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Optimum Solutions of Systems of Differential Equations via Best Proximity Points in *b*-Metric Spaces

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Abstract: This paper deals with the existence of an optimum solution of a system of ordinary differential equations via the best proximity points. In order to obtain the optimum solution, we have developed the best proximity point results for generalized multivalued contractions of *b*-metric spaces. Examples are given to illustrate the main results and to show that the new results are the proper generalization of some existing results in the literature.

Keywords: best proximity points; multivalued mapping; cyclic contractions; *b*-metric spaces; optimum solution

MSC: 47H10; 47H09; 47H04; 41A50; 34A12

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1. Introduction and Preliminaries

Best proximity point theory provides basic tools to find approximate solutions of problems in mathematics and related disciplines, particularly whenever an exact solution does not exist.

For a non-self-mapping $\mathcal{T} : M \to N$, where M and N are two nonempty subsets of a nonempty set Ω, a point $m \in M$ is an exact solution or the fixed point (FP) of $\mathcal{T}$ if $m = \mathcal{T}m$. In the case where M and $\mathcal{T}(M)$ have an empty intersection, then $\mathcal{T}$ has no FP. For such situations, it is better to find a point $m \in M$ such that the distance between m and $\mathcal{T}m$ is minimized. That is,

$$\varsigma(m, \mathcal{T}m) = \varsigma(M, N) \tag{1}$$

where

$$\varsigma(M, N) = \inf_{m \in M, n \in N} \varsigma(m, n),$$

and ς is a metric on Ω. A point m in M that satisfies (1) is called the best proximity point (BPP) of $\mathcal{T}$. In the literature, many mathematicians have contributed to the development of the BPP theory of metric spaces. The main objective of this theory is to develop necessary and sufficient conditions that ensure the existence of best proximity points (BPP(s)) of $\mathcal{T}$ (a non-self-mapping of certain distance space). For more details, one can see the references [1–4].

If $M = N = \Omega$, in (1) (that is, $\mathcal{T}$ is self-mapping), then $\varsigma(m, \mathcal{T}m) = 0$ or $m = \mathcal{T}m$. In this case, m becomes an FP of $\mathcal{T}$. Therefore, BPP theory is a natural generalization of FP theory.

In 1969, Fan [5] provided a remarkable result in BPP theory. After that, many mathematicians have contributed to the development of BPP theory with different proximal contractions [6–8].

One interesting proximal contraction is the $\alpha - \psi-$proximal (α_ψ) contraction by Jleli and Samet [9], and they have developed some BPP(s) results in metric space (in short, MS). Abkar and Gabeleh [10] developed some BPP(s) for Suzuki-type contractions. Hussain et al. [11] generalized the α_ψ contraction to the Suzuki-type α_ψ contraction and developed some BPP(s) results for it.

Recently, Khan et al. [12] generalized the contraction used in [11] and developed some BPP(s) results in the domain of MS.

After the development of fixed points (FP) results for multivalued mappings by Nadler [13] in 1969, many mathematicians extended BPP theory from single-valued mappings to multivalued mappings. For instance, Ali et al. [14] in 2014 extended the α_ψ contraction to α_ψ multivalued contractions and developed some BPP(s) results for them.

Later on, MS was extended to the b-metric space (b-MS) by Bakhtin [15] in 1989 and by Czerwik [16] in 1993. After that, a new area of research for the existence of BPP in b-MS is opened up, and many researchers have developed BPP(s) results for single- as well as multivalued mappings in the domain of b-MS. For more details, one can see the references [17–21].

In this paper, we introduce a new multivalued Suzuki-type α_ψ (cyclic) contractions in the domain of b-MS and develop some BPP(s) results. Examples have been given to explain our main results and to show that our main results are the proper generalization of results given in [12]. As an application of our results, we develop the optimum solution for a system of ordinary differential equations.

Definition 1 ([15])**.** *The mapping $\varsigma : \Omega \times \Omega \to [0, \infty)$ is a b-metric, and (Ω, ς) is called b-MS if the following hold:*

($b1$) $\varsigma(\varkappa_1, \varkappa_2) = 0$ if and only if $\varkappa_1 = \varkappa_2$ for all $\varkappa_1, \varkappa_2 \in \Omega$;
($b2$) $\varsigma(\varkappa_1, \varkappa_2) = \varsigma(\varkappa_2, \varkappa_1)$ for all $\varkappa_1, \varkappa_2 \in \Omega$;
($b3$) There exists a real number $k \geq 1$ such that $\varsigma(\varkappa_1, \varkappa_2) \leq k[\varsigma(\varkappa_1, \varkappa_3) + \varsigma(\varkappa_3, \varkappa_2)]$ for all $\varkappa_1, \varkappa_2, \varkappa_3 \in \Omega$.

Remark 1. *If $k = 1$, then ς becomes a metric.*

In this article, $\mathbb{R}^+$, $\mathbb{R}$, $\mathbb{N}$, $\mathbb{N}_1, 2^\Omega \backslash \emptyset$, denote the set of non-negative reals, reals, positive integers, non-negative integers, and nonempty subsets of Ω, respectively. Define

$$M_0 = \{m \in M : \varsigma(m, n) = \varsigma(M, N) \text{ for some } n \in N\} \text{ and}$$
$$N_0 = \{n \in N : \varsigma(m, n) = \varsigma(M, N) \text{ for some } m \in M\}$$

where $M, N \in 2^\Omega \backslash \emptyset$. If M_0 is nonempty, then (M, N) has a weak $P-$property (shortly as weak P_p) (compare with [22]) if

$$\begin{cases} \varsigma(m_1, n_1) = \varsigma(M, N), \\ \varsigma(m_2, n_2) = \varsigma(M, N), \end{cases} \text{implies } \varsigma(m_1, m_2) \leq \varsigma(n_1, n_2),$$

for all $m_1, m_2 \in M$ and $n_1, n_2 \in N$.

Definition 2. *A mapping $\mathcal{T} : M \cup N \to 2^M \backslash \emptyset \cup 2^N \backslash \emptyset$ is said to be cyclic if $\mathcal{T}(m) \subset N$ for all $m \in M$ and $\mathcal{T}(n) \subset M$ for all $n \in N$.*

In the following, we introduce multivalued $\alpha-$proximal admissibles with respect to p (for short, $m - \alpha_p$) for multivalued mappings (compared with [12]).

Definition 3. *A mapping $\mathcal{T} : M \to 2^N \backslash \emptyset$ is $m - \alpha_p$ if*

$$\begin{cases} \alpha(m_1, m_2) \geq p(m_1, m_2), \\ \varsigma(m_3, n_1) = \varsigma(M, N), \quad \text{implies } \alpha(m_3, m_4) \geq p(m_3, m_4), \\ \varsigma(m_4, n_2) = \varsigma(M, N), \end{cases}$$

for all $m_1, m_2, m_3, m_4 \in M$ and $n_1 \in \mathcal{T}m_1$, $n_2 \in \mathcal{T}m_2$, where $\alpha : M \times M \to [0, \infty)$ and $p : M \times M \to [1, \infty)$.

Remark 2.

(i) *If $\mathcal{T} : M \to 2^N \backslash \emptyset$ is replaced by $\mathcal{T} : M \to N$, then $\mathcal{T}$ is $\alpha-$proximal admissible with respect to p (shortly as α_p) (see [12]).*
(ii) *If $p = 1$ in the Definition 3, then $\mathcal{T}$ is called multivalued $\alpha-$proximal admissible (compare with [14]).*
(iii) *If $p = 1$ and $2^N \backslash \emptyset$ is replaced by N in the Definition 3, then $\mathcal{T}$ is called $\alpha-$proximal admissible (compare with [9]).*
(iv) *If $M = N = \Omega$ in the Definition 3, then $\mathcal{T}$ is called $\alpha-$admissible with respect to p (for short, $\alpha - p$).*

Consider the following class:

Ψ is a class of functions $\psi : [0, \infty) \to [0, \infty)$, such that ψ is monotone increasing and there exist $\mu_0 \in \mathbb{N}$, $a \in (0, 1)$, $b \in [1, \infty)$, and a convergent series of non-negative numbers $\sum_{\mu=1}^{\infty} u_\mu$ such that for any $\Upsilon \geq 0$,

$$b^{\mu+1}\psi^{\mu+1}(\Upsilon) \leq ab^\mu \psi^\mu(\Upsilon) + u_\mu$$

for all $\mu \geq \mu_0$. A function $\psi \in \Psi$ is a "Bianchini-Grandolfi gauge function (also known as (c)-comparison function)".

Lemma 1 ([23]). *If $\psi \in \Psi$, then*

(i) $(\psi^\mu(\Upsilon))_{\mu \in \mathbb{N}}$ *converges to* 0 *as* $\mu \to \infty$ *for all* $\Upsilon \in \mathbb{R}^+$;
(ii) $\psi(\Upsilon) < \Upsilon$, *for any* $\Upsilon \in (0, \infty)$;
(iii) ψ *is continuous at* 0;
(iv) *The series* $\sum_{\mu=0}^{\infty} b^\mu \psi^\mu(\Upsilon)$ *converges for any* $\Upsilon \in \mathbb{R}^+$.

Throughout this article, we denote $k\varsigma^*(\varkappa_1, \varkappa_2) = \varsigma(\varkappa_1, \varkappa_2) - k\varsigma(M, N)$; $CL(\Omega)$ as the closed subsets of Ω; $K(\Omega)$ as the compact subsets of Ω; BPP$(\mathcal{T})$ as the set of BPP(s) of $\mathcal{T}$; and FP$(\mathcal{T})$ as the set of FP(s) of $\mathcal{T}$.

Definition 4. *Let (Ω, ς) be a b-MS and for every $M, N \in 2^\Omega$, the Pompeiu–Hausdorff metric induced by ς is given by*

$$H(M, N) = \begin{cases} \max\{\sup_{m \in M} \varsigma(m, N), \sup_{n \in N} \varsigma(M, n)\}, \text{ if } M \neq N \neq \emptyset \\ 0, \text{ if } M = N = \emptyset \\ +\infty, \text{ otherwise} \end{cases}$$

where $\varsigma(m, N) = \inf\{\varsigma(m, n), n \in N\}$.

