# Passivity Preserving Multipoint Model Order Reduction using Reflective Exploration

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Abstract: Reduced state-space models obtained by model order reduction methods must be accurate over the whole frequency range of interest and must also preserve passivity. In this paper, we propose multipoint reduction technique using reflective exploration for adaptively choosing the expansion points. The projection matrices obtained from the expansion points are merged to form the overall projection matrix. In order to obtain a more compact model the projection matrix is truncated based on its singular values. Finally, the reduced order model is obtained, while ensuring that the passivity of the reduced system is preserved during the reduction process.

## 1 INTRODUCTION

When analyzing and controlling large-scale systems, it is extremely important to develop efficient modeling procedures. The design of a controller for a high-dimensional system may be too time-consuming to implement in practice. In fact it is important that the key dynamic elements be identified and spurious dynamic elements eliminated. Model reduction techniques provide an extremely effective way to address this requirement.

Model order reduction (MOR) techniques are now standard for reducing the complexity of large scale models and the computational cost of the simulations, while retaining the important physical features of the original system (Feldmann and R. Freund, 1995; Gallivan et al., 1996; Odabasioglu et al., 1998; Knockaert and De Zutter, 2000; Freund, 2000; Phillips et al., 2003; Phillips, 2004; Knockaert et al., 2011). Existing approaches based on Krylov subspaces are very efficient.

One of the main concerns regarding MOR algorithms is that the model must be sufficiently accurate not just at a single frequency point but over a whole range of frequencies. This situation typically arises when dealing with microwave circuits. Reduction algorithms that address this concern are the multipoint rational Krylov algorithm (Gallivan et al., 1996; Silveira and Phillips, 2006; Wang et al., 2012) and multipoint expansion using a binary search (Ferranti et al., 2011), which are more accurate but more expensive to set up.

Multipoint projections raise many practical questions while implementation. In this paper, we focus on three points namely;

- the order considered for each expansion point.
- adaptive frequency sampling using reflective exploration (Beyer and Śmieja, 1996).
- obtaining a compact projection matrix.

In this paper, the projection matrices are computed using the PRIMA technique (Odabasioglu et al., 1998), which is known to be an efficient technique for the reduction of large systems. The expansion points are selected adaptively using a reflective exploration technique. It is a sequential sampling algorithm, where the model is improved incrementally using the best possible data at every time step with additional properties allowing it to propose candidate exploration points (Beyer and Śmieja, 1996). An error-based exploration is implemented to find the expansion points. After obtaining the expansion points the corresponding projection matrices are computed using Krylov based MOR technique. The projection matrices are then merged to obtain the overall projection matrix. When the number of expansion points increase, the merged projection matrix also increases and might fail to provide a satisfactory model dimension reduction. In this paper, an adaptive truncation algorithm is proposed to truncate the merged projection matrix based on its singular values, thereby obtaining a more compact reduced order model (ROM) which preserves the system properties. Numerical results validate the proposed technique.

## 2 Brief overview of Multipoint MOR

Here, the PRIMA algorithm (Odabasioglu et al., 1998) is used for obtaining the projection matrices.

### 2.1 PRIMA

Consider a MIMO descriptor system of the form

$$\begin{aligned} \mathbf{E}\dot{x}(t) &= \mathbf{A}x(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}x(t) + \mathbf{D}u(t). \end{aligned} \quad (1)$$

The transfer function is

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}. \quad (2)$$

Let $s_0$ be a suitably chosen expansion point such that the matrix $s_0\mathbf{E} - \mathbf{A}$ is nonsingular. Then the transfer function can be rewritten as:

$$\begin{aligned} \mathbf{H}(s) &= \mathbf{C}(s_0\mathbf{E} - \mathbf{A} + (s - s_0)\mathbf{E})^{-1}\mathbf{B} + \mathbf{D} \\ &= \mathbf{C}(\mathbf{I} + (s - s_0)\mathbf{M})^{-1}\mathbf{R} + \mathbf{D} \end{aligned} \quad (3)$$

where $\mathbf{M} = (s_0\mathbf{E} - \mathbf{A})^{-1}\mathbf{E}$, $\mathbf{R} = (s_0\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$.

