Verification of Nonblockingness in Bounded Petri Nets With Min-Max Basis Reachability Graphs

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Abstract

This paper proposes a semi-structural approach to verify the *nonblockingness* of a Petri net. We construct a structure, called *minimal-maximal basis reachability graph* (min-max-BRG): it provides an abstract description of the reachability set of a net while preserving all information needed to test if the net is *blocking*. We prove that a bounded deadlock-free Petri net is *nonblocking* if and only if its min-max-BRG is *unobstructed*, which can be verified by solving a set of integer constraints and then examining the min-max-BRG. For Petri nets that are not deadlock-free, one needs to determine the set of dead markings. This can be done with an approach based on the computation of *maximal implicit firing sequences* enabled by the markings in the min-max-BRG. The approach we developed does not require the construction of the reachability graph and has wide applicability.

Index Terms

Petri net, basis reachability graph, nonblockingness.

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I. Introduction

As discrete event models, Petri nets are commonly used in the framework of *supervisory control theory* (SCT) [1]–[4]. From the point of view of computational efficiency, Petri nets have several advantages over simpler models such as automata [2], [5], [6]: since states in Petri nets are not explicitly represented in the model in many cases, and structural analysis and linear algebraic approaches can be used without exhaustively enumerating the state space of a system. Therefore, many important problems such as *liveness* enforcement [7], [8] and performance optimization [9] in Petri nets can be efficiently solved using structural-based approaches.

A suite of supervisory control approaches in discrete event systems focuses on an essential property, namely *nonblockingness* [10]–[14]. As defined in [10], nonblockingness is a property prescribing that all reachable states should be *co-reachable* to a set of *final states* representing the completions of pre-specified tasks. Verifying and ensuring the nonblockingness of a system is a problem of primary importance in many applications and should be addressed with state-of-the-art techniques.

The nonblockingness verification (NB-V) problem in automata has been addressed with a variety of approaches. Lin and Wonham [15] derived several sufficient conditions for nonblockingness; however, they are not very suitable for systems that contain complex feedback paths. In [16], [17], a method called *hierarchical interface-based supervisory control*, which consists in breaking up a plant into two subsystems and restricting the interaction between them, is developed to verify if a system is nonblocking. To mitigate the *state explosion* problem, in the framework of *compositional verification* [18] an abstraction approach is proposed in [19] to verify discrete event systems modelled by extended finite-state machines (EFSMs) and such a verification approach is typically designed for large models consisting of several EFSMs that interact both via shared events and variables. A detailed complexity analysis concerning nonblockingness in centralized and modular discrete event systems can be found in [20].

Using Petri net models, the works in [5], [11] study NB-V from the aspect of *Petri net languages*; however, these methods rely on the construction and analysis of the *reachability graph*, which is practically inefficient. Besides, some works [21]–[23] about enforcing nonblockingness (i.e., designing a controller ensuring the nonblockingness of a Petri net) normally address the NB-V in advance; however, they still require building the reachability graph as a prerequisite.

The difficulty of enforcing nonblockingness lies in the fact that the optimal nonblocking supervisory control problem is *NP-hard* [24]. Moreover, the problem of efficiently verifying nonblockingness of a Petri net without constructing its reachability graph remains open to date. By this motivation, in this paper, we aim to develop a method to cope with the NB-V problem in Petri nets.

A state-space abstraction technique in Petri nets, called *basis reachability graph (BRG)* approaches, was recently proposed in [25], [26]. In these approaches, only a subset of the reachable markings, called *basis markings*, are enumerated. This method can be used to solve *marking reachability* [27], *diagnosis* [28], [29], *opacity* [30], *detectability* [31] and *reconfiguration* [32] problems efficiently. Thanks to the BRG, the state explosion problem can be mitigated and the related control problems can be solved efficiently. The BRG-based methods are *semi-structural* since only basis markings are explicitly enumerated in the BRG while all other reachable markings are abstracted by linear algebraic equations.

On the other hand, in our previous work [33] we show that the standard BRG cannot be directly used to solve the NB-V problem due to the possible presence of *livelocks* and *deadlocks*. In particular, livelocks describe an undesirable non-dead repetitive behavior such that the system is bound to evolve along a particular subset of its reachability space. Thus, a Petri net is blocking if a livelock that contains no final markings is reachable. However, the set of markings that form a livelock is usually hard to characterize and is not encoded in the classical BRG of the system. As a countermeasure, preliminary results are presented in [33] to show how it is possible to tailor the BRG to detect livelocks. In more detail, a structure named the *expanded BRG* is proposed, which expands the BRG so that all markings in $R(N, M_0)$ reached by firing a sequence of transitions ending with an explicit transition are included. The set of markings in an expanded BRG is denoted as the *expanded basis marking* set $\mathcal{M}_{\mathcal{B}_\mathcal{E}}$. However, this approach presents two major drawbacks. First, it only applies to deadlock-free nets, which is an undesirable restriction considering that dead non-final markings are one of the causes of blockingness. Second, while the expanded BRG can abstract part of the reachability set, its size can still be very large and its practical efficiency needs to be improved.

When a system is not deadlock-free, a dead marking in the state space characterizes a condition from which the system cannot further advance [34]. If there exists a dead marking that is not final (we call such a state a *non-final deadlock*), the system is blocking.

Inspired by the classical BRG-based methodology, in this paper, we develop a semi-structural approach to tackle the NB-V problem. The contribution consists of three aspects:

– We propose a structure called *minimal-maximal basis reachability graph* (min-max-BRG). In min-max-BRGs, only part of the state space, namely *min-max basis markings*, is encoded and all other markings can be characterized as the integer solutions of a linear constraint set.
– Owing to properties of the min-max-BRG, when a bounded Petri net is known to be deadlock-free, we prove that it is nonblocking if and only if its min-max-BRG consists of all nonblocking nodes (such a min-max-BRG is said to be *unobstructed*), which can be verified by solving a set of integer constraints and then examining the min-max-BRG.
– We generalize the results to arbitrary bounded Petri nets (not necessarily be deadlock-free) and propose a necessary and sufficient condition for NB-V. Numerical results demonstrate the proposed approach.

The rest of the paper is organized as follows. Some basic concepts and formalisms used in the paper are recalled in Section II. Section III dissects the NB-V problem. Section IV introduces the min-max-BRG. Section V investigates how min-max-BRGs can be applied to solving the NB-V problem. Numerical analyses are given in Section VI, while discussions are reported in Section VII. Conclusions and future work are given in Section VIII.

II. Preliminaries

A. Automata and Petri nets

An automaton [35] is a five-tuple $A = (X, \Sigma, \eta, x_0, X_m)$, where X is a set of *states*, Σ is an alphabet of *events*, $\eta : X \times \Sigma \to X$ is a *state transition function*, $x_0 \in X$ is an *initial state* and $X_m \subseteq X$ is a set of *final states* (also called *marker states* in [10]). η can be extended to a function $\eta : X \times \Sigma^* \to X$.

A state $x \in X$ is *reachable* if $x = \eta(x_0, s)$ for some $s \in \Sigma^*$; it is *co-reachable* if there exists $s' \in \Sigma^*$ such that $\eta(x, s') \in X_m$. An automaton is said to be *nonblocking* if every reachable state is co-reachable.

A Petri net [6] is a four-tuple $N = (P, T, Pre, Post)$, where P is a set of m *places* (graphically represented by circles) and T is a set of n *transitions* (graphically represented by bars). $Pre : P \times T \to \mathbb{N}$ and $Post : P \times T \to \mathbb{N}$ ($\mathbb{N} = \{0, 1, 2, \cdots\}$) are the *pre-* and *post- incidence functions* that specify the *arcs* directed from places to transitions, and vice versa in the net, respectively. The *incidence matrix* of N is defined by $C = Post - Pre$. A Petri net is *acyclic* if there are no directed cycles in its underlying digraph.

Given a Petri net $N = (P, T, Pre, Post)$ and a set of transitions $T_x \subseteq T$, the T_x*-induced sub-net* of N is a net resulting by removing all transitions in $T \setminus T_x$ and corresponding arcs from N, denoted as $N_x = (P, T_x, Pre_x, Post_x)$ where $T_x \subseteq T$ and Pre_x ($Post_x$) is the restriction of Pre ($Post$) to P and T_x. The incidence matrix of N_x is denoted by $C_x = Post_x - Pre_x$.

A *marking* M of a Petri net N is a mapping: $P \to \mathbb{N}$ that assigns to each place of a Petri net a non-negative integer number of *tokens*. The number of tokens in a place p at a marking M is denoted by $M(p)$. A Petri net N with an initial marking M_0 is called a *marked net*, denoted by $\langle N, M_0 \rangle$.

For a place $p \in P$, the *set of its input transitions* is defined by ${}^\bullet p = \{t \in T \mid Post(p, t) > 0\}$ and the *set of its output transitions* is defined by $p^\bullet = \{t \in T \mid Pre(p, t) > 0\}$. The notions for ${}^\bullet t$ and $t^\bullet$ are analogously defined. A place $p \in P$ (resp., transition $t \in T$) is said to be a *source place* (resp., *source transition*) if ${}^\bullet p = \emptyset$ (resp., ${}^\bullet t = \emptyset$).

A transition $t \in T$ is *enabled* at a marking M if $M \geq Pre(\cdot, t)$[1], denoted by $M[t\rangle$. If t is enabled at M, the *firing* of t yields marking $M' = M + C(\cdot, t)$, which is denoted as $M[t\rangle M'$. A marking M is *dead* (or is said to be a *deadlock*) if for all $t \in T$, $M \ngeq Pre(\cdot, t)$ holds.

