# Forward refutation for Gödel-Dummett Logics

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**Abstract**
We propose a refutation calculus to check the unprovability of a formula in Gödel-Dummett logics. From refutations we can directly extract countermodels for unprovable formulas, moreover the calculus is designed so to support a forward proof-search strategy that can be understood as a top-down construction of a model.

## 1. Introduction

With the term Gödel-Dummett logics we refer to the family of intermediate logics $\mathrm{GD}_k$ semantically characterised by linear Kripke models of height at most $k$ and the logic GD characterised by linear Kripke models. The logics $\mathrm{GD}_k$ were originally introduced by Gödel [1] to study the logics with $k$-valued matrices semantics, while GD was introduced by Dummett [2] to characterize the logic with infinite valued matrix. Gödel-Dummett logics have been extensively studied for their relations with fuzzy logics [3] and for their computational interpretations [4, 5]. This led to the development of an articulate family of calculi and proof-search strategies for these logics [6, 5, 7, 8, 9, 10].

In this paper we address the problem of defining a logical calculus oriented to generate countermodels for invalid formulas for Gödel-Dummett logics; we exploit the approach based on inverse methods we have developed for Intuitionistic Propositional Logic and the modal logics **K** and **S4** [11, 12, 8, 13]. The inverse method, introduced by Maslov [14], is a saturation based theorem proving technique closely related to (hyper)resolution [15]; it relies on a forward proof-search strategy and can be applied to cut-free calculi enjoying the subformula property. Given a goal, a set of instances of the rules of the calculus at hand is selected; such specialized rules are repeatedly applied in the forward direction, starting from the axioms (i.e., the rules without premises). Proof-search terminates if either the goal is obtained or the set collecting the proved facts saturates (nothing new can be added). The inverse method has been originally applied to Classical Logic and successively extended to some non-classical logics [16, 15, 17, 18]. In all of the mentioned papers, the inverse method has been exploited to prove the validity of a formula in a specific logic. In [12] we launched a new perspective designing a forward calculus to derive the unprovability of a goal formula in Intuitionistic Propositional Logic and to generate countermodels for unprovable formulas. Differently from other approaches to countermodel

*CILC 2022: 37th Italian Conference on Computational Logic, June 29 – July 1, 2022, Bologna, Italy*
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construction for non-classical logics [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29], where countermodels are obtained as a byproduct of a failed proof-search in a direct or refutation calculus, we define refutation calculi directly supporting model extraction and oriented to forward reasoning. Our approach focuses on countermodel construction; indeed, the rules of the refutation calculus are inspired by the Kripke semantics of the logic at hand and the forward refutation-search procedure can be understood as a top-down method to build a countermodel for the given goal formula. Differently from backward proof-search procedures, forward methods re-use sequents and do not replicate them, accordingly the generated models contain few duplications and are in general very concise.

In this paper we present the refutation calculus for Gödel-Dummett logics, we prove its soundness and completeness and we show how to extract countermodels from its derivations.

## 2. Preliminaries

Formulas, denoted by uppercase Latin letters, are built from an infinite set of propositional variables $\mathcal{V} = \{p, q, p_1, p_2, \dots\}$, the constant $\bot$ and the connectives $\land$, $\lor$, $\supset$; moreover, $\neg A$ stands for $A \supset \bot$. Let $G$ be a formula; $\mathrm{Sf}(G)$ is the set of all subformulas of $G$ (including $G$ itself). By $\textsc{Sl}(G)$ and $\textsc{Sr}(G)$ we denote the subsets of *left* and *right* subformulas of $G$ (a.k.a. negative/positive subformulas of $G$ [30]). Formally, $\textsc{Sl}(G)$ and $\textsc{Sr}(G)$ are the smallest subsets of $\mathrm{Sf}(G)$ such that:

- $G \in \textsc{Sr}(G)$;
- $A \odot B \in \textsc{Sx}(G)$ implies $\{A, B\} \subseteq \textsc{Sx}(G)$, where $\odot \in \{\land, \lor\}$ and $\textsc{Sx} \in \{\textsc{Sl}, \textsc{Sr}\}$;
- $A \supset B \in \textsc{Sl}(G)$ implies $B \in \textsc{Sl}(G)$ and $A \in \textsc{Sr}(G)$;
- $A \supset B \in \textsc{Sr}(G)$ implies $B \in \textsc{Sr}(G)$ and $A \in \textsc{Sl}(G)$.

For $\textsc{Sx} \in \{\textsc{Sl}, \textsc{Sr}\}$ we set ($\mathcal{L}^{\supset}$ denotes the set of formulas of the kind $A \supset B$):

$$\textsc{Sx}^{\mathrm{At}}(G) = \textsc{Sx}(G) \cap \mathcal{V} \qquad \textsc{Sx}^{\supset}(G) = \textsc{Sx}(G) \cap \mathcal{L}^{\supset}$$
$$\textsc{Sx}^{\mathrm{At},\supset}(G) = \textsc{Sx}^{\mathrm{At}}(G) \cup \textsc{Sx}^{\supset}(G) \qquad \mathrm{Sf}^{\mathrm{At}}(G) = \textsc{Sl}^{\mathrm{At}}(G) \cup \textsc{Sr}^{\mathrm{At}}(G)$$

A (rooted) Kripke model $\mathcal{K}$ is a quadruple $\langle W, \leq, \rho, V\rangle$ where $W$ is a finite and non-empty set (the set of *worlds*), $\leq$ is a reflexive and transitive binary relation over $W$, the world $\rho$ (the *root* of $\mathcal{K}$) is the minimum of $W$ w.r.t. $\leq$, and $V : W \mapsto 2^{\mathcal{V}}$ (the *valuation* function) is a map obeying the persistence condition: for every pair of worlds $\alpha$ and $\beta$ of $\mathcal{K}$, $\alpha \leq \beta$ implies $V(\alpha) \subseteq V(\beta)$; the triple $\langle W, \leq, \rho\rangle$ is called *(Kripke) frame*. We write $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$; moreover, we write $\beta \geq \alpha$ ($\beta > \alpha$ resp.) to mean that $\alpha \leq \beta$ ($\alpha < \beta$ resp.). A world $\beta$ is an *immediate successor* of $\alpha$ in $\mathcal{K}$ if $\alpha < \beta$ and there is no world $\gamma$ such that $\alpha < \gamma < \beta$.

