Original Manuscript

MMS

Mathematics and Mechanics of Solids
1–17

DOI: 10.1177/10812865221096685
journals.sagepub.com/home/mms
SAGE

Lagrange formal calculus as applied to Lagrange mechanics: An exercise in anachronism

Alberto Maria Bersani
Department of Mechanical and Aerospace Engineering, Sapienza University of Rome, Rome, Italy

Enrico Bersani
International Research Center on Mathematics and Mechanics of Complex Systems (M&MOCS), L'Aquila, Italy

Paolo Caressa
Gestore dei Servizi Energetici S.p.A, Rome, Italy

Received 2 February 2022; accepted 4 April 2022

Abstract
When Lagrange wrote his masterpiece *Mécanique Analytique*, the foundations of analysis were not completely understood: to erect the great building of Analytical Mechanics upon solid foundations, the Piedmontese mathematician tried to lay the foundations of differential calculus in a purely algebraic way, using power series instead of functions, regardless about convergence and uniqueness issues. While this foundation was unsatisfactory as shown by Cauchy some decades later, it can shed light on how Lagrange considered the analytical objects (curves, energies, etc.) he dealt with in Mechanics. In this paper, we review these Lagrangian foundations of analysis, and we try to adopt its obvious modern counterpart, i.e., formal power series, to express some results in Analytical Mechanics related to Helmholtz conditions and Rayleigh description of dissipation. By means of purely algebraic manipulations, we will easily recover results otherwise proved by means of modern analysis.

Keywords
Lagrange formalism, Analytical Mechanics, Helmholtz conditions, Lagrangians, Rayleigh dissipative systems, formal series expansions

Dedicato a Francesco, filosofo naturale

Corresponding author:
Alberto Maria Bersani, Dipartimento di Ingegneria Meccanica e Aerospaziale, Sapienza University of Rome, via A. Scarpa, 16, 00161 Rome, Italy.
Email: alberto.bersani@sbai.uniroma1.it

1. Introduction

The aim of this note is twofold: on one hand, we want to review the concept of differential calculus as Lagrange understood it; on the other hand, we want to take seriously this old point of view to glimpse at how Lagrange understood mechanical concepts and to try to use it, in a modern version, to recover some results in Analytical Mechanics we are currently reflecting upon [1].

To our modern eye, derivatives are Gateaux and Fréchet ones, integrals are Lebesgue and Haar ones, and analytical mechanics speaks the language of symplectic geometry and functional analysis [2]. At the time of Lagrange instead, differential and integral calculus was essentially the one exposed in Euler's influential textbook *Institutiones calculi differentialis* [3] with which Lagrange was not satisfied since its foundations lay on the infinitesimal calculus (which Euler exposed in his textbook appeared in 1748 [4]), still cloaked in metaphysical mists that will dissolve only in the 19th century.

Lagrange proposal for the analytical foundations of differential calculus was largely in the spirit of the Cartesian method, as the adjective *analytique* testifies: this approach has been superseded some decades after by the 19th century theories of Abel and Cauchy and, most of all, Weierstrass, Dedekind and Cantor foundation of real number theory as a fertile and stable ground for sprouting Real Analysis. But we believe, historical interest aside, that Lagrange's viewpoint can give some insight into how the creator of Analytical Mechanics actually thought curves, functions, and differential equations which appears in his immortal mechanical works.

This paper is organized in two parts which *grosso modo* correspond to a historical reconstruction of Lagrange ideas on differential calculus and an application, with modern algebraic notations, of those ideas to simple mechanical concepts.

In the first part of the paper, we recall and describe Lagrange approach, trying to recover the insight of his analytical calculus as opposed to the geometric calculus of Euler, before him, and Cauchy, after him. In particular, we will give a look at variational calculus as expressed in these terms: of course, there are several analyses of Lagrange work on the foundations of calculus on which we will rely (see, e.g., previous works [5–10]).

In the second part of the paper, we attempt to express some simple and modern concepts of Analytical Mechanics *à la* Lagrange by means of the modern formalism which better approximates his symbolic device. We will see that such an approach can provide hints to non-trivial questions. In particular, we will discuss integrability conditions, dealing with Helmholtz characterization of second-order ordinary differential equations which can be expressed as Lagrangian systems [1], next we will focus on the Rayleigh extension of Lagrangian formalism to take care of some dissipative systems [1,11].

Strictly speaking, this is neither a paper about history of Mathematics nor about Mathematics in itself: it is an attempt to understand, in its historical context, the actual view of the founding father of Analytical Mechanics about concepts that we are used today to interpret along viewpoints introduced after decades and centuries of mathematical developments. The aim is to put on some 18th century lens and look at mechanical objects the way Lagrange probably did. We share our reflections since they could be useful to people dealing with Analytical Mechanics to appreciate and even read the Master of our subject. Being not under the spell of philological constraints, we feel free to use modern notations both for the description of Lagrange work and for our exercises on applying his analytical ideas to mechanical problems.

