Analysis of solving a Cauchy Problem, using an eye-tracker

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Abstract

In this paper we present a qualitative analysis based on data collected by means of an eye-tracker tool, concerning the outcome of a mathematical analysis question administered to a group of candidates of university level or higher. One of the research aims is to highlight similarities and differences in the visual observation of the question itself between candidates considered experts and non-experts. This analysis should provide useful information and clues about the way a mathematical problem is tackled (and the underlying cognitive processes used to tackle it) for eventual research use in Mathematics education, also at university level.

Keywords: eye-tracker, Cauchy Problem, heat maps.

Introduction

Research in the field of Mathematics Education, and in the way a student perceives and deals with a mathematical question, has developed considerably in recent decades thanks to the introduction of new investigative methods drawn from research fields such as Psychology, Neuroscience and Linguistics, and based, for example, on analysis of the role of language in Mathematics (Ferrari, 2004) or of creativity in the cognitive process (Schindler & Lilienthal, 2020). In this context, a strong impetus arose from the use of an eye-tracker tool, by which it is possible to analyse the eye movements of a person subjected to a visual stimulus, such as algebraic problems that can be solved by observing a graphic stimulus (Andrà et al., 2015) or geometric problems that can be solved simply by deducing certain properties from the figures (Spagnolo et al., 2021b). The basic hypothesis, which is called the *Eye-Mind Hypothesis* (Just & Carpenter, 1980) and posits that the item a person fixes with his/her eye is closely related to what his/her mind is processing, has recently been tested in the context of research in Mathematics Education by using an eye-tracker tool (Schindler & Lilienthal, 2019). Although eye-tracking methodology is particularly suitable in many areas of Mathematics where the problems assigned (and the students' difficulties in solving them) require visualisation skills, such as Geometry (Epelboim & Suppes, 2001; Schindler & Lilienthal, 2017; Simon et al., 2021), it has also been used in domains such as Algebra (Obersteiner & Tumpek, 2016) or Kinematics (Ferrara & Nemirovsky, 2005). Recent studies have shown that a fundamental role in solving a mathematical problem lies in the student's ability to read (Andrà et al. 2009) the different semiotic representations of mathematical objects (Duval, 2006) and to be able to switch between them, transforming acquired information from one semiotic register to another (Santi, 2011). More recently, in the field of mathematics education, exploratory studies have been carried out on the use of the eye-tracker on secondary school students: for instance, in Spagnolo, et al. (2021a), an item from the international standardized mathematics test OECD PISA 2015 was analysed, regarding a geometrical problem in a real contest. The students were required to model a complex situation, developing a strategy in an unfamiliar context; the results revealed how their attention was often catalysed by elements in the figure that were useless for the solving procedure.

In this study, we have carried out an analysis based on data acquired by means of the eye-tracker instrument, from candidates at university level or above who were set a Mathematical Analysis task. As the number of students to whom the test has been administered so far (see below) is not statistically significant and a quantitative analysis is not yet possible, we consider it profitable to present an analysis of a qualitative nature which can provide interesting insights into the objectives we have set ourselves. One of these aims is to illustrate how the eye-tracker tool can provide useful and valuable insights into the mental processes involved in solving a mathematical problem at university level, in order to understand the difficulties that students encounter in reading mathematical objects (Almfjord & Hallberg, 2020). This information can then be used for two purposes: on the one hand, it provides a "snapshot" of the difficulties that freshmen encounter in the delicate transition between secondary school and tertiary education in the field of science (De Guzman et al, 1998). On the other hand, it can be a useful research tool to suggest new methodologies of teaching and learning in the field of Mathematics Education at first-year university level in a scientific faculty. In addition, the aim of this analysis is to highlight which elements are considered priorities for a student when faced with a mathematical problem, and how these are used to solve the problem. To this end, we tried to highlight similarities and differences between two types of candidates: those who are considered experts (PhD students, high school mathematics professors, students of excellence courses) and candidates who are considered non-experts (students enrolled in the first years of a scientific faculty and who had followed the courses of Geometry and Mathematical Analysis 1), in order to try to identify the different resolution processes implemented by the two different types of candidates when tackling the question, and to deduce the different underlying cognitive paths followed when faced with a problem of a mathematical nature (Andrà et al., 2009; Inglis & Alcock, 2012).

Our research questions can be summarised as follows:

- Are there similarities and/or differences between the elements of a mathematical problem that catch the attention of expert and non-expert candidates?
- What difficulties for candidates are highlighted by these elements in the interpretation of the mathematical objects that make up the question?
- How do these elements influence the resolution process?

The Cauchy problem

The proposed task focuses on the qualitative analysis of the solution to a first-order Cauchy problem. The text says: *Identify the possible graph that represents the solution to the following Cauchy problem.* This kind of differential calculus problem with qualitative graphs is very similar to the questions proposed and analysed in other contexts (such as Ferrari, 2004) where the aim was to provide an argument for deciding which graph to choose among those proposed. In that case, the task was to identify the graph of a primitive of a given function.