Marking M' is *reachable* from M_1 if there exist a sequence of transitions $\sigma = t_1 t_2 \cdots t_n$ and markings $M_2, \cdots, M_n$ such that $M_1[t_1\rangle M_2[t_2\rangle \cdots M_n[t_n\rangle M'$ holds. When $\sigma = \epsilon$, where ϵ denotes the empty sequence, then it holds that $M[\sigma\rangle M$. We denote by T^* the set of all finite sequences of transitions over T. Given a transition sequence $\sigma \in T^*$, $\varphi : T^* \to \mathbb{N}^n$ is a function that associates to σ a vector $\mathbf{y} = \varphi(\sigma) \in \mathbb{N}^n$, called the *firing vector* of σ, i.e., $\mathbf{y}(t) = k$ if transition $t \in T$ appears k times in σ. In particular, it holds that $\varphi(\epsilon) = \mathbf{0}$. Let $\varphi^{-1} : \mathbb{N}^n \to T^*$ be the inverse function of φ, namely for $\mathbf{y} \in \mathbb{N}^n$, $\varphi^{-1}(\mathbf{y}) := \{\sigma \in T^* \mid \varphi(\sigma) = \mathbf{y}\}$. We denote $L(N, M_0) = \{\sigma \in T^* \mid M_0[\sigma\rangle\}$ as the set of all transition sequences of a marked net $\langle N, M_0 \rangle$ that are enabled from M_0.

The reachability set from a marking M is defined as $R(N, M) = \{M' \mid (\exists \sigma \in T^*)\ M[\sigma\rangle M'\}$. In particular, the reachability set of a marked net $\langle N, M_0 \rangle$ is the set $R(N, M_0)$, i.e., the reachability set from the initial marking. A marked net $\langle N, M_0 \rangle$ is said to be *bounded* if there exists an integer $k \in \mathbb{N}$ such that for all $M \in R(N, M_0)$ and for all $p \in P$, $M(p) \leq k$ holds. A marked net $\langle N, M_0 \rangle$ is *deadlock-free* if all markings in $R(N, M_0)$ are not dead.

Proposition 1: [6], [25] Given a marked net $\langle N, M_0 \rangle$ where N is acyclic, $M \in R(N, M_0)$, $M' \in R(N, M_0)$ and a firing vector $\mathbf{y} \in \mathbb{N}^n$, the following holds:

$$M' = M + C \cdot \mathbf{y} \geq \mathbf{0} \Leftrightarrow (\exists \sigma \in \varphi^{-1}(\mathbf{y}))\ M[\sigma\rangle M'. \qquad \square$$

Proposition 1 shows that in acyclic nets, reachability can be characterized (necessary and sufficient condition) in simpler algebraic terms.

Let $G = (N, M_0, \mathcal{M}_\mathcal{F})$ denote a *plant* consisting of a marked net and a finite set of final markings $\mathcal{M}_\mathcal{F} \subseteq R(N, M_0)$. Normally, set $\mathcal{M}_\mathcal{F}$ can be given by explicitly listing all its elements. In practice, set $\mathcal{M}_\mathcal{F}$ in a plant is usually characterized by one or a conjunction of linear constraints, e.g., generalized mutual exclusion constraints (GMECs) [36]. A GMEC is a pair $(\mathbf{w}, k)$, where $\mathbf{w} \in \mathbb{Z}^m$ and $k \in \mathbb{Z}$ ($\mathbb{Z}$ is the set of integers), that defines a set of markings $\mathcal{L}_{(\mathbf{w},k)} = \{M \in \mathbb{N}^m \mid \mathbf{w}^T \cdot M \leq k\}$. For simplicity, in this paper, we assume a single GMEC characterization, i.e., $\mathcal{M}_\mathcal{F} = \mathcal{L}_{(\mathbf{w},k)}$.

Given a plant $G = (N, M_0, \mathcal{M}_\mathcal{F})$ and its reachability graph $\mathcal{R}$, a maximal strongly connected component $\mathcal{G}$ in $\mathcal{R}$ is a maximal subgraph of $\mathcal{R}$ such that for any two markings M_1, M_2 in $\mathcal{G}$, M_1 is reachable from M_2. $\mathcal{G}$ is a *livelock* if for any

[1] We use $A(\cdot, x)$ (resp., $A(x, \cdot)$) to denote the column (resp., row) vector corresponding to the element x in matrix A. The $\geq$ operator on vectors is defined componentwise, i.e., given two vectors $\mathbf{a} = [a(1), ..., a(m)]^T, \mathbf{b} = [b(1), ..., b(m)]^T$, we denote by $\mathbf{a} \geq \mathbf{b}$ the fact that $a(i) \geq b(i)$ holds for all $i = 1, ..., m$.

marking M in $\mathcal{G}$, both of the following conditions hold: (i) $R(N,M) \cap \mathcal{M}_\mathcal{F} = \emptyset$; (ii) all markings $M' \in R(N,M)$ are not dead.

Definition 1: A marking $M \in R(N, M_0)$ of a plant $G = (N, M_0, \mathcal{M}_\mathcal{F})$ is said to be *blocking* if no final marking is reachable from it, i.e., $R(N,M) \cap \mathcal{M}_\mathcal{F} = \emptyset$; otherwise M is said to be *nonblocking*. System G is *nonblocking* if no reachable marking is blocking; otherwise G is *blocking*. $\square$

B. Basis Reachability Graph (BRG) [25]–[27]

Definition 2: Given a Petri net $N = (P, T, Pre, Post)$, transition set T can be partitioned into $T = T_E \cup T_I$, where the disjoint sets T_E and T_I are called the *explicit* transition set and the *implicit* transition set, respectively. A pair $\pi = (T_E, T_I)$ is called a *basis partition* of T if the T_I-induced sub-net of N is acyclic. We denote $|T_E| = n_E$ and $|T_I| = n_I$. Let C_I be the incidence matrix of the T_I-induced sub-net of N. $\square$

For any Petri net, a basis partition of T always exists (e.g., consider an extreme case $\pi = (T_E, T_I)$ with $T_E = T$ and $T_I = \emptyset$). Note that in a BRG with respect to a basis partition (T_E, T_I), the firing information of explicit transitions in T_E is explicitly encoded in the BRG, while the firing information of implicit transitions in T_I is abstracted as firing vectors. Also, the selection of T_E and T_I does not related to the physical meaning of the transitions: the only restriction is that the T_I-induced sub-net is acyclic.

Definition 3: Given a Petri net $N = (P, T, Pre, Post)$, a basis partition $\pi = (T_E, T_I)$, a marking M, and a transition $t \in T_E$, we define

$$\Sigma(M,t) = \{\sigma \in T_I^* | M[\sigma\rangle M', M' \geq Pre(\cdot, t)\}$$

as the set of *explanations* of t at M, and we define

$$Y(M,t) = \{\varphi(\sigma) \in \mathbb{N}^{n_I} | \sigma \in \Sigma(M,t)\}$$

as the set of *explanation vectors*; meanwhile we define

$$\Sigma_{\min}(M,t) = \{\sigma \in \Sigma(M,t) | \nexists \sigma' \in \Sigma(M,t) : \varphi(\sigma') \lneq \varphi(\sigma)\}$$

as the set of *minimal explanations* of t at M, and we define

$$Y_{\min}(M,t) = \{\varphi(\sigma) \in \mathbb{N}^{n_I} | \sigma \in \Sigma_{\min}(M,t)\}$$

as the corresponding set of *minimal explanation vectors*. $\square$

Definition 4: Given a marked net (N, M_0) and a basis partition $\pi = (T_E, T_I)$, its *basis marking set* $\mathcal{M}_\mathcal{B}$ is the smallest subset of reachable markings such that:

- $M_0 \in \mathcal{M}_\mathcal{B}$;
- If $M \in \mathcal{M}_\mathcal{B}$, then for all $t \in T_E$, for all $\mathbf{y} \in Y_{\min}(M,t)$, $M' = M + C_I \cdot \mathbf{y} + C(\cdot, t) \Rightarrow M' \in \mathcal{M}_\mathcal{B}$. $\square$

A marking M in $\mathcal{M}_\mathcal{B}$ is called a *basis marking* of (N, M_0) with respect to $\pi = (T_E, T_I)$.

Definition 5: Given a bounded marked net $\langle N, M_0 \rangle$ and a basis partition $\pi = (T_E, T_I)$, its *basis reachability graph* is a deterministic finite state automaton $\mathcal{B} = (\mathcal{M}_\mathcal{B}, \mathrm{Tr}, \Delta, M_0)$, where the state set $\mathcal{M}_\mathcal{B}$ is the set of basis markings, the event set Tr is the finite set of pairs $(t, \mathbf{y}) \in T_E \times \mathbb{N}^{n_I}$, $\Delta : \mathcal{M}_\mathcal{B} \times \mathrm{Tr} \to \mathcal{M}_\mathcal{B}$ is a transition function, and the initial state is the initial marking M_0. $\square$

We extend in the usual way the definition of transition function to consider a sequence of pairs $\sigma \in \mathrm{Tr}^*$ and write $\Delta(M_1, \sigma) = M_2$ to denote that from M_1 sequence σ yields M_2.

Figure 1: A parameterized plant G with $T_E = \{t_2\}$ marked with shadow (left) and its BRG $\mathcal{B}$ (right).

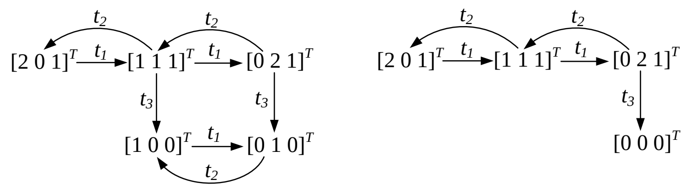

Figure 2: Reachability graph of G in Fig. 1 with $\alpha = 1$ (left) and $\alpha = 2$ (right).

