Modelling spread risk via time-change approach

Scuola di Scienze Statistiche
Curriculum Scienze Attuariali
Dottorato di Ricerca in Scienze Attuariali – XXXV Ciclo

Andrea Giustini
1597419

Tutor
Prof. Luca Passalacqua

Coordinatore
Prof.ssa Giovanna Jona Lasinio

Modelling spread risk via time-change approach

Andrea Giustini

Academic Year 2021-2022

Contents

List of Figures

List of Tables

Introduction

Directive 2009/138/CE of the European parliament and of the council ("Solvency II") requires insurance companies to have a level of Own Funds consistent with the risks to which they are exposed, at least equal to Solvency Capital Requirement (SCR), which is defined as the Value-at-Risk (VaR) of the one-year distribution of the company's Basic Own Funds (BOF), with a probability level of 99.5%.

The approaches proposed for calculating the SCR include, among others, the Standard Formula - a predefined model calibrated on data relating to the European insurance market - and the Internal Model, which should represent undertaking risk profile as accurately as possible. Both approaches require that balance sheet items should be evaluated according to a market consistent method. Therefore, in accordance with the principles of the Directive, valuation of BOF is carried out using risk-neutral probabilities and, on the other hand, valuation of VaR of the one-year distribution of the company's BOF should be based on real-world probabilities of risk factors affecting BOF. In order to comply Directive's principles, calibration of the models should be carried out on the basis of time series of market values, from which both probability distributions can be inferred.

Among the risks covered by SCR, in this thesis we analyzed spread risk, defined as "the sensitivity of the values of assets, liabilities and financial instruments to changes in the level or in the volatility of credit spreads over the risk-free interest rate term structure"[1] . Although many factors can affect the level or the volatility of credit spread, such as counterparty risk, tax effects and liquidity risk (Elton et al. 2001, Dignan 2003 and Driessen 2003), we assume that only counterparty risk is relevant, leaving other components as residuals. Counterparty risk can be defined as the risk that, in the context of a credit transaction, a debtor fails to meet his obligations (repay the principal and/or interest), even partially. The additional return required by the creditor, i.e. the risk premium for default risk, is composed by arrival risk, timing risk, and recovery risk (Schönbucher 2003). Arrival risk is a term for the uncertainty whether a default will occur or not; timing risk refers to the uncertainty about the precise time of default; recovery risk describes the uncertainty about the severity of the losses if a default

[1] Article 105 of the Directive.

has happened.

Considering the above assumptions, in order to manage spread risk we use two intensity-based models: the Duffie-Singleton model (DS model) proposed by Duffie and Singleton 1999 and the model developed by Li, Linetsky, and Mendoza-Arriaga 2016 in the context of electricity spot price modelling, properly adapted to model counterparty risk (as we will see, we call this model GMAC-JCIR model). The latter is set up within the one-dimensional Markovian framework and is based on the DS diffusion process, interspersed with compound Poisson's jumps with exponentially distributed jump size and a subordinated process as a random clock.

The DS model stems from Cox, Ingersoll, and Ross 1985 (CIR) square-root diffusion and has been a warhorse in the stochastic modeling of default risk in financial markets since the seminal work. In this framework, the default time can be thought of as the first jump time of a doubly stochastic Poisson process (Cox process) with stochastic intensity following a diffusion process. The attractiveness of the DS model stems from its dynamics and its analytical tractability. Under certain conditions, the DS process is nonnegative and mean-reverting, which is an important empirical feature observed in credit markets. Its analytical tractability primarly stems from its membership in the class of affine processes, which yields a closed-form solution for the survival probability (Schönbucher 2003) that essentially coincides with the expression for the bond price in CIR interest rate model. These properties lead to analytical pricing for a wide range of credit-sensitive instruments in CIR-based models (Brigo and Mercurio 2001).

A limitation of the CIR-based default intensity model is its inability to capture jumps in credit spread: this fact led a number of authors to introduce jumps into the CIR model (for example, Duffie and Garleanu 2001). In order to preserve analytical tractability, these models have been in the affine class.

On the other hand, a limitation of jump-CIR process (JCIR) is the one-sided nature of their jumps. From the standpoint of financial applications, its sample path behavior is somewhat unnatural because they can never jump down: if the process experiences a large jump up bringing it far away from its long-run mean, the only mechanism for it to return back to its long-run mean is via its continuous mean-reverting drift, with no possibility to jump back down - this is in contrast to the behavior often observed in financial markets where a jump in one direction may be followed by a jump in the opposite direction. This is often observed in energy markets or in credit markets, where the succession of good and bad news about the financial health of an obligor, such as a firm or a sovereign viewed by the markets to be in distress, can result in sharp changes in its market credit spreads over relatively short periods of time.

Mendoza-Arriaga and Linetsky 2014 have introduced more realistic two-sided jump behavior into diffusion intensity models via Bochner's subordination: starting with a nonnegative diffusion intensity model and time change it with a subordinator, they have obtained a jump-diffusion (when the subordinator has a positive drift) or a pure-jump (when the subordinator is driftless) intensity model with two-sided jumps that stays nonnegative. In particular, when the diffusion process is a CIR process, the time-changed model possesses a nonnegative intensity process with two-sided, mean-reverting jumps.

This thesis considers jump-diffusion subordination: in this way, the structure of the process remains state-dependent and numerically tractable due to availability of Laplace transform.

As we pointed out, the aforementioned aspects of the subordinated model allow to obtain both tractability and interesting features in sample paths: process' jumps are state-dependent and contribute to the return to the long-time average level, together with the mean-reversion drift component. This model characteristics are very important because they allow the GMAC-JCIR model to be more flexible than the DS model in describing the behavior of the tail (right-hand side) of the default intensity distribution. As a consequence, as will be seen below, the SCR for spread risk calculated with the GMAC-JCIR model will be more conservative than that calculated with the DS model.

For both models, estimation of model parameters is carried out on time series of market data through maximum likelihood estimation (MLE), where the likelihood function is calculated via particle filter technique (Bolviken and Storvik 2001); this approach allows to estimate both real-world and risk-neutral probability distributions as a whole.

This thesis is organized as follow. Chapter (**1**) introduce the spread risk and its components. Chapter (**2**) provides mathematical notations and tools from a theoretichal point of view, while Chapter (**3**) describes the modelling framework. Chapter (**4**) provides a brief description of the general theory of filtering; here we introduce the filter and we mention the situations in which particles are necessary. After market data presentation in Chapter (**5**), we apply the particle filtering technique to calibrate the models in Chapters (**6**) and (**7**); here we analyze the results of the calibration phase by comparing some statistics related to the market and model time series.

Chapter 1

Spread risk

The purpose of this chapter is to introduce the spread risk.

After providing basic financial definitions, which will be useful in the following, a review of credit spread measures - proposed both in the literature and financial practice - will be presented.

On the basis of credit spread measures, components which affect the level and the volatility of the observed credit spreads in the market will be examined. As we will see, in order to define a model for managing spread risk, only one of these components will be considered in our discussion.

Then, we will treat the main characteristics of spread risk in Solvency II, together with a summary of this regulatory structure. In this framework, the possibility - offered by the Directive - to use alternative methods to the Standard Formula for determining regulatory capital requirements - in particular, the so-called (partial) internal models - will be mentioned. This aspect is relevant in the scope of our discussion, as the methodologies for measuring the spread risk proposed in the following chapters fall under the scope of the internal methods for representing risk profile of an insurance company and calculating regulatory capital requirements.

The chapter concludes with a discussion on the available data for (partial) internal model calibration. It will be clear the importance of a proper choice of the financial product - or products - from which information should be extracted in order to measure and manage spread risk.

1.1 Basic definitions

In its simplest form, a financial contract allows to exchange monetary amounts in different points in time. The first definition is related to the basic type of contract, the risk-free zero-coupond bond.

Definition (**Risk-free zero-coupon bond**). A T-maturity risk-free zero-coupon bond is a contract that *guarantees* its holder the payment of one unit of currency at time T, with no intermediate payments. The contract value at time $t \leq T$ is denoted by $v(t, T)$, with $v(T, T) = 1$.

According the above definition, if we are at time t, a risk-free zero-coupon bond for the maturity T is a contract that establishes the present value of one unit of currency to be paid at time T (the maturity of the contract).

Times t and T can be measured through real numbers expressed as years, from an instant chosen as time origin. The time to maturity of the risk-free zero-coupon bond is defined as $T - t$.

Risk-free zero-coupon-bond prices are the basic quantities in interest-rate theory, then all interest rates can be defined in terms of risk-free zero-coupon-bond prices (Brigo and Mercurio 2001). In moving from risk-free zero-coupon bond prices to interest rates, and vice versa, the following definition is needed for our purposes:

Definition (**Yield to maturity intensity**). The yield to maturity intensity, prevailing at time t for the maturity T, is denoted by $h(t, T)$ and represents the constant rate at which an investment of $v(t, T)$ units of currency at time t accrues continuously to yield a unit amount of currency at maturity T:

$$h(t, T) = -\frac{\ln v(t, T)}{T - t}. \tag{1.1}$$

When a certain type of risk is introduced, for example the possibility that payment obligation may not be satisfied or may be satisfied in part, instead of risk-free zero-coupon bond, the following definition must be considered:

Definition (**Zero-coupon bond**). A T-maturity zero-coupon bond is a contract that *provides* its holder the payment of one unit of currency at time T, with no intermediate payments. The contract value at time $t < T$ is denoted by $\overline{v}(t, T)$.

The the yield-to-maturity intensity is denoted by:

$$\overline{h}(t, T) = -\frac{\ln \overline{v}(t, T)}{T - t}. \tag{1.2}$$

The last useful definition concern the money market account. Roughly speaking, a money-market account represents a (locally) risk-free investments, where profit is accrued continuously at the risk-free rate prevailing in the market at every instant.

Definition (**Money market account**). Let $B(t)$ be the value of a money market account at time $t \geq 0$. Assume $B(0) = 1$ and that the money market account evolves according to the following differential equation:

$$dB(t) = r_t B(t)dt, \qquad B(0) = 1, \tag{1.3}$$

where $r_t \geq 0$. As a consequence,

$$B(t) = \exp\Big(\int_0^t r_s ds \Big). \tag{1.4}$$

The above definition tells us that investing a unit amount in a money market account at time 0 yields at time t the value in (1.4), and r_t is the instantaneous rate at which the bank account accrues. Thus, r_t is usually referred to as instantaneous spot rate, or briefly as short rate, because is the instantaneous rate at which the bank account accrues.

It can be seen that r_t is also the yield to maturity intensity of a risk-free zero-coupon bond when the time to maturity goes to zero.

1.2 Credit spread measures

As we have seen, when a certain type of risk is introduced, a zero-coupon bond contract may be no longer risk-free.

In general, investors need measure to determine how much they are paid for to assume the risk of a credit transaction. These metrics are commonly referred to as credit spreads, because they attempt to measure the performance of a credit asset against the performance of a credit quality benchmark.

The choice of benchmark is arbitrary: if the benchmark is represented by a particular sovereign bond, the performance of a credit asset can be compared to the rate of return of the sovereign bond; otherwise, if the benchmark is represented by an interest rate curve, the performance of a credit asset can be compared to one of the rate of the curve.

In the following, we focus on the most common credit spread measures for fixed rate bonds, of which zero-coupon bond is a special case. For example, credit risk may be embedded within a bond issued both by some corporate or some sovereign, and credit spread measure should enable comparison between the credit quality of:

- securities issued by an issuer, which may differ in terms of maturity, coupon or seniority;

- securities issued by different issuers.

The yield spread is the first and the simplest credit spread measure. Its advantage is its semplicy, as it is defined as the difference between two yields, that of a defaultable bond and that of an associated treasury benchmark.

Definition (Yield spread). The yield spread is the difference between the yield to maturity of a credit risky bond and the yield to maturity of an on-the-run treasury benchmark bond with similar but not necessarily identical maturity.

To overcome the issue of the maturity mismatch, it is possible to use a benchmark yield where the correct maturity yield has been interpolated off the appropriate reference curve. Rather than choose a specific reference benchmark bond, the idea is to use a reference yield curve which can be interpolated.

Definition (Interpolated spread). The Interpolated spread - or I-spread - is the difference between the yield to maturity of a credit risky bond and the linearly interpolated yield to the same maturity on an appropriate reference curve.

The Option Adjusted Spread (OAS) was originally conceived as a measure of the amount of optionality embedded into a callable or puttable bond. However, the calculation methodology has since been borrowed by the credit markets and used for bonds which are not callable and so have no optionality. Used in this sense, the OAS becomes a convenient way to measure the credit risk embedded in a bond. For this reason, within a pure credit context, the OAS is often referred to as the zero volatility spread (ZVS) or Z-Spread.

Definition (Option Adjusted Spread). The Option Adjusted Spread of a fixed rate credit risky bond is the parallel shift to the benchmark curve required in order that the adjusted curve reprices the bond.

Unlike the artificial spread measures described so far, among the traded spread measure there is the asset swap spread. It is convenient to remember that, in a par asset swap package, a credit investor combines a fixed rate asset with a fixed floating interest rate swap in order to remove the interest rate risk of the fixed rate asset. There are two components to the package:

1. At initiation the investor pays par, and in return, receives the bond which is worth its full price.

2. The investor simultaneously enters into an interest rate swap, paying fixed, where the fixed leg cashflows are identical in size and timing to the coupon schedule of the bond. On the floating side of the swap, the investor receives a fixed spread over a benchmark risk-free curve – the asset swap spread. The floating leg of the swap is specified with its own frequency, basis and settlement conventions.

If the asset in the asset swap package defaults, the interest rate swap continues or can be closed out at market and the associated unwind cost is taken by the asset swap buyer. The asset swap buyer also loses the remaining coupons and principal payment of the bond, recovering just some percentage of the face value.

Definition (**Asset swap spread**). The Asset swap spread is the spread over a benchmark curve paid on the floating leg in a par asset swap package.

In the following, we will only use the yield spread definition in order to measure credit spreads. In particular, the credit spread of a zero-coupon bond with maturity T against a risk-free zero-coupon bond with the same maturity (i.e. the benchmark) can be defined in terms of yield to maturity intensity:

$$\overline{s}(t,T) = \overline{h}(t,T) - h(t,T). \tag{1.5}$$

1.3 Credit spread components

In the empirical literature of corporate bond it is shown that many factors can affect the level and the volatility of the credit spreads observed in the market. Even if the default risk is traditionally the main component, according to Elton et al. 2001, Dignan 2003 and Driessen 2003, credit spreads between corporate bond and treasury bonds - which can differ across rating classes - should be positive for the following reasons:

- Credit risk premium - corporate bonds can default and investors require a higher promised payment to compensate for the expected loss from defaults;
- Tax premium — interest payments on corporate and government bonds are typically subject to a different tax regime;
- Liquidity risk premium - in order to take into account the liquidity risk associated with the corporate security.
- Risk premium — the return on corporate bonds is riskier than the return on government bonds, and investors should require a premium for the higher risk;

Credit risk premium - or default risk premium - is intrinsically linked to the payment obligation that the obligor have to honour. Generally speaking, the behaviour of the obligor is regulated by the bankruptcy codes and the contract law: thanks to these, one can speak of the obligor's credit risk, without specifying a particular payment obligation, because the debtor should honor all his payment obligations as much as he is able to. However, obligors who are not bound to bankruptcy code, for example sovereign borrowers and borrowers in countries without a functioning legal system, frequently use the possibility of defaulting only on some of their obligations, sometimes without being in real financial distress; in this cases, the link between the credit risk of and the particular underlying payment obligation should not be ignored. Therefore, a general definition of the default risk covers both the obligor default risk and the particular payment obligation default risk.

Tax premium occurs because the investor in corporate bonds is often subject to a different and higher tax regime than the investor in government bonds. Thus, corporate bonds have to offer a higher pre-tax return to yield the same after-tax return. In Elton et al. 2001 it is shown that taxation explains between 28% and 73% of spreads - depending on rating and maturity - while in Driessen 2003, with a different sample and an other method, this component ranges from 34% to 57%.

Liquidity risk premium arises from situations in which the investor is interested in trading the corporate bond instead of hold it until maturity. Otherwise, liquidity risk premium can be ignored.

Regarding the last component, the risk premium, in Elton et al. 2001 it is shown that corporate bonds require a risk premium because spreads and returns vary systematically with the same factors that affect common stock returns. If investors in common stocks require compensation for this risk, so should investors in corporate bonds. According to Elton, this component occurs because a large part of the risk on corporate bonds is systematic rather than diversifiable.

Hereafter, in order to build a model for measuring and managing spread risk, the following assumption is adopted:

Assumption. *The level of the observed credit spread is affect only by credit risk component; other components are leaving as residuals.*

Schönbucher 2003 identifies the most important components of credit risk with the following:

- **Arrival risk**, as a term for the uncertainty whether a default will occur or not. To allow comparisons, it is specified against a given time horizon, usually one year. The measure of arrival risk is the probability distribution of the (indicator random) variable modelling default, before the time horizon.

- **Timing risk**, as the uncertainty about the precise time of default. Knowledge about the time of default includes knowledge about the arrival risk for all possible time horizons, thus timing risk is more detailed and specific than arrival risk. The measure of the timing risk is the probability distribution of the time of default.

- **Recovery risk**, as the uncertainty about the severity of the losses if a default has happened. In recovery risk, the uncertain quantity is the actual payoff that a creditor receives after a default of the obligor of the payment obligation. Market convention is to measure the recovery risk by a recovery rate, as a fraction of the notional value of the payment obligation. Recovery risk is described by the (conditional upond default) probability distribution of the recovery rate.

Thus, in our framework, change in credit spreads are driven only by changing in the distributions of the credit risk components, i.e., arrival risk, timing risk and recovery risk.

1.4 Spread risk in Solvency II

Directive 2009/138/CE of the European parliament and of the council ("Solvency II") requires insurance companies to have a level of Own Funds consistent with the risks to which they are exposed.

Own Funds should be at least equal to Solvency Capital Requirement (SCR), which is defined as the Value-at-Risk (VaR) of the one-year distribution of the company's Basic Own Funds (BOF), with a probability level of 99.5%.

Company's BOF represents the difference between assets and liabilities and should be evaluated according to a market consistent approach. In particular, "assets shall be valued at the amount for which they could be exchanged between knowledgeable willing parties in an arm's length transaction" (marked-to-market approach), and "liabilities shall be valued at the amount for which they could be transferred, or settled, between knowledgeable willing parties in an arm's length transaction" (current exit value approach).

Therefore, in accordance with the principles of Solvency II, in order to calculate SCR, valuation of BOF is carried out on a market consistent approach; on the other hand, valuation of VaR of the one-year distribution of the company's BOF should be based on real-world assumptions.

The simplest approach proposed by Solvency II for calculating the SCR is represented by the Standard Formula, a predefined model calibrated on data relating to the European insurance market. This model has a modular approach and its structure is shown in Figure (1.1).

