \subseteq \Sigma^1(H, \mathbb{Q}) \subseteq \{[\chi] \in S(H) \mid p_i(H_\chi) = p_i(H) = \Gamma_i \text{ for every } 1 \le i \le m\}.$$

The converse

$$\{[\chi] \in S(H) \mid p_i(H_\chi) = p_i(H) = \Gamma_i \text{ for every } 1 \le i \le m\} \subseteq \Sigma^1(H)$$

is Lemma 4.2.5. $\square$

Lemma 4.2.8. *[11, Proposition 2.7] Let H be a group with a normal subgroup A of type F_∞, let $\chi\colon H \mapsto \mathbb{R}$ be a character such that $\chi(A) = 0$ and let χ_0 be the character induced by χ, $\chi_0\colon \overline{H} = H/A \mapsto \mathbb{R}$.*

(1) If H is of type FP_n, then $[\chi] \in \Sigma^n(H, \mathbb{Z})$ if and only if $[\chi_0] \in \Sigma^n(\overline{H}, \mathbb{Z})$.

(2) If H is of type F_n, then $[\chi] \in \Sigma^n(H)$ if and only if $[\chi_0] \in \Sigma^n(\overline{H})$.

We now have the necessary tools to prove Corollary 4.2.9.

Corollary 4.2.9. *(1) Let $H < \Gamma_1 \times \cdots \times \Gamma_m$ be a finitely presented full subdirect product of limit groups over Droms RAAGs $\Gamma_1, \ldots, \Gamma_m$ where each Γ_i has trivial center. Then where $p_i\colon H \mapsto \Gamma_i$ is the canonical projection.*

(2) If H is a finitely presented residually Droms RAAG, then there exist finitely many subgroups $H_1, \ldots, H_m$ of H such that

$$S(H) \setminus \Sigma^1(H) = \bigcup_{1 \le i \le m} S(H, H_i).$$

Proof. (1) The first equality is a corollary of Proposition 4.2.7. The fact that the hypothesis (1), (2) and (3) from Proposition 4.2.7 hold are established in Proposition 4.2.3, Lemma 3.3.6 and Lemma 3.3.7.

In [72, Lemma 5.10] it is shown that $p_i(H_\chi) = p_i(H)$ is equivalent to

$$\chi(\ker(p_i)) \neq 0.$$

Then $[\chi] \in S(H) \setminus \Sigma^1(H)$ if and only if $[\chi] \in S(H, \ker(p_i))$.

(2) A finitely presented residually Droms RAAG H may be viewed as a full subdirect product $H < \Gamma_0 \times \Gamma_1 \times \cdots \times \Gamma_m$, where $\Gamma_0 = \mathbb{Z}^k$ for some $k \geq 0$ and $\Gamma_1, \ldots, \Gamma_m$ are limit groups over Droms RAAGs with trivial center for $1 \leq i \leq m$. Furthermore, we can assume that $\Gamma_0 \cap H$ has finite index in Γ_0 (see Theorem 2.9.1). By construction, $H \cap \Gamma_0$ is central in H, so by Lemma 1.4.2

$$\{[\chi] \in S(H) \mid \chi(H \cap \Gamma_0) \neq 0\} \subseteq \Sigma^1(H).$$

Thus, if $[\chi] \in S(H) \setminus \Sigma^1(H)$, then we have that $\chi(H \cap \Gamma_0) = 0$ and χ induces a character

$$\chi_0\colon \overline{H} = H/(H \cap \Gamma_0) \mapsto \mathbb{R}.$$

Note that by Lemma 4.2.8

$$[\chi] \in S(H) \setminus \Sigma^1(H) \text{ if and only if } [\chi_0] \in S(\overline{H}) \setminus \Sigma^1(\overline{H}).$$

Since $\overline{H} < \Gamma_1 \times \cdots \times \Gamma_m$ is a full subdirect product, we can apply part (1) to deduce that

$$S(\overline{H}) \setminus \Sigma^1(\overline{H}) = \bigcup_{1 \le i \le m} S(\overline{H}, \ker(\overline{p}_i)),$$

where $\overline{p}_i \colon \overline{H} \mapsto \Gamma_i$ is the canonical projection. Hence, if we take H_i to be the preimage of $\ker(\overline{p}_i)$ in H, we have that

$$S(H) \setminus \Sigma^1(H) = \bigcup_{1 \le i \le m} S(H, H_i).$$

□

The last part of the chapter is related to proving Theorem 4.2.11, where we show that the Virtual Surjection Conjecture implies the discrete case of the Monoidal Virtual Surjection Conjecture, so in particular, the discrete case of the Monoidal Virtual Surjection Conjecture holds in dimension 2.

Theorem 4.2.10. *Let $m \geq 2$, $1 \leq n \leq m$ and let $H < \Gamma_1 \times \cdots \times \Gamma_m$ be a finitely presented full subdirect product of type F_n. Assume further that*

(1) there is a free normal subgroup L_i of Γ_i such that Γ_i / L_i is polycyclic-by-finite for each $1 \leq i \leq m$;

(2) $N = L_1 \times \cdots \times L_m \subseteq H'$;

(3) for each i the Euler characteristic $\chi(\Gamma_i) < 0$ and there is a finite length free resolution of the trivial $\mathbb{Q}\Gamma_i$-module $\mathbb{Q}$ with all modules finitely generated;

(4) every finitely generated subgroup of Γ_i is of type F_n for each $1 \leq i \leq m$;

(5) the Virtual Surjection Conjecture holds in dimension n.

Then

$$[\chi] \in \Sigma^n(H, \mathbb{Q})_{dis} = \Sigma^n(H, \mathbb{Z})_{dis} = \Sigma^n(H)_{dis}$$

if and only if

$$p_{j_1,\ldots,j_n}(H_\chi) = p_{j_1,\ldots,j_n}(H) \quad \text{for all} \quad 1 \leq j_1 < \cdots < j_n \leq m, \tag{4.8}$$

where $p_{j_1,\ldots,j_n} \colon H \mapsto \Gamma_{j_1} \times \cdots \times \Gamma_{j_n}$ is the canonical projection.

Proof. Suppose that $[\chi] \in \Sigma^n(H,\mathbb{Q})_{dis}$. Then, by Proposition 4.2.2,

$$p_{j_1,\dots,j_n}(H_\chi) = p_{j_1,\dots,j_n}(H).$$

For the converse, assume that $\chi\colon H \mapsto \mathbb{R}$ is a discrete character and that

$$p_{j_1,\dots,j_n}(H_\chi) = p_{j_1,\dots,j_n}(H) \quad \text{for all} \quad 1 \le j_1 < \cdots < j_n \le m.$$

Since in general we have that $\Sigma^n(H) \subseteq \Sigma^n(H,\mathbb{Z}) \subseteq \Sigma^n(H,\mathbb{Q})$, it suffices to show that $[\chi] \in \Sigma^n(H)$. If we show that $N_0 = \ker(\chi)$ is of type F_n, then by Theorem 1.4.1 we have that both $[\chi]$ and $[-\chi]$ belong to $\Sigma^n(H)$.

Note that by Lemma 4.2.5 the result holds for $n = 1$. Furthermore, $p_j(H_\chi) = p_j(H)$ is equivalent to $\chi(Ker(p_j)) \neq 0$ (see [72, Lemma 5.10]), so $p_j(H_\chi) = p_j(H)$ if and only if $p_j(H_{-\chi}) = p_j(H)$. Thus, applying Lemma 4.2.5 again we have that $\{[\chi],[-\chi]\} \subseteq \Sigma^1(H)$. Since χ is a discrete character we deduce by Theorem 1.4.1 that N_0 is finitely generated.

Let us write $H = N_0 \rtimes \langle t \rangle$ with $\chi(t) > 0$. Since

$$p_{j_1,\dots,j_n}(H) = p_{j_1,\dots,j_n}(H_\chi) = \bigcup_{i \ge 0} p_{j_1,\dots,j_n}(N_0) p_{j_1,\dots,j_n}(t)^i$$

is a group, $p_{j_1,\dots,j_n}(N_0)$ has finite index in $p_{j_1,\dots,j_n}(H)$. The group H is of type FP_n, so by Theorem 3.3.8, $p_{j_1,\dots,j_n}(H)$ has finite index in $\Gamma_{j_1} \times \cdots \times \Gamma_{j_n}$. Hence, $p_{j_1,\dots,j_n}(N_0)$ has also finite index in $\Gamma_{j_1} \times \cdots \times \Gamma_{j_n}$. In particular, $p_{j_1,\dots,j_n}(N_0)$ has finite index in $p_{j_1}(N_0) \times \cdots \times p_{j_n}(N_0)$.

We consider N_0 as a subdirect product of $p_1(N_0) \times \cdots \times p_m(N_0)$. The group N_0 is finitely generated, so $p_i(N_0)$ is a finitely generated subgroup of Γ_i. Then, condition (4) ensures that $p_i(N_0)$ is of type F_n. Thus, if the Virtual Surjection Conjecture holds in dimension n for the subdirect product N_0, then we deduce that N_0 is of type F_n. ∎

Theorem 4.2.11. *(1) Let $m \ge 2$, $1 \le n \le m$ and let $H < \Gamma_1 \times \cdots \times \Gamma_m$ be a full subdirect product of limit groups over Droms RAAGs $\Gamma_1, \dots, \Gamma_m$ where each Γ_i has trivial center. Suppose that H is of type FP_n, finitely presented and that the Virtual Surjection Conjecture holds in dimension n. Then*

$$[\chi] \in \Sigma^n(H,\mathbb{Q})_{dis} = \Sigma^n(H,\mathbb{Z})_{dis} = \Sigma^n(H)_{dis}$$

if and only if

$$p_{j_1,\dots,j_n}(H_\chi) = p_{j_1,\dots,j_n}(H) \quad \textit{for all} \quad 1 \le j_1 < \cdots < j_n \le m, \tag{4.9}$$

where $p_{j_1,\dots,j_n} \colon H \mapsto \Gamma_{j_1} \times \cdots \times \Gamma_{j_n}$ *is the canonical projection. In particular, since the Virtual Surjection Conjecture holds in dimension* 2*, the result holds for* $n = 2$.

(2) If H *is a finitely presented residually Droms RAAG, then there exist finitely many subgroups* $H_{i,j}$ *of* H*, where* $1 \le i < j \le n$*, such that*

$$S(H)_{dis} \setminus \Sigma^2(H)_{dis} = \bigcup_{1 \le i < j \le m} S(H, H_{i,j})_{dis}.$$

Proof. (1) This follows from Theorem 4.2.10.

(2) A finitely presented residually Droms RAAG H may be viewed as a full subdirect product $H < \Gamma_0 \times \Gamma_1 \times \cdots \times \Gamma_m$, where $\Gamma_0 = \mathbb{Z}^k$ for some $k \ge 0$ and $\Gamma_1, \dots, \Gamma_m$ are limit groups over Droms RAAGs with trivial center for $1 \le i \le m$ and $\Gamma_0 \cap H$ has finite index in Γ_0 (see Theorem 2.9.1). Since $H \cap \Gamma_0$ is central in H, by Lemma 1.4.2

$$\{[\chi] \in S(H) \mid \chi(H \cap \Gamma_0) \ne 0\} \subseteq \Sigma^n(H).$$

If $[\chi] \in S(H) \setminus \Sigma^2(H)$, then we have that $\chi(H \cap \Gamma_0) = 0$ and χ induces a character

$$\chi_0 \colon \overline{H} = H/(H \cap \Gamma_0) \mapsto \mathbb{R}.$$

It follows from Lemma 4.2.8 that

$$[\chi] \in S(H) \setminus \Sigma^2(H) \text{ if and only if } [\chi_0] \in S(\overline{H}) \setminus \Sigma^2(\overline{H}).$$

Since $\overline{H} < \Gamma_1 \times \cdots \times \Gamma_m$, we can apply part (1) to deduce that

$$[\chi_0] \in S(\overline{H})_{dis} \setminus \Sigma^2(\overline{H})_{dis} \text{ if and only if } p_{i,j}(\overline{H}) \ne p_{i,j}(\overline{H}_{\chi_0}) \text{ for some } 1 \le i < j \le m,$$

where $\overline{p}_{i,j} \colon \overline{H} \mapsto \Gamma_i \times \Gamma_j$ is the canonical projection. By [72, Lemma 5.10] this is again equivalent to $\chi_0(\ker(\overline{p}_{i,j})) = 0$. Then, we can define $H_{i,j}$ as the preimage of $\ker(\overline{p}_{i,j})$ in H and obtain that

$$S(H)_{dis} \setminus \Sigma^2(H)_{dis} = \bigcup_{1 \le i \le m} S(H, H_{i,j})_{dis}.$$

□

Chapter 5

Subgroups of the direct product of fundamental groups of graphs of groups with free abelian vertex groups

5.1 Introduction and outline

In this chapter we work with 2-dimensional coherent RAAGs. Recall from Section 1.3.3 that 2-dimensional coherent RAAGs are defined by graphs which are forests, that is they are free products of tree groups. The main example to bear in mind is the RAAG associated to the path with four vertices, P_4.

Tree groups are of special interest to us for a couple of reasons. Droms showed in [46] that tree groups with trivial center are not Droms RAAGs, meaning that they have finitely generated subgroups that are not RAAGs. Therefore, their study becomes more complicated than in the case of Droms RAAGs. Nevertheless, they are coherent RAAGs (see [44]), so the class of tree groups is an appropriate starting point to study the structure of finitely presented subgroups of direct products of coherent RAAGs.

Furthermore, they are essential examples of 3-manifold groups that are also RAAGs. Indeed, Droms proved in [44] that the RAAG GX is the fundamental group of a 3-manifold if and only if each connected component of X is either a tree or a triangle. Hence, GX is the free product of tree groups and free abelian groups of rank three. For instance, the RAAG P_4 is the figure 8 knot group.

In this chapter we generalise Baumslag and Roseblade's result for free groups and we describe the structure of finitely presented subgroups of the direct product of two 2-dimensional coherent RAAGs.

Theorem. *Let S be a finitely presented subgroup of the direct product of two 2-dimensional coherent RAAGs. Then S is virtually H-by-(free abelian), where H is the direct product of two subgroups of 2-dimensional coherent RAAGs.*

The main reason for this new behaviour comes from the fact that tree groups fiber, while free groups and Droms RAAGs with trivial center do not. Thus, on the one hand, these groups have normal subgroups which are not of finite index, and on the other hand, the intersections of the subgroup with each of the factors need not be finitely generated.

Furthermore, we show that these finitely presented subgroups have a good algorithmic behaviour. Namely, we prove the following:

Corollary. *Finitely presented subgroups of the direct product of two 2-dimensional coherent RAAGs have decidable multiple conjugacy problem and membership problem.*

This corollary shows that Bridson's example of a RAAG A and an algorithmically bad finitely presented subgroup of $A \times A$ is not 2-dimensional coherent.

In fact, our results apply to a wider class of groups, the class $\mathcal{A}$, which is the $Z*$-closure of the class $\mathcal{G}$ (see Definition 1.3.3). The class $\mathcal{G}$ is the class of cyclic subgroup separable fundamental groups of graphs of groups with free abelian vertex groups and cyclic edge groups. This class contains 2-dimensional coherent RAAGs and residually finite tubular groups among others.

A *tubular group* is a finitely generated fundamental group of a graph of groups with $\mathbb{Z}^2$ vertex groups and $\mathbb{Z}$ edge groups. Despite their simple definition, they have a surprisingly rich source of diverse behaviour. Tubular groups provide examples of finitely generated 3-manifold groups that are not subgroup separable; of free-by-cyclic groups that do not act properly and semi-simply on a CAT(0) space; of groups that are CAT(0) but not Hopfian, etc (see [58] and references there).

Our main result is the following:

Theorem 5.5.1. Let S be a finitely presented subgroup of $G_1 \times G_2$ where $G_1, G_2 \in \mathcal{A}$ are finitely generated. Then, either

(1) there is $S_0 <_{fi} S$ and a central extension $1 \to \mathbb{Z}^n \to S_0 \to H \to 1$ for some $n \in \mathbb{N}$ and $H \in \mathcal{G}$ finitely generated; or

(2) S is virtually H-by-(free abelian), where H is the direct product of two groups in $\mathcal{A}$.

Furthermore, in the second case S is virtually the kernel of a homomorphism f from $H_1 \times H_2$ to $\mathbb{R}$ for some finitely generated $H_1, H_2 \in \mathcal{A}$. More precisely, H is equal to $M_1 \times M_2$, and either

(1) $M_1 \times M_2 <_{fi} S <_{fi} G_1 \times G_2$; or

(2) S is virtually the kernel $\ker f$ where $f\colon H_1 \times H_2 \mapsto \mathbb{Z}^n$ for some $n \in \{1, 2\}$ and for some finitely generated $H_i \in \mathcal{A}$.

As a consequence, we obtain that a finitely presented subgroup S may be viewed virtually as a kernel of a homomorphism to a free abelian group. Hence, methods from Σ-theory can be used to show in Section 5.7 that S is of type F_∞. Actually, we prove a more general result. Let $\mathcal{D}$ be the class of finitely generated fundamental groups of graphs of groups with free abelian vertex groups and cyclic edge groups such that the groups are not ascending HNN extensions, and let $\mathcal{J}$ be the $Z*$-closure of $\mathcal{D}$.

Proposition 5.7.2. Let S be a finitely presented co-abelian subgroup of $H_1 \times H_2$ with $H_1, H_2 \in \mathcal{J}$. Then S is of type F_∞.

The abelian factor in the description of S is directly related to the edge groups of the decomposition of the groups in our class. In particular, if we consider fundamental groups of graphs of groups in $\mathcal{A}$ with a trivial edge group, we deduce the following theorem and recover the result of Baumslag and Roseblade for direct products of free groups:

Theorem. *Let $\mathcal{A}'$ be the subclass of $\mathcal{A}$ containing the groups which have a non-trivial free product decomposition and let S be a finitely presented subgroup of the direct product of two finitely generated groups in the class $\mathcal{A}'$. Then S is virtually the direct product of two groups in $\mathcal{A}'$.*

We work with 2-dimensional coherent RAAGs because they split as amalgamated free products where the edge groups are infinite cyclic. General coherent RAAGs split as amalgamated free products where the vertex groups are free abelian, but the edge groups might have rank (possibly) greater than one. For 2-dimensional coherent RAAGs we first show that the finitely presented subgroup is an extension of a direct product by a $\mathbb{Z}$-by-$\mathbb{Z}$ group. We then prove that this $\mathbb{Z}$-by-$\mathbb{Z}$ group is a quotient of a Baumslag-Solitar group, and using cyclic subgroup separability and the structural theory of Baumslag-Solitar groups we conclude that the $\mathbb{Z}$-by-$\mathbb{Z}$ group is in fact free abelian. In the general case of coherent RAAGs, the current proof would need to study groups of the form $\mathbb{Z}^m$-by-$\mathbb{Z}^n$. One could probably reduce the problem to the study of the Leary-Minasyan groups, but still we would need to develop some structural results for these groups.

The chapter is organised as follows. In Section 5.2 we introduce the class of groups that we will study and describe some properties of these groups.

In Section 5.3 we review Miller's proof for free groups. The idea is to first show that if S is a finitely presented subgroup of the direct product of two free groups F_1 and F_2, then the subgroups $L_i = S \cap F_i$ are finitely generated, and after that use the fact that non-trivial finitely generated normal subgroups of free groups have finite index to conclude that the direct product $L_1 \times L_2$ has finite index in S.

When considering a finitely presented subgroup S of the direct product of two 2-dimensional coherent RAAGs, say G_1 and G_2, the intersections $S \cap G_1$ and $S \cap G_2$ are not necessarily finitely generated. Furthermore, non-trivial finitely generated normal subgroups of coherent RAAGs do not need to be of finite index. Indeed, coherent RAAGs fiber, that is they admit non-trivial epimorphisms onto $\mathbb{Z}$ with finitely generated kernel.

We address these issues in Section 5.4. We show that although the subgroup L_i may not be finitely generated, a cyclic extension of L_i is (see Proposition 5.4.5). We also characterise finitely generated normal subgroups N of a group G in $\mathcal{G}$: either N is in the center of the group G or G/N is virtually cyclic (see Proposition 5.4.4).

Finally, in Section 5.5 the main result is proved. We show that the quotient $Q_i = G_i/L_i$ is covered by finitely many cosets of the product of two cyclic subgroups and that this covering lifts to a covering of a Baumslag-Solitar group. We use the structure of Baumslag-Solitar groups to deduce that it is free abelian and conclude that Q_i is virtually free abelian.

In Section 5.6 we apply the main theorem to elucidate the algorithmic structure of finitely presented subgroups of the direct product of two 2-dimensional coherent RAAGs. We conclude the chapter with Section 5.7 where we use methods from Σ-theory to prove that these finitely presented subgroups are, in fact, of type F_∞.

5.2 The class of groups $\mathcal{A}$

Given a class of groups, recall from the background that there is a natural way to construct other groups using operations such as taking free products or adding center.

Definition 5.2.1. Let $\mathcal{C}$ be a class of groups. The *$Z*$-closure of* $\mathcal{C}$, denoted by $Z*(\mathcal{C})$, is the union of classes $(Z*(\mathcal{C}))_k$ defined as follows. At level 0, the class

$(Z*(\mathcal{C}))_0$ is $\mathcal{C}$. A group G lies in $(Z*(\mathcal{C}))_k$ if and only if

$$G \cong \mathbb{Z}^m \times (G_1 * \cdots * G_n),$$

where $m \in \mathbb{N} \cup \{0\}$ and the group G_i lies in $(Z*(\mathcal{C}))_{k-1}$ for all $i \in \{1, \ldots, n\}$.