Definition 5 ([12]). *Let (Ω, ς) be an MS, $M, N \in CL(\Omega)$ and $\alpha : M \times M \to [0, \infty)$. $\mathcal{T} : M \to N$ is a Suzuki-type generalized α_ψ contraction if*

$$\varsigma(m_1, \mathcal{T}m_1) - \varsigma(M, N) \leq \alpha(m_1, m_2)\varsigma(m_1, m_2) \tag{2}$$

$$\text{implies} \quad \varsigma(\mathcal{T}m_1, \mathcal{T}m_2) \leq \psi(\Gamma(m_1, m_2)), \tag{3}$$

for all $m_1, m_2 \in M$, *where* $\psi \in \Psi$ *and*

$$\Gamma(m_1, m_2) = \max\left\{\begin{array}{l} \varsigma(m_1, m_2), \varsigma(m_1, \mathcal{T}m_1) - \varsigma(M, N), \\ \varsigma(m_2, \mathcal{T}m_2) - \varsigma(M, N), \varsigma(m_2, \mathcal{T}m_1) - \varsigma(M, N), \\ \dfrac{\varsigma(m_1, \mathcal{T}m_2) - \varsigma(M, N)}{2}, \\ \dfrac{(\varsigma(m_1, \mathcal{T}m_1) - \varsigma(M, N))(\varsigma(m_2, \mathcal{T}m_2) - \varsigma(M, N))}{1 + (\varsigma(m_1, m_2))} \end{array}\right\}.$$

Theorem 1 ([12])**.** *Let* (Ω, ς) *be a complete MS, and* $M, N \in CL(\Omega)$ *with* (M, N) *has* P_p*.* $\mathcal{T} : M \to N$ *is* α_p*, a Suzuki-type generalized* α_ψ *contraction, and for nonempty set* M_0*,* $\mathcal{T}(M_0) \subseteq N_0$*. Also suppose* $\alpha(m_0, m_1) \geq p(m_0, m_1)$ *and* $\varsigma(m_1, \mathcal{T}m_0) = \varsigma(M, N)$ *for some* m_0*,* m_1 *in* M_0*, and* $\mathcal{T}$ *are continuous. Then,* $BPP(\mathcal{T})$ *is singleton.*

In the following, we introduce generalized multivalued Suzuki-type α_ψ contractions in b-MS.

Definition 6. *Let* (Ω, ς) *be a b-MS,* $M, N \in CL(\Omega)$*.* $\mathcal{T} : M \to CL(N)$ *is called a generalized multivalued Suzuki-type* α_ψ *contraction of* ϖ *type if*

$$\varsigma^*(m_1, \mathcal{T}m_1) \leq \alpha(m_1, m_2)\varsigma(m_1, m_2) \text{ implies } H(\mathcal{T}m_1, \mathcal{T}m_2) \leq \psi(\varpi(m_1, m_2)), \tag{4}$$

for all $m_1, m_2 \in M$, *where* $\alpha : M \times M \to [0, \infty)$, $\psi \in \Psi$ *and*

$$\varpi(m_1, m_2) = \max\left\{\begin{array}{l} \varsigma(m_1, m_2), \dfrac{\varsigma(m_1, \mathcal{T}m_1) - k\varsigma(M, N)}{k}, \\ \dfrac{\varsigma(m_2, \mathcal{T}m_2) - k\varsigma(M, N)}{k}, \varsigma(m_2, \mathcal{T}m_1) - \varsigma(M, N), \\ \dfrac{\varsigma(m_1, \mathcal{T}m_2) - k\varsigma(M, N)}{2k^2}, \\ \dfrac{(\varsigma(m_1, \mathcal{T}m_1) - k\varsigma(M, N))(\varsigma(m_2, \mathcal{T}m_2) - k\varsigma(M, N))}{k(1 + k(\varsigma(m_1, m_2))} \end{array}\right\}.$$

Definition 7. *Let* (Ω, ς) *be a b-MS,* $M, N \in CL(\Omega)$*.* $\mathcal{T} : M \to CL(N)$ *is called a generalized multivalued Suzuki-type* α_ψ *contraction of* ξ *type if*

$$\varsigma^*(m_1, \mathcal{T}m_1) \leq \alpha(m_1, m_2)\varsigma(m_1, m_2) \text{ implies } H(\mathcal{T}m_1, \mathcal{T}m_2) \leq \psi(\xi(m_1, m_2)), \tag{5}$$

for all $m_1, m_2 \in M$, *where* $\alpha : M \times M \to [0, \infty)$, $\psi \in \Psi$ *and*

$$\xi(m_1, m_2) = \max\left\{\begin{array}{l} \varsigma(m_1, m_2), \dfrac{\varsigma(m_1, \mathcal{T}m_1) - k\varsigma(M, N)}{k} \\ \dfrac{\varsigma(m_2, \mathcal{T}m_2) - k\varsigma(M, N)}{k}, \dfrac{\varsigma(m_2, \mathcal{T}m_1) - k\varsigma(M, N)}{k} \end{array}\right\}.$$

Definition 8. *Let* (Ω, ς) *be a b-MS,* $M, N \in CL(\Omega)$*.* $\mathcal{T} : M \cup N \to CL(M) \cup CL(N)$ *is called a generalized multivalued Suzuki-type* α_ψ *cyclic contraction of* ϖ *type if*

$$\varsigma^*(m_1, \mathcal{T}m_1) \leq \alpha(m_1, m_2)\varsigma(m_1, m_2) \text{ implies } H(\mathcal{T}m_1, \mathcal{T}m_2) \leq \psi(\varpi(m_1, m_2)),$$

for all $m_1, m_2 \in M \cup N$, *where* $\alpha : M \cup N \times M \cup N \to [0, \infty)$, $\psi \in \Psi$ *and*

$$\varpi(m_1, m_2) = \max\left\{\begin{array}{l} \varsigma(m_1, m_2), \dfrac{\varsigma(m_1, \mathcal{T}m_1) - k\varsigma(M, N)}{k}, \\ \dfrac{\varsigma(m_2, \mathcal{T}m_2) - k\varsigma(M, N)}{k}, \varsigma(m_2, \mathcal{T}m_1) - \varsigma(M, N), \\ \dfrac{\varsigma(m_1, \mathcal{T}m_2) - k\varsigma(M, N)}{2k^2}, \\ \dfrac{(\varsigma(m_1, \mathcal{T}m_1) - k\varsigma(M, N))(\varsigma(m_2, \mathcal{T}m_2) - k\varsigma(M, N))}{k(1 + k(\varsigma(m_1, m_2))} \end{array}\right\}.$$

Definition 9. *Let* (Ω, ς) *be a b-MS,* $M, N \in CL(\Omega)$. $\mathcal{T} : M \cup N \to CL(M) \cup CL(N)$ *is called a generalized multivalued Suzuki-type* α_ψ *cyclic contraction of* ξ *type if*

$$\varsigma^*(m_1, \mathcal{T}m_1) \leq \alpha(m_1, m_2)\varsigma(m_1, m_2) \text{ implies } H(\mathcal{T}m_1, \mathcal{T}m_2) \leq \psi(\varpi(m_1, m_2)),$$

for all $m_1, m_2 \in M \cup N$, *where* $\alpha : M \cup N \times M \cup N \to [0, \infty)$, $\psi \in \Psi$ *and*

$$\xi(m_1, m_2) = \max\left\{\begin{array}{l} \varsigma(m_1, m_2), \dfrac{\varsigma(m_1, \mathcal{T}m_1) - k\varsigma(M, N)}{k} \\ \dfrac{\varsigma(m_2, \mathcal{T}m_2) - k\varsigma(M, N)}{k}, \dfrac{\varsigma(m_2, \mathcal{T}m_1) - k\varsigma(M, N)}{k} \end{array}\right\}.$$

Remark 3.

(i) *If in Definitions 6 and 7,* $\mathcal{T} : M \to CL(N)$ *is replaced by* $\mathcal{T} : M \to N$, *then* $\mathcal{T}$ *is called a generalized Suzuki-type* α_ψ *contraction of* ϖ *type and a generalized Suzuki-type* α_ψ *cyclic contraction of* ξ *type, respectively.*

(ii) *If in Definition 6* $\mathcal{T} : M \to CL(N)$ *is replaced by* $\mathcal{T} : M \to N$, *and* ϖ *is replaced by* ϖ', *where*

$$\varpi'(m_1, m_2) = \max\left\{\begin{array}{l} \varsigma(m_1, m_2), \dfrac{\varsigma(m_1, \mathcal{T}m_1) - k\varsigma(M, N)}{k}, \\ \dfrac{\varsigma(m_2, \mathcal{T}m_2) - k\varsigma(M, N)}{k}, \\ \dfrac{\varsigma(m_2, \mathcal{T}m_1) - k\varsigma(M, N)}{k}, \\ \dfrac{\varsigma(m_1, \mathcal{T}m_2) - k\varsigma(M, N)}{2k^2}, \\ \dfrac{(\varsigma(m_1, \mathcal{T}m_1) - k\varsigma(M, N))(\varsigma(m_2, \mathcal{T}m_2) - k\varsigma(M, N))}{k(1 + k(\varsigma(m_1, m_2))} \end{array}\right\},$$

then $\mathcal{T}$ *is called a generalized Suzuki-type* α_ψ *contraction of* ϖ' *type.*

(iii) *If in Definitions 8 and 9* $\mathcal{T} : M \cup N \to CL(M) \cup CL(N)$ *is replaced by* $\mathcal{T} : M \cup N \to M \cup N$, *then* $\mathcal{T}$ *is called a generalized Suzuki-type* α_ψ *cyclic contraction of* ϖ *type and a generalized Suzuki-type* α_ψ *cyclic contraction of* ξ *type, respectively.*

2. Best Proximity Points Results for Generalized Multivalued Suzuki-Type α_ψ Contractions

The following is our main result of this section.

