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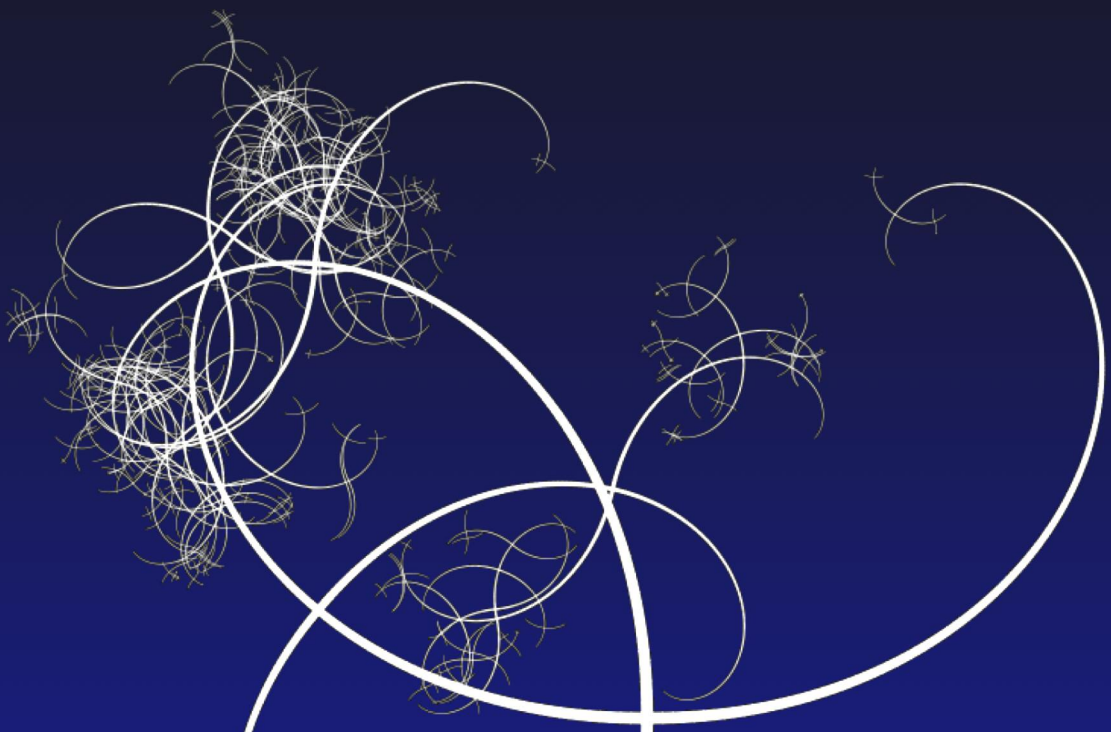

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NUMERICAL EVIDENCE FOR THE APPROXIMATION OF DISSIPATIVE SYSTEMS BY GYROSCOPICALLY COUPLED OSCILLATOR CHAINS

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Inspired by the equations of motion of the gyroscope, where a Lagrangian and conservative system may appear to mimic a dissipative one when focusing on a single degree of freedom for finite time intervals, we introduce a gyroscopic-type coupling between harmonic oscillators. The aim is to propose a tentative scheme of solution for the problem of finding a higher-dimensional Lagrangian system approximating a lower-dimensional dissipative one. Specifically, we consider a certain family of Lagrangian systems, for which the time evolution of the first Lagrangian parameter is conjectured to be a good approximation for the evolution of a one-dimensional linear dissipative system in finite time intervals, up to a fixed precision. The behavior of the selected family of gyroscopic couplings is compared with a given dissipative system, properly optimizing a family of parameters according to a described scheme. Numerical calculations are reported, suggesting the validity of the proposed conjecture.

1. Introduction

A classical problem in mathematical physics is the prediction of the motion of gyroscopes when gravitational loads are applied. In [Arnold 1989] these motions are qualitatively studied by exploiting their Lagrangian structure. In fact, when explicitly formulating the equation of motion of gyroscopes with fixed point and under the gravitational load as done, at first, by Euler, one observes immediately that, in the second order evolution equation, for each Euler angle the first-order time derivatives of the other Euler angles appear; see, for example, [Marsden and Scheurle 1993; Eckardt 2018].