The $q$-th block Krylov-subspace is given by

$$\mathcal{K}_q(M,R) = colspan[\mathbf{R}\ \mathbf{MR}\ \mathbf{M^2R}\ \ldots\ \mathbf{M^{(q-1)}R}]. \quad (4)$$

This yields the projection matrix $V_q$, which is the column orthogonal matrix computed from the Krylov subspace $\mathcal{K}_q(M,R)$, from which using congruence transformation (5) the reduced state-space matrices $(\mathbf{A}_q, \mathbf{E}_q, \mathbf{B}_q, \mathbf{C}_q, \mathbf{D}_q)$ are obtained as:

$$\begin{aligned} \mathbf{A}_q = V_q^T\mathbf{A}V_q, \quad \mathbf{E}_q = V_q^T\mathbf{E}V_q, \\ \mathbf{B}_q = V_q^T\mathbf{B},\ \ \mathbf{C}_q = \mathbf{C}V_q,\ \ \mathbf{D}_q = \mathbf{D}. \end{aligned} \quad (5)$$

### 2.2 Multipoint projection matrix

After model order reduction, the resulting model must not only be accurate at a frequency point but over the whole range of interest and must also preserve passivity. For this the multipoint reduction algorithm is used (Ferranti et al., 2011).

At each expansion point, the projection matrices is computed as described in Section 2.1, i.e., for $N$ expansion points the corresponding projection matrices $V_{q_i}$ $(i = 1, 2, \ldots, N)$ are merged to give;

$$V_{comm} = colspan[V_{q_1}\ V_{q_2}\ \ldots\ V_{q_N}]. \quad (6)$$

The merged projection matrix is not truncated using its singular values during the iterative procedure of the reflective exploration. But the matrix is truncated after all the expansion points are adaptively chosen which is described in the following section.

## 3 Reflective Exploration

The process of selecting expansion points and building the model in an adaptive way is referred to as reflective exploration (Beyer and Śmieja, 1996). Reflective exploration is an effective technique when its very expensive to obtain the model from Electromagnetic (EM) simulators. For the exploration a reflective function is required to select a new expansion point. The reflective function used for the proposed multipoint MOR algorithm is the error norm between the best model and the second best model. As described in (Geest et al., 1999), the algorithm has two loops: an adaptive modeling loop and an adaptive sampling loop.

1. Adaptive Modeling Loop: The algorithm starts with two expansion points selected at $\omega_{min}$ and $\omega_{max}$ of the frequency range of interest. It should be noted that the initial number of sample points that is uniformly distributed along the frequency range of interest can be varied as needed, if prior knowledge of the system is available.

   The reduced order $q$ at these points is equal to the number of input ports of the system. Then the reduced model is obtained with a common projection matrix as explained in Section 2. If the RMS error (7) between the two best models ($I^{th}$ and $(I-1)^{th}$) is more than a threshold $\delta_{mod}$, then the reduced order $q$ is increased by the number of input ports again for all the expansion points. In this paper the threshold is chosen to be $10^{-3}$.

$$Err_{est}^{(I)} = \sqrt{\frac{\sum_{k=1}^{K_s}\sum_{i=1}^{P_{in}}\sum_{j=1}^{P_{out}}\frac{\left|H_{I,(ij)}(s_k) - H_{I-1,(ij)}(s_k)\right|^2}{W_{(ij)}(s_k)}}{P_{in}P_{out}K_s}}$$

$$W_{(ij)}(s_k) = |H_{I,(ij)}(s_k)|^2. \quad (7)$$

   Here, $K_s$, $P_{in}$ and $P_{out}$ are the number of frequency samples considered on a dense grid, input and output ports of the system, respectively.

2. Adaptive Sampling Loop: When the difference in RMS error (7), between the two consecutive models is very small i.e.,

$$\frac{Err_{est}^{(I)} - Err_{est}^{(I-1)}}{Err_{est}^{(I-1)}} < \delta_{samp} \quad (8)$$