The *level of* G, denoted by $l(G)$, is the smallest k for which G belongs to $(Z*(\mathcal{C}))_k$.

The class of groups $\mathcal{A}$ is defined as the $Z*$-closure of the class of groups $\mathcal{G}$, defined as follows.

Definition 5.2.2. Let $\mathcal{G}$ be the class of cyclic subgroup separable fundamental groups of graphs of groups with free abelian vertex groups and cyclic edge groups.

The class of groups $\mathcal{A}$ is the $Z*$-closure of the class $\mathcal{G}$.

Here we are not assuming that the groups in $\mathcal{G}$ are finitely generated.

Remark 5.2.3. Suppose that $G \in \mathcal{G}$ is finitely generated and that all the vertex groups and all the edge groups are infinite cyclic. Then G is a cyclic subgroup separable Generalised Baumslag-Solitar group. Therefore, G is virtually $\mathbb{Z} \times F$ with F free (see [71, Section 2.1]). Hence, if S is a finitely presented subgroup of $G \times A$ where A is any group, then $S \cap ((\mathbb{Z} \times F) \times A)$ has finite index in S and $\mathbb{Z} \times F \in \mathcal{A}$.

Therefore, for the purposes of this chapter we may assume that if $G \in \mathcal{G}$ is finitely generated, then G is a non-trivial free product or if G is freely indecomposable, that there is a vertex group which is a free abelian group of rank greater than 1.

Definition 5.2.4. Let G be a group in the class $\mathcal{G}$. Any splitting of G as a fundamental group of a graph of groups with free abelian vertex groups and cyclic edge groups is called a *standard splitting of* G.

Since subgroups of cyclic subgroup separable groups are again cyclic subgroup separable, then $\mathcal{G}$ is closed under taking subgroups: if $G \in \mathcal{G}$ and $H < G$, then $H \in \mathcal{G}$. In addition, it follows from [30, Theorem 3.6, Proposition 4.1] that cyclic subgroup separability extends from the class $\mathcal{G}$ to the class $\mathcal{A}$.

The class $\mathcal{G}$, and therefore $\mathcal{A}$, contains interesting families of groups. For instance, 2-dimensional coherent RAAGs and residually finite tubular groups.

The RAAG defined by the path with 4 vertices P_4 and which, abusing notation, we also denote by P_4 is a tree group. It is given by the presentation

$$\langle a, b, c, d \mid ab = ba, bc = cb, cd = dc \rangle,$$

and it admits the following splitting as a fundamental group of a graph of groups with free abelian vertex groups and cyclic edge groups:

$$P_4 = \langle a, b\rangle *_{\langle b\rangle} \langle b, c\rangle *_{\langle c\rangle} \langle c, d\rangle.$$

Moreover, coherent RAAGs are cyclic subgroup separable (see, for instance, [48]). Therefore, $P_4 \in \mathcal{G}$.

The RAAG P_4 plays an important role in the theory of 2-dimensional coherent RAAGs as it serves as universe for them. Indeed, in [63, Theorem 7] Kim and Koberda show that any 2-dimensional coherent RAAG embeds in the group P_4. In particular, since the class $\mathcal{G}$ is closed under subgroups, we deduce that 2-dimensional coherent RAAGs also belong to $\mathcal{G}$.

In fact, all tree groups are examples of residually finite tubular groups. Cyclic subgroup separability is a stronger residual property than residual finiteness, but in [58] it is shown that for tubular groups these conditions are equivalent. From this characterisation we have that residually finite tubular groups belong to $\mathcal{G}$.

5.3 Miller's proof and counterexamples in tree groups

Recall that Baumslag and Roseblade's result states that given F_1 and F_2 two finitely generated free groups and S a finitely presented subgroup of $F_1 \times F_2$, then S is free or S is virtually the direct product of two free groups.

We now briefly sketch Miller's strategy to highlight the relevant properties of free groups that are used in his proof of the aforementioned result.

First of all, we can reduce to the case when S is a subdirect product. Indeed, if we consider the projection maps $p_1 \colon S \mapsto F_1$ and $p_2 \colon S \mapsto F_2$, then $p_i(S)$ is a finitely generated free group for $i \in \{1, 2\}$, so we can assume that the projection maps are surjective.

Let us define L_i to be $S \cap F_i$, $i \in \{1, 2\}$. Observe that there is a short exact sequence

$$1 \longrightarrow L_2 \longrightarrow S \xrightarrow{p_1} F_1 \longrightarrow 1.$$

If L_2 is trivial, then S is isomorphic to F_1 and so S is free. A symmetric argument applies if L_1 is trivial.

Now assume that L_1 and L_2 are both non-trivial. Miller then proves, by using Marshall Hall's theorem for free groups, that for $i \in \{1, 2\}$ S is virtually an HNN extension with associated subgroup L_i and since S is finitely presented, then L_i needs to be finitely generated (see [79, Lemma 2, Theorem 1]). A similar argument can be read in the proof of Proposition 5.5.4.

From here one deduces that for $i \in \{1, 2\}$, L_i is a non-trivial finitely generated normal subgroup of F_i, so L_i has finite index in F_i. Hence, since $L_1 \times L_2$ is a subgroup of S, then $L_1 \times L_2$ has finite index in S.

Summarising, the key points in Miller's proof are the following ones:

(1) Subgroups of free groups are free;

(2) if S is finitely presented, then the groups L_1 and L_2 are finitely generated;

(3) finitely generated non-trivial normal subgroups of free groups are of finite index.

In the rest of the section we show that none of the above conditions necessarily hold for tree groups. For that, we use the group P_4.

Firstly, let us give an example to show that non-trivial finitely generated normal subgroups do not need to have finite index in tree groups. Consider the homomorphism $\varphi\colon P_4 \mapsto \mathbb{Z}$ defined as

$$\varphi(a) = \varphi(b) = \varphi(c) = \varphi(d) = 1,$$

and let S be the kernel of that homomorphism. By Corollary 1.4.12, since the graph $\widehat{\mathcal{L}}_\varphi = P_4$ is connected, then S is indeed finitely generated. Hence, S is a finitely generated non-trivial normal subgroup in P_4 but P_4/S is infinite cyclic.

Secondly, subgroups of tree groups do not need to be tree groups or not even RAAGs. In [46], when Droms proves that a RAAG is a Droms RAAG if and only if the associated graph does not contain full subgraphs isomorphic to C_4 or P_4, he considers the homomorphism $\alpha\colon P_4 \mapsto \mathbb{Z}_2$ such that

$$\alpha(a) = \alpha(b) = \alpha(c) = \alpha(d) = 1$$

and shows that the kernel of that homomorphism is not a right-angled Artin group.

Finally, let us give an example of a finitely presented subgroup of $P_4 \times P_4$ such that L_1 is not finitely generated. Suppose that

$$P_4^1 = \langle a, b, c, d\rangle \quad \text{and} \quad P_4^2 = \langle a', b', c', d'\rangle.$$

Consider the homomorphism $f\colon P_4^1 \times P_4^2 \mapsto \mathbb{Z}$ defined as

$$f(a) = f(b) = f(d) = 1, \quad f(c) = 0, \quad f(a') = f(b') = f(c') = f(d') = 1,$$

and denote by S the kernel of that homomorphism. Again by Corollary 1.4.12, S is finitely presented if and only if $\widehat{\mathcal{L}}_f$ is 1-connected and 1-$\mathbb{Z}$-acyclic-dominating. The complex $\widehat{\mathcal{L}}_f$ is the simplicial join of the simplices

It is clearly contractible, so in particular 1-connected. In order the complex to be 1-$\mathbb{Z}$-acyclic-dominating, we need to verify that the simplicial join of

is 0-acyclic and 0-$\mathbb{Z}$-acyclic-dominating in the simplicial join of

but this is obvious. Furthermore, the exact same argument shows that $\widehat{\mathcal{L}}_f$ is n-connected and n-$\mathbb{Z}$-acyclic-dominating for any $n \in \mathbb{N}$, so S is not just finitely presented, but it is of type F_∞ by Corollary 1.4.12.

Nevertheless, $L_1 = S \cap P_4^1$ is not finitely generated. Indeed, the group L_1 is the kernel of the homomorphism $f_{|P_4^1}$, and since $\widehat{\mathcal{L}}_{f_{|P_4^1}}$ is not connected, then L_1 is not finitely generated.

5.4 Alternative properties in the class $\mathcal{G}$

The aim of this section is to study the class $\mathcal{G}$ and to see how the properties of free groups used in Miller's proof generalise for this class. By Remark 5.2.3 we may just consider that if $G \in \mathcal{G}$, then G is a non-trivial free product or if G is freely indecomposable, then there is at least one vertex group having rank greater than 1.

Recall from the previous section that there are three key properties:

(1) Subgroups of free groups are free;

(2) if S is finitely presented, then the groups L_1 and L_2 are finitely generated;

(3) finitely generated non-trivial normal subgroups of free groups are of finite index.

In this section we prove that these properties generalise to the following ones for groups in $\mathcal{G}$:

(1) Subgroups of groups in $\mathcal{G}$ lie in $\mathcal{G}$;

(2) if S is finitely presented, then a cyclic extension of L_i is finitely generated (see Proposition 5.4.5);

(3) if N is a non-trivial finitely generated normal subgroup of a finitely generated group $G \in \mathcal{G}$ with trivial center, then G/N is either finite or virtually cyclic (see Proposition 5.4.4).

The key ingredient to show the above properties is that normal subgroups of groups in $\mathcal{G}$ contain hyperbolic elements. This follows easily from the fact that groups in $\mathcal{G}$ contain elements with a similar behaviour to WPD elements. Tree groups, for instance, always have WPD elements by [81, Proposition 4.8]. We extend this feature to the class $\mathcal{G}$.

If G is the fundamental group of a graph of groups and T is the corresponding Bass-Serre tree, an element g of G satisfies the *weak proper discontinuity condition for the action of* G *on* T (or g is a *WPD element for the action of* G *on* T) if for each vertex group A, $A \cap A^g$ is a finite group. In our case, the vertex groups are torsion-free, so the condition that $A \cap A^g$ is a finite group reduces to $A \cap A^g = 1$.

In order to consider actions with non-trivial kernel, we define relative WPD elements.

Definition 5.4.1. Let G be a group in $\mathcal{G}$, let T be the Bass-Serre tree corresponding to a standard splitting of G and let K be the kernel of the action of G on T. An element g of G is a *relative WPD element for the action of* G *on* T if for each vertex group A, $A \cap A^g$ is K.

In particular, if the action of G on T is faithful, that is $K = 1$, then a relative WPD element is a WPD element.

We first prove the existence of relative WPD elements in groups in the class $\mathcal{G}$.

Lemma 5.4.2. *Let G be a finitely generated group in $\mathcal{G}$, let T be the Bass-Serre tree corresponding to a standard splitting of G and let K be the kernel of the action of G on T. Then G has a relative WPD element for the action of G on T. Moreover, there is a finite index subgroup of G that has center K.*

Proof. Let $G \in \mathcal{G}$ be finitely generated. If G has a non-trivial free product decomposition, then G has a WPD element for the action of G on T (see [81, Proposition 4.8]). Therefore, we can assume that G is freely indecomposable.

Let Y be the underlying graph of the graph of groups associated to the standard splitting of G. Note that since G is finitely generated, then Y is a finite graph. Let Y_0 be a maximal tree of Y, let $\widetilde{Y}$ be a lift of Y in T and $\widetilde{Y}_0$ the corresponding lift of Y_0. Suppose that $t_1, \dots, t_s$ are the stable letters corresponding to the edges in $\widetilde{Y} \setminus \widetilde{Y}_0$ and let C be the intersection $\bigcap_{v \in V(Y_0), g \in \{1, t_1, \dots, t_s\}} G_v^g$, where G_v is the vertex group of $v \in V(Y_0)$.

First, assume that C is infinite cyclic generated by c. Then, since edge groups are cyclic, the group C has finite index in each edge subgroup of a vertex group G_v^g, $v \in V(Y_0)$, $g \in \{1, t_1, \dots, t_s\}$. The vertex groups are abelian, so C is central in the vertex stabilisers of vertices in $\widetilde{Y}_0$. Furthermore, since C has finite index in each edge group, we have that for each stable letter $t \in \{t_1, \dots, t_s\}$ there are $n = n(t_i), m = m(t_i) \in \mathbb{Z}$ such that $(c^m)^t = c^n$. Thus, $\langle t, c \rangle$ is isomorphic to the Baumslag-Solitar group $BS(m, n)$. Since G is cyclic subgroup separable, so is $\langle t, c \rangle$ and by [71, Corollary 7.7] we have that $|m| = |n|$. Therefore, t normalises the subgroup $\langle c^m \rangle$ and t^2 commutes with c^m for each stable letter $t \in \{t_1, \dots, t_s\}$. It follows that there is a power of c, say c^k, that is normalised by t_i and commutes with t_i^2 for all $i \in \{1, \dots, s\}$ and it also commutes with the vertex groups of vertices in Y_0. Without loss of generality, we assume that k is positive and minimal with these properties. Therefore, the subgroup $\langle c^k \rangle$ is normal in G and $K = \langle c^k \rangle$. Since $\langle c^k \rangle$ is the kernel of the action, there is an equivariant epimorphism $G \mapsto G/\langle c^k \rangle$. Now $G/\langle c^k \rangle$ acts on T acylindrically since it has finite edge stabilisers. It follows that $G/\langle c^k \rangle$ has a WPD element and so G contains a relative WPD element for the action of G on T.

Let $\alpha \colon G \mapsto \langle t_1, \dots, t_s \rangle$ be the retract that sends the vertex groups to 1 and let β be the epimorphism $\langle t_1, \dots, t_s \rangle \mapsto \mathbb{Z}_2$ that sends each generator t_i to the generator of the group of order 2. The kernel $H < G$ of the epimorphism $\beta \circ \alpha$ is an index 2 subgroup which contains precisely the set of words w that have an even number of letters $\{t_1, \dots, t_s, {t_1}^{-1}, \dots, {t_s}^{-1}\}$. As we mentioned, $(c^k)^{t_i} = c^{\pm k}$ for $i \in \{1, \dots, s\}$ and c^k commutes with the elements of the vertex groups. It follows that if $w \in H$, then w commutes with c^k. Therefore, H is an index 2 subgroup of G with center $\langle c^k \rangle$.

Assume now that C is trivial. Let $g = \prod_{v \in V(Y_0), j \in \{1, \dots, s\}} a_v \cdot {a_v}^{t_j}$ where a_v is an element of the stabiliser G_v of the vertex $v \in V(Y_0)$.

If $h \in G_v \cap G_v^g$, then h fixes the path from the vertex G_v to the the vertex

gG_v. Since this path contains all the vertices ${G_u}^t$, $u \in V(Y_0)$, $t \in \{1, t_1, \ldots, t_s\}$, it follows that $h \in C$ and so $h = 1$. Therefore, by [81, Propositon 4.8] G contains a hyperbolic WPD element for the given action of G on T. $\square$

We now turn our attention to the study of (finitely generated) normal subgroups of groups in the class $\mathcal{G}$.

Lemma 5.4.3. *Let G be a finitely generated group in $\mathcal{G}$, let T be the Bass-Serre tree corresponding to a standard splitting of G and let K be the kernel of the action of G on T. Suppose that N is a non-trivial normal subgroup of G. Then either $N < K$ or N contains hyperbolic elements and it acts minimally on T.*

Proof. First, suppose that all the elements of N are elliptic and let h be a relative WPD element (see Lemma 5.4.2). Let x be an element in N. The elements x and x^h are elliptic, and if $\mathrm{Fix}(x) \cap \mathrm{Fix}(x^h) = \varnothing$, then xx^h would be hyperbolic (see, for instance, [20, Proposition 2.1 (2)]). Therefore,

$$\mathrm{Fix}(x) \cap \mathrm{Fix}(x^h) \neq \varnothing.$$

Let yG_v be a vertex in T which lies in $\mathrm{Fix}(x) \cap \mathrm{Fix}(x^h)$. Since $yG_v \in \mathrm{Fix}(x)$, then $h^{-1}yG_v \in \mathrm{Fix}(x^h)$. Moreover, by assumption $yG_v \in \mathrm{Fix}(x^h)$. To sum up,

$$x^h \in {G_v}^{y^{-1}} \cap {G_v}^{y^{-1}h} = K,$$

but K is normal in G, so $x \in K$. In conclusion, if all the elements of N are elliptic, then $N < K$.

Second, suppose that N contains a hyperbolic element. Then, the union of the axes of such elements is the unique minimal N-invariant subtree X_0 of T (see [20, Proposition 2.1 (6)]). Since N is normal in G, the N-invariant subtree X_0 is also invariant under the action of G. But T is minimal as a G-tree, so $X_0 = T$. Thus, N acts minimally on T. $\square$

Proposition 5.4.4. *Let G be a finitely generated group in $\mathcal{G}$, let T be the Bass-Serre tree corresponding to a standard splitting of G, and let K be the kernel of the action of G on T. Suppose that N is a non-trivial finitely generated normal subgroup of G. Then either $N < K$ or G/N is virtually cyclic.*

Furthermore, if G/N is virtually $\mathbb{Z}$, then N is a free product of free abelian groups whose ranks are bounded above by r, where r is the maximum of the ranks of the free abelian vertex groups of G. In particular, if G is a residually finite *tubular group, N is a finitely generated non-trivial normal subgroup of G and G/N is virtually $\mathbb{Z}$, then we have that N is a free group.*

Proof. Suppose that N is not contained in K. By Lemma 5.4.3, N acts minimally on T. In addition, N is finitely generated, so by [4, Proposition 7.9], T/N is finite.

Let C be a cyclic edge stabiliser. Then $|N\backslash G/C|$ is finite because the number of edges in T/N is an upper bound for that number.

From the fact that N is normal in G we get that $N\backslash G/C$ and G/NC are isomorphic. Therefore, $\left|G/NC\right|$ needs to be finite. Notice that if an edge group is trivial, then N is of finite index in G. Hence, we further assume that G is freely indecomposable.

The Second Isomorphism Theorem gives us that

$$NC/N \cong C/(N \cap C).$$

It follows that $N \cap C \neq \{1\}$ if and only if N has finite index in NC. Therefore, if N intersects non-trivially an edge group C, then we have that N also has finite index in G.

We are left to consider the case when N intersects trivially each edge group in the standard splitting of G. In this case, for each edge group C

$$NC/N \cong C.$$

Thus, since NC has finite index in G, G/N is virtually $\mathbb{Z}$. Furthermore, since $N < G$ and N intersects trivially each edge group, N gets induced a decomposition as a free product of free abelian groups, where the free abelian groups are the intersections of the (conjugates of the) vertex groups of G with N. Since G is freely indecomposable, each vertex group has an infinite cyclic edge group as subgroup and as N does not intersect any edge group, it follows that the intersection of N with a (conjugate of a) vertex group G_v of G has rank at most the rank G_v minus 1. In the particular case of tubular groups, all the vertex groups are isomorphic to $\mathbb{Z}^2$, so we have that the intersection with N is at most of rank 1 and so N is a free group. $\square$

Our second goal is to find an alternative for the fact that L_1 and L_2 are finitely generated in the case of free groups. For the class $\mathcal{G}$ we prove the following:

Proposition 5.4.5. *Let G be a finitely generated group in $\mathcal{G}$, let T be the Bass-Serre tree corresponding to a standard splitting of G and let K be the kernel of the action of G on T. Let H be any group and let $S < G \times H$ be a finitely presented subdirect product. Define L_1 and L_2 to be $S \cap G$ and $S \cap H$, respectively. Assume that L_1*

is non-trivial and is not contained in K. Then there is $y \in H$ such that $\langle L_2, y\rangle$ is finitely generated.

Proof. As L_1 is non-trivial and is not contained in K, by Lemma 5.4.3 we have that L_1 contains a hyperbolic isometry, say $t \in L_1$.

Since G is cyclic subgroup separable, it follows from [20, Theorem 3.1] that there is a finite index subgroup M in G which is an HNN extension with stable letter t and associated cyclic subgroup $\langle c\rangle$. As G is finitely generated, then M is also finitely generated; suppose that $M = \langle t, s_1, \dots, s_n\rangle$ with $s_1 = c$ and $s_2 = t^{-1}s_1t$.

Let us denote $p_1^{-1}(M)$ by M'. Since M has finite index in G, then M' has finite index in S and since S is finitely presented so is M'. The HNN decomposition of M induces a decomposition of M' as an HNN extension. Let us pick $\hat{s}_i \in S$ such that $p_1(\hat{s}_i) = s_i$ for $i \in \{1, \dots, n\}$ with the restriction that $\hat{s}_2$ needs to be $t^{-1}\hat{s}_1t$. Note that t is an element in S. We then have that

$$M' = \langle L_2, \hat{s}_1, \dots, \hat{s}_n, t \mid t^{-1}\hat{s}_1t = \hat{s}_2, t^{-1}bt = b, \forall b \in L_2, \mathcal{R}'\rangle,$$

where $\mathcal{R}'$ is a set of relations in the elements $L_2 \cup \{\hat{s}_1, \dots, \hat{s}_n\}$. Recall that $s_1 = c$, so that $\hat{s}_1$ is an element of the form cy in S.