The Standard Formula consists of a number of risk sub-modules, whose outcome are aggregated by a correlation matrix to reach a SCR for each module - the outcome of a risk sub-module is usually determined by calculating how a prescribed scenario would affect the insurer's balance sheet; SCR for each module

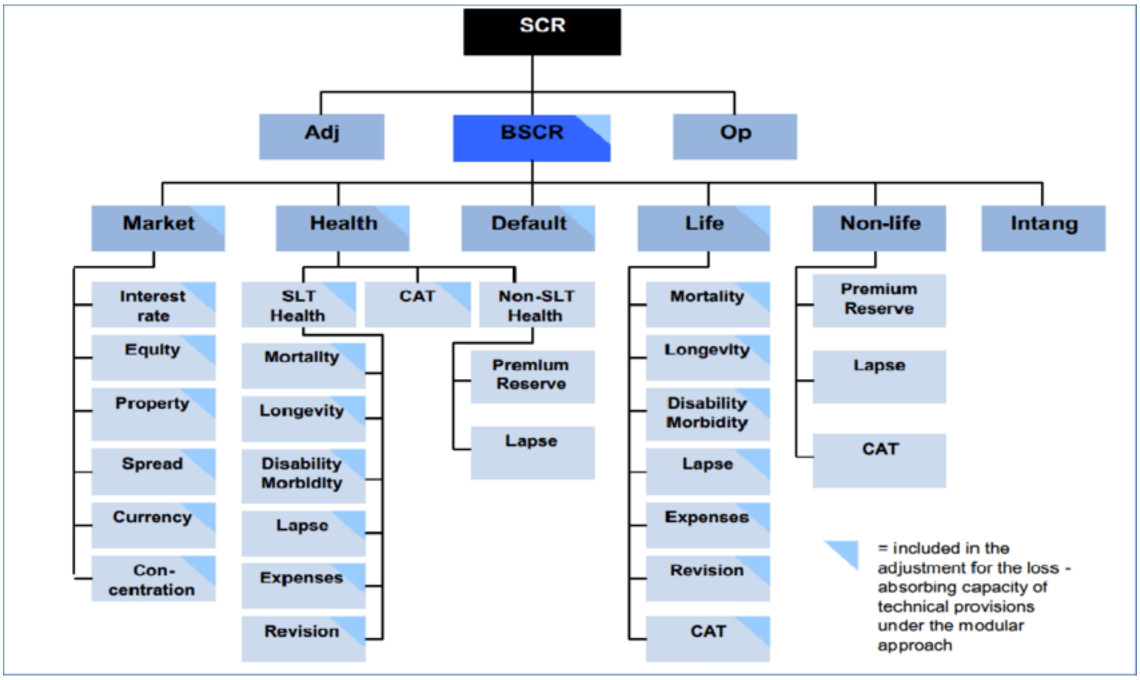

Figure 1.1: Risk sub-modules and risk modules in Solvency II Standard Formula.

are then aggregated in order to calculate the BSCR, which shlould be adjusted to reach the SCR, as a capital requirement for the insurance company.

Among the risks covered by Solvency II, we focus on spread risk, defined by the directive as "the sensitivity of the values of assets, liabilities and financial instruments to changes in the level or in the volatility of credit spreads over the risk-free interest rate term structure".

In the context of Standard Formula, spread risk represent a sub-module of market risk and the related capital requirement shall be equal to the following:

$$SCR_{spread} = SCR_{bonds} + SCR_{cd} + SCR_{sec} \tag{1.6}$$

where

- SCR_{bonds} denotes the capital requirement for spread risk on bonds (government and corporate) and loans;
- SCR_{sec} denotes the capital requirement for spread risk on securitisation positions and covers, in particular, ABS;
- SCR_{cd} denotes the capital requirement for spread risk on credit derivatives, for example CDS and structured products based on synthetic credit instruments.

It should be noted that, for the purposes of this thesis it is useful to focus exclusively on the calculation of SCR related to loans and bonds.

The capital requirement for spread risk on bonds and loans shall be equal to the loss in the basic Own Funds that would result from an instantaneous relative decrease in the value of each bond or loan. The decrease shall depend on the modified duration of the bond or loan - denominated in years - and, if bonds and loans have been subject to a credit assessment by an External Credit Assesment Institution (ECAI), it also shall depend on credit quality step of the bond or loan.

In order to harmonize the different scales provided by the various ECAIs, which follows internal agency assessment parameters, EIOPA, by means of Commission Implementing Regulation 2016/1800, imposed a necessary correspondence between the credit assessments of ECAIs and the objective scale of creditworthiness classes [1].

On the other hand, if bonds and loans are not subject to creditworthiness assessment by an ECAI, stress factors are defined according to their modified duration.

Subject to the approval of the competent supervisory authority, insurance companies can use alternative methodologies for SCR calculation, the so-called (partial) internal model; the choice of using internal methodologies is motivated by different considerations, such as the fact that internal methods should represent - better than Standard Formula - the company risk profile.

In the case of (partial) internal models, data play a fundamental role of as the starting point for the use of appropriate calibration methodologies for parameter estimation.

1.5 Available data

According to principles of the directive, "member States shall ensure that insurance and reinsurance undertakings have internal processes and procedures in place to ensure the appropriateness, completeness and accuracy of the data used in the calculation of their technical provisions".

Thus, data must comply with the principles of:

- *completeness*, i.e. the database that contains data must have all the fundamental and relevant information to manage all the risks to which insurance company is exposed; in addition, the database must be guaranteed in a sufficient level of granularity;

[1] Commission Implementing Regulation (EU) 2016/1800 of 11 October 2016 laying down implementing technical standards with regard to the allocation of credit assessments of external credit assessment institutions to an objective scale of credit quality steps in accordance with Directive 2009/138/EC of the European Parliament and of the Council

- *appropriateness* i.e., whether the data is functional to manage the set of risks;
- *accuracy*, in the sense of correctness and precision, i.e., data must be free of material errors and must be aligned with the valuation date.

Especially in case of (partial) internal models, data play a fundamental role for a proper calibration methodology. In general, there may be different methodologies to infer parameter values of these models.

For our purposes, in order to measure and manage spread risk and to calculate the associated SCR, time series of data should be used. As we can see in the following chapters, this allows for estimation of real-world and risk-neutral parameters of a (partial) internal model; among the financial information, two possible alternatives are available: 1) quotes from Credit Default Swap (CDS); 2) quotes from yield spreads.

As we will see in Chapter 5, for calibration purposes the IHS Markit data provider is used, from which different kinds of data are available via standard license.

Among the services provided, iBoxx is a financial division of IHS Markit that designs, calculates and distributes fixed income indices, including credit spread measures (for details, visit the iBoox documentation website).

The iBoxx product portfolio is global and is supported by multi-source pricing, where IHS Markit's proprietary Evaluated Bond Pricing Service ('EVB') aims to improve pricing accuracy, minimise tracking error and safeguard the independence of the indices. iBoxx coverage spans over 20000 bonds to date and includes corporate, sub-sovereign and sovereign bonds, as well as loans and securitised products.

Chapter 2

Theoretical background

The assumption that credit spreads are influenced solely by credit risk enables us to consider only to the default risk components, i.e., arrival risk, timing risk and recovery risk.

For modelling purposes, stochastic models for credit risk should be taking into account. In general, this models aim to describe the arrival of the default event and to measure its impact on the value of securities subject to credit risk.

In this chapter we will provide mathematical notations and tools that we will need to use in the rest of this thesis in the context of stochastic models for credit risk, that will be presented in Chapter 3.

Considering our purposes, in the presentation of theoretical background, generality is sometimes avoid in favour of simplicity, without sacrificing mathematical correctness. To fully understand the mathematics behind the models, Jacod and Shiryaev 1988, Protter 1990 or the survey by Liptser and Shiryaev 1998 can be considered.

In our framework, all processes and random variables are defined on a filtered probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, where Ω is the set of possible states of nature and the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ represents the information structure of the setup; in particular, $\mathcal{F}_t$ is the information available at time t, and random variables whose realisation is known at time t are said to be measurable with respect to $\{\mathcal{F}_t\}_t$. $\mathbb{P}$ is the probability measure that attaches probabilities to the events in Ω .

In general, when random variables will be introduced, a link with default risk component will be provided.

2.1 Stopping times

In order to model the *arrival risk* of a default event, a random point in time $\tau \in \Re^+$ should be considered. For this random variable we require that, at every point in time t, it should be known if τ has already occurred or not, so we require that τ has to be measurable with respect to $\{\mathcal{F}_t\}_t$:

$$(\omega : \tau(\omega) \leq t) \in \mathcal{F}_t \qquad \forall t \in \Re^+. \tag{2.1}$$

This property defines the random variable τ as a stopping time. An example of stopping time is represented by the time of the first passage of any sequence of random variables $\{X_n\}_n \in \Re$ for a certain level $a \in \Re$.

On the other hand, there are many examples of random times that are not stopping times. For example, consider a Brownian motion $W(t)$ on a fixed time interval $[0, T]$, and let τ be the time at which $W(t)$ attains its maximum. τ is a random point in time but it is impossible to assert at time t if the maximum really was attained; this is possible only at time T — after we have observed the whole path of $W(t)$ over $[0, T]$.

Among the properties of stopping times, the maximum and minimum of a set of stopping times is again a stopping time, as the sum of two stopping times.

In order to represent a stopping time with a stochastic process, we define its indicator process, i.e. a counting process, that jumps from zero to one at the stopping time:

$$N = \mathbf{1}_{\{t \geq \tau\}}.$$

2.2 Point process

If a stopping time is the mathematical description of one event, a point process is a generalisation of stopping times for considering multiple events.

Let $\{T_n\}_n$ a collection of stopping times, with $T_n < T_{n+1}$. The counting process associated with the collection of these stopping times is the random process:

$$N(t) = \sum_{n \geq 1} \mathbf{1}_{\{t \geq T_n\}} = \#\{n \geq 1,\, T_n \geq t\}. \tag{2.2}$$

Point processes should provide a good mathematical framework to analyse timing risk of default event of obligations. In fact, as we will see in intensity-based models, the time of the first jump of a point process can be seen as the time at which a default occurs.

As we can see in (2.2), the process $N(t)$ counts the number of point in time that the process jumps before t. If $T_n > 0\ \forall n$, then $N(t)$ would be a step function that starts at zero and increases by one at each T_n.

The advantage of using $N(t)$ instead of a stopping time is that it is a stochastic process, so the machinery of stochastic process can be used.

There are several examples of counting process. For our purposes, consider a sequence of random variables $\{T_n, n \geq 1\}$, where $T_n \sim \text{Gamma}(\lambda, n)$ and $\lambda > 0$, called the intensity of the process. Then:

$$T_k \leq t \Leftrightarrow N(t) \geq k$$

and

$$\begin{aligned} P\{N(t) = k\} &= P\{T_k \leq t < T_{k+1}\} \\ &= P\{T_k \leq t\} - P\{T_{k+1} \leq t\} = \frac{e^{-\lambda t}(\lambda t)^k}{k!}. \end{aligned}$$

When $N(t) \sim \text{Poisson}(\lambda t)$, $N(t)$ is said to be an homogeneous Poisson process. Poisson process is a non-decreasing, integer-valued process with initial value $N(0) = 0$ whose increments are independent and follow an exponential distribution.

Poisson processes are usually used to model either rare events (for example, in insurance mathematics) or discretely countable events (e.g. radioactive decay, the number of customers in a queue etc.). Both properties also apply to defaults: they are rare and they are discrete.

If the intensity λ of the Poisson process is a (non-constant) deterministic function of time $\lambda(t)$, $N(t)$ is said to be an inhomogeneous Poisson process. Its properties are very similar to the properties of a homogeneous Poisson process and the probability distribution fuction is:

$$P\{N(t) = k\} = e^{-\int_0^t \lambda(s)ds} \frac{\left[\int_0^t \lambda(s)ds\right]^k}{k!}. \tag{2.3}$$

A further generalization of a Poisson process requires that $\lambda(t)$ is a stochastic process: in this case, the process $N(t)$ with stochastic intensity $\lambda(t)$ is called Cox process.

The jumps of a Cox process are clearly correlated via the path taken by $\lambda(t)$. In fact, roughly speaking, if there is a jump at some time T_i, that makes it more likely that $\lambda(t)$ is large around T_i, which in turn means that the next jump is more likely to happen sooner than later; however, this correlation is only indirect, via the default intensity.

If the full realisation of $\lambda(t)$ is known in advance, then the counting process is an inhomogeneous Poisson process. This is also the basic idea of a formal definition of a Cox process. Let:

$$\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{F}_t^N,$$

with the process:

- $\lambda(t)$ adapted to the filtration $\{\mathcal{G}_t\}_t$;
- $N(t)$ adapted to the filtration $\{\mathcal{F}_t^N\}_t$;

then, a point process $N(t)$ with intensity process $\lambda(t)$ is a Cox process if, conditional on the background information $\{\mathcal{G}_t\}_t$, $N(t)$ is an inhomogeneous Poisson process with intensity $\lambda(t)$.

2.3 Pricing rules

In this section we will introduce the fundamental concepts behind pricing, absence of arbitrage and equivalent martingale measures, using a minimum of jargon and technicality. For more technical expositions consider J. Harrison and D. Kreps 1979, J. M. Harrison and S. R. Pliska 1981, Kabanov 2001 and Delbaen and Schachermayer 1998.

The case of the so-called perfect market is considered, meaning that: all investors are price-takers; all parties have the same access to the relevant information; there are no restriction on short-selling; there are no costs for taxes or transactions; all assets are assumed to be perfectly divisible and liquid.

Suppose that the possible evolution of this market, between the points in time 0 and T^*, are described by a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. As usual, $\mathcal{F}_t$ contains all statements which can be made about behavior of prices up to t, while $\mathbb{P}$ is either the "objective" probability of future scenarios or the subjective view of an investor, and it also called real-world probability.

Assets in the market may then be described by a non-anticipating process:

$$\begin{aligned} S : [0, T^*] \times \Omega &\to \Re^{d+1} \\ (t, \omega) &\to \left(S_t^0(\omega), S_t^1(\omega), ..., S_t^d(\omega)\right), \end{aligned} \tag{2.4}$$

where $S_t^i(\omega)$ represents the value of asset i at time t in the market scenario ω, and $S_t^0 := B$ is a numeraire (see definition in Chapter 1).

A contingent claim H with maturity T may be represented by specifying its terminal payoff $H(T; \omega)$ in each market scenario ω.

Let $\mathcal{H}$ be the set of contingent claims of interest. It is natural to assume $S_t^i \in \mathcal{H}$; other non-exhaustive examples are European call options and European put options, as well as path dependent options, where the payoff of the contingent claim can depend on the whole path of the underlying assets.

In order to attribute a value to each contingent claim $H \in \mathcal{H}$, adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, a pricing rule - i.e. a valuation operator - should be defined, as a procedure which attributes to each contingent claim $H \in \mathcal{H}$ a monetary value $\Pi_t(H)$ at each point in time $t \in [0, T]$.

Minimum requirements of a pricing rule include:

- the possibility to compute the value using the information given at t (thus, $\Pi_t(H)$ should be a non-anticipating process);
- positiveness, meaning that a claim with a positive payoff should naturally have a positive value:

$$X \geq 0 \quad q.c. \quad \Rightarrow \quad \Pi_t(X) \geq 0 \quad \forall t \in [0, T^*];$$

- linearity, meaning that the value of a portfolio is given by the sum of the values of its components:

$$\Pi_t\left(\sum_{j=1}^{J} X_j\right) = \sum_{j=1}^{J} \Pi_t(X_j).$$

Another fundamental requirement for a pricing rule is that it does not generate arbitrage opportunities. Roughly speaking, an arbitrage opportunity is a self-financing strategy ϕ, which can lead to a positive terminal gain without any probability of intermediate loss:

$$\forall t \in [0, T^*] \quad \mathbb{P}\{\Pi_t(\phi) \geq 0\} = 1, \qquad \mathbb{P}\{\Pi_T(\phi) \geq \Pi_0(\phi)\} \neq 0.$$

It should be noted that the definition of the arbitrage opportunity involves $\mathbb{P}$, but this probability measure is only used to specify whether the profit is possible or impossible, not to compute its probability of occurring, because only events with probability 0 or 1 are involved. Thus, in the sequel will not require a precise knowledge of probabilities of market scenarios under $\mathbb{P}$.

Under certain condition (see Cont and Tankov 2004), it can be shown that in an arbitrage-free market described by a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, prices are given by the following pricing rule:

$$\Pi_t(H) = \mathbb{E}^{\mathbb{Q}}\{\frac{B_t}{B_T} H | \mathcal{F}_t\} \tag{2.5}$$

where $\mathbb{Q}$ is an equivalent martingale measure such that:

$$\forall A \in \mathfrak{F} \quad \mathbb{Q}(A) = 0 \Leftrightarrow \mathbb{P}(A) = 0, \tag{2.6}$$

$$\mathbb{E}^{\mathbb{Q}}\{\hat{S}_T^i|\mathcal{F}_t\} = \mathbb{E}^{\mathbb{Q}}\{\frac{S_T^i}{B_T}|\mathcal{F}_t\} = \frac{S_t^i}{B_t} = \hat{S}_t^i.$$

Therefore, any arbitrage-free pricing rule is given by an equivalent martingale measure $\mathbb{Q}$, also called risk-neutral measure. Conversely, it can be shown that any risk-neutral measure defines an arbitrage-free pricing rule (Cont and Tankov 2004).

Thus, there is a one-to-one correspondence between arbitrage-free pricing rules and equivalent martingale measures: specifying an arbitrage-free pricing rule on $(\Omega, \mathfrak{F}, \mathbb{P})$ is equivalent to specifying a probability measure $\mathbb{Q} \sim \mathbb{P}$ on market scenarios such that the prices of traded assets are martingales.

Up to now we have assumed that an arbitrage-free pricing rule, i.e. an equivalent martingale measure, does indeed exist, which is not obvious in a given market model. In fact, the above must be interpreted as, if an equivalent martingale measure exists, then the market is arbitrage-free. The converse result, more difficult to show, is sometimes called the Fundamental theorem of asset pricing.

It should be noted that a proper mathematical statement requires a careful specification of this theorem: in fact, for a general unbounded semi-martingale (such as an exponential-Levy models with unbounded jumps) "martingale measure" should be replaced by the notion of "σ-martingale measure", the definition of arbitrage opportunity should be modified to "no free lunch with vanishing risk," etc. For more details on this topic see Delbaen and Schachermayer 1998.

In general, the existence of an equivalent martingale probability measure depends on the market model considered. In the models for credit risk discussed in Chapter 3 the requirement is guaranteed, so it will be possible to evaluate contracts by means of the (2.5).

Appendix

The role of probability measures in Solvency II

As we have seen in Chapter 1, within Solvency II framework assets and liabilities of an insurance company should be evaluated according to a market consistent approach.

Let t be a point in time representing a valuation date, for example the end of the financial year; let A_t and V_t be the market consistent values of assets and liabilities, respectively.

Therefore, Own Fund at time t can be defined as:

$$OF_t = A_t - V_t. \tag{2.7}$$

When a time horizon from t is considered, for example a equal to one year, with $T = t + 1$, capital requirement calculation is based on the quantity OF_T, which is a random variable in t with its own probability distribution.

According to the principle of the Directive, let $\epsilon = 0.5\%$ be a probability such that W_T can be defined as the ϵ-percentile of the distribution of OF_T, as a pessimistic value - a worst case value - of the Own Funds at the end of the time horizon.

Since the SCR covers the unexpected loss, the difference

$$U_T = \mathbb{E}_t\{OF_T\} - W_T \tag{2.8}$$

express the maximum potential loss with respect to the expected value of the Own Funds at the end of the time horizon, considering a confidence level equal to 1 - 99.5%. In addition, since U_T expresses a potential loss which may to occur in one year from t, it should be financially referred to the time of valuation:

$$SCR_t = v(t,T)U_T = v(t,T)[\mathbb{E}_t\{OF_T\} - W_T]. \tag{2.9}$$

As pointed out in the first part of this chapter, $\mathbb{P}$ is the probability measure that represents operators' expectations of the future scenarios of the market, while $\mathbb{Q}$ is a probability measure "imposed" by the valuation model and identified on the basis of the principle of no arbitrage, representing a weighting structure of future events and not a substitute for the natural probability measure.

Thus, the expected value appearing in relation (2.9) should be calculated under the real-world probability measure. On the other hand, because of the principle of the market consistent valuation, for the valuation of assets and liabilities the risk-neutral probability measure should be use.

Generally speaking, when one is interested only in pricing contracts, the risk-neutral probability measure is needed. On the other hand, if one should measure and manage risk and needs to formalize "real" expectations about the future events, both probability measures should be considered.

Chapter 3

Credit risk models

As we have seen in the previous chapters, for modelling spread risk we focus only on credit risk components, i.e. arrival risk, timing risk and recovery risk, leaving the others as residuals. Thus, in this chapter we will provide an overview of credit risk models.

Roughly speaking, credit risk models are of two type: structural models and intensity-based models. In structural models, the default event is defined in terms of the (stochastic) process modeling the assets of an issuer, and default is triggered when the assets hit (or fall below) some boundary. On the other hand, intensity-based models are based on a default-arrival process.

