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A semi-Lagrangian scheme for Hamilton–Jacobi–Bellman equations with oblique derivatives boundary conditions

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Abstract

We investigate in this work a fully-discrete semi-Lagrangian approximation of second order possibly degenerate Hamilton–Jacobi–Bellman (HJB) equations on a bounded domain $\mathcal{O} \subset \mathbb{R}^N$ $(N = 1, 2, 3)$ with oblique derivatives boundary conditions. These equations appear naturally in the study of optimal control of diffusion processes with oblique reflection at the boundary of the domain. The proposed scheme is shown to satisfy a consistency type property, it is monotone and stable. Our main result is the convergence of the numerical solution towards the unique viscosity solution of the HJB equation. The convergence result holds under the same asymptotic relation between the time and space discretization steps as in the classical setting for semi-Lagrangian schemes on $\mathcal{O} = \mathbb{R}^N$. We present some numerical results, in dimensions $N = 1, 2$, on unstructured meshes, that confirm the numerical convergence of the scheme.

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1 Introduction

In this work we deal with the numerical approximation of the following parabolic Hamilton–Jacobi–Bellman (HJB) equation

$$\begin{aligned} \partial_t u + H\left(t, x, Du, D^2u\right) &= 0 \quad \text{in } (0, T] \times \mathcal{O}, \\ L(t, x, Du) &= 0 \quad \text{on } (0, T] \times \partial\mathcal{O}, \\ u(0, x) &= \Psi(x) \quad \text{in } \overline{\mathcal{O}}. \end{aligned} \tag{1}$$

In the system above, $T > 0$, $\mathcal{O} \subset \mathbb{R}^N$ ($N = 1, 2, 3$) is a nonempty smooth bounded open set and H and L are nonlinear functions having the form

$$H(t, x, p, M) = \sup_{a \in A} \left\{ -\frac{1}{2}\mathrm{Tr}\left(\sigma(t, x, a)\sigma(t, x, a)^\top M\right) - \langle \mu(t, x, a), p\rangle - f(t, x, a) \right\}, \tag{2}$$

$$L(t, x, p) = \sup_{b \in B} \left\{ \langle \gamma(x, b), p\rangle - g(t, x, b) \right\}, \tag{3}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^N$, $A \subset \mathbb{R}^{N_A}$ and $B \subset \mathbb{R}^{N_B}$ are nonempty compact sets, $\sigma : [0, T] \times \overline{\mathcal{O}} \times A \to \mathbb{R}^{N \times N_\sigma}$, with $1 \leq N_\sigma \leq N$, $\mu : [0, T] \times \overline{\mathcal{O}} \times A \to \mathbb{R}^N$, $f : [0, T] \times \overline{\mathcal{O}} \times A \to \mathbb{R}$, $\gamma : \partial\mathcal{O} \times \mathcal{V} \to \mathbb{R}^N$, with $\mathcal{V} \subseteq \mathbb{R}^{N_B}$ being an open set containing B, $g : [0, T] \times \partial\mathcal{O} \times B \to \mathbb{R}$, and $\Psi : \overline{\mathcal{O}} \to \mathbb{R}$.

If $A = \{a\}$ and $B = \{b\}$, for some $a \in \mathbb{R}^{N_A}$ and $b \in \mathbb{R}^{N_B}$, and $\gamma(x, b) = n(x)$, with $n(x)$ being the unit outward normal vector to $\overline{\mathcal{O}}$ at $x \in \partial\mathcal{O}$, then (1) reduces to a standard linear parabolic equation with Neumann boundary conditions. In the general case, and after a simple change of the time variable in order to write (1) in backward form, the HJB equation (1) appears in the study of optimal control of diffusion processes with controlled reflection on the boundary $\partial\mathcal{O}$. The existence of such diffusions is related with the so-called Skorokhod problem (see e.g. [22, 45] for its formulation and [21, 38] for its application to show the existence of reflected diffusion processes) and provides a rigorous framework to study stochastic optimal control problems of a class of constrained diffusion processes. We refer the reader to [18, 37] for the first order (or deterministic) case, i.e. $\sigma \equiv 0$, and to [12, 36] for the general second order (or stochastic) case. Let us also mention the contributions [33, 39, 46] which relate singular optimal control problems (see e.g. [26, Chapter VIII] and the references therein), having a smooth value function, and reflected diffusions.

Since the HJB equation (1) is possibly degenerate parabolic, one cannot expect the existence of classical solutions and we have to rely on the notion of viscosity solution (see e.g. [17]). Moreover, as it has been noticed in [35, 37], in general the boundary condition in (1) does not hold in the pointwise sense and we have to consider a suitable weak formulation of it. We refer the reader to [6, 37] and [4, 5, 13, 17, 30], respectively, for well-posedness results for HJB equations with oblique derivatives boundary condition in the first and second order cases.

The study of the numerical approximation of solutions to HJB and, more generally, fully nonlinear second order Partial Differential Equations (PDEs), has made important progress over the last few decades. Most of the related literature consider the case where $\mathcal{O} = \mathbb{R}^N$, or where a Dirichlet boundary condition is imposed on the boundary $\partial\mathcal{O}$ (see e.g. [23–25, 43] and the references therein). Similarly to the case of oblique derivatives boundary conditions, when $\mathcal{O}$ is bounded and Dirichlet type boundary conditions are imposed on $\partial\mathcal{O}$, a suitable weak formulation of the latter is needed to guarantee the well-posedness of the associated HJB equation (see e.g. [7, 8, 32]). In this context, we refer the reader to [1, 32] for a discussion on numerical scheme that capture the behaviour of the viscosity solution at the boundary.

Compared with the cases in the previous paragraph, the numerical approximation of solutions to (1) has been much less explored. In the articles [1, 44] the authors consider (1) in the particular first order case ($\sigma \equiv 0$). Moreover, in the framework of [44], where a finite difference scheme is proposed, the function defining the boundary condition has the particular form $L(t, x, p, b) = \langle n(x), p\rangle$. On the other hand, both references consider Hamiltonians which are not necessarily convex with respect to p. In the recent article [31], the authors consider a fully nonlinear parabolic equation, with $\mathcal{O}$ being a polytopic domain and mixed boundary conditions, and propose a convergent finite element method. Let us also mention the reference [2], where, in the context of mean curvature motion with nonlinear Neumann boundary conditions, the authors propose a discretization that combines a Semi-Lagrangian (SL) scheme in the main part of the domain with a finite difference scheme near the boundary.

The main purpose of this article is to provide a consistent, stable, monotone and convergent SL scheme to approximate the unique viscosity solution to (1). By the results in [4], the latter is well-posed in $C([0, T] \times \overline{\mathcal{O}})$ under the assumptions in Sect. 2 below. Semi-Lagrangian schemes to approximate the solution to (1) when $\mathcal{O} = \mathbb{R}^N$ (see e.g. [14, 19]) can be derived from the optimal control interpretation of (1) and a suitable discretization of the underlying controlled trajectories. These schemes enjoy the feature that they are explicit and stable under an inverse Courant-Friedrichs-Lewy (CFL) condition and, consequently, they allow large time steps. This is an advantage compared to classic finite difference schemes, which require to be implicit in order to allow large time steps. A second important feature is that they permit a simple treatment of the possibly degenerate second order term in H. The scheme that we propose for $\mathcal{O} \neq \mathbb{R}^N$ preserves these two properties and seems to be the first convergent scheme to approximate (1) with the rather general assumptions in Sect. 2. In particular, our results cover the stochastic and degenerate case. Consequently, from the stochastic control point of view, our scheme allows to approximate the so-called value function of the optimal control of a controlled diffusion process with possibly oblique reflection on the boundary $\partial\mathcal{O}$ (see [12]). The main difficulty in devising such a scheme is to be able to obtain a consistency type property at points in the space grid which are near the boundary $\partial\mathcal{O}$ while maintaining the stability. This is achieved by considering a discretization of the underlying controlled diffusion which suitably emulates its reflection at the boundary in the continuous case. We refer the reader to [41] for a related construction of a semi-discrete in time approximation of a second order non-degenerate linear parabolic equation.

The remainder of this paper is structured as follows. In Sect. 2 we state our assumptions, we recall the notion of viscosity solution to (1), and we show the existence of oblique projections onto $\partial\mathcal{O}$ for points near the boundary. In Sect. 3 we provide the SL scheme as well as its probabilistic interpretation (in the spirit of [41]). The latter will play an important role in Sect. 4, which is devoted to show a consistency type property and the stability of the SL scheme. By using the half-relaxed limits technique introduced in [8], we show in Sect. 5 our main result, which is the convergence of solutions to the SL scheme towards the unique viscosity solution to (1). The convergence is uniform in $[0, T] \times \overline{\mathcal{O}}$ and holds under the same asymptotic condition between the space and time steps than in the case $\mathcal{O} = \mathbb{R}^N$. Next, in Sect. 6 we first illustrate the numerical convergence of the SL scheme in the case of a one-dimensional linear equation with homogeneous Neumann boundary conditions. In this case the numerical results confirm that the boundary condition in (1) is not satisfied at every $x \in \partial\mathcal{O}$, but it is satisfied in the viscosity sense recalled in Sect. 2 below. In a second example, we consider a two dimensional degenerate second order nonlinear equation on a circular domain with non-homogeneous Neumann and oblique derivatives boundary conditions. In the last example, we consider a two-dimensional non-degenerate nonlinear equation on a non-smooth domain. Due to the lack of regularity of $\partial\mathcal{O}$, our convergence result does not apply. However, the SL scheme can be successfully applied, which suggests that our theoretical findings could hold for more general domains. This extension as well as the corresponding study in the stationary framework remain as interesting subjects of future research.

2 Preliminaries

As mentioned in the introduction, it will be simpler to describe our approximation scheme when (1) is written in backward form. This can be done by a simple change of the time variable and a possible modification of the time dependency of H. Let us set $\mathcal{O}_T := [0, T) \times \mathcal{O}$ and $\overline{\mathcal{O}}_T = [0, T] \times \overline{\mathcal{O}}$. We consider the HJB equation

$$
\begin{aligned}
-\partial_t u + H\left(t, x, Du, D^2 u\right) &= 0 \quad \text{in } \mathcal{O}_T, \\
L(t, x, Du) &= 0 \quad \text{on } [0, T) \times \partial\mathcal{O}, \\
u(T, x) &= \Psi(x) \quad \text{in } \overline{\mathcal{O}},
\end{aligned}
\tag{HJB}
$$

where H and L are respectively given by (2) and (3).

For notational convenience, throughout this article, we will write $\gamma_b(x) = \gamma(x, b)$ for all $x \in \partial\mathcal{O}$ and $b \in B$. Our standing assumptions for the data in (HJB) are the following.

(H1) $\mathcal{O} \subseteq \mathbb{R}^N$ is a nonempty, bounded domain with boundary $\partial\mathcal{O}$ of class C^3.
(H2) The functions σ, μ, f, g and Ψ are continuous. Moreover, for every $a \in A$, the functions $\sigma(\cdot, \cdot, a)$ and $\mu(\cdot, \cdot, a)$ are Lipschitz continuous, with Lipschitz constants independent of $a \in A$.
(H3) The function γ is of class C^1. We also assume that

$$
(\forall\ (x, b) \in \partial\mathcal{O} \times B) \quad |\gamma_b(x)| = 1 \quad \text{and} \quad \langle n(x), \gamma_b(x)\rangle > 0,
$$

where, for every $x \in \partial\mathcal{O}$, we recall that $n(x)$ denotes the unit outward normal vector to $\overline{\mathcal{O}}$ at x.

2.1 Viscosity solutions

We now recall the notion of viscosity solution to (HJB) (see [4]). We need first to introduce some notation. Given a bounded function $z : \overline{\mathcal{O}}_T \to \mathbb{R}$, its upper semicontinuous (resp. lower semicontinuous) envelope is defined by

$$(\forall\,(t,x) \in \overline{\mathcal{O}}_T) \quad z^*(t,x) := \limsup_{\substack{(s,y)\in\overline{\mathcal{O}}_T,\\ (s,y)\to(t,x)}} z(s,y) \quad \left(\text{resp. } z_*(t,x) := \liminf_{\substack{(s,y)\in\overline{\mathcal{O}}_T,\\ (s,y)\to(t,x)}} z(s,y)\right). \tag{4}$$

Definition 1 (i) An upper semicontinuous function $u_1 : \overline{\mathcal{O}}_T \to \mathbb{R}$ is a *viscosity subsolution* to (HJB) if for any $(t,x) \in \overline{\mathcal{O}}_T$ and $\phi \in C^2(\overline{\mathcal{O}}_T)$ such that $u_1 - \phi$ has a local maximum at (t,x), we have

$$-\partial_t\phi(t,x) + H(t,x,D\phi(t,x),D^2\phi(t,x)) \le 0, \tag{5}$$

if $(t,x) \in \mathcal{O}_T$,

$$\min\left\{-\partial_t\phi(t,x)+H(t,x,D\phi(t,x),D^2\phi(t,x)),\, L(t,x,D\phi(t,x))\right\} \le 0, \tag{6}$$

if $(t,x) \in [0,T) \times \partial\mathcal{O}$ and,

$$u_1(t,x) \le \Psi(x), \tag{7}$$

if $(t,x) \in \{T\} \times \overline{\mathcal{O}}$.

(ii) A lower semicontinuous function $u_2 : \overline{\mathcal{O}}_T \to \mathbb{R}$ is a *viscosity supersolution* to (HJB) if for any $(t,x) \in \overline{\mathcal{O}}_T$ and $\phi \in C^2(\overline{\mathcal{O}}_T)$ such that $u_2 - \phi$ has a local minimum at (t,x), we have

$$-\partial_t\phi(t,x) + H(t,x,D\phi(t,x),D^2\phi(t,x)) \ge 0, \tag{8}$$

if $(t,x) \in \mathcal{O}_T$,

$$\max\left\{-\partial_t\phi(t,x)+H(t,x,D\phi(t,x),D^2\phi(t,x)),\, L(t,x,D\phi(t,x))\right\} \ge 0, \tag{9}$$

if $(t,x) \in [0,T) \times \partial\mathcal{O}$ and,

$$u_2(t,x) \ge \Psi(x), \tag{10}$$

if $(t,x) \in \{T\} \times \overline{\mathcal{O}}$.

(iii) A bounded function $u : \overline{\mathcal{O}}_T \to \mathbb{R}$ is a *viscosity solution* to (HJB) if u^* and u_*, defined in (4), are, respectively, sub- and supersolutions to (HJB).

Remark 1 As shown in [13, Proposition 6], relation (7) can be replaced by

$$\min \left\{ -\partial_t \phi(t,x) + H(t,x,D\phi(t,x),D^2\phi(t,x)), u_1(t,x) - \Psi(x) \right\} \leq 0, \quad (11)$$

if $(t,x) \in \{T\} \times \mathcal{O}$, and

$$\min \left\{ -\partial_t \phi(t,x) + H(t,x,D\phi(t,x),D^2\phi(t,x)), L(t,x,D\phi(t,x)), u_1(t,x) - \Psi(x) \right\} \leq 0, \quad (12)$$

if $(t,x) \in \{T\} \times \partial\mathcal{O}$. Similarly, condition (10) can be replaced by

$$\max \left\{ -\partial_t \phi(t,x) + H(t,x,D\phi(t,x),D^2\phi(t,x)), u_2(t,x) - \Psi(x) \right\} \geq 0, \quad (13)$$

if $(t,x) \in \{T\} \times \mathcal{O}$, and

$$\max \left\{ -\partial_t \phi(t,x) + H(t,x,D\phi(t,x),D^2\phi(t,x)), L(t,x,D\phi(t,x)), u_2(t,x) - \Psi(x) \right\} \geq 0, \quad (14)$$

if $(t,x) \in \{T\} \times \partial\mathcal{O}$.

The following well-posedness result for (HJB) has been shown in [4, Theorem II.1] (see also [12]).

Theorem 1 *Assume* **(H1)–(H3)**. *Then there exists a unique viscosity solution* $u \in C(\overline{\mathcal{O}})$ *to* (HJB).

Remark 2 (i) [Comparison principle and uniqueness] The existence of at most one solution to (HJB) follows from the following comparison principle (see [4, Theorem II.1] and also [12, Proposition 3.4]). If $u_1 : \overline{\mathcal{O}}_T \to \mathbb{R}$ is a bounded viscosity subsolution to (HJB) and $u_2 : \overline{\mathcal{O}}_T \to \mathbb{R}$ is a bounded viscosity supersolution to (HJB), then

$$u_1 \leq u_2 \quad \text{in} \quad \overline{\mathcal{O}}_T.$$

(ii) [Existence] Once a comparison principle has been shown, the existence of a solution to (HJB) follows usually from the existence of sub- and supersolutions to (HJB) and Perron's method. In Sect. 5, we construct sub- and supersolutions to (HJB) as suitable limits of solutions to the approximation scheme that we present in the next section. Together with the comparison principle, this yields an alternative existence proof of solutions to (HJB).
A different and interesting technique to show the existence of a solution to (HJB) is to consider a suitable stochastic optimal control problem, with controlled reflection of the state trajectory at the boundary $\partial\mathcal{O}$, and to show that the associated value function is a viscosity solution to (HJB). This strategy has been followed in [12].

(iii) [Continuity] The continuity of the unique viscosity solution to (HJB) follows directly from the comparison principle and the continuity properties required in the definition of sub- and supersolutions to (HJB). Notice that, as usual for parabolic problems with Neumann type boundary conditions, we do not require any compatibility condition between Ψ and the operator L at the boundary $\partial\mathcal{O}$.