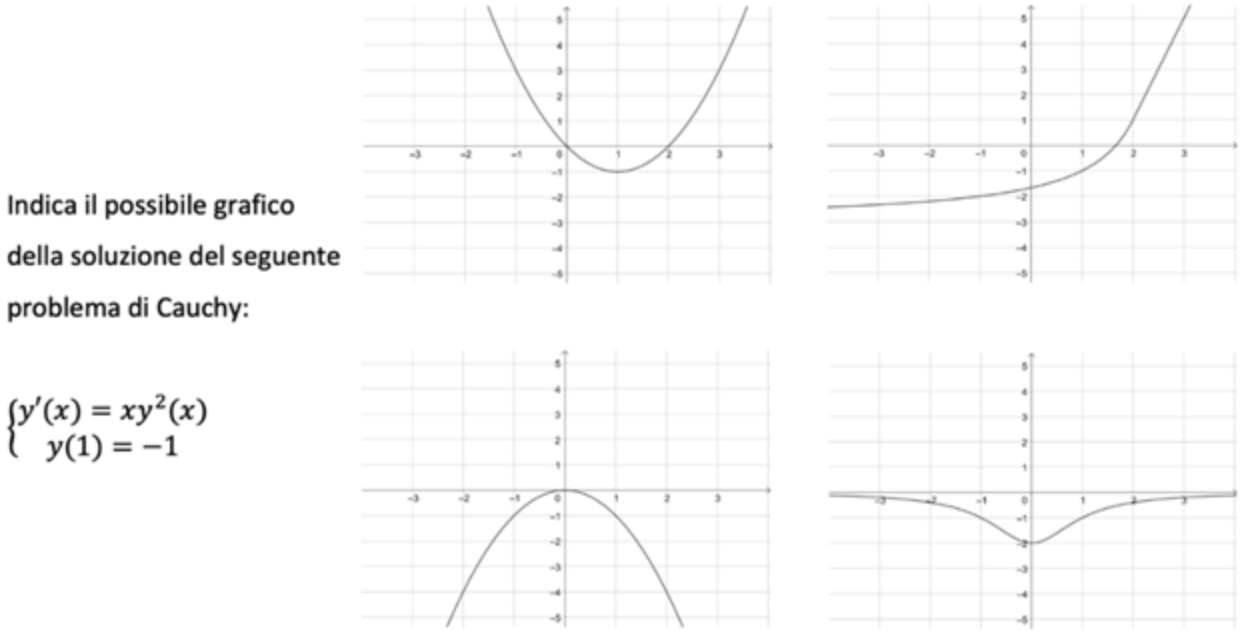


Fig. 1 - Image of the original question as presented to the candidate. Task stimulus: "Identify the possible graph that represents the solution to the following Cauchy problem:"

In its graphic plot it presents three areas of interest, spatially distinct in a relatively clear way. The stimulus, placed to the left of the image, is expressed through a double textual register: above, the common language is used to express the question under examination, while below it is represented in the formulaic-algebraic register, in which the differential equation and relative initial data are expressed. The central/right-hand section of the image displays the four possible answer alternatives, expressed in the figural register.

The cognitive processes required to address the question involve an *understanding* of the query expressed in ordinary language and a subsequent *conversion* from the representation of the mathematical object under consideration, expressed by means of a formula, to the graphical representation. The *compacted* information contained in the symbolic expression of the formula itself must be first *unpacked* and then *distributed* on the spatial extension of the alternative graphs, looking for the right *placement* of this information in the various parts of the graph under examination, in order to *verify* whether there is consistency with what is required.

The graphs of the four alternative answers were plotted on a Cartesian plane provided with a grid. This has been done to allow the candidates to verify whether the possible solution satisfies the initial condition of the Cauchy problem. However, we decided not to make this a decisive factor in the choice of answer, so as not to preclude a cognitive process that did not consider the information contained in the differential equation. For this reason, all four possible solutions satisfy the initial condition.

The choice of the differential equation, together with the preparation of the four alternative answers, was made with the aim of allowing two different solving strategies that can be prepared by the candidate to tackle the question, each based on a different underlying mental process. Such cognitive processes should be intercepted by the analysis of the eye-tracker data, corresponding each of them to specific patterns of eye movements.

In fact, the approaches that can be used are:

1. *explicit integration:* although the absence of pen and paper means this strategy is not particularly easy, the assigned differential equation is essentially integrable and it is possible to obtain the explicit solution by direct mental calculation;
2. *intervals of monotony:* the sign of the first derivative of the solution can be easily evaluated and consequently the intervals of monotony of its graph can be easily deduced. In addition, the obvious presence of a point of relative extreme in $x=0$ is an important indicator for the choice of the correct solution.

As mentioned above, the test is currently being administered to people who are considered experts, such as students of Mathematics, PhD students, high school mathematics teachers, as well as people who are not considered experts, i.e., university students enrolled in the first years of a Bachelor's degree course in a scientific faculty (Physics, Chemistry, Engineering, Statistics) who have followed the course in Mathematical Analysis 1.

Data collection procedure

Eye tracking tools

The instrument we used to conduct the test is Tobii Pro Nano ®, a screen-based eye-tracker that captures gaze data at 60 Hz and is designed for fixation-based studies. It was equipped with a binocular camera capable of achieving a precision of 0.10° RMS and an accuracy of 0.3° under optimal conditions. The method used was eye tracking based on video of the pupillary and corneal reflection with dark and bright pupillary illumination modes. The camera captured images of both eyes for accurate measurement of gaze and eye position in 3D space, as well as pupil diameter. The output data obtained, consisting of timestamp, gaze origin, gaze point and pupil diameter were subsequently analysed by an external processing unit capable of performing gaze calculations, providing a controlled environment to run our tests with dedicated eye tracking processing, thereby improving the performance of the eye-tracker itself.