Definition 6: Given a marked net $\langle N, M_0 \rangle$, a basis partition $\pi = (T_E, T_I)$, and a basis marking $M_b \in \mathcal{M}_{\mathcal{B}}$, we define $R_I(M_b) = \{M \in \mathbb{N}^m \mid (\exists \sigma \in T_I^*)\ M_b[\sigma\rangle M\}$ as the *implicit reach* of M_b. □

The implicit reach of a basis marking M_b is the set of all markings that can be reached from M_b by firing only implicit transitions. Since the T_I-induced sub-net is acyclic, by Proposition 1, it holds that:

$$R_I(M_b) = \{M \in \mathbb{N}^m \mid (\exists \mathbf{y}_I \in \mathbb{N}^{n_I})\ M = M_b + C_I \cdot \mathbf{y}_I\}.$$

III. BRG and Nonblockingness Verification

To efficiently solve the NB-V problem in Petri nets without constructing the reachability graph, we attempted to use the BRG-based approach in [33]. However, as observed in [33], the classical BRG does not necessarily encode all information needed to test nonblockingness. To help clarify, an example is provided in the following.

Example 1: Consider a parameterized plant $G = (N, M_0, \mathcal{M}_{\mathcal{F}})$ in Fig. 1 with $M_0 = [2\ 0\ 1]^{\mathrm{T}}$ and $\mathcal{M}_{\mathcal{F}} = \{M_0\}$. In this net, $Pre(p_2, t_3) = \alpha$ is set to be a parameter ($\alpha \in \mathbb{N}$). Assuming $T_E = \{t_2\}$, the BRG of this net (regardless of the value of α) is also shown in the same figure, where $M_{b0} = M_0$ and $Y_{\min}(M_{b0}, t_2) = \{[1\ 0]^{\mathrm{T}}\}$, which implies that the firing of sequence $\sigma = t_1$ is the prerequisite (the minimal one) of the firing of explicit transition t_2 at marking M_{b0}. The reachability graphs for $\alpha = 1$ and $\alpha = 2$ are shown respectively in Fig. 2.

By inspection of the two reachability graphs, one can verify that G is deadlock-free if $\alpha = 1$ and not deadlock-free if $\alpha = 2$. When $\alpha = 1$, G is blocking due to the livelock composed by two markings $[1\ 0\ 0]^{\mathrm{T}}$ and $[0\ 1\ 0]^{\mathrm{T}}$. When $\alpha = 2$, G is also blocking because of the non-final deadlock $[0\ 0\ 0]^{\mathrm{T}}$. However, these blocking conditions are not captured in the BRG which, in both cases, consists of a unique node $M_{b0} = M_0$ which is also final. On the other hand, note that markings $[1\ 0\ 0]^T$, $[0\ 1\ 0]^T$ and $[0\ 0\ 0]^T$ in Fig. 2 are not co-reachable to the only basis marking $[2\ 0\ 1]^T$ in $\mathcal{B}$. □

Example 1 shows that when all basis markings in the BRG are nonblocking, this does not necessarily imply that all reachable markings in the corresponding plant are nonblocking. Specifically, as we mentioned in Section I, two types of blocking markings should be analyzed, i.e., those are dead but non-final, and those are included in livelocks.

Notice that when tackling the NB-V problem by using the basis marking approach, there may exist some (i) dead and non-final markings, and/or (ii) markings contained in a livelock that are not basis marking. Since such markings do not belong to set $\mathcal{M}_{\mathcal{B}}$, they are not shown in the corresponding BRG. Therefore, the classical structure of BRGs needs to be revised to encode additional information for checking nonblockingness. To this end, in the following, we propose a structure namely min-max-BRG and show how it can be leveraged on solving the NB-V problem.

IV. Min-Max Basis Markings and Min-Max-BRGs

A. Min-Max Basis Markings

To define the min-max-BRG, we first introduce the set of *min-max basis markings*. As two prerequisite concepts, we define *maximal explanations* and *maximal explanation vectors* as follows.

Definition 7: Given a Petri net $N = (P, T, Pre, Post)$, a basis partition $\pi = (T_E, T_I)$, a marking M, and a transition $t \in T_E$, we define

$$\Sigma_{\max}(M, t) = \{\sigma \in \Sigma(M, t) | \nexists \sigma' \in \Sigma(M, t) : \varphi(\sigma') \gneq \varphi(\sigma)\}$$

as the set of maximal explanations of t at M, and

$$Y_{\max}(M, t) = \{\varphi(\sigma) \in \mathbb{N}^{n_I} | \sigma \in \Sigma_{\max}(M, t)\}$$

as the corresponding set of maximal explanation vectors. □

From the standpoint of *partial order set* (poset), the set of maximal explanation vectors $Y_{\max}(M, t)$ is the set of *maximal elements* in the corresponding poset $Y(M, t)$. Note that, as is the case for the set of minimal explanation vectors $Y_{\min}(M, t)$ [25]–[27], $Y_{\max}(M, t)$ may not be a singleton. In fact, there may exist multiple maximal firing sequences $\sigma_I \in T_I^*$ that enable an explicit transition t. Next, we define *min-max basis markings* in an iterative way as follows.

Definition 8: Given a marked net $\langle N, M_0 \rangle$ with a basis partition $\pi = (T_E, T_I)$, its min-max basis marking set $\mathcal{M}_{\mathcal{B}_\mathcal{M}}$ is recursively defined as follows

(a) $M_0 \in \mathcal{M}_{\mathcal{B}_\mathcal{M}}$;

(b) $M \in \mathcal{M}_{\mathcal{B}_\mathcal{M}}$, $t \in T_E, \mathbf{y} \in Y_{\min}(M, t) \cup Y_{\max}(M, t)$, $M' = M + C_I \cdot \mathbf{y} + C(\cdot, t) \Rightarrow M' \in \mathcal{M}_{\mathcal{B}_\mathcal{M}}$.

A marking in $\mathcal{M}_{\mathcal{B}_\mathcal{M}}$ is called a min-max basis marking of the marked net with $\pi = (T_E, T_I)$. □

In practice, the set of min-max basis markings is a smaller subset of reachable markings that contains the initial marking and is closed by reachability through a sequence that contains an explicit transition and one of its maximal or minimal explanations. Meanwhile, note that for a bounded marked net, $\mathcal{M}_\mathcal{B} \subseteq \mathcal{M}_{\mathcal{B}_\mathcal{M}}$ holds. To compute $Y_{\min}(M, t)$, one may refer to Algorithm 1 in [27]. We introduce in Algorithm 1 how to calculate $Y_{\max}(M, t)$ for a given marking M and an explicit transition t. The basic idea is first to iteratively enumerate all explanation vectors in $Y(M, t)$ (not necessarily stored), and then collect the set of maximal elements in $Y(M, t)$.

The computation as to $Y(M, t)$ is presented through lines $1-16$. To put it simply, as a *breadth-first-search* technique, all possible firing vectors $\mathbf{y} \in \mathbb{N}^{n_I}$ such that $\sigma \in \varphi^{-1}(\mathbf{y})$ is an explanation of t at M (i.e., $M[\sigma\rangle M'[t\rangle$) are iteratively searched and enumerated. A detailed description is shown as follows.

Initially, at line 1, the row $A = (M - Pre(\cdot, t))^{\mathrm{T}}$ is either nonnegative or contains at least a negative element. The former implies that t is sufficiently enabled at M (thus $\mathbf{0}_{n_I} \in Y(M, t)$). The latter suggests that the number of tokens in the corresponding place(s) as to M is insufficient. Then, at line 2, we call a subroutine that consists of lines $2-12$ in Algorithm 1 of [27] to process the matrix Γ. Precisely, this procedure enumerates part of the explanation vectors (not all) by iteratively updating Γ, i.e., adding selected rows in $\left[\begin{array}{c|c} C_I^{\mathrm{T}} & I_{n_I \times n_I} \end{array}\right]$ to rows in $\left[\begin{array}{c|c} A & B \end{array}\right]$ that contain negative elements to neutralize them eventually. As a result, if $Y(M, t) \neq \emptyset$, the corresponding explanation vectors are stored individually in the form of row vectors in sub-matrix B.

Algorithm 1 Calculation of $Y_{\max}(M,t)$

Input: A bounded marked net $\langle N, M_0\rangle$, a basis partition $\pi = (T_E, T_I)$, a marking $M \in R(N, M_0)$, and $t \in T_E$

Output: $Y_{\max}(M,t)$

1: $\Gamma := \left[\begin{array}{c|c} C_I^{\mathrm{T}} & I_{n_I\times n_I} \\ \hline A & B \end{array}\right]$ where $A := (M - Pre(\cdot,t))^{\mathrm{T}}$ and $B := \mathbf{0}_{n_I}^{\mathrm{T}}$;

2: Subroutine: update Γ through lines 2−12 in Algorithm 1 of [27];

3: $\alpha :=$ row_size(Γ), $\alpha_{\text{old}} := 0$, and $\alpha'_{\text{old}} := n_I$;

4: **while** $\alpha_{\text{old}} - \alpha \neq 0$ **do**

5:     $\alpha_{\text{old}} := \alpha$;

6:     **for** $k = 1 : n_I$, **do**

7:         **for** $l = (\alpha'_{\text{old}} + 1) : \alpha_{\text{old}}$, **do**

8:             $R := [\Gamma(l,\cdot) + \Gamma(k,\cdot)]$;

9:             **if** $R \geq \mathbf{0}$ and $\nexists \Gamma(i,\cdot) = R$ where $i \in \{(n_I+1), \cdots, \alpha_{\text{old}}\}$, **then**

10:                 $\Gamma_{\text{new}} := \left[\begin{array}{c} \Gamma \\ \hline R \end{array}\right]$;

11:             **end if**

12:         **end for**

13:     **end for**

14:     $\alpha :=$ row_size(Γ_{new}), $\alpha'_{\text{old}} := \alpha_{\text{old}}$, and $\Gamma := \Gamma_{\text{new}}$;

15: **end while**

16: Let $Y(M,t)$ be the set of row vectors in the updated sub-matrix $B = \Gamma((n_I+1) : \alpha, (m+1) : (m+n_I))$;

17: Let $Y_{\max}(M,t)$ be the set of maximal elements in $Y(M,t)$.