With respect to the scope of application, models can be further catalogued in two classes:

- models for the valuation of individual counterparties (which are structural models). In industry there are several examples: EDF RiskCalc model of Moody's KMV (Moody's KMV 2003 and KMV 2005) and the Credit-Grades model (Group 2002, produced by a collaboration between RiskMetrics, Goldman Sachs, JP Morgan, and Deutsche Bank.
- models for the valuation of portfolios (which are both structural and intensity model). In industry there are three major examples, represented by the CreditMetrics models of J.P. Morgan (Morgan 1997, Credit Suisse's CreditRisk+ (CSFB 1997) and McKinsey's CreditPortfolioView (Wilson 1997).

The first part of this chapter consists of the presentation of two particular structural models, the Merton model and the Black-Cox model.

Intensity-based models are the subjects of the second part of this chapter. Here, after providing general considerations about this kind of model, two particular intensity-based models will be presented, which will be the subjects of the rest of this thesis.

3.1 Structural models

This section reviews the valuation of corporate debt in a perfect market setting, where the machinery of option pricing can be brought to use. For our purposes, valuation of zero-coupon bond can be considered as a special case.

The starting point of the models presented in this section is to take as given the evolution of the market value of a firm's assets and to view all corporate securities, including corporate debt, as contingent claims on these assets. This approach dates back to Black and Scholes 1973 and Merton 1974.

Thus, models of this kind are based upon a stochastic process for the firm's value and upon a fundamental approach to valuing defaultable debt providing a link between the prices of equity and all debt instruments issued.

More in particular, firm's value models are built on the premise that there is a fundamental process $V(t)$, usually interpreted as the total value of the assets of the firm that has issued a corporate bond, with V is used to pay off the debt at maturity of the contract. A default occurs at maturity if V is insufficient to pay back the outstanding debt; alternatively (and more realistically) one can assume that a default is already triggered as soon as the value of the collateral V falls below a barrier K. This latter feature is exactly identical to a standard knockout barrier in equity options, and was first used in Black and Cox 1976, therefore we will call models with this kind Black—Cox-type models.

3.1.1 The Merton model

Assume that the time horizon is $[0, T]$ and let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, on which a standard Brownian motion $Z(t)$ is defined.

A bond B issued by a firm should be priced. For this purpose, assume that the firm's value follows a geometric Brownian motion:

$$dV_t = \mu_{\mathrm{V}} V_t dt + \sigma_{\mathrm{V}} V_t dZ_t, \qquad V_0 = V(0), \tag{3.1}$$

so that

$$V_t | V_0 \sim LN\left((\mu_{\mathrm{V}} - \frac{1}{2}\sigma_{\mathrm{V}}^2)t, \sigma_{\mathrm{V}}^2 t \right). \tag{3.2}$$

Assuming that the firm has issued at time 0 two type of claims, an equity and a debt, as a zero-coupon bond with a face value of D and maturity date T.

At time T, if the firm's value V_T is less than D, equity owners, which have limited liability, do not honour the payment obligation represented by the bond.

At best, bond holders take over the remaining asset and receive a "recovery" of V_T instead of the promised payment.

With these assumptions, at date T payoffs of the corporate bond and the equity are:

$$B_T = \min(V_T, D) = D - \max(D - V_T, 0) \tag{3.3}$$

and

$$S_T = \max(V_T - D, 0). \tag{3.4}$$

Thus, corporate debt issued by the firm can be viewed as the difference between a risk-free bond and a put option, while equity can be viewed as a call option on the firm's assets. In particular, both options are written on $V(T)$ with strike-price equal to D.

Therefore, under the Black–Scholes model, values at time t of the claims are:

$$B_t = Dv(t, T) - P(t) \tag{3.5}$$

and

$$S_t = C(t), \tag{3.6}$$

where $P(t)$ and $C(t)$ are the value of the European put and call options defined above.

It can be shown (Lando 2004) that, in Merton model, when the value of assets is greater than the amount of the debt, yield spreads goes to zero as time to maturity goes to 0.

3.1.2 The Black-Cox model

The Black-Cox model is an extension of the Merton model, in which defaults can occur prior to the maturity of the bond.

Even in this model, let $[0, T]$ be the time horizon and $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, on which a standard Brownian motion $Z(t)$ is defined.

A bond B issued by a firm should be priced; the assumptions about the firm's value remain the same as in the Merton model. Nevertheless, defaults can occur prior to the maturity of the bond and they will happen when the level of the asset value hits a lower boundary K, modelled as a deterministic function of time:

$$K_t = \begin{cases} Ke^{-\gamma(T-t)} & t \in [t_0, T) \\ D & t = T \end{cases} \tag{3.7}$$

In this framework, default event is defined as the time of the first passage of the firm's value process $V(t)$ for the lower boundary K_t:

$$\tau = \inf\{t \in (t_0, T] : V_t < K_t\}$$

The payoff of the corporate debt B is then given by:

$$\begin{aligned} B_T &= \min\{V_T, D\}\mathbf{1}_{\tau>T} \\ &= D\mathbf{1}_{\tau>T} - \max\{D - V_T, 0\}\mathbf{1}_{\tau>T}. \end{aligned}$$

Thus, the value of the corporate debt at time $t < T$ is given by:

$$\begin{aligned} B_t &= \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\tau>T}e^{-r(T-t)}D + \mathbf{1}_{\tau\leq T}e^{-r(T-t)}K_\tau|\mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\tau>T}e^{-r(T-t)}D + \mathbf{1}_{\tau=T}e^{-r(T-t)}V_T + \mathbf{1}_{\tau<T}e^{-r(T-t)}Ke^{-\gamma(\tau-t)}|\mathcal{F}_t\right] \end{aligned}$$

Recovery risk is closely related to timing risk in the Black-Cox model. In fact, the default instant defines the payoff recovered by the holder of the risky bond because in this point in time the value of the assets is equal to the value of the barrier, which evolves in a deterministic way.

3.2 Intensity-based models

In intensity-based models, default time is defined as the time of the first jump of a point process. In our framework, the default instant τ coincides with the instant at which a point process $N(t)$ with intensity λ_t makes the first jump:

$$\tau = \inf\{t \in \Re^+ : N(t) > 0\} \tag{3.8}$$

It is possible to show that the value at time t of a contract subject to credit risk with maturity in T and payoff X_T is (Schönbucher 2003):

$$X_t = \mathbb{E}^{\mathbb{Q}}\{e^{-\int_t^T r_u du}\mathbf{1}_{\{\tau>T\}}X_T + e^{-\int_t^\tau r_u du}\mathbf{1}_{\{\tau\leq T\}}g_\tau|\mathcal{F}_t\}, \tag{3.9}$$

where $\mathbb{Q}$ is a martingale measure equivalent to the real-world probability measure $\mathbb{P}$, τ is the instant of default, $\mathbf{1}_A$ is the indicator function of event A, g_τ is the value of the contract in case of default, and the spot rate r_t uniquely determines term structure of interest rates. Assuming that g_τ is a deterministic fraction δ (Recovery Rate) of the value of the contract at the instant immediately preceding the default:

$$g(\tau) = \delta X_{\tau^-} \qquad \delta \in [0, 1), \tag{3.10}$$

the price of the contract X can be rewritten as (Duffie and Singleton 1999):

$$X_t = \mathbb{E}^{\mathbb{Q}}\{e^{-\int_t^T [r_u + (1-\delta)\lambda_u]du} X_T | \mathcal{F}_t\}, \tag{3.11}$$

Let $v(t, t+u)$ be the value at time t of a risk-free zero-coupon bond with maturity $t+u$ and $\overline{v}^R(t, t+u)$ be the value at time t of a zero-coupon bond with maturity $t+u$, issued by an issuer with rating R and conditional on the issuer not going bankrupt in $[0; t]$.

From (3.11), assuming independence between interest rate risk and credit risk:

$$\begin{aligned} \overline{v}^R(t, t+u) &= \mathbb{E}^{\mathbb{Q}}\{e^{-\int_t^{t+u} [r_z + (1-\delta)\lambda_z]dz} | \mathcal{F}_t\} \\ &= v(t, t+u)\mathbb{E}^{\mathbb{Q}}\{e^{-\int_t^{t+\tau} (1-\delta)\lambda_z dz} | \mathcal{F}_t\}. \end{aligned} \tag{3.12}$$

Therefore, the yield-to-maturity intensities are given by:

$$h(t, t+u) = -\frac{1}{u} \log v(t, t+u), \tag{3.13}$$

$$\overline{h}^R(t, t+u) = -\frac{1}{u} \log \overline{v}^R(t, t+u). \tag{3.14}$$

In terms of yield to maturity, yield credit spreads can be defined as:

$$\begin{aligned} \overline{s}^R(t, t+u) &= \overline{h}^R(t, t+u) - h(t, t+u) \\ &= -\frac{1}{u}\mathbb{E}^{\mathbb{Q}}\{e^{-\int_t^{t+u} (1-\delta)\lambda_z dz} | \mathcal{F}_t\} \\ &:= -\frac{1}{u} Q(t, t+u) \end{aligned} \tag{3.15}$$

for every t and $u > 0$. In the rest of this thesis, we omit the reference R of the rating for credit spreads and we assume $\delta = 0$.

According to the above relations, in order to specify an intensity-based model we should specify the evolution of the intensity λ_t.

3.2.1 The Duffie-Singleton model

The first model we point out is a one-dimensional version of the CIR-style model proposed by Duffie and Singleton (DS) in Duffie and Singleton 1999.

Real-world dynamic

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. We define $\mathbb{P}$ as the real-world probability measure. Under $(\Omega, \mathfrak{F}, \mathbb{P})$, in DS we model the instantaneous default intensity as the unique solution to the following stochastic differential equation (SDE):

$$d\lambda_t = \alpha(\gamma - \lambda_t)dt + \rho\sqrt{\lambda_t}dZ_t^{\mathbb{P}}, \tag{3.16}$$

where $\alpha, \gamma, \rho > 0$ and $\lambda_0 = \lambda(0) > 0$. We also impose the Feller condition $2\alpha\gamma \geq \rho^2$, so that zero is an unattainable boundary for the process λ_t.

The dynamics of the deterministic component $f(t, \lambda_t) = \alpha(\gamma - \lambda_t)$ ensures mean reversion of the values towards the long run value γ, with speed of adjustment governed by the strictly positive parameter α.

On the other hand, the form of the stochastic component $g(t, \lambda_t) = \rho\sqrt{\lambda_t}$ avoids the possibility of negative values for all positive values of α and γ, and it characterizes a disturbance term proportional to the value of the intensity itself, which allows great volatility in periods of high spreads.

SDE (3.16) implies for the conditional distribution of $\lambda_{t+u}|\lambda_t$ a non-central chi-squared distribution, making the process stationary. In particular, we have:

$$p(\lambda_{t+u}|\lambda_t) = \frac{\rho^2(1 - e^{-\alpha u})}{4\alpha}\chi_2(d, v) \qquad u \in \Re^+, \tag{3.17}$$

where $\chi_2(d, v)$ is the probability distribution function of a non-central chi-squared distribution with $d = \dfrac{4b\alpha}{\rho^2}$ degrees of freedom and non-centrality parameter $v = \dfrac{4\alpha e^{-\alpha u}}{\rho^2(1 - e^{-\alpha u})}\lambda_t$.

Equivalent measure change

Using the the following market price of risk:

$$\varrho(t, \lambda_t) = -\pi\frac{\sqrt{\lambda_t}}{\rho} \qquad \pi > 0, \tag{3.18}$$

it is possible to show that the dynamic of the process λ_t under the risk-neutral probability $\mathbb{Q}$ is still a mean-reverting process:

$$d\lambda_t = \hat{\alpha}(\hat{\gamma} - \lambda_t)dt + \rho\sqrt{\lambda_t}dZ_t^{\mathbb{Q}}, \tag{3.19}$$

where

$$\hat{\alpha} := \alpha - \pi \tag{3.20}$$

and

$$\hat{\gamma} := \frac{\alpha}{\alpha - \pi}\gamma. \tag{3.21}$$

Pricing formulas

Under (Ω, $\mathfrak{F}$, $\mathbb{Q}$), it can be shown that credit spreads defined in (3.15) are given by (Jeanblanc M. 2009):

$$\overline{s}(t, t+u) = \overline{h}(t, t+u) - h(t, t+u) = \frac{1}{u}a(u; \vec{\mathbf{q}}) + \frac{1}{u}\lambda_t B(u; \vec{\mathbf{q}}), \tag{3.22}$$

with $a(u; \mathbf{q}) = -\log A(u; \mathbf{q})$ and:

$$A(u; \vec{\mathbf{q}}) = \left[\frac{2de^{(\hat{\alpha}+d)u/2}}{(\hat{\alpha}+d)(e^{du}-1)+2d}\right]^{\nu} \tag{3.23}$$

$$B(u; \vec{\mathbf{q}}) = \frac{2(e^{du}-1)}{(\hat{\alpha}+d)(e^{du}-1)+2d} \tag{3.24}$$

$$d = \sqrt{\hat{\alpha}^2 + 2\rho^2} \qquad \nu = \frac{2\hat{\alpha}\hat{\gamma}}{\rho^2} \tag{3.25}$$

3.2.2 The GMAC-JCIR process

The second model we point out is a generalization of the DS model, developed by Li, Linetsky, and Mendoza-Arriaga 2016 in the context of electricity spot price modelling and properly adapted to model credit risk.

Set up within the one-dimensional Markovian framework, the model is based on the Cox-Ingersoll-Ross (CIR) diffusion process, interspersed with compound Poisson's jumps with exponentially distributed jump size and a subordinated process as a random clock. These results allow to obtain both tractability and interesting features in sample paths: process' jumps are state-dependent and contribute to the return to the long-time average level, together with the mean-reversion drift component.

Real-world dynamic

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. We define $\mathbb{P}$ as the real-world probability measure. Under $(\Omega, \mathfrak{F}, \mathbb{P})$, we model the instantaneous default intensity as:

$$\lambda^{\phi}(t) = \lambda(T_t), \tag{3.26}$$

where $\lambda(t)$ is a jump-CIR process (JCIR) and T_t is a random clock.

A JCIR process is the unique solution to the following SDE:

$$d\lambda_t = \alpha(\gamma - \lambda_t)dt + \rho\sqrt{\lambda_t}dZ_t^{\mathbb{P}} + dJ_t \tag{3.27}$$

where $\alpha, \gamma, \rho > 0$ and $\lambda_0 = \lambda(0) > 0$. As in DS, we impose the Feller condition $2\alpha\gamma \geq \rho^2$, so that zero is an unattainable boundary for $\lambda(t)$; $J(t)$ is a compound Poisson process, independent from $Z(t)$, with arrival rate $\omega > 0$ and the jump size which follows an exponential distribution with mean $\mu > 0$.

We choose the random clock T_t as an additive subordinator, i.e. a non-negative and non-decreasing additive process (Sato 1999). In particular, let T_t be a Gamma process (Madan and Chang 1998), independent from $(\lambda_t)_t$; its Lévy measure is given by:

$$\nu(d\tau) = \frac{m^2/\upsilon}{\tau}e^{-\frac{m}{\upsilon}\tau}d\tau, \tag{3.28}$$

where $m = \mathbb{E}[T_1] - \gamma_T$ and $\upsilon =$Var$[T_1]$ are the mean and variance rate of the stochastic part of the Gamma process respectively, while $\gamma_T \geq 0$ is the drift of T. In the rest of this thesis, we adopt the following parametrization:

$$\nu(d\tau) = \frac{C}{\tau}e^{-\eta\tau}d\tau, \tag{3.29}$$

where $C = \dfrac{m^2}{\upsilon}$ and $\eta = \dfrac{m}{\nu}$.

The Laplace transform of the Gamma process is given by:

$$\mathbb{E}[e^{-wT_t}] = e^{\psi(w)t}, \tag{3.30}$$

with the Laplace exponent:

$$\psi(w) = w\gamma_T + C\ln(1 + \frac{w}{\eta}). \tag{3.31}$$

Following Li, Linetsky, and Mendoza-Arriaga 2016, we call $\lambda^{\phi}(t)$ "GMAC-JCIR process". We now calculate the Laplace transform of the GMAC-JCIR process $\lambda^{\phi}(t)$.

First, the Laplace transform of the JCIR process $\lambda(t)$ is well known (Duffie and Singleton 1999):

$$\mathbb{E}_\lambda[e^{-z\lambda_t}] = C(z,t)A(z,t)e^{-B(z,t)\lambda}, \tag{3.32}$$

where:

$$C(z,t) = (e^{-\alpha t} + \frac{(2\alpha + z\rho^2)(1-e^{-\alpha t})}{2\alpha(1+z\mu)})^{-\omega a}, \tag{3.33}$$

$$B(z,t) = \frac{2\alpha z}{2\alpha + (2\alpha + z\rho^2)(e^{\alpha t}-1)}, \tag{3.34}$$

$$A(z,t) = (\frac{2\alpha e^{\alpha t}}{2\alpha + (2\alpha + z\rho^2)(e^{\alpha t}-1)})^b, \tag{3.35}$$

and

$$a = \frac{2\mu}{\rho^2 - 2\mu\alpha}, \quad b = \frac{2\alpha\gamma}{\rho^2}. \tag{3.36}$$

Therefore, the Laplace transform of $\lambda^\phi(t)$ can be written as:

$$\mathbb{E}[e^{-z\lambda^\phi}] = \int_0^{+\infty} \mathbb{E}_x[e^{-z\lambda_u}]q_{s,t}(du). \tag{3.37}$$

Now, let $g_t(du)$ be the transition probability distribution of a Gamma subordinator with zero drift, mean m and variance rate ν, then $g_t(du)$ is given by the following Gamma distribution:

$$g_t(du) = \frac{\eta^{Ct}}{\Gamma(C)}u^{Ct-1}e^{-\eta u}du \tag{3.38}$$

Equation (3.37) can be rewritten as:

$$\mathbb{E}[e^{-z\lambda^\phi}] = \int_0^{+\infty} \mathbb{E}_x[e^{-z\lambda_{\gamma_T(t-s)+u}}]g_{t-s}(du) \tag{3.39}$$

The integral (3.39) can be efficiently computed by the Gauss-Laguerre quadrature. A high level of accuracy can be obtained with a small number of quadrature points.

Availability of the Laplace transform of λ_t^ϕ allows to recover the transition probability density of the process through an efficient numerical Laplace inversion algorithm (see Appendix: Numerical inversion of Laplace transform).

Equivalent measure change

Li, Linetsky, and Mendoza-Arriaga 2016 show that $\lambda^{\phi}(t)$ is a Markov semi-martingale on $(\Omega, \mathfrak{F}, \mathbb{P})$, and derive general explicit conditions under which $\lambda^{\phi}(t)$ is Markov semi-martingale on $(\Omega, \mathfrak{F}, \mathbb{Q})$, where $\mathbb{Q}$ a probability measure equivalent to $\mathbb{P}$.

In general, for the tempered stable family of Lévy subordinators - which includes the Gamma process as a special case - the Lévy measure $\nu(d\tau)$ is given by:

$$\nu(d\tau) = C\tau^{-1-p}e^{-\eta\tau}d\tau. \tag{3.40}$$

Let $(\hat{\alpha}, \hat{\gamma}, \hat{\rho}, \hat{\omega}, \hat{\mu}, \hat{C}, \hat{p}, \hat{\eta}, \hat{\gamma})$ be the parameters which identifies the probability distribution of $\lambda^{\phi}(t)$ under $\mathbb{Q}$, when the random clock belongs to the tempered stable family of Lévy subordinator. Suppose the following conditions are satisfied:

- $\hat{p} = p$
- $\hat{\gamma}\hat{\rho}^2 = \gamma\rho^2$
- $\hat{C}\hat{\rho}^{2\hat{p}} = C\rho^{2p}$

Then $\mathbb{Q}_{|\mathcal{F}_t} \sim \mathbb{P}_{|\mathcal{F}_t}$ for every $t \geq 0$ (Li, Linetsky, and Mendoza-Arriaga 2016).

Pricing formulas

Under GMAC-JCIR process, closed-form expression for the value of a ZCB - so, for credit spread in (3.15) - is not available. Therefore, evaluation of the quantity $Q(t, t+u)$ is carried out through Monte-Carlo simulation (Glasserman 2000).

