A CHARACTERIZATION OF CESÀRO CONVERGENCE

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Abstract. We show that a real bounded sequence (x_n) is Cesàro convergent to ℓ if and only if the sequence of averages with indices in $[\alpha^k, \alpha^{k+1})$ converges to ℓ for all $\alpha > 1$. If, in addition, the sequence (x_n) is nonnegative, then it is Cesàro convergent to 0 if and only if the same condition holds for some $\alpha > 1$.

1. Introduction.

A real sequence $x = (x_n)_{n\ge 1}$ is said to be *Cesàro convergent* to $\ell \in \mathbf{R}$ if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_n = \ell.$$

This is a weaker notion than ordinary convergence: indeed, it involves a kind of smoothing of the original sequence by computing its partial averages. The detailed theory of Cesàro convergence is discussed in Hardy's classic textbook [5]. The vector space w_1 of *strongly Cesàro convergent* sequences x (i.e., the sequences such that $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|x_n-\ell| = 0$ for some $\ell \in \mathbf{R}$), endowed with the norm

$$\|x\| = \sup_{n\ge 1} \frac{1}{2^n} \sum_{2^n \le k < 2^{n+1}} |x_k|,$$

turns out to be a Banach space; see [7]. A characterization of strong Cesàro convergence for bounded sequences with statistical convergence can be found, for example, in [2, Theorem 2.1]; see [3] for extensions with summability methods.

In a different context, it is known that a set A of positive integers has *asymptotic density* 0, that is, $\lim_{n\to\infty}\frac{1}{n}\#(A\cap[1,n]) = 0$ (see, e.g., [6]), if and only if

$$\lim_{n\to\infty} \frac{1}{2^n}\#(A\cap[2^n,2^{n+1})) = 0;$$

see [1, Lemma 3.1] and compare with the proof of [4, Theorem 1.13.3(a)]. Here, $\#S$ stands for the cardinality of a set S. Identifying A with the sequence $(x_n)_{n\ge 1}$ such that $x_n = 1$ if $n \in A$ and $x_n = 0$ otherwise, as in [8], the latter result can be

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rephrased as follows: a $\{0,1\}$-valued sequence $(x_n)_{n\ge 1}$ is Cesàro convergent to 0 if and only if

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{2^n \le k < 2^{n+1}} x_k = 0. \tag{1}$$

This happens, for example, if $(x_n)_{n\ge 1}$ is the sequence associated with $A = \bigcup_{n\ge 1}\{2^n + 1, \ldots, 2^n + n\}$. First of all, we note that the condition that the sequence is $\{0,1\}$-valued cannot be omitted: indeed, the sequence $(x_n)_{n\ge 1}$ defined by

$$x_n = \begin{cases} 1 & \text{if } 2^k \le n < 3\cdot 2^{k-1} \text{ for some } k \ge 1 \\ -1 & \text{otherwise,} \end{cases} \tag{2}$$

satisfies the limit (1) and, on the other hand, $\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^n x_k = \frac{1}{4}$ (computed along the subsequence $(3\cdot 2^k)_{k\ge 1}$). This example implies that, even for bounded sequences, (1) need not be equivalent to Cesàro convergence.

However, replacing the base 2 in condition (1) with *all* bases $\alpha > 1$, gives a characterization of Cesàro convergence for bounded sequences, which is the content of our main result:

Theorem 1.1. *A bounded sequence (x_n) is Cesàro convergent to ℓ if and only if*

$$\lim_{k\to\infty} \frac{1}{\alpha^{k+1}-\alpha^k} \sum_{\alpha^k \le i < \alpha^{k+1}} x_i = \ell \quad \textit{for all } \alpha > 1.$$

However, Theorem 1.1 does not explain why only the base $\alpha = 2$ appears in (1). In this regard, note that a *nonnegative* sequence $(x_n)_{n\ge 1}$ is Cesàro convergent to 0 if and only if it is strongly Cesàro convergent to 0. Hence, condition (1) is justified by the following:

Theorem 1.2. *A bounded nonnegative sequence (x_n) is Cesàro convergent to* 0 (*hence, strongly Cesàro convergent to* 0) *if and only if*

$$\lim_{k\to\infty} \frac{1}{\alpha^{k+1}-\alpha^k} \sum_{\alpha^k \le i < \alpha^{k+1}} x_i = 0 \quad \textit{for some } \alpha > 1.$$

Of course, Theorem 1.2 holds by replacing 0 with an arbitrary ℓ and using the hypothesis that $x_n \ge 0$ with $x_n \ge \ell$ for all n. We chose $\ell = 0$ to ease the exposition.