This device was in turn connected by Ethernet cable to a computer and the data was reprocessed by dedicated software capable of collecting eye tracking data, observing and qualitatively analysing both individual recordings and aggregated data for comparative analysis, such as that used in this work. The latter was made possible thanks to the fact that the software we used (Tobii Pro Lab ®) was able to provide us with useful and powerful analysis tools, such as the video recording of eye movements, the segmentation of the data with the *times of interest*, the calculation of the *areas of interest* and the creation of *heatmap* and *gaze plot* visualisations.

A heatmap is a graphical representation of how the act of looking is distributed over the stimulus, in which warm colors indicate areas where participants either fixated for extended durations or on multiple occasions. Heat map visualizations can incorporate sequential order, time to first fixation and other metrics. The main function of the gaze plot, on the other hand, is to reveal information about the time sequence of looking or where and when the candidate looks at an item. Time spent looking, most commonly expressed as *fixation duration*, is shown by the diameter of the fixation circles. The longer the look, the larger the circle.

The input image was entered through the software and displayed on the screen of a monitor to which the eye-tracker camera was attached and on which the test was carried out. Before starting the test, the software performed a calibration of the instrument for each student: the candidate followed with the eyes the movement of a cursor that lingered on the four vertices of the effective dimensions of the screen (i.e. those in which the input image would have appeared) and the software returned a calibration percentage value that measured how far the candidate's fixations on the cursor were compatible (within a margin of error determined by the sensitivity of the instrument) with the effective dimensions of the screen. In our test we considered valid those calibrations that exceeded 90%. At the end of this procedure, the test started and the monitor showed the stimulus image.

The candidate initially tackled the question without any time limit, while the eye-tracker detected and recorded his/her eye movements. During this phase it was essential that the candidate never took his/her eyes off the screen so as to avoid the loss of the initially accepted calibration. After stating the correct answer from the four options, the candidate was subjected to a voice-recorded interview in which he/she tried to explain the reasons which led to the choice of that solution. Moreover, during this phase, the candidate continued to observe the question on the screen and the eye-tracker never stopped detecting his/her eye movements to allow us, later on, to reconstruct the link between the pattern of eye movements and the justifications given by the candidate in the interview for his/her cognitive choices.

The study was conducted on a total of 4 expert participants (2 high school mathematics professors, 1 student of the Master's Degree in Mathematics and 1 student of an advanced learning course of the Bachelor's Degree in Mathematics, both enrolled at the Sapienza University of Rome) and 9 non-expert participants (4 students of Engineering, 2 students of Physics, 2 students of Chemistry, 1 student of Statistics, all of whom were enrolled in the first academic year at the Sapienza University of Rome). Due to the aforementioned lack of time limits, the duration of the test for each participant was quite heterogeneous. On average, the test with the instrument lasted about forty minutes, half of which was used to answer the next interview. The questions in the interview survey were as follows:

- What did you look at most - the graph or the text of the question?

- Which elements caught your attention most in the text?
- Which elements caught your attention most in the image?
- Which element did you start from when looking for the solution?
- What element enabled you to find the solution?
- Did you first read the text of the question and then look at the picture, or vice versa? Why did you do this?

The software used also made it possible to export the raw data into numerical tables that will be useful for statistical processing, when the amount of data collected also allows us to carry out quantitative analyses.

Test analysis

In this first paper we intend to present a qualitative analysis of the data collected so far, illustrating differences in the recorded eye-tracker data between the observation of an expert candidate and that of a non-expert candidate.

The data at our disposal consisted of:

a) a video sequence, showing the entire animated sequence of eye movements recorded by the instrument while the candidate was trying to answer the question in the first part of the test and while justifying his/her answers during the subsequent interview. This sequence was also used to underline and highlight the passages we considered decisive during the candidate's attempt;
b) the audio recording of the interview and subsequent written transcription. The audio of the interview was synchronised with the above-mentioned video to reconstruct the cognitive process used by the candidate and thus try to interpret it on the basis of eye movements recorded by the eye tracker;
c) graphs containing the heat maps and gaze plots produced by the software for each candidate, both in the question answering phase and in the phase of justifying the chosen strategies. These graphs provided useful and immediate global information on the approaches used by candidates in tackling the proposed problem;
d) a file with the specific numerical data (duration and position of each fixation on the individual points of the screen, duration, and length of the ocular saccades, etc.) in a table that can be exported in Excel® format. This information was not used immediately but is instead considered useful for future quantitative studies, based on statistical elaborations on a significant sample of candidates.