To complete $Y(M,t)$, analogously, from lines 6−12, we add each of the rows in $\left[\begin{array}{c|c} C_I^{\mathrm{T}} & I_{n_I\times n_I}\end{array}\right]$ to rows in $\left[\begin{array}{c|c} A & B\end{array}\right]$ in the updated Γ. If the obtained new row, e.g., $R = [C_I^{\mathrm{T}}(i^*,\cdot) + A(j^*,\cdot) \mid I_{n_I\times n_I}(i^*,\cdot) + B(j^*,\cdot)]$, is nonnegative and does not equal to any of the rows in $\left[\begin{array}{c|c} A & B\end{array}\right]$, it is then recorded in $\left[\begin{array}{c|c} A & B\end{array}\right]$ as a new extended row and matrix Γ will be updated. This act implies that $M + C_I \cdot (I_{n_I\times n_I}(i^*,\cdot) + B(j^*,\cdot))^T - Pre(\cdot,t) \geq \mathbf{0}$. Thus, it is deduced that the vector $(I_{n_I\times n_I}(i^*,\cdot) + B(j^*,\cdot))^T$ is another explanation vector of t at M and it will be recorded in the sub-matrix B.

Iteratively, represented in the sub-matrix B of the updated matrix Γ_{new}, all explanations of M at t can be collected. The computation of $Y(M,t)$ ends when the sub-matrix $\left[\begin{array}{c|c} A & B\end{array}\right]$ of Γ_{new} reaches a fixed point. Finally, at line 17, the set of maximal explanations is obtained by collecting all the maximal rows in $Y(M,t)$. By the following Proposition, we show that Algorithm 1 eventually halts, since $Y_{\max}(M,t)$ is not infinite.

Proposition 2: The set of all transition sequences $L(N, M_0) = \{\sigma \in T^* \mid M_0[\sigma\rangle\}$ of a bounded acyclic net $\langle N, M_0\rangle$ is finite.

Proof: See Appendix in detail. ■

Proposition 2 indicates the finiteness of firing vectors (corresponds to the set of firing sequences) of a bounded acyclic net $\langle N, M_0\rangle$. Moreover, it is inferred that $Y_{\max}(M,t)$ is finite for any $M \in R(N, M_0)$ and $t \in T_E$ of a given bounded marked net $\langle N, M_0\rangle$ with $\pi = (T_E, T_I)$, since $Y_{\max}(M,t) \subseteq Y(M,t)$ holds while $Y(M,t)$ is finite based on Proposition 2.

We analyze the complexity of Algorithm 1, which depends on the complexity of the subroutine at lines 4−15. First, the outer loop (the *while* loop) executes at most $|Y(M,t)|$ times. In such a case, the new row R will be added one by one with each loop, until reach the fixed point and thus we derive $Y(M,t)$. Then, the first-layer inner loop (lines 6−13) executes n_I times with each *while* loop. For the second-layer inner loop (lines 7−12), since in total there will be $|Y(M,t)|$ rows in $[A \mid B]$ and the worst case of the *while* loop executes $|Y(M,t)|$ times, we infer that the second-layer inner loop executes only one time with each first-layer inner loop. In other words, in each execution of the *while* loop, there is(are) only $n_I \cdot 1$ vector(s) computed at line 8. Moreover, to determine whether a non-negative vector R has been visited in Γ at lines 9−11, one can apply a *hash set* data structure to complete it in constant time $O(1)$. In summary, the overall computational complexity of Algorithm 1 is $O(|Y(M,t)| \cdot n_I)$.

B. Min-Max Basis Reachability Graph

Definition 9: Given a bounded marked net $\langle N, M_0\rangle$ and a basis partition $\pi = (T_E, T_I)$, its min-max-BRG is a deterministic finite state automaton $\mathcal{B}_\mathcal{M} = (\mathcal{M}_{\mathcal{B}_\mathcal{M}}, \mathrm{Tr}_\mathcal{M}, \Delta_\mathcal{M}, M_0)$, where $\mathcal{M}_{\mathcal{B}_\mathcal{M}}$ is the set of min-max basis markings, $\mathrm{Tr}_\mathcal{M}$ is a finite set of pairs $(t, \mathbf{y}) \in T_E \times \mathbb{N}^{n_I}$, $\Delta_\mathcal{M}$ is the transition relation $\{(M_1, (t, \mathbf{y}), M_2) \mid t \in T_E; \mathbf{y} \in (Y_{\min}(M_1, t) \cup Y_{\max}(M_1, t)), M_2 = M_1 + C_I \cdot \mathbf{y} + C(\cdot, t)\}$ and M_0 is the initial marking. □

We extend the definition of transition relation $\Delta_\mathcal{M}$ for sequences of pairs $\sigma^+ = (t_1, \mathbf{y}_1), (t_2, \mathbf{y}_2), \cdots, (t_k, \mathbf{y}_k) \in \mathrm{Tr}^*_\mathcal{M}$ and write $(M_1, \sigma^+, M_2) \in \Delta_\mathcal{M}$ to denote that from M_1 sequence σ^+ yields M_2 in $\mathcal{B}_\mathcal{M}$.

According to Definitions 8 and 9, to build a min-max-BRG, one may refer to the construction procedure of a BRG (e.g., see Algorithm 2 in [27]). The difference is that the construction of min-max-BRGs requires taking both minimal and maximal explanation vectors into consideration. On the other hand, comparing with the expanded BRGs in [33] (in which all explanation vectors need to be computed at each node in the expanded BRG), a min-max-BRG can often be more compact (i.e., include fewer nodes and arcs) than an expanded BRG when adopting the same basis partition.

We briefly explain the construction procedure as follows. First, the set $\mathcal{M}_{\mathcal{B}_\mathcal{M}}$ is initialized as $\{M_0\}$. Then, for all untested markings $M \in \mathcal{M}_{\mathcal{B}_\mathcal{M}}$ and for all explicit transitions $t \in T_E$, it is required to check whether there exist explanation vectors $\mathbf{y} \in Y_{\min}(M,t) \cup Y_{\max}(M,t)$: if exist, the corresponding min-max basis marking (i.e., $M' = M + C_I \cdot \mathbf{y} + C(\cdot, t)$) is computed and stored in $\mathcal{M}_{\mathcal{B}_\mathcal{M}}$ (on the condition that M' is not included in the set $\mathcal{M}_{\mathcal{B}_\mathcal{M}}$ before). Moreover, the set of pairs $(t, \mathbf{y})$ and transition relations between M and M' are stored in $\mathrm{Tr}_\mathcal{M}$ and $\Delta_\mathcal{M}$, respectively. Iteratively, the min-max-BRG $\mathcal{B}_\mathcal{M}$ can be constructed. We exemplify this procedure in Example 2.

As for the complexity of constructing the min-max-BRG, in common with the BRG, the upper bound of states in a min-max-BRG is the size of the reachability space of a net (consider $T_E = T$ and $T_I = \emptyset$). Nonetheless, first, the building of a min-max-BRG does not require constructing the reachability graph. Then, our numerical results (e.g., see Section VI and [37]) show that the min-max-BRG can often be more compact in size than that of the reachability graph in the considered cases.

Remark 1: Similar to the BRG, note that the selection of the basis partition may change the computational efficiency of constructing the min-max-BRG. Indeed, there may exist multiple feasible basis partitions that yield BRGs with different sizes. However, in general there is no indicators to evaluate a partition in terms of the size of the corresponding BRG. On the other hand, the work in [27] proved that the size of BRG is nondecreasing (and possibly increasing) when the set T_E increases. In other words, a larger set (in the sense of set containment) T_E implies a larger BRG. Hence, in practice, one may use Algorithm 3 in [27] to obtain a minimal T_E (in the sense of set containment) which is a locally optimal solution. □

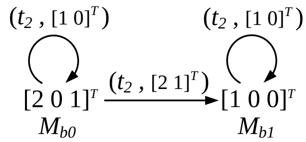

Figure 3: The min-max-BRG $\mathcal{B}_{\mathcal{M}}$ with $T_E = \{t_2\}$.

Example 2: Consider again the parameterized plant $G = (N, M_0, \mathcal{M}_{\mathcal{F}})$ in Fig. 1 (left) with $\alpha = 1$ and $T_E = \{t_2\}$. We briefly introduce how to construct its min-max-BRG $\mathcal{B}_{\mathcal{M}}$. According to Definition 8, $M_{b0} = M_0$ is a min-max basis marking. Next, we compute the minimal and maximal explanation vectors of the explicit transition t_2 at M_{b0} respectively and derive the other potential min-max basis markings. For instance, for t_2, the only minimal explanation vector $\mathbf{y}_{\min} = [1\ 0]^T$ while the only maximal explanation vector $\mathbf{y}_{\max} = [2\ 1]^T$. Since $M_{b0} + C_I \cdot \mathbf{y}_{\min} + C(\cdot, t_2) = [2\ 0\ 1]^T = M_{b0}$, no new min-max basis marking is generated. However, the pair $(t_2, [1\ 0]^T)$ is stored in $\mathrm{Tr}_{\mathcal{M}}$ while the transition relation $(M_{b0}, (t_2, [1\ 0]^T), M_{b0})$ is stored in $\Delta_{\mathcal{M}}$. On the other hand, since $M_{b1} = M_{b0} + C_I \cdot \mathbf{y}_{\max} + C(\cdot, t_2) = [1\ 0\ 0]^T \neq M_{b0}$, let M_{b1} be another min-max basis marking. Corresponding pair and transition relation are also collected. Analogously, $\mathcal{B}_{\mathcal{M}}$ can be constructed which is graphically shown in Fig. 3. □

In the following, we show that the min-max-BRG preserves the reachability information and other non-min-max-basis markings can be algebraically characterized by linear equations.

Proposition 3: Given a marked net $\langle N, M_0 \rangle$ with a basis partition $\pi = (T_E, T_I)$ and a marking $M \in \mathbb{N}^m$, $M \in R(N, M_0)$ if and only if there exists a min-max basis marking $M_b \in \mathcal{M}_{\mathcal{B}_{\mathcal{M}}}$ such that $M \in R_I(M_b)$, where $\mathcal{M}_{\mathcal{B}_{\mathcal{M}}}$ is the set of the min-max basis markings in min-max-BRG of $\langle N, M_0 \rangle$.