2.2 Existence of projections near the boundary

In this section we first study the existence of the projection of x onto $\partial\mathcal{O}$ parallel to γ_b in a neighbourhood of $\partial\mathcal{O}$ and for $b \in B$. These projections play an important role in the construction of our scheme in Sect. 3. The following result is an extension of a result in [28, Section 1.2] to the regularity that we assume in this paper and, more importantly, to the dependence of γ on b. Recall that in **(H3)** $\partial\mathcal{O}$ is assumed to be of class C^3. However, the result in Proposition 1 below is also valid if $\partial\mathcal{O}$ is only of class C^2.

Proposition 1 *Under* **(H3)***, there exists $R > 0$ such that, for any $x \in \mathbb{R}^N$ satisfying $d(x, \partial\mathcal{O}) < R$ and for any $b \in B$, there exist a unique $p^{\gamma_b}(x) \in \partial\mathcal{O}$ and a unique $d^{\gamma_b}(x) \in \mathbb{R}$ such that*

$$x = p^{\gamma_b}(x) + d^{\gamma_b}(x)\gamma_b(p^{\gamma_b}(x)). \tag{15}$$

The mappings $(x, b) \mapsto p^{\gamma_b}(x)$ and $(x, b) \mapsto d^{\gamma_b}(x)$, called respectively the projection onto $\partial\mathcal{O}$ parallel to γ_b and the algebraic distance to $\partial\mathcal{O}$ parallel to γ_b, are of class C^1.

Proof We use the same outline and, as much as possible, the same notations than those in [28].

Let us fix $(s, b_0) \in \partial\mathcal{O} \times B$. Let $g^s : U^s \to \partial\mathcal{O}$ be a C^2 parametrization of $\partial\mathcal{O}$ in a neighbourhood of s, with U^s being an open subset of $\mathbb{R}^{N-1}$, $z_0 \in U^s$, and $g^s(z_0) = s$. By **(H3)** the function

$$U^s \times \mathbb{R} \times \mathcal{V} \ni (z, \lambda, b) \mapsto G^s(z, \lambda, b) = (g^s(z) + \lambda\gamma_b(g^s(z)), b) \in \mathbb{R}^N \times \mathbb{R}^{N_B}$$

is of class C^1. The Jacobian matrix of G^s has the form

$$J^s(z, \lambda, b) = \left(\begin{array}{c|c} J_{z,\lambda}(z, \lambda, b) & J_b(z, \lambda, b) \\ \hline 0_{N_B,N} & I_{N_B} \end{array}\right),$$

where $J_{z,\lambda}(z, \lambda, b)$ coincides with $J(z, \lambda)$ of the Appendix A of [28], that is

$$J_{z,\lambda}(z, \lambda, b) = \left(\begin{array}{c|c|c|c} \partial_{z_1} g^s(z) + \lambda\partial_{z_1}\gamma_b(g^s(z)) & \cdots & \partial_{z_{N-1}} g^s(z) + \lambda\partial_{z_{N-1}}\gamma_b(g^s(z)) & \gamma_b(g^s(z)) \end{array}\right).$$

In particular, for $\lambda = 0$,

$$J_{z,\lambda}(z, 0, b) = \left(\partial_{z_1} g^s(z) \middle| \cdots \middle| \partial_{z_{N-1}} g^s(z) \middle| \gamma_b(g^s(z)) \right)$$

is invertible since its $N-1$ first columns span the tangent space to $\partial\mathcal{O}$ at $g^s(z)$ and, since

$$\langle n(g^s(z)), \gamma_b(g^s(z))\rangle > 0,$$

its last column is non tangent to $\partial\mathcal{O}$. It follows that $J^s(z, 0, b)$ is also invertible, and we can therefore apply the inverse mapping theorem to G^s at $(z_0, 0, b_0)$ to obtain the existence of a neighbourhood V^{s,b_0} of (s, b_0) and C^1 mappings $V^{s,b_0} \ni (x, b) \mapsto p^{\gamma_b}(x) \in \partial\mathcal{O}$ and $V^{s,b_0} \ni (x, b) \mapsto d^{\gamma_b}(x)$ such that (15) holds for every $(x, b) \in V^{s,b_0}$. The compactness of $\partial\mathcal{O} \times B \subset \cup_{(s,b_0)\in\partial\mathcal{O}\times B} V^{s,b_0}$ enables to consider a finite number of $(s_i, (b_0)_i)$, $1 \le i \le k$, such that $\partial\mathcal{O} \times B \subset \cup_{i=1}^k V^{s_i,(b_0)_i}$. Then there exists $\bar{R} > 0$ such that $\{y \in \mathbb{R}^N \mid d(y, \partial\mathcal{O}) < \bar{R}\} \times B \subset \cup_{i=1}^k V^{s_i,(b_0)_i}$. In particular for any x such that $d(x, \partial\mathcal{O}) < \bar{R}$ and any $b \in B$, there exist at least a point $p^{\gamma_b}(x)$ and a scalar $d^{\gamma_b}(x)$ such that (15) holds. We claim that there exists $R \in (0, \bar{R})$ such that for any x satisfying $d(x, \partial\mathcal{O}) < R$ and any $b \in B$, $p^{\gamma_b}(x)$ is unique (and as a consequence $d^{\gamma_b}(x)$ is also unique). Assume that this is not the case. Then (considering for example $R = \frac{1}{k}$) one can build a sequence $(x_k, b_k)_{k\in\mathbb{N}}$ converging (after extracting a subsequence) to some point $(\hat{s}, \hat{b}) \in \partial\mathcal{O} \times B$ and such that for all $k \in \mathbb{N}$, x_k has two distinct projections $p_i^{\gamma_{b_k}}(x_k)$ with associated algebraic distances $d_i^{\gamma_{b_k}}(x_k)$, $i = 1, 2$. At the limit point $\hat{s}$, we consider $G^{\hat{s}}$ which is a local diffeomorphism on a neighbourhood of $(\hat{z}, 0, \hat{b})$ (with $g^{\hat{s}}(\hat{z}) = \hat{s}$). Since $x_k \to \hat{s} \in \partial\mathcal{O}$, then $p_i^{\gamma_{b_k}}(x_k) \to \hat{s}$ and $d_i^{\gamma_{b_k}}(x_k) \to 0$, $i = 1, 2$. Let $z_{i,k}$ be such that $g^{\hat{s}}(z_{i,k}) = p_i^{\gamma_{b_k}}(x_k)$ and $\lambda_{i,k} = d_i^{\gamma_{b_k}}(x_k)$, $i = 1, 2$. Then $(z_{i,k}, \lambda_{i,k}, b_k)_k$, $i = 1, 2$, are distinct sequences that both converge to $(\hat{z}, 0, \hat{b})$ and have the same image $G^{\hat{s}}(z_{i,k}, \lambda_{i,k}, b_k) = (x_k, b_k)$. This contradicts that $G^{\hat{s}}$ is a local diffeomorphism on a neighbourhood of $(\hat{z}, 0, \hat{b})$. □

For any $\varepsilon \ge 0$ let us define

$$D_\varepsilon = \{x \in \overline{\mathcal{O}} \mid d(x, \partial\mathcal{O}) > \varepsilon\}, \tag{16}$$

$$\partial D_\varepsilon = \{x \in \overline{\mathcal{O}} \mid d(x, \partial\mathcal{O}) = \varepsilon\}, \tag{17}$$

$$L_\varepsilon = \{x \in \overline{\mathcal{O}} \mid d(x, \partial\mathcal{O}) \le \varepsilon\}. \tag{18}$$

Now we focus on the existence of classical projections of $x \in L_\varepsilon$ onto ∂D_ε and the regularity of $L_\varepsilon \ni x \mapsto d(x, D_\varepsilon) \in \mathbb{R}$. The following result will be useful in order to study the stability of the scheme in Sect. 4.

Lemma 1 *Assume* **(H3)**. *Then the following hold:*

(i) *There exists $\eta > 0$ such that on L_η, the projection $p_{\partial\mathcal{O}}$ onto $\partial\mathcal{O}$ is well-defined and C^1.*

(ii) *The distance function* $L_\eta \ni x \mapsto d(x, \partial\mathcal{O}) \in \mathbb{R}$ *is* C^3*, and* $Dd(\cdot, \partial\mathcal{O})(x) = -n(p_{\partial\mathcal{O}}(x))$.

Let $\delta \in [0, \eta]$. *Then the following hold:*

(iii) ∂D_δ *is of class* C^3 *and, denoting by* $n_\delta(x)$ *the unit outward normal at* $x \in \partial D_\delta$*, we have* $n_\delta(x) = n(p_{\partial\mathcal{O}}(x))$.
(iv) *For every* $x \in L_\delta$*,* $p = p_{\partial\mathcal{O}}(x) - \delta n(p_{\partial\mathcal{O}}(x))$ *is a projection of* x *onto* ∂D_δ.
(v) *The function* $x \mapsto d(x, \partial D_\delta)$ *is of class* C^3 *on* L_δ *and* $d(x, \partial\mathcal{O}) + d(x, \partial D_\delta) = \delta$ *for every* $x \in L_\delta$.

Proof (i) and (ii) See [27, Lemma 14.16].
(iii) This follows from (ii) and (17).
(iv) & (v) Let us first show that $p \in \partial D_\delta$. We have $d(p, \partial\mathcal{O}) \leq |p - p_{\partial\mathcal{O}}(x)| = \delta$. Thus, $p \in L_\delta$ and, by (i), $p_{\partial\mathcal{O}}(x) = p_{\partial\mathcal{O}}(p)$, which implies that $d(p, \partial\mathcal{O}) = \delta$ and hence $p \in \partial D_\delta$. Since

$$x = p_{\partial\mathcal{O}}(x) - d(x, \partial\mathcal{O})n(p_{\partial\mathcal{O}}(x)),$$

we obtain $d(x, \partial D_\delta) \leq |p - x| = \delta - d(x, \partial\mathcal{O})$. Assume that $d(x, \partial D_\delta) < \delta - d(x, \partial\mathcal{O})$. Then there exists $p' \in \partial D_\delta$ such that $|x - p'| < \delta - d(x, \partial\mathcal{O})$. This implies that

$$\delta = d(p', \partial\mathcal{O}) \leq |p' - p_{\partial\mathcal{O}}(x)| \leq |p' - x| + |x - p_{\partial\mathcal{O}}(x)| < \delta,$$

which is impossible. Thus

$$|p - x| = d(x, \partial D_\delta) = \delta - d(x, \partial\mathcal{O}).$$

The first equality above implies that p is a projection of x onto ∂D_δ. Since $x \in L_\delta$ is arbitrary, the second equality above and (ii) imply that (v) holds. □

3 The fully discrete scheme

We introduce in this section a fully discrete SL scheme that approximates the unique viscosity solution to (HJB). Throughout this section, we assume that **(H1)–(H3)** are fulfilled.

3.1 Discretization of the space domain $\mathcal{O}$

Let us fix $\Delta x > 0$ and consider a polyhedral domain $\mathcal{O}_{\Delta x} \subseteq \mathbb{R}^N$ such that

$$d(\mathcal{O}, \mathcal{O}_{\Delta x}) = \inf \{|x - y| \mid x \in \mathcal{O},\ y \in \mathcal{O}_{\Delta x}\} \leq C(\Delta x)^2, \tag{19}$$

for some $C > 0$. A construction of such a domain $\mathcal{O}_{\Delta x}$ can be found in [9, Section 3] for $N = 2$ or $N = 3$, which explains the dimension constraint $N \leq 3$. However,

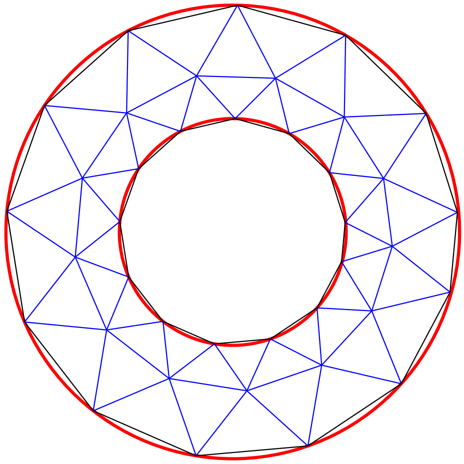

Fig. 1 An example of a domain $\mathcal{O}$ with a hole (boundary drawn with thick red line), an approximating polyhedral domain $\mathcal{O}_{\Delta x}$ (boundary drawn with black line), together with the triangulations $\mathbb{T}_{\Delta x}$(elements with blue and black sides) and $\widehat{\mathbb{T}}_{\Delta x}$ (curved elements with blue and red sides)

the results in the remainder of this article can be extended to $N > 3$, provided that a numerical domain $\mathcal{O}_{\Delta x}$ satisfying (19) exists. Let $\mathbb{T}_{\Delta x}$ be a triangulation of $\mathcal{O}_{\Delta x}$ consisting of simplicial finite elements T with vertices in $\mathcal{G}_{\Delta} = \{x_i \mid i = 1, \ldots, N_{\Delta x}\}$ (for some $N_{\Delta x} \in \mathbb{N}$). We assume that Δx is the mesh size, i.e. the maximum of the diameters of $\mathsf{T} \in \mathbb{T}_{\Delta x}$, all the vertices on $\partial \mathcal{O}_{\Delta x}$ belong to $\partial \mathcal{O}$, at most one face of each element $\mathsf{T} \in \mathbb{T}_{\Delta x}$, with at least one vertex in $\partial \mathcal{O}_{\Delta x}$, intersects $\partial \mathcal{O}_{\Delta x}$, and $\mathbb{T}_{\Delta x}$ satisfies the following regularity condition: there exists $\delta \in (0, 1)$, independent of Δx, such that each $\mathsf{T} \in \mathbb{T}_{\Delta x}$ is contained in a ball of radius $\Delta x/\delta$ and contains a ball of radius $\delta \Delta x$. As in [20], we introduce an auxiliary exact triangulation $\widehat{\mathbb{T}}_{\Delta x}$ of $\overline{\mathcal{O}}$ with vertices in $\mathcal{G}_{\Delta x}$. The boundary elements of $\widehat{\mathbb{T}}_{\Delta x}$ are allowed to be curved, we have

$$\overline{\mathcal{O}} = \bigcup_{\widehat{\mathsf{T}} \in \widehat{\mathbb{T}}_{\Delta x}} \widehat{\mathsf{T}}$$

and $\widehat{\mathbb{T}}_{\Delta x}$ differs from $\mathbb{T}_{\Delta x}$ only in the elements with vertices in $\partial \mathcal{O}$.

Denoting by p_{T} the projection on $\mathsf{T} \in \mathbb{T}_{\Delta x}$, the projection $p_{\Delta x} : \overline{\mathcal{O}} \to \overline{\mathcal{O}}_{\Delta x} \cap \overline{\mathcal{O}}$ is defined by

$$p_{\Delta x}(x) = p_{\mathsf{T}}(x), \quad \text{if } x \in \widehat{\mathsf{T}} \in \widehat{\mathbb{T}}_{\Delta x}$$
$$\text{and the element } \mathsf{T} \in \mathbb{T}_{\Delta x} \text{ has the same vertices than } \widehat{\mathsf{T}}.$$

In Fig. 1 we show an example of a domain $\mathcal{O} \subset \mathbb{R}^2$ together with a polyhedral domain $\mathcal{O}_{\Delta x}$, that approximates $\mathcal{O}$. We observe that in this case, the domain $\mathcal{O}_{\Delta x}$ is not contained in $\mathcal{O}$. In Fig. 2, we show two pairs of elements T and $\widehat{\mathsf{T}}$, which share the same vertices, in two different cases. The pair on the left corresponds to a couple of elements with two vertices on the convex part of the boundary. In this case, the local operator p_{T} projects the points of the curved element $\widehat{\mathsf{T}}$ onto the affine element T. In the other case, the pair of elements on the right has two vertices on a concave part of the domain and therefore, since $\widehat{\mathsf{T}} \subset \mathsf{T}$, $p_{\mathsf{T}}(x) = x$ for any $x \in \widehat{\mathsf{T}}$.