By analysing in detail the video sequence during the test and by cross-analysing the video sequence with the transcript of each candidate's interview, we identified possible common traits (number and duration of fixations on the different areas of interest of the question, quantity and length of saccades between one area and another, patterns of eye movements) of the two different classes of candidates, looking for peculiar characteristic traits. Once this was done, we identified a "typical" candidate representing his/her class and compared the above-mentioned characteristics, in order to highlight similarities and differences. We essentially identified four distinct general phases in the resolution process that are commonly found in observation by both types of candidate:

- *exploration phase*: in this first phase, the candidate quickly observes the textual part of the question to understand the subject he/she has to tackle, while the initial observation of the possible graphic alternatives is reduced to a minimum. This can be deduced from the brief saccades noted between the plain language text and the algebraic text of the formula;
- *calculation phase:* in this phase, the candidate observes a fixed point on the screen for a long time in order not to be *distracted* while carrying out mental calculations. Since in the proposed question the computational element is strictly reserved to the study of a Cauchy problem, by analysing the data, prolonged fixations on the differential equation are detected, with small saccades between its two members. The calculation phase can involve both the attempt at direct integration of the differential equation and the search for intervals of monotony and/or points of relative extremes;
- *comparison phase*: in this phase the candidate begins to look more closely at the four alternative graphs. The eye-tracker data show marked saccades between a single graph and the text of the formula, or saccades between different graphs. The former type indicate a dual purpose of direct comparison between the information obtained from the differential equation and the qualitative trends of the single alternatives, and verification of the passage of the graph itself for the initial data. The latter occur when the candidate is in doubt between two alternatives that he/she considers similar and thus makes observations to discriminate the most plausible solution;
- *verification phase*: in this phase the candidate has already identified/selected the possible solution and takes some extra time to check the validity of his/her choice. During this phase the candidate's gaze is mainly focused on one graph and is therefore characterised by fixations and saccades on the various areas of the chosen graph, and by saccades between it and the text of the question.

More specifically, we identified these phases by characterising them using typical elements used in an eye tracker analysis, which are:

- ***fixation***: point in the visual field where the eyes remain over a relatively long period of time, commonly in the order of tenths of a second. The characteristic parameters used in the eye-tracker analysis are their (temporal) duration and count;
- ***saccade***: rapid transition between two fixations. The characteristic parameter used in the eye-tracker analysis are their (metric) amplitude and frequency;
- ***scanning path (sequence, trajectory):*** the study over time of repeated cycles or trajectories identified by a sequence of successive fixations. It allows, for example, to understand whether the visual interest is focused on the comparison between two objects or two areas of interest in the question, or on some global aspects of part of it.

In the following table we report the qualitative analysis for each phase we identified:

	Experts	**Non-experts**
Exploration phase	Well defined main visual area of interest consists of the textual part of the request, with long eye fixations on the formula expressing Cauchy's problem and short saccades between the first member and the second member of the differential equation	Area of main visual interest not well defined, with long fixations on the formula expressing the Cauchy problem, interspersed with long and frequent saccades between the graphs of the four alternatives, resulting in a messy scanning path
Calculation phase	Initial ocular fixation on the second member of the differential equation, followed by shorter fixations on it interspersed with increasing frequency by rapid saccades. Scanning path following the trend of the graphs of the four alternatives and focusing on the monotonic intervals	Long eye fixations on the differential equation with visual focus on its second member and only sporadic presence of saccades with the graphic part of the question
Comparison phase	Fairly ordered visual pattern, with longer saccades between only two of the alternatives' graphs and shorter saccades on the same graph. Scanning path following the overall pattern (or part of it) of the graph under observation	Chaotic scanning path, expressed by rapid saccades between all four graphical alternatives, with particular attention, expressed by brief fixations, on certain specific points of the graphs (relative extremes, intersections with axes, etc.)
Verification phase	Presence of rapid saccades between the differential equation and a specific graph, with brief ocular fixations on the latter, often detected in different areas than in the previous phase	Presence of rapid and disordered saccades between a specific graph and the differential equation, sometimes alternating with some saccades with a second graph

Expert vs non-expert results

Although these phases are substantially present in the data collected for both expert and non-expert candidates, each of them shows significant differences between the two types of candidates. We now want to illustrate these phase by phase because, in our opinion, they are indicative of the different mental processes underlying the cognitive pathway used to solve the question. This comparison can also be interpreted with some of the categories of the didactic contract (Brousseau G., 1988), which focuses on the expectations, often implicit, that the didactic situation and conventions place on teachers and students.

We can summarise the main differences deduced from our qualitative analysis in the following table:

	Experts	**Non-experts**
Exploration phase	• Initial priority: understanding the request • Gathering information in the formula • Search for the solution more in the input than in the answers	• Initial exploration of the whole content of the question • Gathering information in the formula • Greater difficulty in *reading* a formula
Calculation phase	• Applying the information obtained from the ODE on the monotony intervals of the graphs • Weak *requirement of formal justification*	• Attempt at explicit integration of the Cauchy problem • Strong *requirement of formal justification*
Comparison phase	• Strong process of exclusion • ***Global*** approach: analysis of qualitative chart trends	• Weak process of exclusion • ***Local*** approach: focus on particular points in the graphs
Verification phase	• Predominant irrational-psychological aspect: *performance anxiety* • Search for new characteristics of the chosen graph (e.g. symmetries)	• Predominant irrational-psychological aspect: *frustration* • Stronger process of exclusion between different alternatives • Attempt to determine the analytical trend of the graphs

We now describe the results shown in the table in more detail, together with the interpretation we have derived from the qualitative data obtained using the instrument.