Theorem 2. *Let* (Ω, ς) *be a complete b-MS (b-CMS)* $M, N \in CL(\Omega)$ *with* $M_0 \neq \phi$. *Let* $\mathcal{T} : M \to CL(N)$ *be a generalized multivalued Suzuki-type* α_ψ *contraction of* ϖ *type satisfying:*

1. *For each* $m \in M_0$, $\mathcal{T}(m) \subseteq N_0$ *and* (M, N) *has a weak* P_p;
2. $\mathcal{T}$ *is* $m - \alpha_p$;
3. *There exist elements* m_0 *and* m_1 *in* M_0 *and* $n_1 \in \mathcal{T}m_0$ *such that* $\varsigma(m_1, n_1) = \varsigma(M, N)$ *and* $\alpha(m_0, m_1) \geq p(m_0, m_1)$;
4. $\mathcal{T}$ *is continuous.*

Then, $BPP(\mathcal{T})$ *is nonempty.*

Proof. From (3), there exist m_0 and m_1 in M_0 and $n_1 \in \mathcal{T}m_0$, such that

$$\varsigma(m_1, n_1) = \varsigma(M, N), \ \alpha(m_0, m_1) \geq p(m_0, m_1); \tag{6}$$

if $n_1 \in \mathcal{T}m_1$, then

$$\varsigma(M, N) \leq \varsigma(m_1, \mathcal{T}m_1) \leq \varsigma(m_1, n_1) = \varsigma(M, N),$$

which implies $\varsigma(m_1, \mathcal{T}m_1) = \varsigma(M, N)$. That is, m_1 is the BPP of $\mathcal{T}$. Next, if $n_1 \notin \mathcal{T}m_1$, then,

$$\begin{aligned} \varsigma(m_0, \mathcal{T}m_0) &\leq \varsigma(m_0, n_1) \leq k\varsigma(m_0, m_1) + k\varsigma(m_1, n_1), \\ \varsigma(m_0, \mathcal{T}m_0) &\leq k\varsigma(m_0, m_1) + k\varsigma(M, N), \end{aligned}$$

therefore,

$$k\varsigma^*(m_0, \mathcal{T}m_0) \leq k\varsigma(m_0, m_1).$$

Thus, we get:

$$\varsigma^*(m_0, \mathcal{T}m_0) \leq \varsigma(m_0, m_1) \leq p(m_0, m_1)\varsigma(m_0, m_1) \leq \alpha(m_0, m_1)\varsigma(m_0, m_1).$$

From (4), we get:

$$\begin{aligned} &H(\mathcal{T}m_0, \mathcal{T}m_1) \leq \psi(\varpi(m_0, m_1)) \\ &\leq \psi\left(\max\left\{\begin{array}{l} \varsigma(m_0, m_1), \dfrac{\varsigma(m_0, \mathcal{T}m_0) - k\varsigma(M, N)}{k}, \\ \dfrac{\varsigma(m_1, \mathcal{T}m_1) - k\varsigma(M, N)}{k}, \varsigma(m_1, \mathcal{T}m_0) - \varsigma(M, N), \\ \dfrac{\varsigma(m_0, \mathcal{T}m_1) - k\varsigma(M, N)}{2k^2}, \\ \dfrac{(\varsigma(m_0, \mathcal{T}m_0) - k\varsigma(M, N)(\varsigma(m_1, \mathcal{T}m_1) - k\varsigma(M, N))}{k(1 + k\varsigma(m_0, m_1))} \end{array}\right\}\right) \\ &\leq \psi\left(\max\left\{\begin{array}{l} \varsigma(m_0, m_1), \dfrac{\varsigma(m_0, n_1) - k\varsigma(M, N)}{k}, \\ \dfrac{\varsigma(m_1, \mathcal{T}m_1) - k\varsigma(M, N)}{k}, \varsigma(m_1, n_1) - \varsigma(M, N), \\ \dfrac{\varsigma(m_0, \mathcal{T}m_1) - k^2\varsigma(M, N)}{2k^2}, \\ \dfrac{(\varsigma(m_0, n_1) - k\varsigma(M, N)(\varsigma(m_1, \mathcal{T}m_1) - k\varsigma(M, N))}{k(1 + k\varsigma(m_0, m_1))} \end{array}\right\}\right). \end{aligned}$$

Hence,

$$H(\mathcal{T}m_0, \mathcal{T}m_1) \leq \psi \max\{\varsigma(m_0, m_1), \varsigma(n_1, \mathcal{T}m_1)\}.$$

Consequently,

$$\varsigma(n_1, \mathcal{T}m_1) \leq H(\mathcal{T}m_0, \mathcal{T}m_1) \leq \psi \max\{\varsigma(m_0, m_1), \varsigma(n_1, \mathcal{T}m_1)\}.$$

If $\max\{\varsigma(m_0, m_1), \varsigma(n_1, \mathcal{T}m_1)\} = \varsigma(n_1, \mathcal{T}m_1)$, then

$$\varsigma(n_1, \mathcal{T}m_1) \leq \psi(\varsigma(n_1, \mathcal{T}m_1) < \varsigma(n_1, \mathcal{T}m_1),$$

which is a contradiction. Hence,

$$\varsigma(n_1, \mathcal{T}m_1) \leq \psi(\varsigma(m_0, m_1)). \tag{7}$$

Now for $q > 1$, there exists $n_2 \in \mathcal{T}m_1$ such that

$$\varsigma(n_1, n_2) < q\varsigma(n_1, \mathcal{T}m_1),$$

and using (7), we have

$$\varsigma(n_1, n_2) < q\psi(\varsigma(m_0, m_1)). \tag{8}$$

As $n_2 \in \mathcal{T}m_1 \subseteq N_0$, there exists $m_2 \in M_0$ such that

$$\varsigma(m_2, n_2) = \varsigma(M, N). \tag{9}$$

Note that $m_2 \neq m_1$; otherwise, m_1 becomes the BPP of $\mathcal{T}$. From (6) and (9), we get

$$\begin{aligned}
&\alpha(m_0, m_1) \geq p(m_0, m_1),\\
&\varsigma(m_1, n_1) = \varsigma(M, N),\\
&\varsigma(m_2, n_2) = \varsigma(M, N).
\end{aligned}$$

As $\mathcal{T}$ is $m - \alpha_p$, and (M, N) satisfies the weak P_p, we obtain

$$\alpha(m_1, m_2) \geq p(m_1, m_2),\ \varsigma(m_1, m_2) \leq \varsigma(n_1, n_2). \tag{10}$$

From (8) and (10), we get:

$$\varsigma(m_1, m_2) \leq \varsigma(n_1, n_2) < q\psi(\varsigma(m_0, m_1)). \tag{11}$$

Since ψ is strictly increasing, therefore,

$$\psi\varsigma(m_1, m_2) < \psi(q\psi(\varsigma(m_0, m_1))).$$

Set

$$q_1 = \frac{\psi(q\psi(\varsigma(m_0, m_1)))}{\psi(\varsigma(m_1, m_2))} > 1. \tag{12}$$

If $n_2 \in \mathcal{T}m_2$, then m_2 is the BPP of $\mathcal{T}$ and the proof completes. So, suppose $n_2 \notin \mathcal{T}m_2$, then

$$\varsigma(m_1, \mathcal{T}m_1) \leq \varsigma(m_1, n_2) \leq k\varsigma(m_1, m_2) + k\varsigma(m_2, n_2);$$

therefore,

$$\varsigma^*(m_1, \mathcal{T}m_1) \leq \varsigma(m_1, m_2) \leq p(m_1, m_2)\varsigma(m_1, m_2) \leq \alpha(m_1, m_2)\varsigma(m_1, m_2).$$

From (4), we get:

$$H(\mathcal{T}m_1, \mathcal{T}m_2) \leq \psi(\varpi(m_1, m_2)),$$

where

$$\begin{aligned}
\varpi(m_1, m_2) &= \max\left\{\begin{array}{l}
\varsigma(m_1, m_2), \dfrac{\varsigma(m_1, \mathcal{T}m_1) - k\varsigma(M, N)}{k},\\
\dfrac{\varsigma(m_2, \mathcal{T}m_2) - k\varsigma(M, N)}{k}, \varsigma(m_2, \mathcal{T}m_1) - \varsigma(M, N),\\
\dfrac{\varsigma(m_1, \mathcal{T}m_2) - k\varsigma(M, N)}{2k^2},\\
\dfrac{(\varsigma(m_1, \mathcal{T}m_1) - k\varsigma(M, N)(\varsigma(m_2, \mathcal{T}m_2) - k\varsigma(M, N))}{k(1 + k\varsigma(m_1, m_2))}
\end{array}\right\}\\
&\leq \max\{\varsigma(m_1, m_2), \varsigma(n_2, \mathcal{T}m_2)\}.
\end{aligned}$$

Hence,

$$H(\mathcal{T}m_1, \mathcal{T}m_2) \leq \psi \max\{\varsigma(m_1, m_2), \varsigma(n_2, \mathcal{T}m_2)\}.$$

This implies

$$\varsigma(n_2, \mathcal{T}m_2) \leq H(\mathcal{T}m_1, \mathcal{T}m_2) \leq \psi \max\{\varsigma(m_1, m_2), \varsigma(n_2, \mathcal{T}m_2)\}.$$

If $\max\{\varsigma(m_1, m_2), \varsigma(n_2, \mathcal{T}m_2)\} = \varsigma(n_2, \mathcal{T}m_2)$, then

$$\varsigma(n_2, \mathcal{T}m_2) \leq \psi(\varsigma(n_2, \mathcal{T}m_2) < \varsigma(n_2, \mathcal{T}m_2),$$

which is a contradiction. Hence,

$$\varsigma(n_2, \mathcal{T}m_2) \leq \psi(\varsigma(m_1, m_2)).$$

Now, again for $q_1 > 1$, there exists $n_3 \in \mathcal{T}m_2$ such that

$$\varsigma(n_2, n_3) < q_1\varsigma(n_2, \mathcal{T}m_2) \leq q_1\psi(\varsigma(m_1, m_2)).$$

From above and (12), we get

$$\varsigma(n_2, n_3) \leq \psi(q\psi(\varsigma(m_0, m_1)). \tag{13}$$

As $n_3 \in \mathcal{T}m_2 \subseteq N_0$, there exists $m_3 \in M_0$ such that

$$\varsigma(m_3, n_3) = \varsigma(M, N). \tag{14}$$

Note that $m_3 \neq m_2$; otherwise, m_2 becomes the BPP of $\mathcal{T}$. From (6) and (14), we get

$$\begin{aligned} &\alpha(m_1, m_2) \geq p(m_1, m_2), \\ &\varsigma(m_2, n_2) = \varsigma(M, N), \\ &\varsigma(m_3, n_3) = \varsigma(M, N). \end{aligned}$$

As $\mathcal{T}$ is $m - \alpha_p$ and (M, N) satisfies the weak P_p, we obtain

$$\alpha(m_2, m_3) \geq p(m_2, m_3),\ \varsigma(m_2, m_3) \leq \varsigma(n_2, n_3), \tag{15}$$

From (13) and (15), we get

$$\varsigma(m_2, m_3) \leq \psi(q\psi(\varsigma(m_0, m_1)). \tag{16}$$

Since ψ is strictly increasing, therefore

$$\psi(\varsigma(m_2, m_3)) \leq \psi^2(q\psi(\varsigma(m_0, m_1)).$$

Continuing this, we obtain sequences $\{m_\mu\} \subseteq M_0$ and $\{n_\mu\} \subseteq N_0$, such that

$$\begin{aligned} \alpha(m_\mu, m_{\mu+1}) &\geq p(m_\mu, m_{\mu+1}), \\ \varsigma(m_{\mu+1}, n_{\mu+1}) &= \varsigma(M, N), \\ \varsigma(m_{\mu+2}, n_{\mu+2}) &= \varsigma(M, N), \end{aligned} \tag{17}$$

$$\varsigma(n_{\mu+1}, n_{\mu+2}) < q_\mu(\psi(\varsigma(m_\mu, m_{\mu+1}))), \tag{18}$$

where

$$q_\mu = \frac{\psi^\mu(q\psi(\varsigma(m_0, m_1)))}{\psi(\varsigma(m_\mu, m_{\mu+1}))} > 1, \tag{19}$$

for all $\mu \in \mathbb{N}$. Using (19) in (18) we get

$$\varsigma(n_{\mu+1}, n_{\mu+2}) < \psi^\mu(q\psi(\varsigma(m_0, m_1))), \tag{20}$$

for all $\mu \in \mathbb{N}$. Since $\mathcal{T}$ is $m - \alpha_p$ and (M, N) satisfies the weak P_p, we obtain

$$\alpha(m_{\mu+1}, m_{\mu+2}) \geq p(m_{\mu+1}, m_{\mu+2}),\ \varsigma(m_{\mu+1}, m_{\mu+2}) \leq \varsigma(n_{\mu+1}, n_{\mu+2}). \tag{21}$$