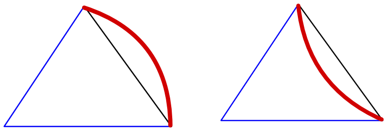

Fig. 2 Examples of two pairs of elements T (with blue and black sides) and $\widehat{\mathsf{T}}$ (with blue and thick red sides) that share the same vertices. In both cases, $p_{\Delta x}$ is defined on the curved element $\widehat{\mathsf{T}}$. On the right, $p_{\Delta x}(x) = x$ for all $x \in \widehat{\mathsf{T}}$ while, on the left, $p_{\Delta x}(x) = p_{\mathsf{T}}(x) \neq x$ for all $x \in \widehat{\mathsf{T}} \backslash \mathsf{T}$

Set $\mathcal{I}_{\Delta x} = \{1, \dots, N_{\Delta x}\}$ and denote by $\{\psi_i \,|\, i \in \mathcal{I}_{\Delta x}\}$ the linear finite element $\mathbb{P}_1$ basis function on $\mathcal{T}_{\Delta x}$. More precisely, for each $i \in \mathcal{I}_{\Delta x}$, $\psi_i : \mathcal{O}_{\Delta x} \to \mathbb{R}$ is a continuous function, affine on each $\mathsf{T} \in \mathcal{T}_{\Delta x}$, $0 \leq \psi_i \leq 1$, $\psi_i(x_i) = 1$, $\psi_i(x_j) = 0$ for all $i, j \in \mathcal{I}_{\Delta x}$ with $i \neq j$, and $\sum_{i=1}^{N_{\Delta x}} \psi_i(x) = 1$ for all $x \in \mathcal{O}_{\Delta x}$. For any $\phi : \mathcal{G}_{\Delta x} \to \mathbb{R}$ its linear interpolation $I[\phi]$ on the mesh $\widehat{\mathbb{T}}_{\Delta x}$ is defined by

$$I\left[\phi\right](x) := \sum_{i=1}^{N_{\Delta x}} \psi_i(p_{\Delta x}(x))\phi(x_i), \quad \text{for all } x \in \overline{\mathcal{O}}. \tag{20}$$

Lemma 2 *Let $\phi \in C^2(\overline{\mathcal{O}})$ and denote by $\phi|_{\mathcal{G}_{\Delta x}}$ its restriction to $\mathcal{G}_{\Delta x}$. Then there exists a constant $C_\phi > 0$, independent of Δx, such that*

$$\sup_{x \in \overline{\mathcal{O}}} \left|\phi(x) - I\left[\phi|_{\mathcal{G}_{\Delta x}}\right](x)\right| \leq C_\phi (\Delta x)^2. \tag{21}$$

Proof Let $x \in \overline{\mathcal{O}}$ and let $\mathsf{T} \in \mathbb{T}_{\Delta x}$ and $\widehat{\mathsf{T}} \in \widehat{\mathbb{T}}_{\Delta x}$ be two elements having the same vertices and such that $x \in \widehat{\mathsf{T}}$. By the triangular inequality

$$|\phi(x) - I\left[\phi|_{\mathcal{G}_{\Delta x}}\right](x)| \leq |\phi(x) - \phi(p_{\mathsf{T}}(x))| + |\phi(p_{\mathsf{T}}(x)) - I\left[\phi|_{\mathcal{G}_{\Delta x}}\right](x)|. \tag{22}$$

Using that ϕ is Lipschitz, we deduce from (19) the existence of $C_1 > 0$, independent of Δx and $x \in \overline{\mathcal{O}}$, such that $|\phi(x) - \phi(p_{\mathsf{T}}(x))| \leq C_1(\Delta x)^2$. In addition, by standard error estimates for $\mathbb{P}_1$ interpolation (see for instance [16]) and (20), there exists $C_2 > 0$, independent of Δx and $x \in \overline{\mathcal{O}}$, such that $|\phi(p_{\mathsf{T}}(x)) - I\left[\phi|_{\mathcal{G}_{\Delta x}}\right](x)| \leq C_2(\Delta x)^2$. Relation (21) follows from these two estimates and (22). □

3.2 A semi-Lagrangian scheme

Let $\Delta t > 0$, set $N_{\Delta t} := \lfloor T/\Delta t \rfloor$, $\mathcal{I}_{\Delta t} := \{0, \dots, N_{\Delta t}\}$ and $\mathcal{I}^*_{\Delta t} := \mathcal{I}_{\Delta t} \backslash \{N_T\}$. We define the time grid $\mathcal{G}_{\Delta t} := \{t_k \,|\, t_k = k\Delta t, \;\; k \in \mathcal{I}_{\Delta t}\}$.

Given $(k, i) \in \mathcal{I}^*_{\Delta t} \times \mathcal{I}_{\Delta x}$, $a \in A$, and $\ell = 1, \dots, N_\sigma$, we define the discrete characteristics

$$y^{\pm,\ell}_{k,i}(a) = x_i + \Delta t \mu\left(t_k, x_i, a\right) \pm \sqrt{N_\sigma \Delta t} \sigma^\ell(t_k, x_i, a). \tag{23}$$

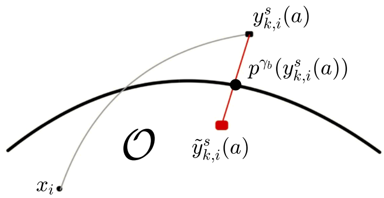

Fig. 3 Reflection: reflected characteristic $\tilde{y}^s_{k,i}(a)$ (red square) starting from x_i (black circle), which exits from $\mathcal{O}$ and arrives in $y^s_{k,i}(a)$ (black square). The red segment represents the oblique direction γ_b and the black circle the projected point $p^{\gamma_b}(y^s_{k,i}(a))$

For any $\delta > 0$ we set

$$(\partial\mathcal{O})_\delta := \{x \in \mathbb{R}^N \mid d(x, \partial\mathcal{O}) < \delta\}.$$

By Proposition 1, there exist $R > 0$ and two C^1 functions $(\partial\mathcal{O})_R \times B \ni (x, b) \mapsto p^{\gamma_b}(x) \in \partial\mathcal{O}$ and $(\partial\mathcal{O})_R \times B \ni (x, b) \mapsto d^{\gamma_b}(x) \in \mathbb{R}$, uniquely determined, such that

$$x = p^{\gamma_b}(x) + d^{\gamma_b}(x)\gamma_b(p^{\gamma_b}(x)), \quad \text{for all } (x, b) \in (\partial\mathcal{O})_R \times B. \tag{24}$$

Set $\mathcal{I} = \{+, -\} \times \{1, \dots, N_\sigma\}$ and let $\bar{c} > 0$ be a fixed constant. From (24), there exists $\overline{\Delta t} > 0$ such that for all $\Delta t \in [0, \overline{\Delta t}]$, $(k, i) \in \mathcal{I}^*_{\Delta t} \times \mathcal{I}_{\Delta x}$, $a \in A$, $b \in B$, and $s \in \mathcal{I}$, the *reflected characteristic*

$$\tilde{y}^s_{k,i}(a, b) := \begin{cases} y^s_{k,i}(a) & \text{if } y^s_{k,i}(a) \in \overline{\mathcal{O}}, \\ p^{\gamma_b}(y^s_{k,i}(a)) - \bar{c}\sqrt{\Delta t}\,\gamma_b(p^{\gamma_b}(y^s_{k,i}(a))) & \text{otherwise} \end{cases} \tag{25}$$

is well-defined.

Remark 3 The introduction of parameter $\bar{c}$ is inspired from [42] and its role is to obtain a reflection of the characteristic $y^s_{k,i}(a)$ inside $\mathcal{O}$, whose distance to $\partial\mathcal{O}$, in the direction $\gamma_b(p^{\gamma_b}(y^s_{k,i}(a)))$, is of order $\sqrt{\Delta t}$. This property will play a key role in the proofs of our main results (see Remark 5).

In Fig. 3 we illustrate how the reflected characteristic is computed from the projection $p^{\gamma_b}(y^s_{k,i}(a))$ of $y^s_{k,i}(a)$ onto $\partial\mathcal{O}$ parallel to γ_b.

Let us set

$$\tilde{d}^s_{k,i}(a, b) := \begin{cases} 0 & \text{if } y^s_{i,k}(a) \in \overline{\mathcal{O}}, \\ d^{\gamma_b}(y^s_{k,i}(a)) + \bar{c}\sqrt{\Delta t} & \text{otherwise}, \end{cases} \tag{26}$$

$$\tilde{g}^s_{k,i}(a, b) := \begin{cases} 0 & \text{if } y^s_{k,i}(a) \in \overline{\mathcal{O}}, \\ g\left(t_k, p^{\gamma_b}\left(y^s_{k,i}(a)\right), b\right) & \text{otherwise}. \end{cases} \tag{27}$$

Notice that if $y^s_{k,i}(a) \notin \overline{\mathcal{O}}$, then (24), (25), and (26) imply that

$$\tilde{y}^s_{k,i}(a,b) = y^s_{k,i}(a) - \tilde{d}^s_{k,i}(a,b)\gamma_b\left(p^{\gamma_b}(y^s_{k,i}(a))\right). \tag{28}$$

For $(k,i) \in \mathcal{I}^*_{\Delta t} \times \mathcal{I}_{\Delta x}$ and $\Phi : \mathcal{G}_{\Delta x} \to \mathbb{R}$, let us define $\mathcal{S}_{k,i}[\Phi] : A \times B \to \mathbb{R}$ by

$$\mathcal{S}_{k,i}[\Phi](a,b) := \frac{1}{2N_\sigma}\sum_{s\in\mathcal{I}}\left[I[\Phi](\tilde{y}^s_{k,i}(a,b)) + \tilde{d}^s_{k,i}(a,b)\tilde{g}^s_{k,i}(a,b)\right] + \Delta t f(t_k, x_i, a) \tag{29}$$

and set

$$S_{k,i}[\Phi] := \inf_{a\in A,\, b\in B} \mathcal{S}_{k,i}[\Phi](a,b). \tag{30}$$

Given $U \in B(\mathcal{G}_{\Delta t} \times \mathcal{G}_{\Delta x})$, let us denote $U_k = \{U_{k,i}\}_{i\in\mathcal{G}_{\Delta x}}$. In the remainder of this work, we will consider the following fully discrete SL scheme to approximate the solution to (HJB):

$$\begin{aligned} U_{k,i} &= S_{k,i}\left[U_{k+1}\right], \quad \text{for } (k,i) \in \mathcal{I}^*_{\Delta t} \times \mathcal{I}_{\Delta x}, \\ U_{N_{\Delta t},i} &= \Psi(x_i), \qquad \text{for } i \in \mathcal{I}_{\Delta x}, \end{aligned} \tag{HJB$_{\text{disc}}$}$$

The main difference with respect to the SL scheme when $\mathcal{O} = \mathbb{R}^d$ (see e.g. [14]) is the presence of the terms $\tilde{y}^s_{k,i}(a,b)$, rather than $y^s_{k,i}(a)$, and $\tilde{d}^s_{k,i}(a,b)\tilde{g}^s_{k,i}(a,b)$ in the definition of $\mathcal{S}_{k,i}[\Phi](a,b)$. These additional terms will make appear the boundary condition in the expansion of $S_{k,i}[\phi|_{\mathcal{G}_{\Delta x}}] - \phi(x_i)$, where $\phi : \overline{\mathcal{O}} \to \mathbb{R}$ is smooth, at grid points x_i where $y^s_{k,i}(a) \notin \mathcal{O}$ for some $a \in A$ and $s \in \mathcal{I}$ (see Proposition 3 below).

3.3 Probabilistic interpretation of the scheme

The fully-discrete SL scheme to approximate the solution to (HJB) in the unbounded case, i.e. $\mathcal{O} = \mathbb{R}^d$, has a natural interpretation in terms of a discrete time, finite state, Markov control process (see e.g. [14, Section 3]). We show below that a similar interpretation holds for (HJB$_{\text{disc}}$). The latter will play an important role in the stability analysis of (HJB$_{\text{disc}}$) presented in the next section. Given $k \in \mathcal{I}^*_{\Delta t}$, $a \in A$, and $b \in B$, let us define the controlled transition law

$$p_{k,i,j}(a,b) := \frac{1}{2N_\sigma}\sum_{s\in\mathcal{I}}\psi_j(\tilde{y}^s_{k,i}(a,b)), \quad \text{for all } i,\ j \in \mathcal{I}_{\Delta x}. \tag{31}$$

We say that $(\pi_k)_{k\in\mathcal{I}^*_{\Delta t}}$ is a $N_{\Delta t}$-*policy* if for all $k \in \mathcal{I}^*_{\Delta t}$ we have $\pi_k : \mathcal{G}_{\Delta x} \to A \times B$. The set of $N_{\Delta t}$-policies is denoted by $\Pi_{N_{\Delta t}}$. Let us fix $k \in \mathcal{I}^*_{\Delta t}$ and, for notational convenience, set $\mathfrak{X}_k = \mathcal{G}_{\Delta x}^{N_{\Delta t}-k+1}$. Associated to $x_i \in \mathcal{G}_{\Delta x}$ and $\pi \in \Pi_{N_{\Delta t}}$, there exists a probability measure $\mathbb{P}^{k,x_i,\pi}$ on $2^{\mathfrak{X}_k}$ (the powerset of $\mathfrak{X}_k$) and a Markov chain $\{X_m \mid m = k, \ldots, N_{\Delta t}\}$, with state space $\mathcal{G}_{\Delta x}$, such that

$$\mathbb{P}^{k,x_i,\pi}(X_k = x_i) = 1 \quad \text{and} \quad \mathbb{P}^{k,x_i,\pi}(X_{m+1} = x_j \mid X_m = x_i) = p_{m,i,j}(\pi_m(x_i)), \tag{32}$$

for $m = k, \ldots, N_{\Delta t} - 1$. Now, consider a family $\{\xi_{k+1}, \ldots, \xi_{N_{\Delta t}}\}$ of $\mathbb{R}^{N_\sigma}$-valued independent random variables, which are also independent of $\{X_m \mid m = k, \ldots, N_{\Delta t}\}$, and with common distribution given by

$$\mathbb{P}(\xi_m = \pm e_\ell) = \frac{1}{2N_\sigma}, \quad \text{for } m = k+1, \ldots, N_{\Delta t} \text{ and } \ell = 1, \ldots, N_\sigma,$$

where e_ℓ denotes the ℓ-th canonical vector of $\mathbb{R}^{N_\sigma}$. By a slight abuse of notation (see (23)), for $m = k, \ldots, N_{\Delta t} - 1$, $x_i \in \mathcal{G}_{\Delta x}$, and $a \in A$, let us set

$$y_m(x_i, a) = x_i + \Delta t \mu(t_m, x_i, a) + \sqrt{N_\sigma \Delta t}\sigma(t_m, x_i, a)\xi_{m+1}. \tag{33}$$

For $m = k, \ldots, N_{\Delta t} - 1$, $x_i \in \mathcal{G}_{\Delta x}$, $a \in A$, and $b \in B$, define the random variable

$$h(t_m, x_i, a, b) = \begin{cases} 0 & \text{if } y_m(x_i, a) \in \overline{\mathcal{O}}, \\ \left(d^{\gamma_b}(y_m(x_i, a)) + \bar{c}\sqrt{\Delta t}\right) g(t_m, p^{\gamma_b}(y_m(x_i, a)), b) & \text{otherwise.} \end{cases} \tag{34}$$

For all $i \in \mathcal{I}_{N_{\Delta x}}$ and $\pi \in \Pi_{N_{\Delta t}}$, let us define

$$\begin{aligned} J_{k,i}(\pi) &= \mathbb{E}_{\mathbb{P}^{k,x_i,\pi}}\left(\textstyle\sum_{m=k}^{N_{\Delta t}-1}\left[\Delta t f(t_m, X_m, \alpha_m) + h(t_m, X_m, \alpha_m, \beta_m)\right] + \Psi\left(X_{N_{\Delta t}}\right)\right), \\ J_{N_{\Delta t},i}(\pi) &= \Psi(x_i), \end{aligned}$$

where, for notational convenience, we have denoted, respectively, by α_m and β_m the first N_A and the last N_B coordinates of $\pi_m(X_m)$. Notice that, by construction and (29), we have that

$$J_{k,i}(\pi) = \mathcal{S}_{k,i}[J_{k+1}(\pi)](\alpha_k, \beta_k).$$

Moreover, setting

$$\begin{aligned} \hat{U}_{k,i} &= \inf_{\pi \in \Pi_{N_{\Delta t}}} \; J_{k,i}(\pi), \\ \hat{U}_{N_{\Delta t},i} &= \Psi(x_i), \end{aligned}$$

for all $i \in \mathcal{G}_{\Delta x}$, the dynamic programming principle (see e.g. [29, Theorem 12.1.5]) implies that $\{\hat{U}_{k,i} \mid k \in \mathcal{I}_{\Delta t},\ i \in \mathcal{I}_{\Delta x}\}$ satisfies (HJB_{disc}). Since the latter has a unique solution, we deduce that $U_{k,i} = \hat{U}_{k,i}$ for all $k \in \mathcal{I}_{\Delta t}$ and $i \in \mathcal{I}_{\Delta x}$.

Remark 4 When $\mathcal{O} = \mathbb{R}^d$, the semi-Lagrangian scheme studied in [14] can be described in terms of a Markov chain with controlled transition probabilities

$$p_{k,i,j}(a) = \frac{1}{2N_\sigma} \sum_{s\in\mathcal{I}} \psi_j(y^s_{k,i}(a)), \quad \text{for all } i,\ j \in \mathbb{Z}^d \tag{35}$$

(since there is no boundary, there is no control $b \in B$) and allows to approximate the value function of a stochastic optimal control problem with state space $\mathbb{R}^d$. In the case of a bounded domain and oblique reflection on the boundary, the characteristic $y^s_{k,i}(a)$ is replaced by the reflected one $\tilde{y}^s_{k,i}(a, b)$ (see Fig. 3), which yields (32). In this manner, scheme (HJB$_{\text{disc}}$) can thus be interpreted as a Markov chain discretization of a stochastic control problem with state space $\overline{\mathcal{O}}$ and oblique reflection at the boundary (see e.g. [12]).

We refer the reader to [34, Chapter 5, Section 7] for a related Markov chain discretization of a stochastic control problem with reflection leading to a finite difference scheme.

4 Properties of the fully discrete scheme

In this section, we establish some basic properties of (HJB$_{\text{disc}}$).