Exploration phase

The main difference between expert and non-expert candidates is the *area of interest* on which visual attention is focused. Since, in this initial phase, the candidate does not yet have any indication either of the question or of the information useful for solving it, visual attention is strongly correlated with *cognitive attention* in the process of *gathering information* for the construction of the resolution strategy and therefore for the acquisition of knowledge. The expert candidate's gaze is easily captured first by the common language text and then by the algebraic text of the formula, while only a fleeting glance is directed at the graphs of the four alternatives. This can be explained by the fact that the initial priority of the expert candidate is knowledge of the request in the question under examination (contained in the textual part), rather than the technical aspects useful to solve it (contained in the formula) or initial exploration of the four alternatives. This confirms the fact that, in general, the expert candidate searches for the solution to a multiple-choice question more in the input than in the available alternatives (Andrà et al., 2009). In the following interview, an experienced candidate states: "As soon as I saw the question prompt, I only took a quick initial look at a graph, but then I immediately focused on the Cauchy problem".

On the contrary, the non-expert candidate is initially only struck by the word **Cauchy** in the textual part (a feature perhaps linked to a linguistic aspect, since it is the only non-Italian word in the text, or to the capital letter identifying a name and therefore a categorisation), and then focuses attention with long fixations and short saccades concentrated on the formulas constituting the Cauchy problem, followed by long saccades among the four graphic alternatives. This more *disordered* behaviour of the non-expert candidate can be explained by the fact that the formulas represent the *most complicated stimulus* for the non-expert, as the rules for *reading a formula* require a more complex semiotic register (Andrà et al., 2009). There is more overall visual attention from students since a formula *condenses* the information it contains in a *more compact but more cryptic way.* This is also confirmed in the interview when a non-expert candidate states: "As soon as I saw the text of Cauchy's problem in the question, I got scared for a moment..." and then continues "...then I started to look at the graphs to see what kind of alternatives I had".

Calculation phase

The heat map of the question shows that a large portion of the time spent observing the question is devoted to the differential equation, and the video analysis of the eye-tracker data shows that this occurs mainly in the initial part of the cognitive process.

This is also corroborated by the subsequent interview, in which most of the non-expert candidates stated that they had at least attempted to solve Cauchy's problem explicitly. For example, one of them stated, "At first, I tried to solve the differential equation in my head, but then I gave up because it was too difficult without writing anything down..."

This manifests itself through long fixations of the differential equation itself, with small saccades between the various terms that make up the equation. We deduce that at this stage candidates try to *unpack* the information contained in the differential equation by means of mental calculation processes. The expert candidate, however, tends to process the data by trying to identify the monotonic intervals and/or possible relative extreme points, while the non-expert candidate's first reaction is to search for direct integration of the differential equation by obtaining the *explicit* analytical expression of the solution.

This difference in approach, backed up by the explicit statements of the interviewees, can be deduced from the eye-tracker analysis in relation to the comparison phase with the alternative graphs (see the description of the next phase). This seems to suggest that for non-expert candidates, one of the most dangerously binding clauses of the didactic contract applies, namely the *requirement of formal justification*, according to which the solution of a given mathematical problem is *correct* only if calculations and operations (or, more generally, formal procedures) are performed. Alternative solutions that do not make use of traditional calculations are considered incomplete or even incorrect. Our data show that even for experienced candidates there is a very early stage in which the formal solution of the differential equation is sought, demonstrating that this clause is well *established* not only in a purely scholastic environment. An expert candidate in the interview admitted: "As soon as I realised the nature of the question, I tried to solve the differential equation...".

Comparison phase

This is a tricky stage in the analysis of eye-tracker data, especially for non-expert candidates, as eye movements become more rapid and uncertain. Since the absence of pen and paper usage makes the task of calculating the explicit solution more complex, candidates who decided to use this strategy have difficulty in finding matches in the alternative graphs due to the lack of useful information. The visible consequence on the eye-tracker data is that the candidate produces many quick saccades between all four alternatives, alternating with fixations on some specific points on the graph (typically the relative extreme points or the intersections with the Cartesian axes), thus highlighting a substantially *local* type of approach.