Since M' is finitely generated, there are $a_1, \dots, a_k$ in L_2 such that

$$M' = \langle a_1, \dots, a_k, \hat{s}_1, \dots, \hat{s}_n, t \mid t^{-1}\hat{s}_1t = \hat{s}_2, t^{-1}bt = b, \forall b \in L_2, \mathcal{R}'\rangle.$$

Let D be the subgroup $\langle a_1, \dots, a_k, \hat{s}_1, \dots, \hat{s}_n\rangle$. The subgroup L_2 is finitely generated as a normal subgroup of M' because M is finitely presented. But since t acts trivially by conjugation, then L_2 is generated by the a_i together with their conjugates by words in $\{\hat{s}_1, \dots, \hat{s}_n\}$. Hence, L_2 is a subgroup of D. In conclusion,

$$M' = \langle D, t \mid t^{-1}\hat{s}_1t = \hat{s}_2, t^{-1}bt = b, \forall b \in L_2\rangle.$$

Since M' is finitely presented, by [79, Lemma 2] we deduce that $\langle L_2, \hat{s}_1\rangle$ is finitely generated. Finally, L_2 is contained in H and $\hat{s}_1 = cy$ with $c \in G$ and $y \in H$, so we have that the image of $\langle L_2, \hat{s}_1\rangle$ under the natural projection $\pi_2 \colon G \times H \mapsto H$ is finitely generated, that is $\langle L_2, y\rangle$ is finitely generated. $\square$

5.5 Main result

The main goal of this section is to prove the following:

Theorem 5.5.1. *Let S be a finitely presented subgroup of $G_1 \times G_2$ where $G_1, G_2 \in \mathcal{A}$ are finitely generated. Then, either*

(1) there is $S_0 <_{fi} S$ and a central extension $1 \to \mathbb{Z}^n \to S_0 \to H \to 1$ for some $n \in \mathbb{N}$ and $H \in \mathcal{G}$ finitely generated; or

(2) S is virtually H-by-(free abelian), where H is the direct product of two groups in $\mathcal{A}$.

Furthermore, in the second case S is virtually the kernel of a homomorphism f from $H_1 \times H_2$ to $\mathbb{R}$ for some finitely generated $H_1, H_2 \in \mathcal{A}$. More precisely, H is equal to $M_1 \times M_2$, and either

(1) $M_1 \times M_2 <_{fi} S <_{fi} G_1 \times G_2$; or

(2) S is virtually the kernel $\ker f$ where $f \colon H_1 \times H_2 \mapsto \mathbb{Z}^n$ for some $n \in \{1, 2\}$ and for some finitely generated $H_i \in \mathcal{A}$.

Let G_1 and G_2 be two finitely generated groups in $\mathcal{A}$ and let $S < G_1 \times G_2$ be a finitely presented subgroup. Suppose that $G_i = \mathbb{Z}^{n_i} \times H_i$ for some $n_i \in \mathbb{N} \cup \{0\}$. Then S can be viewed as a subgroup of $\mathbb{Z}^n \times H_1 \times H_2$ for some $n \in \mathbb{N} \cup \{0\}$.

Assume that $n \neq 0$ and define L to be $S \cap \mathbb{Z}^n$. If L is the trivial group, then S is isomorphic to $\pi(S)$, where by π we mean the projection homomorphism $\pi \colon \mathbb{Z}^n \times H_1 \times H_2 \mapsto H_1 \times H_2$.

If L is not the trivial group, then by the basis extension property for free abelian groups up to finite index we may assume that $\mathbb{Z}^n$ is of the form $L \oplus R$. Since $S \cap R = 1$, then the subgroup S is isomorphic to a subgroup T of $L \times H_1 \times H_2$. But $L \subseteq T$, so in fact $T = L \times (T \cap (H_1 \times H_2))$. As S is finitely presented, so is $T \cap (H_1 \times H_2)$.

In conclusion, in order to show Theorem 5.5.1 it suffices to prove it for finitely presented subgroups of the direct product of two groups in $\mathcal{A}$ with trivial center. That is, we may assume that $n_1 = n_2 = 0$.

Moreover, the group S is a subdirect product of $p_1(S) \times p_2(S)$. Let us understand the subgroup $p_i(S)$ of G_i for $i \in \{1, 2\}$. Suppose that $G_i \in \mathcal{A}$ has level n_i. Then G_i is

$$H_1^i * \cdots * H_{m_i}^i$$

for some $H_j^i \in (Z * \mathcal{G})_{n_i - 1}$. Then either $p_i(S)$ is a cyclic subgroup separable (and, in particular, residually finite) non-trivial free product or $p_i(S)$ is a subgroup of H_j^i for some $j \in \{1, \ldots, m_i\}$. Again H_j^i is of the form

$$\mathbb{Z}^{k_{i,j}} \times (H_1^{i,j} * \cdots * H_{m_{i,j}}^{i,j})$$

for some $H_l^{i,j} \in (Z * \mathcal{G})_{n_i - 2}$. By the above argument we may assume that $k_{i,j} = 0$.

By inductively repeating the same argument we arrive to the conclusion that there are three different cases to consider:

(1) S is a finitely presented subdirect product of $G_1 \times G_2$, where G_1 and G_2 are finitely generated residually finite free products;

(2) S is a finitely presented subdirect product of $G_1 \times G_2$, where G_1 is a finitely generated residually finite free product and G_2 is a finitely generated group in $\mathcal{G}$;

(3) S is a finitely presented subdirect product of $G_1 \times G_2$, where G_1 and G_2 are finitely generated and belong to $\mathcal{G}$.

In addition, we may suppose that S is a full subdirect product of $G_1 \times G_2$. Indeed, for $i \in \{1,2\}$ recall that L_i is defined to be $S \cap G_i$. Then there are short exact sequences

$$1 \longrightarrow L_i \longrightarrow S \longrightarrow G_j \longrightarrow 1,$$

so if L_i is trivial, then S is isomorphic to G_j.

Remark 5.5.2. Let S be a subdirect product of $G_1 \times G_2$. Then

$$G_1/L_1 \cong S/(L_1 \times L_2) \cong G_2/L_2.$$

Indeed, since S surjects onto G_1, we have an epimorphism $\pi \colon S \mapsto G_1/L_1$ with kernel $L_1 \times L_2$.

5.5.1 Case (1).

We can assume that G_1 and G_2 are residually finite free products.

Theorem 5.5.3. *Let $G_1 \times G_2$ be the direct product of two finitely generated residually finite groups that admit a non-trivial free product decomposition. Let S be a finitely presented full subdirect product in $G_1 \times G_2$ and define L_i to be $S \cap G_i$, $i \in \{1,2\}$. Then L_i has finite index in G_i, $i \in \{1,2\}$, and so $L_1 \times L_2$ has finite index in S. In particular, for $i \in \{1,2\}$, if $G_i \in \mathcal{G}$, then $L_i \in \mathcal{G}$ and it is finitely generated.*

We will first show that the groups L_1 and L_2 are finitely generated.

Proposition 5.5.4. *Let $A \times K$ be the direct product of a group A with K, where K is a finitely generated residually finite group that admits a non-trivial free product decomposition. Suppose that S is a subdirect product in $A \times K$ which intersects K non-trivially. Then $L = S \cap A$ is finitely generated.*

Proof. By hypothesis since S intersects K non-trivially, there is a non-trivial element t in $S \cap K$. Since K has a non-trivial free product decomposition, it acts minimally on a tree T with trivial edge stabilisers. Moreover, K is residually finite, so the trivial group is closed in the pro-finite topology. Thus, by [20, Theorem 3.1] there is a finite index subgroup M in K which is a free product of the form $B * \langle t \rangle$. Since K is finitely generated, so is M. Let $\{t, s_1, \ldots, s_n\}$ be a generating set for M. For $i \in \{1, \ldots, n\}$ let us pick $\hat{s_i} \in p_2^{-1}(s_i)$ and let $M' = p_2^{-1}(M)$.

Note that M' is of finite index in S since M is of finite index in K. The free product decomposition of M induces a splitting of M' of the form:

$$M' = \langle L, t, \hat{s_1}, \ldots, \hat{s_n} \mid t^{-1}bt = b, \hat{s_i}^{-1} b \hat{s_i} = \phi_i(b), i \in \{1, \ldots, n\}, \forall b \in L \rangle,$$

where ϕ_i is the automorphism of L induced by conjugation by $\hat{s}_i$.

Now since S is finitely presented and M' is of finite index in S, we have that M' is also finitely presented. Suppose that M' is generated by elements $a_1, \ldots, a_k$ in L together with the elements $t, \hat{s}_1, \ldots, \hat{s}_n$. Let D be the group $\langle a_1, \ldots, a_k, \hat{s_1}, \ldots, \hat{s_n} \rangle$. The group L is finitely generated as a normal subgroup of M' since M is finitely presented. But t acts trivially, so L is generated by the a_i together with their conjugates by words in $\{\hat{s}_1, \ldots, \hat{s}_n\}$. Hence, L is a subgroup of D. Moreover,

$$M' = \langle D, t \mid t^{-1}bt = b, \forall b \in L \rangle.$$

Finally, since M' is finitely presented, by [79, Lemma 2] we have that L is finitely generated. $\square$

We now address the proof of Theorem 5.5.3.

Proof of Theorem 5.5.3. By Proposition 5.5.4, L_1 and L_2 are finitely generated. Hence, for $i \in \{1, 2\}$, L_i is a non-trivial finitely generated normal subgroup of G_i and since by assumption G_i admits a non-trivial free product decomposition, it follows from [6] that L_i has finite index in G_i. Therefore, $L_1 \times L_2$ has finite index in S. $\square$

5.5.2 Case (2).

It suffices to check the result of Theorem 5.5.1 for $G_1 \times G_2$, where G_1 is a residually finite free product and G_2 is a group in $\mathcal{G}$.

Theorem 5.5.5. *Let G_1 be a finitely generated residually finite group that decomposes as a non-trivial free product and let G_2 be a finitely generated group in $\mathcal{G}$. Let*

S be a finitely presented full subdirect product in $G_1 \times G_2$. Then either

(1) there is $S_0 <_{fi} S$ and a central extension $1 \to \mathbb{Z} \to S_0 \to H \to 1$ with $H <_{fi} G_i$ for some $i \in \{1,2\}$; or

(2) S is virtually H-by-(free abelian), where H is the direct product of two subgroups of G_1 and G_2.

Furthermore, in the second case H is equal to $L_1 \times L_2$, where $L_i = S \cap G_i$ for $i \in \{1,2\}$ and either

(1) $L_1 \times L_2 <_{fi} S <_{fi} G_1 \times G_2$; or

(2) S is virtually the kernel $\ker f$ where $f\colon H_1 \times H_2 \mapsto \mathbb{Z}$ for some $H_i <_{fi} G_i$, $i \in \{1,2\}$.

Proof. Let T be the Bass-Serre tree corresponding to a standard splitting of G_2 and let $K = \langle c \rangle$ be the kernel of the action of G_2 on T. Suppose that $L_2 < K$. By Lemma 5.4.2 there is a finite index subgroup in G_2, say H_2, with center $\langle c \rangle$. Then $S_0 = S \cap (G_1 \times H_2)$ has finite index in S. In this case, the short exact sequence

$$1 \longrightarrow L_2 \longrightarrow S_0 \longrightarrow p_1(S_0) \longrightarrow 1$$

is a central extension.

We now need to deal with the case when L_2 is not contained in K. By Proposition 5.4.5 there is $e \in G_1$ such that $\langle L_1, e \rangle$ is finitely generated. Then $\langle L_1, e \rangle$ has finite index in G_1 (see [27, Theorem 4.1]). Therefore, G_1/L_1 is finite or virtually $\mathbb{Z}$. Since $S/(L_1 \times L_2) \cong G_1/L_1 \cong G_2/L_2$ (see Remark 5.5.2), then $S/(L_1 \times L_2)$ is finite or virtually $\mathbb{Z}$.

Suppose that G_1/L_1 and G_2/L_2 are virtually $\mathbb{Z}$. Then, for $i \in \{1,2\}$, there is H_i a finite index subgroup in G_i such that $H_i/L_i \cong \mathbb{Z}$. If we define S_0 to be $S \cap (H_1 \times H_2)$, then S_0 has finite index in S, S_0 is moreover normal in $H_1 \times H_2$ and $(H_1 \times H_2)/S_0$ is virtually $\mathbb{Z}$, so there is S_1 a finite index subgroup of S_0 (and, hence, of S) and H_1', H_2' two finite index subgroups in G_1 and G_2, respectively, such that

$$(H_1' \times H_2')/S_1 \cong \mathbb{Z}.$$

□

5.5.3 Case (3).

We can assume that the groups belong to $\mathcal{G}$, and by the previous cases that they are freely indecomposable. Therefore, it suffices to prove the following:

Theorem 5.5.6. *Let G_i be a freely indecomposable finitely generated group in $\mathcal{G}$ for $i \in \{1,2\}$. Let S be a finitely presented full subdirect product of $G_1 \times G_2$. Then either*

(1) there is $S_0 <_{fi} S$ and a central extension $1 \to \mathbb{Z} \to S_0 \to H \to 1$ with $H <_{fi} G_i$ for some $i \in \{1,2\}$; or

(2) S is virtually H-by-(free abelian), where H is the direct product of two subgroups of G_1 and G_2.

Furthermore, in the second case H is equal to $L_1 \times L_2$, where $L_i = S \cap G_i$ for $i \in \{1,2\}$ and either

(1) $L_1 \times L_2 <_{fi} S <_{fi} G_1 \times G_2$; or

(2) S is virtually the kernel $\ker f$ *where $f \colon H_1 \times H_2 \mapsto \mathbb{Z}$ for some $H_i <_{fi} G_i$, $i \in \{1,2\}$; or*

(3) S is virtually the kernel $\ker f$ *where $f \colon H_1 \times H_2 \mapsto \mathbb{Z}^2$ for some $H_i <_{fi} G_i$, $i \in \{1,2\}$. In this case L_i is the free product of finitely generated free abelian groups for $i \in \{1,2\}$.*

With the aim of sheeding light to the proof, we first show the steps of the proof for the case $G_1 = G_2 = P_4$ and we then explain the technicalities that we will have to overcome in the general case.

Hence, suppose that S is a finitely presented full subdirect product of $P_4^1 \times P_4^2$. For $i \in \{1,2\}$ we can take

$$\langle a_i, b_i\rangle *_{\langle b_i\rangle} \langle b_i, c_i\rangle *_{\langle c_i\rangle} \langle c_i, d_i\rangle$$

to be a standard splitting of P_4^i.

By Remark 5.5.2,

$$P_4^1/L_1 \cong S/(L_1 \times L_2) \cong P_4^2/L_2,$$

so it suffices to prove that P_4^2/L_2 is finite, virtually $\mathbb{Z}$ or virtually $\mathbb{Z}^2$.

Let T_i be the Bass-Serre tree associated to the above splitting of P_4^i, $i \in \{1,2\}$. The action of P_4^i on T_i is faithful. Thus, from Lemma 5.4.3 we get that L_2 acts minimally on T_2 and Proposition 5.4.5 implies that there is $e \in P_4^2$ such that $\langle L_2, e\rangle$ is finitely generated. This group contains L_2, so it acts minimally on T_2. Hence, by [4, Proposition 7.9] the graph $T_2/\langle L_2, e\rangle$ is finite. Then, since the number of edges in that graph has to be finite, we get that

$$\left|\langle L_2, e\rangle\backslash P_4^2/\langle b_2\rangle\right| < \infty \quad \text{and} \quad \left|\langle L_2, e\rangle\backslash P_4^2/\langle c_2\rangle\right| < \infty.$$

In particular, there are $z_1, \ldots, z_m \in P_4^2$ such that

$$P_4^2 = \dot{\bigcup}_{j\in\{1,\ldots,m\}} \langle L_2, e\rangle z_j \langle b_2\rangle.$$

Simmetrically, there are $y_1, \ldots, y_o \in P_4^2$ such that

$$P_4^2 = \dot{\bigcup}_{j\in\{1,\ldots,o\}} \langle L_2, e\rangle y_j \langle c_2\rangle. \tag{5.1}$$

Since the set $\{z_1, \ldots, z_m\}$ is finite, there exist $n_1 < n_2 \in \mathbb{N}$ and $j_0 \in \{1, \ldots, m\}$ such that

$$c_2^{n_1} = le^{m_1} z_{j_0} b_2^{k_1} \quad \text{and} \quad c_2^{n_2} = l' e^{m_2} z_{j_0} b_2^{k_2},$$

for some $l, l' \in L_2$ and $k_1, m_1, k_2, m_2 \in \mathbb{Z}$.

Equating z_{j_0} in the previous equations and using the facts that L_2 is normal in P_4^2 and b_2 and c_2 commute, we have that there is $\hat{l} \in L_2$ such that

$$\hat{l} e^{m_2 - m_1} = c_2^{n_2 - n_1} b_2^{k_1 - k_2}. \tag{5.2}$$

As a consequence, if $m_2 \neq m_1$ we obtain another double coset decomposition of P_4^2 of the form

$$P_4^2 = \dot{\bigcup}_{j\in\{1,\ldots,m'\}} \langle L_2, c_2^n b_2^k\rangle f_j \langle b_2\rangle,$$

for some $f_1, \ldots f_{m'} \in P_4^2$, $k \in \mathbb{Z}$ and $n \in \mathbb{N}$.

The next goal is to achieve a decomposition of P_4^2 as a disjoint union of single cosets. For each $j \in \{1, \ldots, m'\}$ and $t \in \mathbb{Z}$, $b_2^t f_j$ lies in P_4^2. Therefore, there are $t_1 \neq t_2 \in \mathbb{N}$ such that $b_2^{t_1} f_j$ and $b_2^{t_2} f_j$ lie in the same double coset. Repeating the same argument as above (we will explain this carefully in the main proof), we get that there is $\tilde{l} \in L_2$ such that

$$\tilde{l}(c_2^n b_2^k)^m b_2^t f_j = f_j b^p \tag{5.3}$$

for some $m, t, p \in \mathbb{Z}$. Again, if $p \neq 0$ we take cosets as we did in the previous case to get that there are $q_1, \ldots, q_r \in P_4^2$ such that

$$P_4^2 = \dot{\bigcup_{j\in\{1,\ldots,r\}}} \langle L_2, w, w'\rangle q_j$$

for some $w, w' \in \langle b_2, c_2 \rangle$. In particular, w and w' commute. Summarising, $\langle L_2, w, w' \rangle$ has finite index in P_4^2 and $\langle L_2, w, w' \rangle / L_2$ is abelian.

This first part of the proof is the same for the general case. The complication arises when in (5.2) we have that $m_1 = m_2$ or in (5.3) we have that $p = 0$. These two cases imply that there are $n \in \mathbb{N}$, $m \in \mathbb{Z}$ such that

$$c_2^n = b_2^m \quad \text{in} \quad P_4^2 / L_2.$$

Since c_2 commutes with b_2 and d_2 in P_4^2, then c_2^n commutes with b_2 and d_2 in P_4^2/L_2. Moreover, $c_2^n L_2 = b_2^m L_2$, so $c_2^n L_2$ commutes with $a_2 L_2$. That is c_2^n lies in the center of P_4^2/L_2.

Taking cosets in (5.1), there are $\hat{y}_1, \dots, \hat{y}_u \in P_4^2$ such that

$$P_4^2 = \dot{\bigcup}_{j \in \{1,\dots,u\}} \langle L_2, e \rangle \hat{y}_j \langle c_2^n \rangle.$$

Now, in P_4^2/L_2 the element c_2^n commutes with $\hat{y}_j$ for all $j \in \{1, \dots, u\}$. As a consequence,

$$P_4^2 = \dot{\bigcup}_{j \in \{1,\dots,u\}} \langle L_2, e, c_2^n \rangle \hat{y}_j.$$

Finally, $\langle L_2, e, c_2^n \rangle$ has finite index in P_4^2 and $\langle L_2, e, c_2^n \rangle / L_2$ is abelian because c_2^n lies in the center of P_4^2/L_2.

Observe that in the argument we have just used that we are working with an amalgamated free product. Indeed, assume that G is a finitely generated amalgamated free product with free abelian vertex groups and infinite cyclic edge groups $\langle c_1 \rangle, \dots, \langle c_k \rangle$. Suppose that N is a non-trivial normal subgroup of G and that for each $i \in \{2, \dots, k\}$ there are $n(i) \in \mathbb{N}$, $m(i) \in \mathbb{Z}$ such that

$$c_1^{n(i)} = c_i^{m(i)} \quad \text{in} \quad G/N.$$

By taking n to be $n(2)n(3)\cdots n(k)$,

$$c_1^n = c_i^{\bar{m}(i)} \quad \text{in} \quad G/N$$

for some $\bar{m}(i) \in \mathbb{Z}$, $i \in \{2, \dots, k\}$.

Hence, in G/N the element c_1^n commutes with the elements of all the vertex groups. Therefore, since G is an amalgamated free product (there are no stable letters), $c_1^n \in Z(G/N)$ and the argument is identical to the above one.

The technicality that we have to overcome appears when G has stable letters. Suppose that G is the fundamental group of a graph of groups with free abelian vertex groups, infinite cyclic edge groups and stable letters. Let t be a stable letter such that $t^{-1}c_1t = c_2$ where $\langle c_1\rangle$ and $\langle c_2\rangle$ are two infinite cyclic edge stabilisers.