Now, to prove $\{m_\mu\}$ is a Cauchy sequence in M, let $\varepsilon > 0$ be given. Since

$$\sum_{\mu=1}^{\infty} k^\mu \psi^\mu (q\psi(\varsigma(m_1, m_0)) < \infty,$$

there exists some positive integer $h = h(\varepsilon)$ such that

$$\sum_{l \geq h}^{\infty} k^l \psi^l (q\psi(\varsigma(m_1, m_0)) < \varepsilon.$$

Using the triangular inequality, we obtain

$$\begin{array}{l} \varsigma(m_\mu, m_\lambda) \leq k\varsigma(m_\mu, m_{\mu+1}) + k^2\varsigma(m_{\mu+1}, m_{\mu+2}) \\ + \cdots + k^{\lambda-\mu}\varsigma(m_{\lambda-1}, m_\lambda). \end{array}.$$

This implies

$$\begin{aligned} \varsigma(m_\mu, m_\lambda) \quad &\leq \quad k\psi^{\mu-1}(q\psi(\varsigma(m_1, m_0)) + k^2\psi^\mu(q\psi(\varsigma(m_1, m_0)) \\ &\quad\quad + \cdots\cdots + k^{\lambda-2}\psi^{\lambda-1}(q\psi(\varsigma(m_1, m_0)) \\ &\leq \quad \frac{1}{k^{\mu-2}}(k^{\mu-1}\psi^{\mu-1}(q\psi(\varsigma(m_1, m_0)) + k^\mu\psi^\mu(q\psi(\varsigma(m_1, m_0)) \\ &\quad\quad\quad + \cdots\cdots + k^{\lambda-2}\psi^{\lambda-2}(q\psi(\varsigma(m_1, m_0))) \\ &\leq \quad \frac{1}{k^{\mu-2}} \sum_{i=\mu}^{\lambda-1} k^i\psi^i(q\psi(\varsigma(m_1, m_0)) \\ &\leq \quad \frac{1}{k^{\mu-2}} \sum_{i \geq h}^{\infty} k^i\psi^i(q\psi(\varsigma(m_1, m_0)) \\ &< \quad \frac{1}{k^{\mu-2}}\varepsilon \leq \varepsilon, \end{aligned}$$

for all $\lambda > \mu > h' > h$, where $h' = \max\{2, h\}$. Thus, $\{m_\mu\}$ is a Cauchy sequence in M. Similarly, $\{n_\mu\}$ is a Cauchy sequence in N. Since (Ω, ς) is complete and M and N are closed, there exist $m^* \in M$ and $n^* \in N$ such that $m_\mu \to m^*$ and $n_\mu \to n^*$ as $\mu \to \infty$, respectively. Since $\varsigma(m_\mu, n_\mu) \to \varsigma(M, N)$ for all $\mu \in \mathbb{N}$. We conclude

$$\lim_{\mu \to \infty} \varsigma(m_\mu, n_\mu) = \varsigma(m^*, n^*) = \varsigma(M, N).$$

Continuity of $\mathcal{T}$ implies

$$\lim_{\mu \to \infty} H(\mathcal{T}m_\mu, \mathcal{T}m^*) = 0.$$

As $n_{\mu+1} \in \mathcal{T}m_\mu$:

$$\varsigma(n^*, \mathcal{T}m^*) \leq k\varsigma(n^*, n_{\mu+1}) + k\varsigma(n_{\mu+1}, \mathcal{T}m^*) \leq k\varsigma(n^*, n_{\mu+1}) + kH(\mathcal{T}m_\mu, \mathcal{T}m^*).$$

Letting $\mu \to \infty$, we get:

$$\varsigma(n^*, \mathcal{T}m^*) \leq 0,$$

which implies $n^* \in \overline{\mathcal{T}m^*} = \mathcal{T}m^*$. Furthermore,

$$\varsigma(M, N) \leq \varsigma(m^*, \mathcal{T}m^*) \leq \varsigma(m^*, n^*) = \varsigma(M, N);$$

hence,

$$\varsigma(m^*, \mathcal{T}m^*) = \varsigma(M, N),$$

which implies m^* is a BPP($\mathcal{T}$). $\square$

Next is the single-valued version of Theorem 2.

Theorem 3. *Let (Ω, ς) be a b-CMS, $M, N \in CL(\Omega)$ with $M_0 \neq \phi$. Let $\mathcal{T} : M \to N$ be a generalized Suzuki-type α_ψ contraction of ϖ type satisfying the following:*

1. *For each $m \in M_0$, we have $\mathcal{T}(m) \in N_0$, and (M, N) has the P_p;*
2. *$\mathcal{T}$ is α_p;*
3. *There exist elements m_0 and m_1 in M_0 such that $\varsigma(m_1, \mathcal{T}m_0) = \varsigma(M, N)$ and $\alpha(m_0, m_1) \geq p(m_0, m_1)$;*
4. *$\mathcal{T}$ is continuous.*

Then, $BPP(\mathcal{T})$ is nonempty.

Proof. The proof follows from Theorem 2. □

Corollary 1. *Let (Ω, ς) be a b-CMS, $M, N \in CL(\Omega)$ with $M_0 \neq \phi$. Let $\mathcal{T} : M \to N$ be a generalized Suzuki-type α_ψ contraction of ϖ' type satisfying the following:*

1. *For each $m \in M_0$, we have $\mathcal{T}(m) \in N_0$, and (M, N) has the P_p;*
2. *$\mathcal{T}$ is α_p;*
3. *There exist elements m_0 and m_1 in M_0 such that $\varsigma(m_1, \mathcal{T}m_0) = \varsigma(M, N)$ and $\alpha(m_0, m_1) \geq p(m_0, m_1)$;*
4. *$\mathcal{T}$ is continuous.*

Then, $BPP(\mathcal{T})$ is singleton.

Proof. The existence of BPP directly follows from Theorem 2. For uniqueness, suppose on the contrary that m_1 and m_2 are two distinct BPP(s). Then,

$$\begin{aligned} \varsigma(m_1, \mathcal{T}m_1) &= \varsigma(M, N), \\ \varsigma(m_2, \mathcal{T}m_2) &= \varsigma(M, N). \end{aligned}$$

Then, P_p implies

$$\varsigma(m_1, m_2) = \varsigma(\mathcal{T}m_1, \mathcal{T}m_2). \tag{22}$$

Now,

$$\begin{aligned} \varsigma^*(m_1, \mathcal{T}m_1) &= \frac{1}{k}(\varsigma(m_1, \mathcal{T}m_1) - k\varsigma(M, N)) \\ &= \frac{1}{k}(\varsigma(M, N) - k\varsigma(M, N)) \leq 0 \leq \alpha(m_1, m_2)\varsigma(m_1, m_2), \end{aligned}$$

which implies

$$\varsigma(\mathcal{T}m_1, \mathcal{T}m_2) \leq \psi(\varpi'(m_1, m_2)).$$

It further implies

$$\varsigma(m_1, m_2) \leq \psi(\leftrightarrow'(m_1, m_2)) \leq \psi(\varsigma(m_1, m_2)) < \varsigma(m_1, m_2),$$

which is a contradiction. Hence, BPP$(\mathcal{T})$ is singleton. □

Remark 4. *If we take $b = 1$ and $p = 1$, then Theorem 1 becomes the corollary of Corollary 1.*

Now, we prove the following result without the assumption of continuity of the mapping $\mathcal{T}$.

Theorem 4. *Let (Ω, ς) be a b-CMS, $M, N \in K(\Omega)$ with $M_0 \neq \phi$. Let $\mathcal{T} : M \to K(N)$ be a generalized multivalued Suzuki-type α_ψ contraction of ξ type satisfying the following:*

1. *For each $m \in M_0$, we have $\mathcal{T}(m) \subseteq N_0$, and (M, N) has a weak P_p;*
2. *$\mathcal{T}$ is $m - \alpha_p$;*

3. *There exist elements m_0 and m_1 in M_0 and $n_1 \in \mathcal{T}m_0$ such that $\varsigma(m_1, n_1) = \varsigma(M, N)$ and $\alpha(m_0, m_1) \geq p(m_0, m_1) \geq 2k$;*
4. *If $\{m_\mu\}$ is a sequence in M such that $\alpha(m_\mu, m_{\mu+1}) \geq p(m_\mu, m_{\mu+1}) \geq 2k$ and $m_\mu \to m \in M$ as $\mu \to \infty$, then there exists a subsequence $\{m_{\mu_l}\}$ of $\{m_\mu\}$ such that $\alpha(m_{\mu_l}, m) \geq p(m_{\mu_l}, m) \geq 2k$ for all $l \geq 1$.*

Then, $BPP(\mathcal{T})$ is nonempty.