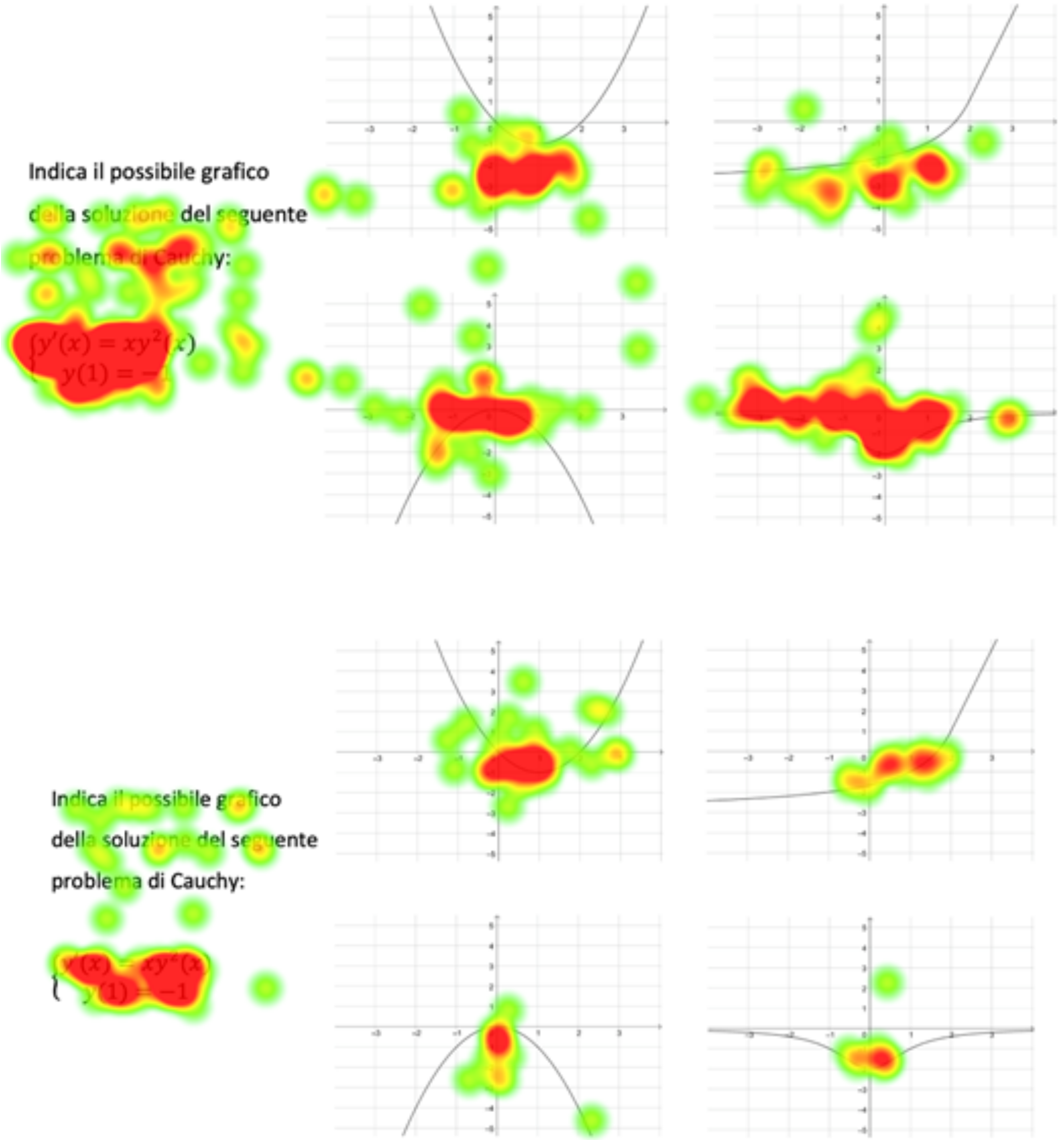
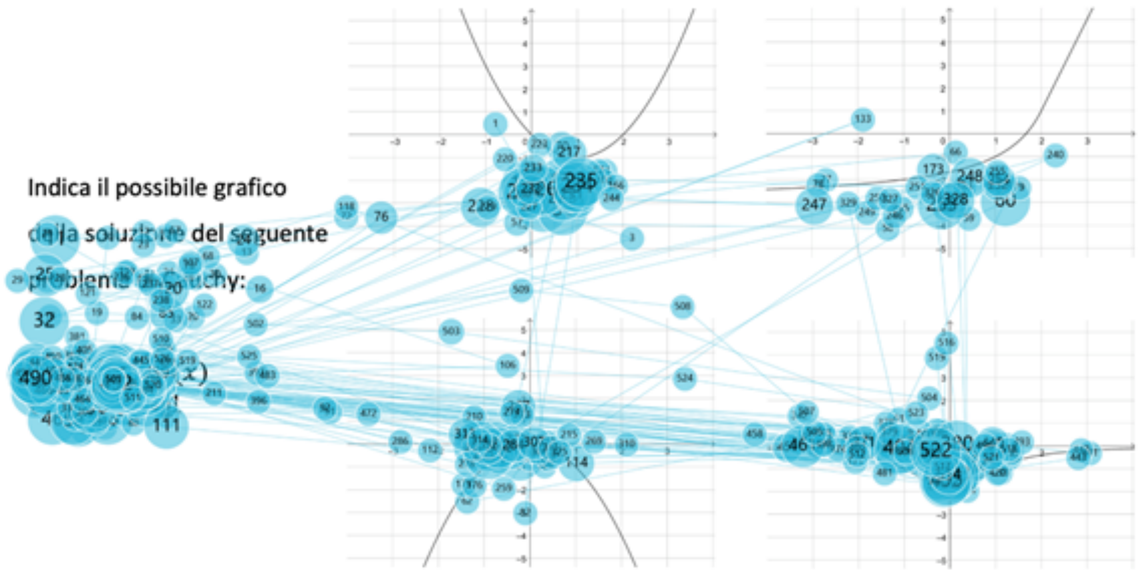


Fig 2 – Heat Map showing Global vs Local nature between the experts (up) and non-experts (down) trend of the fixations

The verification of the passage of the graphs for the point defined by the initial data is almost completely absent (few fixations and not on all four graphs). This is due to the fact that the lack of information resulting from a failed integration of the differential equation (need for formal justification) has *distracted* attention from the initial condition of the Cauchy problem (purely geometric information). This triggers a failed attempt at an *exclusion* process between the four alternatives. One non-expert candidate, when asked explicitly about the initial datum, stated: "Ah no, I didn't even think to check the initial datum because I focused mainly on the differential equation!" At this stage the eye-tracker shows short fixations and quick saccades involving substantially all four graphs.