We are under the condition that there are $m \in \mathbb{N}$, $n \in \mathbb{Z}$ such that $c_1^m = c_2^n$ in G/N. Therefore,

$$t^{-1}c_1^n t = c_1^m \quad \text{in} \quad G/N.$$

Then, in G/N, $\langle t, c_1\rangle$ is isomorphic to a quotient of the Baumslag-Solitar group $BS(n,m)$, say $BS(n,m)/K$. The point of the proof will be to show that $m = n$, and as a consequence, that c_1^n commutes also with t in G/N. For that, by using the double coset decomposition (5.1) we will deduce a decomposition of $BS(n,m)$ of the form

$$BS(n,m) = \dot{\bigcup} K\langle t, c_1\rangle a_j,$$

and specific properties of Baumslag-Solitar groups will give us that $m = n$.

As we have mentioned in the introduction of the chapter, if we worked with general coherent RAAGs (not necessarily 2-dimensional), then the argument would lead to a $\mathbb{Z}^m$-by-$\mathbb{Z}^n$ group in (5.1) for arbitrary $m, n \in \mathbb{N}$. Note that in the proof it is a key property to have $m = n = 1$. Firstly, when passing from a double coset decomposition to a single coset decomposition we have used that non-trivial subgroups in $\mathbb{Z}$ have finite index. Secondly, when G has stable letters, we make use of the fact that the edge groups have rank 1 to conclude that the problem reduces to a quotient of a Baumslag-Solitar group.

Proof. For $i \in \{1,2\}$, let T_i be the Bass-Serre tree corresponding to a standard splitting of G_i and let $K_i = \langle c_i\rangle$ be the kernel of the action of G_i on T_i. By Lemma 5.4.2 there is a finite index subgroup, say H_i, in G_i with center $\langle c_i\rangle$ and G_i has a relative WPD element.

By Lemma 5.4.3 either $L_i < K_i$ or L_i acts minimally on T_i. If $L_i < K_i$ for some $i \in \{1,2\}$, say $L_2 < K_2$, then $S_0 = S \cap (G_1 \times H_2)$ has finite index in S and there is a central short exact sequence

$$1 \longrightarrow L_2 \longrightarrow S_0 \longrightarrow p_1(S_0) \longrightarrow 1.$$

We now deal with the case when L_i is not contained in K_i for $i \in \{1,2\}$. In this case, we show that G_2/L_2 is finite, virtually $\mathbb{Z}$ or virtually $\mathbb{Z}^2$. Since

$$G_2/L_2 \cong S/(L_1 \times L_2) \cong G_1/L_1,$$

we then obtain that $S/(L_1 \times L_2)$ is finite, virtually $\mathbb{Z}$ or virtually $\mathbb{Z}^2$.

Let us denote T_2 by T. Then L_2 acts minimally on T. By Proposition 5.4.5 there is $e \in G_2$ such that $\langle L_2, e\rangle$ is finitely generated. This group contains L_2, so it also acts minimally on T. Furthermore, it is finitely generated, so by [4, Proposition 7.9] the graph $T/\langle L_2, e\rangle$ is finite. Then for every cyclic edge stabiliser Γ_e we have that

$$\left|\langle L_2, e\rangle \backslash G_2/\Gamma_e\right|$$

is finite since it is bounded above by the number of edges in $T/\langle L_2, e\rangle$. Then, there are $z_1, \ldots, z_m \in G_2$ such that

$$G_2 = \dot{\bigcup}_{j\in\{1,\ldots,m\}} \langle L_2, e\rangle z_j \Gamma_e. \tag{5.4}$$

Suppose that $C = \langle c\rangle$ is a cyclic edge stabiliser and $C < A$ is a free abelian vertex stabiliser.

Assume first that there is $a \in A < G_2$ such that $c^n \neq a^m$ in G_2/L_2 for all $n, m \in \mathbb{Z}$. Since the set $\{z_1, \ldots, z_m\}$ is finite, there are two different powers of a that are in the same double coset, that is there exist $n_1 \neq n_2 \in \mathbb{N}$ and $j_0 \in \{1, \ldots, m\}$ such that

$$a^{n_1} = l_2 e^{k_1} z_{j_0} c^{m_1} \quad \text{and} \quad a^{n_2} = {l_2}' e^{k_2} z_{j_0} c^{m_2},$$

for some $l_2, {l_2}' \in L_2$, $k_1, m_1, k_2, m_2 \in \mathbb{Z}$.

Equating z_{j_0} in the previous equations and using the fact that L_2 is normal in G_2 we deduce that there is $l \in L_2$ such that

$$l e^{k_2-k_1} = a^{n_2} c^{m_1-m_2} a^{-n_1}.$$

Note that since by the standing assumption $c^n \neq a^m$ in G_2/L_2, then we have that $k_1 \neq k_2$ (and so without loss of generality we assume that $k_2 > k_1$). Indeed, otherwise $a^{n_1-n_2}$ would be equal to $c^{m_1-m_2}$ modulo L_2. Since $a, c \in A$ we deduce that $e^{k_2-k_1}$ is congruent to an element of A, say a', modulo L_2. If we define $z_{s,j}$ to be $e^s z_j$ for $s \in \{0, \ldots, k_2 - k_1 - 1\}$ and $j \in \{1, \ldots, m\}$, then we have that

$$G_2 = \dot{\bigcup_{j\in\{1,\ldots,m\}, s\in\{0,\ldots,k_2-k_1-1\}}} \left\langle L_2, e^{k_2-k_1}\right\rangle z_{s,j} \langle c\rangle =$$

$$\dot{\bigcup_{j\in\{1,\ldots,m\}, s\in\{0,\ldots,k_2-k_1-1\}}} \left\langle L_2, a'\right\rangle z_{s,j} \langle c\rangle.$$

The next goal is to obtain a decomposition of G_2 as a disjoint union of single cosets.

The element $c^t z_{s,j}$ lies in G_2 for all $t \in \mathbb{Z}$, $j \in \{1, \ldots, m\}, s \in \{0, \ldots, k_2 - k_1 - 1\}$. Therefore, there are distinct natural numbers t_1 and t_2 and $s_0 \in \{0, \ldots, k_2 - k_1 - 1\}$, $j_0 \in \{1, \ldots, m\}$ such that

$$c^{t_1} z_{s,j} = l(a')^{s_1} z_{s_0,j_0} c^{m_1} \quad \text{and}$$

$$c^{t_2} z_{s,j} = l'(a')^{s_2} z_{s_0,j_0} c^{m_2},$$

for some $l, l' \in L_2$, $s_1, m_1, s_2, m_2 \in \mathbb{Z}$.

Equating the z_{s_0,j_0} and using the normality of L_2 we deduce that there is $l'' \in L_2$ such that

$$l'' c^{-t_2} (a')^{s_2 - s_1} c^{t_1} z_{s,j} = z_{s,j} c^{m_1 - m_2}.$$

Denote the element $c^{-t_2}(a')^{s_2-s_1}c^{t_1} \in A$ by $a'' \in A$. Then the previous equation is of the form

$$l'' a'' z_{s,j} = z_{s,j} c^{m_1 - m_2},$$

where $l'' \in L_2$ and $a'' \in A$. Again by the standing assumption we have that m_1 and m_2 are different and by taking further cosets as we did in the previous case, there are $f_1, \ldots, f_r \in G_2$ such that

$$G_2 = \dot{\bigcup}_{j \in \{1, \ldots, r\}} \langle L_2, a', a'' \rangle f_j.$$

Summarising, the subgroup $\langle L_2, a', a'' \rangle$ has finite index in G_2 and so $L_2 A$ has finite index in G_2. Moreover, by the Second Isomorphism Theorem $L_2 A / L_2$ is isomorphic to $A/(A \cap L_2)$, and since A is abelian, so is $L_2 A / L_2$. Therefore, under the standing assumption we have that G_2 / L_2 is virtually free abelian.

We now deal with the case when for each vertex stabiliser A and each edge stabiliser $C = \langle c \rangle < A$, for each $a \in A$ there are $n, m \in \mathbb{Z}$ such that

$$a^n = c^m \quad \text{in} \quad G_2/L_2. \tag{5.5}$$

In particular, for any two edge stabilisers $\langle c_1 \rangle$ and $\langle c_2 \rangle$ there are $w_1, w_2 \in \mathbb{Z}$ such that ${c_1}^{w_1} = {c_2}^{w_2}$ modulo L_2.

Recall the double coset splitting of G_2,

$$G_2 = \dot{\bigcup}_{j \in \{1, \ldots, m'\}} \langle L_2, e \rangle z_j \Gamma_e. \tag{5.6}$$

Suppose that $\Gamma_e = \langle \gamma_e \rangle$. Note that for each $j \in \{1, \dots, m\}$

$$\langle L_2, e\rangle z_j \langle \gamma_e \rangle = \langle L_2, e\rangle z_j \langle \gamma_e \rangle z_j^{-1} z_j = \langle L_2, e\rangle \langle {\gamma_e}^{z_j^{-1}} \rangle z_j.$$

Observe that $\langle {\gamma_e}^{z_j^{-1}} \rangle$ is an edge stabiliser, so by assumption there are two numbers $n_j, m_j \in \mathbb{Z}$ such that

$$\left({\gamma_e}^{z_j^{-1}}\right)^{n_j} = {\gamma_e}^{m_j} \quad \text{in} \quad G_2/L_2.$$

We again take cosets as in the previous cases to get that

$$G_2 = \dot{\bigcup}_{j\in\{1,\dots,m\}} \langle L_2, e, \gamma_e \rangle q_j, \tag{5.7}$$

for some $q_1, \dots, q_m \in G_2$.

Let us distinguish two cases. First, suppose that e is elliptic. Then $e \in A$ for a vertex stabiliser A. Let $\langle c \rangle$ be an edge stabiliser such that $c \in A$. We are now in the same situation as in the previous case since $e, c \in A$: $L_2 A$ has finite index in G_2 and so we have again that G_2/L_2 is virtually free abelian.

Finally, we need to deal with the case when e is hyperbolic. By [20, Corollary 3.2] there is M a finite index subgroup of G_2 such that M is an HNN extension with stable letter e and amalgamated subgroup $M \cap C$, where C is an edge stabiliser. If c is a generator of C, then $M \cap C = \langle c^r \rangle$ for some $r \in \mathbb{N} \cup \{0\}$. Let us denote c^r by c_1. Since M has finite index in G_2, then M is finitely presented and admits a presentation of the form

$$\langle b_1, \dots, b_k, c_1, e, c_2 \mid \mathcal{R}, e^{-1} c_1 e = c_2 \rangle,$$

where $\mathcal{R}$ is a set of relations in the words $\{b_1, \dots, b_k, c_1, c_2\}$ and c_2 is a power of a generator of another edge stabiliser. Then we are in the following situation:

$$\begin{tikzcd}
 & G_2 \arrow[d, dash] & \\
 & ML_2 \arrow[dl, dash] \arrow[dr, dash] & \\
L_2 \arrow[ddr, dash] & & M \arrow[d, dash] \\
 & & \langle e, c_1 \rangle (M \cap L_2) \arrow[dl, dash] \\
 & M \cap L_2 &
\end{tikzcd}$$

From the double coset representation (5.7) applied to c_1 we have that

$$G_2 = \dot{\bigcup}_{j\in\{1,\ldots,\bar{m}\}} \langle L_2, e, c_1\rangle q_j. \tag{5.8}$$

The group L_2 is a subgroup of ML_2 and the elements e, c_1 belong to $M < ML_2$. Then there are $z_1, \ldots, z_t \in ML_2$ such that

$$ML_2 = \dot{\bigcup}_{j\in\{1,\ldots,t\}} \langle L_2, e, c_1\rangle z_j.$$

Thus, there are $\overline{z_1}, \ldots, \overline{z_t} \in M$ such that

$$M = \dot{\bigcup}_{j\in\{1,\ldots,t\}} \langle M \cap L_2, e, c_1\rangle \overline{z_j}.$$

Assume that the elements $\overline{z_1}, \ldots, \overline{z_s}$ belong to $\langle e, c_1\rangle(M \cap L_2)$ and that $\overline{z_{s+1}}, \ldots, \overline{z_t}$ do not belong to $\langle e, c_1\rangle(M \cap L_2)$. Then we have the coset decomposition

$$\langle e, c_1\rangle(M \cap L_2) = \dot{\bigcup}_{j\in\{1,\ldots,s\}} \langle M \cap L_2, e, c_1\rangle \overline{z_j}. \tag{5.9}$$

By the standing assumption (5.5) we have that $e^{-1}c_1^m e = c_1^n$ modulo L_2 for some $m, n \in \mathbb{Z}$. Therefore, the group $\langle e, c_1\rangle(M \cap L_2)/(M \cap L_2)$ is isomorphic to the quotient of the Baumslag-Solitar group

$$BS(m, n) = \langle x, t \mid t^{-1}x^m t = x^n\rangle.$$

That is, there is N a normal subgroup of $BS(m, n)$ and an isomorphism

$$f \colon \langle e, c_1\rangle(M \cap L_2)/(M \cap L_2) \mapsto BS(m, n)/N$$

with $f(eM \cap L_2) = tN, f(c_1 M \cap L_2) = xN$.

By the decomposition given in (5.9) there are elements $a_1, \ldots, a_s \in BS(m, n)$ such that

$$BS(m, n) = \dot{\bigcup}_{j\in\{1,\ldots,s\}} N\langle t, x\rangle a_j. \tag{5.10}$$

If m is equal to n, then c_1^m commutes with e modulo L_2. Then, from the decomposition given in (5.8) we have that G_2/L_2 is virtually free abelian. If m equals $-n$, then c_1^m commutes with e^2 modulo L_2. From the decomposition (5.8) we get that there are elements $r_1, \ldots, r_g \in G_2$ such that

$$G_2 = \dot{\bigcup}_{j\in\{1,\ldots,g\}} \langle L_2, e^2, {c_1}^m\rangle r_j,$$

so G_2/L_2 is virtually abelian.

Let us deal with the case when m is not equal to $\pm n$. In this case, our aim is to show that there is $q \in \mathbb{N}$ such that $x^q \in N$ and using the isomorphism f we deduce that $c_1^q \in M \cap L_2 < L_2$. Notice that if $c_1^q \in L_2$ for $q \neq 0$, then it follows from the decomposition (5.8) that G_2/L_2 is virtually cyclic.

Note that we can further assume that $\gcd(m,n) = 1$. Otherwise, if $\gcd(m,n) = d$, then $m = dm'$ and $n = dn'$ for some $m', n' \in \mathbb{Z}$ with $\gcd(m',n') = 1$ and $e^{-1}({c_1}^d)^{m'}e = ({c_1}^d)^{n'}$ modulo L_2. Moreover, we may assume that either $m,n > 0$ or $m < 0$ and $n > 0$.

Considering the normal closure $\langle\langle x\rangle\rangle$ in $BS(m,n)$, let us prove that for each $g \in \langle\langle x\rangle\rangle$ there is $N = N(g) \in \mathbb{N}$ such that $(x^N)^g = x^N$. If $g \in \langle\langle x\rangle\rangle$, then

$$g = (g_0x^{\pm 1}g_0^{-1})(g_1x^{\pm 1}g_1^{-1})\dots(g_nx^{\pm 1}g_n^{-1})$$

for some $g_i \in BS(m,n)$, $i \in \{0,\dots,n\}$. For each $i \in \{0,\dots,n\}$ let

$$x_i = \max\big\{\text{number of } t\text{'s in } g_i, \text{number of } t^{-1}\text{'s in } g_i\big\},$$

and let $S = \max\big\{x_i \mid i \in \{0,\dots,n\}\big\}$. Then

$$\left(x^{|m|^S|n|^S}\right)^g = x^{|m|^S|n|^S}.$$

We now consider two cases based on whether or not N is contained in $\langle\langle x\rangle\rangle$.

Suppose that N is not contained in $\langle\langle x\rangle\rangle$, that is there is an element h in N which is not in $\langle\langle x\rangle\rangle$. Since $BS(m,n) = \langle\langle x\rangle\rangle\langle t\rangle$, we can write $h = gt^k$ for some $g \in \langle\langle x\rangle\rangle$ and $k \in \mathbb{Z} \setminus \{0\}$. By the previous paragraph, there is S such that $\left(x^{|m|^S|n|^S}\right)^g = x^{|m|^S|n|^S}$. Let us take M to be $\max\{S, |k|\}$.

If m and n are greater than 0, then

$$\left(x^{m^Mn^M}\right)^h = x^{m^{M-k}n^kn^M} \text{ if } k > 0, \quad \left(x^{m^Mn^M}\right)^h = x^{n^{M-k}m^km^M} \text{ if } k < 0.$$

So in $BS(m,n)/N$,

$$x^{m^Mn^M}N = x^{m^{M-k}n^kn^M}N \quad \text{or} \quad x^{m^Mn^M}N = x^{n^{M-k}m^km^M}N.$$

That is,

$$x^{n^Mm^{M-k}(m^k-n^k)} \in N \quad \text{or} \quad x^{m^Mn^{M-k}(n^k-m^k)} \in N.$$

Since m is not equal to $\pm n$, then $m^k - n^k \neq 0$ and $n^k - m^k \neq 0$. Hence, there is

$q \in \mathbb{N}$ such that $x^q \in N$.

If $m < 0$ and $n > 0$,

$$\left(x^{|m|^M n^M}\right)^h = (x^{-1})^{|m|^{M-k} n^k n^M} \text{ if } k > 0, \quad \left(x^{|m|^M n^M}\right)^h = (x^{-1})^{n^{M-k}|m|^k |m|^M} \text{ if } k < 0.$$

Therefore, as in the previous case,

$$x^{n^M |m|^{M-k}(|m|^k - n^k)} \in N \quad \text{or} \quad x^{|m|^M n^{M-k}(n^k - |m|^k)} \in N.$$

Since m is not equal to $\pm n$, we again have that there is $q \in \mathbb{N}$ such that $x^q \in N$.

Therefore, we are left to consider the case when N is a subgroup of $\langle\langle x \rangle\rangle$. Let us first show that $\langle\langle x \rangle\rangle / N$ is virtually cyclic. For that, we show that $\langle N, x \rangle$ has finite index in $\langle\langle x \rangle\rangle$. Let $g \in \langle\langle x \rangle\rangle$. By the decomposition (5.10) there is an element $n \in N$, some $m, k \in \mathbb{Z}$ and $j \in \{1, \ldots, s\}$ such that $g = nt^m a_j x^k$. Observe that $n^{-1} g x^{-k}$ is an element of $\langle\langle x \rangle\rangle$ and so $t^m a_j$ also belongs to $\langle\langle x \rangle\rangle$. The sum of the powers of t is 0 in every element of $\langle\langle x \rangle\rangle$, and since a_j is a fixed element, then m also needs to be a fixed number, say k_j. Therefore, from the above observation and the decomposition (5.10) we get that

$$\langle\langle x \rangle\rangle = \dot{\bigcup} N y_i \langle x \rangle,$$

where $y_i = t^{k_i} a_i \in \langle\langle x \rangle\rangle$. In conclusion, $\langle\langle x \rangle\rangle / N$ is virtually cyclic.

Let us denote $\langle\langle x \rangle\rangle$ by H and its commutator subgroup by H'. Then we are in the following situation:

$$\begin{tikzcd}[arrows=dash]
 & \langle\langle x \rangle\rangle = H \arrow[d] & \\
 & H'\langle N, x \rangle \arrow[dl] \arrow[d] \arrow[ddr] & \\
\langle N, x \rangle \arrow[d] & H'N \arrow[dl] \arrow[dr] & \\
N & & H'
\end{tikzcd}$$

Since H/N is virtually cyclic, then H/N is either finite, or finite-by-(infinite cyclic) or finite-by-(infinite dihedral) (see, for instance, [57, Lemma 11.4]). If H/N is finite, then N has finite index in H, so $x^q \in N$ for some $q \in \mathbb{N}$.

If H/N is finite-by-(infinite cyclic), then there is an epimorphism f from H to $\mathbb{Z}$. Therefore, $H' \subseteq \ker f$ and f factors through the abelianisation of H, which is

$\mathbb{Z}[\frac{1}{mn}]$ (see Lemma 5.5.7). Thus, we get an epimorphism

$$\varphi\colon \mathbb{Z}[\frac{1}{mn}] \mapsto \mathbb{Z}.$$

Without loss of generality we may assume that $mn \geq 1$. Suppose that $mn \neq 1$. Since φ is an epimorphism, there are $a \in \mathbb{Z}$, $k \in \mathbb{N}$ such that $\varphi\left(\frac{a}{(mn)^k}\right) = 1$. But

$$1 = \varphi\left(\frac{a}{(mn)^k}\right) = \varphi\left((mn)\frac{a}{(mn)^{k+1}}\right) = (mn)\varphi\left(\frac{a}{(mn)^{k+1}}\right),$$

and this is a contradiction because $\varphi\left(\frac{a}{(mn)^{k+1}}\right) \in \mathbb{Z}$ and $mn \neq 1$ (the case $mn = 1$ is covered in the case when m is equal to $\pm n$).