The specific case of the gyroscope's motion gives an interesting example of evolutionary equations which turn out to be the Euler–Lagrange stationarity condition for an action functional. It is interesting to notice that, when considering the equation for a single Euler angle, an apparent "energy subtraction or injection" from one

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degree of freedom towards the other degrees of freedom emerges. Indeed, the time derivatives of the other degrees of freedom cause an acceleration (or deceleration) of any degree of freedom.

Gyroscopic couplings do not appear only in the equations for the spinning top; in fact, these are pervasive in many Lagrangian systems in which an exchange of energy occurs among different degrees of freedom. We mention here, for instance, some related studies about phenomena occurring in piezomechanical structures; see [Alessandroni et al. 2004; Andreaus and dell'Isola 2004; Darleux et al. 2022; Giorgio et al. 2009; Maurini et al. 2004; Olive and Auffray 2021; Rosi et al. 2010; Shen et al. 2010].

Note that the idea of accounting for a coupling between micro and macro phenomena by means of gyroscopic effect was already conceived in [Berezovski et al. 2020]. While it is well known that a variational approach is possible for describing dissipative phenomena by using the Hamilton–Rayleigh dissipation potential (see, for instance, [Abali et al. 2016; Capobianco et al. 2021; Baroudi et al. 2019]), we believe that the purely conservative Lagrangian approach chosen in [Berezovski et al. 2020] may often be preferable for its computational advantages.

Focusing on a single equation of the Euler–Lagrange system for the gyroscopic motion, in properly chosen finite time intervals, implies only apparently the appearance of dissipation phenomena. Of course, in longer time intervals and due to the Hamiltonian nature of the system (see the Poincaré recurrence theorem, for instance, in [Arnold 1989]), the subtracted energy will be given back to each degree of freedom. However, the so-called Poincaré recurrence time may be so long that Hamiltonian systems can be suitable for describing dissipation at appropriate time scales.

These considerations suggest an approach toward the goal of constructing an enlarged Lagrangian system whose first Lagrange coordinate, in its evolution, is intended to approximate the motion of a given dissipative system with one degree of freedom. We will call the original dissipative system the *master system*, and the added one the *slave system*. Of course the two systems must be suitably coupled, and the slave system should be conservative. In the specific example examined in this paper, (i) the master system is a harmonic oscillator; (ii) the slave system harmonic oscillators; and (iii) the coupling between master and slave systems is gyroscopic in nature. Both the gyroscopic-like couplings and the stiffnesses of the slave system must be optimized to reproduce, as closely as possible, the evolution of the master system when dissipation is considered.

We hope that this case study can serve as a springboard for the study of a more general problem: Given a dissipative master system (i.e., a set of non-Hamiltonian evolution equations), is it possible to find a higher-dimensional Lagrangian system, consisting of a nondissipative slave system and a slave/master coupling, such that

some part of its dynamics shadows the evolution of the master system within a prescribed error over a time interval of interest? Despite the specificity of the gyroscopic-like coupling selected in solving the example problem considered here, the encouraging numerical calculations that are reported seem to indicate that the general case deserves further attention.

To be more precise, we start by considering a nonconservative one-dimensional autonomous system in normal form with prescribed initial conditions

$$\begin{cases} \ddot{x} = f(x, \dot{x}), \\ x(t_0) = x_0, \\ \dot{x}(t_0) = \dot{x}_0. \end{cases} \tag{1}$$

It is well known that precise conditions need to hold for a second order dynamical system to be Lagrangian, the so-called Helmholtz conditions, which in the case of autonomous normal systems can be formulated as follows (see [Bersani and Caressa 2021] for a complete discussion in the general case):

Theorem 1.1. *Given a set of differential equations*

$$\ddot{x}_k = f_k(\boldsymbol{x}, \dot{\boldsymbol{x}}), \quad k = 1, 2, \ldots, N,$$

necessary and sufficient condition for the existence of a Lagrangian function $L = L(t, \boldsymbol{x}, \dot{\boldsymbol{x}})$ *such that*

$$\ddot{x}_k - f_k = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k} - \frac{\partial L}{\partial x_k}, \quad k = 1, 2, \ldots, N,$$

are

(I) $\partial f_k/\partial \dot{x}_h + \partial f_h/\partial \dot{x}_k = 0$,

(II) $\partial f_h/\partial \dot{x}_k - \partial f_k/\partial \dot{x}_h = \frac{1}{2}\sum_{l=1}^{n} \dot{x}_l \partial/\partial x_l(\partial f_h/\partial x_k - \partial f_k/\partial x_h)$,

for each $h, k = 1, \ldots, N$.