The valuation $V$ is extended into a *forcing* relation, denoted by $\Vdash$, between worlds of $\mathcal{K}$ and formulas as follows:

$$\mathcal{K}, \alpha \Vdash p \text{ iff } p \in V(\alpha), \forall p \in \mathcal{V} \qquad \mathcal{K}, \alpha \nVdash \bot$$
$$\mathcal{K}, \alpha \Vdash A \land B \text{ iff } \mathcal{K}, \alpha \Vdash A \text{ and } \mathcal{K}, \alpha \Vdash B \qquad \mathcal{K}, \alpha \Vdash A \lor B \text{ iff } \mathcal{K}, \alpha \Vdash A \text{ or } \mathcal{K}, \alpha \Vdash B$$
$$\mathcal{K}, \alpha \Vdash A \supset B \text{ iff } \forall \beta \geq \alpha, \mathcal{K}, \beta \Vdash A \text{ implies } \mathcal{K}, \beta \Vdash B.$$

We sometimes write $\alpha \Vdash A$ instead of $\mathcal{K}, \alpha \Vdash A$, leaving understood the model $\mathcal{K}$ at hand when it is clear from the context. By $\alpha \Vdash \Gamma$ we mean that $\alpha \Vdash A$ for every $A \in \Gamma$. A formula $A$ is *valid* in the frame $\langle W, \leq, \rho \rangle$ iff for every valuation $V$, $\rho \Vdash A$ in the model $\langle W, \leq, \rho, V \rangle$. Propositional Intuitionistic Logic (IPL) is the set of formulas valid in all frames. Accordingly, if there is a model $\mathcal{K}$ such that $\rho \nVdash A$ (here and below $\rho$ designates the root of $\mathcal{K}$), then $A$ is not IPL-valid; we call $\mathcal{K}$ a *countermodel* for $A$. We write $\Gamma \Vdash A$ iff, for every model $\mathcal{K}$, $\rho \Vdash \Gamma$ implies $\rho \Vdash A$; thus, $A$ is IPL-valid iff $\emptyset \Vdash A$.

Given a frame $\langle W, \leq, \rho \rangle$, the *height* $\mathrm{h}(\alpha)$ of $\alpha \in W$, is defined as follows:

$$\mathrm{h}(\alpha) = \begin{cases} 0 & \text{if } \alpha \text{ is a maximal world of } W \text{ w.r.t. } \leq \\ 1 + \max\{\, \mathrm{h}(\beta) \mid \alpha < \beta \,\} & \text{otherwise} \end{cases}$$

The *height of* $\mathcal{K}$, denoted by $\mathrm{h}(\mathcal{K})$, is the height of its root.

We say that a *Kripke frame* $\langle W, \leq, \rho \rangle$ *is linear* iff $\leq$ is a linear order over $W$; i.e., for every pair of worlds $\alpha$ and $\beta$, either $\alpha \leq \beta$ or $\beta \leq \alpha$.

Given a formula $G$ we say that a Kripke model $\mathcal{K} = \langle W, \leq, \rho, V \rangle$ is *$G$-separable* iff, for every pair of worlds $\alpha$ and $\beta$ in $W$, the following separation property holds:

- if $\alpha < \beta$, then there is $p \in \mathrm{S{\scriptstyle L}}^{\mathrm{At}}(G) \cap \mathrm{S{\scriptstyle R}}^{\mathrm{At}}(G)$ such that $\mathcal{K}, \alpha \nVdash p$ and $\mathcal{K}, \beta \Vdash p$.

Let $\Theta$ be a set of formulas and let us consider the formulas $P$ and $N$ defined by the following grammar, where $A \in \Theta$ and $F$ is any formula

$$\begin{array}{lcl} P & ::= & A \mid P \wedge P \mid F \vee P \mid P \vee F \mid F \supset P \\ N & ::= & A \mid N \vee N \mid F \wedge N \mid N \wedge F \end{array}$$

The *positive closure* of $\Theta$, denoted by $\mathcal{C}l^+(\Theta)$, is the smallest set containing the formulas $P$; the *negative closure* of $\Theta$, denoted by $\mathcal{C}l^-(\Theta)$, is the smallest set containing the formulas $N$. The following properties can be easily proved:

($\mathcal{C}l1$) If $\Theta_1 \subseteq \Theta_2$, then $\mathcal{C}l^+(\Theta_1) \subseteq \mathcal{C}l^+(\Theta_2)$ and $\mathcal{C}l^-(\Theta_1) \subseteq \mathcal{C}l^-(\Theta_2)$.

($\mathcal{C}l2$) If $\mathcal{K}, \alpha \Vdash A$, for every $A \in \Theta$, and $P \in \mathcal{C}l^+(\Theta)$, then $\mathcal{K}, \alpha \Vdash P$.

($\mathcal{C}l3$) If $\mathcal{K}, \alpha \nVdash A$, for every $A \in \Theta$, and $N \in \mathcal{C}l^-(\Theta)$, then $\mathcal{K}, \alpha \nVdash N$.

### The logics $\mathrm{GD}_k$ and $\mathrm{GD}$

In this paper we consider the Gödel-Dummett logics $\mathrm{GD}_k$ ($k \geq 0$) and $\mathrm{GD}$ defined as follows (see [31]):

- $\mathrm{GD}_k$ is the set of formulas valid in linear models $\mathcal{K}$ such that $\mathrm{h}(\mathcal{K}) \leq k$;
- $\mathrm{GD} = \bigcap_{k \geq 0} \mathrm{GD}_k$.

We remark that $\mathrm{IPL} \subset \mathrm{GD} \subset \cdots \subset \mathrm{GD}_2 \subset \mathrm{GD}_1 \subset \mathrm{GD}_0 = \mathrm{CPL}$, where CPL is the set of classically valid formulas.

## 3. The $\mathrm{GD}$-refutation calculus

The forward refutation calculus $\mathbf{R}_{\mathrm{GD}}(G)$ is a calculus to infer the unprovability of a formula $G$ (the *goal formula*) in $\mathrm{GD}_k$ and it is designed to support forward refutation-search (for a presentation of forward calculi we refer to [15]). The calculus acts on $\mathbf{R}_{\mathrm{GD}}(G)$-*sequents*[1] having the form $\Gamma \not\Rightarrow_k \Lambda\,;\,\Delta$ where:

- $k \geq 0$, $\Gamma \subseteq \mathrm{Sl}^{\mathrm{At},\supset}(G)$, $\Lambda \subseteq \mathrm{Sl}^{\mathrm{At}}(G) \cap \mathrm{Sr}^{\mathrm{At}}(G)$, and $\Delta \subseteq \mathrm{Sr}^{\mathrm{At},\supset}(G)$;
- if $k = 0$, then $\Lambda = \emptyset$.

The *rank* of $\sigma = \Gamma \not\Rightarrow_k \Lambda\,;\,\Delta$, denoted by $\mathrm{Rn}(\sigma)$, is $k$. We will see that, whenever there exists a refutation $\mathcal{D}$ of $\sigma$ in the calculus $\mathbf{R}_{\mathrm{GD}}(G)$, from $\mathcal{D}$ we can extract a model containing a world $\alpha$ such that $\mathrm{h}(\alpha) = k$ and:

- $\mathcal{K}, \alpha \Vdash \bigwedge \Gamma$ and $\mathcal{K}, \alpha \nVdash \bigvee \Delta$; moreover, for every $A \supset B \in \Gamma$, $\mathcal{K}, \alpha \nVdash A$;
- if $k > 0$, then $\Lambda$ is the set of propositional variables forced in the immediate successor of $\alpha$ and not in $\alpha$.