If H/N is finite-by-(infinite dihedral), again there is an epimorphism ϕ from $\mathbb{Z}[\frac{1}{mn}]$ to the abelianisation of the infinite dihedral group,

$$C_2 \times C_2 = \{(1,0),(0,1),(1,1),(0,0)\}.$$

Since ϕ is an epimorphism,

$$(1,0) = \phi\left(\frac{a_1}{(mn)^{k_1}}\right) \quad \text{and} \quad (0,1) = \phi\left(\frac{a_2}{(mn)^{k_2}}\right),$$

for some $a_1, a_2 \in \mathbb{Z}$, $k_1, k_2 \in \mathbb{N}$. Suppose that $mn \neq 1$ and let us distinguish two cases.

If k_1 equals k_2, then

$$a_1\phi\left(\frac{1}{(mn)^{k_1}}\right) = (1,0) \quad \text{and} \quad a_2\phi(\frac{1}{(mn)^{k_1}}) = (0,1),$$

which is not possible. If $k_1 \neq k_2$ (assume that $k_1 > k_2$), then

$$(0,1) = \phi\left(\frac{a_2}{(mn)^{k_2}}\right) = a_2(mn)^{k_1-k_2}\phi\left(\frac{1}{(mn)^{k_1}}\right),$$

and for the same reason this is not possible.

We have just proved that G_i/L_i is either finite, virtually $\mathbb{Z}$ or virtually $\mathbb{Z}^2$. If G_i/L_i is finite, since G_i is finitely generated, so is L_i.

If G_i/L_i is virtually $\mathbb{Z}$, there is H_i a finite index subgroup in G_i such that $H_i/L_i \cong \mathbb{Z}$. If we define S_0 to be $S \cap (H_1 \times H_2)$, then S_0 has finite index in S, S_0 is normal in $H_1 \times H_2$ and $(H_1 \times H_2)/S_0$ is virtually $\mathbb{Z}$, so there is S_1 a finite index subgroup of S_0 and H_1', H_2' two finite index subgroups in G_1 and G_2, respectively, such that

$$(H_1' \times H_2')/S_1 \cong \mathbb{Z}.$$

Finally, we deal with the case when G_i/L_i is virtually $\mathbb{Z}^2$. Since the group L_i is a subgroup of G_i, it acts on the Bass-Serre tree T_i and so it inherits a graph of groups decomposition. We claim that the intersection of L_i with each edge group is trivial and so the decomposition of L_i is in fact a non-trivial free product decomposition. Let us prove it for $i = 2$ being the case $i = 1$ analogous. If Γ_e is a cyclic edge stabiliser of T_2 such that $L_2 \cap \Gamma_e \neq 1$, then from (5.4) we obtain that G_2/L_2 is virtually $\mathbb{Z}$ contradicting our assumption. □

The following lemma is a well-known fact on Baumslag-Solitar groups but we add the proof here for completeness.

Lemma 5.5.7. *Let $BS(m,n) = \langle x, t \mid t^{-1}x^m t = x^n\rangle$ such that $\gcd(m,n) = 1$ and denote the normal closure of x in $BS(m,n)$ by $\langle\langle x\rangle\rangle$. Then $H_1(\langle\langle x\rangle\rangle; \mathbb{Z})$ is isomorphic to $\mathbb{Z}\left[\frac{1}{mn}\right]$.*

Proof. Let us denote $\langle\langle x\rangle\rangle$ by L. If $m \in \{1, -1\}$ or $n \in \{1, -1\}$, then L is free abelian and isomorphic to $\mathbb{Z}\left[\frac{1}{n}\right]$ (see [39]). Thus, we may assume that $m, n \in \mathbb{Z} \setminus \{0, 1, -1\}$.

Let $x_i = t^{-i}xt^i$ for $i \in \mathbb{Z}$. Then $x_{i+1}^m = x_i^n$ and L has a decomposition as a two-way infinite amalgamated free product:

$$\cdots *_{\langle x_{-1}\rangle} \langle x_{-1}, x_0 \mid x_{-1}^n = x_0^m\rangle *_{\langle x_0\rangle} \langle x_0, x_1 \mid x_0^n = x_1^m\rangle *_{\langle x_1\rangle} \cdots$$

Let us show that $H_1(L;\mathbb{Z})$ is locally cyclic. A good reference for properties of locally cyclic groups that we will use now is [55, Chapter 19].

For that we have to prove that every subgroup generated by two elements in $H_1(L;\mathbb{Z})$ is cyclic. Let us define ρ_l to be the element

$$\prod_{0\leq i,j\leq l} x_{i-j}^{k_1^{l+i-j}k_2^{l-i+j}\binom{l}{i}\binom{l}{j}}.$$

For $k \in \mathbb{N}$, $x_k = {\rho_k}^{m^{2k}}$ and $x_{-k} = {\rho_k}^{n^{2k}}$ and $\rho_{l+k}^{(mn)^k} = \rho_l$. From here follows that every 2-generated subgroup is cyclic.

In locally cyclic groups either every element is of finite order (periodic) or no element other than the identity is (aperiodic). In the group $H_1(L;\mathbb{Z})$ the element $x_0 = x$ has infinite order. Indeed, from the presentation of L we obtain a presentation of $H_1(L;\mathbb{Z})$ given as follows. It is the quotient of the free abelian group generated by the elements x_i by the relations $\left\{x_{i+1}^m x_i^{-n}\right\}$. If x_0 has finite order in

$H_1(L;\mathbb{Z})$, then there is $k \in \mathbb{N}$ such that

$$x_0^k = h_1^{-1}(x_{i_1+1}^m x_{i_1}^{-n})^{\epsilon_1} h_1 h_2^{-1}(x_{i_2+1}^m x_{i_2}^{-n})^{\epsilon_2} h_2 \cdots h_l^{-1}(x_{i_l+1}^m x_{i_l}^{-n})^{\epsilon_l} h_l,$$

for some element h_i of the free abelian group and $\epsilon_i \in \{1,-1\}$. Summing the powers of the x_0's (taking into account the sign) we get that $k = mq$ or $k = nq$ for some $q \in \mathbb{Z}$.

Suppose that $k = mq$ and that k is the minimum element in $\mathbb{N}$ such that $x_0^k = 1$ in $H_1(L;\mathbb{Z})$. The other case is symmetric. Then $x^{mq} \in [L,L]$, so $(x^{mq})^t$ is also an element in $[L,L]$. That is, $x^{nq} \in [L,L]$. Since $\gcd(n,m) = 1$, then there are $a, b \in \mathbb{Z}$ such that $1 = ma + nb$. The elements x^{amq} and x^{bnq} lie in $[L,L]$, so $x^{|q|} \in [L,L]$ and this is a contradiction because k was the minimum.

As a consequence, $H_1(L;\mathbb{Z})$ is isomorphic to a subgroup of $(\mathbb{Q},+)$ where the isomorphism

$$\varphi\colon H_1(L;\mathbb{Z}) \mapsto \varphi(H_1(L;\mathbb{Z})) \subseteq \mathbb{Q}$$

sends

$$\varphi(x_k) = \left(\frac{m}{n}\right)^k \quad \text{for } k \in \mathbb{N} \cup \{0\},$$

$$\varphi(x_{-k}) = \left(\frac{n}{m}\right)^k \quad \text{for } k \in \mathbb{N}.$$

Let us show that $\varphi(H_1(L;\mathbb{Z})) = \mathbb{Z}\left[\frac{1}{mn}\right]$. The inclusion $\varphi(H_1(L;\mathbb{Z})) \subseteq \mathbb{Z}\left[\frac{1}{mn}\right]$ is obvious from the definition of φ. For the other inclusion, since $\gcd(m,n) = 1$, then for each $k \in \mathbb{N}$ we have that $\gcd(m^k, n^k) = 1$. Thus, for each $k \in \mathbb{N}$ there are $a_k, b_k \in \mathbb{Z}$ such that

$$1 = a_k m^k + b_k n^k.$$

As a result,

$$\frac{1}{(mn)^k} = \frac{a_k}{n^k} + \frac{b_k}{m^k}, \quad \frac{1}{n^k} = b_k + a_k\left(\frac{m}{n}\right)^k, \quad \frac{1}{m^k} = a_k + b_k\left(\frac{n}{m}\right)^k.$$

In conclusion,

$$\frac{1}{(mn)^k} = 2a_k b_k + a_k^2\left(\frac{m}{n}\right)^k + b_k^2\left(\frac{n}{m}\right)^k,$$

so $\frac{1}{(mn)^k} \in \varphi(H_1(L;\mathbb{Z}))$. $\square$

5.6 Algorithmic problems

In this section we study algorithmic problems for finitely presented subgroups of the direct product of two 2-dimensional coherent RAAGs. Our approach follows closely the one in [28, Section 7]. We focus on the multiple conjugacy problem and the membership problem.

The multiple conjugacy problem has been mentioned in Chapter 2 but let us recall it. The *multiple conjugacy problem* for a finitely generated group G (given by a finite generating set) asks if there is an algorithm that, given a natural number l and two l-tuples of elements in the generators of G, say $x = (x_1, \dots, x_l)$ and $y = (y_1, \dots, y_l)$, can determine if there exists $g \in G$ such that $g^{-1}x_i g = y_i$ in G for $i \in \{1, \dots, l\}$.

If G is a finitely generated group (given by a finite generating set) and H is a finitely generated subgroup of G (given by a finite set of words in the generators of G), the *membership problem* asks if there is an algorithm that decides whether or not a given element g in G as a word in the generators belongs to H.

Suppose that A_1 and A_2 are two 2-dimensional coherent RAAGs given by a standard splitting and let S be a finitely presented subgroup of $A_1 \times A_2$ (given by a finite presentation as words in the generators of $A_1 \times A_2$).

The group $G_i = p_i(S)$ is a finitely generated subgroup of A_i and from [60, Corollary 1.3] one can effectively describe the induced standard splitting and the presentation of G_i. If all the elements of G_i are elliptic, since G_i is finitely generated, then G_i is a subgroup of a conjugate of a vertex group. In particular, it is a free abelian group. Otherwise, if G_i contains a hyperbolic element, then there is T_i' a minimal G_i-invariant subtree of T_i. In this case, either G_i acts faithfully on T_i' or the intersection of the edge groups of G_i is non-trivial. Note that the second scenario just happens if and only if G_i is a subgroup of $\mathbb{Z} \times F_2$, and in this case we can compute a finite presentation of $\mathbb{Z}$ and F_2. To sum up, we can algorithmically decide under which of the following situations we are:

(1) $G_1 \times G_2 = \mathbb{Z}^n$ for some $n \in \{1, 2, 3, 4\}$.

(2) $G_1 \times G_2 < (\mathbb{Z} \times F_2) \times (\mathbb{Z} \times F_2)$.

(3) $G_1 \times G_2 < (\mathbb{Z} \times F_2) \times G_2$ and G_2 acts faithfully on T_2'.

(4) G_1 and G_2 act faithfully on T_1' and T_2', respectively.

The decidability of the multiple conjugacy problem and the membership problem for finitely presented subgroups of S in the first two cases are covered in [28]. Hence, from now on we just focus on cases (3) and (4).

We first show a couple of results that will be helpful to prove the decidability of the multiple conjugacy and the membership problems.

Lemma 5.6.1. *Let H_1 and H_2 be coherent RAAGs (given by two finite presentations) and let $\pi\colon H_1 \times H_2 \mapsto H_1$ be the natural projection homomorphism. Suppose that S is a finitely presented subgroup of $H_1 \times H_2$ given by a finite generating set X (the elements of X are tuples of words in the generators of H_1 and H_2). Then it is algorithmically decidable whether $\pi_{|S}$ is an isomorphism or not.*

Proof. [60, Corollary 1.3] gives us a finite presentation for $\pi(S)$. By using Tietze transformations we may assume that the presentation is $\langle x_1, \dots, x_s \mid r_1, \dots, r_t \rangle$ where $X = \{x_1, \dots, x_s\}$ and the normal subgroup generated by the r_j is the kernel of the surjection $x \mapsto \pi(x)$ from the free group on X to $\pi(S)$. Then $\pi_{|S}$ is an isomorphism if and only if $r_j = 1$ in S, and these equalities can be tested using the solution to the word problem for $H_1 \times H_2$. $\square$

Lemma 5.6.2. *There is an algorithm that, given two finite presentations for F_m and G_2 (where G_2 is a 2-dimensional coherent RAAG) and a finitely presented subdirect product $S < F_m \times G_2$ given by a finite generating set Y (the elements of Y are tuples of words in the generators of F_m and G_2), will output a finite presentation $\langle Y \mid R \rangle$ for S.*

Proof. The proof is a consequence of the proof of [28, Theorem 3.7]. For convenience of the reader, we sketch it here.

By [28, Theorem 2.2, Remark 2.3] it is sufficient to find presentations for

(1) $G_1 = F_m$,

(2) G_2,

(3) $Q = G_2/L_2$ where $L_2 = G_2 \cap S$,

(4) explicit epimorphisms $F_m \mapsto G_2/L_2$ and $G_2 \mapsto G_2/L_2$, and

(5) a finite set of generators for π_2 of the presentation for Q as $\mathbb{Z}Q$-module.

By using Tietze transformations we may take $p_i(Y)$ to be the generating set of G_i, $i \in \{1, 2\}$. Thus, we express G_i as a quotient of the free group on Y. Suppose that $\langle Y \mid r_1(Y), \dots, r_m(Y) \rangle$ is a finite presentation for G_1.

Let us obtain a finite presentation of Q. The images in G_2 of the words $r_j(Y)$ normally generate L_2. Then by adding these words as relations to the presentation of G_2 we get a presentation of Q.

The epimorphism $G_1 \mapsto Q$ is induced by the identity map on Y and $G_2 \mapsto Q$ is the natural quotient map.

Finally, we need a finite set of π_2-generators for the presentation for Q as a $\mathbb{Z}Q$-module. We first search for a finite index free abelian normal subgroup Q' of Q and an isomorphism $Q' \mapsto P$, where P is a free abelian group given by a presentation $\mathcal{P}$ (also, P is either trivial, $\mathbb{Z}$ or $\mathbb{Z}^2$). If X is the $K(P,1)$-complex obtained from the presentation of P, then $\pi_2(X^{(2)})$ is trivial. We replace our presentation for Q by a new presentation $\mathcal{Q}$ that contains $\mathcal{P}$ as a sub-presentation. Let K be the presentation complex associated to $\mathcal{Q}$, $\hat{K}$ the regular cover corresponding to the subgroup $P = Q'$ and let Z be the preimage of $X^{(2)} \subseteq K$ in $\hat{K}$. Then Z consists of one copy of $X^{(2)}$ at each vertex of $\hat{K}$. There is then an exact homotopy sequence

$$\cdots \to \mathbb{Z}Q \otimes_{\mathbb{Z}Q'} \pi_2(X^{(2)}) \to \pi_2(\hat{K}) \to \pi_2(\hat{K}, Z) \to 0.$$

Thus, $\pi_2(K) \cong \pi_2(\hat{K}) \cong \pi_2(\hat{K}, Z) \cong H_2(\hat{K}/Z)$, and the last isomorphism holds because $\hat{K}/Z$ is simply-connected. Hence, $\pi_2(K)$ is generated as a $\mathbb{Z}Q$-module by any finite set C that maps onto a generating set for $H_2(\hat{K}/Z)$ and it can be found by a naive search over finite sets of identity sequences over Q. $\square$

5.6.1 Multiple conjugacy problem in case (3)

In case (3) we view S as a subgroup of $\mathbb{Z} \times F_2 \times G_2$. Moreover, we may assume that the projection homomorphism $S \mapsto G_2$ is surjective. Let us define L_1, L_2 and L_3 to be $S \cap \mathbb{Z}$, $S \cap F_2$ and $S \cap G_2$, respectively.

Lemma 5.6.3. *If L_i is as above for $i \in \{1,2,3\}$, then we can algorithmically decide whether L_i is trivial or not.*

Proof. The group L_1 is trivial if and only if $S \cong \pi_1(S)$, where

$$\pi_1 \colon \mathbb{Z} \times F_2 \times G_2 \mapsto F_2 \times G_2,$$

the group L_2 is trivial if and only if $S \cong \pi_2(S)$, where

$$\pi_2 \colon \mathbb{Z} \times F_2 \times G_2 \mapsto \mathbb{Z} \times G_2,$$

and the group L_3 is trivial if and only if $S \cong \pi_3(S)$, where

$$\pi_3 \colon \mathbb{Z} \times F_2 \times G_2 \mapsto \mathbb{Z} \times G_2.$$

These three conditions may be checked by Lemma 5.6.1 and Lemma 5.6.2. $\square$

Suppose that L_2 or L_3 is trivial, say L_2. Then S is isomorphic to $\pi_2(S)$ which is a subgroup of $\mathbb{Z} \times G_2$. In particular, the multiple conjugacy problem is decidable in $\pi_2(S)$ because it is a subgroup of a coherent RAAG. Thus, the multiple conjugacy problem is decidable for S.

Now assume that L_2 and L_3 are non-trivial. If L_1 is trivial, then S is isomorphic to $\pi_1(S) < F_2 \times G_2$. The subgroup $\pi_1(S)$ is a full subdirect product of $F_m \times G_2$. The groups F_m and G_2 are CAT(0) (see [35, Corollary 9.5]) and have unique roots (see, for instance, [80, Lemma 6.3]). By Theorem 5.5.1 the quotient $(F_m \times G_2)/(L_1 \times L_2)$ is virtually abelian, so there is G a finite index subgroup in $F_m \times G_2$ such that $L_1 \times L_2 < G$ and $G/(L_1 \times L_2)$ is abelian. The group G is CAT(0) for being a finite index subgroup of a CAT(0) group and so it is bicombable. Then, since $L_1 \times L_2 < \pi_1(S) \cap G < G$ and $(\pi_1(S) \cap G)/(L_1 \times L_2)$ is abelian, it follows from Proposition 2.9.3 that $G \cap \pi_1(S)$ has decidable multiple conjugacy problem. The group $\pi_1(S)$ has unique roots and $G \cap \pi_1(S)$ has finite index in $\pi_1(S)$, so from Lemma 2.9.4 we conclude that $\pi_1(S)$ has decidable multiple conjugacy problem.

If L_1 is non-trivial, then $\mathbb{Z} \cap S$ has finite index in $\mathbb{Z}$, so there is $S_0 = (\mathbb{Z} \cap S) \times (S \cap (F_2 \times G_2))$ a finite index subgroup of S. In particular, $S_1 = S \cap (F_2 \times G_2) < F_2 \times G_2$ is finitely presented, so again by Theorem 5.5.1, S_0 has a finite index subgroup S_0' of the form $(M_1 \times M_2)$-by-(free abelian). By the exact same argument S_0' has decidable multiple conjugacy problem, so again using Lemma 2.9.4, S has decidable multiple conjugacy problem.

5.6.2 Membership problem in case (3)

Suppose that S is a finitely presented subgroup of $\mathbb{Z} \times F_2 \times A_2$ (given by a finite presentation) and let H be a finitely presented subgroup of S (given by a finite generating set of words in the generators of S). A solution to the membership problem for $H \subseteq \mathbb{Z} \times F_2 \times A_2$ provides a solution for $H \subseteq S$. Let g be an element in $\mathbb{Z} \times F_2 \times A_2$ given as words in the generators of the factors, and thus we write $g = (g_1, g_2, g_3)$. Again let us define L_1, L_2 and L_3 to be $H \cap \mathbb{Z}$, $H \cap F_2$ and $H \cap A_2$, respectively.

By Lemma 5.6.3 we can algorithmically decide whether $L_i = 1$ or not for $i \in \{1, 2, 3\}$. Suppose that L_2 or L_3 is trivial, say $L_2 = 1$. Then $H \cong \pi_2(H)$. The group $\pi_2(H)$ is a finitely presented subgroup of a coherent RAAG. In particular, the membership problem is decidable for $\pi_2(H)$, so we have an algorithm that determines if $(g_1, g_3) \in \pi_2(H)$. If it does not, then $g \notin H$. If it does, then enumerating equalities $g^{-1}w = 1$ we find a word w in the generators of H so that $g^{-1}w$ projects to 1 (here we are using the decidability of the word problem in $\mathbb{Z} \times A_2$). Since $L_2 = 1$, then

$g \in H$ if and only if $g^{-1}w = 1$ in H and this equality can be checked using the solution to the word problem in $\mathbb{Z} \times F_2 \times A_2$.

If $L_1 = 1$, by the same argument the problem restricts to the decidability of the membership problem of $\pi_1(H)$ in $F_2 \times A_2$. We discuss this in Section 5.6.4. If L_1, L_2 and L_3 are non-trivial, the argument is also analogous to the one in Section 5.6.4.

5.6.3 Multiple conjugacy problem in case (4)

In this case $S < G_1 \times G_2$ is a finitely presented subdirect product and G_1 and G_2 act faithfully on T_1' and T_2', respectively. Applying Lemma 5.6.1 we can algorithmically decide whether $S \cap G_i$ is trivial or not for $i \in \{1, 2\}$. If $S \cap G_i$ is trivial for some i, say $i = 1$, then S is isomorphic to a finitely presented subgroup of G_2, so the multiple conjugacy problem is decidable.

If $S \cap G_1$ and $S \cap G_2$ are non-trivial, Theorem 5.5.1 implies that the group $(G_1 \times G_2)/(L_1 \times L_2)$ is virtually abelian. Thus, there is G a finite index subgroup in $G_1 \times G_2$ such that $L_1 \times L_2 < G < G_1 \times G_2$ and $G/(L_1 \times L_2)$ is abelian. The groups G_1 and G_2 are CAT(0), so G is also CAT(0). Moreover, $L_1 \times L_2 < G \cap S < G$ and $(G \cap S)/(L_1 \times L_2)$ is abelian. By Proposition 2.9.3, $G \cap S$ has decidable multiple conjugacy problem, so S has decidable multiple conjugacy problem (see Lemma 2.9.4).

