# A CHARACTERIZATION OF CONVEX FUNCTIONS

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Abstract. Let $D$ be a convex subset of a real vector space. It is shown that a radially lower semicontinuous function $f : D \to \mathbf{R} \cup \{+\infty\}$ is convex if and only if for all $x, y \in D$ there exists $\alpha = \alpha(x, y) \in (0, 1)$ such that $f(\alpha x + (1 - \alpha)y) \le \alpha f(x) + (1 - \alpha)f(y)$.

## 1. Introduction.

Let $X$ be a vector space over the real field and fix a convex set $D \subseteq X$. A function $f : D \to \mathbf{R}$ is convex whenever

$$f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y) \tag{1}$$

for all $\lambda \in [0, 1]$ and for all $x, y \in D$. In addition, $f$ is said to be

(1) *$\alpha$-convex*, for some fixed $\alpha \in (0, 1)$, if

$$f(\alpha x + (1 - \alpha)y) \le \alpha f(x) + (1 - \alpha)f(y)$$

for all $x, y \in D$;

(2) *midconvex* if

$$f\left(\tfrac{1}{2}(x + y)\right) \le \tfrac{1}{2}(f(x) + f(y))$$

for all $x, y \in D$.

It is clear that convex functions are $\alpha$-convex. In addition, it easily follows by the Daróczy–Páles identity, see [1, Lemma 1], that $\alpha$-convex functions are midconvex, see also [2] and [3, Proposition 4]. Finally, we recall that if $f$ is midconvex then inequality (1) holds for all $x, y \in D$ and *rationals* $\lambda \in [0, 1]$.

It is well known that, assuming the axiom of choice, there exist nonconvex functions that are $\alpha$-convex. On the other hand, a *continuous* real function $f$ is convex if and only if it is $\alpha$-convex if and only if it is midconvex. We refer the reader to [4] and references therein for a textbook exposition on convex functions.

In this note we provide another characterization of convexity, assuming that the function $f$ is *radially lower semicontinuous*, i.e.,

$$f(x) \le \liminf_{t \downarrow 0} f(x + t(y - x))$$

for all $x, y \in D$. Our main result is as follows:

**Theorem 1.** *Let $D$ be a convex subset of a real vector space. Then, a radially lower semicontinuous function $f : D \to \mathbf{R} \cup \{+\infty\}$ is convex if and only if, for all $x, y \in D$, there exists $\lambda = \lambda(x, y) \in (0, 1)$ that satisfies inequality (1).*

2010 *Mathematics Subject Classification.* Primary: 25B62. Secondary: 26A51, 52A07.
*Key words and phrases.* Convexity, real vector spaces, (radially) lower semicontinuity.

The following corollary, which has been conjectured by Miroslav Pavlović in the case of real-valued continuous functions, is immediate:

**Corollary 2.** *Let $I \subseteq \mathbf{R}$ be a nonempty interval. Then, a lower semicontinuous function $f : I \to \mathbf{R} \cup \{+\infty\}$ is convex if and only if, for all $x, y \in I$, there exists $\lambda = \lambda(x, y) \in (0, 1)$ that satisfies inequality (1).*

It is worth noting that the lower semicontinuous assumption cannot be relaxed too much since there exists a nonconvex function $f : I \to \mathbf{R} \cup \{+\infty\}$ that is lower semicontinous everywhere in $I$ except at countably many points and such that, for all $x, y \in I$, there exists $\lambda = \lambda(x, y) \in (0, 1)$ that satisfies inequality (1). Indeed, it is sufficient to choose $I = \mathbf{R}$ and let $f$ be the indicator function of the rationals, i.e., $f(x) = 1$ if $x$ is rational and $f(x) = 0$ otherwise.

## 2. Proof of Theorem 1.

We need only to prove the "if" part. Moreover, the claim is obvious if $D$ is empty, hence let us suppose hereafter that $D \neq \emptyset$.

Fix $x, y \in D$ and define the set

$$S := \{\lambda \in [0, 1] : f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y)\}.$$

Notice that $S = [0, 1]$ provided that $f(x) = +\infty$ or $f(y) = +\infty$. Hence, let us assume hereafter that $f(x) < +\infty$ and $f(y) < +\infty$.