In the particular case of one degree of freedom, condition (II) identically vanishes, while condition (I) reduces to

$$\frac{\partial f}{\partial \dot{x}} = 0,$$

so we recover the well known fact that an autonomous system with one degree of freedom is Lagrangian if its potential does not depend on the velocity.

These conditions are generally violated by dissipative systems, in which terms depending on the velocities appear. However, two related questions arise in connection with such a system: whether a different system having the same trajectories and a Lagrangian formulation exists, and whether one can somehow approximate the dissipative system by a conservative one, which is by definition Lagrangian. We refer to [Bersani and Caressa 2021] for the first problem; for the second, some

progress has been made using classical elastic chains in [Bersani et al. 2022].[1] Here, by contrast, we approach the second question using gyroscopic approximating systems.

In principle, we could deal with Cauchy problems (1) for general second-order equations in normal form. However, it is well known that if $f(x, \dot{x})$ satisfies suitable hypotheses for $(x, \dot{x}) \in \mathbb{R}^2$ (for example continuity and sublinearity), the problem (1) has a global solution. Here, as in [Bersani et al. 2022], we will focus on linear equations, for which an existence and uniqueness theorem in global form is guaranteed. In the future we hope to extend our research to nonlinear systems, where only local existence can, in general, be guaranteed, and problems concerning the extendability of the solutions could arise.

To be specific, we ask: given $\varepsilon > 0$ and a compact time interval $[t_0, t_1]$, is it possible to find a skew-symmetric matrix A and a symmetric matrix S such that the first component $q_1(t)$ of the solution vector $\boldsymbol{q}(t)$ of the Lagrangian system

$$\begin{cases} \ddot{\boldsymbol{q}} + A \cdot \dot{\boldsymbol{q}} + S \cdot \boldsymbol{q} = 0, \\ \boldsymbol{q}(t_0) = \boldsymbol{q}_0, \\ \dot{\boldsymbol{q}}(t_0) = \dot{\boldsymbol{q}}_0, \end{cases} \tag{2}$$

satisfies

$$\|q_1 - x\|_\infty := \sup_{[t_0, t_1]} |q_1(t) - x(t)| < \varepsilon, \tag{3}$$

where $x(t)$ is the solution of system (1)?

In Section 2, we describe in detail the gyroscopic system we are studying and write it in a Hamiltonian form suitable for our computations. In Section 3, we illustrate the results of our numerical simulations aimed at optimizing the coefficients of A and S to force inequality (3) to hold. In particular, we focus on two special forms for the matrices involved which make computations more feasible. Finally, in Section 4, we draw some conclusions from our work. Details about the numerical scheme are reported in the supplemental material (see Appendix).

2. A gyroscopic system

Let us fix a dimension N and consider two matrices $G, K \in M_N(\mathbb{R})$, such that

(1) G is skew-symmetric (hence G^2 is a symmetric negative semidefinite matrix);

(2) K is symmetric and definite positive.

[1] It seems that the more general problem of a master system having k degrees of freedom elastically coupled with a slave one having $N \gg k$ degrees of freedom could also be treated in a similar way. The metamaterial designs developed in [Turco et al. 2022; Turco 2021a; 2021b] are based on this principle.

Next define the Lagrangian function

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \tfrac{1}{2}\|\dot{\boldsymbol{q}}\|^2 - \langle G \cdot \dot{\boldsymbol{q}}, \boldsymbol{q}\rangle - \tfrac{1}{4}\langle G^2 \cdot \boldsymbol{q}, \boldsymbol{q}\rangle - \tfrac{1}{2}\langle K \cdot \boldsymbol{q}, \boldsymbol{q}\rangle,$$

where $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \boldsymbol{u}^T \cdot \boldsymbol{v}$ denotes the Euclidean scalar product and $\|\boldsymbol{u}\| = \sqrt{\langle \boldsymbol{u}, \boldsymbol{u}\rangle}$ the Euclidean norm in $\mathbb{R}^N$.