Proposition 2 *The following hold:*

(i) (Monotonicity) *For all $U, V : \mathcal{G}_{\Delta x} \to \mathbb{R}$ with $U \leq V$, we have*

$$\mathcal{S}_{k,i}[U] \leq \mathcal{S}_{k,i}[V], \quad \text{for } k \in \mathcal{I}^*_{\Delta t} \text{ and } i \in \mathcal{I}_{\Delta x}.$$

(ii) (Commutation by constant) *For any $c \in \mathbb{R}$ and $U : \mathcal{G}_{\Delta x} \to \mathbb{R}$,*

$$\mathcal{S}_{k,i}[U + c] = \mathcal{S}_{k,i}[U] + c, \quad \text{for } k \in \mathcal{I}^*_{\Delta t} \text{ and } i \in \mathcal{I}_{\Delta x}.$$

Proof Both assertions follow directly from (29) and (HJB$_{\text{disc}}$). □

We show in Proposition 3 below a consistency result for (HJB$_{\text{disc}}$). For this purpose, let us set

$$\begin{aligned} \mathcal{H}(t, x, p, M, a) &= -\frac{1}{2}\mathrm{Tr}\left(\sigma(t, x, a)\sigma(t, x, a)^\top M\right) - \langle \mu(t, x, a), p\rangle - f(t, x, a), \\ &\qquad \text{for } (t, x, p, M, a) \in \overline{\mathcal{O}}_T \times \mathbb{R}^N \times \mathbb{R}^{N\times N_\sigma} \times A, \qquad (36) \\ \text{Ł}(t, x, p, b) &= \langle \gamma(x, b), p\rangle - g(t, x, b), \\ &\qquad \text{for } (t, x, p, b) \in [0, T] \times \partial\mathcal{O} \times \mathbb{R}^N \times B, \qquad (37) \end{aligned}$$

and for all $k \in \mathcal{I}^*_{\Delta t}$, $i \in \mathcal{I}_{\Delta x}$, $s \in \mathcal{I}$, $q \in \mathbb{R}^N$, $a \in A$, and $b \in B$, define

$$\tilde{\text{Ł}}^s_{k,i}(q, a, b) := \begin{cases} 0 & \text{if } y^s_{k,i}(a) \in \overline{\mathcal{O}}, \\ \text{Ł}\left(t_k, p^{\gamma_b}(y^s_{k,i}(a)), q, b\right) & \text{otherwise.} \end{cases} \tag{38}$$

The following consistency result relates the operator $S_{k,i}$ with the Hamiltonian and the boundary condition in (HJB).

Proposition 3 (Consistency) *Let $\phi \in C^3\left(\overline{\mathcal{O}}\right)$ and denote by $\phi|_{\mathcal{G}_{\Delta x}}$ its restriction to $\mathcal{G}_{\Delta x}$. Then the following hold:*

(i) *For all $k \in \mathcal{I}^*_{\Delta t}$, $i \in \mathcal{I}_{\Delta x}$, $a \in A$, and $b \in B$, we have*

$$
\begin{aligned}
&\mathcal{S}_{k,i}[\phi|_{\mathcal{G}_{\Delta x}}](a,b) - \phi(x_i) \\
&\quad = -\Delta t \mathcal{H}(t_k, x_i, D\phi(x_i), D^2\phi(x_i), a) \\
&\qquad - \frac{1}{2N_\sigma}\sum_{s\in\mathcal{I}} \tilde{d}^s_{k,i}(a,b)\left(\tilde{\text{Ł}}^s_{k,i}(D\phi(x_i), a, b) - \sqrt{\Delta t} K^s_{k,i}(a,b)\right) \\
&\qquad + O\left(\Delta t\sqrt{\Delta t} + (\Delta x)^2\right),
\end{aligned}
\tag{39}
$$

where the set of constants $\{K^s_{k,i}(a,b) \,|\, k \in \mathcal{I}^*_{\Delta t},\ i \in \mathcal{I}_{\Delta x},\ s \in \mathcal{I},\ a \in A,\ b \in B\}$ is bounded, independently of $(\Delta t, \Delta x)$.

(ii) *For all $k \in \mathcal{I}^*_{\Delta t}$ and $i \in \mathcal{I}_{\Delta x}$, we have*

$$
\begin{aligned}
&S_{k,i}[\phi|_{\mathcal{G}_{\Delta x}}] - \phi(x_i) \\
&\quad = -\sup_{a\in A,\ b\in B}\Bigg\{\Delta t \mathcal{H}(t_k, x_i, D\phi(x_i), D^2\phi(x_i), a) \\
&\qquad + \tfrac{1}{2N_\sigma}\sum_{s\in\mathcal{I}} \tilde{d}^s_{k,i}(a,b)\left(\tilde{\text{Ł}}^s_{k,i}(D\phi(x_i), a, b) - \sqrt{\Delta t} K^s_{k,i}(a,b)\right)\Bigg\} \\
&\qquad + O\left(\Delta t\sqrt{\Delta t} + (\Delta x)^2\right).
\end{aligned}
$$

Proof In what follows, we denote by $C > 0$ a generic constant, which is independent of k, i, s a, b, Δt and Δx. Since assertion (ii) follows directly from (i), we only show the latter.

For every $s \in \mathcal{I}$, (23) and (26) imply that $0 \le \tilde{d}^s_{k,i}(a,b) \le C\sqrt{\Delta t}$. Thus, by (23), (28), and a second order Taylor expansion of ϕ around x_i, for every $\ell = 1, \ldots, N_\sigma$, we have

$$
\begin{aligned}
&\phi\left(\tilde{y}^{\pm,\ell}_{k,i}(a,b)\right) \\
&= \phi(x_i) + \Delta t \langle D\phi(x_i), \mu(t_k, x_{i,a})\rangle + \frac{N_\sigma \Delta t}{2}\langle D^2\phi(x_i)\sigma^\ell(t_k, x_{i,a}), \sigma^\ell(t_k, x_{i,a})\rangle \\
&\quad \pm\sqrt{N_\sigma \Delta t}\langle D\phi(x_i), \sigma^\ell(t_k, x_i, a)\rangle - \tilde{d}^{\pm,\ell}_{k,i}(a,b)\left\langle D\phi(x_i), \tilde{\gamma}^{\pm,\ell}_{k,i}(a,b)\right\rangle \\
&\quad + \frac{\left(\tilde{d}^{\pm,\ell}_{k,i}(a,b)\right)^2}{2}\left\langle D^2\phi(x_i)\tilde{\gamma}^{\pm,\ell}_{k,i}(a,b), \tilde{\gamma}^{\pm,\ell}_{k,i}(a,b)\right\rangle \\
&\quad \mp\sqrt{N_\sigma \Delta t}\tilde{d}^{\pm,\ell}_{k,i}(a,b)\left\langle D^2\phi(x_i)\tilde{\gamma}^{\pm,\ell}_{k,i}(a,b), \sigma^\ell(t_k, x_{i,a})\right\rangle \\
&\quad + O\left(\Delta t\sqrt{\Delta t}\right),
\end{aligned}
$$

where, for every $s \in \mathcal{I}$,

$$\tilde{\gamma}^s_{k,i}(a,b) := \begin{cases} 0 & \text{if } y^s_{k,i}(a) \in \overline{\mathcal{O}}, \\ \gamma_b\left(p^{\gamma_b}(y^s_{k,i}(a))\right) & \text{otherwise.} \end{cases}$$

This implies that

$$\begin{aligned} &\tfrac{1}{2}\phi\left(\tilde{y}^{+,\ell}_{k,i}(a,b)\right) + \tfrac{1}{2}\phi\left(\tilde{y}^{-,\ell}_{k,i}(a,b)\right) \\ &\quad = \phi(x_i) + \Delta t\,\langle D\phi(x_i), \mu(t_k,x_i,a)\rangle + \tfrac{N_\sigma \Delta t}{2}\left\langle D^2\phi(x_i)\sigma^\ell(t_k,x_i,a), \sigma^\ell(t_k,x_i,a)\right\rangle \\ &\qquad - \tilde{d}^{+,\ell}_{k,i}(a,b)\left(\left\langle D\phi(x_i), \tilde{\gamma}^{+,\ell}_{k,i}(a,b)\right\rangle - \sqrt{\Delta t}K^{+,\ell}_{k,i}(a,b)\right) \\ &\qquad - \tilde{d}^{-,\ell}_{k,i}(a,b)\left(\left\langle D\phi(x_i), \tilde{\gamma}^{-,\ell}_{k,i}(a,b)\right\rangle - \sqrt{\Delta t}K^{-,\ell}_{k,i}(a,b)\right) + O\left(\Delta t\sqrt{\Delta t}\right), \end{aligned} \tag{40}$$

where

$$\begin{aligned} K^{\pm,\ell}_{k,i}(a,b) := \frac{\tilde{d}^{\pm,\ell}_{k,i}(a,b)}{2\sqrt{\Delta t}}\langle D^2\phi(x_i)\tilde{\gamma}^{\pm,\ell}_{k,i}(a,b), \tilde{\gamma}^{\pm,\ell}_{k,i}(a,b)\rangle \\ \mp\sqrt{N_\sigma}\langle D^2\phi(x_i)\tilde{\gamma}^{\pm,\ell}_{k,i}(a,b), \sigma^\ell(t_k,x_i,a)\rangle. \end{aligned} \tag{41}$$

Multiplying (40) by $1/N_\sigma$ and taking the sum over $s \in \mathcal{I}$, we obtain

$$\begin{aligned} &\tfrac{1}{2N_\sigma}\sum_{s\in\mathcal{I}}\phi(\tilde{y}^s_{k,i}(a,b)) \\ &= \phi(x) + \Delta t\langle D\phi(x_i), \mu(t_k,x_i,a)\rangle + \tfrac{\Delta t}{2}\mathrm{Tr}\left(\sigma(t_k,x_i,a)\sigma(t_k,x_i,a)^T D^2\phi(x_i)\right) \\ &\quad - \tfrac{1}{2N_\sigma}\sum_{s\in\mathcal{I}}\tilde{d}^s_{k,i}(a,b)\left(\left\langle D\phi(x_i), \tilde{\gamma}^s_{k,i}(a,b)\right\rangle - \sqrt{\Delta t}K^s_{k,i}(a,b)\right) \\ &\quad + O\left(\Delta t\sqrt{\Delta t}\right), \end{aligned}$$

which, by Lemma 2, yields

$$\begin{aligned} &\tfrac{1}{2N_\sigma}\sum_{s\in\mathcal{I}}I[\phi|_{\mathcal{G}_{\Delta x}}](\tilde{y}^s_{k,i}(a,b)) \\ &= \phi(x) + \Delta t\langle D\phi(x_i), \mu(t_k,x_i,a)\rangle + \tfrac{\Delta t}{2}\mathrm{Tr}\left(\sigma(t_k,x_i,a)\sigma(t_k,x_i,a)^T D^2\phi(x_i)\right) \\ &\quad - \tfrac{1}{2N_\sigma}\sum_{s\in\mathcal{I}}\tilde{d}^s_{k,i}(a,b)\left(\left\langle D\phi(x_i), \tilde{\gamma}^s_{k,i}(a,b)\right\rangle - \sqrt{\Delta t}K^s_{k,i}(a,b)\right) \\ &\quad + O\left(\Delta t\sqrt{\Delta t} + (\Delta x)^2\right). \end{aligned}$$

The result follows from the previous expression, (29), (36) and (38). □

Remark 5 Assertion (i) in the previous consistency result shows that the first and second main terms in the expansion of $\mathcal{S}_{k,i}[\phi|_{\mathcal{G}_{\Delta x}}](a,b) - \phi(x_i)$ involve the function $\mathcal{H}$, multiplied by Δt, and the boundary terms $\tilde{\mathrm{L}}^s_{k,i}$, multiplied by $\tilde{d}^s_{k,i}(a,b)$, respectively. Notice that if $y^s_{k,i}(a,b) \notin \mathcal{O}$, then the presence of $\bar{c}$ in (26) implies that $\tilde{d}^s_{k,i}(a,b)$ is of order $\sqrt{\Delta t}$. This last property will be crucial in order to establish the stability of the scheme (see Lemma 3 and Proposition 4) and, in Proposition 5, that the upper and lower half-relaxed limits of solutions to the scheme, as the discretization parameters tend to zero, are viscosity sub- and supersolutions to (1), respectively. In turn, by the comparison principle in Remark 2(i), this will imply that solutions to the scheme converge to the unique viscosity solution to (HJB) (see Theorem 2).

For $k \in \mathcal{I}^*_{\Delta t}$ and $a \in A$, let us define

$$(\forall\, k \in \mathcal{I}^*_{\Delta t}, \forall\, a \in A) \quad \Gamma_k(a) := \{x_i \in \mathcal{G}_{\Delta x} \mid \exists\, s \in \mathcal{I},\ y^s_{k,i}(a) \notin \overline{\mathcal{O}}\}, \tag{42}$$

and recall from Sect. 3.3 that given $x_i \in \mathcal{G}_{\Delta x}$ and a policy $\pi \in \Pi_{N_{\Delta t}}$, the Markov chain $\{X_m \mid m = k, \dots, N_{\Delta t}\}$ is defined by the transition probabilities (32). As in Sect. 3.3, we denote by α_m and β_m ($m = k, \dots, N_{\Delta t} - 1$), respectively, the first N_A and the last N_B coordinates of $\pi_m(X_m)$. Finally, given $D \subset \mathbb{R}^d$, we denote by $\mathbb{I}_D$ the indicator function of D, i.e. $\mathbb{I}_D(x) = 1$, if $x \in D$, and $\mathbb{I}_D(x) = 0$, otherwise.

The following technical result will be useful to establish the stability of (HJB_{disc}).

Lemma 3 *The following holds:*

$$\sup_{k \in \mathcal{I}^*_{\Delta t},\ i \in \mathcal{I}^*_{\Delta x}, \pi \in \Pi_{N_{\Delta t}}} \mathbb{E}_{\mathbb{P}^{k,x_i,\pi}} \left(\sum_{m=k}^{N_T - 1} \mathbb{I}_{\Gamma_m(\alpha_m)}\big(X_m\big) \right) \leq \frac{C}{\sqrt{\Delta t}}, \tag{43}$$

where $C > 0$ is independent of $(\Delta t, \Delta x)$ as long as Δt is small enough and $(\Delta x)^2/\Delta t$ is bounded.

Proof The argument of the proof is inspired from [41, Lemma 1]. Let $\varepsilon > 0$, set

$$\begin{aligned} D_\varepsilon &= \{x \in \overline{\mathcal{O}} \mid d(x, \partial\mathcal{O}) > \varepsilon\}, \quad \partial D_\varepsilon = \{x \in \overline{\mathcal{O}} \mid d(x, \partial\mathcal{O}) = \varepsilon\}, \\ L_\varepsilon &= \{x \in \overline{\mathcal{O}} \mid d(x, \partial\mathcal{O}) \leq \varepsilon\}, \end{aligned}$$

and define $\overline{\mathcal{O}} \ni x \mapsto w_\varepsilon(x) = d^2(x, D_\varepsilon) \in \mathbb{R}$. By Lemma 1(v), there exists $\eta > 0$ such that $w_\eta \in C^3(\overline{\mathcal{O}} \setminus \partial D_\eta)$ with bounded third order derivatives on the connected components of $\overline{\mathcal{O}} \setminus \partial D_\eta$. Let us fix this η and, for notational convenience, let us write $w = w_\eta$. Let $M > 0$ and, for any $k \in \mathcal{I}_{\Delta t}$, define

$$\overline{\mathcal{O}} \ni x \mapsto W_k(x) = \begin{cases} M(T - t_k) + w(x) & \text{if } k \in \mathcal{I}^*_{\Delta t}, \\ 0 & \text{if } k = N_{\Delta t} \end{cases} \in \mathbb{R}. \tag{44}$$

By (29), with $f \equiv 0$ and $g \equiv 0$, for all $a \in A$ and $b \in B$, we have

$$\mathcal{S}_{k,i}[W_{k+1}|_{\mathcal{G}_{\Delta x}}](a,b) - W_k(x_i) = -M\Delta t + \mathcal{S}_{k,i}[w|_{\mathcal{G}_{\Delta x}}](a,b) - w(x_i), \tag{45}$$

$$= -M\Delta t + \frac{1}{2N_\sigma}\sum_{s\in\mathcal{I}} I[w](\tilde{y}^s_{k,i}(a,b)) - w(x_i). \tag{46}$$

Moreover, assumption **(H2)** implies the existence of $\overline{C} > 0$ such that

$$\sup\left\{ |y^s_{k,i}(a) - x_i| \;\middle|\; k \in \mathcal{I}^*_{\Delta t},\ i \in \mathcal{I}_{\Delta x},\ a \in A,\ s \in \mathcal{I} \right\} \leq \overline{C}\sqrt{\Delta t}. \tag{47}$$

Now, let us fix $k \in \mathcal{I}^*_{\Delta t}$, $i \in \mathcal{I}_{\Delta x}$, $a \in A$, and $b \in B$. We have the following cases.

(i) $\underline{x_i \notin \Gamma_k(a) \text{and} d(x_i, \partial D_\eta) \geq \overline{C}\sqrt{\Delta t}}$. The first condition implies that $y^s_{k,i}(a) \in \overline{\mathcal{O}}$, for any $s \in \mathcal{I}$, and, hence, (25) yields $\tilde{y}^s_{k,i}(a,b) = y^s_{k,i}(a)$. The condition $d(x_i, \partial D_\eta) \geq \overline{C}\sqrt{\Delta t}$, (47), and standard error estimates for $\mathbb{P}_1$ interpolation (see for instance [16]), imply that

$$I[w](\tilde{y}^s_{k,i}(a,b)) = w(\tilde{y}^s_{k,i}(a,b)) + O((\Delta x)^2) = w(y^s_{k,i}(a)) + O((\Delta x)^2).$$

Since, by second order Taylor expansion, $\frac{1}{2N_\sigma}\sum_{s\in\mathcal{I}} w(y^s_{k,i}(a)) - w(x_i) = O(\Delta t)$, (46) yields

$$\mathcal{S}_{k,i}[W_{k+1}|_{\mathcal{G}_{\Delta x}}](a,b) - W_k(x_i) = -M\Delta t + O\left(\Delta t + (\Delta x)^2\right). \tag{48}$$

(ii) $\underline{x_i \notin \Gamma_k(a) \text{and} d(x_i, \partial D_\eta) < \overline{C}\sqrt{\Delta t}}$. Condition $d(x_i, \partial D_\eta) < \overline{C}\sqrt{\Delta t}$ and (47) imply that $w(x_i) = O(\Delta t)$ and, for any $s \in \mathcal{I}$, $d^2(y^s_{k,i}(a), \partial D_\eta) = O(\Delta t)$. Since the cardinality of $\mathcal{J} := \{j \in \mathcal{I}_{\Delta x} \mid \psi_j(y^s_{k,i}(a)) > 0\}$ is independent of Δx and, for all $j \in \mathcal{J}$, $|y^s_{k,i}(a) - x_j| = O(\Delta x)$, we deduce that

$$\begin{aligned} I[w](y^s_{k,i}(a)) &= \textstyle\sum_{j\in\mathcal{J}} \psi_j(y^s_{k,i}(a)) w(x_j) \\ &\leq \textstyle\sum_{j\in\mathcal{J}} \psi_j(y^s_{k,i}(a)) d^2(x_j, \partial D_\eta) \\ &= \textstyle\sum_{j\in\mathcal{J}} \psi_j(y^s_{k,i}(a)) d^2(y^s_{k,i}(a), \partial D_\eta) + O((\Delta x)^2) \\ &= O(\Delta t + (\Delta x)^2). \end{aligned}$$

Thus, since $\tilde{y}^s_{k,i}(a,b) = y^s_{k,i}(a)$, (46) implies that (48) still holds.