Proof. From Theorem 2, we have:

$$\varsigma(n_1, \mathcal{T}m_1) \leq \psi(\varsigma(m_0, m_1)), \tag{23}$$

as $\mathcal{T}m_1$ is compact; therefore, there exists $n_2 \in \mathcal{T}m_1$ such that

$$\varsigma(n_1, n_2) = \varsigma(n_1, \mathcal{T}m_1). \tag{24}$$

Using (23) in (24), we get

$$\varsigma(n_1, n_2) \leq \psi(\varsigma(m_0, m_1)). \tag{25}$$

By assumption (1), we have $\mathcal{T}m_1 \subseteq N_0$, so there exists $m_2 \neq m_1 \in M_0$ such that

$$\varsigma(m_2, n_2) = \varsigma(M, N); \tag{26}$$

otherwise, m_1 is the BPP of $\mathcal{T}$. From (6) and (26), we get

$$\begin{aligned} &\alpha(m_0, m_1) \geq p(m_0, m_1), \\ &\varsigma(m_1, n_1) = \varsigma(M, N), \\ &\varsigma(m_2, n_2) = \varsigma(M, N). \end{aligned}$$

As $\mathcal{T}$ is $m - \alpha_p$ and (M, N) satisfies the weak P_p, we obtain

$$\alpha(m_1, m_2) \geq p(m_1, m_2), \ \varsigma(m_1, m_2) \leq \varsigma(n_1, n_2),$$

so

$$\varsigma(m_1, m_2) \leq \varsigma(n_1, n_2) < \psi(\varsigma(m_0, m_1)).$$

Continuing in a similar way as in Theorem 2, we get sequences $\{m_\mu\}$ in M_0 and $\{n_\mu\}$ in N_0 such that

$$\alpha(m_\mu, m_{\mu+1}) \geq p(m_\mu, m_{\mu+1}) \text{ and } m_\mu \neq m_{\mu+1},$$
$$n_\mu \in \mathcal{T}m_{\mu-1} \text{ and } n_\mu \notin \mathcal{T}m_\mu,$$

$$\begin{aligned} \varsigma(m_\mu, n_\mu) &= \varsigma(M, N) \text{ and} \\ \varsigma(m_\mu, m_{\mu+1}) &\leq \varsigma(n_\mu, n_{\mu+1}) \leq \psi(\varsigma(m_{\mu-1}, m_\mu))). \end{aligned} \tag{27}$$

Along similar lines as in Theorem 2, we can prove that $\{m_\mu\}$ and $\{n_\mu\}$ are Cauchy sequences in M and N, respectively. Since (Ω, ς) is complete and M and N are closed, there exist $m^* \in M$ and $n^* \in N$ such that $m_\mu \to m^*$ and $n_\mu \to n^*$ as $\mu \to \infty$, respectively, and $\varsigma(m^*, n^*) = \varsigma(M, N)$. Now, we show that m^* is the BPP of $\mathcal{T}$. If there exists a subsequence $\{m_{\mu_l}\}$ of $\{m_\mu\}$ such that $\mathcal{T}m_{\mu_l} = \mathcal{T}m^*$ for all $l \geq 1$, then

$$\begin{aligned} \varsigma(M, N) &\leq \varsigma(m_{\mu_l+1}, \mathcal{T}m_{\mu_l}) \leq \varsigma(m_{\mu_l+1}, n_{\mu_l+1}) = \varsigma(M, N), \\ \varsigma(M, N) &\leq \varsigma(m_{\mu_l+1}, \mathcal{T}m^*) \leq \varsigma(M, N) \text{ for all } l \geq 1. \end{aligned}$$

Letting $l \to \infty$, we obtain

$$\varsigma(M, N) \leq \varsigma(m^*, \mathcal{T}m^*) \leq \varsigma(M, N).$$

Hence, m^* is the BPP of $\mathcal{T}$. Thus, we may assume $\mathcal{T}m_\mu \neq \mathcal{T}m^*$ for all $\mu \in \mathbb{N}$. From assumption (4), we have a subsequence $\{m_{\mu_l}\}$ of $\{m_\mu\}$ such that $\alpha(m_{\mu_l}, m^*) \geq p(m_{\mu_l}, m^*) \geq 2k$ for all $l \geq 1$. For $n_{\mu_l+1} \in \mathcal{T}m_{\mu_l}$

$$\begin{aligned} \varsigma(m_{\mu_l}, \mathcal{T}m_{\mu_l}) &\leq \varsigma(m_{\mu_l}, n_{\mu_l+1}) \leq k\varsigma(m_{\mu_l}, m_{\mu_l+1}) + k\varsigma(m_{\mu_l+1}, n_{\mu_l+1}), \\ \varsigma(m_{\mu_l}, \mathcal{T}m_{\mu_l}) &\leq k\varsigma(m_{\mu_l}, m_{\mu_l+1}) + k\varsigma(M, N); \end{aligned}$$

therefore,

$$\varsigma^*(m_{\mu_l}, \mathcal{T}m_{\mu_l}) \leq \varsigma(m_{\mu_l}, m_{\mu_l+1}) \tag{28}$$

and

$$\begin{aligned} k\varsigma^*(m_{\mu_l+1}, \mathcal{T}m_{\mu_l+1}) &= \varsigma(m_{\mu_l+1}, \mathcal{T}m_{\mu_l+1}) - k\varsigma(M, N) \\ &\leq k\varsigma(m_{\mu_l+1}, m_{\mu_l+2}) + k\varsigma(m_{\mu_l+2}, \mathcal{T}m_{\mu_l+1}) - k\varsigma(M, N) \\ &\leq k\varsigma(m_{\mu_l+1}, m_{\mu_l+2}) + k\varsigma(m_{\mu_l+2}, n_{\mu_l+2}) - k\varsigma(M, N). \end{aligned}$$

Using $\varsigma(m_{\mu_l+2}, n_{\mu_l+2}) = \varsigma(M, N)$ and (27), we get

$$\varsigma^*(m_{\mu_l+1}, \mathcal{T}m_{\mu_l+1}) \leq \varsigma(m_{\mu_l+1}, m_{\mu_l+2}) < \varsigma(m_{\mu_l}, m_{\mu_l+1}), \tag{29}$$

and adding (28) and (29), we get

$$\varsigma^*(m_{\mu_l}, \mathcal{T}m_{\mu_l}) + \varsigma^*(m_{\mu_l+1}, \mathcal{T}m_{\mu_l+1}) < 2\varsigma(m_{\mu_l}, m_{\mu_l+1}).$$

Now, for $\alpha(m_{\mu_l}, m^*) \geq p(m_{\mu_l}, m^*) \geq 2k$, if for some $l \in \mathbb{N}$,

$$\varsigma^*(m_{\mu_l}, \mathcal{T}m_{\mu_l}) \geq \alpha(m_{\mu_l}, m^*)\varsigma(m_{\mu_l}, m^*) \tag{30}$$

and

$$\varsigma^*(m_{\mu_l+1}, \mathcal{T}m_{\mu_l+1}) \geq \alpha(m_{\mu_l+1}, m^*)\varsigma(m_{\mu_l+1}, m^*) \tag{31}$$

holds, then we get

$$\varsigma^*(m_{\mu_l}, \mathcal{T}m_{\mu_l}) \geq \alpha(m_{\mu_l}, m^*)\varsigma(m_{\mu_l}, m^*) \geq 2k\varsigma(m_{\mu_l}, m^*)$$

and

$$\varsigma^*(m_{\mu_l+1}, \mathcal{T}m_{\mu_l+1}) \geq \alpha(m_{\mu_l+1}, m^*)\varsigma(m_{\mu_l+1}, m^*) \geq 2k\varsigma(m_{\mu_l+1}, m^*).$$

By triangular inequality,

$$\begin{aligned} 2\varsigma(m_{\mu_l}, m_{\mu_l+1}) &\leq 2k\varsigma(m_{\mu_l}, m^*) + 2k\varsigma(m_{\mu_l+1}, m^*) \\ &\leq \varsigma^*(m_{\mu_l}, \mathcal{T}m_{\mu_l}) + \varsigma^*(m_{\mu_l+1}, \mathcal{T}m_{\mu_l+1}) < 2\varsigma(m_{\mu_l}, m_{\mu_l+1}), \end{aligned}$$

which is a contradiction. Hence, either

$$\varsigma^*(m_{\mu_l+1}, \mathcal{T}m_{\mu_l+1}) \leq \alpha(m_{\mu_l+1}, m^*)\varsigma(m_{\mu_l+1}, m^*) \tag{32}$$

or

$$\varsigma^*(m_{\mu_l}, \mathcal{T}m_{\mu_l}) \leq \alpha(m_{\mu_l}, m^*)\varsigma(m_{\mu_l}, m^*) \tag{33}$$

holds for infinitely many $l \in \mathbb{N}$. If (32) holds for infinitely many $l \in \mathbb{N}$, then from (5), we get

$$H(\mathcal{T}m_{\mu_l}, \mathcal{T}m^*) \leq \psi(\xi(m_{\mu_l}, m^*)).$$

For $n_{\mu_l+1} \in \mathcal{T}m_{\mu_l}$, we have $\varsigma(n_{\mu_l+1}, \mathcal{T}m^*) \leq H(\mathcal{T}m_{\mu_l}, \mathcal{T}m^*)$; therefore,

$$\varsigma(n_{\mu_l+1}, \mathcal{T}m^*) \leq \psi\xi(m_{\mu_l}, m^*)), \tag{34}$$

where

$$\begin{aligned}\xi(m_{\mu_l}, m^*) &= \max\left\{\begin{array}{c}\varsigma(m_{\mu_l}, m^*), \dfrac{\varsigma(m_{\mu_l}, \mathcal{T}m_{\mu_l}) - k\varsigma(M,N)}{k}, \\ \dfrac{\varsigma(m^*, \mathcal{T}m^*) - k\varsigma(M,N)}{k}, \dfrac{\varsigma(m^*, \mathcal{T}m_{\mu_l}) - k\varsigma(M,N)}{k}\end{array}\right\} \\ &\leq \max\left\{\begin{array}{c}\varsigma(m_{\mu_l}, m^*), \varsigma(m_{\mu_l}, m_{\mu_l+1}), \\ \dfrac{\varsigma(m^*, \mathcal{T}m^*) - k\varsigma(M,N)}{k}, \varsigma(m^*, m_{\mu_l+1})\end{array}\right\},\end{aligned}$$

if

$$\begin{aligned}&\max\left\{\varsigma(m_{\mu_l}, m^*), \varsigma(m_{\mu_l}, m_{\mu_l+1}), \frac{\varsigma(m^*, \mathcal{T}m^*) - k\varsigma(M,N)}{k}, \varsigma(m^*, m_{\mu_l+1})\right\} \\ = \ &\frac{\varsigma(m^*, \mathcal{T}m^*) - k\varsigma(M,N)}{k}.\end{aligned}$$

Then, from (34), we have

$$\varsigma(n_{\mu_l+1}, \mathcal{T}m^*) \leq \psi\left(\frac{\varsigma(m^*, \mathcal{T}m^*) - k\varsigma(M,N)}{k}\right). \tag{35}$$

By triangular inequality, we have

$$\frac{1}{k}(\varsigma(m_{\mu_l+1}, \mathcal{T}m^*) - k\varsigma(\mathcal{T}m_{\mu_l}, m_{\mu_l+1})) \leq \varsigma(\mathcal{T}m_{\mu_l}, \mathcal{T}m^*) \leq \varsigma(n_{\mu_l+1}, \mathcal{T}m^*).$$

Using the fact that $n_{\mu_l+1} \in \mathcal{T}m_{\mu_l}$ and by (35)

$$\frac{1}{k}(\varsigma(m_{\mu_l+1}, \mathcal{T}m^*) - k\varsigma(M,N)) \leq \psi\left(\frac{\varsigma(m^*, \mathcal{T}m^*) - k\varsigma(M,N)}{k}\right).$$