Proof: (only if) This part of the proof follows from Corollary 1 in [27].

(if) Since $M \in R_I(M_b)$, according to Definition 6, there exists a firing sequence $\sigma \in T_I^*$ such that $M_b[\sigma\rangle M$. On the other hand, there exists another firing sequence $\sigma' \in T^*$ such that $M_0[\sigma'\rangle M_b$, which implies that $M_0[\sigma'\sigma\rangle M$ and concludes the proof. ■

In summary, a marking M is reachable from M_0 if and only if it belongs to the implicit reach of a min-max basis marking M_b and thus M can be characterized by a linear equation, i.e., $M = M_b + C_I \cdot \mathbf{y}_I$, where $\mathbf{y}_I = \varphi(\sigma_I)$, $\sigma_I \in T_I^*$ and $M_b[\sigma_I\rangle M$.

V. Verifying Nonblockingness of Bounded Plants Using Min-Max-BRGs

In this section, we investigate how min-max-BRGs can be applied to solving the NB-V problem.

A. Unobstructiveness of Min-Max-BRGs

This subsection generalizes the notion of unobstructiveness that is given in [33] for a BRG to a min-max-BRG. Such a property is essential to establish our method since it is strongly related to the nonblockingness of a Petri net. First, we define the set of *i-coreachable min-max basis markings*, denoted by $\mathcal{M}_{\mathrm{i_{co}}}$, from which at least one of the final markings in $\mathcal{M}_{\mathcal{F}}$ is reachable by firing implicit transitions only.

Definition 10: Consider a bounded plant $G = (N, M_0, \mathcal{M}_{\mathcal{F}})$ with the set of min-max basis markings $\mathcal{M}_{\mathcal{B}_{\mathcal{M}}}$ in its min-max-BRG. The set of i-coreachable min-max basis markings of $\mathcal{M}_{\mathcal{B}_{\mathcal{M}}}$ is defined as $\mathcal{M}_{\mathrm{i_{co}}} = \{M_b \in \mathcal{M}_{\mathcal{B}_{\mathcal{M}}} | R_I(M_b) \cap \mathcal{M}_{\mathcal{F}} \neq \emptyset\}$.□

Proposition 4: Given a set of final markings defined by a single GMEC $\mathcal{L}_{(\mathbf{w},k)}$ and a min-max basis marking M_b, M_b belongs to $\mathcal{M}_{\mathrm{i_{co}}}$ if and only if the following set of integer constraints is feasible.

$$\begin{cases} M_b + C_I \cdot \mathbf{y}_I = M; \\ \mathbf{w}^T \cdot M \leq k; \\ \mathbf{y}_I \in \mathbb{N}^{n_I}; \\ M \in \mathbb{N}^m. \end{cases} \tag{1}$$

Proof: (only if) Since $M_b \in \mathcal{M}_{\mathrm{i_{co}}}$, according to Definition 10, $R_I(M_b) \cap \mathcal{M}_{\mathcal{F}} \neq \emptyset$. Therefore, integer constraints (1) meets feasible solution $\mathbf{y}_I$.

(if) The state equation $M_b + C_I \cdot \mathbf{y}_I = M$ provides necessary and sufficient conditions for reachability since the implicit sub-net is acyclic (see Proposition 1). Moreover, $M \in \mathcal{L}_{(\mathbf{w},k)}$ is a final marking. Therefore, the statement holds. ■

Remark 2: For simplicity, up to now, we assume a single GMEC characterization ($\mathcal{L}_{(\mathbf{w},k)}$) for the final marking set; however, our approach can be further generalized to:

(a) $\mathcal{M}_{\mathcal{F}}$ defined by the conjunction of r GMECs (namely an *AND-GMEC*), i.e., $\mathcal{L}_{AND} = \{M \in \mathbb{N}^m \mid \mathbf{W}^T \cdot M \leq \mathbf{k}\} = \bigcap_{(\mathbf{w}_i,k_i)\in(\mathbf{W},\mathbf{k})} \mathcal{L}_{(\mathbf{w}_i,k_i)}$, where $\mathbf{W} = [\mathbf{w}_1 \ \mathbf{w}_2 \cdots \mathbf{w}_r] \in \mathbb{Z}^{m\times r}$ and $\mathbf{k} = [k_1 \ k_2 \cdots k_r]^T \in \mathbb{N}^r$, where $(\mathbf{w}_i, k_i) \in (\mathbf{W}, \mathbf{k})$ implies that $(\mathbf{w}_i, k_i) = (\mathbf{W}(i,\cdot), \mathbf{k}(i,\cdot))$ and $i \in \{1, 2, \cdots, r\}$. In such a case, the set $\mathcal{M}_{\mathrm{i_{co}}}$ can be computed in the same way by revising the constraint $\mathbf{w}^T \cdot M \leq k$ in constraints (1) as $\mathbf{W}^T \cdot M \leq \mathbf{k}$;

(b) $\mathcal{M}_{\mathcal{F}}$ defined by the union of s GMECs (namely an *OR-GMEC*), i.e., $\mathcal{L}_{OR} = \bigcup_{i\in\{1,2,\ldots,s\}} \mathcal{L}_{(\mathbf{w}_i,k_i)}$ where $\mathbf{w}_i \in \mathbb{N}^m$ and $k_i \in \mathbb{N}$. Then, the set $\mathcal{M}_{\mathrm{i_{co}}}$ can be settled by revising the constraint $\mathbf{w}^T \cdot M \leq k$ in constraints (1) as a disjunctive form of constraints, which can be transformed into its equivalent conjunctive form. □

The notion of unobstructiveness in a min-max-BRG is given in Definition 11. In the following, we show how the unobstructiveness of a min-max-BRG is related to the nonblockingness of the corresponding Petri net.

Definition 11: Given a min-max-BRG $\mathcal{B}_{\mathcal{M}} = (\mathcal{M}_{\mathcal{B}_{\mathcal{M}}},$
$\mathrm{Tr}_{\mathcal{M}}, \Delta_{\mathcal{M}}, M_0)$ and a set of i-coreachable min-max basis markings $\mathcal{M}_{\mathrm{i_{co}}} \subseteq \mathcal{M}_{\mathcal{B}_{\mathcal{M}}}$, $\mathcal{B}_{\mathcal{M}}$ is said to be *unobstructed* if for all $M_b \in \mathcal{M}_{\mathcal{B}_{\mathcal{M}}}$ there exist a marking $M_b' \in \mathcal{M}_{\mathrm{i_{co}}}$ in $\mathcal{B}_{\mathcal{M}}$ and a firing sequence $\sigma^+ \in \mathrm{Tr}_{\mathcal{M}}^*$ such that $(M_b, \sigma^+, {M_b}') \in \Delta_{\mathcal{M}}$. Otherwise it is *obstructed*. □

To determine the unobstructiveness of min-max-BRG $\mathcal{B}_{\mathcal{M}}$, it is only required to check if all min-max basis markings are co-reachable to some i-coreachable min-max basis markings in $\mathcal{B}_{\mathcal{M}}$. This can be done by using some search algorithm (e.g., *Dijkstra*) in the underlying digraph of the min-max-BRG, whose complexity is polynomial in the size of $\mathcal{B}_{\mathcal{M}}$. An example is illustrated in the following to help clarify.

Example 3: Consider again the parameterized plant $(N, M_0, \mathcal{M}_{\mathcal{F}})$ in Fig. 1 (left) with $\alpha = 1$, $T_E = \{t_2\}$ and $\mathcal{M}_{\mathcal{F}} = \mathcal{L}_{(\mathbf{w},k)}$ where $\mathbf{w} = [1\ 1\ 0\ 0]^{\mathrm{T}}$ and $k = 1$. We explain how to verify the unobstructiveness of its min-max-BRG $\mathcal{B}_{\mathcal{M}}$ shown in Fig. 3. By solving the linear constraint (1) in Proposition 4, we conclude that $\mathcal{M}_{\mathrm{i_{co}}} = \{[2\ 0\ 1]^{\mathrm{T}}\}$. Since there is no directed path from M_{b1} to M_{b0}, M_{b1} is not co-reachable to the only marking in $\mathcal{M}_{\mathrm{i_{co}}}$. Thus, the min-max-BRG $\mathcal{B}_{\mathcal{M}}$ in Fig. 3 is obstructed. □

Proposition 5: Given a plant $G = (N, M_0, \mathcal{M}_{\mathcal{F}})$, its min-max-BRG $\mathcal{B}_{\mathcal{M}}$ is unobstructed if and only if all min-max basis markings are nonblocking.

Proof: (only if) By Definition 11, for all $M_b \in \mathcal{M}_{\mathcal{B}_{\mathcal{M}}}$, there exist a marking $M_b' \in \mathcal{M}_{\mathrm{i_{co}}}$ and a sequence of pairs σ^+ such that $(M_b, \sigma^+, {M_b}') \in \Delta_{\mathcal{M}}$. By Definition 10, M_b' is co-reachable to a final marking $M_f \in \mathcal{M}_{\mathcal{F}}$; thus, in the reachability space, M_b is also co-reachable to M_f, implying that M_b is nonblocking.