The rules of $\mathbf{R}_{\mathrm{GD}}(G)$ are displayed in Fig. 1. We point out that the formulas introduced in the conclusion of the rules in the left (side of the sequents) must belong to $\mathrm{Sl}(G)$ and the formulas introduced in the right must belong to $\mathrm{Sr}(G)$. An $\mathbf{R}_{\mathrm{GD}}(G)$-sequent $\sigma$ is *saturated* if none of the rules $L\supset$ and $R\supset$ can be applied to $\sigma$. As a consequence of the side condition, the application of the rule $\mathrm{Succ}$ is delayed until a saturated sequent is get. The successor rule $\mathrm{Succ}$ moves the propositional variables in $\Lambda'$ from the left side of sequent to the right side; in countermodel construction, an application of the $\mathrm{Succ}$ rule corresponds to a downward expansion of a model, obtained by adding a new root $\rho'$ below the current root $\rho$; the propositional variables in $\Lambda'$ are forced in $\rho$ and not in $\rho'$. Note that, given a rule and the sequent in the premise, we can build different instances of the rule according with the non-deterministic choices described in the side-condition of the rule. E.g., we can generate a different instance of $L\supset$ having $\Gamma \not\Rightarrow_0 \cdot\,;\,\Delta$ in the premise, for every $A \supset B \in \mathrm{Sl}(G)$ such that $A \supset B \notin \Gamma$ and $A \in \mathcal{C}l^{-}(\Delta)$. This also holds for the axiom-rule; we get a different axiom for every possible partition $(\Gamma^{\mathrm{At}}, \Delta^{\mathrm{At}})$ of $\mathrm{Sf}^{\mathrm{At}}(G)$. A *proof tree* of the calculus $\mathbf{R}_{\mathrm{GD}}(G)$ is a tree having $\mathbf{R}_{\mathrm{GD}}(G)$-sequents as nodes and built according to the rules of $\mathbf{R}_{\mathrm{GD}}(G)$ (see e.g. [30] for a formal definition). Note that all the proof trees of $\mathbf{R}_{\mathrm{GD}}(G)$ are linear. We introduce some definitions:

- $\mathcal{D}$ is an $\mathbf{R}_{\mathrm{GD}}(G)$-*refutation of* $\sigma$ iff $\mathcal{D}$ is a proof tree of $\mathbf{R}_{\mathrm{GD}}(G)$ having $\sigma$ as root sequent; the rank of $\mathcal{D}$ is the rank of $\sigma$ ($\mathrm{Rn}(\mathcal{D}) = \mathrm{Rn}(\sigma)$).
- $\mathcal{D}$ is an $\mathbf{R}_{\mathrm{GD}}(G)$-*refutation of* $G$ iff $\mathcal{D}$ is an $\mathbf{R}_{\mathrm{GD}}(G)$-refutation of $\Gamma \not\Rightarrow_k \Lambda\,;\,\Delta$ and $G \in \mathcal{C}l^{-}(\Delta \cup \Lambda)$.
- $\vdash^k_G G$ iff there is an $\mathbf{FRJ}(G)$-refutation $\mathcal{D}$ of $G$ such that $\mathrm{Rn}(\mathcal{D}) \leq k$.

[1] In refutation calculi sequents are sometimes called *anti-sequents* (see,e.g., [27]).

$$\frac{}{\Gamma^{\mathrm{At}} \not\Rightarrow_0 \cdot\,;\, \Delta^{\mathrm{At}}, \bot}\ \mathrm{Ax} \qquad \begin{array}{l}\Gamma^{\mathrm{At}} \cup \Delta^{\mathrm{At}} = \mathrm{Sf}^{\mathrm{At}}(G)\\ \Gamma^{\mathrm{At}} \cap \Delta^{\mathrm{At}} = \emptyset\end{array}$$

$$\frac{\Gamma \not\Rightarrow_0 \cdot\,;\, \Delta}{A \supset B, \Gamma \not\Rightarrow_0 \cdot\,;\, \Delta}\ L\supset \quad \begin{array}{l}A \supset B \notin \Gamma \cup \Delta\\ A \in \mathcal{C}l^-(\Delta)\end{array} \qquad \frac{\Gamma \not\Rightarrow_k \Lambda\,;\, \Delta}{A \supset B, \Gamma \not\Rightarrow_k \Lambda\,;\, \Delta}\ L\supset \quad \begin{array}{l}A \supset B \notin \Gamma \cup \Delta\\ A \in \mathcal{C}l^-(\Delta \cup \Lambda)\\ B \in \mathcal{C}l^+(\Gamma \cup \Lambda)\end{array}$$

$$\frac{\Gamma \not\Rightarrow_k \Lambda\,;\, \Delta}{\Gamma \not\Rightarrow_k \Lambda\,;\, \Delta, A \supset B}\ R\supset \quad \begin{array}{l}A \supset B \notin \Delta \cup \Delta\\ A \in \mathcal{C}l^+(\Gamma)\\ B \in \mathcal{C}l^-(\Delta \cup \Lambda)\end{array}$$

$$\frac{\Gamma \not\Rightarrow_k \Lambda\,;\, \Delta}{\Gamma \setminus \Lambda' \not\Rightarrow_{k+1} \Lambda'\,;\, \Delta, \Lambda}\ \mathrm{Succ} \quad \begin{array}{l}\Gamma \not\Rightarrow_k \Lambda\,;\, \Delta \text{ is saturated}\\ \emptyset \subset \Lambda' \subseteq \Gamma \cap \mathcal{V}\end{array}$$

**Figure 1:** The refutation calculus $\mathbf{R}_{\mathbf{GD}}(G)$.

**Example 1** Let us consider the following formula $G$:

$$G \;=\; A \vee (p \supset r) \vee B \vee (C \supset (p \vee \neg p))$$

$$A \;=\; \neg(q \wedge r) \qquad B \;=\; (\neg\neg p \wedge (p \supset q)) \supset q \qquad C \;=\; B \supset (\neg\neg p \wedge q)$$

We search for an $\mathbf{R}_{\mathbf{GD}}(G)$-derivation building a database of proved sequents according with the naive recipe of [15]: we start by inserting all the axioms; then we enter a loop where, at each iteration, we apply all the possible rules to the sequents collected in previous steps. The loop ends if either a sequent $\Gamma \not\Rightarrow_k \Lambda\,;\, \Delta$ with $G \in \mathcal{C}l^-(\Delta \cup \Lambda)$ is generated or no new sequent can be added to the database (the database is saturated). Fig. 2 shows the fragment of the database containing the sequents needed to get the $\mathbf{R}_{\mathbf{GD}}(G)$-derivation of $G$. In the example, we denote with $\sigma_{(j)}$ the sequent at line $(j)$ of Fig. 2. As an example, the sequent $\sigma_{(2)}$ is obtained by applying the rule $R\supset$ to the sequent $\sigma_{(1)}$, i.e,:

$$\frac{p,\ q,\ r \not\Rightarrow_0 \cdot\,;\, \bot}{p,\ q,\ r \not\Rightarrow_0 \cdot\,;\, \bot,\ \neg p}\ R\supset$$

recalling that $\neg p = p \supset \bot$; note that, $p \in \mathcal{C}l^+(\{p,\ q,\ r\})$ and $\bot \in \mathcal{C}l^-(\{\bot\})$. As for sequent $\sigma_{(3)}$ it is obtained by applying the $L\supset$ rule to $\sigma_{(2)}$:

$$\frac{p,\ q,\ r \not\Rightarrow_0 \cdot\,;\, \bot,\ \neg p}{p,\ q,\ r \not\Rightarrow_0 \cdot\,;\, \bot,\ \neg p,\ A}\ L\supset$$