The proofs, which are completely elementary, follow in the next sections.

2. Proof of Theorem 1.1.

For each $\alpha > 1$ and $j \in \mathbf{N}$, define

$$I_{\alpha,j} := [\alpha^{j-1}, \alpha^j) \cap \mathbf{N}.$$

Assume by convention that $\frac{1}{\#I_{\alpha,j}} := 0$ if $I_{\alpha,j}$ is empty. Thus, for each $\alpha > 1$, we have $\lim_{j\to\infty} \frac{1}{\#I_{\alpha,j}} \cdot (\alpha^j - \alpha^{j-1}) = 1$. Moreover, for each $n, j \in \mathbf{N}^+$, set

$$a_n := \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{and} \quad b_j := \frac{1}{\#I_{\alpha,j}} \sum_{i\in I_{\alpha,j}} x_i.$$

Finally, note that we can suppose without loss of generality that $\ell = 0$.

IF PART. Suppose that $\lim_{j\to\infty} b_j = 0$. Fix $\varepsilon > 0$, so that there exists $t_0 \in \mathbf{N}^+$ such that $|b_t| < \varepsilon$ for all $t \ge t_0$. Define $\theta := \sup_n |x_n|$ and assume that $\theta > 0$; otherwise the statement is trivially true. Also fix $\alpha > \max\{1, \frac{2\theta}{\varepsilon}\}$, and $n, k \in \mathbf{N}^+$ such that $n \ge \alpha^{t_0} \max\{1, \frac{\theta}{\varepsilon}, \frac{1}{\theta}\}$ and $n \in I_{\alpha,k}$. Then

$$na_n = \sum_{i=1}^{n} x_i = \sum_{j=1}^{k-1} w_j b_j + \sum_{i\in I_{\alpha,k}\cap [1,n]} x_i, \tag{3}$$

where $w_j := \#I_{\alpha,j}$ for each $j \in \mathbf{N}^+$. Note that $\sum_{j\le t-1} w_j \le \alpha^t$ for all $t \in \mathbf{N}^+$. Since $n \ge \alpha^{t_0}$, we have $k-1 \ge t_0$, so that

$$a_n = \frac{1}{n}\left(\sum_{j=1}^{t_0-1} w_j b_j + \sum_{j=t_0}^{k-1} w_j b_j + \sum_{i\in I_{\alpha,k}\cap [1,n]} x_i\right).$$

Considering that a_n is the average of all b_j (with $j \le k-1$), each repeated w_j times, and the remaining x_i (with $i \in I_{\alpha,k} \cap [1,n]$), we obtain that

$$\begin{aligned}
|a_n| &\le \frac{1}{n}\left(\sum_{j=1}^{t_0-1} w_j |b_j| + \sum_{j=t_0}^{k-1} w_j |b_j| + \sum_{i\in I_{\alpha,k}\cap [1,n]} |x_i|\right)\\
&\le \frac{1}{n}\left(\sum_{j=1}^{t_0-1} w_j \theta + \sum_{j=t_0}^{k-1} w_j \varepsilon + \sum_{i\in [n(1-1/\alpha),n]} \theta\right)\\
&\le \frac{1}{n}\left(\theta\alpha^{t_0} + n\varepsilon + \#[n(1-1/\alpha),n]\cdot\theta\right)\\
&\le \frac{\theta\alpha^{t_0}}{n} + \varepsilon + \frac{\theta}{\alpha} + \frac{1}{n} \le \frac{\theta\alpha^{t_0}}{n} + \varepsilon + \frac{2\theta}{\alpha} \le \varepsilon+\varepsilon+\varepsilon.
\end{aligned} \tag{4}$$

In the second line, we used that $I_{\alpha,k}\cap[1,n]$ is contained in $[\alpha^{k-1},\alpha^k]\cap[1,n]$, which is, in turn, contained in $[n(1-\frac{1}{\alpha}), n]$; also, in the last line, we used the inequalities $n \ge \alpha^{t_0}\frac{\theta}{\varepsilon}$, $\frac{1}{n} \le \frac{\theta}{\alpha^{t_0}} \le \frac{\theta}{\alpha}$, and $\alpha \ge \frac{2\theta}{\varepsilon}$.