Letting $l \to \infty$ and using $\psi(\Upsilon) < \Upsilon$, we get

$$\frac{\varsigma(m^*, \mathcal{T}m^*) - k\varsigma(M,N)}{k} < \frac{\varsigma(m^*, \mathcal{T}m^*) - k\varsigma(M,N)}{k},$$

which is a contradiction. Hence,

$$\varsigma(n_{\mu_l+1}, \mathcal{T}m^*) \leq \psi(\max\{\varsigma(m_{\mu_l}, m^*), \varsigma(m_{\mu_l}, m_{\mu_l+1}), \varsigma(m^*, m_{\mu_l+1})\}).$$

Letting $l \to \infty$, we get

$$n^* \in \mathcal{T}m^*.$$

Hence,

$$\varsigma(M,N) \leq \varsigma(m^*, \mathcal{T}m^*) \leq \varsigma(m^*, n^*) = \varsigma(M,N).$$

Hence,

$$\varsigma(m^*, \mathcal{T}m^*) = \varsigma(M,N),$$

which implies $m^* \in \mathrm{BPP}(\mathcal{T})$. Similarly, if (33) holds for infinitely many $l \in \mathbb{N}$, the conclusion holds. □

Theorem 5. *Let (Ω, ς) be a b-CMS, $M, N \in CL(\Omega)$ with $M_0 \neq \phi$. Let $\mathcal{T} : M \to N$ be a generalized Suzuki-type α_ψ contraction of ξ type satisfying the following:*

1. *For each $m \in M_0$, we have $\mathcal{T}(m) \in N_0$, and (M,N) has the P_p;*
2. *$\mathcal{T}$ is α_p;*
3. *There exist elements m_0 and m_1 in M_0 and $n_1 = \mathcal{T}m_0$ such that $\varsigma(m_1, n_1) = \varsigma(M,N)$ and $\alpha(m_0, m_1) \geq p(m_0, m_1) \geq 2k$;*

4. *If $\{m_\mu\}$ is a sequence in M such that $\alpha(m_\mu, m_{\mu+1}) \geq p(m_\mu, m_{\mu+1}) \geq 2k$ and $m_\mu \to m \in M$ as $\mu \to \infty$, then there exists a subsequence $\{m_{\mu_l}\}$ of $\{m_\mu\}$ such that $\alpha(m_{\mu_l}, m) \geq p(m_{\mu_l}, m) \geq 2k$ for all $l \geq 1$.*

   Then, $BPP(\mathcal{T})$ is singleton.

Proof. The existence of BPP(s) follows from Theorem 4, and the uniqueness follows from Theorem 3. □

We give an example to illustrate the above theorems.

Example 1. *Let $\Omega = \mathbb{R}^2$, $\varsigma(P_1, P_2) = |x_1 - x_2|^2 + |y_1 - y_2|^2$, where*

$$P_1(x_1, y_1), P_2(x_2, y_2) \in \Omega.$$

Then, ς is a b-metric with $k = 2$. Let

$$M = \{(1, 2^\mu) : \mu \in \mathbb{N}_1\}, N = \left\{\left(0, \frac{1}{2^\mu}\right) : \mu \in \mathbb{N}_1\right\} \cup \{(0,0)\},$$

which implies

$$\varsigma(M, N) = 1.$$

Define mapping $\mathcal{T} : M \to 2^N \backslash \emptyset$ as

$$\mathcal{T}(1, 2^\mu) = \left\{\left(0, \frac{1}{2^a}\right) : 0 \leq a \leq \mu\right\}.$$

We have

$$M_0 = \{(1,1)\} \text{ and } N_0 = \{(0,1)\},$$

which implies

$$\mathcal{T}(M_0) \subseteq N_0.$$

$$\alpha(\varkappa_1, \varkappa_2) = \begin{cases} \varsigma(\varkappa_1, \varkappa_2) \text{ if } \varkappa_1 \neq \varkappa_2, \\ 2 \text{ otherwise}, \end{cases}, \psi(\Upsilon) = \frac{9}{10}\Upsilon, \text{ and } p(\varkappa_1, \varkappa_2) = 2.$$

Let $P_1 = (1, 2^{\mu_1})$, $P_2 = (1, 2^{\mu_2}) \in M$, where $\mu_2 > \mu_1$.
Now,

$$\mathcal{T}(P_1) = \left\{\left(0, \frac{1}{2^{\mu_1}}\right), \cdots, (0,1)\right\} \text{ and } \mathcal{T}(P_2) = \left\{\left(0, \frac{1}{2^{\mu_2}}\right), \cdots, (0,1)\right\}.$$

It implies

$$H(\mathcal{T}(P_1), \mathcal{T}(P_2)) = \left(\frac{1}{2^{\mu_1}} - \frac{1}{2^{\mu_2}}\right)^2 = \left(\frac{2^{\mu_2 - \mu_1} - 1}{2^{\mu_2}}\right)^2,$$

as $\mu_2 - \mu_1 \leq \mu_2$ it implies $\left(\frac{2^{\mu_2 - \mu_1} - 1}{2^{\mu_2}}\right)^2 < \frac{1}{4}$; therefore,

$$H(\mathcal{T}(P_1), \mathcal{T}(P_2)) < \frac{1}{4}. \tag{36}$$

Now, consider

$$\varsigma(P_1, P_2) = (2^{\mu_2} - 2^{\mu_1})^2 \geq 1;$$

it implies

$$\varpi(x, y) \geq 1.$$

Therefore,

$$\psi(\varpi(x, y)) \geq \frac{9}{10}, \tag{37}$$

(36) and (37) implies

$$H(\mathcal{T}(P_1), \mathcal{T}(P_2)) < \psi(\varpi(x,y)).$$

Therefore, $\mathcal{T}$ is generalized multivalued Suzuki-type α_ψ contraction of ϖ type. Note that $\mathcal{T}(M_0) \subseteq N_0$ and (M,N) satisfies a weak P_p. Furthermore, $\mathcal{T}$ is clearly $m - \alpha_p$. Theorem 2 implies $\mathcal{T}$ has a BPP, which is $(1,1)$.

Now, we give an example that satisfies all the conditions of Theorem 3, whereas Theorem 1 will not be applicable.

Example 2. *Let $\Omega = \{1,2,3,4,5\}$, such that*

$$\begin{aligned} \varsigma(1,2) &= 1, \varsigma(1,3) = 5, \varsigma(1,4) = 4, \varsigma(1,5) = 8, \varsigma(2,3) = 3, \\ \varsigma(2,4) &= 6, \varsigma(2,5) = 9, \varsigma(3,4) = 7, \varsigma(3,5) = 10, \varsigma(4,5) = 13, \\ \varsigma(x,y) &= \varsigma(y,x) \text{ and } \varsigma(x,x) = 0 \text{ for all } x,y \text{ in } \Omega. \end{aligned}$$

ς is not metric because

$$\varsigma(1,3) = 5 \nleq 1 + 3 = \varsigma(1,2) + \varsigma(2,3).$$

For $k = \frac{5}{4}$, (Ω, ς) is a b-MS.

Suppose $M = \{2,4\}$ and $N = \{1,3,5\}$. Define $\mathcal{T} : M \to N$ by

$$\mathcal{T}(2) = 1, \mathcal{T}(4) = 3.$$

$$\alpha(\varkappa_1, \varkappa_2) = \begin{cases} \varsigma(\varkappa_1, \varkappa_2) \text{ if } \varkappa_1 \neq \varkappa_2, \\ 0 \text{ otherwise}, \end{cases}, \psi(\Upsilon) = \frac{9}{10}\Upsilon, \text{ and } p(\varkappa_1, \varkappa_2) = 2.$$

Note that

$$\varsigma(M,N) = 1, \; M_0 = \{2\} \text{ and } N_0 = \{1\}.$$

Here, we discuss different cases. Case (i), $x = 2$, $y = 4$, is as follows:

$$\varsigma(2, \mathcal{T}2) - k\varsigma(M,N) = \varsigma(2,1) - k\varsigma(M,N) = 1 - \frac{5}{4} = -\frac{1}{4} \leq 36 = \alpha(2,4)\varsigma(2,4),$$

and

$$\varsigma(\mathcal{T}2, \mathcal{T}4) = \varsigma(1,3) = 5 \leq \frac{9}{10}(6) = \psi(\varpi'(2,4)).$$

Case (ii), $x = 4$, $y = 2$, is as follows:

$$\varsigma(4, \mathcal{T}4) - k\varsigma(M,N) = \varsigma(4,3) = 7 - \frac{5}{4} = \frac{23}{4} \leq 36 = \alpha(4,2)\varsigma(4,2),$$

and

$$\varsigma(\mathcal{T}4, \mathcal{T}2) = \varsigma(3,1) = 5 \leq \frac{9}{10}(6) = \psi(\varpi'(4,2)).$$

Therefore $\mathcal{T}$ is a Suzuki-type generalized α_ψ contraction of ϖ' type. Note that $\mathcal{T}(M_0) \subseteq N_0$, and the pair (M,N) has P_p. Furthermore, $\mathcal{T}$ is clearly α_p. All axioms of Theorems 3, hold. Therefore, $\mathcal{T}$ has a unique BPP, which is 2.

However, if we define the usual metric $d(x,y) = |x - y|$ on Ω, then Theorem 1 is not applicable. For instance, if $x = 2$, $y = 4$, then

$$d(2, \mathcal{T}2) - d(M,N) = d(2,1) - d(M,N) = 0 \leq 4 = \alpha(2,4)d(2,4),$$

whereas

$$d(\mathcal{T}2, \mathcal{T}4) = d(1,3) = 2 \nleq \frac{9}{10}(2) = \psi(\Gamma(2,4)).$$

Therefore, our results are the proper generalization of the results already exist in the literature.

3. Best Proximity Points Results for Generalized Multivalued Suzuki-Type α_ψ Cyclic Contractions

In this section, we derive the existence of BPP(s) for generalized multivalued Suzuki-type α_ψ cyclic contractions.