In order to calibrate the model on historical data (Chapter 7), number of Monte-Carlo simulations (equal to 1000) was selected taking into account numerical accuracy - measured by a comparison between simulated values and "true" values, setting jump and subordinator parameters almost equal to 0 (so, we had DS model) - and overall computational time of calibration procedure.

In Li, Linetsky, and Mendoza-Arriaga 2016 is available an exact simulation algorithm, which can be used to generate sample paths from the GMAC-JCIR process. For a complete discussion on simulation algorithm of Lévy process, see Cont and Tankov 2004.

Given $\lambda^{\phi}(s) = \lambda$, in order to simulate the random variable $\lambda^{\phi}(t)$ $(t > s)$, the following steps must be carried out:

1. Draw a value for the subordinator T_t. We denote the realization by ι;

2. Simulate the random variable $\lambda^{\phi}(\iota)$ (i.e. the value of the JCIR process at time ι) given it starts at λ:

   - Simulate N from the Poisson distribution with parameter $\omega\iota$. N gives the total number of jumps on the interval $[0, \iota]$;
   - Simulate N independent random variable, $U_1, ..., U_N$, uniformly distributed on the interval $[0, \iota]$. These variables mark the jump times;
   - Simulate N independent random variable, $J_1, ..., J_N$, from the Exponential distribution with mean μ. These variables correspond to the jump sizes;
   - Let Y denote the process and set $U_{N+1} = \iota$. Simulate Y_{U_1} given $Y_0 = \lambda$. Then, for $i = 2, ..., N+1$, simulate Y_{U_i} given the value of the process at time U_{i-1} is $Y_{U_{i1}} + J_{i1}$.

3. Set $\lambda^{\phi}(t) = Y_{N+1}$.

Appendix

Numerical inversion of Laplace transform

Numerical inversion of Laplace transform of the GMAC-JCIR process can be performed by the Euler algorithm due to Abate and Whitt 1992. In this appendix we give a brief description of the algorithm.

The object is to calculate values of a real-valued function $f(x)$ from its Laplace transform:

$$\hat{f}(s) = \int_0^{+\infty} e^{-sx} f(x) dx, \tag{3.41}$$

where s is a complex number.

The method uses Euler summation and is based on Bromwich contour inversion integral, which can be expressed as the integral of a real-valued function of real variable by choosing a specific contour.

Letting the contour be any vertical line $s = a$ such that $\hat{f}(s)$ has no singularities on or to the right of it, we obtain:

$$\begin{aligned} f(x) &= \frac{1}{2\pi i} \int_{a-\infty}^{a+\infty} e^{sx} \hat{f}(s) ds = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(a+iu)x} \hat{f}(a+iu) du \\ &= \frac{2e^{ax}}{\pi} \int_0^{+\infty} \mathrm{Re}(\hat{f}(a+iu)) \cos ut \, du. \end{aligned} \tag{3.42}$$

We numerically evaluate (3.42) by means of trapezoidal rule:

$$f(x) = f_h(x) = \frac{he^{ax}}{\pi} \mathrm{Re}(\hat{f})(a) + \frac{2he^{ax}}{\pi} \sum_{k=1}^{+\infty} \mathrm{Re}(\hat{f})(a + ihk) \cos(kht). \tag{3.43}$$

Letting $h = \frac{\pi}{2x}$ and $a = \frac{A}{2x}$:

$$f_h(x) = \frac{e^{A/2}}{2x} \mathrm{Re}(\hat{f})(\frac{A}{2x}) + \frac{e^{A/2}}{x} \sum_{k=1}^{+\infty} (-1)^k \mathrm{Re}(\hat{f})(\frac{A + 2k\pi i}{2x}). \tag{3.44}$$

In order to numerically evaluate the latter formula, which involves an infinite sum, we use Euler summation.

Euler summation can be described as the weighted average of the last m partial sums by a binomial probability distribution with parameters m and $p = 1/2$. In particular, let $s_n(x)$ by the approximation $f_h(x)$ with the infinite sum truncated to n terms, we apply the Euler summation to m terms after an initial n, so that the approximation of (3.43) is

$$E(m,n,x) = \sum_{k=0}^{m} \binom{m}{k} 2^{-m} s_{n+k}(x). \tag{3.45}$$

Laplace inversion in intensity-based models

For calibration purposes, in (3.44) and (3.45) we imposed $A = 18.4$, $n = 150$ and $m = 11$.

The numerical accuracy of the algorithm can be seen through a numerical inversion of the Laplace transform of the GMAC-JCIR process, obtained by setting the contribution of the jump and the subordinator processes to zero (in this way, the probability distribution function should coincide with the density of DS process).

In figures (3.1) and (3.2), the yellow solid line shows the probability distribution function and the cumulative distribution function of the GMAC-JCIR process obtained by inverting the Laplace transform, setting the contibution of the jump and the subordinator components to zero (so, it is the case of a numerical inversion of the DS process); on the other hand, the black solid line shows the probability distribution function and the cumulative distribution function of the DS process, obtained through the corresponding closed-form expressions.

Red lines show the probability distribution function and the cumulative distribution function of the GMAC-JCIR process, which has the same parameters as the DS model, with the exception of γ, which has been modified so that the two distributions have the same expected value. The other parameters are: $\omega = 1$; $\mu = 0.0002$; $C = 3$; $\eta = 0.1$. We can see the larger extension of the right tail of the distribution with respect to the DS model.

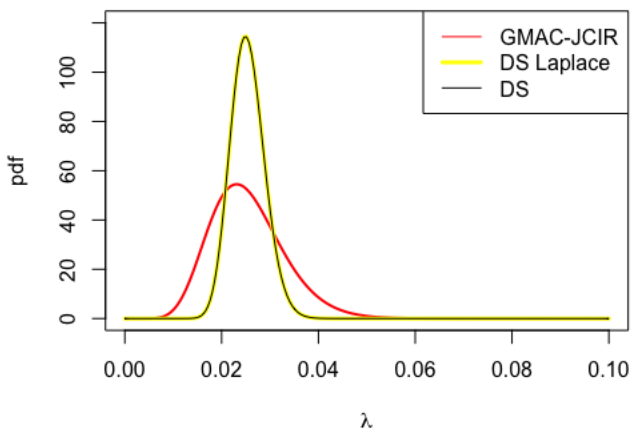

Figure 3.1: Probability distribution functions of the GMAC-JCIR process and of the DS process, the latter obtained by inverting the Laplace transform and by using its closed-form expression.

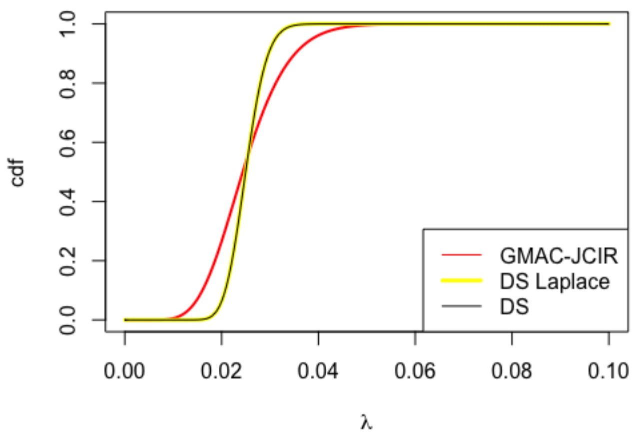

Figure 3.2: Cumulative distribution functions of the GMAC-JCIR process and of the DS process, the latter obtained by inverting the Laplace transform and by using its closed-form expression.

Chapter 4

Particle filtering technique

State space modelling provides a unified methodology for treating a wide range of problems in time series analysis. It is assumed that the development over time of a dynamic system is determined by an unobserved series of quantities with which are associated a series of observations, where the relation between the unobserved and the observed ones is specified by the state space model. In general, the main purpose of state space analysis is to infer the relevant properties of unobserved quantity from the characteristics of the observations (Durbin and Koopman 2001).

Within state space modelling, the Kalman filter is an estimator of the state of a linear dynamic system perturbed by white noise. The estimation is based on the measurement of observable variables that are linearly related to the state disturbed by the white noise. As reported by Grewal and Andrews 2001, the Kalman filter is considered one of the greatest discovery of the twentieth century in the field of statistical estimation theory.

Among its most known application there are the control of complex dynamic systems such as continuous manufacturing processes, aircraft, ships or spacecraft. The Kalman filter allows to control variables that drive these dynamic systems inferring missing information from observable measures. It is also used for prediction purposes, for instance, widely applied to control flows of rivers during flood, trajectories of asteroids or commodities prices.

Kalman 1960 discovered this useful tool combining the notion of state variables to the Wiener filtering problem. In 1960 it was presented at the Ames research center of NASA and one of its extension was implemented as part of the Apollo onboard guidance system. Subsequently, it became very popular and started to be widely used in "modern" control systems, in tracking and navigation of all sort of vehicles and also for predictive analysis.

Several extension of the Kalman filter have been developed over years aiming at its application in case the assumptions of linearity and Gaussian distribution of the disturbances are released. For instance, the Extended Kalman filter introduced by McGee and Schmidt 1985 and the Unscented Kalman filter by Wan and Merwe 2000 allows to manage non-Gaussian nonlinear problems either linearising the observation and the state equation or applying functional approximation to the iterative formulas of the Kalman filter.

Moreover, further developments in filtering techniques have led to the definition of the Particle filter: the term particle appeared for the first time in the work by Kitagawa 1996. The Particle filter allows to manage highly nonlinear and non-Gaussian problems where the Extended Kalman filter and the Unscented Kalman filter fail to provide reasonable estimates.

From a practical point of view, as we will see below, filtering in state-space models concerns computing a series of linked numerical integrals, where output from one is input to the other. Particle filtering, in particular, can be regarded as a technique for solving these integrals by discrete approximations, based on particles. In the following we only consider the case of deterministic particle filtering. In contrast to Monte-Carlo filtering, it could be the preferred method when the state process has low dimension and high numerical accuracy is desired (Bolviken and Storvik 2001).

In our framework, the dynamic system for which measurements are available is represented by the market, where the measurements represented by credit spreads can happen in discrete time instants. From the independence between interest rate risk and credit risk, at every time the state of the dynamic system is determined only via the instantaneous default intensity: in this way, changes in the dynamic system are governed by real-world probability distribution of the instantaneous default intensity, while the risk-neutral probability distribution allows to take into account the relationship between the state of the system and the measurements, i.e. the observed credit spreads.

4.1 Exact filter

Suppose in a dynamic system we deal with a state variable which follows a Markov stochastic process $\lambda^{\phi}(t)$, observed indirectly through a discrete-time process:

$$(\mathbf{s_t})_\mathbf{t} := (s(t, t+\tau))_t = (\overline{h}(t, t+\tau) - h(t, t+\tau))_t, \tag{4.1}$$

for some $\tau > 0$. We also denote the collection of the observations until T as:

$$\mathbf{s_{t_0:T}} := \{s(t_0, t_0+\tau), ..., s(T, T+\tau)\}. \tag{4.2}$$

Let:

- $p(\lambda_t^\phi|\lambda_{t-1}^\phi)$ be the *state equation*, which represents the transition probability distribution of the state variable $\lambda^\phi(t)$ and describes the dynamic characteristics of the system at time t. The state equation is governed by the real-world probability distribution;

- $p(\mathbf{s_t}|\lambda_t^\phi)$ be the *observation equation*, which represents the likelihood and describes the relationship between the latent variable and the observed ones at time t. The observation equation is governed by the risk-neutral probability distribution.

The above equations uniquely define the dynamic system as:

$$\begin{cases} p(\lambda_t^\phi|\lambda_{t-1}^\phi) \\ p(\mathbf{s_t}|\lambda_t^\phi). \end{cases} \tag{4.3}$$

The exact filter for the process $\lambda^\phi(t)$ can be written as a set of recursive integration equations. Starting with the prior distribution of the state variable at time $t_0 = 0$:

$$p(\lambda_0^\phi|\mathbf{s_0}) := p(\lambda_0^\phi), \tag{4.4}$$

we can calculate recursively the following equations for all $t \geq 1$ (Bolviken and Storvik 2001), which represent, respectively, the *predictive distribution* and the *posterior distribution*:

$$p(\lambda_t^\phi|\mathbf{s_{1:t-1}}) = \int_{\Re^+} p(\lambda_t^\phi|\lambda_{t-1}^\phi)p(\lambda_{t-1}^\phi|\mathbf{s_{1:t-1}})d\lambda_{t-1}^\phi \tag{4.5}$$

$$p(\lambda_t^\phi|\mathbf{s_{1:t}}) = C_t^{-1}p(\mathbf{s_t}|\lambda_t^\phi)p(\lambda_t^\phi|\mathbf{s_{1:t-1}}), \tag{4.6}$$

Thus, starting from the prior distribution, we have an increase of knowledge about the unknown parameter λ, i.e. the state variable, when an observation is made, as the "new" information is represented by the posterior density. In this sense, the procedure is dynamic: when a new observation is obtained, the posterior distribution is used as the initial distribution in order to update the information and to calculate a new posterior distribution.

This approach is very computationally advantageous because it allows estimates to be updated in real time without having to retain the entire sample of observed values.

The normalisation constants C_t is defined as follows:

$$C_t = \int_{\Re+} p(\mathbf{s_t}|\lambda_t^{\phi})p(\lambda_t^{\phi}|\mathbf{s_{1:t-1}})d\lambda_t^{\phi} \tag{4.7}$$

and produces the log-likelihood function of the observations $(\mathbf{s_t})_\mathbf{t}$ through:

$$\log(p(\mathbf{s_{t_0:T}})) = \sum_{t=t_0}^{T} \log(C_t) \tag{4.8}$$

4.1.1 Estimation of the state variable

Variable estimation is the problem of determining the trajectory of the unobserved state of a dynamic system, given its observable measurements.

Let t^* be a point in time and $\mathbf{s_{t_0:t^*}} := \{\mathbf{s_{t_0}}, ..., \mathbf{s_{t^*}}\}$ be the vector of measurement. The estimation of the state variable at time t can be referred to:

- the prediction problem, if $t > t^*$;
- the filtering problem, if $t = t^*$;
- the smoothing problem, if $t < t^*$.

The prediction and filtering problems can be defined as an "on-line" problems, upon the conditional density of the state variable given all measurements until t^*; they are relevant in monitoring, control and optimization process.

On the other hand, the smoothing problem can be defined as an "off-line" problem, because it is defined as the estimation of past states with a knowledge of the history of all measurements collected at both past and future instances. For further discussion of the this problem, which will not be considered in the following, see Ungarala 2012.

As one might expect, prediction problem is based on predictive distribution (4.5), while filtering problem on posterior distribution (4.6).

For state estimation purposes, we need to identify an evaluation criterion of decision procedures. From decision-theoretic foundations in Bayesian framework (for details, see Robert 2007), in order to estimate the state variable in prediction and filtering problems, we choose the expected value of the above probability distributions, which is equivalent to assume a quadratic loss function for point estimation.

4.2 Deterministic particle filter

Computation of the integral in equations (4.5), (4.6) and (4.7) can be difficult. When all distributions are Gaussian and the relationship between the observed variables and the latent one is represented by a linear function, then there are closed-form solutions to the recursive equations (Kalman 1960, Kalman and Bucy 1961, Kalman and Bucy 1963).

In general, when the integrals must be calculated with numerical techniques or the relationship is not linear, the recursive equations do not admit a closed-form solution. In this case we must deal with particle filter, a methodology that allows to calculate numerically the integrals through discrete approximations, based on the definition of points (particles) over the integration set.

In the context of deterministic particle filter, i.e. when the points are defined through a non-stochastic approach, evaluations of integrals can be efficiently carried out by Gaussian quadrature (Quarteroni, Sacco, and Saleri 2000); the simplest among these rules is the Gauss-Legendre method. In this way, quadrature filters are constructed by replacing the densities in (4.5) and (4.6) by a particle approximation based on the quadrature formulas.

Let ${}_{(i)}\lambda_t^{\phi}$ be the particles (abscissas) defined over the integration set within the Gauss-Legendre method and ${}_{(i)}\epsilon$ be the correspondent positive wheigts, for $i = 1, ..., N$. Let $\hat{p}$ be some discrete analogue to the exact density p, then we have the following recursive scheme:

$$\hat{p}({}_{(i)}\lambda_0^{\phi}|\mathbf{s_0}) = {}_{(i)}\epsilon \; p({}_{(i)}\lambda_0^{\phi}) \tag{4.9}$$

$$\hat{p}({}_{(i)}\lambda_t^{\phi}|\mathbf{s_{1:t-1}}) = \sum_{j=1}^{N} p({}_{(i)}\lambda_t^{\phi}|{}_{(j)}\lambda_t^{\phi}) \; \hat{p}({}_{(j)}\lambda_{t-1}^{\phi}|\mathbf{s_{1:t-1}}), \tag{4.10}$$

$$\hat{p}({}_{(i)}\lambda_t^{\phi}|\mathbf{s_{1:t}}) = \frac{p(\mathbf{s_t}|{}_{(i)}\lambda_t^{\phi}) \;\; {}_{(i)}\epsilon \; \hat{p}({}_{(i)}\lambda_t^{\phi}|\mathbf{s_{1:t-1}})}{\hat{C}_t} \tag{4.11}$$

for $t \geq 1$ and $i = 1, ..., N$, where:

$$\hat{C}_t = \sum_{i=1}^{N} p(\mathbf{s_t}|{}_{(i)}\lambda_t^{\phi}) \; {}_{(i)}\epsilon \; \hat{p}({}_{(i)}\lambda_t^{\phi}|\mathbf{s_{1:t-1}}), \tag{4.12}$$

Then, the approximate likelihood function is defined as:

$$\log \hat{p}(\mathbf{s_{1:t}}) = \sum_{k=1}^{t} \log \hat{C}_k \tag{4.13}$$

We now point out that, with the particle filtering technique, we approximate the probability distribution of an absolutely continuous random variable with the probability distribution of a discrete one, with support on the particles and probability distribution given by ϵp.

4.2.1 An illustrative example (Doucet 1998)

Suppose we deal with the Gaussian one-dimensional dynamic system:

$$\begin{cases} x_t = 0.5x_{t-1} + 25\dfrac{x_{t-1}}{1+x_{t-1}} + 8\cos(1.2t) + e_t \\ z_t = 0.05x_t^2 + v_t, \end{cases} \tag{4.14}$$

where:

- $e_t \perp v_t$, with e_t i.i.d. $\sim N(0, 10)$ and v_t i.i.d. $\sim N(0, 1)$.
- $x_0 \sim N(0, 5)$.

Then, the state equation and the observation equation are:

$$\begin{cases} p(x_t|x_{t-1}) \sim N\left(0.5x_{t-1} + 25\dfrac{x_{t-1}}{1+x_{t-1}} + 8\cos(1.2t), 10\right) \\ p(z_t|x_t) \sim N(0.05x_t^2, 1) \end{cases}$$

Now, suppose we have observed a path of $z(t)$ from $t = 1, ..., T = 100$ and we don't know the true value of parameters governing the dynamic system (4.14). In this regard, we implement the deterministic particle filter in order to:

- calibrate the model;
- solve the filtering problem (i.e. estimate the entire path of the state variable).

Model calibration

According to the above assumption, for calibration purpose suppose we deal with the following dynamic system:

$$\begin{cases} p(x_t|x_{t-1}) \sim N\left(p_1x_{t-1} + p_2\dfrac{x_{t-1}}{1+x_{t-1}} + p_3\cos(1.2t), 10\right) \\ p(z_t|x_t) \sim N(q_1x_t^2, 1). \end{cases} \tag{4.15}$$

Then, we want to estimate the parameters:

$$\psi = (\vec{\mathbf{p}}, \vec{\mathbf{q}}) = (p_1, p_2, p_3, q_1)$$

Where $\vec{\mathbf{p}} = (p_1, p_2, p_3)$ represents the real-world parameters and $\vec{\mathbf{q}} = q_1$ the risk-neutral parameter.