(iii) $\underline{x_i \in \Gamma_k(a)}$. Let $0 < \delta < \eta$. Since μ and σ are bounded, there exists $\overline{\Delta t} > 0$, independent of k, i and a, such that

$$\Gamma_k(a) \subseteq L_\delta \subset L_\eta, \tag{49}$$

if $\Delta t \leq \overline{\Delta t}$. By (45) and Proposition 3(i), with $f \equiv 0$ and $g \equiv 0$, we have

$$\mathcal{S}_{k,i}[W_{k+1}|_{\mathcal{G}_{\Delta x}}](a,b) - W_k(x_i)$$

$$= -M\Delta t - \frac{1}{2N_\sigma}\sum_{s\in\mathcal{I}} \tilde{d}^s_{k,i}(a,b)\left\langle Dw(x_i), \gamma_b\left(p^{\gamma_b}\left(y^s_{k,i}(a)\right)\right)\right\rangle$$
$$+O\left(\Delta t + (\Delta x)^2\right). \tag{50}$$

By Lemma 1(v), for any $x \in L_\eta$, we have $d\left(x, \partial D_\eta\right) = \eta - d(x, \partial\mathcal{O})$. Thus, Lemma 1(ii) implies that $Dd\left(x, \partial D_\eta\right) = n(p_{\partial\mathcal{O}}(x))$, and hence

$$Dw(x_i) = 2d\left(x_i, \partial D_\eta\right) Dd\left(x_i, \partial D_\eta\right) = 2d\left(x_i, \partial D_\eta\right) n(p_{\partial\mathcal{O}}(x)). \tag{51}$$

On the other hand, in view of [28, Proposition 1.1(v)], there exists $C > 0$ such that $|d^{\gamma_b}(x_i)| \leq Cd(x_i, \partial\mathcal{O})$. Thus,

$$|p^{\gamma_b}(x_i) - p_{\partial\mathcal{O}}(x_i)| \leq |p^{\gamma_b}(x_i) - x_i| + |x_i - p_{\partial\mathcal{O}}(x_i)| = |d^{\gamma_b}(x_i)| + d(x_i, \partial\mathcal{O})$$
$$\leq (C+1)d(x_i, \partial\mathcal{O}).$$

Since $x_i \in \Gamma_k(a)$, we have $d(x_i, \partial\mathcal{O}) = O(\sqrt{\Delta t})$ and hence $|p^{\gamma_b}(x_i) - p_{\partial\mathcal{O}}(x_i)| = O(\sqrt{\Delta t})$. Proposition 1 implies that γ_b and p^{γ_b} are Lipschitz and hence, for any $s \in \mathcal{I}$,

$$\gamma_b\left(p^{\gamma_b}\left(y^s_{k,i}(a)\right)\right) = \gamma_b\left(p^{\gamma_b}(x_i)\right) + O\left(\sqrt{\Delta t}\right) = \gamma_b\left(p_{\partial\mathcal{O}}(x_i)\right) + O\left(\sqrt{\Delta t}\right). \tag{52}$$

Since, for all $s \in \mathcal{I}$, $\tilde{d}^s_{k,i}(a,b) = O(\sqrt{\Delta t})$, from (50)–(52) we obtain

$$\mathcal{S}_{k,i}[W_{k+1}|_{\mathcal{G}_{\Delta x}}](a,b) - W_k(x_i)$$
$$= -M\Delta t - \frac{1}{N_\sigma}\sum_{s\in\mathcal{I}} d\left(x_i, \partial D_\eta\right)\tilde{d}^s_{k,i}(a,b)\left\langle n(p_{\partial\mathcal{O}}(x_i)), \gamma_b\left(p_{\partial\mathcal{O}}(x_i)\right)\right\rangle$$
$$+ O\left(\Delta t + (\Delta x)^2\right). \tag{53}$$

Noticing that $d\left(x_i, \partial D_\eta\right) \geq \eta - \delta > 0$ and that, by **(H3)**, $\nu := \min_{x\in\partial\mathcal{O}, b\in B}\langle\gamma_b(x), n(x)\rangle$ is strictly positive, we get

$$\mathcal{S}_{k,i}[W_{k+1}|_{\mathcal{G}_{\Delta x}}](a,b) - W_k(x_i) \leq -M\Delta t - \frac{\nu(\eta-\delta)}{N_\sigma}\sum_{s\in\mathcal{I}}\tilde{d}^s_{k,i}(a,b)$$
$$+O\left(\Delta t + (\Delta x)^2\right).$$

Since $\tilde{d}^s_{k,i}(a,b) > 0$ implies that $\tilde{d}^s_{k,i}(a,b) \geq \bar{c}\sqrt{\Delta t}$ (see (26)), there exists $C > 0$, independent of $k \in \mathcal{I}^*_{\Delta t}$, $i \in \mathcal{I}_{\Delta x}$, $a \in A$, and $b \in B$, such that

$$\mathcal{S}_{k,i}[W_{k+1}|_{\mathcal{G}_{\Delta x}}](a,b) - W_k(x_i) \leq -M\Delta t - C\sqrt{\Delta t} + O\left(\Delta t + (\Delta x)^2\right). \tag{54}$$

As long as $(\Delta x)^2/\Delta t$ is bounded, we have that $O\left(\Delta t + (\Delta x)^2\right) = O(\Delta t)$. Thus, from cases (i)-(iii) we can choose M large enough such that

$$\mathcal{S}_{k,i}[W_{k+1}|_{\mathcal{G}_{\Delta x}}](a,b) - W_k(x_i) \leq -C\sqrt{\Delta t}\mathbb{I}_{\Gamma_k(a)}(x_i). \tag{55}$$

Now, set $q_k(x_i,a,b) = W_k(x_i) - \mathcal{S}_{k,i}[W_{k+1}|_{\mathcal{G}_{\Delta x}}](a,b)$. Then the probabilistic interpretation of the operator $\mathcal{S}_{k,i}$ (see Sect. 3.3) implies that, for any policy $\pi \in \Pi_{N_{\Delta t}}$,

$$W_k(x_i) = \mathbb{E}_{\mathbb{P}^{k,x_i,\pi}}\left(\sum_{m=k}^{N_T-1} q_m\big(X_m,\alpha_m,\beta_m\big) + w\big(X_{N_T}\big)\right).$$

Since (55) implies that $q_k(x_i,a,b) \geq C\sqrt{\Delta t}\mathbb{I}_{\Gamma_k(a)}(x_i)$ for $k \in \mathcal{I}^*_{\Delta t}$, $i \in \mathcal{I}_{\Delta x}$, $a \in A$ and $b \in B$, we deduce that for any policy $\pi \in \Pi_{N_{\Delta t}}$ we have

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^{k,x_i,\pi}}\left(\textstyle\sum_{m=k}^{N_T-1}\mathbb{I}_{\Gamma_m(\alpha_m)}\big(X_m\big)\right) &\leq \frac{1}{C\sqrt{\Delta t}}\mathbb{E}_{\mathbb{P}^{k,x_i,\pi}}\left(\sum_{m=k}^{N_T-1} q_m\big(X_m,\alpha_m,\beta_m\big)\right)\\ &= \frac{W_k(x_i) - \mathbb{E}_{\mathbb{P}^{k,x_i,\pi}}\big(w\big(X_{N_T}\big)\big)}{C\sqrt{\Delta t}}.\end{aligned}$$

Finally, using that W_k and w are bounded, (43) follows. □

Proposition 4 (Stability) *The fully discrete scheme* ($\mathrm{HJB}_{\mathrm{disc}}$) is stable, i.e. there exists $C > 0$ such that

$$\max_{k\in\mathcal{I}^*_{\Delta t},\, i\in\mathcal{I}_{\Delta x}} |U_{k,i}| \leq C, \tag{56}$$

where C is independent of $(\Delta t, \Delta x)$ as long as Δt is small enough and $(\Delta x)^2/\Delta t$ is bounded.

Proof Let us fix $k \in \mathcal{I}^*_{\Delta t}$ and $i \in \mathcal{I}_{\Delta x}$. Then the probabilistic interpretation of the scheme in Sect. 3.3 and the definition of h in (34) imply the existence of a constant $C > 0$ such that

$$\begin{aligned}|U_{k,i}| \leq \sup_{\pi\in\Pi_{N_{\Delta t}}} \mathbb{E}_{\mathbb{P}^{k,x_i,\pi}}&\left(\sum_{m=k}^{N_{\Delta t}-1}\Big[\Delta t\big|f(t_m,X_m,\alpha_m)\big|\right.\\ &\left.+\ \big|h(t_m,X_m,\alpha_m,\beta_m)\big|\Big] + \big|\Psi\big(X_{N_{\Delta t}}\big)\big|\right)\\ \leq \|\Psi\|_\infty + T\|f\|_\infty + C\sqrt{\Delta t}\|g\|_\infty &\sup_{\pi\in\Pi_{N_{\Delta t}}}\mathbb{E}_{\mathbb{P}^{k,x_i,\pi}}\left(\sum_{m=k}^{N_{\Delta t}-1}\mathbb{I}_{\Gamma_m(\alpha_m)}(X_m)\right).\end{aligned}$$

Thus, (56) follows from Lemma 3. □

5 Convergence analysis

In this section we provide the main result of this article which is the convergence of solutions to ($\mathrm{HJB}_{\mathrm{disc}}$) to the unique viscosity solution of (HJB). The proof is based on the half-relaxed limits technique introduced in [8] and the properties of solutions to ($\mathrm{HJB}_{\mathrm{disc}}$) investigated in Sect. 4.

Let $\Delta t > 0$, let $\Delta x > 0$ and let $(U_k)_{k=0}^{N_{\Delta t}}$ be the solution to ($\mathrm{HJB}_{\mathrm{disc}}$) associated to the discretization parameters Δt and Δx. Let us define an extension of $(U_k)_{k=0}^{N_{\Delta t}}$ to $\overline{\mathcal{O}}_T$ by

$$(\forall\ (t,x)\in\overline{\mathcal{O}}_T)\quad u_{\Delta t,\Delta x}(t,x):=I[U_{\lfloor t/\Delta t\rfloor}](x), \tag{57}$$

where we recall that the interpolation operator $I[\cdot]$ is defined in (20). Now, let $(\Delta t_n,\Delta x_n)_{n\in\mathbb{N}}\subseteq(0,+\infty)^2$ be such that $\lim_{n\to\infty}(\Delta t_n,\Delta x_n)=(0,0)$ and the sequence $(\Delta x_n/\Delta t_n)_{n\in\mathbb{N}}$ is bounded. For every $(t,x)\in\overline{\mathcal{O}}_T$, let us define

$$\begin{aligned}\overline{u}(t,x)&:=\limsup_{\substack{n\to\infty\\ \overline{\mathcal{O}}_T\ni(s_n,y_n)\to(t,x)}} u_{\Delta t_n,\Delta x_n}(s_n,y_n),\\ \underline{u}(t,x)&:=\liminf_{\substack{n\to\infty\\ \overline{\mathcal{O}}_T\ni(s_n,y_n)\to(t,x)}} u_{\Delta t_n,\Delta x_n}(s_n,y_n).\end{aligned} \tag{58}$$

From Proposition 4 we deduce that $\overline{u}\colon\overline{\mathcal{O}}_T\to\mathbb{R}$ and $\underline{u}\colon\overline{\mathcal{O}}_T\to\mathbb{R}$ are well-defined and bounded. Moreover, from [3, Chapter V, Lemma 1.5], we have that $\overline{u}$ and $\underline{u}$ are, respectively, upper and lower semicontinuous functions.

Proposition 5 *Assume that $(\Delta x_n)^2/\Delta t_n\to0$, as $n\to\infty$. Then $\overline{u}$ and $\underline{u}$ are, respectively, viscosity sub- and supersolutions to* (HJB).

Proof We only show that $\overline{u}$ is a viscosity subsolution to (HJB), the proof that $\underline{u}$ is a viscosity supersolution being similar. Let $(\bar t,\bar x)\in\overline{\mathcal{O}}_T$ and $\phi\in C^\infty(\overline{\mathcal{O}}_T)$ be such that $\overline{u}(\bar t,\bar x)=\phi(\bar t,\bar x)$ and $\overline{u}-\phi$ has a maximum at $(\bar t,\bar x)$. Then by [3, Chapter V, Lemma 1.6] there exists a subsequence of $(u_{\Delta t_n,\Delta x_n})_{n\in\mathbb{N}}$, which for simplicity is still labelled by $n\in\mathbb{N}$, and a sequence $(s_n,y_n)_{n\in\mathbb{N}}\subseteq\overline{\mathcal{O}}_T$ such that $(u_{\Delta t_n,\Delta x_n})_{n\in\mathbb{N}}$ is uniformly bounded, $u_{\Delta t_n,\Delta x_n}-\phi$ has a local maximum at (s_n,y_n), and, as $n\to\infty$, $(s_n,y_n)\to(\bar t,\bar x)$ and $u_{\Delta t_n,\Delta x_n}(s_n,y_n)\to\overline{u}(\bar t,\bar x)$. Moreover, by modifying the test function ϕ, we can assume that $u_{\Delta t_n,\Delta x_n}-\phi$ has a global maximum at (s_n,y_n), i.e. setting $\xi_n:=u_{\Delta t_n,\Delta x_n}(s_n,y_n)-\phi(s_n,y_n)$, we have

$$(\forall\ (t,x)\in\overline{\mathcal{O}}_T)\quad u_{\Delta t_n,\Delta x_n}(t,x)\le\phi(t,x)+\xi_n,\quad \text{with } \xi_n\to0. \tag{59}$$

We distinguish now the following cases.

(i) $\underline{(\bar t,\bar x)\in[0,T)\times\mathcal{O}}$. In this case, for all n large enough, by (19), we have $y_n\in\overline{\mathcal{O}}_{\Delta x_n}$. Let $k:\mathbb{N}\to\mathcal{I}^*_{\Delta t_n}$ be such that $s_n\in[t_{k(n)},t_{k(n)+1})$. As $n\to\infty$, we have $t_{k(n)}\to\bar t$ and, from (57) and (59), with $t=t_{k(n)+1}$, we have

$$(\forall\ x\in\overline{\mathcal{O}})\quad I[U_{k(n)+1}](x)\le\phi(t_{k(n)+1},x)+\xi_n. \tag{60}$$

From Proposition 2, we obtain

$$(\forall\, i \in \mathcal{I}_{\Delta x}) \quad S_{k_n,i}[U_{k(n)+1}] \leq S_{k_n,i}[\Phi_{k(n)+1}] + \xi_n, \tag{61}$$

where, for all $k \in \mathcal{I}_{\Delta t}$, we have denoted $\Phi_k := \phi(t_k, \cdot)|_{\mathcal{G}_{\Delta x_n}}$. In particular, by ($\mathrm{HJB}_{\mathrm{disc}}$) we get

$$(\forall\, i \in \mathcal{I}_{\Delta x}) \quad U_{k(n),i} \leq S_{k_n,i}[\Phi_{k(n)+1}] + \xi_n. \tag{62}$$

The monotonicity of the interpolation operator (20) yields

$$\left(\forall\, x \in \overline{\mathcal{O}}\right) \quad u_{\Delta t_n, \Delta x_n}(s_n, x) \leq \sum_{i \in \mathcal{I}_{\Delta x_n}} \psi_i\left(p_{\Delta x_n}(x)\right) S_{k_n,i}[\Phi_{k(n)+1}] + \xi_n, \tag{63}$$

and hence, by taking $x = y_n$ and using the definition of ξ_n, we obtain

$$\phi(s_n, y_n) \leq \sum_{i \in \mathcal{I}_{\Delta x_n}} \psi_i(y_n) S_{k_n,i}[\Phi_{k(n)+1}]. \tag{64}$$

Since $(\bar{t}, \bar{x}) \in [0, T) \times \mathcal{O}$ and A, B are compacts, if n is large enough, for all $a \in A$, $b \in B$ and for all $s \in \mathcal{I}$ we have $\tilde{d}^s_{k_n,i}(a, b) = 0$ for all $i \in \mathcal{I}_{\Delta x}$ such that $\psi_i(y_n) > 0$. Using Proposition 3(ii) and inequality (64), we get

$$\begin{aligned}
\phi(s_n, y_n) \leq \sum_{i \in \mathcal{I}_{\Delta x_n}} \psi_i(y_n) \Bigg[& \phi(t_{k(n)+1}, x_i) \\
& - \Delta t_n \sup_{a \in A} \mathcal{H}\left(t_{k(n)}, x_i, D\phi(t_{k(n)+1}, x_i), D^2\phi(t_{k(n)+1}, x_i), a\right) \Bigg] \\
& + O\left(\Delta t_n \sqrt{\Delta t_n} + (\Delta x_n)^2\right).
\end{aligned}$$

Then following the same arguments than those in [15, Theorem 3.1] (see also [23, Theorem 4.22]) we conclude that

$$-\partial_t \phi(\bar{t}, \bar{x}) + H(\bar{t}, \bar{x}, D\phi(\bar{t}, \bar{x}), D^2\phi(\bar{t}, \bar{x})) \leq 0, \tag{65}$$

and hence (5) holds.