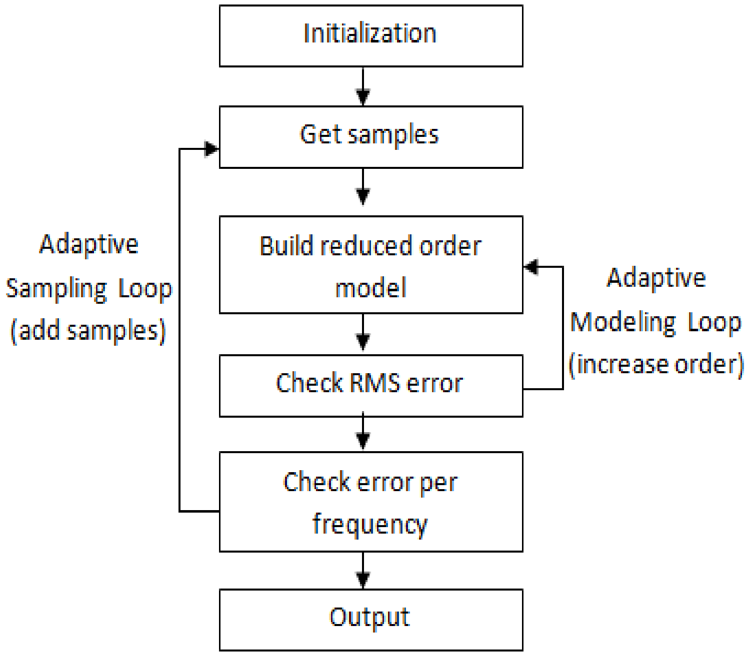


Figure 1: Reflective Exploration.

(in this paper the difference is considered to be less than 10%), then a new expansion point is selected. For selecting the new expansion point, the error per frequency is first computed by taking the L1 norm of the frequency response of the best model ($H_I$) and the original model ($H_{act}$), and then the frequency at which $Err_{s_k}$ is maximum is considered as the new expansion point.

$$Err_{s_k} = \|H_{act)}(s_k) - H_I(s_k)\|; \quad k = 1, \ldots, K_s. \tag{9}$$

This process is iteratively repeated until the RMS error between the original frequency response and the reduced model is $10^{-3}$. Figure 1 shows a flowchart of the reflective exploration algorithm.

# 4 Model compacting

After obtaining the best reduced order model from the iterative procedure, it might be possible to further compact the model with the information obtained from the singular values $\Sigma$ (10) of the $V_{comm}$ (6).

The economy-size *svd* is computed for the common projection matrix $V_{comm}$ (6), to obtain the singular values $\Sigma$ of the merged projection matrix. In matlab the economy-sized *svd* is computed as shown:

$$\mathbf{U}\Sigma\mathbf{V}^T = svd(\mathbf{V}_{comm}, 0) \tag{10}$$

Here, **U** and **V** are orthogonal matrices, which are known as the left and right singular values. The diagonal of $\Sigma$ gives the singular values of the system.

$$\sigma = diag(\Sigma). \tag{11}$$

The reduced order for the system is defined based on the first $q_{comm}$ significant singular values of $\mathbf{V}_{comm}$, which is computed by adaptively setting a threshold to the ratio of the singular values to the largest singular value as shown in Fig.2. The ROM obtained by the truncation of the merged projection matrix with respect to the singular value, is compared with the best model obtained from reflective exploration. If the RMS error is less than $10^{-4}$, then we shall truncate the singular values, else we keep the model with the reduced order obtained using the reflective exploration.

The compact projection matrix $\mathbf{Q_{comm}}$ is equal to the left singular value **U** where the column is truncated to a size $q_{comm}$ based on the significance of the singular values.

$$\mathbf{Q_{comm}} = \mathbf{U}(:, 1 : q_{comm}). \tag{12}$$

After truncation it can be noted that, on average per expansion point an order of $q_{samp}$ (13) is required to guarantee the desired accuracy at that expansion point.

$$q_{samp} = q_{comm}/N. \tag{13}$$

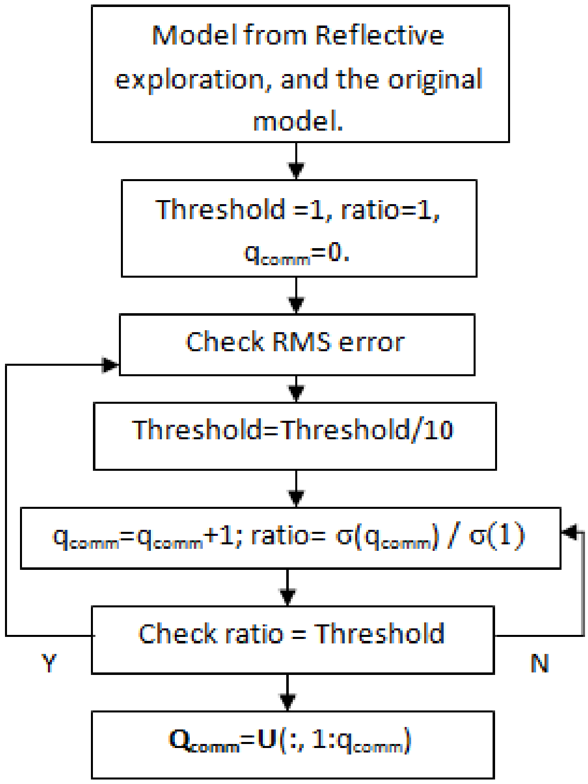


Figure 2: Truncation of the projection matrix.