5.6.4 Membership problem in case (4)

Suppose that S is a finitely presented subgroup of $A_1 \times A_2$ (given by a finite presentation) and let H be a finitely presented subgroup of S (given by a finite generating set of words in the generators of S). The problem restricts to the decidability of the membership problem for $H < A_1 \times A_2$. Let g be an element in $A_1 \times A_2$ given as words in the generators of the factors, thus we write $g = (g_1, g_2)$. Let us define L_1 and L_2 to be $A_1 \cap H$ and $A_2 \cap H$, respectively. The group L_i is trivial if and only if $H \cong \pi_j(H)$ for $j \neq i \in \{1, 2\}$. This can be decided by Lemma 5.6.1.

Suppose that $L_1 = 1$ or $L_2 = 1$, say $L_1 = 1$. The membership problem for H in this case is decidable. Indeed, then $H \cong \pi_2(H)$. The group $\pi_2(H)$ is a finitely presented subgroup of a coherent RAAG. In particular, the membership problem is decidable for $\pi_2(H)$, so we have an algorithm that determines if $g_2 \in \pi_2(H)$. If it does not, then $g \notin H$. If it does, then enumerating equalities $g^{-1}w = 1$ we find a word w in the generators of H so that $g^{-1}w$ projects to 1 (here we are using the decidability of the word problem in A_2). Since $L_1 = 1$, then $g \in H$ if and only if $g^{-1}w = 1$ in H and this equality can be checked using the solution to the word problem in $A_1 \times A_2$.

Now suppose that L_1 and L_2 are non-trivial. We can determine algorithmically if $g_i \in H_i = \pi_i(H)$. If $g_i \notin H_i$, then $g \notin H$. Otherwise, we replace $A_1 \times A_2$ by $H_1 \times H_2$. By Theorem 5.5.1 the quotient group $Q = (H_1 \times H_2)/(L_1 \times L_2)$ is virtually free abelian.

Let $\phi\colon H_1 \times H_2 \mapsto Q$ be the quotient map. Virtually free abelian groups are subgroup separable, so if $\phi(g) \notin \phi(H)$, then there is a finite quotient of Q that separates g from H. But since $L = \ker\phi \subseteq H$, then $\phi(g) \in \phi(H)$ if and only if $g \in H$. If $\phi(g) \notin \phi(H)$, then an enumeration of finite quotients of $H_1 \times H_2$ provides an effective procedure. We run this procedure in paralell with an enumeration of $g^{-1}w$ for words w in the generators of H that will terminate if $g \in H$.

5.7 Finitely presented subgroups are of type F_∞

We apply the subgroup structure Theorem 5.5.1 and the results about Σ-invariants for direct products and fundamental groups of graphs of groups summarised in Section 1.4.2 to prove that finitely presented subgroups of the direct product of two finitely generated groups in $\mathcal{A}$ are of type F_∞. In fact, we prove the same result for a wider class of groups.

Let $\mathcal{D}$ be the class of finitely generated fundamental groups of graphs of groups with free abelian vertex groups and cyclic edge groups. Moreover, we ask the groups not to be ascending HNN extensions. Let $\mathcal{J}$ be the $Z*$-closure of $\mathcal{D}$.

Recall that by Remark 5.2.3 we may assume that if $G \in \mathcal{G}$ is finitely generated, then G is a non-trivial free product or if G is freely indecomposable, then there is a vertex group of G that has rank greater than 1. Thus, the class $\mathcal{G}$ is contained in $\mathcal{D}$.

Proposition 5.7.1. *Let $G \in \mathcal{J}$. Then $\Sigma^1(G) = \Sigma^\infty(G)$.*

Proof. Since in general $\Sigma^\infty(G) \subseteq \Sigma^1(G)$, we have to show that $\Sigma^1(G) \subseteq \Sigma^\infty(G)$. Let $[\chi] \in \Sigma^1(G)$.

We use induction on the level of G. Suppose that $G \in \mathcal{J}_0$. That is G lies in $\mathcal{D}$. Then, the restriction of χ to any edge group is non-zero (see Theorem 1.4.5), so in particular, the restriction of χ to every vertex group is non-zero. Since for every finitely generated free abelian group A we have that $S(A) = \Sigma^\infty(A)$, by Theorem 1.4.3 and Theorem 1.4.4 we get that $[\chi] \in \Sigma^\infty(G, \mathbb{Z})$ and $[\chi] \in \Sigma^2(G)$. Finally, as

$$\Sigma^\infty(G) = \Sigma^2(G) \cap \Sigma^\infty(G, \mathbb{Z}),$$

we deduce that $[\chi] \in \Sigma^\infty(G)$.

Suppose now that G has level $k \geq 1$ and that the result holds for groups in $\mathcal{J}_{k-1}$. Then

$$G = \mathbb{Z}^m \times (G_1 * \cdots * G_n),$$

where each G_i has level at most $k-1$.

If $m = 0$ we get that $n = 1$ (see Theorem 1.4.5), so the result follows from the inductive hypothesis. If $m \geq 1$, assume first that there is $c \in \mathbb{Z}^m$ such that $\chi(c) \neq 0$. Then $[\chi] \in \Sigma^\infty(G)$ (see Lemma 1.4.2). Let us deal with the case when $\chi|_{\mathbb{Z}^m} = 0$. We set μ to be the restriction of χ to $G_1 * \cdots * G_n$. Since $[\chi] \in \Sigma^1(G)$, then $[\mu] \in \Sigma^1(G_1 * \cdots * G_n)$. Again, by Theorem 1.4.5, $n = 1$ and $[\mu] \in \Sigma^\infty(G_1)$. But since $\chi|_{\mathbb{Z}^m} = 0$ this is equivalent to $[\chi] \in \Sigma^\infty(G)$. $\square$

Proposition 5.7.2. *Let S be a finitely presented co-abelian subgroup of $H_1 \times H_2$ with $H_1, H_2 \in \mathcal{J}$. Then S is of type F_∞.*

Proof. Let $\chi = (\chi_1, \chi_2)\colon H = H_1 \times H_2 \mapsto \mathbb{R}$ be a character such that $\chi(S) = 0$. Since S is finitely presented, then $[\chi] \in \Sigma^2(H)$ (Theorem 1.4.1). We aim to show that $[\chi] \in \Sigma^\infty(H)$ so that S is of type F_∞ by Theorem 1.4.1.

(1) Suppose first that $\chi_1 \neq 0$ and $\chi_2 \neq 0$. If $[\chi_1] \notin \Sigma^1(H_1)$ and $[\chi_2] \notin \Sigma^1(H_2)$, then $[\chi] \notin \Sigma^2(H)$ (Theorem 1.4.8), which is a contradiction. Hence, for at least one i we have that $[\chi_i] \in \Sigma^1(H_i) = \Sigma^\infty(H_i)$. Thus, by Theorem 1.4.8, $[\chi] \in \Sigma^n(H)$ for every n, that is $[\chi] \in \Sigma^\infty(H)$.

(2) Suppose that χ_1 or χ_2 is the zero character. Without loss of generality, assume that $\chi_1 = 0$. Then $[\chi] \in \Sigma^n(H) \iff [\chi_2] \in \Sigma^n(H_2)$ (see Lemma 1.4.7). Since $[\chi] \in \Sigma^2(H)$ we have that $[\chi_2] \in \Sigma^2(H_2) \subseteq \Sigma^1(H_2) = \Sigma^\infty(H_2)$. Hence, $[\chi] \in \Sigma^\infty(H)$. $\square$

Finally, we prove our result.

Theorem 5.7.3. *Let S be a finitely presented subgroup of $G_1 \times G_2$ where $G_1, G_2 \in \mathcal{A}$ are finitely generated. Then S is of type F_∞.*

Proof. There are two cases to consider by Theorem 5.5.1. In the first case, there is S_0 a finite index subgroup of S and a central extension $1 \to \mathbb{Z}^n \to S_0 \to H \to 1$ for some $n \in \mathbb{N}$ and $H \in \mathcal{G}$ finitely generated. Since $\mathbb{Z}^n$ and H are of type F_∞, then S_0 is of type F_∞. The group S_0 has finite index in S, so S is of type F_∞.

In the second case, there is S_0 a finite index subgroup in S (in particular, S_0 is finitely presented) such that S_0 is the kernel of a homomorphism $H_1 \times H_2 \mapsto \mathbb{Z}^n$ for some $n \in \{1, 2\}$ and some finitely generated $H_i \in \mathcal{A}$. By Proposition 5.7.2, S_0 is of type F_∞. Thus, S is of type F_∞. $\square$

Chapter 6

Subgroups of direct products of finitely many fundamental groups of graphs of groups with free abelian vertex groups

6.1 Introduction and outline

In this last chapter we study subgroups of direct products of arbitrarily many 2-dimensional coherent RAAGs.

Recall that after Baumslag and Roseblade's result for finitely presented subgroups of the direct product of two free groups, Bridson, Howie, Miller and Short conducted the general study of subgroups of direct products of arbitrarily many free groups (and, more generally, of limit groups).

In Chapter 5 we generalised Baumslag and Roseblade's result by describing the structure of finitely presented subgroups of the direct product of two 2-dimensional coherent RAAGs. In this chapter we start the study for finitely presented subgroups of the direct product of finitely many 2-dimensional coherent RAAGs. The path taken so far is to follow the steps in [27] that we also used in Chapter 2 to characterise finitely presented residually Droms RAAGs. One of the key properties we used of finitely presented full subdirect products of limit groups over Droms RAAGs is the fact that they virtually contain a term of the lower central series, that is they are a nilpotent extension of a direct product of limit groups over Droms RAAGs (see Theorem 2.6.4).

In the case of 2-dimensional coherent RAAGs we prove that such subdirect products are a polycyclic extension of a direct product. In fact, our result applies to a wider class $\mathcal{G}'$. Remember that the class $\mathcal{G}$ defined in Chapter 5 is the class of cyclic subgroup separable fundamental groups of graphs of groups with free abelian vertex groups and cyclic edge groups and such a splitting is called a standard splitting of G. Nevertheless, we have seen in Chapter 5 that this class also gathers groups that do not have a faithful action on its Bass-Serre tree.

In this chapter we do not deal with this case, so we define a subclass $\mathcal{G}'$ of $\mathcal{G}$. If G is a group in $\mathcal{G}$, then G lies in $\mathcal{G}'$ if G has a standard splitting such that the action on the associated Bass-Serre tree is faithful. A short remark to make here is that the general case where we would consider finitely presented subgroups of the direct product of finitely generated groups in $\mathcal{A}$ (see Definition 5.2.2) follows from this particular case where we consider finitely presented full subdirect products of finitely generated groups in $\mathcal{G}'$. For the general case one should also consider cyclic subgroup separable fundamental groups of graphs of groups with finitely generated abelian vertex groups and finite cyclic edge groups, since these are precisely the splittings that we get when we quotient a group in $\mathcal{G}$ by the kernel of its action on the Bass-Serre tree. These groups are, however, just virtually cyclic subgroup separable free products.

The arguments restricted to the class $\mathcal{G}'$ developed in this chapter are already quite technical, where the theory of spectral sequences and homology of groups are involved. Therefore, we have decided to stick to this case. The main result of this chapter is the following.

Theorem 6.4.15. Let $G_1, \ldots, G_n$ be finitely generated groups in $\mathcal{G}'$ and suppose that S is a finitely presented full subdirect product of $G_1 \times \cdots \times G_n$. If L_i is $S \cap G_i$ for $i \in \{1, \ldots, n\}$, then G_i/L_i is virtually (finitely generated nilpotent)-by-(finitely generated free abelian).

Although we do not yet get a complete characterisation of finitely presented full subdirect products, we do get the following necessary condition:

Corollary 6.4.8. Let $G_1, \ldots, G_n$ be finitely generated groups in $\mathcal{G}'$ and let S be a finitely presented full subdirect product of $G_1 \times \cdots \times G_n$. Then $p_{i,j}(S)$ is virtually the kernel of $H_i \times H_j \mapsto \mathbb{Z}^{n_{i,j}}$ for some $n_{i,j} \in \mathbb{N} \cup \{0\}$ and $H_k <_{fi} G_k$.

6.2 Subgroups of direct products of limit groups

The aim of this section is to summarise the steps taken in [27] (or Chapter 2) to study subgroups of direct products of limit groups over free groups (or over Droms RAAGs).

Let $\Gamma_1, \ldots, \Gamma_n$ be limit groups over Droms RAAGs where each Γ_i has trivial center and let S be a finitely presented full subdirect product of $\Gamma_1 \times \cdots \times \Gamma_n$. Recall that the projection map $S \mapsto \Gamma_i \times \Gamma_j$ is denoted by $p_{i,j}$, the kernel of $p_i \colon S \mapsto \Gamma_i$ is K_i and $N_{i,j}$ is the group $p_j(K_i)$. By Proposition 3.2.7 $N_{i,j}$ has finite index in Γ_j. Thus, the group N_j, which is

$$N_{1,j} \cap \cdots \cap N_{j-1,j} \cap N_{j+1,j} \cap \cdots \cap N_{n,j},$$

has finite index in Γ_j, and it follows from Lemma 2.6.1 that N_j/L_j is nilpotent. In addition, since $N_{j,i} \times N_{i,j} < p_{i,j}(S) < \Gamma_i \times \Gamma_j$, then $p_{i,j}(S)$ has finite index in $\Gamma_i \times \Gamma_j$.

Another key property we used has to do with finiteness properties of kernels of epimorphisms $\Gamma_1 \times \cdots \times \Gamma_n \mapsto \mathbb{Z}$. Recall Theorem 2.7.1:

Theorem. *Let $\Gamma_1, \ldots, \Gamma_n$ be limit groups over Droms RAAGs such that Γ_i has trivial center, and let N be the kernel of an epimorphism $\Gamma_1 \times \cdots \times \Gamma_n \mapsto \mathbb{Z}$. Then there is a subgroup of finite index $N_0 \subseteq N$ such that at least one of the homology groups $H_k(N_0; \mathbb{Q})$ has infinite dimension.*

These two results were the main ingredients needed to conclude that if S is a subgroup of the direct product of n limit groups over Droms RAAGs of type w$FP_n(\mathbb{Q})$, then the nilpotent part actually needs to be finite and so the subgroup is virtually a direct product of limit groups over Droms RAAGs.

6.3 Counterexamples in tree groups

In Section 5.3 we showed that if f is the homomorphism $P_4^1 \times P_4^2 \mapsto \mathbb{Z}$ defined by

$$f(a) = f(b) = f(d) = 1, \quad f(c) = 0, \quad f(a') = f(b') = f(c') = f(d') = 1$$

and if S is the kernel of that homomorphism, then S is finitely presented. Observe that S is a full subdirect product of $P_4^1 \times P_4^2$ and that $N_{2,1} = L_1$ and $N_{1,2} = L_2$. Thus,

$$P_4^1/N_{2,1} \cong P_4^2/N_{1,2} \cong \mathbb{Z},$$

so $N_{i,j}$ is of infinite index in P_4^j. Moreover, S is of type w$FP_2(\mathbb{Q})$, so for each $S_0 <_{fi} S$, $H_i(S_0; \mathbb{Q})$ is finite dimensional for $0 \leq i \leq 2$. In conclusion, neither of the properties discussed in the previous section holds for all 2-dimensional coherent RAAGs.

6.4 Alternative properties in $\mathcal{G}'$

In this section we study the class $\mathcal{G}'$ and we see how the previous two properties may be modified in this case. Let us start with the second property, since it requires less work.

Theorem 6.4.1. *Let $G_1, \ldots, G_n$ be finitely generated groups in the class $\mathcal{G}$ and let ϕ be an epimorphism $G_1 \times \cdots \times G_n \mapsto \mathbb{Z}^{n+1}$. Then $\ker \phi$ is not of type FP_n.*

Proof. For $i \in \{1, \ldots, n\}$ let $\langle b_i \rangle$ be a cyclic edge group of a standard splitting of G_i. Then there is $\chi \in Hom(\mathbb{Z}^{n+1}, \mathbb{Z})$ such that $(\chi \circ \phi)(b_i) = 0$ for all $i \in \{1, \ldots, n\}$. In addition, ϕ is an epimorphism, so we can ensure that $\chi \circ \phi \neq 0$.

If $\ker \phi$ were of type FP_n, then $\ker(\chi \circ \phi)$ would also be of type FP_n as it is a finitely generated abelian extension of $\ker \phi$. By Theorem 1.4.1,

$$[\chi \circ \phi] \in \Sigma^n(G_1 \times \cdots \times G_n; \mathbb{Z})$$

and it follows from Theorem 1.4.8 that $[(\chi \circ \phi)_i] \in \Sigma^1(G_i)$ for some $i \in \{1, \ldots, n\}$. However, this contradicts Theorem 1.4.5 since $(\chi \circ \phi)_i$ is trivial in an edge group of G_i. □

There is an alternative proof for the case of tree groups that uses the theory of Σ-invariants for RAAGs:

Theorem 6.4.2. *Let $GX_1, \ldots, GX_n$ be tree groups and let ϕ be an epimorphism $GX_1 \times \cdots \times GX_n \mapsto \mathbb{Z}^{n+1}$. Then $\ker \phi$ is not of type FP_n.*

Proof. Let $\widehat{X}_i$ be the flag complex associated to GX_i, $i \in \{1, \ldots, n\}$ and let us define $\widehat{X}$ to be $\widehat{X}_1 * \cdots * \widehat{X}_n$ so that $\widehat{X}$ is the flag complex associated to $GX_1 \times \cdots \times GX_n$.

By Corollary 1.4.14 it suffices to show that $\widehat{X}$ is not $n-(n-1)$-acyclic. Let $\langle b_i \rangle$ be an edge group in a standard splitting of GX_i. It suffices to show that the flag complex $\widehat{X} - \{b_1, \ldots, b_n\}$ is not $(n-1)$-acyclic.

Note that $\widehat{X} - \{b_1, \ldots, b_n\}$ is $Y_1 * \cdots * Y_n$ where Y_i is $X_i - \{b_i\}$. Let us prove by induction on n that

$$\widetilde{H_{n-1}}(Y_1 * \cdots * Y_n; \mathbb{Z}) \neq 0.$$

If n equals 1, then $\widetilde{H_0}(Y_1; \mathbb{Z}) = \mathbb{Z}$. Now suppose that $n \geq 2$ and that

$$\widetilde{H_{k-1}}(Y_1 * \cdots * Y_k; \mathbb{Z}) \neq 0 \quad \text{for} \quad k \leq n-1.$$

By Lemma 1.4.11 we have that

$$\widetilde{H_{n-2}}(Y_1 * \cdots * Y_{n-1};\mathbb{Z}) \otimes_{\mathbb{Z}} \widetilde{H_0}(Y_n;\mathbb{Z}) \subseteq \widetilde{H_{n-1}}(Y_1 * \cdots * Y_n;\mathbb{Z}).$$

Since $\widetilde{H_0}(Y_n;\mathbb{Z}) = \mathbb{Z}$, then $\widetilde{H_{n-2}}(Y_1 * \cdots * Y_{n-1};\mathbb{Z}) \otimes_{\mathbb{Z}} \widetilde{H_0}(Y_n;\mathbb{Z})$ is

$$\widetilde{H_{n-2}}(Y_1 * \cdots * Y_{n-1};\mathbb{Z}),$$

and this group is non-trivial by inductive hypothesis. In particular, the group $\widetilde{H_{n-1}}(Y_1 * \cdots * Y_n;\mathbb{Z})$ is also non-trivial. Hence, $\widehat{X}$ is not $n-(n-1)$-acyclic, so $\ker\phi$ is not of type FP_n. □

Recall from Section 6.2 that for limit groups over Droms RAAGs with trivial center, N_j has finite index in Γ_j and N_j/L_j is a finitely generated nilpotent group. For finitely generated groups in $\mathcal{G}'$ we prove that G_j/N_j is virtually free abelian and that N_j/L_j is finitely generated nilpotent.

Lemma 6.4.3. *Let S be a finitely presented full subdirect product of $G_1 \times \cdots \times G_n$ where each G_i is either a finitely generated residually finite free product or G_i is a finitely generated group in $\mathcal{G}'$. In the former case K_i is finitely generated. In the latter case, for each cyclic edge group $\langle b_i\rangle$ in a standard splitting of G_i, there is $n_i \in \mathbb{N}\cup\{0\}$ and a lift $\overline{b_i^{n_i}} \in S$ of $b_i^{n_i}$ such that $\langle K_i, \overline{b_i^{n_i}}\rangle$ is finitely generated.*

Proof. For $i \in \{1,\dots,n\}$ there is a short exact sequence

$$1 \longrightarrow K_i \longrightarrow S \longrightarrow G_i \longrightarrow 1.$$

The argument is the same as in Theorem 5.5.3 and Proposition 5.4.5. For the sake of completeness, we sketch them here. Suppose that G_i is a finitely generated residually finite free product. By hypothesis there is a non-trivial element t in $S \cap G_i$. By [20, Theorem 3.1] there is a finite index subgroup M in G_i which is a free product of the form $B * \langle t\rangle$. Since G_i is finitely generated, so is M. Let $\{t, s_1, \dots, s_n\}$ be a generating set for M. For $j \in \{1,\dots,n\}$ let us pick $\hat{s_j} \in p_i^{-1}(s_j)$ and let $M' = p_i^{-1}(M)$. Hence, M' is of finite index in S and

$$M' = \langle K_i, t, \hat{s_1}, \dots, \hat{s_n} \mid t^{-1}bt = b, \hat{s_j}^{-1}b\hat{s_j} = \phi_j(b), j \in \{1,\dots,n\}, \forall b \in K_i\rangle,$$

where ϕ_j is the automorphism of K_i induced by conjugation by $\hat{s_j}$.