Theorem 6. *Let (Ω, ς) be a b-CMS, $M, N \in CL(\Omega)$ with $M_0 \neq \phi$. Let $\mathcal{T} : M \cup N \to CL(M) \cup CL(N)$ be a generalized multivalued Suzuki-type α_ψ cyclic contraction of ϖ type satisfying the following:*

(i) For every $m \in M_0$, $\mathcal{T}(m) \subseteq N_0$, and for every $n \in N_0$, $\mathcal{T}(n) \subseteq M_0$. (M, N) has the weak P_p;
(ii) $\mathcal{T}$ is $m - \alpha_p$;
(iii) For m_0, m_1 in M_0 and $n_1 \in \mathcal{T}m_0$ such that $\varsigma(m_1, n_1) = \varsigma(M, N)$ and $\alpha(m_0, m_1) \geq p(m_0, m_1)$ for n_0 and n_1 in N_0 and $m_1 \in \mathcal{T}n_0$, such that $\varsigma(m_1, n_1) = \varsigma(M, N)$ and $\alpha(n_0, n_1) \geq p(n_0, n_1)$;
(iv) $\mathcal{T}$ is continuous.

Then, there exist $m^* \in M$ such that $\varsigma(m^*, \mathcal{T}m^*) = \varsigma(M, N)$ and $n^* \in N$ such that $\varsigma(n^*, \mathcal{T}n^*) = \varsigma(M, N)$.

Proof. Consider the restrictions $\mathcal{T}' : M \to CL(N)$ and $\mathcal{T}'' : N \to CL(M)$ of $\mathcal{T}$ on M and N, defined as

$$\mathcal{T}'(m) = \mathcal{T}(m) \text{ for all } m \in M \text{ and } \mathcal{T}''(n) = \mathcal{T}(n) \text{ for all } n \in N,$$

respectively. Then, $\mathcal{T}'$ and $\mathcal{T}''$ satisfy all the conditions of Theorem 2. Hence, by Theorem 2, with mappings $\mathcal{T}'$ and $\mathcal{T}''$, there exist $m^* \in M$ such that

$$\varsigma(m^*, \mathcal{T}'m^*) = \varsigma(m^*, \mathcal{T}m^*) = \varsigma(M, N)$$

and $n^* \in N$ such that $\varsigma(n^*, \mathcal{T}''n^*) = \varsigma(n^*, \mathcal{T}n^*) = \varsigma(M, N)$. This completes the proof. □

Theorem 7. *Let (Ω, ς) be a b-CMS, $M, N \in CL(\Omega)$ with $M_0 \neq \phi$. Let $\mathcal{T} : M \cup N \to M \cup N$ be a generalized Suzuki-type α_ψ cyclic contraction of ϖ type satisfying the following conditions:*

(i) For every $m \in M_0$, $\mathcal{T}(m) \in N_0$, and for every $n \in N_0$, $\mathcal{T}(n) \in M_0$; (M, N) has the P_p;
(ii) $\mathcal{T}$ is $m - \alpha_p$;
(iii) For m_0, m_1 in M_0 and $n_1 = \mathcal{T}m_0$ such that $\varsigma(m_1, n_1) = \varsigma(M, N)$ and $\alpha(m_0, m_1) \geq p(m_0, m_1)$ and for n_0 and n_1 in N_0 and $m_1 = \mathcal{T}n_0$, such that $\varsigma(m_1, n_1) = \varsigma(M, N)$ and $\alpha(n_0, n_1) \geq p(n_0, n_1)$;
(iv) $\mathcal{T}$ is continuous.
Then there exist $m^* \in M$ such that $\varsigma(m^*, \mathcal{T}m^*) = \varsigma(M, N)$ and $n^* \in N$ such that $\varsigma(n^*, \mathcal{T}n^*) = \varsigma(M, N)$.

Proof. Following along similar lines of Theorem 6, we will obtain the required results. □

In the following, we derived some fixed-points theorems from our main results.
If we take $M = N = \Omega$ in Theorems 2 and 4, then we have the following results.

Theorem 8. *Let (Ω, ς) be a b-CMS and $\mathcal{T} : \Omega \to CL(\Omega)$ be an $\alpha - p$ such that*

$$\varsigma(\varkappa_1, \mathcal{T}\varkappa_1) \leq k\alpha(\varkappa_1, y)\varsigma(\varkappa_1, \varkappa_2) \text{ implies } H(\mathcal{T}\varkappa_1, \mathcal{T}\varkappa_2) \leq \psi(\varpi(\varkappa_1, \varkappa_2)),$$

for all $\varkappa_1, \varkappa_2 \in \Omega$ where $\psi \in \Psi$ satisfying the following:

(i) There exists $\varkappa_0 \in \Omega$ such that $\alpha(\varkappa_0, \mathcal{T}\varkappa_0) \geq p(\varkappa_0, \mathcal{T}\varkappa_0)$;
(ii) $\mathcal{T}$ is continuous.

Then, $FP(\mathcal{T})$ is nonempty.

Theorem 9. *Let (Ω, ς) be a b-CMS and $\mathcal{T} : \Omega \to K(\Omega)$ be an $\alpha - p$ such that*

$$\varsigma(\varkappa_1, \mathcal{T}\varkappa_1) \leq k\alpha(\varkappa_1, \varkappa_2)\varsigma(\varkappa_1, \varkappa_2) \text{ implies } H(\mathcal{T}\varkappa_1, \mathcal{T}\varkappa_2) \leq \psi(\xi(\varkappa_1, \varkappa_2)),$$

for all $\varkappa_1, \varkappa_2 \in \Omega$, where $\psi \in \Psi$ satisfies the following:

(i) *There exists $\varkappa_0 \in \Omega$ such that $\alpha(\varkappa_0, \mathcal{T}\varkappa_0) \geq p(\varkappa_0, \mathcal{T}\varkappa_0) \geq 2k$;*
(ii) *If $\{\varkappa_\mu\}$ is a sequence in Ω such that $\alpha(\varkappa_\mu, \varkappa_{\mu+1}) \geq p(\varkappa_\mu, \varkappa_{\mu+1}) \geq 2k$ and $\varkappa_\mu \to \varkappa \in \Omega$ as $\mu \to \infty$, then there exists a subsequence $\{\varkappa_{\mu_l}\}$ of $\{\varkappa_\mu\}$ such that $\alpha(\varkappa_{\mu_l}, \varkappa) \geq p(\varkappa_{\mu_l}, \varkappa) \geq 2k$ for all $l \geq 1$.*

Then, $FP(\mathcal{T})$ is nonempty.

If we take $\psi(\Upsilon) = q\Upsilon$ in Theorems 8 and 9, where $0 \leq q < 1$, then we can conclude the following theorems.

Theorem 10. *Let (Ω, ς) be a b-CMS and $\mathcal{T} : \Omega \to CL(\Omega)$ be an $\alpha - p$ such that*

$$\varsigma(\varkappa_1, \mathcal{T}\varkappa_1) \leq k\alpha(\varkappa_1, \varkappa_2)\varsigma(\varkappa_1, \varkappa_2) \text{ implies } H(\mathcal{T}\varkappa_1, \mathcal{T}\varkappa_2) \leq q(\varpi(\varkappa_1, \varkappa_2)),$$

for all $\varkappa_1, \varkappa_2 \in \Omega$, where $q \in [0, 1)$ satisfies the following:

(i) *There exists $\varkappa_0 \in \Omega$ such that $\alpha(\varkappa_0, \mathcal{T}\varkappa_0) \geq p(\varkappa_0, \mathcal{T}\varkappa_0)$;*
(ii) *$\mathcal{T}$ is continuous.*

Then, $FP(\mathcal{T})$ is nonempty.

Theorem 11. *Let (Ω, ς) be a b-CMS and $\mathcal{T} : \Omega \to K(\Omega)$ be an $\alpha - p$ such that*

$$\varsigma(\varkappa_1, \mathcal{T}\varkappa_1) \leq k\alpha(\varkappa_1, \varkappa_2)\varsigma(\varkappa_1, \varkappa_2) \text{ implies } H(\mathcal{T}\varkappa_1, \mathcal{T}\varkappa_2) \leq q(\xi(\varkappa_1, \varkappa_2)),$$

for all $\varkappa_1, \varkappa_2 \in \Omega$, where $q \in [0, 1)$ satisfies the following:

(i) *There exists $\varkappa_0 \in \Omega$ such that $\alpha(\varkappa_0, \mathcal{T}\varkappa_0) \geq p(\varkappa_0, \mathcal{T}\varkappa_0) \geq 2k$;*
(ii) *If $\{\varkappa_\mu\}$ is a sequence in Ω such that $\alpha(\varkappa_\mu, \varkappa_{\mu+1}) \geq p(\varkappa_\mu, \varkappa_{\mu+1}) \geq 2k$ and $\varkappa_\mu \to \varkappa \in \Omega$ as $\mu \to \infty$, then there exists a subsequence $\{\varkappa_{\mu_l}\}$ of $\{\varkappa_\mu\}$ such that $\alpha(\varkappa_{\mu_l}, \varkappa) \geq p(\varkappa_{\mu_l}, \varkappa) \geq 2k$ for all $l \geq 1$.*

Then, $FP(\mathcal{T})$ is nonempty.

4. Applications to Differential Equations

BPP theory plays an important role in approximating many problems, especially in the fields of differential equations and integral equations. For more details, one can see [24–26]. In this section, we obtain the optimum solution of system of differential equations by applying our obtained results. Consider the following system of differential equations:

$$\begin{aligned} \frac{d\rho}{d\sigma} &= \varrho(\sigma, \rho);\ \rho(\sigma_0) = \rho_1, \\ \frac{d\eta}{d\sigma} &= \varphi(\sigma, \eta);\ \eta(\sigma_0) = \eta_1, \end{aligned} \tag{38}$$

where $(\sigma_0, \rho_0) \in \mathbb{R}^2$ and (σ, ρ_1), (σ, η_1) are the points in

$$S = \Big\{(\sigma, \rho) \in \mathbb{R}^2 :\ |\sigma - \sigma_0| \leq a,\ |\rho - \rho_0| \leq b\Big\}, \tag{39}$$

for some $a, b > 0$. The b-metric is given as follows:

$$\varsigma(\rho(\sigma), \wp(\sigma)) = \|\rho - \wp\|^2,$$

where

$$\|\rho - \wp\| = \max_{t \in [\sigma_0 - a, \sigma_0 + a]} |\rho(t) - \wp(t)|.$$

Define

$$C_a = \{\rho \in C[\sigma_0 - a, \sigma_0 + a] : |\rho(\sigma) - \rho_0| \leq b\},$$
$$M = \{\rho \in C_a : \rho(\sigma_0) = \rho_1\} \tag{40}$$

and

$$N = \{\rho \in C_a : \rho(\sigma_0) = \eta_1\}. \tag{41}$$

Then, for any $\rho \in M$ and $\wp \in N$, $\|\rho - \wp\|^2 \geq |\eta_1 - \rho_1|^2$ and $\varsigma(M, N) = |\eta_1 - \rho_1|^2$.