Here, n is the number of expansion points.

Once the compact projection matrix $Q_{comm}$ is computed, it is applied to the original system (1) and a reduced system (5) is obtained through congruence transformation.

# 5 Passivity Preservation

For transient behavior, stability and passivity are the fundamental properties to be guaranteed by the system, as known that, while a passive system is also stable, the reverse is not necessarily true. A passive system denotes a system that is incapable of generating energy, and hence one that can only absorb energy from the sources used to excite it (Anderson and Vongpanitlerd, 1973). Passivity is an important property to satisfy because stable, but not passive macromodels can produce unstable systems when connected to other stable, even passive, loads.

If the descriptor state space model in (1) satisfies the following properties (Odabasioglu et al., 1998):

$$\begin{aligned} \mathbf{E} &= \mathbf{E}^T \geq 0 \\ \mathbf{A} + \mathbf{A}^T &\leq 0 \\ \mathbf{B} &= \mathbf{C}^T, \end{aligned} \tag{14}$$

then it ensures the passivity of the admittance model $\mathbf{Y}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$ and with congruence transformation the passivity of the model is preserved,

$$\begin{aligned} \mathbf{E}_r(\mathbf{g}) &= \mathbf{Q}_{comm}{}'\mathbf{E}(\mathbf{g})\mathbf{Q}_{comm} \geq 0 \\ \mathbf{A}_r(\mathbf{g}) &= \mathbf{Q}_{comm}{}'\mathbf{A}(\mathbf{g})\mathbf{Q}_{comm} \leq 0 \\ \mathbf{B}_r(\mathbf{g}) &= \mathbf{Q}_{comm}{}'\mathbf{B}(\mathbf{g}) \\ \mathbf{C}_r(\mathbf{g}) &= \mathbf{Q}_{comm}\mathbf{C}(\mathbf{g}). \end{aligned} \tag{15}$$

If the system fails to have the state-space properties described in (14) then the technique of Linear Matrix Inequalities (LMI's) (Knockaert et al., 2011) has to be used, from which the solution obtained fromLMI's (Boyd et al., 1994; Gahinet and Apkarian, 1993) gives a descriptor state space format that preserves positive-realness and bounded realness of the system.

Solving the LMI can be replaced by equivalently solving an ARE, which is known to be a more efficient approach (Gahinet et al., 1995; Balas et al., 2005) as the number of operations required to solve a Riccati equation is $O(n^3)$, while the cost of solving an equivalent LMI is $O(n^6)$. Thus for high orders it is advisable to solve using ARE as it is computationally cheaper in comparison with LMI.

# 6 Numerical Results

Some pertinent numerical examples are used to demonstrate the accuracy and efficiency of the proposed technique. The numerical simulations were performed on a Windows 7 platform on Intel$^{(R)}$ Core$^{(TM)}$2 Duo P8700 2.53 GHz machine with 2 GB RAM and has been implemented in Matlab R2012b.

## 6.1 Example 1: Lossy Transmission Line

For this example, a 1 cm long two conductor lossy transmission line with the following per-unit-length matrices

$$
\begin{aligned}
r_{pul} &= \begin{bmatrix} 75 & 15 \\ 15 & 75 \end{bmatrix} \Omega/\text{m} \\
l_{pul} &= \begin{bmatrix} 494.6 & 63.3 \\ 63.3 & 494.6 \end{bmatrix} \text{nH/m} \\
g_{pul} &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \text{S/m} \\
c_{pul} &= \begin{bmatrix} 62.8 & -4.9 \\ 4.9 & 62.8 \end{bmatrix} \text{pF/m.} \qquad (16)
\end{aligned}
$$

for a frequency range of $[1\ Khz - 1\ GHz]$, is modeled as described in (Knockaert and De Zutter, 2000). The original state-space order of the system is 1202 with 4 ports.