The Euler–Lagrange equations induced by this Lagrangian are straightforward to compute in vector notation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} - \frac{\partial L}{\partial \boldsymbol{q}} = \ddot{\boldsymbol{q}} + 2G \cdot \dot{\boldsymbol{q}} + \tfrac{1}{2}G^2 \cdot \boldsymbol{q} + K \cdot \boldsymbol{q} = 0.$$

Hence the equations of motion of the Lagrangian system described by L are

$$\ddot{\boldsymbol{q}} + 2G \cdot \dot{\boldsymbol{q}} + \left(\tfrac{1}{2}G^2 + K\right) \cdot \boldsymbol{q} = 0; \tag{4}$$

hence they are in the form of the gyroscopic system (2), since $\frac{1}{2}G^2 + K$ is still a symmetric matrix.

Moving to the canonical formalism and writing the corresponding Hamiltonian function for the system (4) is straightforward too: let us define

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}} = \dot{\boldsymbol{q}} + G \cdot \boldsymbol{q} \implies \dot{\boldsymbol{q}} = \boldsymbol{p} - G \cdot \boldsymbol{q}.$$

Therefore (since $G^T = -G$ we have $G^T G = -G^2$)

$$\begin{aligned} H(\boldsymbol{q}, \boldsymbol{p}) &= \langle \boldsymbol{p}, \dot{\boldsymbol{q}}\rangle - L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \langle \boldsymbol{p}, \boldsymbol{p} - G \cdot \boldsymbol{q}\rangle - \tfrac{1}{2}\langle \boldsymbol{p} - G \cdot \boldsymbol{q}, \boldsymbol{p} - G \cdot \boldsymbol{q}\rangle \\ &\quad + \langle G \cdot (\boldsymbol{p} - G \cdot \boldsymbol{q}), \boldsymbol{q}\rangle + \tfrac{1}{4}\langle G^2 \cdot \boldsymbol{q}, \boldsymbol{q}\rangle + \tfrac{1}{2}\langle K \cdot \boldsymbol{q}, \boldsymbol{q}\rangle \\ &= \tfrac{1}{2}\langle \boldsymbol{p}, \boldsymbol{p}\rangle - \langle \boldsymbol{p}, G \cdot \boldsymbol{q}\rangle + \langle G \cdot \boldsymbol{q}, \boldsymbol{p}\rangle - \tfrac{1}{2}\langle G \cdot \boldsymbol{q}, G \cdot \boldsymbol{q}\rangle \\ &\quad + \langle G \cdot \boldsymbol{p}, \boldsymbol{q}\rangle - \langle G^2 \cdot \boldsymbol{q}, \boldsymbol{q}\rangle + \tfrac{1}{4}\langle G^2 \cdot \boldsymbol{q}, \boldsymbol{q}\rangle + \tfrac{1}{2}\langle K \cdot \boldsymbol{q}, \boldsymbol{q}\rangle \\ &= \tfrac{1}{2}\|\boldsymbol{p}\|^2 + \langle G \cdot \boldsymbol{p}, \boldsymbol{q}\rangle - \tfrac{1}{4}\langle G^2 \cdot \boldsymbol{q}, \boldsymbol{q}\rangle + \tfrac{1}{2}\langle K \cdot \boldsymbol{q}, \boldsymbol{q}\rangle. \end{aligned}$$

Thus, the Hamilton equations associated with H are readily written as

$$\dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}} = \boldsymbol{p} - G \cdot \boldsymbol{q}, \qquad \dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}} = -G \cdot \boldsymbol{p} + \tfrac{1}{2}G^2 \cdot \boldsymbol{q} - K \cdot \boldsymbol{q}.$$