Theorem 12. *Let S, M, and N be as defined in (39), (40), and (41), respectively, and let $\rho_1 < \eta_1$. Suppose ϱ and φ are continuous functions defined on S satisfying the following:*

(1) $|\varrho(\sigma, \wp) - \varphi(\sigma, \rho)| \leq K|\rho - \wp| - \frac{1}{\beta}|\eta_1 - \rho_1|$ for some $K > 0$ whenever $K|\rho - \wp| \geq \frac{1}{\beta}|\eta_1 - \rho_1|$;

(2) $\varrho(\sigma, \wp) \geq \varphi(\sigma, \rho)$, if $\sigma \leq \sigma_0$ and $\varrho(\sigma, \wp) \leq \varphi(\sigma, \rho)$, if $\sigma \geq \sigma_0$, whenever $K|\rho - \wp| \leq \frac{1}{\beta}|\eta_1 - \rho_1|$.

Define $\mathcal{T} : M \cup N \to M \cup N$ as follows:

$$\begin{aligned} \mathcal{T}(\rho(\sigma)) &= \eta_1 + \int_{\sigma_0}^{\sigma} \varphi(t, \rho(t))dt, \ \rho \in M, \\ \mathcal{T}(\wp(\sigma)) &= \rho_1 + \int_{\sigma_0}^{\sigma} \varphi(t, \wp(t))dt, \ \wp \in N, \end{aligned} \tag{42}$$

satisfying the following:

(i) For each $\rho(\sigma) \in M_0$, we have $\mathcal{T}\rho(\sigma) \in N_0$, and for each $\wp(\sigma) \in N_0$, we have $\mathcal{T}(\wp(\sigma)) \in M_0$; (M, N) has the P_p;

(ii) There exist elements $\rho_0(\sigma)$ and $\eta_1(\sigma)$ in M_0 and $\wp_1(\sigma) = \mathcal{T}\rho_0(\sigma)$ such that $\varsigma(\eta_1(\sigma), \wp_1(\sigma)) = \varsigma(M, N)$ and there exist elements $\wp_0(\sigma)$ and $\wp_1(\sigma)$ in N_0 and $\eta_1(\sigma) = \mathcal{T}\wp_0(\sigma)$, such that $\varsigma(\eta_1(\sigma), \wp_1(\sigma)) = \varsigma(M, N)$. The b-metric is given as follows:

$$\varsigma(\rho(\sigma), \wp(\sigma)) = \|\rho - \wp\|^2.$$

Then, for any

$$\beta < \min\left\{a, \frac{b - |\eta_1 - \rho_0|}{P}, \frac{b - |\rho_1 - \rho_0|}{P}, \frac{1}{K}, \frac{|\eta_1 - \rho_1|}{Q}\right\},$$

where P is the bound for both ϱ and φ and

$$Q = \sup\left\{|\varrho(\sigma, \wp) - \varphi(\sigma, \rho)| : K|\rho - \wp| \leq \frac{1}{\beta}|\eta_1 - \rho_1|\right\},$$

(38) has an optimum solution; that is, there exists $\rho^* \in M$ such that $\varsigma(\rho^*, \mathcal{T}\rho^*) = |\eta_1 - \rho_1|^2$, and there exists $\wp^* \in N$ such that $\varsigma(\wp^*, \mathcal{T}\wp^*) = |\eta_1 - \rho_1|^2$.

Proof. Let $\rho \in M_0$; then, $\mathcal{T}(\rho(\sigma_0)) = \eta_1$ and

$$\begin{aligned} |\mathcal{T}(\rho(\sigma)) - \rho_0| &= \left|\eta_1 - \rho_0 + \int_{\sigma_0}^{\sigma} \varphi(t, \rho(t))dt\right| \\ &\leq |\eta_1 - \rho_0| + \left|\int_{\sigma_0}^{\sigma} \varphi(t, \rho(t))dt\right| \\ &\leq |\eta_1 - \rho_0| + P|\sigma - \sigma_0| \\ &\leq |\eta_1 - \rho_0| + \beta P \leq b. \end{aligned}$$

This implies $\mathcal{T}(\rho(\sigma)) \in N$. Hence, $\mathcal{T}(M) \subseteq N$. Similarly, we can prove $\mathcal{T}(N) \subseteq M$. To prove that $\mathcal{T}$ is a generalized Suzuki-type α_ψ cyclic contraction of ϖ type. Take $\rho \in M, \wp \in N$, and assume $\sigma \geq \sigma_0$,

$$|\mathcal{T}\rho(\sigma) - \mathcal{T}\wp(\sigma)|^2 = \left|\rho_1 - \eta_1 + \int_{\sigma_0}^{\sigma} (\varrho(t, \wp(t)) - \varphi(t, \rho(t)))dt\right|^2. \tag{43}$$

Now,

$$\int_{\sigma_0}^{\sigma} (\varrho(t, \wp(t)) - \varphi(t, \rho(t)))\varsigma t = \int_{[\sigma_0, \sigma]} (\varrho(t, \wp(t)) - \varphi(t, \rho(t)))dt,$$

where $\int_{[\sigma_0,\sigma]} (\varrho(t, \wp(t)) - \varphi(t, \rho(t)))dt$ is the Lebesgue integral of $(\varrho(t, \wp(t)) - \varphi(t, \rho(t)))$ over the interval $[\sigma_0, \sigma]$. Now, let

$$\begin{aligned} C_1 &= \left\{t \in [\sigma_0, \sigma_0 + \beta] : K|\rho(t) - \wp(t)| > \frac{1}{\beta}|\eta_1 - \rho_1|\right\}, \\ C_2 &= \left\{t \in [\sigma_0, \sigma_0 + \beta] : K|\rho(t) - \wp(t)| \leq \frac{1}{\beta}|\eta_1 - \rho_1|\right\}. \end{aligned}$$

Since ρ and $\wp$ are continuous functions, we have both C_1 and C_2 are disjoint measurable sets. Therefore,

$$\int_{\sigma_0}^{\sigma} (\varrho(t, \wp(t)) - \varphi(t, \rho(t)))dt = \begin{array}{l} \int_{C_1} (\varrho(t, \wp(t)) - \varphi(t, \rho(t)))dt + \\ \int_{C_2} (\varrho(t, \wp(t)) - \varphi(t, \rho(t)))dt. \end{array}$$

Hence, from (43), we have

$$\begin{aligned} |\mathcal{T}(\rho(\sigma)) - \mathcal{T}(\wp(\sigma))|^2 &= \left| \begin{array}{c} \rho_1 - \eta_1 + \int_{C_1} (\varrho(t, \wp(t)) - \varphi(t, \rho(t)))\varsigma t + \\ \int_{C_2} (\varrho(t, \wp(t)) - \varphi(t, \rho(t)))\varsigma t \end{array} \right|^2 \\ &\leq \left(\begin{array}{c} \left|\rho_1 - \eta_1 + \int_{C_1} (\varrho(t, \wp(t)) - \varphi(t, \rho(t)))\varsigma t\right| + \\ \left|\int_{C_2} (\varrho(t, \wp(t)) - \varphi(t, \rho(t)))\varsigma t\right| \end{array} \right)^2. \end{aligned}$$

In C_2, $K|\rho(t) - \wp(t)| \leq \frac{1}{\beta}|\eta_1 - \rho_1|$, for $\sigma \geq \sigma_0$ by condition (2), we get $\varrho(t, \wp(t)) \leq \varphi(t, \rho(t))$, so $\int_{C_2} (\varrho(t, \wp(t)) - \varphi(t, \rho(t))) \leq 0$ and

$$\begin{aligned} \left|\int_{C_2} (\varrho(t, \wp(t)) - \varphi(t, \rho(t)))dt\right| &\leq \int_{C_2} |(\varrho(t, \wp(t)) - \varphi(t, \rho(t)))|dt \\ &\leq Q\int_{C_2} dt \leq Q|\sigma - \sigma_0| \\ &\leq Q\beta < |\eta_1 - \rho_1|. \end{aligned}$$

Therefore,

$$|\mathcal{T}(\rho(\sigma)) - \mathcal{T}(\wp(\sigma))|^2 \leq \left(|\rho_1 - \eta_1| + \int_{C_1} |(\varrho(t,\wp(t)) - \varphi(t,\rho(t)))|dt\right)^2.$$

Hence,

$$\begin{aligned}
\varsigma(\mathcal{T}(\rho(\sigma)),\mathcal{T}(\wp(\sigma))) &= \sup\left(|\mathcal{T}(\rho(\sigma)) - \mathcal{T}(\wp(\sigma))|^2\right) \\
&\leq \left(\begin{array}{c} |\rho_1 - \eta_1| + \\ \int_{C_1}(K|\rho(t) - \wp(t)| - \frac{1}{\beta}|\eta_1 - \rho_1|)dt \end{array}\right)^2, \\
&\leq \left(\begin{array}{c} |\rho_1 - \eta_1| + \beta \max_{t\in[\sigma_0-\beta,\sigma_0+\beta]} \\ (K|\rho(t) - \wp(t)| - \frac{1}{\beta}|\eta_1 - \rho_1|) \end{array}\right)^2 \\
&\leq (|\rho_1 - \eta_1| + K\beta\|\rho - \wp\| - |\eta_1 - \rho_1|)^2 \\
&\leq (K\beta\|\rho - \wp\|)^2 \leq K\beta\|\rho - \wp\|^2,
\end{aligned}$$

which implies

$$\varsigma(\mathcal{T}(\rho(\sigma)),\mathcal{T}(\wp(\sigma))) \leq K\beta\varsigma(\rho(\sigma),\wp(\sigma)) \leq K\beta\varpi(\rho(\sigma),\wp(\sigma)).$$

Hence, $\mathcal{T}$ is a generalized Suzuki-type α_ψ cyclic contraction of ϖ type. $\mathcal{T}$ is clearly α_p. Thus, all axioms of Theorem 7 hold for $\alpha(x) = x$ and $p(x) = x$. Therefore, by Theorem 7, there exists $\rho^* \in M$ such that $\varsigma(\rho^*,\mathcal{T}\rho^*) = |\eta_1 - \rho_1|^2$ and $\wp^* \in N$ such that $\varsigma(\wp^*,\mathcal{T}\wp^*) = |\eta_1 - \rho_1|^2$. This completes the proof. □

5. Conclusions

In this article, we have established multivalued generalized Suzuki-type $\alpha - \psi -$proximal (cyclic) contractions of b-metric spaces along with the provision of the existence of BPP(s) of multivalued generalized Suzuki-type $\alpha - \psi -$proximal (cyclic) contractions of b-metric spaces. Examples are given to explain our results and to show that our results are the proper generalization of the already existing results in the literature. In the end, we have developed the optimum solution for a system of ordinary differential equations with initial data. In the future, these results can further be investigated in the context of partially ordered asymmetric distance spaces and Riesz spaces with some example applications.

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