(ii) <u>$(\bar{t}, \bar{x}) \in [0, T) \times \partial\mathcal{O}$</u>. If

$$L(\bar{t}, \bar{x}, D\phi(\bar{t}, \bar{x})) \leq 0 \quad \text{or} \quad -\partial_t \phi(\bar{t}, \bar{x}) + H(\bar{t}, \bar{x}, D\phi(\bar{t}, \bar{x}), D^2\phi(\bar{t}, \bar{x})) \leq 0,$$

holds, then (6) holds. Thus, let us suppose that

$$L(\bar{t}, \bar{x}, D\phi(\bar{t}, \bar{x})) > 0 \quad \text{and} \quad -\partial_t \phi(\bar{t}, \bar{x}) + H(\bar{t}, \bar{x}, D\phi(\bar{t}, \bar{x}), D^2\phi(\bar{t}, \bar{x})) > 0. \tag{66}$$

Letting $k : \mathbb{N} \to \{0, \ldots, N_T - 1\}$ as in (i), we have $t_{k(n)} \to \bar{t}$, (63) holds true, and hence,

$$\phi(s_n, y_n) \leq \sum_{i \in \mathcal{I}_{\Delta x_n}} \psi_i\big(p_{\Delta x_n}(y_n)\big) S_{k_n,i}[\Phi_{k(n)+1}]. \tag{67}$$

On the one hand, from Proposition 3(ii) we get

$$\begin{aligned} 0 \leq &\sum_{i \in \mathcal{I}_{\Delta x_n}} \psi_i(p_{\Delta x_n}(y_n)) \Bigg(\Delta t_n \partial_t \phi(t_{k(n)}, x_i) \\ &- \sup_{\substack{a \in A,\\ b \in B}} \Bigg\{ \Delta t_n \mathcal{H}(t_{k(n)}, x_i, D\phi(t_{k(n)+1}, x_i), D^2\phi(t_{k(n)+1}, x_i), a) \\ &+ \tfrac{1}{2N_\sigma} \sum_{s \in \mathcal{I}} \tilde{d}^s_{k,i}(a,b) \left(\tilde{Ł}^s_{k(n),i}(D\phi(t_{k(n)+1}, x_i), a, b) - \sqrt{\Delta t}_n K^s_{k(n),i}(a,b) \right) \Bigg\} \Bigg) \\ &+ O\left(\Delta t_n \sqrt{\Delta t}_n + (\Delta x_n)^2 \right) \end{aligned}$$

and hence, for all $a \in A$ and $b \in B$, we have

$$\begin{aligned} &\sum_{i \in \mathcal{I}_{\Delta x_n}} \psi_i\big(p_{\Delta x_n}(y_n)\big) \Bigg\{ - \Delta t_n \partial_t \phi(t_{k(n)}, x_i) \\ &+ \Delta t_n \mathcal{H}(t_{k(n)}, x_i, D\phi(t_{k(n)+1}, x_i), D^2\phi(t_{k(n)+1}, x_i), a) \\ &+ \frac{1}{2N_\sigma} \sum_{s \in \mathcal{I}} \tilde{d}^s_{k,i}(a,b) \left(\tilde{Ł}^s_{k(n),i)}(D\phi(t_{k(n)+1}, x_i), a, b) - \sqrt{\Delta t}_n K^s_{k(n),i}(a,b) \right) \Bigg\} \\ &+ O\left(\Delta t_n \sqrt{\Delta t}_n + (\Delta x_n)^2 \right) \leq 0. \end{aligned} \tag{68}$$

On the other hand, since A is compact, there exists $\bar{a} \in A$ such that

$$H(\bar{t}, \bar{x}, D\phi(\bar{t}, \bar{x}), D^2\phi(\bar{t}, \bar{x})) = \mathcal{H}(\bar{t}, \bar{x}, D\phi(\bar{t}, \bar{x}), D^2\phi(\bar{t}, \bar{x}), \bar{a})$$

and

$$\begin{aligned} &\sum_{i \in \mathcal{I}_{\Delta x_n}} \psi_i\big(p_{\Delta x_n}(y_n)\big) \big(-\partial_t \phi(t_{k(n)}, x_i) \\ &\qquad + \mathcal{H}(t_{k(n)}, x_i, D\phi(t_{k(n)+1}, x_i), D^2\phi(t_{k(n)+1}, x_i), \bar{a}) \big) \\ &\qquad \to -\partial_t \phi(\bar{t}, \bar{x}) + H(\bar{t}, \bar{x}, D\phi(\bar{t}, \bar{x}), D^2\phi(\bar{t}, \bar{x})), \quad \text{as } n \to \infty. \end{aligned} \tag{69}$$

Let us set $\tilde{d}^*_n = \max\left\{ \tilde{d}^s_{k_n,i}(\bar{a}) \;\middle|\; s \in \mathcal{I},\ i \in \mathcal{I}_{\Delta x_n} \right\}$ and take $a = \bar{a}$ and an arbitrary $b \in B$ in (68). If there exists a subsequence, still labelled by n, such that $\tilde{d}^*_n = 0$, then dividing (68) by Δt_n, and letting $n \to \infty$, (69) yields

$$-\partial_t \phi(\bar{t}, \bar{x}) + H(\bar{t}, \bar{x}, D\phi(\bar{t}, \bar{x}), D^2\phi(\bar{t}, \bar{x})) \leq 0,$$

which contradicts (66). Otherwise, by (26), for all $n \in \mathbb{N}$, large enough, we have $\tilde{d}_n^* \geq \bar{c}\sqrt{\Delta t_n}$. Notice that the second relation in (66) and (69) imply that, for $n \in \mathbb{N}$ large enough,

$$\begin{aligned} 0 < \sum_{i \in \mathcal{I}_{\Delta x_n}} & \psi_i\big(p_{\Delta x_n}(y_n)\big)\big(-\partial_t \phi(t_{k(n)}, x_i) \\ & + \mathcal{H}(t_{k(n)}, x_i, D\phi(t_{k(n)+1}, x_i), D^2\phi(t_{k(n)+1}, x_i), \bar{a})\big). \end{aligned} \tag{70}$$

Therefore, inequality (68) with $a = \bar{a}$ implies that for all $b \in B$

$$\begin{aligned} \sum_{i \in \mathcal{I}_{\Delta x_n}} \psi_i\big(p_{\Delta x_n}(y_n)\big)\Big\{ & \sum_{s \in \mathcal{I}} \tilde{d}^s_{k_n,i}(\bar{a}, b)\Big(\tilde{Ł}^s_{k_n,i}(D\phi(t_{k(n)+1}, x_i), \bar{a}, b) \\ - & \sqrt{\Delta t}_n K^s_{k(n),i}(\bar{a}, b)\Big)\Big\} + O\left(\Delta t_n \sqrt{\Delta t}_n + (\Delta x_n)^2\right) < 0. \end{aligned} \tag{71}$$

Since the set $\mathcal{I} = \{+, -\} \times \{1, \dots, d\}$ is finite, there exist $\hat{s} \in \mathcal{I}$, $\{q^s \mid s \in \mathcal{I}\backslash\{\hat{s}\}\} \subseteq [0, 1]$, and $i(n) \in \mathcal{I}_{\Delta x_n}$ such that, up to some subsequence, $\tilde{d}_n^* = \tilde{d}^{\hat{s}}_{k(n),i(n)}(\bar{a})$ and, for all $s \in \mathcal{I}\backslash\{\hat{s}\}$, $\tilde{d}^s_{k(n),i(n)}(\bar{a})/\tilde{d}_n^* \to q^s$. Recall that $\tilde{d}_n^* \geq \bar{c}\sqrt{\Delta t_n}$ and $(\Delta x_n)^2/\Delta t_n \to 0$ as $n \to \infty$. Dividing (71) by $\tilde{d}_n^*$ and taking the limit $n \to \infty$ yields

$$\begin{gathered} (\forall\, b \in B) \quad \left(\sum_{s \in \mathcal{I}\backslash\{\hat{s}\}} q^s + 1\right) Ł(\bar{t}, \bar{x}, D\phi(\bar{t}, \bar{x}), b) \leq 0 \\ \text{and hence} \quad (\forall\, b \in B) \quad Ł(\bar{t}, \bar{x}, D\phi(\bar{t}, \bar{x}), b) \leq 0. \end{gathered}$$

Thus, $L(\bar{t}, \bar{x}, D\phi(\bar{t}, \bar{x})) \leq 0$, which contradicts (66).

(iii) $\underline{(\bar{t}, \bar{x}) \in \{T\} \times \overline{\mathcal{O}}}$. Let us first assume that $(\bar{t}, \bar{x}) \in \{T\} \times \mathcal{O}$. Thus, for $n \in \mathbb{N}$ large enough, we have $y_n \in \mathcal{O}$. By taking a subsequence, if necessary, it suffices to consider the cases $s_n \in [0, T)$, for all $n \in \mathbb{N}$, and $s_n = T$, for all $n \in \mathbb{N}$. In the first case, proceeding as in (i), we get

$$-\partial_t \phi(\bar{t}, \bar{x}) + H(\bar{t}, \bar{x}, D\phi(\bar{t}, \bar{x}), D^2\phi(\bar{t}, \bar{x})) \leq 0. \tag{72}$$

In the second case, (57) implies that $u_{\Delta t_n, \Delta x_n}(s_n, y_n) = I[\Psi|_{\mathcal{G}_{\Delta x}}](y_n)$ and hence letting $n \to \infty$ we get

$$\overline{u}(\bar{t}, \bar{x}) = \Psi(\bar{x}). \tag{73}$$

Now, assume that $(\bar{t}, \bar{x}) \in \{T\} \times \partial\mathcal{O}$. As before, it suffices to consider the cases $s_n \in [0, T)$, for all $n \in \mathbb{N}$, and $s_n = T$ for all $n \in \mathbb{N}$. If $s_n \in [0, T)$, then, proceeding as in (ii), we get

$$L(\bar{t}, \bar{x}, D\phi(\bar{t}, \bar{x})) \leq 0 \quad \text{or} \quad -\partial_t \phi(\bar{t}, \bar{x}) + H(\bar{t}, \bar{x}, D\phi(\bar{t}, \bar{x}), D^2\phi(\bar{t}, \bar{x})) \leq 0. \tag{74}$$

Finally, if $s_n = T$, for all $n \in \mathbb{N}$, we have $u_{\Delta t_n, \Delta x_n}(s_n, y_n) = I[\Psi|_{\mathcal{G}_{\Delta x}}](y_n)$ and hence (73) holds.

Altogether, (72) and (73) imply that (11) holds if $(\bar{t}, \bar{x}) \in \{T\} \times \mathcal{O}$, and (74) and (73) imply that (12) holds if $(\bar{t}, \bar{x}) \in \{T\} \times \partial\mathcal{O}$.

Thus, from cases (i)–(iii) and Remark 1 we obtain that $\overline{u}$ is a subsolution to (HJB). □

Theorem 2 *Assume (**H1**)–(**H3**) and that $(\Delta x_n)^2/\Delta t_n \to 0$, as $n \to \infty$. Then*

$$u_{\Delta t_n, \Delta x_n} \to u \quad \textit{uniformly in } \overline{\mathcal{O}}_T,$$

where u is the unique continuous viscosity solution to (HJB).

Proof By (58) we have $\underline{u} \leq \overline{u}$ in $\overline{\mathcal{O}}_T$ and, by Proposition 5 and the comparison principle for sub- and super solutions to (HJB) (see Remark 2(i)), we obtain that $\underline{u} \geq \overline{u}$ in $\overline{\mathcal{O}}_T$. Thus, $u = \underline{u} = \overline{u}$ and the result follows from [3, Chapter V, Lemma 1.9]. □

Remark 6 The previous result shows that our scheme is convergent under the same conditions on the time and space steps than those in standard SL schemes when $\mathcal{O} = \mathbb{R}^d$ (see e.g. [23]). More precisely, compared to a standard explicit finite difference scheme, the SL scheme is stable and convergent under an inverse CFL condition $\Delta x = O((\Delta t)^2)$, which means that large time steps are allowed. A standard explicit finite difference scheme would require a parabolic CFL condition $\Delta t = O((\Delta x)^2)$ to be stable, whereas a standard implicit finite difference, which does not need a CFL condition to be stable, would require to solve a linear system at each time step.

6 Numerical results

In this section, we present some numerical experiments in order to show the performance of the scheme. We consider first a one-dimensional linear parabolic equation, with homogeneous Neumann boundary conditions, and both the first and second order cases. In the former, the boundary conditions are not satisfied in the pointwise sense at every point in the boundary, but they hold in the viscosity sense (see Definition 1). The second example deals with a degenerate second order nonlinear equation on a smooth two-dimensional domain. We consider both non-homogeneous Neumann and oblique derivatives boundary conditions. In the last example, we approximate the solution to a non-degenerate second order nonlinear equation with mixed Dirichlet and homogeneous Neumann boundary conditions on a non-smooth domain. Because of the presence of Dirichlet boundary conditions and corners, the scheme has to be modified and the convergence result in Sect. 4 does not apply. However, the scheme can be successfully applied to solve the problem numerically.

The equations in the first two tests have known analytical solutions. This will allow to compute the errors of solutions to the scheme and to perform a numerical convergence analysis. In the examples dealing with two-dimensional domains, we have considered unstructured triangular meshes, constructed with the Matlab2019 function `initmesh`.

Our theoretical findings in Sects. 4 and 5 show that the scheme is stable for $\Delta t \sim (\Delta x)^q$ for every $q \in (0, 2]$ and convergent for $q \in (0, 2)$.[1] Let us justify, heuristically, that among these choices the rate of convergence is maximized for $q = 1$. Indeed, for internal nodes where the characteristics do not exit, the scheme coincides with the standard semi-Lagrangian scheme for which the local truncation (or one time step) error is of order $(\Delta x)^2 + (\Delta t)^2$ (see [14]), which yields a global in time truncation error of order $(\Delta x)^2/\Delta t + \Delta t$. On the other hand, for nodes with characteristics exiting the domain, Proposition 3 yields a local truncation error of order $(\Delta x)^2 + (\Delta t)^{3/2}$. Since Lemma 3 provides a bound of order $1/\sqrt{\Delta t}$ on the expected number of time steps where the characteristic exits the domain, we obtain a global truncation error of order $(\Delta x)^2/\sqrt{\Delta t}) + \Delta t$ at these nodes. Summing up, the global truncation error is of order $(\Delta x)^2/\Delta t + \Delta t$ which is maximized by choosing $\Delta t \sim \Delta x$ and the heuristic optimal rate of convergence is 1.

In the first test, we consider the relations $\Delta t \sim \Delta x$ and $\Delta t \sim (\Delta x)^{3/2}$ between the time and space steps. Choosing larger time steps, i.e. $\Delta t \sim (\Delta x)^q$, with $q \in (0, 1)$ may decrease accuracy, but not stability (see Remark 6).

Let us comment on the implementation of the scheme. We will consider three examples, the first one deals with a backward HJB equation and the second and third ones deal with a forward HJB equation. Notice that, since u solves (1) if and only if $u(T - \cdot, \cdot)$ solves (HJB), with H being replaced by $H(T - \cdot, \cdot, \cdot, \cdot)$, we can compute an approximation $U_{k,i}$ ($k \in \mathcal{I}_{\Delta t}, i \in \mathcal{I}_{\Delta x}$) of u with the following forward and explicit scheme

$$
\begin{aligned}
U_{k+1,i} &= S_{N-k,i}\big[U_k\big], && \text{for } (k,i) \in \mathcal{I}^*_{\Delta t} \times \mathcal{I}_{\Delta x}, \\
U_{0,i} &= \Psi(x_i), && \text{for } i \in \mathcal{I}_{\Delta x}.
\end{aligned} \tag{75}
$$

In order to compute $S_{k,i}(U)$ for $k \in \mathcal{I}^*$, $i \in \mathcal{I}_{\Delta x}$ and $U : \mathcal{G}_{\Delta x} \to \mathbb{R}$, in the backward and forward schemes (HJB$_{\text{disc}}$) and (75), respectively, we optimize by simply comparing the values of $\mathcal{S}_{k,i}[U](a, b)$, where (a, b) vary in a mesh defined over $A \times B$. The mesh size is chosen small enough in order to ensure that the dominant error is given by the truncation error, due to the discrete operator $\mathcal{S}_{k,i}[U](a, b)$ (see Proposition 3(i)). More sophisticated optimization algorithms can be considered, as long as they are derivatives free, since, because of the presence of the basis functions ψ_i, the function $\mathcal{S}_{k,i}[U](\cdot, \cdot)$ is at most Lipschitz continuous. Observe that, by (25), (26), (28), and (29), if the discrete characteristic $y^s_{k,i}(a) \notin \mathcal{O}$, for some $s \in \mathcal{I}$, then we need to compute $\tilde{y}^s_{k,i}(a, b)$ which depends on $\bar{c}$, $d^{\gamma_b}(y^s_{k,i}(a))$ and $p^{\gamma_b}(y^s_{k,i}(a))$. The parameter $\bar{c}$ is chosen empirically in (0, 1) and its values are specified in all the tests below. On the other hand, except for some particular cases, $d^{\gamma_b}(y^s_{k,i}(a))$ and $p^{\gamma_b}(y^s_{k,i}(a))$ cannot be

[1] By $\Delta t \sim (\Delta x)^q$, we mean here that Δt is of the order of $(\Delta x)^q$, i.e. there exists $c > 0$ such that $\Delta t = c(\Delta x)^q$.

computed explicitly and they have to be approximated by numerically solving equation (15). Recall that, by Proposition 1 and its proof, Eq. (15) is locally well-posed around the boundary as soon as a smooth parametrization of $\partial\mathcal{O}$ is available. Once $\tilde{y}^s_{k,i}(a,b)$ is available, one computes its projection onto the triangulation $\mathbb{T}_{\Delta x}$.