(if) For any min-max basis marking M_b in $\mathcal{B}_{\mathcal{M}}$, there exists a firing sequence $\sigma \in T^*$ and a final marking $M_f \in \mathcal{M}_{\mathcal{F}}$ such that $M_b[\sigma\rangle M_f$. We write $\sigma = \sigma_1 t_{i_1} \cdots \sigma_k t_{i_k} \sigma_{k+1}$ where all $\sigma_i \in T_I^*, t_{i_j} \in T_E, j = 1, \ldots, k$. Following the procedure in the proof of Theorem 3.8 in [25], we can repeatedly move transitions in each σ_j ($j \in \{1, \ldots, k\}$) to somewhere after t_{i_j} to obtain a new sequence $\sigma_{\min,1} t_{i_1} \sigma_{\min,2} t_{i_2} \cdots \sigma_{\min,\mathrm{k}} t_{i_k} \sigma'_{k+1}$ such that $M_b[\sigma_{\min,1} t_{i_1}\rangle M_{b,1}[\sigma_{\min,2} t_{i_2}\rangle \cdots [\sigma_{\min,\mathrm{k}} t_{i_k}\rangle M_{b,k}[\sigma'_{k+1}\rangle M_f$, where each $\sigma_{\min,\mathrm{j}} \in T_I^*$ is a minimal explanation of t_{i_j} at $M_{b,j} \in \mathcal{M}_{\mathcal{B}_\mathcal{M}}$ for $j = 1, \ldots k$ and $\sigma'_{k+1} \in T_I^*$. Thus, we have $M_{b,k} \in \mathcal{M}_{\mathrm{i_{co}}}$, which implies $\mathcal{B}_\mathcal{M}$ is unobstructed. ∎

By Proposition 5, we can relate the property of unobstructiveness to the property of the min-max basis markings of being nonblocking. In the following subsections, we show how to use the min-max-BRG to also verify if there exist non-basis markings that are blocking. Section V-B initially proposes a solution to NB-V problem regarding deadlock-free plants, since there exist classes of Petri nets for which the test of deadlock-freeness can be excluded for reachability graph construction [38]. As a generalization, in Section V-C, we show that the min-max-BRG-based technique can also be applied (with minor changes) for NB-V of plants that are not deadlock-free.

B. Verifying Nonblockingness of Deadlock-Free Plants

In this subsection, we focus on deadlock-free plants. An intermediate result is shown in Proposition 6.

Proposition 6: Given a bounded marked net $\langle N, M_0 \rangle$ with basis partition $\pi = (T_E, T_I)$, for all $M \in R(N, M_0)$, for all $t \in T_E$, for all $\sigma \in \Sigma(M, t)$ with $M[\sigma t\rangle \overline{M}$, the following implication holds:

$$\begin{aligned} &(\forall \sigma' \in \Sigma(M,t))\ \varphi(\sigma) - \varphi(\sigma') = \tilde{\mathbf{y}} \geq \mathbf{0} \Rightarrow \\ &\qquad\qquad (\exists \sigma'' \in \varphi^{-1}(\tilde{\mathbf{y}}))\ M[\sigma' t \sigma''\rangle \overline{M} \end{aligned} \tag{2}$$

Proof: Let $M' \in \mathbb{N}^m$ such that $M[\sigma' t\rangle M'$. Then it holds that:

$$\begin{cases} \overline{M} = M + C_I \cdot \varphi(\sigma) + C(\cdot, t) \\ M' = M + C_I \cdot \varphi(\sigma') + C(\cdot, t) \end{cases} \tag{3}$$

From Equation (3) we conclude that $\overline{M} - M' = C_I(\varphi(\sigma) - \varphi(\sigma'))$, which implies $\overline{M} = M' + C_I \cdot \tilde{\mathbf{y}}$ and $\tilde{\mathbf{y}} \in \mathbb{N}^n$. This indicates:

$$\exists \sigma'' \in \varphi^{-1}(\tilde{\mathbf{y}}) : M'[\sigma''\rangle \overline{M} \tag{4}$$

and thus $M[\sigma' t\rangle M'[\sigma''\rangle \overline{M}$ that concludes the proof. ∎

By Proposition 6, if markings $\overline{M}$ and M are both reachable from a marking M by firing an explicit transition t but the explanation vector to reach M is smaller than that of to reach $\overline{M}$, then $\overline{M}$ is also reachable from M. As a consequence, if M' is nonblocking, then M'' is nonblocking as well, since there exists a firing sequence $\sigma'' \in \varphi^{-1}(\tilde{\mathbf{y}})$ such that $M''[\sigma''\rangle M'$. According to this proposition, we next show that the unobstructiveness of the min-max-BRG is a necessary and sufficient condition for nonblockingness of a net in the considered class.

Lemma 1: Consider a bounded deadlock-free marked net $\langle N, M_0 \rangle$ with a basis partition $\pi = (T_E, T_I)$. For all markings $M \in R(N, M_0)$, there exists a firing sequence σt, where $\sigma \in T_I^*$ and $t \in T_E$, such that $M[\sigma t\rangle$ holds.

Proof: We prove this statement by contradiction. Assume the system is deadlock-free and there exists a marking M from which no explicit transition can eventually fire. Since the implicit sub-net of the system is bounded and acyclic, by Proposition 2, the maximal length of sequences enabled at M and composed by only implicit transitions is finite. Hence, from M, after the firing of such maximal sequences of implicit transitions, the net reaches a deadlock, which is a contradiction. ∎

The result in Lemma 1 can be applied to both BRG and min-max-BRG. However, it does not imply that the marking reached after the firing of the explicit transition is a basis marking, as we have shown in Example 1. Hence, it does not rule out the presence of livelocks in the BRG.

Lemma 2: Consider a bounded deadlock-free marked net $\langle N, M_0\rangle$ with a basis partition $\pi = (T_E, T_I)$. For all markings $M \in R(N, M_0)$, for all explicit transition $t \in T_E$, the following holds:

$$\sigma \in \Sigma(M,t) \Rightarrow (\exists \sigma' \in \Sigma_{\max}(M,t))\ \varphi(\sigma') \geq \varphi(\sigma).$$

Proof: If $\sigma \notin \Sigma_{\max}(M,t)$, according to Definition 7, there exists an explanation $\sigma' \in \Sigma_{\max}(M,t))$ such that $\varphi(\sigma') > \varphi(\sigma)$; otherwise $\varphi(\sigma') = \varphi(\sigma)$, hence the result holds. ■

Lemma 3: Consider a bounded deadlock-free marked net $\langle N, M_0\rangle$ with a basis partition $\pi = (T_E, T_I)$ and its min-max-BRG $\mathcal{B}_{\mathcal{M}}$. For all markings $M \in R(N, M_0)$, there exists $M_b \in \mathcal{M}_{\mathcal{B}_{\mathcal{M}}}$ such that $M_b \in R(N, M)$.

Proof: Due to Lemma 1, there exists a firing sequence σt, where $\sigma \in T_I^*$ and $t \in T_E$, such that $M[\sigma t\rangle$, which implies that $\sigma \in \Sigma(M,t)$. By Lemma 2, there exists a maximal explanation $\sigma' \in \Sigma_{\max}(M,t)$ such that $\varphi(\sigma') \geq \varphi(\sigma)$. Let $\varphi(\sigma')-\varphi(\sigma) = \mathbf{y}$ and $M[\sigma' t\rangle M'$. By Definition 8, we have $M' \in \mathcal{M}_{\mathcal{B}_{\mathcal{M}}}$. According to Proposition 6, there exists a firing sequence $\sigma'' \in \varphi^{-1}(\mathbf{y})$ such that $M[\sigma t \sigma''\rangle M'$, which implies that $M' \in R(N, M)$. ■

We point out that Lemma 3 holds for min-max-BRGs but not for BRGs (e.g., see Example 1). In other words, when a net is deadlock-free, for any (non-)basis marking in a min-max-BRG there always exists a maximal explanation that leads to a min-max basis marking. However, in a conventional BRG, not any reachable marking is guaranteed to be co-reachable to a basis marking. Based on Lemma 3, in the following, we show how the min-max-BRG can be leveraged for NB-V.

Theorem 1: A bounded deadlock-free plant $G = (N, M_0, \mathcal{M}_{\mathcal{F}})$ is nonblocking if and only if its min-max-BRG $\mathcal{B}_{\mathcal{M}}$ is unobstructed.

Proof: (only if) Since the net is nonblocking, all reachable markings, including all min-max basis markings, are nonblocking. By Proposition 5, its min-max-BRG $\mathcal{B}_{\mathcal{M}}$ is unobstructed.

(if) Consider an arbitrary marking $M \in R(N, M_0)$. By Lemma 3, there exists a min-max basis marking $M_b \in \mathcal{M}_{\mathcal{B}_{\mathcal{M}}}$ such that $M_b \in R(N, M)$, i.e., there exists a firing sequence $\sigma \in T^*$ such that $M[\sigma\rangle M_b$. Since the min-max-BRG $\mathcal{B}_{\mathcal{M}}$ is unobstructed, according to Proposition 5, all min-max basis markings including M_b are nonblocking, which implies that marking M is co-reachable to a nonblocking marking. Hence, G is nonblocking. ■

By Theorem 1, for a deadlock-free net, one can use an arbitrary basis partition to construct the min-max-BRG to verify its nonblockingness. Since the existence of a livelock that contains all blocking markings implies the existence of at least a blocking min-max basis marking M_b in $\mathcal{B}_{\mathcal{M}}$, the potential livelock problem mentioned in Section III is avoided.

C. Verifying Nonblockingness of Plants with Deadlocks

In this subsection, we generalize the results in Section V-B to systems that are not deadlock-free. Notice that a dead marking $M \in R(N, M_0)$ can either be non-final (i.e., $M \notin \mathcal{M}_{\mathcal{F}}$) or final (i.e., $M \in \mathcal{M}_{\mathcal{F}}$).

Theorem 2: A bounded plant $G = (N, M_0, \mathcal{M}_{\mathcal{F}})$ is nonblocking if and only if its min-max-BRG $\mathcal{B}_{\mathcal{M}}$ is unobstructed and all its dead markings are final.

Proof: (only if) When all reachable markings are nonblocking, all dead markings (if any exists) and all min-max basis markings are also nonblocking. Hence, all dead markings are final and by Proposition 5, the min-max-BRG $\mathcal{B}_{\mathcal{M}}$ is unobstructed.