**Claim 1.** $S$ is a nonempty compact subset of $[0, 1]$.

*Proof.* Note that $\{0, 1\} \subseteq S \subseteq [0, 1]$, hence $S$ is nonempty and bounded.

Let $(\lambda_n)$ be a sequence in $S$ such that $\lambda_n \uparrow \lambda > 0$. Define the real sequence $(t_n)$ by $t_n := 1 - \frac{\lambda_n}{\lambda}$ for each $n$ and note that $t_n \downarrow 0$. Since $f$ is radially lower semicontinuous, it follows that

$$\begin{aligned} f(\lambda x + (1 - \lambda)y) &\le \liminf_{n\to\infty} f(\lambda x + (1 - \lambda)y) + t_n(y - \lambda x - (1 - \lambda)y))) \\ &= \liminf_{n\to\infty} f(\lambda_n x + (1 - \lambda_n)y) \\ &\le \liminf_{n\to\infty} \lambda_n f(x) + (1 - \lambda_n)f(y) = \lambda f(x) + (1 - \lambda)f(y). \end{aligned}$$

Similarly, given a sequence $(\lambda_n)$ in $S$ such that $\lambda_n \downarrow \lambda < 1$, let $(s_n)$ be the sequence defined by $s_n := 1 - \frac{1-\lambda_n}{1-\lambda}$ for each $n$ and note that $s_n \downarrow 0$. Then

$$\begin{aligned} f(\lambda x + (1 - \lambda)y) &\le \liminf_{n\to\infty} f(\lambda x + (1 - \lambda)y) + s_n(x - \lambda x - (1 - \lambda)y))) \\ &= \liminf_{n\to\infty} f(\lambda_n x + (1 - \lambda_n)y) \le \lambda f(x) + (1 - \lambda)f(y). \end{aligned}$$

Therefore $S$ is also closed, proving the claim. □

**Claim 2.** $(\lambda_1, \lambda_2) \cap S \neq \emptyset$ for all $\lambda_1, \lambda_2 \in S$ with $\lambda_1 < \lambda_2$.

*Proof.* Fix $\lambda_1, \lambda_2 \in S$ with $\lambda_1 < \lambda_2$ (note that this is possible since $\{0, 1\} \subseteq S$) and define

$$a = \lambda_1 x + (1 - \lambda_1)y \quad \text{and} \quad b = \lambda_2 x + (1 - \lambda_2)y.$$

Hence $a, b \in D$ and, by hypothesis, there exists $\lambda = \lambda(a, b) \in (0, 1)$ such that

$$f(\lambda a + (1 - \lambda)b) \le \lambda f(a) + (1 - \lambda)f(b).$$

At this point, define $\lambda' := \lambda\lambda_1 + (1-\lambda)\lambda_2$ and observe that

$$\begin{aligned} f(\lambda' x + (1-\lambda')y) &= f(\lambda a + (1-\lambda)b) \le \lambda f(a) + (1-\lambda)f(b) \\ &\le \lambda(\lambda_1 f(x) + (1-\lambda_1)f(y)) + (1-\lambda)(\lambda_2 f(x) + (1-\lambda_2)f(y)) \\ &= \lambda' f(x) + (1-\lambda')f(y). \end{aligned}$$

Therefore $\lambda' \in S$. The claim follows by the fact that $\lambda_1 < \lambda' < \lambda_2$. □

To complete the proof of Theorem 1, let us suppose for the sake of contradiction that

$$\Lambda := [0,1] \setminus S \neq \emptyset.$$

Note that $\Lambda$ is open since it can be written as $(0,1) \cap S^c$ and $S^c$ is open by Claim 1. Fix $\lambda \in \Lambda$. Hence, there is a maximal open interval $(\lambda_1, \lambda_2)$ containing $\lambda$ and contained in $\Lambda$. Then $\lambda_1, \lambda_2 \in S$ while $(\lambda_1, \lambda_2) \cap S = \emptyset$, contradicting Claim 2.

This shows that $f$ is necessarily convex.

**Acknowledgments.** The author is grateful to the editor and two anonymous referees for their remarks that allowed a substantial improvement of the presentation.

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