where $A = \neg(q \wedge r) = (q \wedge r) \supset \bot$ , note that $\bot \in \mathcal{C}l^-(\{\bot\})$ and $q \wedge r \in \mathcal{C}l^+(\{p,\ q,\ r\})$. Sequent $\sigma_{(5)}$ is obtained applying Succ to $\sigma_{(4)}$ by moving $r$ from left to right; similarly, $\sigma_{(7)}$

$$G \;=\; A \vee (p \supset r) \vee B \vee (C \supset (p \vee \neg p))$$
$$A \;=\; \neg(q \wedge r) \qquad B \;=\; (\neg\neg p \wedge (p \supset q)) \supset q \qquad C \;=\; B \supset (\neg\neg p \wedge q)$$
$$\textsc{Sl}^{\mathrm{At}}(G) \;=\; \{\, p,\, q,\, r \,\} \qquad \textsc{Sl}^{\supset}(G) \;=\; \{\, C,\, \neg\neg p,\, p \supset q \,\}$$
$$\textsc{Sr}^{\mathrm{At}}(G) \;=\; \{\, p,\, q,\, r \,\} \qquad \textsc{Sr}^{\supset}(G) \;=\; \{\, A,\, p \supset r,\, B,\, C \supset (p \vee \neg p),\, \neg p \,\}$$

| | | |
|---|---|---|
| (1) | $p,\, q,\, r \;\not\Rightarrow_0\; \cdot\,;\, \bot$ | Ax |
| (2) | $p,\, q,\, r \;\not\Rightarrow_0\; \cdot\,;\, \bot,\, \neg p$ | $R \supset$ (1) |
| (3) | $p,\, q,\, r \;\not\Rightarrow_0\; \cdot\,;\, \bot,\, \neg p,\, A$ | $R \supset$ (2) |
| (4) | $\neg\neg p,\, p,\, q,\, r \;\not\Rightarrow_0\; \cdot\,;\, \bot,\, \neg p,\, A \quad (*)$ | $L \supset$ (3) |
| (5) | $\neg\neg p,\, p,\, q \;\not\Rightarrow_1\; r\,;\, \bot,\, \neg p,\, A$ | Succ (4) |
| (6) | $\neg\neg p,\, p,\, q \;\not\Rightarrow_1\; r\,;\, \bot,\, \neg p,\, A,\, p \supset r \quad (*)$ | $R \supset$ (5) |
| (7) | $\neg\neg p \;\not\Rightarrow_2\; p,\, q\,;\, \bot,\, \neg p,\, A, p \supset r,\, r$ | Succ (6) |
| (8) | $p \supset q,\, \neg\neg p \;\not\Rightarrow_2\; p,\, q\,;\, \bot,\, \neg p,\, A, p \supset r,\, r$ | $L \supset$ (7) |
| (9) | $p \supset q,\, \neg\neg p \;\not\Rightarrow_2\; p,\, q\,;\, \bot,\, \neg p,\, A, p \supset r,\, r,\, B$ | $R \supset$ (8) |
| (10) | $C,\, p \supset q,\, \neg\neg p \;\not\Rightarrow_2\; p,\, q\,;\, \bot,\, \neg p,\, A, p \supset r,\, r,\, B$ | $L \supset$ (9) |
| (11) | $C,\, p \supset q,\, \neg\neg p \;\not\Rightarrow_2\; p,\, q\,;\, \bot,\, \neg p,\, A, p \supset r,\, r,\, B,\, C \supset (p \vee \neg p) \quad (*)$ | $R \supset$ (10) |

**Figure 2:** Building the $\mathbf{R}_{\mathbf{GD}}(G)$-refutation of $G$; p-sequents are marked by (*).

is obtained applying Succ to $\sigma_{(6)}$ by moving $p$ and $q$ from left to right and moving $r$ to the rightmost zone. We have marked with $*$ the premises of Succ that, as we discuss later, play a role in the construction of the countermodel. Note that sequent $\sigma_{(11)}$ meets the property $G \in \mathcal{C}l^{-}(\Delta \cup \Lambda)$. The tree-like structure of the $\mathbf{R}_{\mathbf{GD}}(G)$*-refutation of* $G$ is displayed in the left of Fig. 3. $\Diamond$

We introduce the following relations between $\mathbf{R}_{\mathbf{GD}}(G)$-sequents:

- $\sigma_1 \overset{\mathcal{R}}{\mapsto}_0 \sigma_2$ iff $\mathcal{R}$ is a rule of $\mathbf{R}_{\mathbf{GD}}(G)$ having premise $\sigma_1$ and conclusion $\sigma_2$;
- $\sigma_1 \mapsto_0 \sigma_2$ iff there exists a rule $\mathcal{R}$ such that $\sigma_1 \overset{\mathcal{R}}{\mapsto}_0 \sigma_2$;
- $\sigma_1 \bar{\mapsto}_0 \sigma_2$ iff there exists a rule $\mathcal{R} \neq$ Succ such that $\sigma_1 \overset{\mathcal{R}}{\mapsto}_0 \sigma_2$;
- $\mapsto_*$ (resp. $\bar{\mapsto}_*$) is the reflexive and transitive closure of $\mapsto$ (resp. $\bar{\mapsto}_*$).

The following properties can be easily proved ($||\Theta||$ denotes the cardinality of the set $\Theta$)

**Lemma 1** Let $\sigma_1 = \Gamma_1 \not\Rightarrow_{k_1} \Lambda_1 \,;\, \Delta_1$ and $\sigma_2 = \Gamma_2 \not\Rightarrow_{k_2} \Lambda_2 \,;\, \Delta_2$ be two $\mathbf{R}_{\text{GD}}(G)$-sequents such that $\sigma_1 \mapsto_* \sigma_2$. Then:

(i) $k_1 \leq k_2$.

(ii) $\Gamma_1 \cap \mathcal{L}^{\supset} \subseteq \Gamma_2 \cap \mathcal{L}^{\supset}$ and $\Gamma_2 \cap \mathcal{V} \subseteq \Gamma_1$. Moreover, if $k_1 = k_2$ then $\Gamma_1 \subseteq \Gamma_2$ and $\Gamma_2 \cap \mathcal{V} = \Gamma_1 \cap \mathcal{V}$.

(iii) If $k_1 = k_2$, then $\Lambda_1 = \Lambda_2$ and $\Delta_1 \subseteq \Delta_2$. If $k_1 < k_2$, then $\Delta_1 \cup \Lambda_1 \subseteq \Delta_2$ and $\Lambda_2 \subseteq \Gamma_1$.

(iv) $k_2 \leq k_1 + ||\Gamma_1 \cap \mathcal{V}||$.

By Lemma 1, we get:

**Proposition 1** The relation $\mapsto_0$ on $\mathbf{R}_{\text{GD}}(G)$-sequents is terminating.

*Proof.* Each application of rules $L \supset$ and $R \supset$ introduces a new subformula of $G$ in the conclusion, thus $\bar{\mapsto}_0$ is terminating, Accordingly, an infinite $\mapsto_0$-chain starting from $\Gamma \not\Rightarrow_k \Lambda \,;\, \Delta$ should contain infinitely many applications of rule $\mathrm{Succ}$. This is not possible, since every application of rule $\mathrm{Succ}$ increases by 1 the rank of a sequent and, by Lemma 1(iv), the rank of any sequent in the chain is bounded by $k + ||\Gamma \cap \mathcal{V}||$. We conclude that $\mapsto_0$ is terminating. □

## 4. Soundness

Soundness of $\mathbf{R}_{\text{GD}}(G)$ is stated as follows:

**Theorem 1 (Soundness of $\mathbf{R}_{\text{GD}}(G)$)** $\vdash_G^k G$ implies $G \notin \mathrm{GD}_k$.

To prove this, we show that from an $\mathbf{R}_{\text{GD}}(G)$-refutation $\mathcal{D}$ of $G$ we can extract a countermodel $\mathrm{Mod}(\mathcal{D})$ for $G$ such that $\mathrm{h}(\mathrm{Mod}(\mathcal{D})) = \mathrm{Rn}(\mathcal{D})$.