The second equality is just the Euler–Lagrange equations (4) written in terms of canonical coordinates, since

$$\begin{aligned} 0 &= \dot{\boldsymbol{p}} - (-G \cdot \boldsymbol{p} + \tfrac{1}{2}G^2 \cdot \boldsymbol{q} - K \cdot \boldsymbol{q}) \\ &= (\dot{\boldsymbol{q}} + G \cdot \boldsymbol{q})^{\cdot} + G(\dot{\boldsymbol{q}} + G \cdot \boldsymbol{q}) - \tfrac{1}{2}G^2 \cdot \boldsymbol{q} + K \cdot \boldsymbol{q} = \ddot{\boldsymbol{q}} + 2G \cdot \dot{\boldsymbol{q}} + (\tfrac{1}{2}G^2 + K) \cdot \boldsymbol{q}. \end{aligned}$$

We can express Hamilton equations as a block-matrix linear system as

$$\begin{pmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{p}} \end{pmatrix} = \begin{pmatrix} -G & I \\ \frac{1}{2}G^2 - K & -G \end{pmatrix} \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{pmatrix}.$$

We will use this canonical reduction to a first order system for our original gyroscopic system to perform some numerical computations.

3. Numerical computations

The Hamiltonian system we have written depends on $\frac{1}{2}n(n+1)+\frac{1}{2}n(n-1)=n^2$ parameters, namely the entries of the matrices G and K.

To lower the complexity of the problem we will make the assumption that G *is tridiagonal* and that K *is diagonal*. The latter assumption is not severe, since a change of coordinates can diagonalize K. We will write $G=\mathrm{tridiag}(g_1,\dots,g_{n-1})$ and $K=\mathrm{diag}(k_1,\dots,k_n)$, where

$$\mathrm{tridiag}(g_1,\dots,g_{n-1}) := \begin{pmatrix} 0 & g_1 & 0 & \cdots & 0 & 0 \\ -g_1 & 0 & g_2 & \cdots & 0 & 0 \\ 0 & -g_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & g_{n-1} \\ 0 & 0 & 0 & \cdots & -g_{n-1} & 0 \end{pmatrix}.$$

The square of G is the symmetric matrix

$$G^2 = \begin{pmatrix} -g_1^2 & 0 & g_1g_2 & 0 & \cdots & 0 & 0 \\ 0 & -g_1^2-g_2^2 & 0 & g_2g_3 & \cdots & 0 & 0 \\ g_1g_2 & 0 & -g_2^2-g_3^2 & 0 & \cdots & 0 & 0 \\ 0 & g_2g_3 & 0 & -g_3^2-g_4^2 & \cdots & 0 & 0 \\ 0 & 0 & g_3g_4 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & g_{n-3}g_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & g_{n-2}g_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & -g_{n-2}^2-g_{n-1}^2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -g_{n-1}^2 \end{pmatrix}.$$

with entries

$$[G^2]_{i,j} = \begin{cases} -g_1^2 & \text{if } i=j=1, \\ -g_{i-1}^2-g_i^2 & \text{if } i=j=2,\dots,n-1, \\ -g_{n-1}^2 & \text{if } i=j=n, \\ g_ig_{i+1} & \text{if } i=1,\dots,n-2,\ j=i+2, \\ g_{i-2}g_{i-1} & \text{if } i=3,\dots,n,\ j=i-2, \\ 0 & \text{otherwise.} \end{cases}$$

With these definitions, the gyroscopic Hamiltonian system of the previous section may be written componentwise as (for $r = 2, \dots, n-1$ and $h = 3, \dots, n-2$):

$$
\begin{aligned}
\dot{q}_1 &= p_1 - g_1 q_2,\\
\dot{q}_r &= p_r + g_{r-1} q_{r-1} - g_r q_{r+1},\\
\dot{q}_n &= p_n + g_{n-1} q_{n-1},\\
\dot{p}_1 &= -\big(\tfrac{1}{2}g_1^2 + k_1\big) q_1 + \tfrac{1}{2} g_1 g_2 q_3 - g_1 p_2,\\
\dot{p}_2 &= -\big(\tfrac{1}{2}g_1^2 + \tfrac{1}{2}g_2^2 + k_2\big) q_2 + \tfrac{1}{2} g_2 g_3 q_4 + g_1 p_1 - g_2 p_3,\\
\dot{p}_h &= -\big(\tfrac{1}{2}g_{h-1}^2 + \tfrac{1}{2}g_h^2 + k_h\big) q_h + \tfrac{1}{2} g_{h-2} g_{h-1} q_{h-2} + \tfrac{1}{2} g_h g_{h+1} q_{h+2}\\
&\qquad + g_{h-1} p_{h-1} - g_h p_{h+1},\\
\dot{p}_{n-1} &= -\big(\tfrac{1}{2}g_{n-2}^2 + \tfrac{1}{2}g_{n-1}^2 + k_{n-1}\big) q_{n-1} + \tfrac{1}{2} g_{n-3} g_{n-2} q_{n-3} + g_{n-2} p_{n-2} - g_{n-1} p_n,\\
\dot{p}_n &= -\big(\tfrac{1}{2}g_{n-1}^2 + k_n\big) q_n + \tfrac{1}{2} g_{n-2} g_{n-1} q_{n-2} + g_{n-1} p_{n-1}.
\end{aligned}
$$