Calibration is carried out through maximum likelihood estimation:

$$\psi^* = \psi_{mv} = \text{argmax}_{\psi \in \Psi} \log\{\hat{p}(z_{1:T})\} \qquad \Psi := \left(\Re^+\right)^4, \tag{4.16}$$

where $\log\{\hat{p}(z_{1:T})\}$ is calculated by particle filtering technique with the following assumption:

- $[-25, 35]$ is the integration set;
- $N = 100$ is the number of particles over the integration set;
- the N particles and the corresponding weights are defined by the Gauss-Legendre method over the integration set.

The starting point of Cobyla algorithm (for details, see Chapter 5), chosen as the numerical optimization procedure, is represented by the point $\psi_0 = (1.13, 11.99, 6.62, 0.06)$. Setting a maximum number of iterations, equal to 1000, the result of the algorithm is:

$$\psi^* = (0.57, 26.38, 9.28, 0.04) \tag{4.17}$$

Filtering

Once the calibration procedure has been completed, we consider the problem involving the estimation of the state variable. Using the parameters in (4.17), as we have seen in this section, we estimate the state variable through the expected value of the posterior density in the context of the particle filtering equations. A comparison between the "real" and the "estimated" path of the state variable are displayed in Figure (4.1).

In addition, a comparison between the "real" and the "estimated" path of the observations are displayed in Figure (4.2). The estimated ones are calculated through the estimated state variable, as $\hat{z}_t = q_1^* \hat{x}_t$.

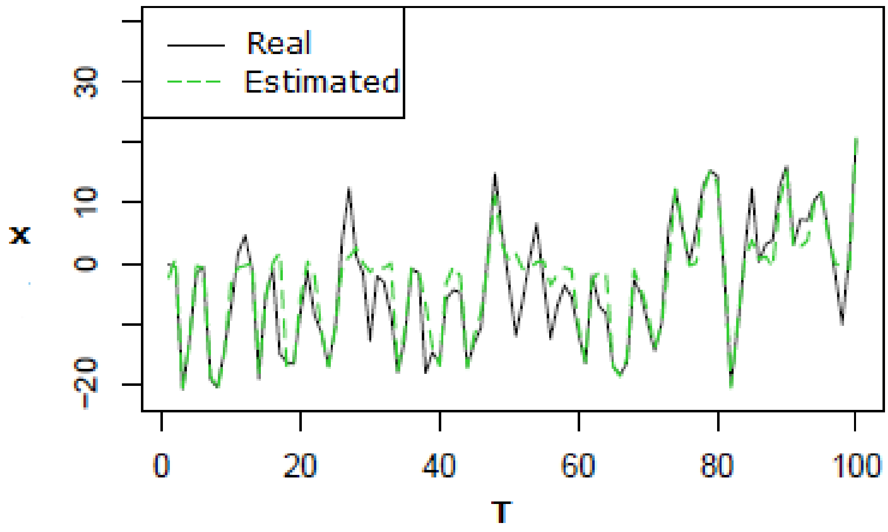

Figure 4.1: Comparison between the "real" and the "estimated" path of state variable.

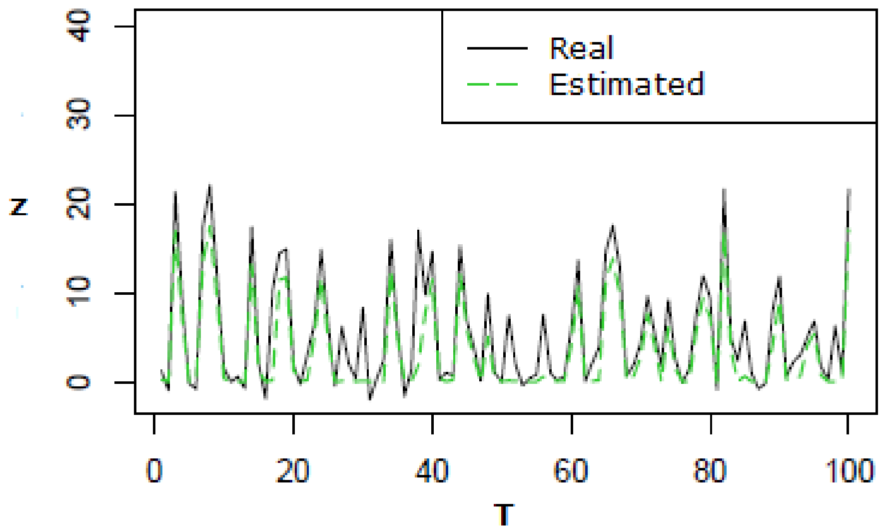

Figure 4.2: Comparison between the "real" and the "estimated" path of the observations.

Chapter 5

Calibration on historical data

The purpose of this chapter is to provide the tools used to analyze the estimation process of the intensity-based models presented in the previous chapter.

Data for model estimation are presented, as well as the tools for determining the goodness of fit of the models to these data.

In the final part of this chapter is proposed a practical application of the calibration in the Solvency II framework for SCR calculation.

5.1 Calibration procedure

The calibration procedure of intensity-based models presented in Chapter 3 via deterministic particle filtering technique, consists of maximization of the likelihood function (4.13); as we have seen, it depends on parameters of both probability measures. Therefore, the calibration procedure can be defined as:

$$\max_{\psi \in \Psi} \log(\hat{p}(\mathbf{s_{1:t}})), \tag{5.1}$$

where $\psi = (\psi^{\mathbb{P}}, \psi^{\mathbb{Q}})$.

We recall usuful considerations on state equations and observation equations of intensity-based models we have considered.

For the state equation $p(\lambda_t^{\phi} | \lambda_{t-1}^{\phi})$ of the DS model, a closed-form expression is available (see Chapter 3) and the real-world parameters are:

$$\psi_{m1}^{\mathbb{P}} = (\alpha, \gamma, \rho) \tag{5.2}$$

On the other hand, for GMAC-JCIR process, the state equation $p(\lambda_t^{\phi} | \lambda_{t-1}^{\phi})$ is derived by Laplace inversion of (3.39) through the algorithm proposed by Abate and Whitt 1992 (see Chapter 3).

By Li, Linetsky, and Mendoza-Arriaga 2016, we also note that the parameters of the GMAC-JCIR model are identified up to a constant; hence, we set $\gamma_T = 1$ for the jump-diffusion specification to fix the scale. Then, the state equation depends on real-world parameters, identified by:

$$\psi^{\mathbb{P}}_{m2} = (\alpha, \gamma, \rho, \omega, \mu, m, \nu) \tag{5.3}$$

With regard the observation equations, for both models we assume for $p(\mathbf{s_t}|\lambda^{\phi}_t)$ a Gaussian distribution:

$$p(\mathbf{s_t}|\lambda^{\phi}_t) \sim N(\bar{\mathbf{s}}_t, \Sigma), \tag{5.4}$$

where $\bar{\mathbf{s}}_t$ is defined by (3.15) and $\Sigma = \sigma^2 I$ is a diagonal matrix.

The risk-neutral parameters of the DS model are identified by:

$$\psi^{\mathbb{Q}}_{m1} = (\hat{\alpha}, \hat{\gamma}, \hat{\rho}, \sigma^2). \tag{5.5}$$

Unlike the DS model, for the GMAC-JCIR model the quantity Q in (3.15) cannot be obtained in closed form, so the model credit spreads $\bar{\mathbf{s}}_t$ can be derived through Monte-Carlo simulation. The observation equation depends on risk-neutral parameters, identified by:

$$\psi^{\mathbb{Q}}_{m2} = (\hat{\alpha}, \hat{\gamma}, \hat{\rho}, \hat{\omega}, \hat{\mu}, \hat{C}, \hat{\eta}, \sigma^2). \tag{5.6}$$

We emphasize that $\psi^{\mathbb{P}}$ and $\psi^{\mathbb{Q}}$ have to meet the conditions for which $\mathbb{Q}_{|\mathcal{F}_t} \sim \mathbb{P}_{|\mathcal{F}_t}$.

For the maximization problem, we assume the following:

- $P(\lambda^{\phi}_t \in [0, 0.1]) \sim= 1$;

- Over the integration set $[0, 0.1]$, we define $N = 256$ particles through the Gauss-Legendre method;

- $\lambda^{\phi}_0 \sim \text{Unif}[0; 0.1]$.

The calibration procedure is implemented in R. For the solution of the optimum is used the library "nloptr", in particular the derivative-free algorithm Cobyla (Constrained Optimization By Linear Approximations) proposed by Powell 2007. The choice is justified by the fact that the likelihood function is very irregular and a gradient optimization algorithms do not allow stability of the solution. The maximum number of iterations has been set equal to 1000.

5.2 Historical data

Data used for the calibration purpose of the two models are obtained from the Markit provider; in particular, the iBoxx EUR indices of the corporate financial sector and rating "A" are considered, represented by the following codes DE000A0JZA12, DE000A0JZA38, DE000A0JZA53, DE000A0JZA79.

The observation of the daily term structure of credit spreads, with maturities equal to 2, 4, 6 and 8 years, starts on January 3, 2007 and ends on December 31, 2021, so the latter represents the valuation date. Therefore, the number of input observations is equal to 15520.

Figure (5.1) shows the surface of the spreads observed in the market; Figure (5.2) shows that the correlation is very high between the spreads when maturities are near, while it decreases when credit spreads with short and long maturities are considered.

In Table (5.1) there are means, volatilities and quantiles at 99.5% calculated over the entire period of observation, while in Tables (5.2)-(5.5) the same calculation is presented considering sub-periods of annual amplitude.

P-values of the Ljung–Box test on the first difference of historical series (Matteson and Ruppert 2015) are almost equal to zero. Therefore, at the significance level of 0.5%, the (weak) stationarity hypothesis of the Ljung-Box test is rejected. If stationarity was verified, the time series would be characterized by a single covariance matrix; otherwise, one would have to define a covariance matrix for each time instant.

However, for simplicity we assume that the times series are governed by a single covariance matrix. In order to investigate a possible dimensional reduction, we apply a principal component analysis, which shows that one factor explains, approximately, 98.3% of the total variability; on the other hand, the first two principal components, explain the 99.7% (see Figure (5.3)).

For this reason we have proceed to model credit spreads using a one-factor model; we should also have taken into account multifactorial models (as also noted by Duffie and Singleton 1999).

Figures (5.4 - 5.7) show the trend of first difference of credit spreads (black continuous line) over the observation period; it can be seen that volatility is significantly lower in the more recent past. In the same figures on the right-hand scale, the values of the observed credit spread are also represented (red line); the volatility of credit spreads is higher in the periods when they are higher. This aspect is consistent with the assumptions of the intensity-based models we have considered.

To measure the goodness of fit to the input data, a graphical comparison between the market and model time series for each time to maturity will be made, as well as an analysis of the histograms and QQ-Plots of the residuals, defined as the difference between market values and model values. In addition, coefficient of determination (R^2) and root mean square error mean square error (RMSE) will be evaluated:

$$R^2 = 1 - \frac{RSS}{TSS} = 1 - \frac{\sum_{i=1}^N e_i^2}{\sum_{i=1}^N (\mathbf{s_i} - \sum_{i=1}^N \frac{\mathbf{s_i}}{N})}, \tag{5.7}$$

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^N e_i^2}, \tag{5.8}$$

where $e_i = \overline{\mathbf{s}}_\mathbf{i} - \mathbf{s_i}$.

5.3 SCR calculation

Once the calibration has been performed, an out-of sample analysis can be made through a comparison between:

- the value of the spread risk SCR of a ZCB for each maturities, calculated within a (partial) internal model framework as the VaR of the predictive probability distribution of the price of the 1-year ZCB;
- the corresponding SCR curve, obtained through the Standard Formula applied to the A rating, calculating the statistics of the 1-year distribution of the spread structure for the remaining times to maturity.

Calculation of the spread SCR of a ZCB within (partial) internal model framework can be made taking into account that the spread risk component at time $t + \Delta t$ is obtained by modifying the formula (3.15). We note that:

$$Q(t + \Delta t, t + \Delta t + \tau) = f(\lambda_{t+\Delta t}, \psi^{\mathbb{Q}}), \tag{5.9}$$

so, it is a random variable depending on risk-neutral parameters.

Then, spread SCR whitin (partial) internal model framework can be defined as:

$$SCR^{IM} = \frac{\mathbb{E}[Q(t + \Delta t, t + \Delta t + \tau)] - Q^*(t + \Delta t, t + \Delta t + \tau)}{Q(t, t + \tau)} \tag{5.10}$$

where $Q^*(t + \Delta t, t + \Delta t + \tau)$ is the quantile at 0.5% of the distribution of $Q^*(t + \Delta t, t + \Delta t + \tau)$.

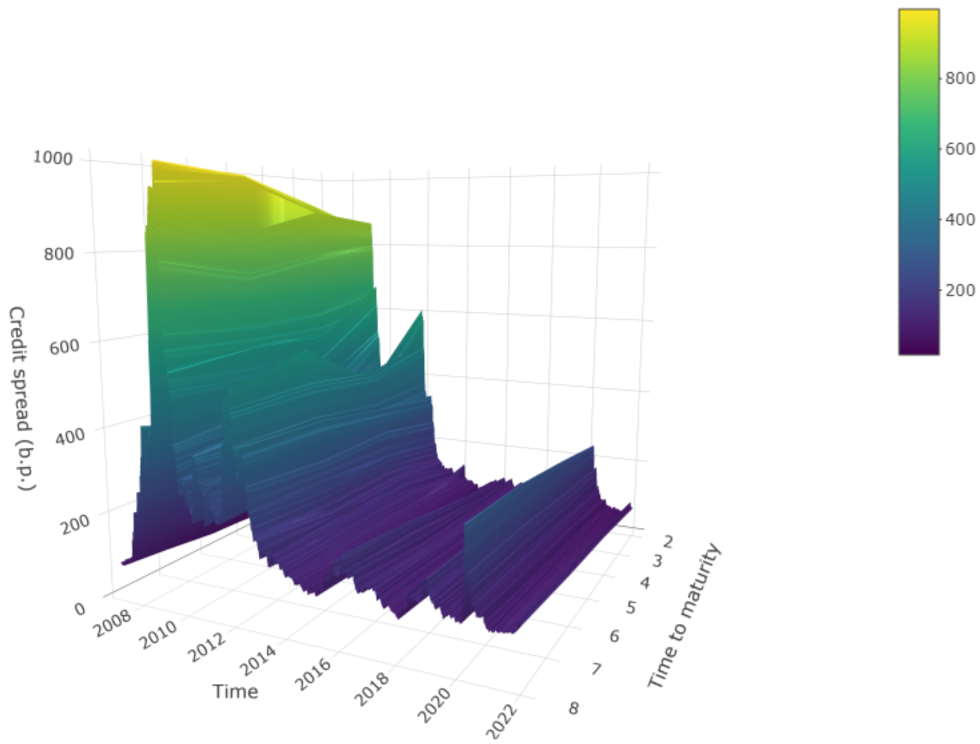

Figure 5.1: Credit spreads surface trend.

Time to maturity	**Mean**	**St. Dev.**	**Quantile**
2	166.8381	163.219	827.456
4	176.8903	150.7311	863.793
6	201.498	165.1883	908.1179
8	215.6286	165.8232	939.1012

Table 5.1: Means, standard deviations and quantiles of confidence interval equal to 99.5% of historical credit spreads.

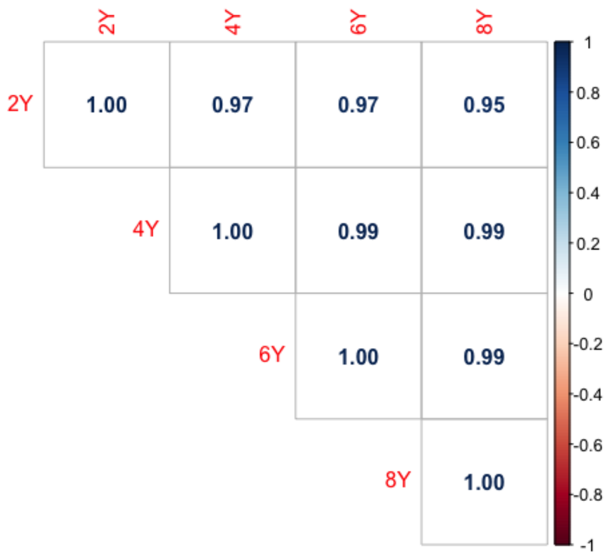

Figure 5.2: Correlation of market credit spreads.

Year	Mean	St. Dev.	Quantile
2007	46	27	110
2008	315	197	692
2009	560	186	859
2010	246	56	344
2011	329	129	597
2012	254	96	498
2013	104	9	122
2014	84	15	132
2015	76	10	98
2016	91	8	109
2017	82	10	100
2018	78	13	110
2019	79	11	117
2020	102	45	236
2021	61	5	83

Table 5.2: Means, standard deviations and quantiles of confidence interval equal to 99.5% of historical credit spreads for time to maturity equal to 2 years.

Year	Mean	St. Dev.	Quantile
2007	77	43	176
2008	388	178	754
2009	518	212	907
2010	223	37	291
2011	264	88	435
2012	273	82	452
2013	139	13	166
2014	95	12	117
2015	93	12	117
2016	108	11	137
2017	88	12	109
2018	92	19	135
2019	101	15	149
2020	118	45	265
2021	78	6	102

Table 5.3: Means, standard deviations and quantiles of confidence interval equal to 99.5% of historical credit spreads for time to maturity equal to 4 years.

Year	Mean	St. Dev.	Quantile
2007	97	47	198
2008	444	208	847
2009	547	229	979
2010	275	40	367
2011	335	113	558
2012	284	88	473
2013	143	11	175
2014	108	9	129
2015	114	17	147
2016	127	19	171
2017	102	14	126
2018	112	23	160
2019	117	18	174
2020	133	47	291
2021	92	7	116

Table 5.4: Means, standard deviations and quantiles of confidence interval equal to 99.5% of historical credit spreads for time to maturity equal to 6 years.

Year	Mean	St. Dev.	Quantile
2007	129	58	256
2008	488	221	941
2009	548	240	994
2010	283	39	362
2011	315	95	512
2012	302	82	469
2013	157	13	187
2014	115	10	136
2015	132	22	177
2016	150	19	196
2017	118	16	145
2018	124	21	168
2019	127	18	182
2020	145	45	300
2021	107	8	134

Table 5.5: Means, standard deviations and quantiles of confidence interval equal to 99.5% of historical credit spreads for time to maturity equal to 8 years.

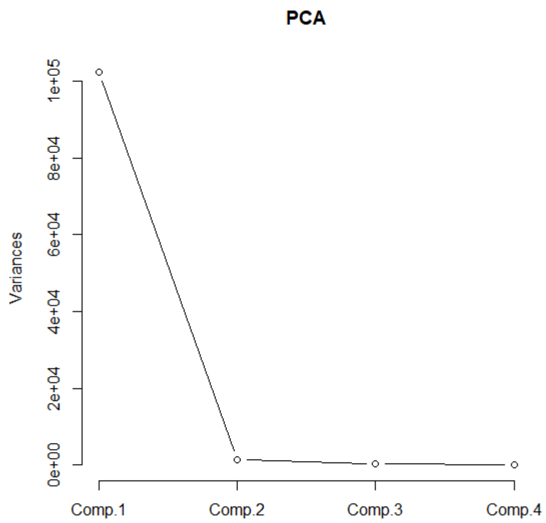

Figure 5.3: Sree plot of a principal component analysis on historical credit spreads.

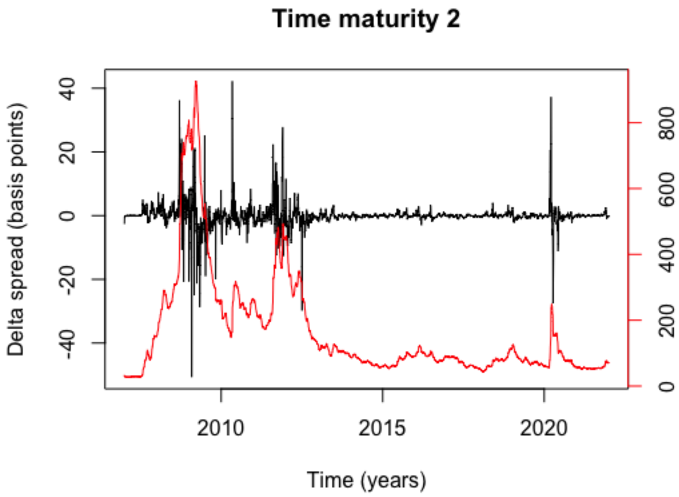

Figure 5.4: Historical trend of credit spreads (red line) and their first difference (black line) for time to maturity equal to 2 years.