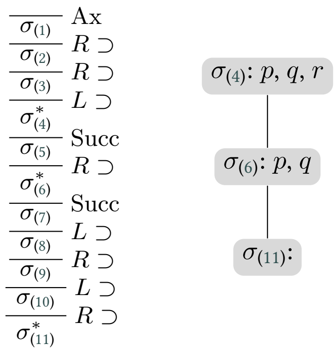
Let $\mathcal{D}$ an $\mathbf{R}_{\text{GD}}(G)$-refutation and let $\sigma$ be a sequent occurring in $\mathcal{D}$; $\sigma$ is a *p-sequent* (*prime sequent*) iff $\sigma$ is saturated or $\sigma$ is the root sequent of $\mathcal{D}$. Let $\mathrm{Mod}(\mathcal{D}) = \langle \mathrm{P}(\mathcal{D}), \leq, \rho, V \rangle$ where:

- $\mathrm{P}(\mathcal{D})$ is the set of all p-sequents occurring in $\mathcal{D}$;
- for every $\sigma_1, \sigma_2 \in \mathrm{P}(\mathcal{D})$, $\sigma_1 \leq \sigma_2$ iff $\sigma_2 \mapsto_* \sigma_1$;
- $\rho$ is the root of $\mathcal{D}$;
- $V$ maps a p-sequent $\Gamma \not\Rightarrow_k \Lambda \,;\, \Delta$ to the set $\Gamma \cap \mathcal{V}$.

Then, since $\mathbf{R}_{\text{GD}}(G)$-refutations are linear, $\mathrm{Mod}(\mathcal{D})$ is a linear model; note that, by Lemma1(ii), $\sigma_1 \leq \sigma_2$ implies $V(\sigma_1) \subseteq V(\sigma_2)$, hence the definition of $V$ is sound. We call $\mathrm{Mod}(\mathcal{D})$ the *model extracted from* $\mathcal{D}$. For every sequent $\sigma$ occurring in $\mathcal{D}$, let $\phi(\sigma)$ be the p-sequent in $\mathcal{D}$ immediately below $\sigma$, namely:

$$\phi(\sigma) = \sigma_p \qquad \text{iff} \quad \sigma_p \in \mathrm{P}(\mathcal{D}) \text{ and } \sigma \,\bar{\mapsto}_*\, \sigma_p$$

It is easy to check that:




sequents labeled by $(*)$ are p-sequents
$\sigma_{(j)}$ refers to the sequent at line $(j)$

$$\phi(\sigma_j) = \begin{cases} \sigma_{(4)} & j = 1,2,3,4 \\ \sigma_{(6)} & j = 5,6 \\ \sigma_{(11)} & j = 7,8,9,10,11 \end{cases}$$

**Figure 3:** The $\mathbf{R}_{\mathrm{GD}}(G)$-derivation of $G$ and the extracted countermodel.

- p-sequents are fixed points of $\phi$, i.e., $\sigma_p \in \mathrm{P}(\mathcal{D})$ implies $\phi(\sigma_p) = \sigma_p$;
- $\phi$ is a surjective map from the set of sequents of $\mathcal{D}$ onto $\mathrm{P}(\mathcal{D})$;
- $\sigma_1 \mapsto_* \sigma_2$ implies $\phi(\sigma_2) \leq \phi(\sigma_1)$;
- $\mathrm{h}(\phi(\sigma)) = \mathrm{Rn}(\sigma)$.

We call $\phi$ the *map associated with* $\mathcal{D}$; note that $\mathrm{Mod}(\mathcal{D})$ is $G$-separable.

**Example 2** The model $\mathrm{Mod}(\mathcal{D}_G)$ and the related map $\phi$ are shown in Fig. 3. The bottom world is the root and $\sigma < \sigma'$ iff the world $\sigma$ is drawn below $\sigma'$. For each $\sigma$, we display the set $V(\sigma)$. As an example, $V(\sigma_4) = \{p, q, r\}$. It is easy to check that $\sigma_{(11)} \nVdash G$. ♢

The following lemma is the main step to prove the soundness theorem:

**Lemma 2** Let $\mathcal{D}$ be an $\mathbf{R}_{\mathrm{GD}}(G)$-refutation, let $\mathrm{Mod}(\mathcal{D})$ be the model extracted from $\mathcal{D}$ and $\phi$ the map associated with $\mathcal{D}$. For every sequent $\sigma = \Gamma \not\Rightarrow_k \Lambda\,;\,\Delta$ occurring in $\mathcal{D}$, the following properties hold.

(i) For every $C \in \Gamma$, $\mathrm{Mod}(\mathcal{D}), \phi(\sigma) \Vdash C$. Moreover, if $C = A \supset B$, then $\mathrm{Mod}(\mathcal{D}), \phi(\sigma) \nVdash A$.

(ii) For every $C \in \Delta \cup \Lambda$, $\mathrm{Mod}(\mathcal{D}), \phi(\sigma) \nVdash C$.

*Proof.* By induction on the height of $\sigma$ in $\mathcal{D}$, taking into account the closure properties $(\mathcal{C}l1)$-$(\mathcal{C}l2)$ and Lemma 1. □

Let $\vdash^k_G G$. Then, there exists an $\mathbf{R}_{\mathrm{GD}}(G)$-refutation $\mathcal{D}$ of $\sigma = \Gamma \not\Rightarrow_{k'} \Lambda\,;\,\Delta$ such that $k' \leq k$ and $G \in \mathcal{C}l^-(\Delta \cup \Lambda)$. Let $\mathrm{Mod}(\mathcal{D}) = \langle P, \leq, \rho, V\rangle$ and $\phi$ the associated map. We have $\mathrm{h}(\mathrm{Mod}(\mathcal{D})) = k' \leq k$ and $\phi(\sigma) = \rho$; by Lemma 2(ii), we get $\mathrm{Mod}(\mathcal{D}), \rho \nVdash C$, for every $C \in \Delta \cup \Lambda$. Since $G \in \mathcal{C}l^-(\Delta \cup \Lambda)$, by property $(\mathcal{C}l3)$ of negative closures $\mathrm{Mod}(\mathcal{D}), \rho \nVdash G$, hence $G \notin \mathrm{GD}_k$. This proves the soundness of $\mathbf{R}_{\mathrm{GD}}(G)$ (Theorem 1).