As stated in Section 1, we want to use the q_1-component of the solution of this system to approximate $x(t)$, the solution of (1). As the main system, we are dealing with the simple and well known damped harmonic oscillator $\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0$.

The numerical computation we implement aims to optimize both the g_r and the k_h towards the minimum of $\|x(t) - q_1(t)\|_\infty$. We consider the constants $\gamma = 0.5$ and $\omega = 4$ and fix $[t_0, t_1] := [0, 5]$ as the time interval, with initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$. The initial conditions of the gyroscopic system are 0 for all the initial Lagrangian coordinates and velocities, except for q_1, for which we take $q_1(0) = 1$, the same as for the damped system.

The gyroscopic system has been calibrated, with the double simulated annealing algorithm (see [Xiang et al. 1997; Bersani et al. 2022]) for $N = 3, 4, \dots, 10$ degrees of freedom: the resulting trajectories are plotted against the damped one in Figure 1.

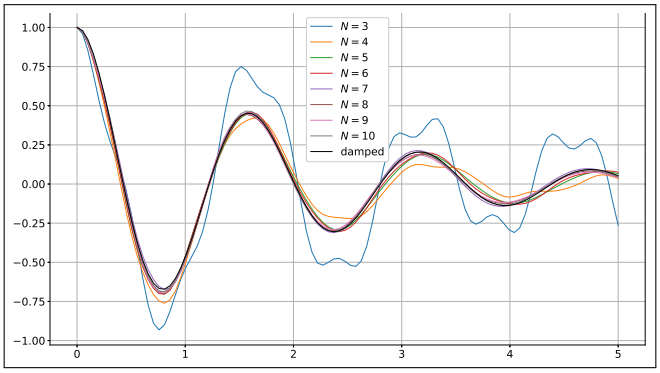

Figure 1. Approximation of a damped harmonic motion via the gyroscopic system for different numbers of degrees of freedom N, when we optimize the entries of both K and G.

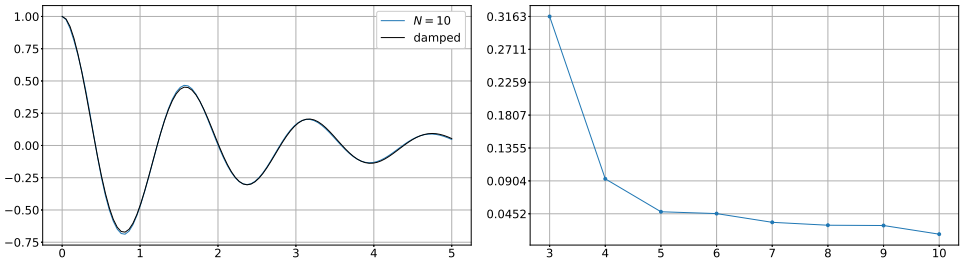

Figure 2. Optimization of entries of both G and K. Left: approximation of the damped oscillator by a gyroscopic system with $N = 10$ degrees of freedom. Right: approximation error as a function of degrees of freedom.

It is clear that for $N = 3$ the result is too rough and that to obtain a reasonable approximation one needs, in this case, at least a gyroscopic system with four degrees of freedom. The case of 10 degrees of freedom, on the other hand, is quite satisfactory. Indeed, in this case, the approximation error decreases up to $\|x(t) - q_1(t)\|_\infty = 0.0171$. The comparison plot of $x(t)$ with $q_1(t)$ for $N = 10$ is shown in Figure 2, left.