In addition, it should be noted that the non-expert candidate has greater difficulty in selecting useful information and discarding useless information. For instance, the value of the function at a stationary point becomes a discriminating element (and therefore a visual check of the height on the y-axis of such a point), rather than the value on the x-axis where such a point is located, as admitted by one of the non-expert candidates: "In the first graph the minimum is -1, while in the fourth -3...". Again, this is backed up when he tries to extract information about the solution by observing that "the quadratic term in the y suggested that the solution might be a parabola..." and thus focusing his visual attention on the two parabolic trends of the alternative answers.

The visual exploration of the expert candidate, on the contrary, appears significantly different: first, the process of exclusion is much more marked, so that the saccades mostly involve a couple of graphic alternatives left available. This is a clear sign that the *distractors* for the expert candidate are less effective and that therefore the number of possible answers available to him is significantly lower than that for the non-expert candidate.



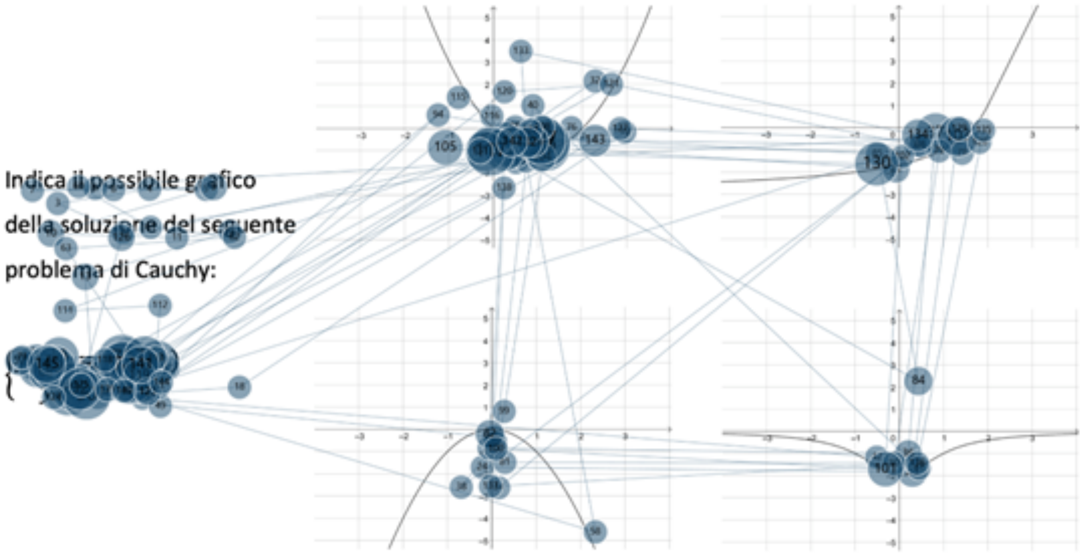


Fig 3 – Gaze Plot showing Global vs Local nature between the expert (up) and non-expert (down) trend of the fixations

The heat map and the gaze plot also show that the approach is a *global* one: the expert candidate prefers quick saccades that follow the global trend of the graph, rather than long fixations on specific points of the graph. As mentioned above, this is a consequence of the fact that in the calculation phase the expert candidate prefers a qualitative analysis of the differential equation, obtaining useful information on the monotonic intervals of the solution. In fact, one expert candidate states: "I noticed the symmetry of the second member of the differential equation and understood that the solution had to be decreasing for negative values of x and vice versa..."

Verification phase

This is one of the most difficult phases to analyse because, as we will now briefly illustrate, both in the cases of the non-expert and expert candidate, the link between ocular observations and the underlying cognitive process is accompanied by a predominant irrational-psychological aspect.

As a direct consequence of the previous phase, verification of the presumed solution to the question found by the non-expert candidate appears more disordered. The saccades between the chosen graph and the algebraic text of the formula are rapid and chaotic and indeed, not infrequently, there are still saccades between two different graphs, a sign of an *awareness* (justified or not) that the motivations which led to the choice are weak and that the process is more the result of *randomness* than of a methodical cognitive process. One of the candidates stated: 'I was in doubt between two possible answers until the very end, and in the end, I instinctively chose the one that seemed best to me...', without being able to provide a logical argument for his choice.

At this stage the *frustration* of the non-expert candidate arises, who, contrary to events in the previous phase, only now triggers a *process of exclusion* of some graphs, as if to check that the solution chosen is the least excludable. In fact, it happened quite regularly that the verification phase for the non-expert candidate turned into a second check step and that the solution originally selected was changed at the last moment. Similarly, it occurred (not infrequently) that in the interview phase, while explaining the reasons for his choice, the non-expert candidate realised the incorrect reasoning made during the test phase and corrected himself: "Now that I am talking about it, I realise that I was wrong before...". The saccades in the explanation phase appear less chaotic and more methodical with consequently more marked fixations on essential elements of the graph, a sign that some cognitive processes supported by using spoken language appear more solid for the non-expert candidate.

It must be pointed out that, in some cases, the exclusion process is substantially based on the attempt, where possible, at an analytical formulation of the alternative graphs (switch from graphic to algebraic register) and of a subsequent verification that this hypothesised trend could satisfy the differential equation in a formal way (need for formal justification). It is also symptomatic that the two curves to the left of the question were assumed as exact parabolas. In other words, the *parabolic curve* (determined by the gaze) coincides with an analytical expression of a *parabola* (determined by the mental process). More than one non-expert candidate states: "I tried to substitute a parabola in the differential equation to see if it was verified".