## 5. Completeness

We prove the completeness of $\mathbf{R}_{\mathrm{GD}}(G)$:

**Theorem 2 (Completeness of $\mathbf{R}_{\mathrm{GD}}(G)$)** $G \notin \mathrm{GD}_k$ implies $\vdash^k_G G$.

The proof goes along the following lines. First we show that we can use a $G$-separable countermodel of $G$ of height $k$ to build an $\mathbf{R}_{\mathrm{GD}}(G)$-refutation of $G$ with rank $k$ at most. Then, we prove that, given a formula $G \notin \mathrm{GD}_k$, there exists a $G$-separable model $\mathcal{K} = \langle K, \leq, \rho, V\rangle$ of height at most $k$ such that $\mathcal{K}, \rho \nVdash G$.

The following properties of saturated sequents can be easily proved.

**Lemma 3** Let $\sigma = \Gamma \not\Rightarrow_k \Lambda\,;\, \Delta$ be a saturated $\mathbf{R}_{\mathrm{GD}}(G)$-sequent. Then:

(i) If $k = 0$ and $A \supset B \in \mathrm{S}_{\mathrm{L}}(G)$ and $A \in \mathcal{C}l^{-}(\Delta)$, then $A \supset B \in \Gamma$.

(ii) If $A \supset B \in \mathrm{S}_{\mathrm{L}}(G)$ and $A \in \mathcal{C}l^{-}(\Delta \cup \Lambda)$ and $B \in \mathcal{C}l^{+}(\Gamma \cup \Lambda)$, then $A \supset B \in \Gamma$.

(iii) If $A \supset B \in \mathrm{S}_{\mathrm{R}}(G)$ and $A \in \mathcal{C}l^{+}(\Gamma)$ and $B \in \mathcal{C}l^{-}(\Delta \cup \Lambda)$, then $A \supset B \in \Delta$.

**Lemma 4** For every $\mathbf{R}_{\mathrm{GD}}(G)$-sequent $\sigma$, there exists a unique saturated $\mathbf{R}_{\mathrm{GD}}(G)$-sequent $\sigma'$ such that $\sigma \mapsto\!\!\!\bar{\ }_* \sigma'$.

*Proof.* Let $\mathcal{S}_G$ be the set of all the $\mathbf{R}_{\mathrm{GD}}(G)$-sequents and let us consider the Abstract Reduction System $\mathcal{A}_G = \langle \mathcal{S}_G, \bar{\mapsto} \rangle$ (see e.g. [32]). By Proposition 1, $\mathcal{A}_G$ is terminating; the irreducible elements of $\mathcal{A}_G$ are the saturated sequents. Moreover, one can easily check that $\mathcal{A}_G$ is locally confluent; indeed, if $\sigma \bar{\mapsto} \sigma_1$ and $\sigma \bar{\mapsto} \sigma_2$, there exists $\sigma'$ such that $\sigma_1 \bar{\mapsto} \sigma'$ and $\sigma_2 \bar{\mapsto} \sigma'$. By Newman's Lemma [32], $\mathcal{A}_G$ is confluent, and this proves the assertion. □

Let $\sigma$ be an $\mathbf{R}_{\mathrm{GD}}(G)$-sequent; by $\sigma^*$ we denote the unique saturated $\mathbf{R}_{\mathrm{GD}}(G)$-sequent such that $\sigma \bar{\mapsto}_* \sigma^*$ (thus, if $\sigma$ is saturated, we have $\sigma^* = \sigma$).

Let $\mathcal{K} = \langle W, \leq, \rho, V\rangle$ be a $G$-separable model. For every $\alpha \in W$, we define the saturated $\mathbf{R}_{\mathrm{GD}}(G)$-sequent $\mathrm{Sat}_G(\alpha)$ associated with $\alpha$ by induction on $\mathrm{h}(\alpha)$.

- $\mathrm{h}(\alpha) = 0$.

$$\mathrm{Sat}_G(\alpha) \;=\; (\,\Gamma^{\mathrm{At}} \not\Rightarrow_0 \cdot\,;\, \Delta^{\mathrm{At}}\,\bot\,)^* \qquad \begin{array}{rcl} \Gamma^{\mathrm{At}} &=& \{\, p \in \mathrm{S}_{\mathrm{L}}^{\mathrm{At}}(G) \mid \mathcal{K}, \alpha \Vdash p \,\} \\ \Delta^{\mathrm{At}} &=& \{\, p \in \mathrm{S}_{\mathrm{R}}^{\mathrm{At}}(G) \mid \mathcal{K}, \alpha \nVdash p \,\} \end{array}$$

- $\mathrm{h}(\alpha) > 0$.

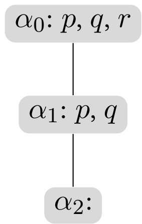
Let $\beta$ be the immediate successor of $\alpha$, let $\mathrm{Sat}_G(\beta) = \Gamma \not\Rightarrow_k \Lambda\,;\, \Delta$ and let

$$\Lambda_\beta \;=\; \{\, p \in \mathrm{S}_{\mathrm{L}}^{\mathrm{At}}(G) \cap \mathrm{S}_{\mathrm{R}}^{\mathrm{At}}(G) \mid \mathcal{K}, \beta \Vdash p \text{ and } \mathcal{K}, \alpha \nVdash p \,\}$$

Note that $\Lambda_\beta$ is not empty (indeed, $\mathcal{K}$ is $G$-separable). We set:

$$\mathrm{Sat}_G(\alpha) \;=\; (\,\Gamma \setminus \Lambda_\beta \not\Rightarrow_{k+1} \Lambda_\beta\,;\, \Delta, \Lambda\,)^*$$

**Example 3** Let $G$ be defined as in Ex. 1. Below we display a $G$-separable model $\mathcal{K}$ consisting of three worlds $\alpha_0$, $\alpha_1$, $\alpha_2$; for each $\alpha_j$, the saturated set $\mathrm{Sat}_G(\alpha_j)$ coincides with one of the saturated sequents occurring in the refutation in Fig. 2.




$$\mathrm{Sat}_G(\alpha_0) = \neg\neg p,\ p,\ q,\ r \not\Rightarrow_0 \cdot\,;\ \bot,\ \neg p,\ A \qquad (\sigma_{(4)})$$

$$\mathrm{Sat}_G(\alpha_1) = \neg\neg p,\ p,\ q \not\Rightarrow_1 r\,;\ \bot,\ \neg p,\ A,\ p \supset r \qquad (\sigma_{(6)})$$

$$\mathrm{Sat}_G(\alpha_2) = C,\ p \supset q,\ \neg\neg p \not\Rightarrow_2 p,\ q\,;\ \bot,\ \neg p,\ A, p \supset r,\ r,\ B,\ C \supset (p \vee \neg p) \qquad (\sigma_{(11)})$$

◊

**Lemma 5** Let $\mathcal{K} = \langle W, \leq, \rho, V\rangle$ be a $G$-separable model, let $\alpha \in W$ and $\mathrm{Sat}_G(\alpha) = \Gamma \not\Rightarrow_k \Lambda\,;\,\Delta$. Then:

(i) $k = \mathrm{h}(\alpha)$.

(ii) If $p \in \mathrm{SL}^{\mathrm{At}}(G)$ and $\mathcal{K}, \alpha \Vdash p$, then $p \in \Gamma$.

(iii) If $p \in \mathrm{SR}^{\mathrm{At}}(G)$ and $\mathcal{K}, \alpha \nVdash p$, then $p \in \Delta \cup \Lambda$.