The group M' is finitely generated and assume that it is generated by elements $a_1,\dots,a_k$ in K_i together with the elements $t, \hat{s}_1, \dots, \hat{s}_n$. Let D be the group

$\langle a_1, \ldots, a_k, \hat{s_1}, \ldots, \hat{s_n} \rangle$. Then we get that

$$M' = \langle D, t \mid t^{-1}bt = b, \forall b \in K_i \rangle.$$

In conclusion, it follows from [79, Lemma 2] that K_i is finitely generated.

Now suppose that G_i is a finitely generated group in $\mathcal{G}'$. Then $S \cap G_i$ contains a hyperbolic isometry, say $t \in S \cap G_i$. It follows from [20, Theorem 3.1] that there is a finite index subgroup M in G_i which is an HNN extension with stable letter t and associated cyclic subgroup $\langle b_i^{n_i} \rangle$. Then there is M' a finite index subgroup in S of the form

$$M' = \langle K_i, \hat{s}_1, \ldots, \hat{s}_n, t \mid t^{-1}\hat{s}_1 t = \hat{s}_2, t^{-1}bt = b, \forall b \in K_i, \mathcal{R}' \rangle,$$

where $\mathcal{R}'$ is a set of relations in the elements $K_i \cup \{\hat{s}_1, \ldots, \hat{s}_n\}$ and $\hat{s}_1$ is a lift of $b_i^{n_i}$. Since M' is finitely generated, there are $a_1, \ldots, a_k$ in K_i such that

$$M' = \langle a_1, \ldots, a_k, \hat{s}_1, \ldots, \hat{s}_n, t \mid t^{-1}\hat{s}_1 t = \hat{s}_2, t^{-1}bt = b, \forall b \in K_i, \mathcal{R}' \rangle.$$

Let D be the subgroup $\langle a_1, \ldots, a_k, \hat{s}_1, \ldots, \hat{s}_n \rangle$. Hence,

$$M' = \langle D, t \mid t^{-1}\hat{s}_1 t = \hat{s}_2, t^{-1}bt = b, \forall b \in K_i \rangle,$$

so it follows from [79, Lemma 2] that $\langle K_i, \hat{s}_1 \rangle$ is finitely generated. $\square$

Remark 6.4.4. Suppose that K_i is not finitely generated. By the previous lemma

$$\langle K_i, (x_1^i, \ldots, x_{i-1}^i, b_i^{n_i}, x_{i+1}^i, \ldots, x_n^i) \rangle$$

is finitely generated for some $n_i \in \mathbb{N} \cup \{0\}$ and $(x_1^i, \ldots, x_{i-1}^i, b_i^{n_i}, x_{i+1}^i, \ldots, x_n^i) \in S$. In particular, $(x_1^i, \ldots, x_{i-1}^i, b_i^{n_i}, x_{i+1}^i, \ldots, x_n^i)^k$ is not an element in K_i for any $k \in \mathbb{N}$. Note that there is a short exact sequence

$$1 \to K_i \to \langle K_i, (x_1^i, \ldots, x_{i-1}^i, b_i^{n_i}, x_{i+1}^i, \ldots, x_n^i) \rangle \to$$

$$\to \langle (x_1^i, \ldots, x_{i-1}^i, b_i^{n_i}, x_{i+1}^i, \ldots, x_n^i) \rangle \to 1.$$

The associated Lyndon-Hochschild-Serre spectral sequence is a 2-column spectral sequence, so E^∞ coincides with E^2. Since $H_1(\langle K_i, (x_1^i, \ldots, x_{i-1}^i, b_i^{n_i}, x_{i+1}^i, \ldots, x_n^i) \rangle; \mathbb{Q})$ is finite dimensional, then

$$E_{0,1}^\infty = E_{0,1}^2 = H_0(\langle (x_1^i, \ldots, x_{i-1}^i, b_i^{n_i}, x_{i+1}^i, \ldots, x_n^i) \rangle; H_1(K_i; \mathbb{Q}))$$

is also finite dimensional.

Remark 6.4.5. The group $\langle K_i, (x^i_1, \dots, x^i_{i-1}, b_i^{n_i}, x^i_{i+1}, \dots, x^i_n)\rangle$ is finitely generated for some $n_i \in \mathbb{N} \cup \{0\}$, so for $j \neq i$, $\langle N_{i,j}, x^i_j\rangle$ is finitely generated as it is the image by the homomorphism p_j.

Lemma 6.4.6. *Let G be a finitely generated group in $\mathcal{G}'$ and let N be a normal subgroup in G such that G/N is virtually $\mathbb{Z}$. Suppose that for any edge group $\langle b\rangle$ in a standard splitting of G, $N \cap \langle b\rangle = 1$. Then N is finitely generated.*

Proof. Let T be the Bass-Serre tree corresponding to the splitting of G. Since N is a subgroup of G, it also acts on T. If we check that T/N is finite, then N is finitely generated because the vertex groups are finitely generated free abelian.

For that, it suffices to show that the number of edges in T/N is finite. That is, it is enough to show that

$$\sum |N\backslash G/\langle b\rangle| < \infty,$$

where the sum is taken over the edge groups in the splitting of G.

For any edge group $\langle b\rangle$ in the splitting of G, as N is normal in G we have that

$$|N\backslash G/\langle b\rangle| = |G/N\langle b\rangle|.$$

Note that $N\langle b\rangle$ has finite index in G because by assumption G/N is virtually $\mathbb{Z}$ and $N \cap \langle b\rangle = 1$. $\square$

Let us show that G_j/N_j is virtually free abelian.

Lemma 6.4.7. *Let S be a finitely presented full subdirect product of $G_1 \times \cdots \times G_n$ where each G_i is either a finitely generated residually finite free product or G_i is a finitely generated group in $\mathcal{G}'$. Then, for $i \neq j \in \{1, \dots, n\}$, $G_j/N_{i,j}$ is virtually $\mathbb{Z}^m$ for some $m \in \{0, 1, 2\}$.*

Proof. The proof is the same as in the case $n = 2$ and this is done in Section 5.5. $\square$

Corollary 6.4.8. *Let $G_1, \dots, G_n$ be finitely generated groups in $\mathcal{G}'$ and let S be a finitely presented full subdirect product of $G_1 \times \cdots \times G_n$. Then $p_{i,j}(S)$ is virtually a kernel of $H_i \times H_j \mapsto \mathbb{Z}^{n_{i,j}}$ for some $n_{i,j} \in \mathbb{N} \cup \{0\}$ and $H_k <_{fi} G_k$.*

Proof. For $j \neq i \in \{1, \dots, n\}$, $N_{j,i} \times N_{i,j}$ is a subgroup of $p_{i,j}(S)$. By Lemma 6.4.7 there are two groups $H_i <_{fi} G_i$ and $H_j <_{fi} G_j$ such that $(H_i \times H_j)/(N_{j,i} \times N_{i,j})$ is free abelian. Therefore, there is a finite index subgroup of $p_{i,j}(S)$, $K_{i,j}$, such that $(H_i \times H_j)/K_{i,j}$ is free abelian of finite rank. $\square$

Remark 6.4.9. From the proof of Lemma 6.4.7 we get that for each $i \neq j \in \{1, \dots, n\}$ there is $H^{i,j}$ a finite index subgroup in G_j such that $N_{i,j} < \langle N_{i,j}, x_j^i \rangle < H^{i,j}$ and we have the following options:

(1) $N_{i,j}$ is finitely generated; or

(2) $\langle N_{i,j}, x_j^i \rangle$ has finite index in $H^{i,j}$ and $\langle N_{i,j}, x_j^i \rangle / N_{i,j} \cong \mathbb{Z}$; or

(3) $\langle N_{i,j}, x_j^i \rangle$ is a free group, $H^{i,j}/N_{i,j} \cong \mathbb{Z}^2$ and $H^{i,j}/\langle N_{i,j}, x_j^i \rangle \cong \mathbb{Z}$.

Lemma 6.4.10. *Let S be a finitely presented full subdirect product of $G_1 \times \cdots \times G_n$ where each G_i is either a finitely generated residually finite free product or G_i is a finitely generated group in $\mathcal{G}'$. Then, for $i \in \{1, \dots, n\}$ and $1 \leq j_1 < \cdots < j_s \leq n$, $G_i/(N_{j_1,i} \cap \cdots \cap N_{j_s,i})$ is virtually free abelian.*

Proof. Let us prove it by induction on s. If s equals 1, then by Lemma 6.4.7, $G_i/N_{j_1,i}$ is virtually free abelian.

Now suppose that $s \geq 2$. For each $k \in \{1, \dots, s\}$, by Lemma 6.4.7, $G_i/N_{j_k,i}$ is virtually free abelian, so there is H_{j_k} a finite index subgroup in G_i such that $H_{j_k}/N_{j_k,i}$ is free abelian. In order to make notation easier let us denote $N_{j_k,i}$ by N_{j_k} and assume that

$$H_{j_k}/N_{j_k} = \langle b^{n_{j_k}} N_{j_k}, x_{j_k} N_{j_k} \rangle$$

with $[b^{n_{j_k}}, x_{j_k}] \in N_{j_k}$. Note that there is a short exact sequence

$$1 \to (N_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_s})/(N_{j_1} \cap N_{j_2} \cap \cdots \cap N_{j_s}) \to \tag{6.1}$$

$$\to (H_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_s})/(N_{j_1} \cap N_{j_2} \cap \cdots \cap N_{j_s}) \to$$

$$\to (H_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_s})/(N_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_s}) \to 1.$$

The group $(H_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_s})/(N_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_s})$ is a finite index subgroup of H_{j_1}/N_{j_1}. Since $H_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_s}$ has finite index in H_{j_1}, then there is $m_{j_1} \in \mathbb{N}$ such that

$$(b^{n_{j_1}})^{m_{j_1}} \in H_{j_1} \cap \cdots \cap H_{j_s} \quad \text{and} \quad x_{j_1}^{m_{j_1}} \in H_{j_1} \cap \cdots \cap H_{j_s}.$$

Therefore, there is H a finite index subgroup in $H_{j_1} \cap \cdots \cap H_{j_s}$ such that

$$N_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_s} < H \quad \text{and}$$

$$H/(N_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_s}) = \langle b^{n_{j_1} m_{j_1}}, {x_{j_1}}^{m_{j_1}} \rangle.$$

For the argument that follows we may assume that $m_{j_1} = 1$. That is, from now on,

$$H/(N_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_s}) = \langle b^{n_{j_1}} N_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_s}, x_{j_1} N_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_s} \rangle.$$

Let us consider the short exact sequence

$$1 \to (N_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_s})/(N_{j_1} \cap N_{j_2} \cap \cdots \cap N_{j_s}) \to \tag{6.2}$$

$$\to H/(N_{j_1} \cap N_{j_2} \cap \cdots \cap N_{j_s}) \to H/(N_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_s}) \to 1.$$

Let us first show that (6.2) right splits. For that we need to show that $[b^{n_{j_1}}, x_{j_1}]$ is an element in N_{j_k} for all $k \in \{2, \ldots, s\}$. Remember that

$$H_{j_k}/N_{j_k} = \langle b^{n_{j_k}} N_{j_k}, x_{j_k} N_{j_k} \rangle$$

with $[b^{n_{j_k}}, x_{j_k}] \in N_{j_k}$. Moreover, we may suppose that n_{j_1} is a multiple of n_{j_k} so that $[b^{n_{j_1}}, x_{j_k}] \in N_{j_k}$.

Since x_{j_1} is an element in H and H is a subgroup of $H_{j_1} \cap \cdots \cap H_{j_s}$, then $x_{j_1} \in H_{j_k}$, so there are $t_k, l_k \in \mathbb{Z}$ such that $x_{j_1}(b^{n_{j_k}})^{t_k} {x_{j_k}}^{l_k} \in N_{j_k}$. Hence, as N_{j_k} is a normal subgroup in H_{j_k} and $b^{n_{j_1}} \in H_{j_k}$ we have that

$$[x_{j_1}(b^{n_{j_k}})^{t_k} {x_{j_k}}^{l_k}, b^{n_{j_1}}] \in N_{j_k}.$$

Recall that we could assume that $[{x_{j_k}}^{l_k}, b^{n_{j_1}}] \in N_{j_k}$, so we get that $[x_{j_1}, b^{n_{j_1}}] \in N_{j_k}$.

In conclusion, the short exact sequence (6.2) right splits, so if we prove that the action of the group $H/(N_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_s})$ on

$$(N_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_s})/(N_{j_1} \cap N_{j_2} \cap \cdots \cap N_{j_s})$$

is trivial, then we obtain that

$$H/(N_{j_1} \cap \cdots \cap N_{j_s}) \cong$$

$$(N_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_s})/(N_{j_1} \cap \cdots \cap N_{j_s}) \times \langle b^{n_{j_1}} N_{j_1} \cap \cdots \cap N_{j_s}, x_{j_1} N_{j_1} \cap \cdots \cap N_{j_s} \rangle.$$

By inductive hypothesis $(N_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_s})/(N_{j_1} \cap \cdots \cap N_{j_s})$ is virtually free abelian. Therefore, $H/(N_{j_1} \cap \cdots \cap N_{j_s})$ is virtually free abelian.

Hence, the last goal is to prove that the action is trivial. For that it suffices to show that if $x \in N_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_s}$, then

$$[x, b^{n_{j_1}}] \in N_{j_1} \cap \cdots \cap N_{j_s} \quad \text{and} \quad [x, x_{j_1}] \in N_{j_1} \cap \cdots \cap N_{j_s}.$$

Let us fix $k \in \{2, \dots, s\}$. Since $x \in N_{j_1} \cap H_{j_2} \cap \dots \cap H_{j_s}$, then x is an element in H_{j_k}, so there are $w_k, f_k \in \mathbb{Z}$ such that $x(b^{n_{j_k}})^{w_k} {x_{j_k}}^{f_k} \in N_{j_k}$. It can be shown as in the previous argument that $[x, b^{n_{j_1}}] \in N_{j_k}$. It remains to check that $[x, x_{j_1}] \in N_{j_k}$.

Since $[x(b^{n_{j_k}})^{w_k} {x_{j_k}}^{f_k}, x_{j_1}]$ is an element in N_{j_k}, in order to show that $[x, x_{j_1}]$ lies in N_{j_k} it suffices to check that $[x_{j_k}, x_{j_1}] \in N_{j_k}$. But again x_{j_1} is an element in H_{j_k}, so $x_{j_1}(b^{n_{j_k}})^{g_k} {x_{j_k}}^{i_k} \in N_{j_k}$ for some $g_k, i_k \in \mathbb{Z}$. Thus,

$$[x_{j_1}(b^{n_{j_k}})^{g_k}(x_{j_k})^{i_k}, x_{j_k}] \in N_{j_k}$$

and so $[x_{j_1}, x_{j_k}] \in N_{j_k}$. □

The second aim is to prove that N_j/L_j is a finitely generated nilpotent group. Since by Lemma 2.6.1 the group N_j/L_j is nilpotent, it suffices to show that it is finitely generated. Recall that we are assuming that S is a finitely presented full subdirect product of $G_1 \times \dots \times G_n$ and for $1 \leq j_1 < \dots < j_s \leq n$, $p_{j_1,\dots,j_s}$ is the homomorphism

$$p_{j_1,\dots,j_s} \colon S \mapsto G_{j_1} \times \dots \times G_{j_s}.$$

Lemma 6.4.11. *Let $G_1, \dots, G_n$ be groups and let S be a subdirect product of the direct product $G_1 \times \dots \times G_n$. Then, for $j \neq i \in \{1, \dots, n\}$, $G_j/N_{i,j} \cong G_i/N_{j,i}$.*

Proof. Assume without loss of generality that $i < j$. Let us define $\varphi \colon G_j \mapsto G_i/N_{j,i}$ in the following way: let $x_j \in G_j$. Since $p_j \colon S \mapsto G_j$ is an epimorphism, there is $(x_1, \dots, x_i, \dots, x_j, \dots, x_n) \in S$. Then, we define $\varphi(x_j)$ to be $x_i N_{j,i}$.

Note that φ is well-defined. Indeed, suppose that there are two elements in S of the form $(x_1, \dots, x_i, \dots, x_j, \dots, x_n)$ and $(x'_1, \dots, x'_i, \dots, x_j, \dots, x'_n)$. Then $x'_i x_i^{-1} \in N_{j,i}$.

Clearly φ is an epimorphism, so we just need to check that $\ker \varphi = N_{i,j}$. The inclusion $N_{i,j} \subseteq \ker \varphi$ is routine. Now assume that $(x_1, \dots, x_i, \dots, x_j, \dots, x_n)$ is an element in S such that $x_i \in N_{j,i}$. Then, there is an element in S of the form $(x'_1, \dots, x_i, \dots, x'_{j-1}, 1, x'_{j+1}, \dots, x'_n)$. Therefore, $x_j \in N_{i,j}$. □

Lemma 6.4.12. *Let $G_1, \dots, G_n$ be groups and let S be a subgroup of $G_1 \times \dots \times G_n$. For $1 \leq j_1 < \dots < j_s \leq n$ and $k, i \in \{1, \dots, n\} \setminus \{j_1, \dots, j_s\}$,*

$$\frac{p_i(\ker(p_{j_1,\dots,j_s}))}{p_i(\ker(p_{j_1,\dots,j_s,k}))} \cong \frac{p_k(\ker(p_{j_1,\dots,j_s}))}{p_k(\ker(p_{j_1,\dots,j_s,i}))}.$$

Proof. In order to make notation easier, we prove it for $j_l = l, l \in \{1, \dots, s\}$, $i = s+1$, $k = s + 2$.

We define $\varphi\colon p_{s+1}(\ker(p_{1,\dots,s})) \mapsto p_{s+2}(\ker(p_{1,\dots,s}))/p_{s+2}(\ker(p_{1,\dots,s,s+1}))$ in the following way: if $x_{s+1} \in p_{s+1}(\ker(p_{1,\dots,s}))$, we define

$$\varphi(x_{s+1}) = x_{s+2}p_{s+2}(\ker(p_{1,\dots,s,s+1})) \quad \text{such that}$$

$$(1,\dots,1,x_{s+1},x_{s+2},*,\dots,*) \in S.$$

The map φ is well-defined. Indeed, if

$$(1,\dots,1,x_{s+1},x_{s+2},*,\dots,*) \in S \quad \text{and} \quad (1,\dots,1,x_{s+1},x'_{s+2},*,\dots,*) \in S,$$

then $x'_{s+2}{x_{s+2}}^{-1} \in p_{s+2}(\ker(p_{1,\dots,s,s+1}))$. The map φ is clearly an epimorphism, so it suffices to check that $\ker\varphi = p_{s+1}(\ker(p_{1,\dots,s,s+2}))$. It is clear that $p_{s+1}(\ker(p_{1,\dots,s,s+2})) \subseteq \ker\varphi$. Now suppose that

$$(1,\dots,1,x_{s+1},x_{s+2},*,\dots,*) \in S \quad \text{and} \quad (1,\dots,1,1,x_{s+2},*,\dots,*) \in S.$$

Hence, $(1,\dots,1,x_{s+1},1,*,\dots,*)$ is an element in S, so $x_{s+1} \in p_{s+1}(\ker(p_{1,\dots,s,s+2}))$. □

We next prove that $p_i(\ker(p_{j_1,\dots,j_s}))/p_i(\ker(p_{j_1,\dots,j_s,k}))$ is finitely generated. In fact, we prove that it is (finitely generated abelian)-by-(virtually finitely generated abelian).

Theorem 6.4.13. *Let S be a finitely presented full subdirect product of the direct product $G_1\times\cdots\times G_n$ where each G_i is either a finitely generated residually finite free product or G_i is a finitely generated group in $\mathcal{G}'$. Then, for $1 \le j_1 < \cdots < j_s \le n$ and $i \in \{1,\dots,n\}\setminus\{j_1,\dots,j_s\}$, $p_i(\ker(p_{j_1,\dots,j_s}))/p_i(\ker(p_{j_1,\dots,j_s,k}))$ is (finitely generated abelian)-by-(virtually finitely generated abelian).*

Proof. We prove it by induction on s. If s equals 1, we denote j_1 by j. Then we have to prove that $N_{j,i}/p_i(\ker(p_{j,k}))$ is (finitely generated abelian)-by-(virtually finitely generated abelian). There is a short exact sequence

$$1 \to (N_{k,i}\cap N_{j,i})/p_i(\ker(p_{j,k})) \to N_{j,i}/p_i(\ker(p_{j,k})) \to$$

$$\to N_{j,i}/(N_{k,i}\cap N_{j,i}) \to 1$$

and by Lemma 6.4.10, $N_{j,i}/(N_{k,i}\cap N_{j,i})$ is virtually finitely generated abelian. Hence, it is enough to show that $(N_{k,i}\cap N_{j,i})/p_i(\ker(p_{j,k}))$ is finitely generated abelian. Note that by the definition of $p_i(\ker(p_{j,k}))$, the group is abelian, so it suffices to show that it is finitely generated.