The sampling starts by considering two expansion points at $\omega_{min}$ and $\omega_{max}$. The reduced order for the first iteration is equal to 4, the number of ports of the system. Then, as briefed in Section 3, the frequency responses is computed using a merged projection matrix (6) formed from the two expansion points. For the next iteration, the frequency response is computed for the same expansion points with an increased order of 8, i.e.: it is increased by the number of ports. Then the difference in response between the two models is computed using (7). The error obtained is 2.147, which is significantly greater than $10^{-3}$, the threshold set for the RMS error. Therefore, the algorithm increases the order of the expansion points and again computes the RMS error. Since the difference in the RMS error in the successive iterations is less than 10%, the algorithm checks for the new expansion point by computing the L1 norm of the best model and the original model. As shown in Fig.3, the norm of the frequency responses of the best model and the actual response gives the error per frequency and the new expansion point is considered at the frequency at which the error is maximum.

Then, the frequency response which is the admittance parameter $\mathbf{Y(s)}$, is again computed with all the expansion points, with a reduced order of 12 per expansion point. Similarly in this manner the sampling process is iterated till the RMS error (7) is less than the $10^{-3}$, the accuracy threshold value set.

Figure 4 plots the RMS error (7) between the iterated models when new expansion points are added during reflective exploration.

Figure 5, shows the magnitude of the admittance parameter $\mathbf{Y}_{11}$ obtained during the reflective exploration for different iterations. A best model of dimension 96 is obtained with 4 expansion points within a CPU time of 15.23 secs.

Then the model is compacted based on the truncation of the singular values of the common projection matrix. With the truncation algorithm described in Section 4, we obtain a model of order 85 by adaptively choosing a threshold of $10^{-4}$ in 0.715 secs as shown in Table 1.

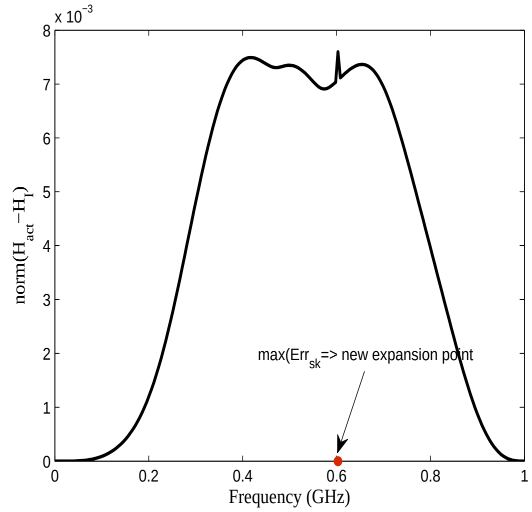


Figure 3: Example 1: Error per frequency used to select the new expansion point for the adaptive sampling loop.

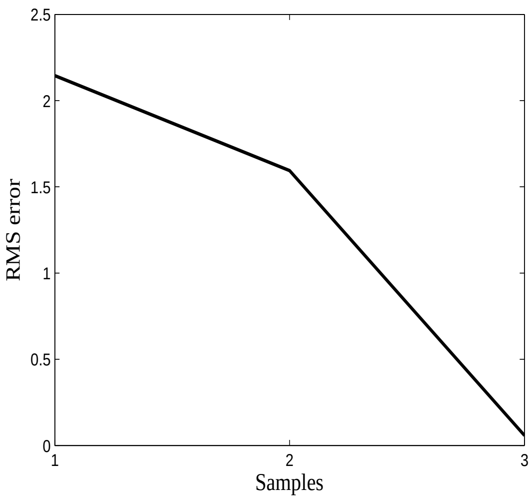


Figure 4: Example 1: RMS error between the iterated models during the addition of new expansion points.

Figure 6 plots the magnitude of $\mathbf{Y}_{11}$ for the original and the reduced model using 4 expansion points with a reduced order of 22 per expansion point.

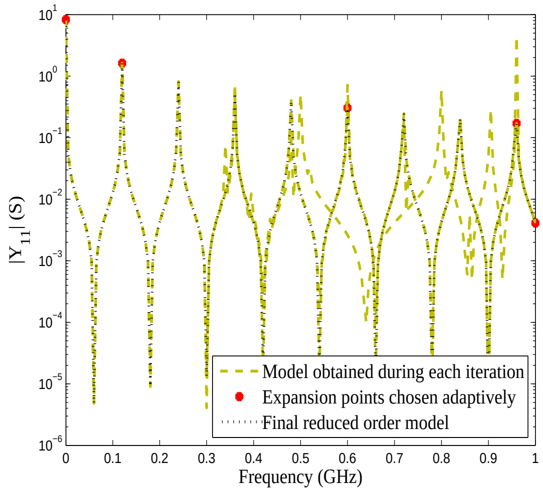


Figure 5: Example 1: Magnitude of $\mathbf{Y}_{11}$ for each iteration with the adaptively chosen expansion points.