(iv) There exists an $\mathbf{R}_{\mathrm{GD}}(G)$-refutation of $\mathrm{Sat}_G(\alpha)$.

(v) If $C \in \mathrm{SL}(G)$ and $\mathcal{K}, \alpha \Vdash C$, then $C \in \mathcal{C}l^+(\Gamma)$.

(vi) If $C \in \mathrm{SR}(G)$ and $\mathcal{K}, \alpha \nVdash C$, then $C \in \mathcal{C}l^-(\Delta \cup \Lambda)$.

*Proof.* Points (i)-(iv) easily follow by induction on $\mathrm{h}(\alpha)$. We prove (v) and (vi) by a main induction hypothesis (IH1) on $\mathrm{h}(\alpha)$ and a secondary induction hypothesis (IH2) on $|C|$. Note that, by point (i), we have $k = \mathrm{h}(\alpha)$.

(C1) $\mathrm{h}(\alpha) = 0$.

We have $k = 0$, hence $\Lambda = \emptyset$. Let $C \in \mathrm{SL}(G)$ such that $\mathcal{K}, \alpha \Vdash C$; we show $C \in \mathcal{C}l^+(\Gamma)$. If $C \in \mathcal{V}$, by point (ii) we get $p \in \Gamma$, hence $p \in \mathcal{C}l^+(\Gamma)$. Let $C = A \wedge B$. Then, $\mathcal{K}, \alpha \Vdash A$ and $\mathcal{K}, \alpha \Vdash B$. By (IH2), we get $A \in \mathcal{C}l^+(\Gamma)$ and $B \in \mathcal{C}l^+(\Gamma)$, hence $A \wedge B \in \mathcal{C}l^+(\Gamma)$. The case $C = A \vee B$ is similar. Let $C = A \supset B$. If $\mathcal{K}, \alpha \Vdash B$, by (IH2) we get $B \in \mathcal{C}l^+(\Gamma)$, hence $A \supset B \in \mathcal{C}l^+(\Gamma)$. Let us assume $\mathcal{K}, \alpha \nVdash B$. Then $\mathcal{K}, \alpha \nVdash A$ hence, by (IH2), we get $A \in \mathcal{C}l^-(\Delta)$. By point Lemma 3(i) it follows that $A \supset B \in \Gamma$, hence $A \supset B \in \mathcal{C}l^+(\Gamma)$. This concludes the proof of (v).

Let $C \in \mathrm{S_R}(G)$ such that $\mathcal{K}, \alpha \nVdash C$; we show $C \in \mathcal{C}l^-(\Delta)$. If $C \in \mathcal{V}$, by point (iii) we get $C \in \Delta$, hence $C \in \mathcal{C}l^-(\Delta)$. Let $C = A \wedge B$. Then, $\mathcal{K}, \alpha \nVdash A$ or $\mathcal{K}, \alpha \nVdash B$. According to the case, by (IH2) we get $A \in \mathcal{C}l^-(\Delta)$ or $B \in \mathcal{C}l^-(\Delta)$, hence $A \wedge B \in \mathcal{C}l^-(\Delta)$. The case $C = A \vee B$ is similar. Let $C = A \supset B$. We have $\mathcal{K}, \alpha \Vdash A$ and $\mathcal{K}, \alpha \nVdash B$. By (IH2), we get $A \in \mathcal{C}l^+(\Gamma)$ and $B \in \mathcal{C}l^-(\Delta)$. By Lemma 3(iii) it follows that $A \supset B \in \Delta$, hence $A \supset B \in \mathcal{C}l^+(\Delta)$. This concludes the proof of (vi).

(C2) $\mathrm{h}(\alpha) > 0$.

Let $\beta$ be the immediate successor of $\alpha$ (thus, $\mathrm{h}(\beta) = \mathrm{h}(\alpha) - 1$) and let:

$$\begin{array}{rcl} \mathrm{Sat}_G(\beta) & = & \Gamma' \not\Rightarrow_{k-1} \Lambda' \,;\, \Delta' \\ \Lambda_\beta & = & \{\, p \in \mathrm{S_L}^{\mathrm{At}}(G) \cap \mathrm{S_L}^{\mathrm{At}}(G) \mid \mathcal{K}, \beta \Vdash p \text{ and } \mathcal{K}, \alpha \nVdash p \,\} \end{array}$$

We have:

$$\mathrm{Sat}_G(\alpha) \;=\; (\,\Gamma' \setminus \Lambda_\beta \not\Rightarrow_k \Lambda_\beta \,;\, \Delta', \Lambda'\,)^* \qquad \Gamma' \setminus \Lambda_\beta \subseteq \Gamma \qquad \Delta' \cup \Lambda' \subseteq \Delta$$

Let $C \in \mathrm{S_L}(G)$ such that $\mathcal{K}, \alpha \Vdash C$; we show $C \in \mathcal{C}l^+(\Gamma)$. The cases $C \in \mathcal{V}$, $C = A \wedge B$ and $C = A \vee B$ can be proved as in the case (C1). Let $C = A \supset B$. If $\mathcal{K}, \alpha \Vdash B$ then, by (IH2), $B \in \mathcal{C}l^+(\Gamma)$, which implies $A \supset B \in \mathcal{C}l^+(\Gamma)$. Let us assume $\mathcal{K}, \alpha \nVdash B$; we show that $A \supset B \in \Gamma$. Since $\alpha < \beta$, it holds that $\mathcal{K}, \beta \Vdash A \supset B$. By (IH1), $A \supset B \in \mathcal{C}l^+(\Gamma')$, hence $B \in \mathcal{C}l^+(\Gamma')$ or $A \supset B \in \Gamma'$. In the latter case, since $A \supset B \in \Gamma' \setminus \Lambda_\beta$ and $\Gamma' \setminus \Lambda_\beta \subseteq \Gamma$, we get $A \supset B \in \Gamma$. Let us consider the former case (namely, $B \in \mathcal{C}l^+(\Gamma')$). From $\Gamma' \setminus \Lambda_\beta \subseteq \Gamma$, it follows that $\Gamma' \subseteq \Gamma \cup \Lambda_\beta$, hence $B \in \mathcal{C}l^+(\Gamma \cup \Lambda_\beta)$. Since $\mathcal{K}, \alpha \Vdash A \supset B$ and $\mathcal{K}, \alpha \nVdash B$, it holds that $\mathcal{K}, \alpha \nVdash A$ hence, by (IH2), $A \in \mathcal{C}l^-(\Delta \cup \Lambda)$. We can apply Lemma 3(ii), and infer that $A \supset B \in \Gamma$. Having proved $A \supset B \in \Gamma$, we get $A \supset B \in \mathcal{C}l^+(\Gamma)$, and this concludes the proof of point (v).