Assume that $N_{k,i}N_{j,i}$ is finitely generated. There is a short exact sequence

$$1 \to (N_{k,i} \cap N_{j,i})/p_i(\ker(p_{j,k})) \to N_{k,i}N_{j,i}/p_i(\ker(p_{j,k})) \to N_{k,i}N_{j,i}/(N_{k,i} \cap N_{j,i}) \to 1 \tag{6.3}$$

and $N_{k,i}N_{j,i}/p_i(\ker(p_{j,k}))$ is finitely generated by assumption. The associated Lyndon-Hochschild-Serre spectral sequence converging to $H_*(N_{k,i}N_{j,i}/p_i(\ker(p_{j,k}));\mathbb{Z})$ has

$$E^2_{p,q} = H_p(N_{k,i}N_{j,i}/(N_{k,i} \cap N_{j,i}); H_q((N_{k,i} \cap N_{j,i})/p_i(\ker(p_{j,k}));\mathbb{Z})).$$

Note that $E^\infty_{0,1}$ needs to be a finitely generated $\mathbb{Z}$-module and

$$E^\infty_{0,1} = E^2_{0,1} =$$

$$= \frac{H_0(N_{k,i}N_{j,i}/(N_{k,i} \cap N_{j,i}); H_1((N_{k,i} \cap N_{j,i})/p_i(\ker(p_{j,k}));\mathbb{Z}))}{\mathrm{im}(H_2(N_{k,i}N_{j,i}/(N_{k,i} \cap N_{j,i}); H_0((N_{k,i} \cap N_{j,i})/p_i(\ker(p_{j,k}));\mathbb{Z})))}.$$

The group $N_{k,i}N_{j,i}/(N_{k,i} \cap N_{j,i})$ is virtually finitely generated abelian, so in particular, it is of type $FP_\infty(\mathbb{Z})$. Therefore,

$$\mathrm{im}(H_2(N_{k,i}N_{j,i}/(N_{k,i} \cap N_{j,i}); H_0((N_{k,i} \cap N_{j,i})/p_i(\ker(p_{j,k}));\mathbb{Z})))$$

is a finitely generated $\mathbb{Z}$-module. As a consequence,

$$H_0(N_{k,i}N_{j,i}/(N_{k,i} \cap N_{j,i}); H_1((N_{k,i} \cap N_{j,i})/p_i(\ker(p_{j,k}));\mathbb{Z}))$$

has to be a finitely generated $\mathbb{Z}$-module.

The group $H_0(N_{k,i}N_{j,i}/(N_{k,i} \cap N_{j,i}); H_1((N_{k,i} \cap N_{j,i})/p_i(\ker(p_{j,k}));\mathbb{Z}))$ is the quotient of

$$H_1((N_{k,i} \cap N_{j,i})/p_i(\ker(p_{j,k}));\mathbb{Z})$$

by the action of $N_{k,i}N_{j,i}/(N_{k,i} \cap N_{j,i})$. But the group $(N_{k,i} \cap N_{j,i})/p_i(\ker(p_{j,k}))$ is central in $N_{k,i}N_{j,i}/p_i(\ker(p_{j,k}))$, so the action of $N_{k,i}N_{j,i}/(N_{k,i} \cap N_{j,i})$ on the group $(N_{k,i} \cap N_{j,i})/p_i(\ker(p_{j,k}))$ (and hence on $H_1((N_{k,i} \cap N_{j,i})/p_i(\ker(p_{j,k}));\mathbb{Z})$) is trivial. Therefore,

$$H_0(N_{k,i}N_{j,i}/(N_{k,i} \cap N_{j,i}); H_1((N_{k,i} \cap N_{j,i})/p_i(\ker(p_{j,k}));\mathbb{Z})) =$$

$$H_1((N_{k,i} \cap N_{j,i})/p_i(\ker(p_{j,k}));\mathbb{Z}),$$

so $H_1((N_{k,i} \cap N_{j,i})/p_i(\ker(p_{j,k}));\mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module. We have previously noticed that $(N_{k,i} \cap N_{j,i})/p_i(\ker(p_{j,k}))$ is abelian, so this implies that

$(N_{k,i} \cap N_{j,i})/p_i(\ker(p_{j,k}))$ is finitely generated.

Now assume that $N_{k,i}N_{j,i}$ is not finitely generated. Then, by Lemma 6.4.7 either $G_i/N_{k,i}N_{j,i}$ is virtually $\mathbb{Z}$ or $\mathbb{Z}^2$.

First, let us deal with the case when $G_i/N_{k,i}N_{j,i}$ is virtually $\mathbb{Z}^2$ so that $N_{k,i}$ and $N_{j,i}$ have finite index in $N_{k,i}N_{j,i}$. In order to make it more clear, we suppose that $i = 1, k = 2, j = 3$. By Lemma 6.4.3 there is $k \in \mathbb{N} \cup \{0\}$ and an element $(b_1^k, x_2^1, x_3^1, \dots, x_n^1) \in S$ such that $\langle K_1, (b_1^k, x_2^1, x_3^1, \dots, x_n^1)\rangle$ is finitely generated.

Let us show that there is $x \in N_{2,1}$ such that $[x, b_1] = 1$. Let $\langle c_1\rangle$ be another edge group such that $[b_1, c_1] = 1$. From the proof of Lemma 6.4.7 we get that either $b_1^n c_1^m \in N_{2,1}$ for some $n \in \mathbb{N}, m \in \mathbb{Z}$ or

$$G_1 = \dot{\bigcup}_{j\in\{1,\dots,f\}} \langle N_{2,1}, b_1, c_1\rangle z_j.$$

Let A be a free abelian vertex group such that $\langle b_1\rangle < A$ and there is $a \in A$ and $a \notin \langle b_1, c_1\rangle$. Since the set $\{z_1, \dots, z_f\}$ is finite, there are $n_1 < n_2 \in \mathbb{N}$ and j_0 in $\{1, \dots, f\}$ such that

$$a^{n_1} = y_1 b_1^{k_1} c_1^{l_1} z_{j_0} \quad \text{and} \quad a^{n_2} = y_2 b_1^{k_2} c_1^{l_2} z_{j_0},$$

for some $k_1, k_2, l_1, l_2 \in \mathbb{Z}$ and $y_1, y_2 \in N_{2,1}$. Equating z_{j_0} in the previous equations we get that there is $x \in \langle a, b_1, c_1\rangle$ such that $x \in N_{2,1}$. In particular, $[x, b_1] = 1$.

Since x is an element in $N_{2,1}$ and by hypothesis $N_{2,1} \cap N_{3,1}$ has finite index in $N_{2,1}$, there is $e \in \mathbb{N}$ such that $x^e \in N_{2,1} \cap N_{3,1}$. In what follows we may assume that $e = 1$. By definition, this means that there are elements

$$(x, 1, z_3, z_4, \dots, z_n) \in S \quad \text{and} \quad (x, y_2, 1, y_4, \dots, y_n) \in S.$$

Suppose that $[z_3, x_3^1] = 1$. Note that z_3 is an element in $N_{1,3} \subseteq \langle N_{1,3}, x_3^1\rangle$. This group is a finitely generated free group (see Remark 6.4.9), so $z_3 = (x_3^1)^n$ for some $n \in \mathbb{Z}$. In particular, $(x_3^1)^n \in N_{1,3}$. Remember that $\langle N_{1,3}, x_3^1\rangle$ is finitely generated (see Remark 6.4.5), so if $(x_3^1)^n \in N_{1,3}$, then $N_{1,3}$ is also finitely generated. This is a contradiction because $G_3/N_{1,3} \cong G_1/N_{3,1}$ (see Lemma 6.4.11) and we are assuming that this is virtually $\mathbb{Z}^2$. In conclusion, z_3 and x_3^1 do not commute. In the same way it can be shown that $[y_2, x_2^1] \neq 1$.

For each $m \in \mathbb{Z}$ we can conjugate $(x, 1, z_3, \dots, z_n)$ by $(b_1^k, x_2^1, x_3^1, \dots, x_n^1)^m$ to get that

$$(x, 1, {z_3}^{(x_3^1)^m}, \dots, {z_n}^{(x_n^1)^m}) \in S.$$

In the same way, for each $l \in \mathbb{Z}$ we can conjugate $(x, y_2, 1, y_4, \dots, y_n)$ by the element

$(b_1^k, x_2^1, x_3^1, \ldots, x_n^1)^l$ to get that

$$(x, {y_2}^{(x_2^1)^l}, 1, {y_4}^{(x_4^1)^l}, \ldots, {y_n}^{(x_n^1)^l}) \in S.$$

Then, for all $m, l \in \mathbb{Z}$

$$(1, {y_2}^{(x_2^1)^l}, {z_3}^{(x_3^1)^m}, {z_4}^{(x_4^1)^m} {y_4}^{(x_4^1)^l}, \ldots, {z_n}^{(x_n^1)^m} {y_n}^{(x_n^1)^l}) \in K_1,$$

but the subspace generated by

$$\{(1, {y_2}^{(x_2^1)^l}, {z_3}^{(x_3^1)^m}, {z_4}^{(x_4^1)^m} {y_4}^{(x_4^1)^l}, \ldots, {z_n}^{(x_n^1)^m} {y_n}^{(x_n^1)^l}) \mid m, l \in \mathbb{Z}\}$$

is infinite dimensional in $H_1(K_1; \mathbb{Q})$ and this is a contradiction because by Remark 6.4.4 we have that $H_0(\langle (b_1^k, x_2^1, \ldots, x_n^1) \rangle; H_1(K_1; \mathbb{Q}))$ is finite dimensional.

Second, let us deal with the case when $G_i/N_{k,i}N_{j,i}$ is virtually $\mathbb{Z}$ and $N_{k,i}N_{j,i}$ is not finitely generated. By Lemma 6.4.6 there is an edge group $\langle b_i \rangle$ in the standard splitting of G_i such that $b_i^m \in N_{k,i}N_{j,i}$ for some $m \in \mathbb{N}$. Without loss of generality we may assume that $m = 1$. In order to make notation easier, let us assume again that $i = 1, k = 2, j = 3$.

Since b_1 is an element in $N_{2,1}N_{3,1}$, there are elements in S of the form $(n_{2,1}, 1, n_{2,3}, \ldots, n_{2,n})$ and $(n_{3,1}, n_{3,2}, 1, n_{3,4}, \ldots, n_{3,n})$ such that $n_{2,1}n_{3,1} = b_1$. From Lemma 6.4.3 we get that there is $k \in \mathbb{N} \cup \{0\}$ such that $\langle K_1, (b_1^k, x_2^1, \ldots, x_n^1) \rangle$ is finitely generated for some $(b_1^k, x_2^1, \ldots, x_n^1) \in S$. Therefore, the group

$$\langle K_1, (b_1^k, n_{3,2}^k, n_{2,3}^k, (n_{2,4}n_{3,4})^k, \ldots, (n_{2,n}n_{3,n})^k) \rangle$$

is also finitely generated.

By taking the images of that group under the homomorphisms p_2 and p_3, respectively, we get that $\langle N_{1,2}, n_{3,2}^k \rangle$ and $\langle N_{1,3}, n_{2,3}^k \rangle$ are finitely generated. Note that we have the chain

$$N_{1,2} < \langle N_{1,2}, n_{3,2}^k \rangle < N_{1,2}N_{3,2} < G_2,$$

and by Lemma 6.4.7, $G_2/N_{1,2}$ is virtually $\mathbb{Z}^m$ for some $m \in \{0, 1, 2\}$. Therefore, there are three options:

(i) $N_{1,2}$ has finite index in $\langle N_{1,2}, n_{3,2}^k \rangle$, so $N_{1,2}$ is finitely generated. In particular, by Proposition 5.4.4, $G_2/N_{1,2}$ is finite or virtually $\mathbb{Z}$. Therefore, either $N_{1,2}N_{3,2}$ has finite index in G_2 or $N_{1,2}$ has finite index in $N_{1,2}N_{3,2}$. In both of the cases $N_{1,2}N_{3,2}$ is finitely generated.

(ii) $N_{1,2}N_{3,2}$ has finite index in G_2. Thus, $N_{1,2}N_{3,2}$ is finitely generated.

(iii) $\langle N_{1,2}, n_{3,2}^k \rangle$ has finite index in $N_{1,2}N_{3,2}$. Hence, $N_{1,2}N_{3,2}$ is finitely generated.

We can apply the same argument for the case $N_{1,3}$. In summary, we get that $N_{1,2}N_{3,2}$ and $N_{1,3}N_{2,3}$ are finitely generated.

Remember that we want to prove that $(N_{k,i} \cap N_{j,i})/p_i(\ker(p_{j,k}))$ is finitely generated and we are under the assumptions that $N_{k,i}N_{j,i}$ is infinitely generated and that $G_i/N_{k,i}N_{j,i}$ is virtually $\mathbb{Z}$. We have just shown that in this case $N_{i,k}N_{j,k}$ and $N_{i,j}N_{k,j}$ are finitely generated.

The group $N_{i,k}N_{j,k}$ is finitely generated, so as in (6.3) we obtain that the quotient group $(N_{i,k} \cap N_{j,k})/p_k(\ker(p_{i,j}))$ is finitely generated. There is a short exact sequence

$$1 \to (N_{i,k} \cap N_{j,k})/p_k(\ker(p_{i,j})) \to N_{j,k}/p_k(\ker(p_{i,j})) \to$$

$$\to N_{j,k}/(N_{i,k} \cap N_{j,k}) \to 1,$$

and both $(N_{i,k} \cap N_{j,k})/p_k(\ker(p_{i,j}))$ and $N_{j,k}/(N_{i,k} \cap N_{j,k})$ are finitely generated. Hence, the group $N_{j,k}/p_k(\ker(p_{i,j}))$ is finitely generated, so $N_{j,i}/p_i(\ker(p_{j,k}))$ is also finitely generated (see Lemma 6.4.12). The same argument as in (6.3) can be used in order to prove that this implies that $(N_{j,i} \cap N_{k,i})/p_i(\ker(p_{j,k}))$ is finitely generated. In conclusion, the group $N_{j,i}/p_i(\ker(p_{j,k}))$ is (finitely generated abelian)-by-(virtually finitely generated abelian). In particular, every subgroup is finitely generated.

Now suppose that $s \geq 2$. Observe that there is a chain

$$p_i(\ker(p_{j_1,\dots,j_s,k})) < p_i(\ker(p_{j_1,\dots,j_s})) \cap p_i(\ker(p_{j_1,\dots,j_{s-1},k})) <$$

$$p_i(\ker(p_{j_1,\dots,j_s})) \cap N_{k,i} < p_i(\ker(p_{j_1,\dots,j_s})),$$

and the quotient group

$$p_i(\ker(p_{j_1,\dots,j_s}))/(p_i(\ker(p_{j_1,\dots,j_s})) \cap p_i(\ker(p_{j_1,\dots,j_{s-1},k})))$$

embeds in $p_i(\ker(p_{j_1,\dots,j_{s-1}}))/p_i(\ker(p_{j_1,\dots,j_{s-1},k}))$. Therefore, by inductive hypothesis this group is (finitely generated abelian)-by-(virtually finitely generated abelian). Hence,

$$(p_i(\ker(p_{j_1,\dots,j_s})) \cap N_{k,i})/(p_i(\ker(p_{j_1,\dots,j_s})) \cap p_i(\ker(p_{j_1,\dots,j_{s-1},k})))$$

is finitely generated. Note that if we show that the group

$$(p_i(\ker(p_{j_1,\dots,j_s})) \cap p_i(\ker(p_{j_1,\dots,j_{s-1},k})))/p_i(\ker(p_{j_1,\dots,j_s,k}))$$

is finitely generated, we would have that $(p_i(\ker(p_{j_1,\dots,j_s})) \cap N_{k,i})/p_i(\ker(p_{j_1,\dots,j_s,k}))$ is finitely generated. Moreover, by the definition of $p_i(\ker(p_{j_1,\dots,j_s,k}))$, it is abelian. Hence, since we have the short exact sequence

$$1 \to (p_i(\ker(p_{j_1,\dots,j_s})) \cap N_{k,i})/p_i(\ker(p_{j_1,\dots,j_s,k})) \to p_i(\ker(p_{j_1,\dots,j_s}))/p_i(\ker(p_{j_1,\dots,j_s,k})) \to$$

$$\to p_i(\ker(p_{j_1,\dots,j_s}))/(p_i(\ker(p_{j_1,\dots,j_s})) \cap N_{k,i}) \to 1,$$

we would have that $p_i(\ker(p_{j_1,\dots,j_s}))/p_i(\ker(p_{j_1,\dots,j_s,k}))$ is (finitely generated abelian)-by-(virtually finitely generated abelian). To sum up, it suffices to show that

$$(p_i(\ker(p_{j_1,\dots,j_s})) \cap p_i(\ker(p_{j_1,\dots,j_{s-1},k})))/p_i(\ker(p_{j_1,\dots,j_s,k}))$$

is finitely generated.

If $N_{k,i}N_{j_s,i}$ is finitely generated, then we can use the same argument as in (6.3) to prove that this group is finitely generated. Thus, we may assume that $N_{k,i}N_{j_s,i}$ is infinitely generated and that $G_i/N_{k,i}N_{j_s,i}$ is virtually $\mathbb{Z}$. Therefore, $N_{i,k}N_{j_s,k}$ and $N_{i,j_s}N_{k,j_s}$ are finitely generated, so the group

$$(p_k(\ker(p_{j_1,\dots,j_s})) \cap p_k(\ker(p_{j_1,\dots,j_{s-1},i})))/p_k(\ker(p_{j_1,\dots,j_s,i}))$$

is finitely generated (see the argument in (6.3)). Hence, from the short exact sequence

$$1 \to (p_k(\ker(p_{j_1,\dots,j_s})) \cap p_k(\ker(p_{j_1,\dots,j_{s-1},i})))/p_k(\ker(p_{j_1,\dots,j_s,i})) \to$$

$$\to p_k(\ker(p_{j_1,\dots,j_s}))/p_k(\ker(p_{j_1,\dots,j_s,i})) \to$$

$$\to p_k(\ker(p_{j_1,\dots,j_s}))/(p_k(\ker(p_{j_1,\dots,j_s})) \cap p_k(\ker(p_{j_1,\dots,j_{s-1},i}))) \to 1$$

we get that $p_k(\ker(p_{j_1,\dots,j_s}))/p_k(\ker(p_{j_1,\dots,j_s,i}))$ is finitely generated. We can apply Lemma 6.4.12 to obtain that $p_i(\ker(p_{j_1,\dots,j_s}))/p_k(\ker(p_{j_1,\dots,j_s,k}))$ is finitely generated. In conclusion, again as in (6.3), we get that

$$(p_i(\ker(p_{j_1,\dots,j_s})) \cap p_i(\ker(p_{j_1,\dots,j_{s-1},k})))/p_i(\ker(p_{j_1,\dots,j_s,k}))$$

is finitely generated. $\square$

Theorem 6.4.14. *Let S be a finitely presented full subdirect product of $G_1 \times \cdots \times G_n$ where each G_i is either a finitely generated residually finite free product or G_i is a finitely generated group in $\mathcal{G}'$. Then, for each $i \in \{1, \ldots, n\}$, N_i/L_i is finitely generated.*

Proof. We prove it for $n = 1$ so that the notation is easier. Note that we have the following diagram:

$$\begin{array}{c} N_1 = N_{2,1} \cap N_{3,1} \cap \cdots \cap N_{n,1} \\ | \\ p_1(\ker(p_{2,3})) \cap N_{4,1} \cap \cdots \cap N_{n,1} \\ | \\ p_1(\ker(p_{2,3,4})) \cap N_{5,1} \cap \cdots \cap N_{n,1} \\ \vdots \\ p_1(\ker(p_{2,3,4,\ldots,n-1})) \cap N_{n,1} \\ | \\ L_1 = p_1(\ker(p_{2,3,4,\ldots,n})) \end{array}$$

and for $i \in \{2, \ldots, n-1\}$ each quotient

$$(p_1(\ker(p_{2,\ldots,i})) \cap N_{i+1} \cap \cdots \cap N_{n,1})/(p_1(\ker(p_{2,\ldots,i+1})) \cap N_{i+2} \cap \cdots \cap N_{n,1})$$

embeds in $p_1(\ker(p_{2,\ldots,i}))/p_1(\ker(p_{2,\ldots,i+1}))$. In particular, by Theorem 6.4.13

$$(p_1(\ker(p_{2,\ldots,i})) \cap N_{i+1} \cap \cdots \cap N_{n,1})/(p_1(\ker(p_{2,\ldots,i+1})) \cap N_{i+2} \cap \cdots \cap N_{n,1})$$

is finitely generated. In conclusion, N_1/L_1 is finitely generated. $\square$

Theorem 6.4.15. *Let $G_1, \ldots, G_n$ be finitely generated groups in $\mathcal{G}'$ and suppose that S is a full subdirect product of $G_1 \times \cdots \times G_n$. If L_i is $S \cap G_i$ for $i \in \{1, \ldots, n\}$, then G_i/L_i is virtually (finitely generated nilpotent)-by-(finitely generated free abelian).*

Proof. It is a direct consequence of Lemma 6.4.10 and Theorem 6.4.13. $\square$

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