Table 1: Example 1: Adaptive truncation for model compacting.

| Threshold | RMS Error | Dimension of ROM |
|---|---|---|
| $10^{-1}$ | $2.23 \times 10^{-1}$ | 61 |
| $10^{-2}$ | $1.703 \times 10^{-1}$ | 71 |
| $10^{-3}$ | $6.96 \times 10^{-2}$ | 77 |
| $10^{-4}$ | $9.72 \times 10^{-4}$ | 85 |

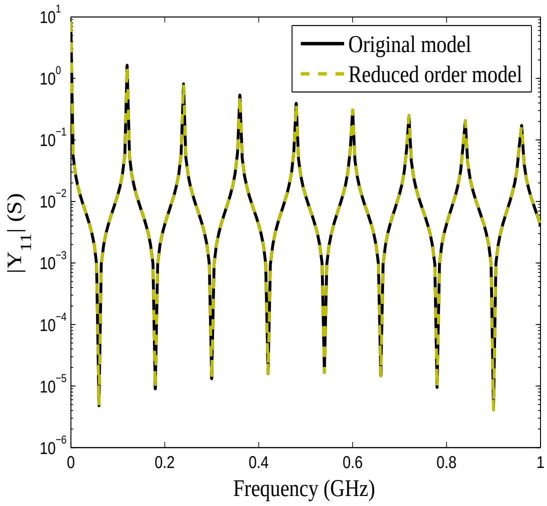


Figure 6: Example 1: Magnitude of $\mathbf{Y}_{11}$ after model compacting.

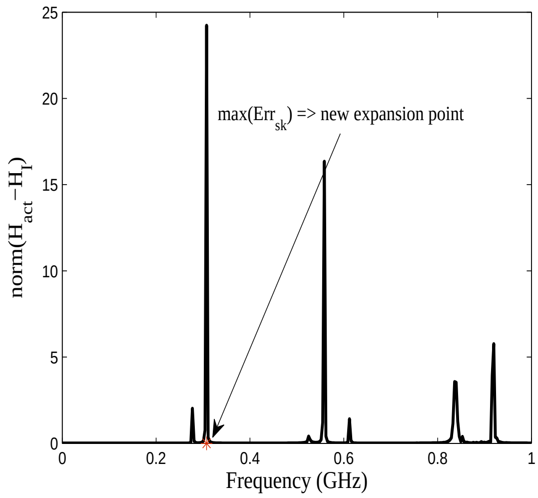


Figure 7: Example 2: Error per frequency used to select the new expansion point for the adaptive sampling loop.

## 6.2 Example 2: Lossless Transmission Line

For this example, a 20 cm lossless uniform coupled microstrip structure with two strips with the following per-unit-length matrices (Khalaj-Amirhosseini, 2006),

$$l_{pul} = \begin{bmatrix} 425.6 & 74.83 \\ 74.83 & 425.6 \end{bmatrix} \text{nH/m}$$

$$c_{pul} = \begin{bmatrix} 174.9 & 14.25 \\ 14.25 & 174.9 \end{bmatrix} \text{pF/m.} \qquad (17)$$

for a frequency range of interest $[1\ KHz - 1\ GHz]$, is modeled. The original system has an order of 1604 with 4 ports.

Similar to Example 1, two expansion points at $\omega_{min}$ and $\omega_{max}$ are considered. The reduced order for the first iteration is equal to 4, the number of ports. Then as briefed in Section 3, the frequency responses are computed using a merged projection matrix (6). Then similar to the previous case the difference in response between the two models (7) is computed and as the error is bigger than the threshold set, the algorithm checks for the next expansion point using the adaptive sampling loop as shown in Fig.11.

Similarly in this manner the sampling process is iterated till the RMS error (7) is less than threshold value of $10^{-3}$.

Figure 8 plots the RMS error (7) of the best two models during each iteration of the reflective exploration algorithm.

Figure 12, shows the frequency responses obtained during the reflective exploration for different

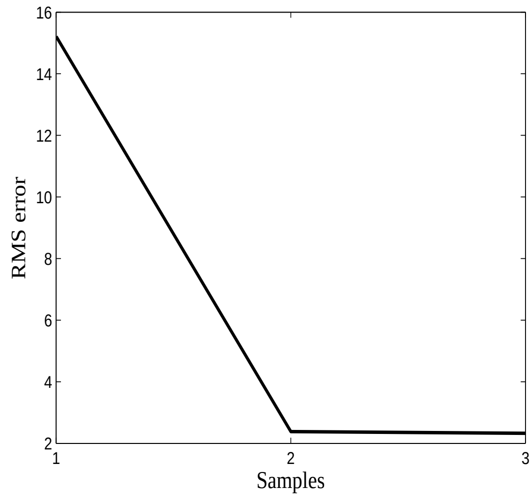


Figure 8: Example 2: RMS error between the iterated models during the addition of new expansion points.