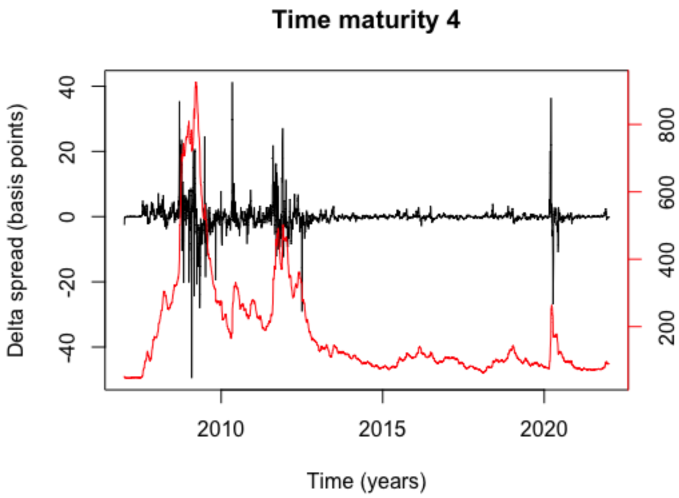

Figure 5.5: Historical trend of credit spreads (red line) and their first difference (black line) for time to maturity equal to 4 years.

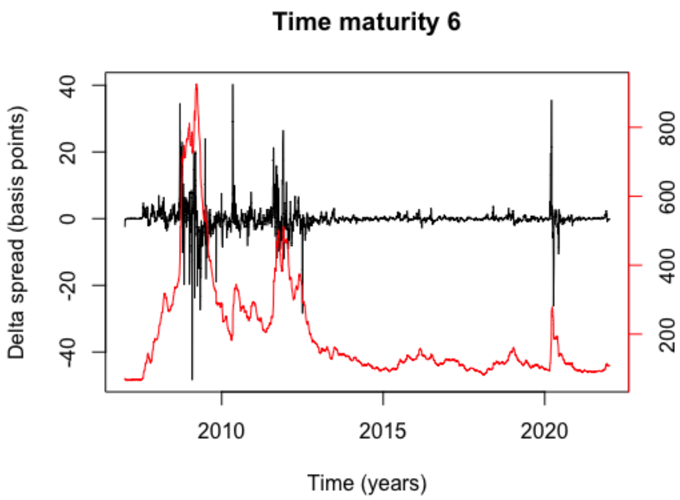

Figure 5.6: Historical trend of credit spreads (red line) and their first difference (black line) for time to maturity equal to 6 years.

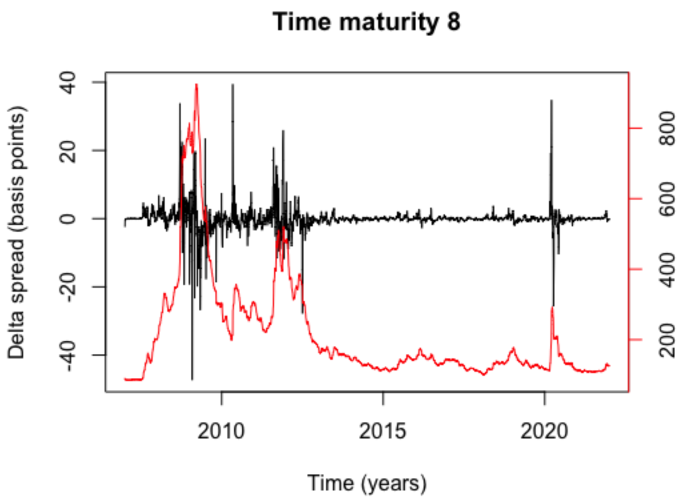

Figure 5.7: Historical trend of credit spreads (red line) and their first difference (black line) for time to maturity equal to 8 years.

Chapter 6

Calibration of DS model

6.1 Goodness of fit on historical data

The initial value, upper and lower bounds, as well as the set of parameter result of the optimization procedure are reported in Table (6.1). The optimization algorithm stopped at 1000 iterations and the computational time was equal to 1 hour, 37 minutes and 26 seconds.

It can be seen that the value of the γ parameter is stands at about 46 b.p.; this value is reasonable when compared with the averages of the observed time series (see Chapter 5), especially with shorter maturity.

Regarding goodness of fit, Figures (6.1 - 6.4) show the comparison between model and market values, while Tables (6.2-6.3) and Figures (6.5-6.6) provide a summary of the values of means and standard deviations of the time series of the market and model time series: means are very close, while standard deviations are more close when shorter maturities are considered.

The overall prediction error is measured by the Root Mean Squared Error (RMSE), reported in Table (6.4) for all maturities: the goodness of fit of the model is better when central maturities are considered. This is evident even from Table (6.5), where coefficients of determination are reported: R^2 is close to 99% for central maturities, greater than others.

On the other hand, Figures (6.7-6.14) show histograms and QQ-plots of the residual distributions for the various maturities. It can be seen that residuals are inconsistent with the hypothesis of Gaussian distribution.

6.2 SCR calculation

From Figure (6.16), it can be seen that the calculation of SCR made within (partial) internal model framework is almost equal than that of the Standard

	Plow	Pupp	Par0	ParOptim
α	0.001	5	0.2	0.2871
γ	0.001	0.08	0.003	0.0046
ρ	0.01	0.8	0.03	0.0472
$\hat{\alpha}$	0.001	0.2	0.04	0.0088
$\hat{\gamma}$	0.001	0.4	0.09	0.2247
σ	0.00001	0.03	0.001	0.0023

Table 6.1: Initial values (Par0), lower constraints (Plow), upper constraints (Pupp) and output of estimation of model parameters (ParOptim).

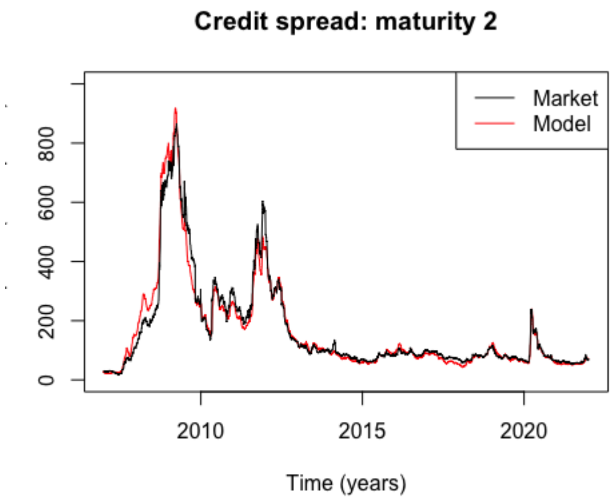

Figure 6.1: Historical series of model credit spreads (red line) and market credit spreads (black line) for time to maturity equal to 2 years.

Formula.

Figure (6.15) show the probability distributions function of the 1-year credit spreads structure for different times to maturity.

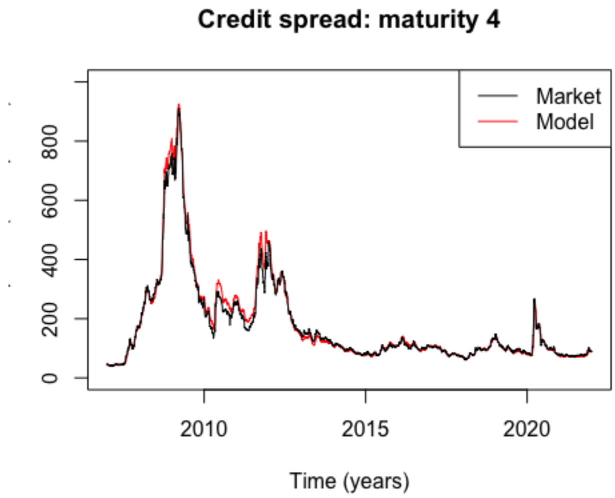

Figure 6.2: Historical series of model credit spreads (red line) and market credit spreads (black line) for time to maturity equal to 4 years.

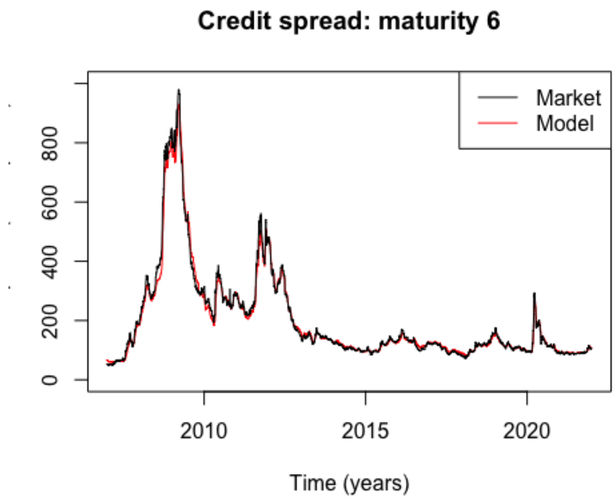

Figure 6.3: Historical series of model credit spreads (red line) and market credit spreads (black line) for time to maturity equal to 6 years.

Table 6.2: Means of historical market and model values

Maturity	Means MKT (b.p)	Means MDL (b.p.)
2	166.83	164.45
4	176.89	181.85
6	201.49	198.51
8	215.63	214.39

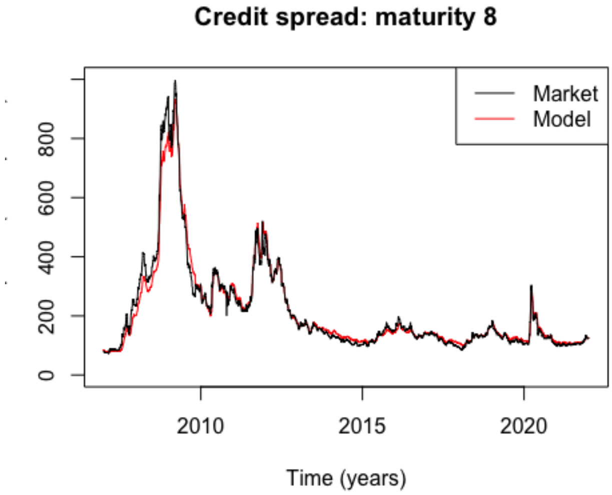

Figure 6.4: Historical series of model credit spreads (red line) and market credit spreads (black line) for time to maturity equal to 8 years.

Table 6.3: Standard deviations of historical market and model values

Maturity	SD MKT (b.p)	SD MDL (b.p.)
2	163.21	163.09
4	150.73	160.97
6	165.18	158.45
8	165.82	155.57

Table 6.4: RMSE of the residuals

Maturity	RMSE (b.p)
2	30.32
4	17.64
6	17.18
8	25.13

Table 6.5: Coefficient of determination

Maturity	R^2
2	96.54%
4	98.62%
6	98.91%
8	97.70%

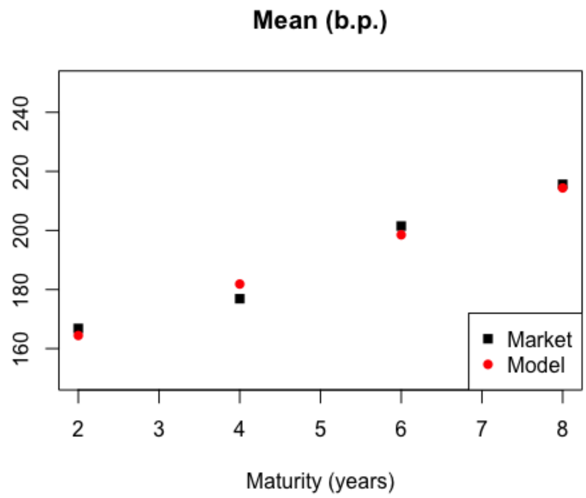

Figure 6.5: Mean analysis of historical and model credit spreads

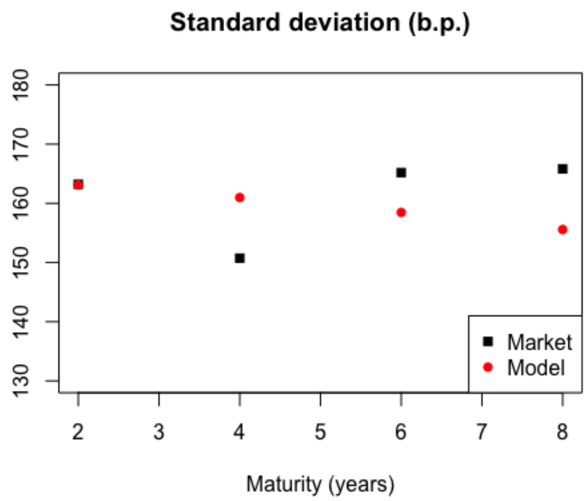

Figure 6.6: Standard deviation analysis of historical and model credit spreads

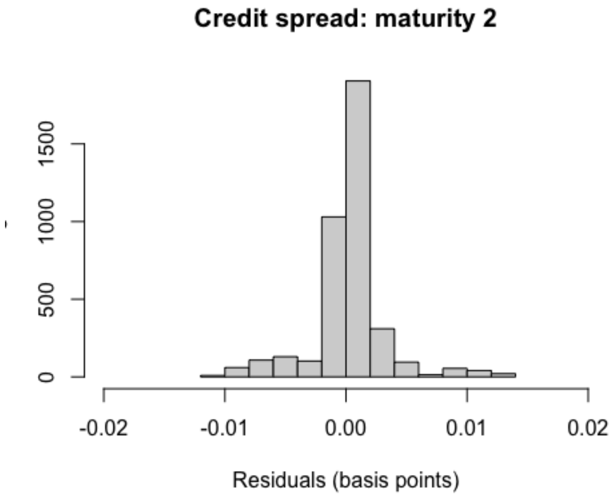

Figure 6.7: Distribution of the residuals for time to maturity equal to 2: histogram

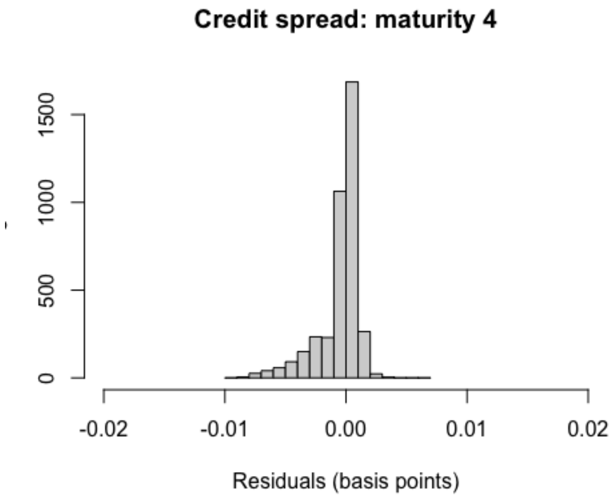

Figure 6.8: Distribution of the residuals for time to maturity equal to 4: histogram

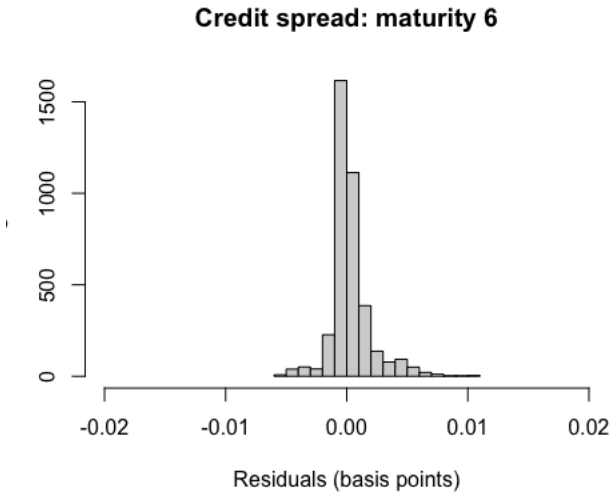

Figure 6.9: Distribution of the residuals for time to maturity equal to 6: histogram

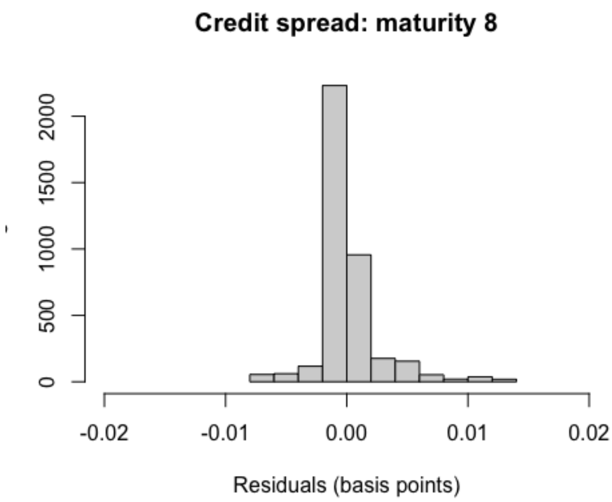

Figure 6.10: Distribution of the residuals for time to maturity equal to 8: histogram

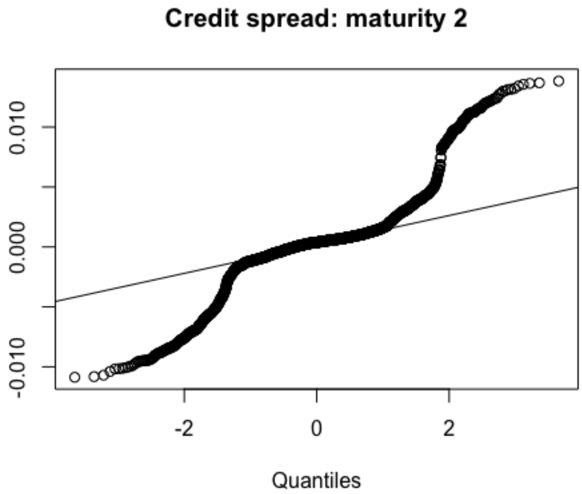

Figure 6.11: Distribution of the residuals for time to maturity equal to 2: qq-plot

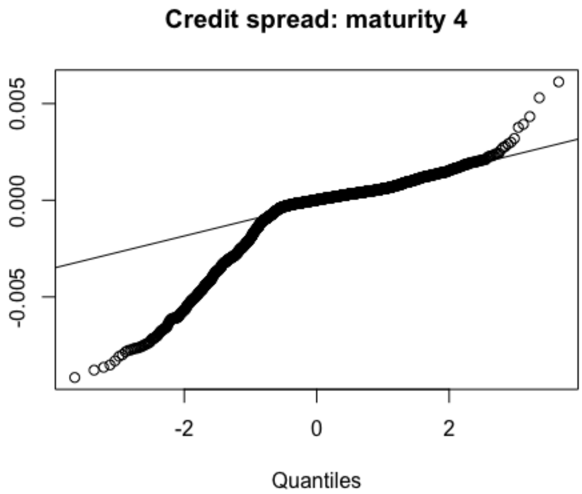

Figure 6.12: Distribution of the residuals for time to maturity equal to 4: qq-plot

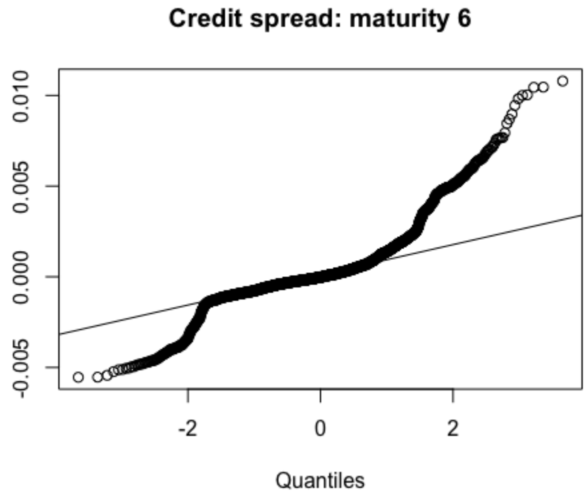

Figure 6.13: Distribution of the residuals for time to maturity equal to 6: qq-plot

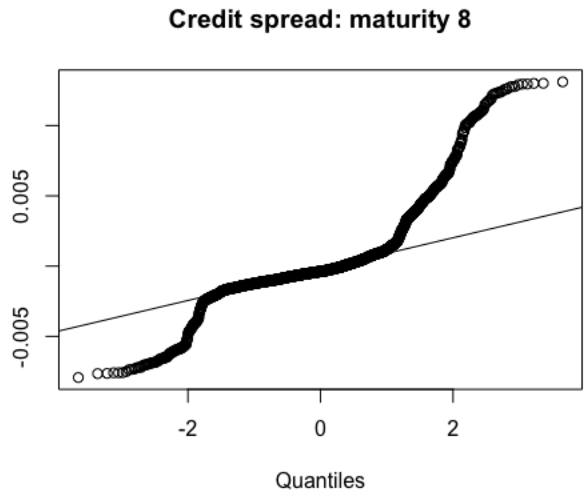

Figure 6.14: Distribution of the residuals for time to maturity equal to 8: qq-plot

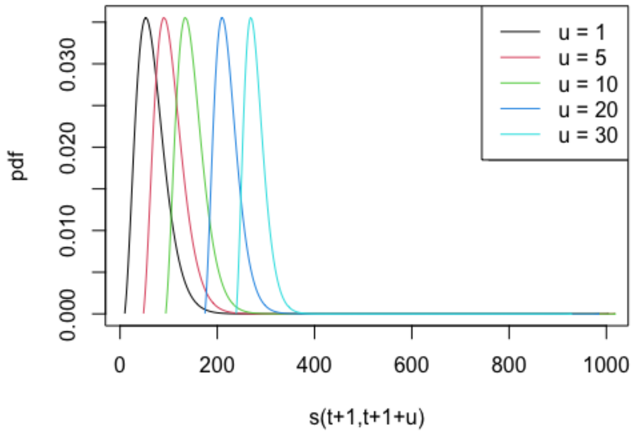

Figure 6.15: Predictive probability distribution functions of $s(t, t+u)$ for different times to maturity u.