Most importantly, as shown in Figure 2, right and Table 1, the error curve plotted as a function of the number of degrees of freedom shows a decreasing trend. This supports the intuition that as the number of degrees of freedom increases, the accuracy of the approximation improves.

Note that we used an efficient optimization algorithm, which is rather expensive in computational terms, while simpler algorithms are much less efficient in approximating the damped system with the gyroscopic one up to the same accuracy.

To decrease the complexity, at least in this specific case, one could try to set the matrix K as a fixed matrix, not subject to optimization, halving the parameters to be optimized, reduced to only the super-diagonal elements of G. However, the numerical calculations we performed here seem to indicate that in general the optimization of the elements of the G-matrix alone is not sufficient to hope for an arbitrary reduction in the difference between the trajectories.

Indeed, when we fix the entries of K and optimize only those of G, we find that the approximation error *does not* show a decreasing trend as the degrees of freedom increase, as shown in Table 2 and Figure 3, right, where the approximation error

N	3	4	5	6	7	8	9	10
error	0.3163	0.0932	0.0480	0.0454	0.0334	0.0294	0.0290	0.0171

Table 1. Approximation error as a function of degrees of freedom, when we optimize the entries of both K and G.

N	3	4	5	6	7	8	9	10
error	0.2638	0.2394	0.2454	0.2454	0.2450	0.2450	0.2449	0.2449

Table 2. Approximation error as a function of degrees of freedom. Case of fixed K and optimization on G entries only.

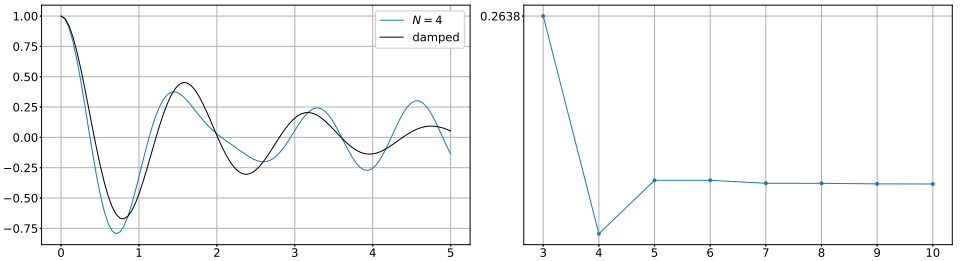

Figure 3. Case of fixed K and optimization on G entries only. Left: approximation of the damped oscillator by a gyroscopic system with $N = 4$ degrees of freedom. Right: approximation error as a function of degrees of freedom.

as a function of degrees of freedom is illustrated. In Figure 3, left, the best fitting we found in this case, corresponding to 4 degrees of freedom for the gyroscopic system, is plotted against the actual solution of the damped oscillator.

We consider the final case where K is a scalar matrix (i.e., a nonzero matrix multiple of the identity matrix). Thus, we have a problem that is computationally much simpler than the general one, since in this case we optimize only N parameters (the nonzero entries of G and the diagonal identical entries of K), instead of $2N-1$. Our numerical simulations show that, although with about three times the error of the general case, the gyroscopic system can still approximate the damped system with an error that has a decreasing trend as the number of degrees of freedom increases. Figure 4, left, plots the best-fitting case, corresponding to $N = 10$. The error as a function of the number of degrees of freedom is reported in Table 3 and Figure 4, right.

Therefore, numerical simulations seem to suggest that in the case of G tridiagonal and K diagonal, for the considered gyroscopic coupling, one could reduce the optimization to N parameters to approximate the dissipative system, within the fixed time interval and the desired accuracy.

N	3	4	5	6	7	8	9	10
error	0.3592	0.1428	0.1216	0.0978	0.0970	0.0683	0.0844	0.0527

Table 3. Approximation error as a function of degrees of freedom: case of optimization of G and K entries, K being scalar.