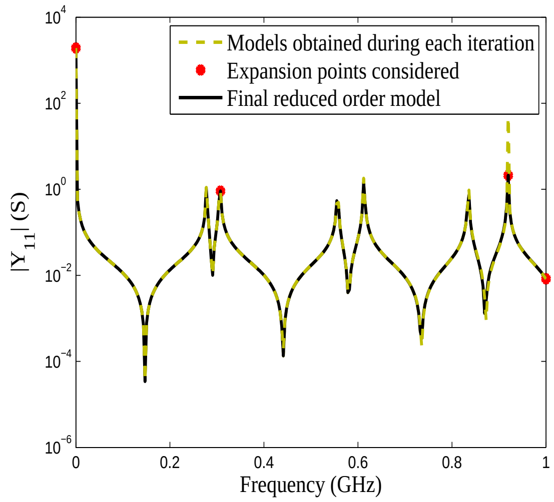


Figure 9: Example 2: Magnitude of $Y_{11}$ for each iterative step with the adaptively chosen expansion points.

iterations. The best model has dimension 64 and is obtained with 4 expansion points within a CPU time of 14.8 secs.

Finally, the model is compacted based on the truncation of the singular values of the common projection matrix. With the truncation algorithm described in Section 4, we obtain a model of order 35 by adaptively choosing a threshold of $10^{-2}$ in 0.16 secs as shown in Table 2.

Figure 13 plots the magnitude of $Y_{11}$ for the original and the reduced model using 4 expansion points with a reduced order of 9 per expansion point..

Table 2: Example 2: Adaptive truncation for model compacting.

| Threshold | RMS Error | Dimension of ROM |
|---|---|---|
| $10^{-1}$ | 8.04 | 29 |
| $10^{-2}$ | $8.645 \times 10^{-4}$ | 35 |

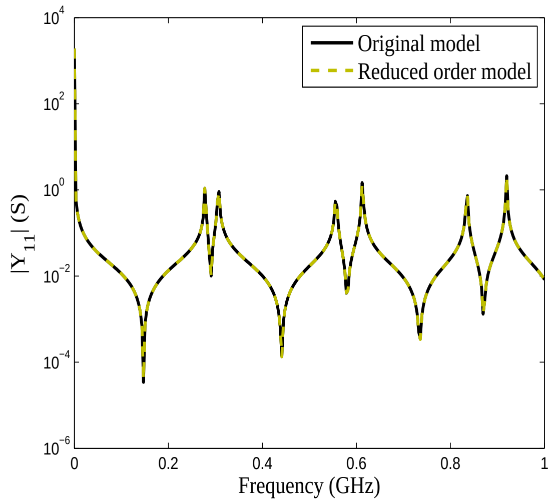


Figure 10: Example 2: Magnitude of $Y_{11}$ with the original response after model compacting.

## 6.3 Example 3: Modified Nodal Analysis

A modified nodal analysis (MNA) formulation for a 22 port circuit as given in the Niconet benchmark collections [1] is considered for this example. The original system has an order of 4863 with 22 ports for a frequency range of interest $[1\ KHz - 5\ GHz]$.

Similar to the previous examples, two expansion points at $\omega_{min}$ and $\omega_{max}$ are considered. The reduced order for the first iteration is equal to 22, the number of ports. Then as briefed in Section 3, the frequency responses are computed using a merged projection matrix (6). Then similar to the previous case the difference in response between the two models (7) is computed and as the error is bigger than the threshold set, the algorithm checks for the next expansion point using the adaptive sampling loop as shown in Fig.11.