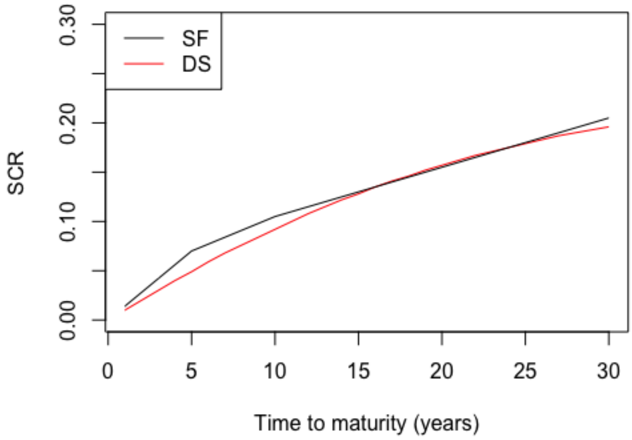

Figure 6.16: Comparison between SCR of a (partial) internal model and SCR calculated via Standard Formula.

Chapter 7

Calibration of GMAC-JCIR model

7.1 Goodness of fit on historical data

The initial value, upper and lower bounds, as well as the set of parameter result of the optimization procedure are reported in Table (7.1). As in the DS model, the optimization algorithm stopped at 1000 iterations, but the computational time was 5 hours, 37 minutes and 42 seconds.

Goodness of fit is similar than the DS model. Figures (7.1 - 7.4) show the comparison between model and market values, while Tables (7.2-7.3) and Figures (7.5-7.6) provide a summary of the values of means and standard deviations of the time series of the market and model time series. RMSE and R^2 are reported in Table (7.4) (7.5) for all maturities; as in DS model, the goodness of fit is better when central maturities are considered.

Figures (7.7-7.14) show histograms and QQ-plots of the residual distributions for the various maturities. Even in this model, residuals are not consistent with Gaussian distribution.

7.2 SCR calculation

From Figure (7.16), it can be seen that the calculation of SCR made within (partial) internal model framework is greater than that of the Standard Formula, therefore higher than that of the DS model.

Figure (7.15) shows the probability distributions function of the 1-year credit spreads structure for different time to maturities.

	Plow	Pupp	Par0	ParOptim
α	0.001	5	0.2	0.1508
γ	0.001	0.08	0.003	0.0050
ρ	0.01	0.8	0.03	0.0256
ω	0	5	1	2.1411
μ	0	0.1	0.00002	0.0002
C	0	5	3	4.5069
η	0	0.5	0.1	0.1794
$\hat{\alpha}$	0.001	0.2	0.04	0.0777
$\hat{\gamma}$	0.001	0.4	0.09	0.0843
$\hat{\omega}$	0	1	0.03	0.0329
$\hat{\mu}$	0	0.5	0.015	0.0294
$\hat{\eta}$	0	30	15	15.1462
σ	0.00001	0.03	0.001	0.0023

Table 7.1: Initial values (Par0), lower constraints (Plow), upper constraints (Pupp) and output of estimation of model parameters (ParOptim).

Table 7.2: Means of historical market and model values

Maturity	Means MKT (b.p)	Means MDL (b.p.)
2	166.83	165.49
4	176.89	182.29
6	201.49	198.70
8	215.63	214.48

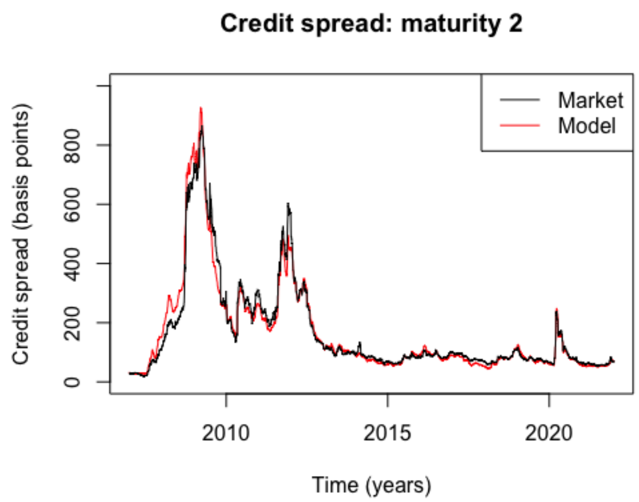

Figure 7.1: Historical series of model credit spreads (red line) and market credit spreads (black line) for time to maturity equal to 2 years.

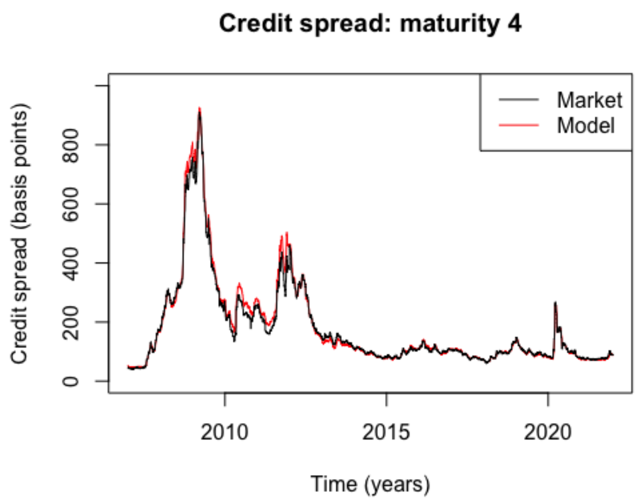

Figure 7.2: Historical series of model credit spreads (red line) and market credit spreads (black line) for time to maturity equal to 4 years.

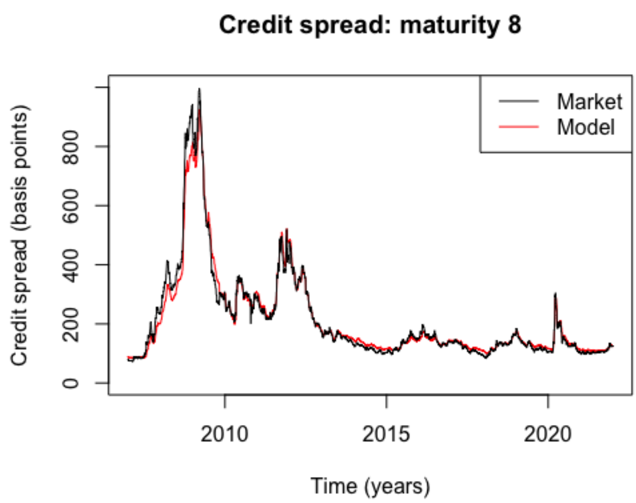

Figure 7.3: Historical series of model credit spreads (red line) and market credit spreads (black line) for time to maturity equal to 6 years.

Table 7.3: Standard deviations of historical market and model values

Maturity	SD MKT (b.p)	SD MDL (b.p.)
2	163.21	164.56
4	150.73	160.74
6	165.18	157.16
8	165.82	153.67

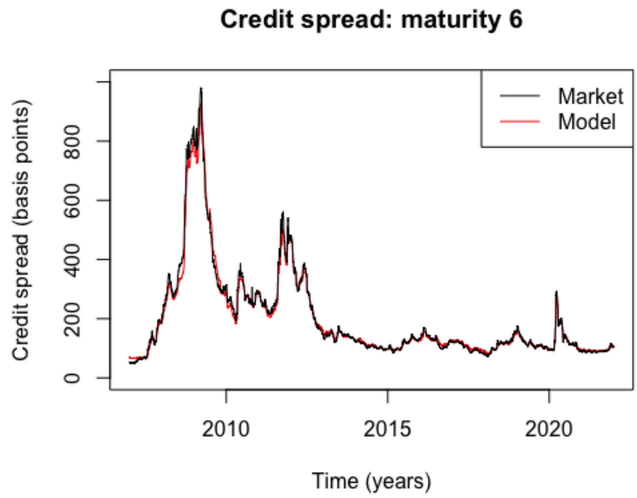

Figure 7.4: Historical series of model credit spreads (red line) and market credit spreads (black line) for time to maturity equal to 8 years.

Table 7.4: RMSE of the residuals

Maturity	RMSE (b.p)
2	30.25
4	17.54
6	17.14
8	25.47

Table 7.5: Coefficient of determination

Maturity	R^2
2	96.56%
4	98.66%
6	98.92%
8	97.64%

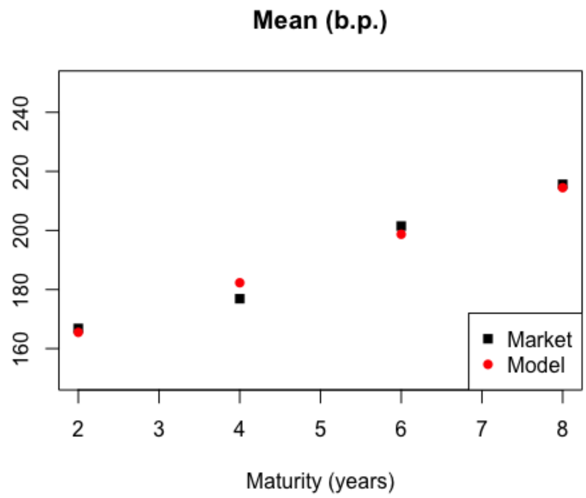

Figure 7.5: Mean analysis of historical and model credit spreads

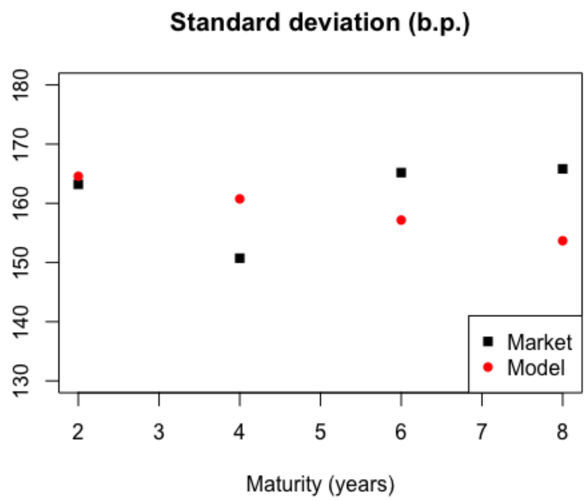

Figure 7.6: Standard deviation analysis of historical and model credit spreads

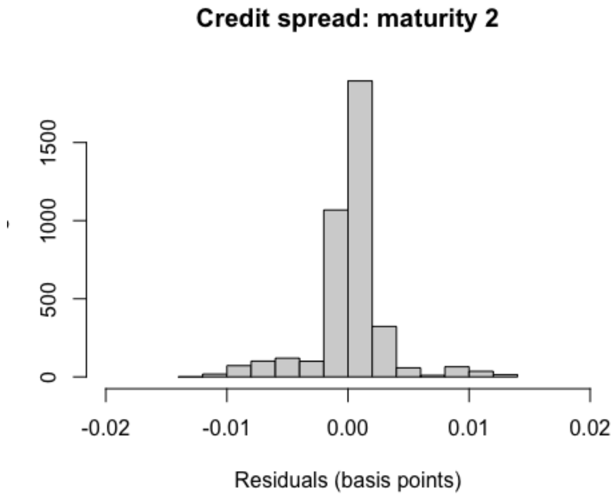

Figure 7.7: Distribution of the residuals for time to maturity equal to 2: histogram

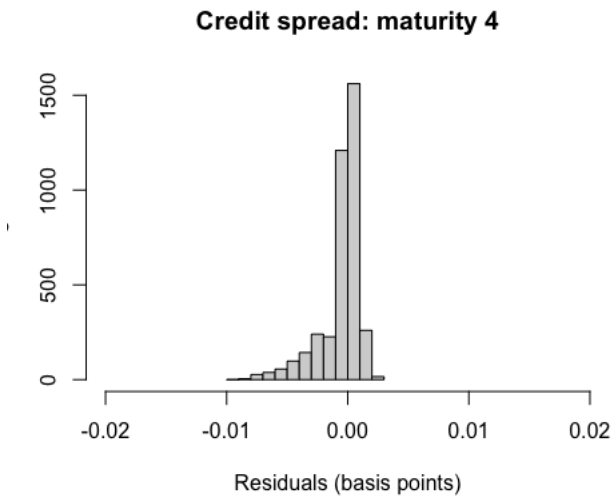

Figure 7.8: Distribution of the residuals for time to maturity equal to 4: histogram

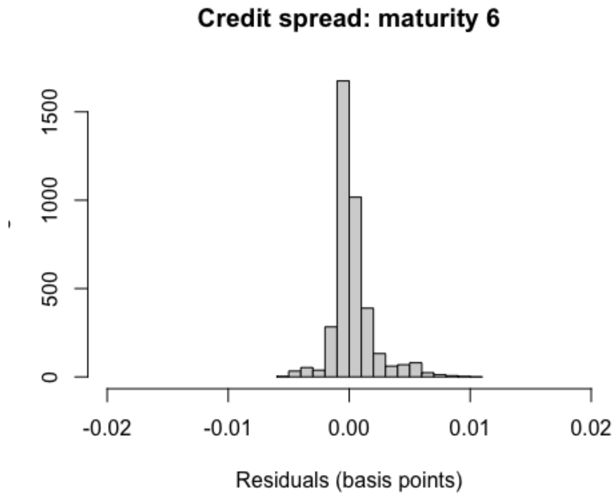

Figure 7.9: Distribution of the residuals for time to maturity equal to 6: histogram

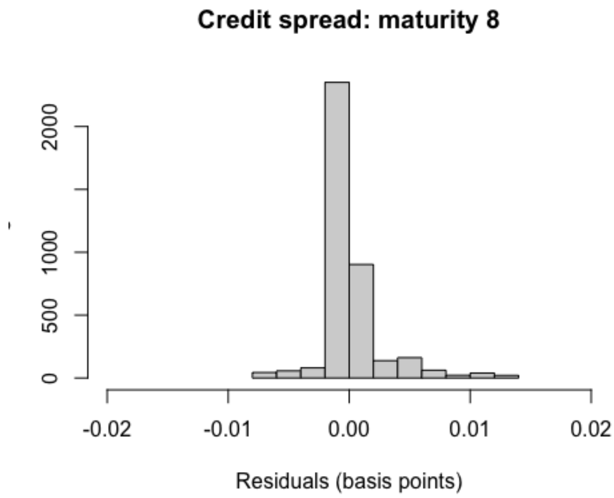

Figure 7.10: Distribution of the residuals for time to maturity equal to 8: histogram

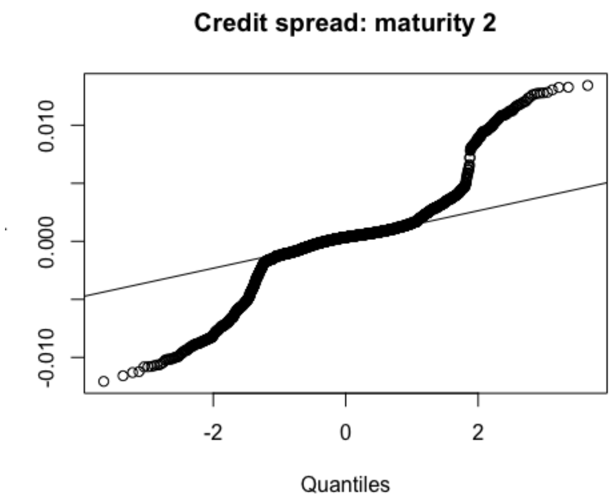

Figure 7.11: Distribution of the residuals for time to maturity equal to 2: qq-plot

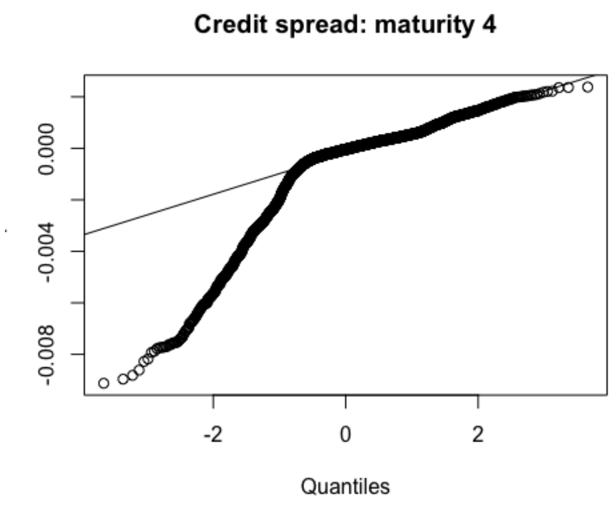

Figure 7.12: Distribution of the residuals for time to maturity equal to 4: qq-plot

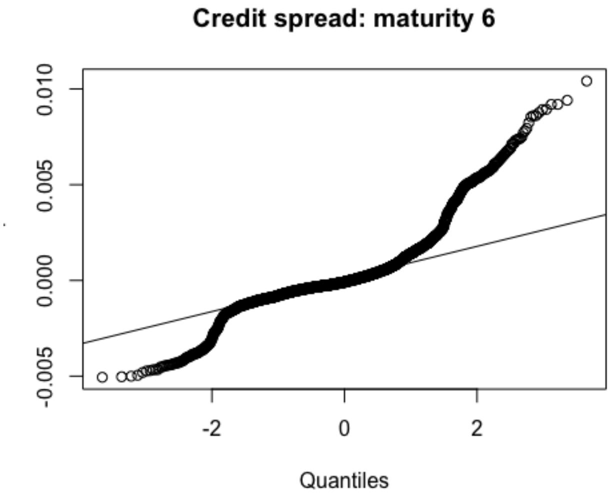

Figure 7.13: Distribution of the residuals for time to maturity equal to 6: qq-plot

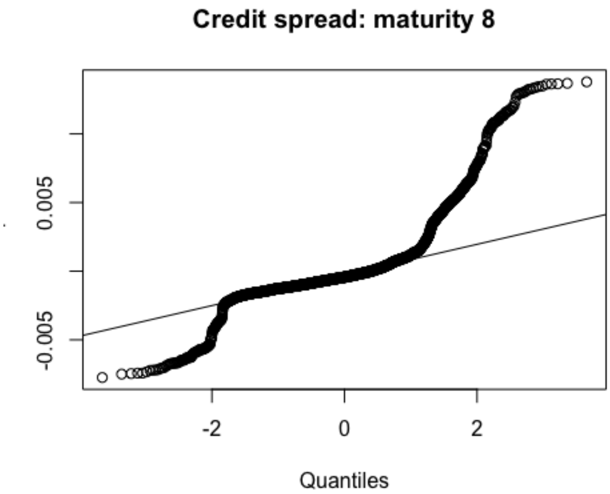

Figure 7.14: Distribution of the residuals for time to maturity equal to 8: qq-plot

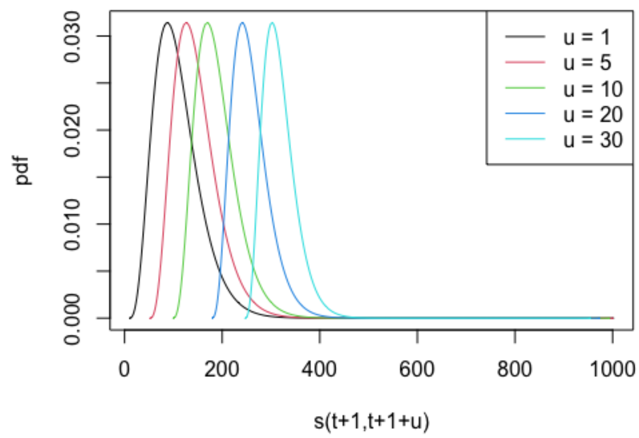

Figure 7.15: Predictive probability distribution functions of $s(t, t+u)$ for different times to maturity u.