6.1 A linear problem

The purpose of this simple example, which is an adaptation to the time-dependent case of [17, Example 7.3], is twofold: first, to illustrate that solutions to ($\mathrm{HJB}_{\mathrm{disc}}$) capture the correct behaviour of the solution at the boundary; and second, to show that time steps smaller than $\Delta t \sim \Delta x$ do not improve the performance of the scheme.

Let $\varepsilon > 0$, set $\lambda^{\pm}_{\varepsilon} = (1 \pm \sqrt{1+4\varepsilon})/2\varepsilon$, and define

$$
\begin{aligned}
f_\varepsilon(t,x) &= \frac{3-t}{2}\left(1 + \frac{e^{\lambda^+_\varepsilon x}\left(e^{\lambda^-_\varepsilon}-1\right)}{e^{\lambda^+_\varepsilon}-e^{\lambda^-_\varepsilon}}\left(1-\varepsilon\lambda^+_\varepsilon\right) + \frac{e^{\lambda^-_\varepsilon x}\left(1-e^{\lambda^+_\varepsilon}\right)}{e^{\lambda^+_\varepsilon}-e^{\lambda^-_\varepsilon}}\left(1-\varepsilon\lambda^-_\varepsilon\right)\right) \\
&\quad + \frac{1}{2}\left(x + \frac{e^{\lambda^+_\varepsilon x}\left(e^{\lambda^-_\varepsilon}-1\right)}{e^{\lambda^+_\varepsilon}-e^{\lambda^-_\varepsilon}} + \frac{e^{\lambda^-_\varepsilon x}\left(1-e^{\lambda^+_\varepsilon}\right)}{e^{\lambda^+_\varepsilon}-e^{\lambda^-_\varepsilon}}\right), \\
u_\varepsilon(t,x) &= \frac{3-t}{2}\left(x + \frac{e^{\lambda^-_\varepsilon}-1}{\lambda^+_\varepsilon\left(e^{\lambda^+_\varepsilon}-e^{\lambda^-_\varepsilon}\right)}e^{\lambda^+_\varepsilon x} + \frac{1-e^{\lambda^+_\varepsilon}}{\lambda^-_\varepsilon\left(e^{\lambda^+_\varepsilon}-e^{\lambda^-_\varepsilon}\right)}e^{\lambda^-_\varepsilon x}\right),
\end{aligned}
$$

for $(t,x) \in [0,1]^2$. Then u_ε is the unique classical solution to

$$
\begin{aligned}
&-\partial_t u - \varepsilon\partial^2_x u + \partial_x u = f_\varepsilon && \text{in } [0,1)\times(0,1), \\
&\partial_x u(\cdot,0) = \partial_x u(\cdot,1) = 0 && \text{in } [0,1), \\
&u(1,\cdot) = u_\varepsilon(1,\cdot) && \text{in } [0,1]
\end{aligned}
\tag{76}
$$

and

$$
u_\varepsilon(t,x) \xrightarrow[\varepsilon\to 0]{} u_0(t,x) := \frac{3-t}{2}\left(x+e^{-x}\right), \quad \text{uniformly on } [0,1]^2.
$$

Notice that, setting

$$
f_0(t,x) = \frac{3-t}{2}(1-e^{-x}) + \frac{1}{2}(x-e^{-x}) \quad \text{for } (t,x)\in[0,1]^2,
$$

u_0 solves the PDE

$$
\begin{aligned}
&-\partial_t u + \partial_x u = f_0 && \text{in } [0,1)\times(0,1), \\
&\partial_x u(\cdot,0) = \partial_x u(\cdot,1) = 0 && \text{in } [0,1), \\
&u(1,\cdot) = u_0(1,\cdot) && \text{in } [0,1],
\end{aligned}
\tag{77}
$$

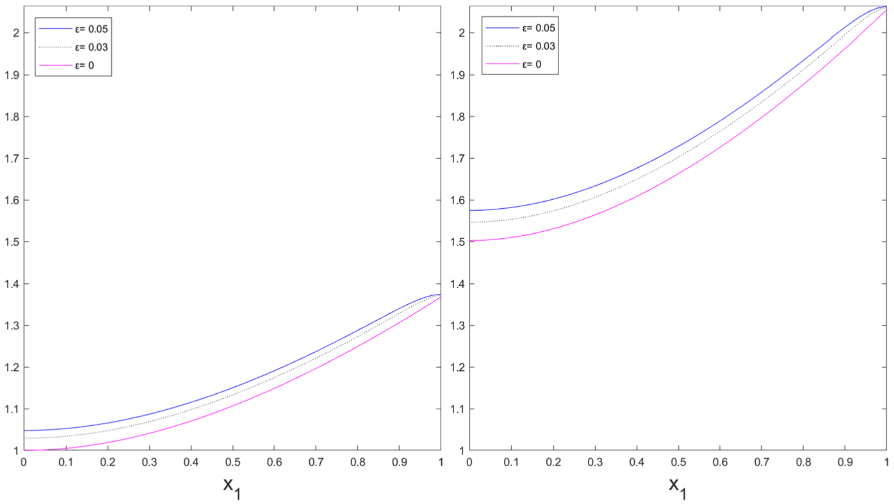

Fig. 4 Exact final condition $u_\varepsilon(1,\cdot)$ (left) and numerical approximations of $u_\varepsilon(0,\cdot)$ (right) for $\varepsilon = 0.05$, $\varepsilon = 0.03$, and $\varepsilon = 0$, with step sizes $\Delta x = 6.25 \times 10^{-3}$ and $\Delta t = \Delta x/2$

Table 1 Errors and convergence rates for problem (76) with $\varepsilon = 0.05$ and $\bar{c} = 0.5$

Δx	$\Delta t = 2(\Delta x)^{3/2}$				$\Delta t = \Delta x/2$			
	E_∞	E_1	p_∞	p_1	E_∞	E_1	p_∞	p_1
5.00×10^{-2}	2.68×10^{-2}	2.66×10^{-2}	–	–	2.02×10^{-2}	1.89×10^{-2}	–	–
2.50×10^{-2}	1.49×10^{-2}	1.33×10^{-3}	0.85	1.00	1.11×10^{-2}	1.05×10^{-3}	0.86	0.85
1.25×10^{-2}	8.31×10^{-3}	6.67×10^{-3}	0.84	1.00	4.86×10^{-3}	4.51×10^{-3}	1.19	1.22
6.25×10^{-3}	6.17×10^{-3}	4.82×10^{-3}	0.84	0.47	2.81×10^{-3}	2.73×10^{-3}	0.79	0.72

the boundary condition being satisfied in the viscosity sense but not in the pointwise sense. Indeed, $\partial_x u_0(t,1) > 0$ and $-\partial_t u_0(t,1) + \partial_x u_0(t,1) - f_0(t,1) \le 0$ for all $t \in [0,1]$.

Using (HJB_{disc}), we approximate u_ε for $\varepsilon = 0.05$, $\varepsilon = 0.03$, and $\varepsilon = 0$. For these choices, we plot in Fig. 4 the final data $u_\varepsilon(1,\cdot)$ and the approximation of $u_\varepsilon(0,\cdot)$ computed with the steps sizes $\Delta x = 3.125 \cdot 10^{-3}$ and $\Delta t = \Delta x/2$. The plot on the right, shows that the numerical solution correctly captures the behaviour of u_0 at the boundary point $x = 1$. Let us point out that, in general, this is not the case for finite difference schemes, which need a special treatment in order to capture the correct behaviour of the viscosity solution at the boundary (see e.g. [1, Section 5.1.2]). Denote by U^ε the approximation of u_ε and consider the errors

$$E_\infty = \max_{i \in \mathcal{I}_{\Delta x}} |U^\varepsilon_{0,i} - u_\varepsilon(0,x_i)|, \quad E_1 = \Delta x \sum_{i \in \mathcal{I}_{\Delta x}} |U^\varepsilon_{0,i} - u_\varepsilon(0,x_i)|.$$

In Tables 1 and 2 we show the values of E_∞ and E_1 as well as the corresponding

Table 2 Errors and convergence rates for problem (76) with $\varepsilon = 0$ and $\bar{c} = 0.05$

Δx	$\Delta t = 2(\Delta x)^{3/2}$				$\Delta t = \Delta x/2$			
	E_∞	E_1	p_∞	p_1	E_∞	E_1	p_∞	p_1
5.00×10^{-2}	2.47×10^{-2}	1.91×10^{-2}	–	–	2.26×10^{-2}	1.86×10^{-2}	–	–
2.50×10^{-2}	1.14×10^{-2}	1.01×10^{-2}	1.12	0.92	1.15×10^{-2}	9.97×10^{-3}	0.97	0.90
1.25×10^{-2}	5.86×10^{-3}	5.73×10^{-3}	0.96	0.82	5.88×10^{-3}	5.42×10^{-3}	0.97	0.88
6.25×10^{-3}	3.49×10^{-3}	3.27×10^{-3}	0.75	0.81	3.04×10^{-3}	2.97×10^{-3}	0.95	0.87

convergence rates p_∞ and p_1 for $\Delta t \sim (\Delta x)^{3/2}$, $\Delta t \sim \Delta x$, and different values of ε and $\overline{c}$. The proof of Lemma 3 suggests to take large $\overline{c}$ for large diffusion terms in order to preserve stability. With this choice, the larger the value of ε, the more the characteristics are reflected further into $\mathcal{O}$. In both tables, we observe an order of convergence close to 1 and, as expected, we do not see an improvement by choosing $\Delta t \sim (\Delta x)^{3/2}$ instead of $\Delta t \sim \Delta x$.

6.2 Nonlinear problem on a circular domain

Let $T = 1$, $\mathcal{O} = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid |x| < 1\}$, and

$$
\begin{aligned}
\sigma(t,x) &= \sqrt{2}(\sin(x_1 + x_2), \cos(x_1 + x_2))\\
f(t,x) &= \left(\frac{1}{2} - t\right)\sin(x_1)\sin(x_2) + \left(\frac{3}{2} - t\right)\Big(\sqrt{\cos^2(x_1)\sin^2(x_2) + \sin^2(x_1)\cos^2(x_2)}\\
&\qquad - 2\sin(x_1 + x_2)\cos(x_1 + x_2)\cos(x_1)\cos(x_2)}\Big),\\
g(t,x) &= \left(\frac{3}{2} - t\right)(x_1\cos(x_1)\sin(x_2) + x_2\sin(x_1)\cos(x_2)).
\end{aligned}
$$

Then $\overline{\mathcal{O}}_T \ni (t, x_1, x_2) \mapsto \bar{u}(t, x_1, x_2) = \left(\frac{3}{2} - t\right)\sin(x_1)\sin(x_2)$ is the unique classical solution to

$$
\begin{aligned}
\partial_t u - \tfrac{1}{2}\mathrm{Tr}(\sigma\sigma^\top D^2 u) + |Du| &= f \quad \text{in } \mathcal{O}_T,\\
\langle n, Du\rangle &= g \quad \text{in } [0,T) \times \partial\mathcal{O},\\
u(0,x) &= \bar{u}(0,x) \quad \text{in } \overline{\mathcal{O}}.
\end{aligned}
\tag{78}
$$

In Fig. 5, we show the numerical solution U to the above forward degenerate HJB equation at the final time $T = 1$, computed on an unstructured triangular mesh $\mathcal{G}_{\Delta x}$ with mesh size $\Delta x = 1.25 \cdot 10^{-1}$. On the left, we plot the result together with the contour lines. On the right, we plot the approximation together with the mesh used to compute it.

Given an element $\hat{T}$ of the triangulation, we denote by $x_{\hat{T}}$ its barycentre and by $|\hat{T}|$ its area. We show in Tables 3 and 4 the errors

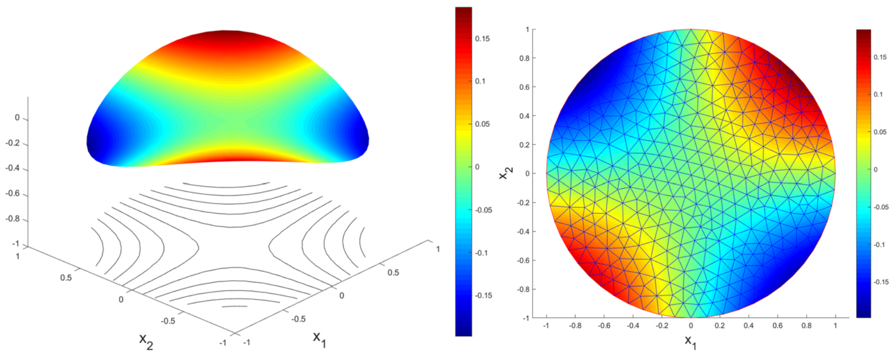

Fig. 5 Numerical solution at time $T = 1$ of problem in Sect. 6.2 with Neumann boundary condition, computed with $\Delta x = 0.125$ and $\Delta t = \Delta x/2$

Table 3 Errors and convergence rates for the approximation of (78) with $\bar{c} = 0.25$

Δx	$\Delta t = \Delta x$				$\Delta t = \Delta x/2$			
	E_∞	E_1	p_∞	p_1	E_∞	E_1	p_∞	p_1
2.50×10^{-1}	2.73×10^{-1}	2.95×10^{-1}	–	–	1.22×10^{-1}	1.07×10^{-1}	–	–
1.25×10^{-1}	1.24×10^{-1}	1.12×10^{-1}	1.14	1.40	5.54×10^{-2}	4.57×10^{-2}	1.14	1.24
6.25×10^{-2}	5.55×10^{-2}	4.72×10^{-2}	1.16	1.24	2.39×10^{-2}	2.11×10^{-2}	1.21	1.11
3.125×10^{-2}	2.49×10^{-2}	2.16×10^{-2}	1.16	1.13	1.22×10^{-2}	1.10×10^{-2}	0.97	0.94

Table 4 Errors and convergence rates for the approximation of (78) with $\bar{c} = 0.5$

Δx	$\Delta t = \Delta x$				$\Delta t = \Delta x/2$			
	E_∞	E_1	p_∞	p_1	E_∞	E_1	p_∞	p_1
2.50×10^{-1}	2.65×10^{-1}	2.55×10^{-1}	–	–	1.18×10^{-1}	1.02×10^{-1}	–	–
1.25×10^{-1}	1.23×10^{-1}	1.12×10^{-1}	1.11	1.19	5.60×10^{-2}	4.72×10^{-2}	1.08	1.11
6.25×10^{-2}	5.74×10^{-2}	5.06×10^{-2}	1.10	1.15	2.64×10^{-2}	2.27×10^{-2}	1.08	1.06
3.125×10^{-2}	2.70×10^{-2}	2.39×10^{-2}	1.09	1.08	1.22×10^{-2}	1.10×10^{-2}	1.11	1.05

$$E_\infty = \max_{i \in \mathcal{I}_{\Delta x}} |U_{N_T,i} - \bar{u}(t_{N_T}, x_i)|, \quad E_1 = \sum_{\hat{T} \in \mathbb{T}_{\Delta x}} |\hat{T}| \left| I[U_{N_T,(\cdot)}](x_{\hat{T}}) - \bar{u}(t_{N_T}, x_{\hat{T}}) \right|, \tag{79}$$

and the corresponding convergence rates p_∞ and p_1. In each table, we specify in the first column the mesh size Δx. To obtain the results shown in Tables 3 and 4, we have chosen $\bar{c}$ in (25) and (26) as $\bar{c} = 0.25$ and $\bar{c} = 0.5$, respectively. For both choices of $\bar{c}$, we observe similar errors and an analogue behaviour of the convergence rates. As in the previous example, an order of convergence close to 1 is obtained.