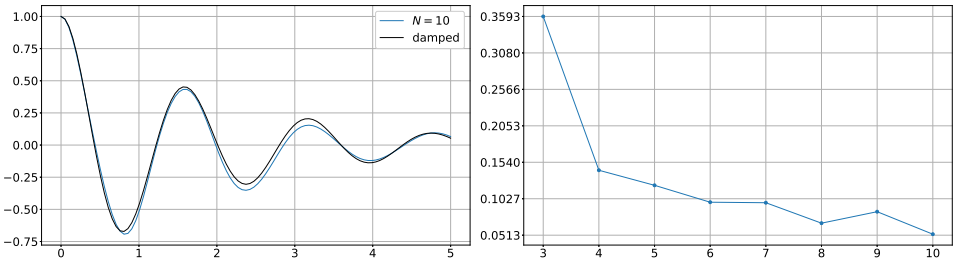

Figure 4. case of optimization of G and K entries, K being scalar. Left: approximation of the damped oscillator by a gyroscopic system with $N = 10$ degrees of freedom. Right: approximation error as a function of the number of degrees of freedom.

4. Conclusions

The problem of finding a Lagrangian higher-dimensional system that can be used to approximate a lower-dimensional dissipative system has both a theoretical interest, since its solution may aid in understanding the fundamental nature of dissipative phenomena, and practical applications, since it may yield interesting developments in methods for formulating numerical codes for the study and prediction of many types of natural phenomena. In this paper, given a 1-dimensional dissipative system, we investigated the possibility of constructing a triplet: a master system, a slave system, and a coupling scheme. The effect of the interaction in the slave system seems to be effective in producing apparent dissipation, mimicking the master system thanks to the designed coupling scheme, at least in given finite time intervals. The coupling scheme used is based on gyroscopic Lagrangian terms and is found to be suitable, at least in the linear case that has been treated.

The mentioned choice has been motivated by the very interesting characteristics shown by the equations describing the gyroscopic phenomena and it is possible to say that the conjecture on which we have based the presented investigation seems quite well-grounded. In fact, it was possible to optimize the stiffnesses of the slave system and the coefficients needed in the specific class of gyroscopic couplings used to approximate the linear damping in the chosen linear master system with an error that numerically appears to be reducible to any small amount. The reported positive results seem to motivate further investigations. In particular, one should address the case of master systems with dimension greater than one, more complicated gyroscopic couplings, nonlinear master systems coupled to linear slave systems via linear gyroscopic couplings, and the effects of nonlinearities in gyroscopic coupling terms. The future investigation should in our opinion involve the need of reconciling the Lagrangian extension for the coupled master and slave systems with the Hamilton–Rayleigh approach. This approach has been systematically exploited

for describing a variety of phenomena. This includes phenomena occurring in porous media [dell'Isola et al. 2009; 2019; dell'Isola and Hutter 1997; Giorgio et al. 2017a], thermomechanical phenomena [Giorgio 2021; Cuomo 2017; Altenbach et al. 2012], biomechanical phenomena in bone growth [Giorgio et al. 2017b], friction phenomena in mechanical systems [Ciallella et al. 2021; Giorgio et al. 2009; Giorgio 2020; Scerrato et al. 2015; Giorgio and Scerrato 2017; Spagnuolo and Cazzani 2021], damage phenomena Placidi and Barchiesi 2018; Timofeev et al. 2021; Placidi et al. 2021; 2018; Contrafatto and Cuomo 2006; Cuomo et al. 2014]. In addition, the study of the theoretical motivations behind the numerical evidence shown should be pursued, with the aim of both understanding the limitations to the extension to more general systems of the approximation scheme studied and exploring the possibility of a constructive a priori estimation results about the number of degrees of freedom needed to achieve the desired accuracy.

Appendix: Code used to perform numerical computations

To perform numerical computations we used the `scipy` Python package under the Python 3.8 compiler. We used the simulated annealing optimization function which implements a generalized annealing [Xiang et al. 1997] by using its default parameters both for annealing temperature, number of iterations etc.

The code can be found on the GitHub page of one of the authors[2] and it can be executed in a Python 3 environment which provides standard libraries `numpy`, `scipy` and `matplotlib`, easily installed via the `pip` tool.

The script optimizes parameters of a coupled system to approximate a damped one: results are either written on png files or plotted interactively; further information is printed. The functions defined in this script may be easily engineered to run several classes of simulations and collect results.

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