(if) If the min-max-BRG $\mathcal{B}_{\mathcal{M}}$ is unobstructed, all min-max basis markings are nonblocking, by Proposition 5. Consider an arbitrary marking $M \in R(N, M_0)$. By Proposition 3, there exist a min-max basis marking $M_b \in \mathcal{M}_{\mathcal{B}_{\mathcal{M}}}$ in the min-max-BRG of the system and an implicit firing sequence $\sigma_I \in T_I^*$ such that $M_b[\sigma_I\rangle M$.

We prove that marking M is nonblocking by contradiction. In fact, if we assume that M is blocking, since all dead markings are final, M is neither dead nor co-reachable to a deadlock in the system. Suppose that from M no explicit transition can eventually fire: following the argument of the proof of Lemma 1, a dead marking will be reached, leading to a contradiction. Therefore, there exist $\sigma_I' \in T_I^*$ and $t \in T_E$ such that $M[\sigma_I' t\rangle$ and thus $M_b[\sigma_I \sigma_I' t\rangle$. Also, there exists a maximal explanation $\sigma' \in \Sigma_{\max}(M_b, t)$ such that $\varphi(\sigma') \geq \varphi(\sigma_I \sigma_I')$. According to Proposition 6, it follows that M is co-reachable to a min-max basis marking, which implies that M is nonblocking, another contradiction, which concludes the proof. ■

According to Theorem 2, determining the nonblockingness of a plant G can be addressed by two steps: (1) determine if there exists a reachable non-final dead marking; if not, then (2) determine the unobstructiveness of a min-max-BRG of it.

Since step (2) has already been discussed in the previous section, we only need to study step (1). Next, we show how to determine the existence of non-final dead markings by using the min-max-BRG. Denote the set of non-final dead markings as $\mathcal{D}_{\rm nf}$. Then, we define the set of *maximal implicit firing sequences* and the corresponding set of vectors as follows.

Definition 12: Given a bounded marked net $\langle N, M_0\rangle$ with basis partition $\pi = (T_E, T_I)$ and a marking $M \in R(N, M_0)$, we define

$$\Sigma_{\rm I,max}(M) = \{\sigma \in T_I^* | (M[\sigma\rangle) \wedge (\nexists \sigma' \in T_I^* : M[\sigma'\rangle, \varphi(\sigma') \gneqq \varphi(\sigma))\}$$

as the set of maximal implicit firing sequences at M, and

$$Y_{\rm I,max}(M) = \{\varphi(\sigma) \in \mathbb{N}^{n_I} | \sigma \in \Sigma_{\rm I,max}(M)\}$$

as the corresponding set of maximal implicit firing vectors. □

Proposition 7: Given a bounded marked net $\langle N, M_0\rangle$ with basis partition $\pi = (T_E, T_I)$, let $\mathcal{M}_{\mathcal{B}_\mathcal{M}}$ be its min-max basis marking set. Marking $M \in R(N, M_0)$ is dead if and only if there exist $M_b \in \mathcal{M}_{\mathcal{B}_\mathcal{M}}$ and $\sigma \in \Sigma_{\rm I,max}(M_b)$ such that for all $t \in T_E, \sigma \notin \Sigma_{\max}(M_b, t)$ and $M_b[\sigma\rangle M$.

Proof: (if) Since $\sigma \in \Sigma_{\rm I,max}(M_b)$, there does not exist an implicit transition $t_I \in T_I$ such that $M[t_I\rangle$. On the other hand, since for all $t \in T_E, \sigma \notin \Sigma_{\max}(M_b, t)$, i.e., there does not exist an explicit transition $t' \in T_E$ such that $M[t'\rangle$, which implies that M is dead.

(only if) Since M is dead, there does not exist $t \in T$ such that $M[t\rangle$. Therefore, there exists $\sigma \in \Sigma_{\rm I,max}(M_b)$ such that for all $t \in T_E, \sigma \notin \Sigma_{\max}(M_b, t)$ and $M_b[\sigma\rangle M$. ■

Proposition 7 shows the relation between dead markings and min-max basis markings in a bounded system, i.e., all reachable dead markings can be obtained by firing a maximal implicit firing sequence σ from a min-max basis marking M_b where for all $t \in T_E$, σ is not a maximal explanation of t. Next, we introduce Algorithm 2 to verify if there exist non-final dead markings in a plant.

In Algorithm 2, first, from lines 1−4, we add an explicit transition t_0 to N with $Pre(\cdot, t_0) = Post(\cdot, t_0) = \mathbf{0}$ and derive a new plant $(N', M_0, \mathcal{M}_\mathcal{F})$. Obviously, t_0 is enabled from any reachable marking and, since its firing does not modify the marking, it holds that $R(N, M_0) = R(N', M_0)$. Hence, for all $M_b \in \mathcal{M}_{\mathcal{B}_\mathcal{M}}$, we conclude that $Y_{\rm I,max}(M_b) = Y_{\max}(M_b, t_0)$, i.e., the set of maximal implicit firing vectors at M_b can be determined by computing maximal explanation of t_0 at M_b based on Algorithm 1.

Then, we determine if, for all $t \in T_E$, the obtained firing vector $\mathbf{y} \in Y_{\rm I,max}(M_b)$ is not an explanation of t at M_b. Implemented in lines 5−16, this consists in checking if, for all $t \in T_E$, t is disabled at marking $M' = M_b + C_I \cdot \mathbf{y}$: since no implicit transition can fire at M', the only transitions that can possibly fire are those explicit ones. If no explicit transition is enabled at M', according to Proposition 7, marking M' is dead. Further, M' will be added into the set $\mathcal{D}_{\rm nf}$ if it is dead and not final. Note that Algorithm 2 also tests if a min-max basis marking $M_b' \in \mathcal{M}_{\mathcal{B}_\mathcal{M}}$ is dead. Since $Y_{\max}(M_b', t_0) = \{\mathbf{0}\}$ and for all $t \in (T_E' \setminus \{t_0\})$, $Y_{\max}(M_b', t) = \emptyset$, M_b' will be added to $\mathcal{D}_{\rm nf}$ if it is not final. When the algorithm terminates, if

Algorithm 2 Verification of $\mathcal{D}_{\rm nf}$

Input: A bounded plant $(N, M_0, \mathcal{M}_{\mathcal{F}})$ with $\pi = (T_E, T_I)$ and its min-max basis marking set $\mathcal{M}_{\mathcal{B}_{\mathcal{M}}}$

Output: "$\mathcal{D}_{\rm nf} = \emptyset$"/ "$\mathcal{D}_{\rm nf} \neq \emptyset$"

1: $\mathcal{D}_{\rm nf} := \emptyset$, $T' := T \cup \{t_0\}$ and $T'_E := T_E \cup \{t_0\}$;

2: $Pre' := [\mathbf{0}; Pre]$ and $Post' := [\mathbf{0}; Post]$;

3: $N' := (P, T', Pre', Post')$ and $\pi' := (T'_E, T_I)$;

4: Construct a bounded plant $(N', M_0, \mathcal{M}_{\mathcal{F}})$ with basis partition π';

5: **for all** $M \in \mathcal{M}_{\mathcal{B}_{\mathcal{M}}}$, **do**

6:     **for all** $\mathbf{y} \in Y_{\max}(M, t_0)$, **do**

7:         $M' := M + C_I \cdot \mathbf{y}$;

8:         **if** M' is dead and $M' \notin \mathcal{M}_{\mathcal{F}}$, **then**

9:             $\mathcal{D}_{\rm nf} := \mathcal{D}_{\rm nf} \cup \{M'\}$;

10:             Output "$\mathcal{D}_{\rm nf} \neq \emptyset$" and Return;

11:         **end if**

12:     **end for**

13: **end for**

14: **if** $\mathcal{D}_{\rm nf} = \emptyset$, **then**

15:     Output "$\mathcal{D}_{\rm nf} = \emptyset$" and Return.

16: **end if**

Table I: Analysis of the reachability graph, expanded BRG from [33] and min-max-BRG for the plant in Fig. 4 with

$$T_E = \{t_3, t_6, t_{11}, t_{13}\}.$$

Run	λ	μ	$\|R(N, M_0)\|$	Time (s)	$\|\mathcal{M}_{\mathcal{B}_\varepsilon}\|$	Time (s)	$\|\mathcal{M}_{\mathcal{B}_\mathcal{M}}\|$	Time (s)	$\mathcal{D}_{\rm nf} = \emptyset$?	Time (s)	Unobstructed?	Time (s)	NB?	$\|\mathcal{M}_{\mathcal{B}_\mathcal{M}}\|/\|\mathcal{M}_{\mathcal{B}_\varepsilon}\|$	$\|\mathcal{M}_{\mathcal{B}_\mathcal{M}}\|/\|R(N, M_0)\|$
1	5	1	102	< 1	31	0.2	11	0.04	Yes	0.03	Yes	1.9	Yes	35.5%	10.8%
2	5	2	384	1	191	0.7	37	0.2	Yes	0.1	Yes	6	Yes	19.3%	9.6%
3	5	3	688	2	405	1	68	0.4	Yes	0.4	Yes	12	Yes	16.8%	9.9%
4	6	1	840	4	449	2	81	0.5	Yes	0.5	Yes	15	Yes	18.0%	9.6%
5	6	2	12066	431	9117	302	1171	23	Yes	16	No	251	No	12.8%	9.7%
6	6	3	88681	24354	75378	20944	9985	833	No	58	-	-	No	13.2%	11.3%
7	6	4	-	o.t.	-	o.t.	22095	4517	Yes	1099	Yes	3502	Yes	-	-
8	6	5	-	o.t.	-	o.t.	31147	10082	No	955	-	-	No	-	-
9	6	6	-	o.t.	-	o.t.	41817	18295	No	1618	-	-	No	-	-
10	6	7	-	o.t.	-	o.t.	45458	21229	No	1754	-	-	No	-	-

[*] The computing time is denoted by *overtime* (o.t.) if the program does not terminate within 28,800 seconds (8 hours).