Similarly in this manner the sampling process is

[1] http://www.win.tue.nl/niconet/niconet.html

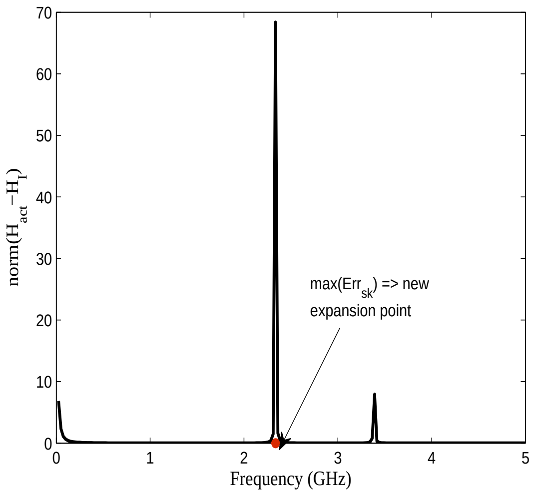


Figure 11: Example 3: Error per frequency used to select the new expansion point for the adaptive sampling loop.

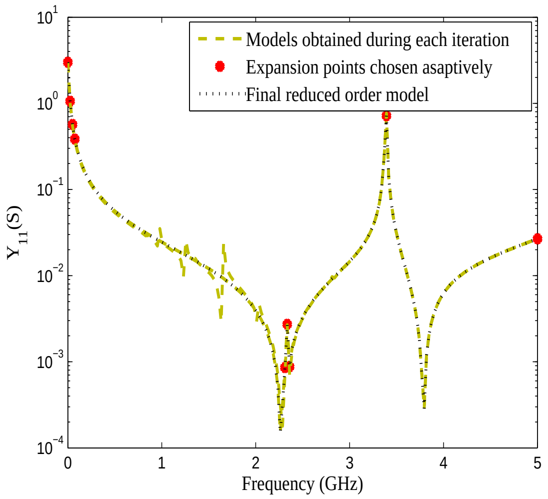


Figure 12: Example 2: Magnitude of $Y_{11}$ for each iterative step with the adaptively chosen expansion points.

iterated till the RMS error (7) is less than threshold value of $10^{-3}$.

Figure 12, shows the frequency responses obtained during the reflective exploration for different iterations. The best model has dimension 1122 and is obtained with 9 expansion points within a CPU time of 2048.5 secs.

Finally, the model is compacted based on the truncation of the singular values of the common projection matrix. As shown in Table 3, with the truncation algorithm described in Section 4, we obtain a model of order 202 by adaptively choosing a

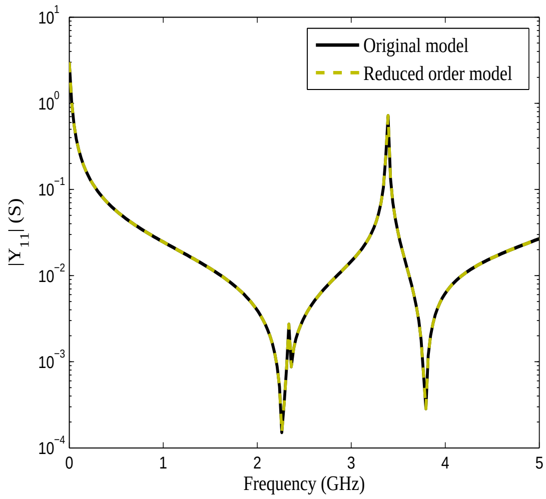


Figure 13: Example 3: Magnitude of $Y_{11}$ with the original response after model compacting.

threshold of $10^{-2}$ in 4.38 secs.

Table 3: Example 3: Adaptive truncation for model compacting.

| Threshold | RMS Error | Dimension of ROM |
|---|---|---|
| $10^{-1}$ | 15.09 | 88 |
| $10^{-2}$ | $2.645 \times 10^{-3}$ | 202 |

Figure 13 plots the magnitude of $Y_{11}$ for the original and the reduced model using 9 expansion points with a reduced order of 22 per expansion point..

From the examples described it can be illustrated that the proposed technique is able to capture the behavior of the system accurately and is able to preserve passivity of the original model by construction.

# 7 Conclusion

For model order reduction its important that the model is accurate over the whole frequency range of interest and must also preserve passivity. Passivity preservation is important since that interconnection of merely stable systems does not necessarily yield a stable system, while interconnection of passive systems yield a passive and hence stable system. In the literature several multipoint reduction algorithms with passivity preservation has been proposed to overcome this. In this paper, the adaptive selection of expansion point is based on a reflective exploration technique. The projection matrices obtained from the expansion points are merged to obtain the overall projection matrix for the system. To get a more compact model the merged projection matrix is truncated based on its singular values. Reduced order models are obtained by a congruence transformation, which preserves the passivity the system. The numerical examples validate the proposed technique.

## ACKNOWLEDGEMENTS

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