By the arbitrariness of ε, we conclude that $\lim_{n\to\infty} a_n = 0$.

ONLY IF PART. Suppose that $\lim_{n\to\infty} a_n = 0$ and fix $\alpha > 1$. For each $j \in \mathbf{N}$, define $\iota_j := \lceil \alpha^{j-1} \rceil$ and note that $\iota_j = \min I_{\alpha,j}$ if j is sufficiently large. Reasoning as in (3), we obtain that

$$(\iota_{k+1}-\iota_k)b_k=\sum_{i\in I_{\alpha,k}}x_i=\sum_{i<\iota_{k+1}}x_i-\sum_{i<\iota_k}x_i=a_{\iota_{k+1}-1}(\iota_{k+1}-1)-a_{\iota_k-1}(\iota_k-1)$$

whenever k is sufficiently large. This implies that

$$\lim_{k\to\infty}b_k=\lim_{k\to\infty}a_{\iota_{k+1}-1}\cdot\lim_{k\to\infty}\frac{\iota_{k+1}-1}{\iota_{k+1}-\iota_k}-\lim_{k\to\infty}a_{\iota_k-1}\cdot\lim_{k\to\infty}\frac{\iota_k-1}{\iota_{k+1}-\iota_k}=0,$$

completing the proof.

3. Proof of Theorem 1.2.

IF PART. The proof proceeds along the same lines as the previous result. Here, we let $\alpha>1$ be *any* fixed number (in particular, not necessarily greater than $2\theta/\varepsilon$) and $n,k\in\mathbf{N}^+$ are still taken such that $n\ge\alpha^{t_0}\max\{1,\frac{\theta}{\varepsilon},\frac{1}{\theta}\}$ and $n\in I_{\alpha,k}$. In addition, we have the following upper estimate:

$$\sum_{i\in I_{\alpha,k}\cap[1,n]}|x_i|\le\sum_{i\in I_{\alpha,k}}x_i=b_k\le\varepsilon(\alpha^k-\alpha^{k-1})\le\varepsilon\alpha^k\le\varepsilon\alpha n,$$

where the first inequality depends on the fact the sequence has nonnegative terms. Therefore, in place of the chain of inequalities (4), we obtain that

$$|a_n|\le\frac{\theta\alpha^{t_0}}{n}+\varepsilon+\varepsilon$$

which is smaller than $3\alpha\varepsilon$ if n is sufficiently large.

ONLY IF PART. This follows by Theorem 1.1.

4. Concluding Remarks.

In light of Theorem 1.2, one may hope for a characterization of the (upper or lower) asymptotic density of $A\subseteq\mathbf{N}$ in terms of (superior or inferior) limit of the block averages $\frac{1}{2^n}\#(A\cap[2^n,2^{n+1}))$, which was the original motivation for this work. However, this is not possible: the reason is along the same lines as the example given in (2). Indeed, denote the upper and lower asymptotic density of a set $A\subseteq\mathbf{N}$ by

$$\mathsf{d}^\star(A):=\limsup_{n\to\infty}\frac{1}{n}\#(A\cap[1,n])\quad\text{and}\quad\mathsf{d}_\star(A):=\liminf_{n\to\infty}\frac{1}{n}\#(A\cap[1,n]),$$

respectively, see e.g. [6], and define

$$A_s:=\bigcup\nolimits_{n\ge1}\{2^n+\lfloor s2^{n-1}\rfloor+t:t=1,\dots,2^{n-1}\}$$

for each $s \in [0,1]$. It follows that $\lim_n \frac{1}{2^n}\#(A_s \cap [2^n, 2^{n+1})) = \frac{1}{2}$. On the other hand, none of the A_s admits asymptotic density: in fact,

$$\mathsf{d}_\star(A_s) = \frac{1}{2+s} < \frac{2}{3+s} = \mathsf{d}^\star(A_s)$$

for each $s \in [0,1]$.

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