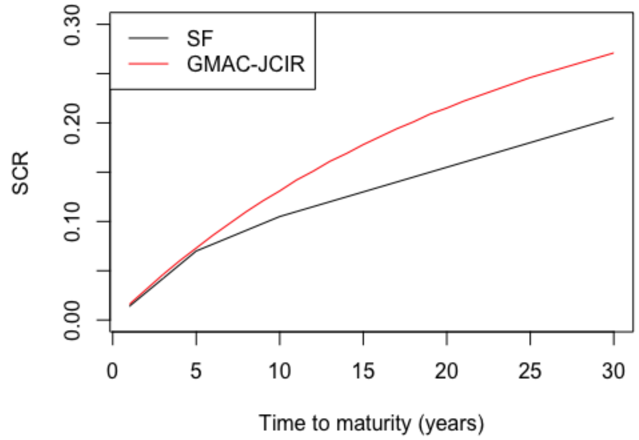

Figure 7.16: Comparison between SCR of a (partial) internal model and SCR calculated via Standard Formula.

Conclusions

The purpose of this thesis is to calibrate a model for measuring and managing spread risk in a partial internal model framework, as a possible alternative to the Standard Formula. Then, for this type of calibration particularly relevant is the behavior of the model on the tails of the distribution of the risk driver, that contributes to the tails of the loss distribution and then to the SCR calculation.

Results shown in the previous chapters highlighted that the use of particle filtering is able to capture the large number of observations used for calibration purpose and that both models had a satisfactory goodness of the estimates. As pointed out, the GMAC-JCIR model provided the same goodness of fit to the data as the DS model but - even if the projected data (SCR) seem to be reasonable when compared with the Standard Formula - it allowed for more conservative results in the description of tail events, which is relevant in the insurance context.

Regarding the tail events, GMAC-JCIR model seems to be preferred when a comparison between observed credit spreads and Value-at-risk of predictive probability distributions is made (Figures (7.17-7.20) and Table (7.6)).

The particle filter was found to be a valid and versatile tool for estimating the models, and has a large number of advantages: from a computational point of view, once the estimate is updated, the memory location that contains its value can be overwritten; on the other hand, the link between state variables and observations allows to take into account a wide range of models for describing time series, both stationary and non-stationary.

However, the use of filters also involves a number of difficulties. Indeed, one is the choice of the depth of the time series, which can substantially influence the result of the estimation. In addition, the computational time for calibration is much longer than with classical estimation procedure, referring to a single date. Finally, the implementation of particle filtering requires numerical calculation of integrals, whose stability and speed depends on the type of implementation carried out.

Further in-depth analysis inherent in the study of the dependence of the solution on the depth of the time series and its stability with respect to the presence of local minima will be explored further in future research.

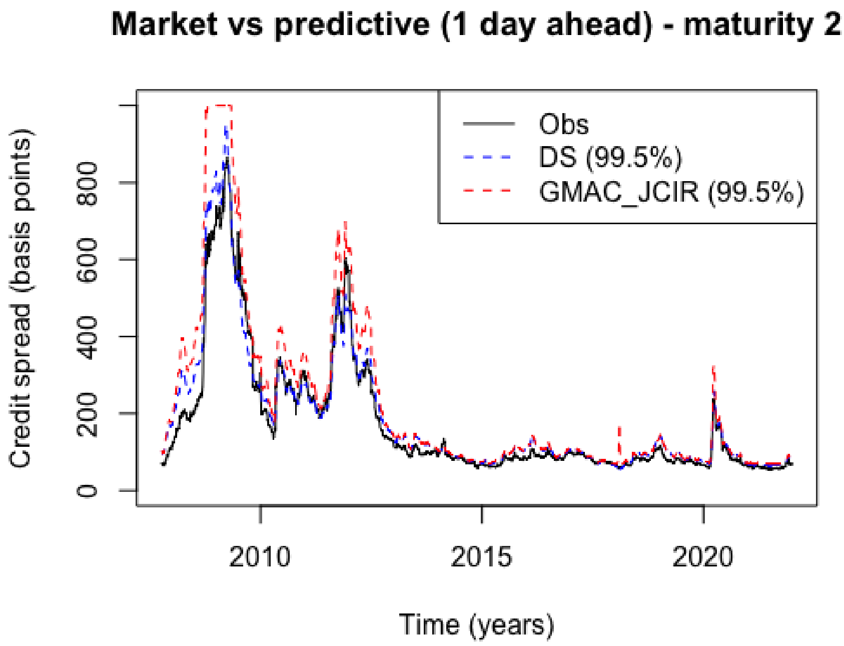

Figure 7.17: Comparison between observed credit spreads and 99.5% quantile of the day-ahed predictive distribution for time to maturity equal to 2 years

Table 7.6: Number and incidence of violations at confidence level equal to 99.5%

tau	DS (#)	GMAC-JCIR (#)	DS (%)	GMAC-JCIR (%)
2	682	65	17.58%	1.68%
4	91	27	2.35%	0.70%
6	503	30	12.97%	0.77%
8	556	174	14.33%	4.49%

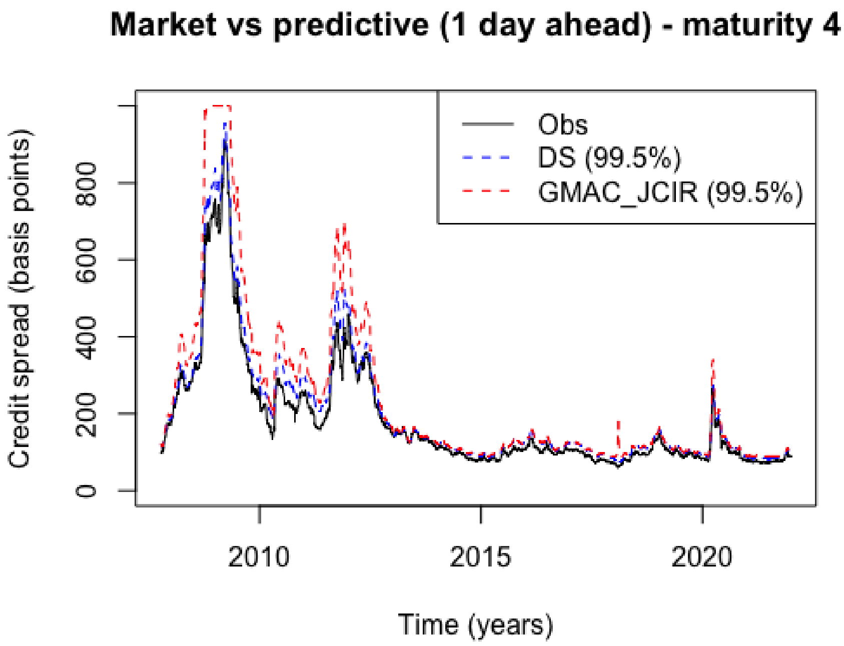

Figure 7.18: Comparison between observed credit spreads and 99.5% quantile of the day-ahed predictive distribution for time to maturity equal to 4 years

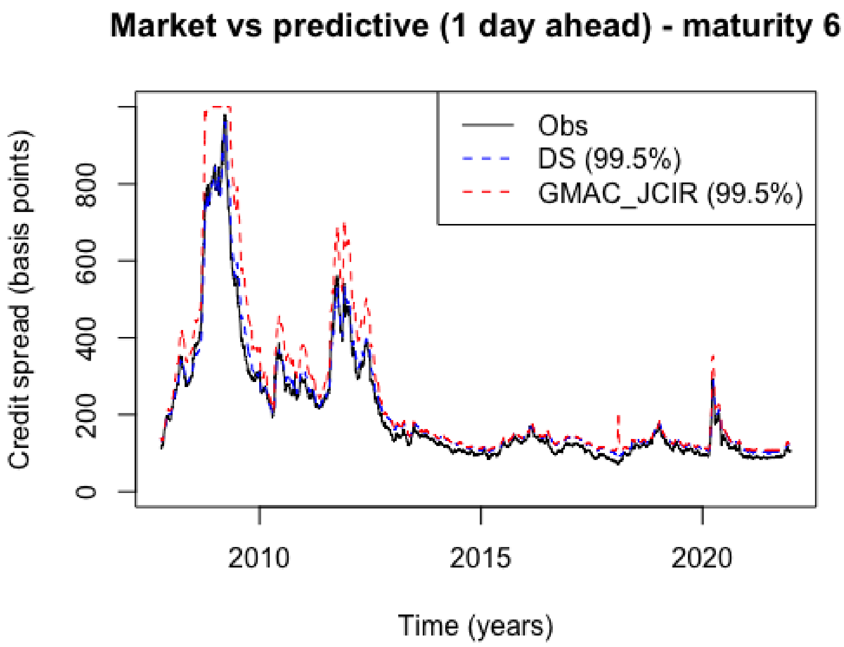

Figure 7.19: Comparison between observed credit spreads and 99.5% quantile of the day-ahed predictive distribution for time to maturity equal to 6 years

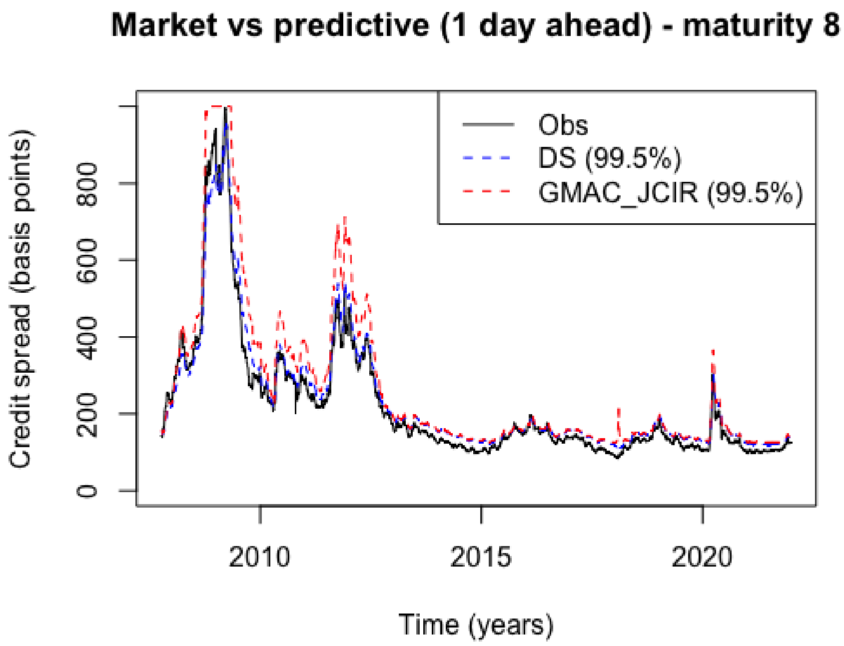

Figure 7.20: Comparison between observed credit spreads and 99.5% quantile of the day-ahed predictive distribution for time to maturity equal to 8 years

Bibliography

Abate and Whitt (1992). *The Fourier-series method for inverting transforms of probability distributions*. Queueing Syst.

Black and Cox (1976). "Valuing corporate securities: some effects of bond indenture provisions". In: *The Journal of Finance*.

Black and Scholes (1973). "The pricing of options and corporate liabilities". In: *Journal of Political Economy*.

Bolviken and Storvik (2001). "Deterministic and Stochastic Particle Filters in State-Space Models". In: *Sequential Monte Carlo Methods in Practice*. Springer. Chap. 5, pp. 97–116.

Brigo and Mercurio (2001). *Interest rate models: Theory and practice*. Springer.

Cappé, Moulines, and Ryden (2005). *Inference in Hidden Markov Models*. Springer.

Castellani, De Felice, and Moriconi (2005). *Manuale di finanza I*. il Mulino.

— (2006). *Manuale di finanza III*. il Mulino.

Cont and Tankov (2004). *Financial Modelling with jump processes*. Chapman & Hall/CRC.

Cox, Ingersoll, and Ross (1985). *A theory of the term structure of interest rates*. Econometrica.

CSFB (1997). *Credit Risk+: A Credit Risk Management Framework*.

De Felice and Moriconi (2011). *Una nuova finanza d'impresa: le imprese di assicurazioni, Solvency II, le autorità di vigilanza*. il Mulino.

Delbaen, F. and W. Schachermayer (1998). *The fundamental theorem of asset pricing for unbounded stochastic processes*. Math. Ann.

Deutsch (1965). *Estimation theory*. Prentice-Hall.

Dignan (2003). "Nondefault components of investment-grade bond spread". In: *Financial Analysts Journal*.

Doucet (1998). "On sequential simulation-based methods for Bayesian filtering". In: *Technical Report CUED/F-INFENG/TR 310*.

Driessen (2003). "Is default event risk priced in corporate bonds?" In: *Financial Analysts Journal*.

Duffie and Garleanu (2001). "Risk and valuation of collateralized debt obligations". In: *Financial Analysts Journal*.

Duffie and Singleton (1999). “Modelling term structure of defaultable bond”. In: *The Review of Financial Studies.*

Durbin and Koopman (2001). *Time Series Analysis by State Space Methods.* Oxford University Press.

Elton (1999). “Expected return, realized return, and asset pricing tests”. In: *Journal of Finance.*

Elton et al. (2001). “Explaining the rate spread on corporate bond”. In: *The Journal of Finance.*

Glasserman (2000). *Monte Carlo Methods in Financial Engineering.* Springer.

Gorovoi and Linetsky (2004). “Black’s model of interest rates as options, eigenfunction expansion and japanese interest rates”. In: *Department of industtial engineering and mangement science, McCormick school of engineering and applied science, Northwestern university.*

Grewal and Andrews (2001). *Kalman Filtering Theory and Practice Using MATLAB.* John Wiley and Sons.

Group, RiskMetrics (2002). *Credit Grades Technical Document.* C.F. Finger.

Harrison, J. and D. Kreps (1979). *Martingales and arbitrage in multiperiod security markets.* J. Economic Theory.

Harrison, J. M. and S. R. Pliska (1981). *Martingales and stochastic integrals in the theory of continuous trading.* Stochastic Process.

Harrison and Kreps (1979). “Martingales and arbitrage in multiperiod security markets”. In: *Economic Theory.*

Harrison and Pliska (1981). “Martingales and stochastic integrals in the theory of continuos trading”. In: *Process. Appl.*

International Actuarial Association (2004). *Guidance paper on the use of internal models for regulatory purpose.*

Isaacson and Keller (1966). *Analysis of Numerical Methods.* Wiley.

ISVAP (2008a). *Regolamento 16.*

— (2008b). *Regolamento 19.*

— (2008c). *Regolamento 21.*

Jacod and Shiryaev (1988). *Limit Theorems for Stochastic Processes.* Springer.

Jazwinski (1970). *Stochastic processes and filtering theory.* Academic Press.

Jeanblanc M. Yor M., Chesney M (2009). *Mathematical methods for financial markets.* Springer Finance.

Kabanov (2001). *Arbitrage theory.* Cambridge University Press.

Kalman (1960). “A new approach to linear filtering and prediction problems”. In: *Trans. ASME.*

Kalman and Bucy (1961). “New result in lenear filtering and prediction theory”. In: *Trans. ASME.*

— (1963). "New methods in wiener filtering theory". In: *Proc. Symp. Appl. Random Function Theory and Probability.*

Kitagawa (1996). *Monte carlo filter and smoother for non-gaussian nonlinear state space models.* Journal of computational and graphical statistics.

KMV, Moody's (2005). *The Moody's KMV EDF RiskCalc v3.1 Model.*

Lando, D. (2004). *Credit Risk Modeling Theory and Applications.* Princeton University Press.

Li, Linetsky, and Mendoza-Arriaga (2016). *Modelling electricity prices: a time change approach.* Quantitative Finance.

Liptser and Shiryaev (1998). *Elements of the general theory of stochastic processes.* Yu.V. Prokhorod and A.N. Shiryaev (eds), Probability Theory III. Springer Encyclopedia of Mathematical Sciences, Vol. 45, Springer, Berlin, pp. 111-157.

Madan, Carr and Chang (1998). *The variance Gamma process and option pricing.* Eur. Finance Rev.

Matteson and Ruppert (2015). *Statistics and Data Analysis for Financial Engineering.* Springer.

McGee and Schmidt (1985). *Discovery of the Kalman filter as a practical tool for aerospace and industry.*

Mendoza-Arriaga and Linetsky (2014). "Time-changed CIR default intensities with two-sided mean-reverting jumps". In: *Institute of Mathematical Statistics.*

Merton (1974). "On the pricing of corporate debt: the risk structure of interest rates". In: *Journal of Finance.*

Moody's KMV P. Crosbie, J. Bohn (2003). *Modeling Default Risk: modeling methodology.*

Morgan, J.P. (1997). *Credit Metrics Technical Document.*

Musiela and Rutkowski (2005). *Martingale methods in financial modelling.* Springer.

O'Kane and Sen (2004). *Credit spreads explained.* Tech. rep. Lehman Brothers.

Parliament, European and the Council (2009). *DIRECTIVE 2009/138/EC.*

Powell (2007). "A view of algorithms for optimization without derivatives". In: *Cambridge University Technical Report.*

Protter (1990). *Stochastic Integration and Diflerential Equations.* Springer.

Quarteroni, Sacco, and Saleri (2000). *Matematica numerica.* Springer.

Robert (2007). *The Bayesian choice: from decision-theoretic foundations to computational implementation.* Springer.

Sato (1999). *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press.

Schönbucher (2003). *Credit derivatives pricing models: models, pricing and implementation.* Wiley.

Stratonovich (1959). “On the theory of optimal nonlinear filtration of random functions”. In: *Theor. Probability Appl.*

— (1960). “Conditional Markov Processes”. In: *Theor. Probability Appl.*

Ungarala (2012). “Fixed Interval Smoothing of Nonlinear/Non-Gaussian Dynamic Systems in Cell Space”. In: *Department of Chemical and Biomedical Engineering, Cleveland State University.*

Wan and Van Der Merwe (2000). *The unscented Kalman filter for nonlinear estimation.* Adaptive Systems for Signal Processing, Communications, and Control Symposium 2000. AS-SPCC. The IEEE 2000.

Wiener (1949). *The Extrapolation, Interpolation and Smoothing of stationary time series*. Wiley.

— (1958). *Nonlinear problems in random theory*. Wiley.

Wilson, T. (1997). *Portfolio credit risk I-II. Risk.*