The verification phase of the expert candidate is more detailed. The saccades take place more neatly between the graph chosen as the presumed solution and the text of the formula and the check is made by searching for elements not previously considered (e.g., the search for possible *symmetry* properties of the graph), although this process has not always proved fruitful (e.g., in the analysis of asymptotes). It must be said that the need for verification on the part of the expert candidate was often excessive, a sign of the fact that, presumably, the expert candidate has greater *performance anxiety* than the non-expert candidate, since the expectations of a correct answer to the question are much higher for him/her. One states that: "After choosing what I believed to be the solution I looked for other indirect verifications that would give me further confirmation...".

Conclusions

The work presented in this paper is part of a general project still in progress, through which we intend to administer a series of Mathematical Analysis questions to a group of candidates of university level or higher and to analyse the results by means of data elaborated by the eye-tracker which are then compared with the subsequent interview given by the candidates themselves.

Being in its initial phase and not yet having statistically sufficient data to carry out a quantitative analysis, we have presented in this first work a qualitative analysis of one of the questions of the complete test, as it already provides interesting insights for the purpose of the whole project. The chosen question consists of a task to determine which graph offers the solution to a Cauchy problem, by means of a qualitative analysis of the differential equation. This request implies the ability to read different representations (formula and graphical) of the same mathematical object (a solution to a Cauchy problem) and transform the information obtained from one representation to another via a change of semiotic register.

We have identified four distinct phases of the cognitive process (exploration phase, calculation phase, comparison phase, verification phase) that are substantially common between the two types of candidates, thus indicating a common investigation methodology acquired after high school studies, with a non-negligible component of the *need for formal justification* that appears rooted in both types of candidates. However, each phase presents substantial differences in the parameters associated with eye movements between an expert and a non-expert. These differences can be summarised as follows:

- the expert candidate searches for the solution mainly in the question, while the non-expert candidate searches among the four possible alternatives;

- the expert candidate is left with fewer alternative answers to search through than the non-expert candidate, since in the former case there is a more effective process of *a priori* exclusion for alternatives considered unacceptable at first glance;

- the stimulus expressed in the form of a formula seems to be a much more difficult factor for non-expert candidates than for expert candidates, a difficulty highlighted by more disordered behaviour of the eye movements in the saccades between the formula and alternative graphs;

- the main strategy adopted by the expert candidate is that of identifying the intervals of monotony deduced from the differential equation and verified in the alternatives, by means of a sequence of *global* fixations of the graphs. The non-expert candidate, on the contrary, prefers first an attempt at direct integration (long fixations and saccades on the text of Cauchy's problem), followed by fixations of *local* nature in some points of the graph considered useful for the determination of the solution of the problem;

- the non-expert candidate has more difficulty than the expert candidate in eliminating information that is useless for the question in the graphic register, presenting several fixations on irrelevant elements of the trends of the alternative answers;

- once the possible solution has been identified, the expert candidate tends to carry out a verification by comparing characteristics of the chosen graph with the text of the question prompt. On the contrary, the verification phase of the non-expert candidate is mainly devoted to comparing the characteristics of the chosen graph with those of one (or more than one) of its alternatives among the various distractors, thus implementing a process of *a posteriori* exclusion, which evidently stimulates the candidate to prefer the *least improbable* answer;

- the psychological element is strong for the expert candidate in the verification phase, since there is a more marked *anxiety about performance* than for the non-expert candidate, linked to *awareness* of the higher *expectations* placed on him/her.

To summarise, expert candidates' reason first about the stimulus and then about the alternatives by looking for the monotonic intervals of the function; the strategies are also different in the verification phase. Non-expert candidates are influenced by the four alternatives and do not know how to find the right information from the differential equation. This may be because they do not have the possibility of finding an algebraic solution of the differential equation.

It seems that teaching practices, both at the end of secondary school and in the first year of university where mathematics is a service course, follow a more algorithmic approach and problem-solving skills are neglected. Usually, the qualitative solution of a differential equation is not taken into account as the focus is more often on solving it algebraically.

The solving process may therefore be influenced by the students' lack of aptitude for problem-solving activities and unfamiliarity with tasks of this type which lead candidates to look immediately at the 4 alternatives. This also demonstrates a detachment between the algebraic equation of the differential equation and the graph of its solution: although for all these students it is easy to deduce some graphical properties of a function written in an algebraic register (a conversion in Duval's sense), this process is complex if the function is written within a differential equation. This is confirmed by the Eye-Mind hypothesis, which shows the disordered behaviour of eye movements in the saccades between the formula and the proposed alternatives.

The same fact, evident to expert candidates, that the derivative is positive for positive x and vice versa, is a *transparent* representation to expert candidates and *opaque* to non-expert candidates.

All this seems to suggest a different cognitive pathway in dealing with a question of a mathematical nature for an expert candidate as compared with a non-expert, and that this diversity is influenced by various factors into which we believe further investigation is useful. This will be continued through analysis of other questions of the test currently being administered, and then consolidated through quantitative analysis of the eye-tracker data at our disposal as soon as there is sufficient data to be considered statistically relevant.

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