Table 5 Errors and convergence rates for the approximation of (80) with $\bar{c} = 0.25$

Δx	$\Delta t = \Delta x$				$\Delta t = \Delta x/2$			
	E_∞	E_1	p_∞	p_1	E_∞	E_1	p_∞	p_1
2.50×10^{-1}	3.06×10^{-1}	4.38×10^{-1}	–	–	1.50×10^{-1}	2.08×10^{-1}	–	–
1.25×10^{-1}	1.56×10^{-1}	2.25×10^{-1}	0.97	0.96	7.96×10^{-2}	1.17×10^{-1}	0.91	0.83
6.25×10^{-2}	8.10×10^{-2}	1.21×10^{-1}	0.95	0.89	4.36×10^{-2}	6.84×10^{-2}	0.88	0.77
3.125×10^{-2}	4.47×10^{-2}	7.17×10^{-2}	0.86	0.75	2.58×10^{-2}	4.26×10^{-2}	0.76	0.68

Table 6 Errors and convergence rates for the approximation of (80) with $\bar{c} = 0.5$

Δx	$\Delta t = \Delta x$				$\Delta t = \Delta x/2$			
	E_∞	E_1	p_∞	p_1	E_∞	E_1	p_∞	p_1
2.50×10^{-1}	2.94×10^{-1}	3.81×10^{-1}	–	–	1.42×10^{-1}	1.69×10^{-1}	–	–
1.25×10^{-1}	1.49×10^{-1}	1.88×10^{-1}	0.98	1.02	7.22×10^{-2}	8.56×10^{-2}	0.98	0.98
6.25×10^{-2}	7.55×10^{-2}	9.33×10^{-2}	0.98	1.01	3.79×10^{-2}	4.63×10^{-2}	0.93	0.89
3.125×10^{-2}	3.95×10^{-2}	5.02×10^{-2}	0.93	0.89	2.12×10^{-2}	2.75×10^{-2}	0.84	0.75

Next, we consider the same problem but with oblique derivatives boundary conditions. More precisely, for $x = (x_1, x_2) \in \partial\mathcal{O}$ we set

$$\gamma(x) = (x_1 \cos(\pi/6) + x_2 \sin(\pi/6), x_2 \cos(\pi/6) - x_1 \sin(\pi/6))$$

and

$$\begin{aligned}\tilde{g}(t, x) = \left(\frac{3}{2} - t\right) \big[&(x_1 \cos(\pi/6) + x_2 \sin(\pi/6)) \cos(x_1) \sin(x_2) \\ &+ (x_2 \cos(\pi/6) - x_1 \sin(\pi/6)) \sin(x_1) \cos(x_2)\big] \quad \text{in } [0, T) \times \partial\mathcal{O}.\end{aligned}$$

Then $\bar{u}$ is the unique classical solution to

$$\begin{aligned} \partial_t u - \tfrac{1}{2}\mathrm{Tr}(\sigma\sigma^\top D^2 u) + |Du| &= f \quad \text{in } \mathcal{O}_T, \\ \langle \gamma, Du \rangle &= \tilde{g} \quad \text{in } [0, T) \times \partial\mathcal{O}, \\ u(0, x) &= \bar{u}(0, x) \quad \text{in } x \in \overline{\mathcal{O}}. \end{aligned} \tag{80}$$

The solution $\bar{u}$ is approximated by using the same unstructured meshes as in the previous case. We show in Tables 5 and 6 the errors (79) computed with $\bar{c} = 0.25$ and $\bar{c} = 0.5$, respectively. As in the previous case, we observe similar errors and an analogue behaviour of the convergence rates for both choices of $\bar{c}$. We also observe a slight degradation of the errors and convergence rates in the case of oblique derivatives boundary conditions. This could be explained by the need to approximate the solution to (15) every time a characteristic exits the domain.

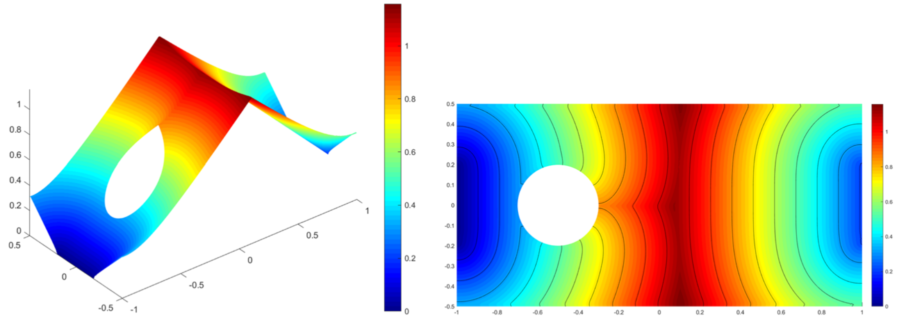

Fig. 6 Solution at time $T = 3$ with $\Delta x = 0.01$, $\Delta t = \Delta x$, and $\overline{c} = 0.25$

6.3 Nonlinear problem on a non-smooth domain with mixed Dirichlet–Neumann boundary conditions

In this last example, we deal with a problem of exiting from a bounded rectangular domain with a circular obstacle inside of it. We model this problem by considering a modification of (1) including mixed Dirichlet-Neumann boundary conditions, with a large time horizon T in order to reach a stationary solution. We consider the space domain

$$\mathcal{O} = \Big((-1, 1) \times (-0.5, 0.5)\Big) \setminus \{x \in \mathbb{R}^2 \,|\, |x - (-0.5, 0)| \leq 0.2\},$$

a control set $A = \{a \in \mathbb{R}^2 \,|\, |a| = 1\}$, a drift $\mu(t, x, a) = a$, a diffusion coefficient $\sigma(t, x, a) = 0.1 I_2$, where I_2 is the identity matrix of size 2, a running cost $f \equiv 1$, and an initial condition $\Psi \equiv 0$. We impose constant Dirichlet boundary conditions on some parts of $\partial\mathcal{O}$, representing the exits of the domain, in order to model some exit costs. More precisely, Dirichlet boundary conditions (or exit costs) $u = 0$ and $u = 0.2$ are imposed on $\partial\mathcal{O}_1 = \{x = (x_1, x_2) \in \partial\mathcal{O} \,|\, x_1 = -1, |x_2| \leq 0.2\}$ and $\partial\mathcal{O}_2 = \{x = (x_1, x_2) \in \partial\mathcal{O} \,|\, x_1 = 1, |x_2| \leq 0.2\}$, respectively. We also consider homogeneous Neumann boundary conditions on the remaining part of the boundary. Because of the mixed boundary conditions, we approximate the solution to the HJB equation by combining scheme (HJB_{disc}), to deal with the Neumann boundary condition, and the scheme proposed in [11], to deal with the Dirichlet boundary condition. In the latter, if the characteristic exits through $\partial\mathcal{O}_1 \cup \partial\mathcal{O}_2$, we approximate the solution by extrapolation using an additional layer of coarser elements inside $\overline{\mathcal{O}}$, with a side belonging to $\partial\mathcal{O}_1 \cup \partial\mathcal{O}_2$, and the Dirichlet condition. For a more detailed discussion, including the size of the elements in the coarser grid, which ensures stability of the scheme, we refer the reader to [10, Section 5]. Let us point out that this approximation has been shown to be more accurate with respect to the methods proposed in [11, 40].

We show in Fig. 6 the numerical approximation computed on an unstructured mesh of size $\Delta x = 0.01$, a time step $\Delta t = \Delta x$ and final time $T = 3$. Figure 7 displays the quiver plot of $-Du$ at time $T = 3$.

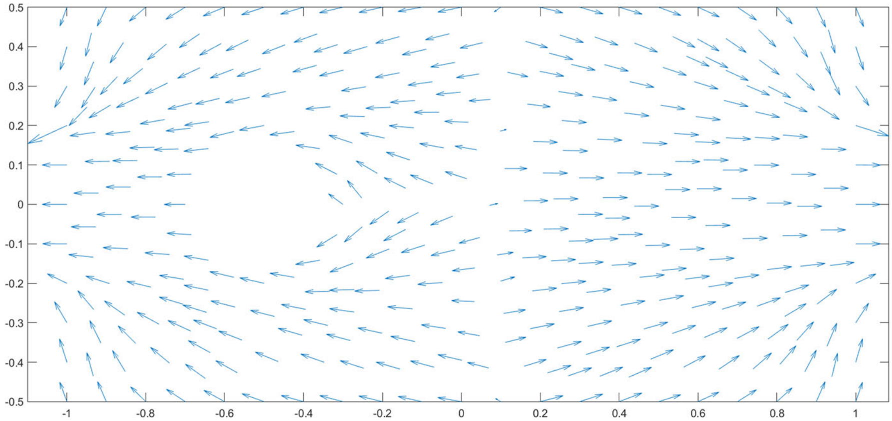

Fig. 7 Quiver plot of $-Du$ at time $T = 3$

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References

1. Abgrall, R.: Numerical discretization of boundary conditions for first order Hamilton–Jacobi equations. SIAM J. Numer. Anal. **41**(6), 2233–2261 (2003). https://doi.org/10.1137/S0036142998345980
2. Achdou, Y., Falcone, M.: A semi-Lagrangian scheme for mean curvature motion with nonlinear Neumann conditions. Interfaces Free Bound. **14**(4), 455–485 (2012)
3. Bardi, M., Dolcetta, I.C.: Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations. Birkauser (1996)
4. Barles, G.: Fully nonlinear Neumann type boundary conditions for second-order elliptic and parabolic equations. J. Differ. Equ. **106**(1), 90–106 (1993). https://doi.org/10.1006/jdeq.1993.1100
5. Barles, G.: Nonlinear Neumann boundary conditions for quasilinear degenerate elliptic equations and applications. J. Differ. Equ. **154**(1), 191–224 (1999). https://doi.org/10.1006/jdeq.1998.3568
6. Barles, G., Lions, P.L.: Fully nonlinear Neumann type boundary conditions for first-order Hamilton–Jacobi equations. Nonlinear Anal. **16**(2), 143–153 (1991). https://doi.org/10.1016/0362-546X(91)90165-W
7. Barles, G., Rouy, E.: A strong comparison result for the Bellman equation arising in stochastic exit time control problems and its applications. Commun. Partial Differ. Equ. **23**(11–12), 1995–2033 (1998). https://doi.org/10.1080/03605309808821409
8. Barles, G., Souganidis, P.E.: Convergence of approximation schemes for fully nonlinear second order equations. Asymptot. Anal. **4**(3), 271–283 (1991)
9. Barrett, J.W., Elliott, C.M.: Finite element approximation of the Dirichlet problem using the boundary penalty method. Numer. Math. **49**(4), 343–366 (1986). https://doi.org/10.1007/BF01389536
10. Bonaventura, L., Calzola, E., Carlini, E., Ferretti, R.: Second order fully semi-Lagrangian discretizations of advection–diffusion–reaction systems. J. Sci. Comput. **88**(1), 1–29 (2021). https://doi.org/10.1007/s10915-021-01518-8
11. Bonaventura, L., Ferretti, R., Rocchi, L.: A fully semi-Lagrangian discretization for the 2D incompressible Navier–Stokes equations in the vorticity-streamfunction formulation. Appl. Math. Comput. **323**, 132–144 (2018). https://doi.org/10.1016/j.amc.2017.11.030

12. Bouchard, B.: Optimal reflection of diffusions and barrier options pricing under constraints. SIAM J. Control Optim. **47**(4), 1785–1813 (2008). https://doi.org/10.1137/070697161
13. Bourgoing, M.: Viscosity solutions of fully nonlinear second order parabolic equations with L^1 dependence in time and Neumann boundary conditions. Discrete Contin. Dyn. Syst. **21**(3), 763–800 (2008). https://doi.org/10.3934/dcds.2008.21.763
14. Camilli, F., Falcone, M.: An approximation scheme for the optimal control of diffusion processes. RAIRO Modél. Math. Anal. Numér. **29**(1), 97–122 (1995)
15. Carlini, E., Falcone, M., Ferretti, R.: Convergence of a large time-step scheme for mean curvature motion. Interfaces Free Bound. **12**(4), 409–441 (2010)
16. Ciarlet, P.G., Lions, J.L. (eds.): Handbook of Numerical Analysis. Finite Element Methods. Part 1, vol. II. North-Holland, Amsterdam (1991)
17. Crandall, M.G., Ishii, H., Lions, P.L.: User's guide to viscosity solutions of second order partial differential equations. Bull. Am. Math. Soc. (N.S.) **27**(1), 1–67 (1992). https://doi.org/10.1090/S0273-0979-1992-00266-5
18. Day, M.V.: Neumann-type boundary conditions for Hamilton–Jacobi equations in smooth domains. Appl. Math. Optim. **53**(3), 359–381 (2006). https://doi.org/10.1007/s00245-005-0852-z
19. Debrabant, K., Jakobsen, E.R.: Semi-Lagrangian schemes for linear and fully non-linear diffusion equations. Math. Comput. **82**(283), 1433–1462 (2013). https://doi.org/10.1090/S0025-5718-2012-02632-9
20. Deckelnick, K., Hinze, M.: Convergence of a finite element approximation to a state-constrained elliptic control problem. SIAM J. Numer. Anal. **45**(5), 1937–1953 (2007). https://doi.org/10.1137/060652361
21. Dupuis, P., Ishii, H.: SDEs with oblique reflection on nonsmooth domains. Ann. Probab. **21**(1), 554–580 (1993)
22. Dupuis, P., Ramanan, K.: Convex duality and the Skorokhod problem. I, II. Probab. Theory Relat. Fields **115**(2), 153–195 (1999). https://doi.org/10.1007/s004400050269
23. Falcone, M., Ferretti, R.: Semi-Lagrangian Approximation Schemes for Linear and Hamilton–Jacobi Equations. MOS-SIAM Series on Optimization (2013)
24. Feng, X., Glowinski, R., Neilan, M.: Recent developments in numerical methods for fully nonlinear second order partial differential equations. SIAM Rev. **55**(2), 205–267 (2013). https://doi.org/10.1137/110825960
25. Feng, X., Jensen, M.: Convergent semi-Lagrangian methods for the Monge–Ampère equation on unstructured grids. SIAM J. Numer. Anal. **55**(2), 691–712 (2017). https://doi.org/10.1137/16M1061709
26. Fleming, W.H., Soner, H.M.: Controlled Markov Processes and Viscosity Solutions, Stochastic Modelling and Applied Probability, vol. 25, 2nd edn. Springer, New York (2006)
27. Gilbarg, D., Trudinger, N.S.: Elliptic partial differential equations of second order. In: Classics in Mathematics. Springer-Verlag, Berlin (2001). Reprint of the 1998 edition
28. Gobet, E.: Efficient schemes for the weak approximation of reflected diffusions. Monte Carlo Methods Appl. **7**(1–2), 193–202 (2001). https://doi.org/10.1515/mcma.2001.7.1-2.193. (**Monte Carlo and probabilistic methods for partial differential equations (Monte Carlo, 2000)**)
29. Hinderer, K., Rieder, U., Stieglitz, M.: Dynamic Optimization. Universitext, Springer, Cham (2016). https://doi.org/10.1007/978-3-319-48814-1 . (**Deterministic and stochastic models**)
30. Ishii, H., Sato, M.H.: Nonlinear oblique derivative problems for singular degenerate parabolic equations on a general domain. Nonlinear Anal. **57**(7–8), 1077–1098 (2004). https://doi.org/10.1016/j.na.2004.04.003
31. Jaroszkowski, B., Jensen, M.: Finite element approximation of Hamilton–Jacobi–Bellman equations with nonlinear mixed boundary conditions (2021). Preprint arXiv:2105.09585
32. Jensen, M., Smears, I.: On the notion of boundary conditions in comparison principles for viscosity solutions. In: Hamilton–Jacobi–Bellman Equations, Radon Series on Computational and Applied Mathematics, vol. 21, pp. 143–154. De Gruyter, Berlin (2018)
33. Kruk, L.: Optimal policies for n-dimensional singular stochastic control problems. I. The Skorokhod problem. SIAM J. Control Optim. **38**(5), 1603–1622 (2000). https://doi.org/10.1137/S0363012998347535
34. Kushner, H.J., Dupuis, P.: Numerical methods for stochastic control problems in continuous time. In: Applications of Mathematics (New York), vol. 24, 2nd edn. Springer-Verlag, New York (2001). https://doi.org/10.1007/978-1-4613-0007-6. (**Stochastic Modelling and Applied Probability**)

35. Lions, P.L.: Generalized Solutions of Hamilton–Jacobi Equations, Research Notes in Mathematics, vol. 69. Pitman (Advanced Publishing Program), Boston (1982)
36. Lions, P.L.: Optimal control of diffusion processes and Hamilton–Jacobi–Bellman equations. I. The dynamic programming principle and applications. Commun. Partial Differ. Equ. **8**(10), 1101–1174 (1983). https://doi.org/10.1080/03605308308820297
37. Lions, P.L.: Neumann type boundary conditions for Hamilton–Jacobi equations. Duke Math. J. **52**(4), 793–820 (1985). https://doi.org/10.1215/S0012-7094-85-05242-1
38. Lions, P.L., Sznitman, A.S.: Stochastic differential equations with reflecting boundary conditions. Commun. Pure Appl. Math. **37**(4), 511–537 (1984). https://doi.org/10.1002/cpa.3160370408
39. Menaldi, J.L., Robin, M.: On some cheap control problems for diffusion processes. Trans. Am. Math. Soc. **278**(2), 771–802 (1983). https://doi.org/10.2307/1999183
40. Milstein, G., Tretyakov, M.: Numerical solution of the Dirichlet problem for nonlinear parabolic equations by a probabilistic approach. IMA J. Numer. Anal. **21**, 887–917 (2001)
41. Milstein, G.N.: Application of the numerical integration of stochastic equations for the solution of boundary value problems with Neumann boundary conditions. Teor. Veroyatnost. i Primenen. **41**(1), 210–218 (1996)
42. Milstein, G.N., Tretyakov, M.V.: A probabilistic approach to the solution of the Neumann problem for nonlinear parabolic equations. IMA J. Numer. Anal. **22**(4), 599–622 (2002). https://doi.org/10.1093/imanum/22.4.599
43. Neilan, M., Salgado, A.J., Zhang, W.: Numerical analysis of strongly nonlinear PDEs. Acta Numer. **26**, 137–303 (2017). https://doi.org/10.1017/S0962492917000071
44. Rouy, E.: Numerical approximation of viscosity solutions of first-order Hamilton–Jacobi equations with Neumann type boundary conditions. Math. Models Methods Appl. Sci. **2**(3), 357–374 (1992). https://doi.org/10.1142/S0218202592000223
45. Skorokhod, A.V.: Stochastic equations for diffusion processes in a bounded region. Theor. Prob. Appl. **6**(3), 264–274 (1961). https://doi.org/10.1137/1106035
46. Soner, H.M., Shreve, S.E.: Regularity of the value function for a two-dimensional singular stochastic control problem. SIAM J. Control Optim. **27**(4), 876–907 (1989). https://doi.org/10.1137/0327047