Let $C \in \mathrm{S_R}(G)$ such that $\mathcal{K}, \alpha \nVdash C$; we show $C \in \mathcal{C}l^-(\Delta \cup \Lambda)$. The cases $C \in \mathcal{V}$, $C = A \wedge B$ and $C = A \vee B$ can be proved as in the case (C1). Let $C = A \supset B$; we show that $A \supset B \in \Delta \cup \Lambda$. Since $K, \alpha \nVdash A \supset B$, there exists $\gamma \in W$ such that $\alpha \leq \gamma$ and $\mathcal{K}, \gamma \Vdash A$ and $\mathcal{K}, \gamma \nVdash B$. If $\gamma = \alpha$, by (IH2) we get $A \in \mathcal{C}l^+(\Gamma)$ and $B \in \mathcal{C}l^-(\Delta \cup \Lambda)$. By Lemma 3(iii), it follows that $A \supset B \in \Delta$. Let us assume $\alpha < \gamma$. Then, $\beta \leq \gamma$, hence $\mathcal{K}, \beta \nVdash A \supset B$. By (IH1), we get $A \supset B \in \mathcal{C}l^-(\Delta' \cup \Lambda')$, which implies $A \supset B \in \Delta' \cup \Lambda'$. Since $\Delta' \cup \Lambda' \subseteq \Delta$, we get $A \supset B \in \Delta$. Having proved that $A \supset B \in \Delta$, it follows that $A \supset B \in \mathcal{C}l^-(\Delta \cup \Lambda)$, and this concludes the proof of point (vi). □

To conclude the proof of completeness, we need to prove that:

**Lemma 6** If $G \notin \mathrm{GD}_k$, then there exists a countermodel $\mathcal{K}$ for $G$ such that $\mathrm{h}(\mathcal{K}) \leq k$ and $\mathcal{K}$ is $G$-separable.

*Proof.* We give a sketch of the proof. Let us assume $G \notin \mathrm{GD}_k$. Then, there exists a model $\mathcal{K}_1 = \langle W_1, \leq_1, \rho_1, V_1 \rangle$ such that $\mathcal{K}_1, \rho_1 \nVdash G$ and $\mathrm{h}(\rho_1) \leq k$. We define the countermodel $\mathcal{K}$ in two steps. Firstly, we define the model $\mathcal{K}_2$ obtained from $\mathcal{K}_1$ by adding to each set $V_1(\alpha)$ the propositional variables in $\text{Sl}^{\text{At}}(G) \setminus \text{Sr}^{\text{At}}(G)$. Secondly, we get $\mathcal{K}$ by filtrating $\mathcal{K}_2$. The model $\mathcal{K}_2 = \langle W_2, \leq_2, \rho_2, V_2 \rangle$ is defined as follows:

$$
\begin{array}{l}
W_2 \; = \; W_1 \quad \leq_2 = \leq_1 \quad \rho_2 \; = \; \rho_1 \\
\forall \alpha \in W_1,\ V_2(\alpha) \; = \; \Big( V_1(\alpha) \,\cup\, (\text{Sl}^{\text{At}}(G) \setminus \text{Sr}^{\text{At}}(G)) \Big) \setminus (\text{Sr}^{\text{At}}(G) \setminus \text{Sl}^{\text{At}}(G))
\end{array}
$$

By induction on $|C|$, we can prove that:

(1) for every $\alpha \in W_1$ and $C \in \text{Sl}(G)$, $\mathcal{K}_1, \alpha \Vdash C$ implies $\mathcal{K}_2, \alpha \Vdash C$;

(2) for every $\alpha \in W_1$ and $C \in \text{Sr}(G)$, $\mathcal{K}_1, \alpha \nVdash C$ implies $\mathcal{K}_2, \alpha \nVdash C$.

Let us introduce the following relation between worlds of $W_2$:

$$\alpha \sim \beta \quad \text{iff} \quad V_2(\alpha) \cap \text{Sf}^{\text{At}}(G) \; = \; V_2(\beta) \cap \text{Sf}^{\text{At}}(G)$$

It is easy to check that:

- $\sim$ is an equivalence relation;
- If $\alpha \leq_2 \beta$ and $\alpha' \sim \alpha$ and $\beta' \sim \beta$ then $\alpha' \sim \beta'$ or $\alpha' <_2 \beta'$.

We turn $\mathcal{K}_2$ into a $G$-separable model $\mathcal{K}$ by collapsing $\sim$-equivalent worlds. For $\alpha \in W_2$, let $[\alpha]$ denote the equivalence class of $\alpha$ (w.r.t. $\sim$) and let $W$ be the quotient of $W_2$. By the above properties, the model $\mathcal{K} = \langle W, \leq, \rho, V \rangle$ can be defined as follows:

$$
\begin{array}{l}
\leq = \{ ([\alpha], [\beta]) \mid \alpha \leq_2 \beta \} \qquad \rho \; = \; [\rho_2] \\
\forall \alpha \in W_2,\ V([\alpha]) \; = \; V_2(\alpha) \cap \text{Sf}^{\text{At}}(G)
\end{array}
$$

By induction on $|C|$, we can prove that:

(3) For every $\alpha \in W_2$ and $C \in \text{Sf}(G)$, $\mathcal{K}_2, \alpha \Vdash C$ iff $\mathcal{K}, [\alpha] \Vdash C$.

We show that $\mathcal{K}$ is $G$-separable. Let $[\alpha] < [\beta]$. Then, $\alpha \leq_2 \beta$ and $\alpha \not\sim \beta$. Thus, that there exists $p \in V_2(\beta) \setminus V_2(\alpha)$, and this implies $p \in \text{Sl}(G) \cap \text{Sr}(G)$. Since $\mathcal{K}_1, \rho_1 \nVdash G$ and $G \in \text{Sr}(G)$, by (2) and (3) we get $\mathcal{K}, \rho \nVdash G$, hence $\mathcal{K}$ is a countermodel for $G$. Finally, we observe that $\text{h}(\mathcal{K}) \leq \text{h}(\mathcal{K}_2) = \text{h}(\mathcal{K}_1) = k$. □

Let us assume $G \notin \text{GD}_k$. By Lemma 6, there exists a model $\mathcal{K} = \langle K, \leq, \rho, V \rangle$ such that $\mathcal{K}, \rho \nVdash G$, $\text{h}(\rho) \leq k$ and $\mathcal{K}$ is $G$-separable. Let $\text{Sat}_G(\rho) = \Gamma \not\Rightarrow_{k'} \Lambda\,;\,\Delta$. By Lemma 5(i), $k' = \text{h}(\rho) \leq k$ and there exists an $\mathbf{R}_{\text{GD}}(G)$-refutation of $\text{Sat}_G(\rho)$. Since $\mathcal{K}, \rho \nVdash G$, by Lemma 5(vi) we get $G \in \mathcal{C}l^{-}(\Delta \cup \Lambda)$. We conclude $\vdash^{k}_{G} G$, and this proves the completeness theorem. As a corollary, we get

**Theorem 3** $G \notin \text{GD}$ iff there exists an $\mathbf{R}_{\text{GD}}(G)$-refutation of $G$.

## 6. Conclusions

In this paper we have introduced a forward calculus $\mathbf{R}_{\text{GD}}(G)$ to derive the non-validity of a goal formula $G$ in Gödel-Dummett logics. From an $\mathbf{R}_{\text{GD}}(G)$-refutation of $G$ we can extract a countermodel for $G$. As for the proof-search strategy, we have presented the naive forward strategy of [15], we leave as future work the investigation of clever strategies (e.g., using subsumption to reduce redundancies as those discussed in [8]) and the implementation of the calculus exploiting the full-fledged Java Framework JTabWb [33]. The refinement of the forward proof-search strategy and the implementation are key step to compare our approach with the ones presented in [34, 9, 10]. We also aim to extend our approach to other intermediate logics.

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