$\mathcal{D}_{\rm nf} \neq \emptyset$, we conclude that the plant is blocking; otherwise, the unobstructiveness verification procedure (mentioned in Section V-A) of the min-max-BRG should be further executed.

The complexity of Algorithm 2 depends on the two *for* loops (lines 5−13). First, there are $|\mathcal{M}_{\mathcal{B}_\mathcal{M}}|$ and $|Y_{\max}(M, t_0)|$ iterations in lines 5 and 6, respectively. In line 8, to verify M' is dead, one may need to test if $M' \ngeq Pre'(\cdot, t)$ for all $t \in T_E$ (no need to test transitions in T_I since no implicit transition is enabled at M'), which requires $|T_E|$ iterations. In summary, the worst-case time complexity of Algorithm 2 is $O(|\mathcal{M}_{\mathcal{B}_\mathcal{M}}| \cdot |Y_{\max}(M, t_0)| \cdot |T_E|)$.

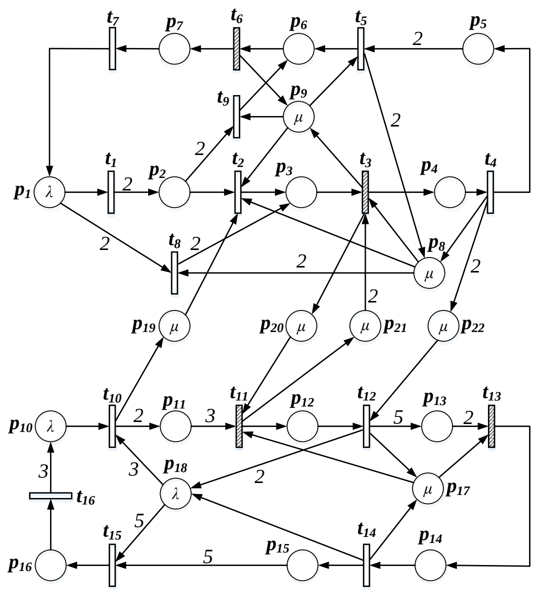

Figure 4: A parameterized manufacturing example.

VI. Case Study

We use a parameterized plant (chosen from [27]) depicted in Fig. 4 to test the efficacy and efficiency of our method in this section. Let $M_0 = [\lambda\ 0\ 0\ 0\ 0\ 0\ 0\ \mu\ \mu\ \lambda\ 0\ 0\ 0\ 0\ 0\ 0\ \mu\ \lambda\ \mu\ \mu\ \mu\ \mu]^{\mathrm{T}}$. Consider $T_E = \{t_3, t_6, t_{11}, t_{13}\}$ (marked as shadow bars). Also, we set $\mathcal{M}_{\mathcal{F}} = \mathcal{L}_{(\mathbf{w},k)} = \{M \in \mathbb{N}^m \mid \mathbf{w}^{\mathrm{T}} \cdot M \leq k\}$, where $\mathbf{w} = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0]^{\mathrm{T}}$ and $k = 3$ (for run 4) or $k = 4$ (for runs $8-10$) or $k = 5$ (for runs $1-3$) or $k = 7$ (for runs $5-6$) or $k = 15$ (for run 7), to test nonblockingness of this plant for all cases.

We run several simulations on a laptop with Intel i7-5500U 2.40 GHz processor and 8 GB RAM. Table I shows, for different values of the parameters λ and μ, the sizes of the reachability graph $|R(N, M_0)|$, of the expanded BRG $|\mathcal{M}_{\mathcal{B}_{\mathcal{E}}}|$ [33] and of min-max-BRG $|\mathcal{M}_{\mathcal{B}_{\mathcal{M}}}|$ as well as the time required to compute them. We also show the ratios of $|\mathcal{M}_{\mathcal{B}_{\mathcal{M}}}|$ to $|\mathcal{M}_{\mathcal{B}_{\mathcal{E}}}|$ and $|\mathcal{M}_{\mathcal{B}_{\mathcal{M}}}|$ to $|R(N, M_0)|$. It can be verified that $|\mathcal{M}_{\mathcal{B}_{\mathcal{M}}}| \ll |\mathcal{M}_{\mathcal{B}_{\mathcal{E}}}|$ and $|\mathcal{M}_{\mathcal{B}_{\mathcal{M}}}| \ll |R(N, M_0)|$ in all cases. Note that the size of min-max-BRG depends on the net structure, initial resource distribution and choice of basis partition $\pi = (T_E, T_I)$. Also in Table I, we show the simulation results of determining if there exist non-final dead markings based on Algorithm 2 (columns $10-11$), and verifying unobstructiveness (the set of i-coreachable markings $\mathcal{M}_{\mathrm{i_{co}}}$ of a min-max-BRG can be obtained by using two free MATLAB integer linear programming problems solver toolboxes namely YALMIP [39] and lpsolve [40]) for all cases if necessary (columns $12-13$). Moreover, the nonblockingness of the system for all cases are listed in column 14. The test cases show that min-max-BRG-based technique achieves practical efficiency when coping with the NB-V problem in this considered case. Additional case studies are also considered in [37], which consists of three Petri net benchmarks taken from the literature.

VII. Discussions

We propose the min-max-BRG to ensure that the essential features of a system, from which a blocking condition may originate, are captured in the abstracted model. As a non-trivial task, it is necessary to formally characterize and validate the proposed approach with a series of theoretical results. When tackling the NB-V problem, the min-max-BRG-based approach is general and can be directly applied to arbitrary bounded plants (the only restriction is that the T_I-induced sub-net is acyclic).

This is a major practical advantage with respect to other abstraction approaches that are based on particular structures or symmetries, and require significant analysis of the model in a preliminary stage before they can be applied.

On the other hand, we briefly compare the min-max-BRG-based approach and the reachability-graph-based ones. With the conventional reachability-graph-based method, the NB-V requires first constructing the reachability graph (enumerating and storing all reachable markings), i.e., a digraph with $|R(N, M_0)|$ nodes, for further examination. Our approach needs to build the min-max-BRG $\mathcal{B}_{\mathcal{M}}$, which requires computing the maximal explanation vectors. Although by Algorithm 1 such computation needs computing all explanation vectors, which is equivalent to compute all reachable markings, only markings computed through minimal/maximal explanation vectors are stored. As a result, a digraph with $|\mathcal{M}_{\mathcal{B}_{\mathcal{M}}}|$ nodes is built for further checking. Since the simulation results (i.e., Section VI and [28]) indicate that $|\mathcal{M}_{\mathcal{B}_{\mathcal{M}}}| \ll |R(N, M_0)|$ in some considered cases, the developed approach shows practical efficiency compared with the reachability-graph-based ones.

Further, as a potential advantage, when it comes to a related problem of NB-V, i.e., *nonblocking enforcement*, which consists of designing a *supervisor* (an online control agent) to ensure that the controlled plant does not reach a blocking marking, a supervisor designed based on the min-max-BRG can also be more compact than that of a reachability-graph-based one.

VIII. Conclusions and Future Work

In this paper, we studied the problem of nonblockingness verification of a plant. A semi-structural method using min-max-BRG is developed, which can be used to determine the nonblockingness of a system modelled by bounded Petri nets by first determining the existence of non-final deadlocks and later checking the unobstructiveness of the corresponding min-max-BRG. The proposed approach does not require the construction of the reachability graph and has wide applicability. As for future work, we will investigate necessary and sufficient conditions for verifying nonblockingness in unbounded nets. As a final comment, we point out that the proposed approach may be adapted to address other problems of interest for Petri net models, e.g., the synthesis of a min-max-BRG-based supervisor to enforce that the closed-loop system is nonblocking.

Appendix

Proof of Proposition 2:

First we observe that a bounded net cannot have source transitions. Thus, following [41] the graph of an acyclic bounded net with diameter[2] d can be topologically sorted as follows:

$$P_0 \prec T_1 \prec P_1 \prec T_2 \prec \cdots \prec T_k \prec P_k$$

where $k = \lceil d/2 \rceil$ and

$$\begin{aligned} P_0 &= \{p \in P \mid {}^\bullet p = \emptyset\} \\ T_i &= \{t \in T \setminus \cup_{j=1}^{i-1} T_j \mid {}^\bullet t \cap P_{i-1} \neq \emptyset\} \quad (i = 1, \ldots k) \\ P_i &= \{p \in P \setminus \cup_{j=1}^{i-1} P_j \mid {}^\bullet p \cap T_i \neq \emptyset\} \quad (i = 1, \ldots k) \end{aligned}$$

In other words, P_0 is the set of source places, T_i is the set of transitions whose minimal distance from a source place is $2i-1$ and P_i is the set of places whose minimal distance from a source place is $2i$. Note that when d is odd all maximal paths end with a transition and P_k is empty: all other sets are always not empty.

[2]The distance between two nodes of a graph is the length of shortest path between them. The diameter of graph is the maximal distance between pairs of nodes.

We now consider a net N' obtained from N removing all pre-arcs except those going from places in tier P_{i-1} to transitions in tier T_i. This net is still acyclic and has no source transitions hence it is bounded. Since we have only removed pre-arcs, which constrain the firing of transitions, the language of the original net is contained in the language of $\langle N', M_0\rangle$, i.e.,

$$L(N, M_0) \subseteq L(N', M_0).$$

We are left to prove that the latter set is finite. We observe that in net N' for any firing sequence σ there exists an ordered firing sequence $\sigma_{ord} = \sigma_1\sigma_2 \cdots \sigma_k$ where $\sigma_i \in T_i^*$ (for $i = 1, \ldots, k$) and $\varphi(\sigma_{ord}) = \varphi(\sigma)$ because the firing of a transition in tier T_j cannot disable transitions in tier T_i for $i < j$. Obviously in an ordered sequence the length of any substring σ_i (for $i = 1, \ldots, k$) is bounded since each transition firing decreases the token content of set P_{i-1} and thus the set of ordered sequences is finite. Since $L(N', M_0)$ contains only permutations of ordered sequences, this set is also finite.

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