2. Lagrange avoidance of calculus metaphysics

Joseph Louis Lagrange masterpiece, *Méchanique Analytique*, aims at developing all mechanical theories into a single framework whose cornerstone is the "principle of virtual velocities," as he called it, that now is known as principle of virtual works: when he wrote his book, in the 1780s, Lagrange considered this principle more fundamental than the least action one [12,13], which was proposed by Euler and Maupertuis and later formulated by Hamilton and Schwinger as it is still today employed as theoretical foundation for mechanical theories [14,15].

The adjective *analytique* was actually intended to mean *algebraic*, and it is used along the Cartesian tradition: Descartes, a century and a half before Lagrange, applied his method to ancient geometry, providing an algebraic description of *loci*, which now we would term as a particular case of plane real algebraic geometry [16,17].

The "analytical taste" which characterizes all Lagrange works, even and especially his mechanical memoirs and treatises, is actually a rigorous mind-set, such as we will find in Weierstrass a century later, in which the development of mathematical theories is framed inside a finitistic description of mathematical terms. It is an attempt to rigor in times when infinitesimals and other metaphysical notions abounded in differential calculus texts.

Lagrange is interested in dealing with differential and integral calculus in these same rigorous and finitistic terms, with no reference to infinitesimals, fluxions, etc. as it can be seen in his masterpiece, the *Théorie Des Fonctions Analytiques* [18], whose first edition was published in Paris in 1797 but which indeed is the fruit of a long reflection upon the foundations of Analysis (as we call it today!) which stems from the Turin and Berlin years [9,10].

Even the new foundations for variational calculus, settled by Lagrange when he was 19 and completed in the following years and applied to questions of Mechanics [16], share that algebraic flavor which is typical of his way to intend mathematical rigor: of course, according to nowadays standards, Lagrange treatises are no more rigorous, but they still stand as the most elegant and formally correct pieces of Mathematics of 18th century.

2.1. Lagrange's intensional concept of a function

Being probably the most rigorous and formally accurate mathematician of his age, Lagrange was not satisfied with the status of differential and integral calculus that, since Newton, Leibniz and Bernoulli, relied on the infinitesimal analysis, a subject not correctly understood until 19th century.

There are several studies which describe and provide interpretations of Lagrange attempts to lay the foundations of differential calculus on algebraic bases, e.g., previous works [5–10]. Here, we will review, in modern notations, some fundamental concepts put forth by Lagrange for his foundation of calculus.

The first sentence of Lagrange 1806 textbook *Leçons sur le calcul des fonctions* [19][1] is explicit about Lagrange's intentions:

> Function calculus has the same object as the differential calculus taken in the broadest sense, but it is not subject to the difficulties encountered in the principles and in the ordinary course of this calculus: it serves more to link the differential calculus immediately to algebra, which can be said was a separate science so far.[2]

Lagrange notices that Euler considered Leibniz infinitesimals as being null quantities, incurring in $0/0$ expressions; he also mentions Mac Laurin and d'Alembert intuitive concepts of limit, which lead to the same difficulty (since a general definition of limit will be given by Cauchy a quarter of century later) [5,8]. It is worth to remind that Lagrange pays always much attention to the history of the subjects he studies, summarizing and criticizing previous efforts: this is done for the concept of function and for differential calculus in the introduction of his *Théorie* [18].

Moreover, Lagrange considers infinitesimals and limits as *metaphysics*, while he thinks that "analysis must have no other metaphysics than that which consists in the first principles and in the first fundamental operations of the calculus"[3] [19, p. 2].

On the contrary, Lagrange states that the truly science of functions is algebra: on extending algebraic operations to embrace function developments, Lagrange believes that differential calculus may be well founded as (polynomial) algebra is. Remember that Lagrange published important studies on the theory of algebraic equations, paving the way to Cauchy and most of all to Galois [16].

Lagrange definition of a function obviously avoids any set-theoretical flavor: he defines a function $f(x_1, ..., x_n)$ depending on variables $x_1, ..., x_n$ as an algebraic expression involving such variables and possibly numeric or symbolic constants: this expression is formed by elementary functions, variables, and constants connected via algebraic operations.

We recognize in Lagrange point of view the approach of modern computability theory, started with Alonzo Church's λ-calculus in 1936, which identifies functions with λ-expressions formed inductively in terms of elementary functions and composition rules [20]. In other terms, while, after 150 years of set theory, our concept of a function is *extensional*, Lagrange had an *intensional* concept in the sense specified by Church [20, §I.2]: functions are not sets, they are not identified with their graphs, rather they are symbolic algorithms which can be computed once unknowns are assigned to specific values. This is the approach of Church's theory, which has several applications in modern logic and computer science.

It is worth to underline that for Lagrange a function is not a numerical object: for us a function is "built on numbers," since we identify it with its graph and we usually think at it as a vector in an infinite dimensional vector space, where its argument is the coordinate index and its value the coordinate itself. On the contrary, for Lagrange, a function is not a numerical object but quite a symbolic one.

Lagrange fundamental assumption about functions is the following one (we state it in the case $n = 1$): given a function $f(x)$, if y is another indeterminate not depending on x, then one can always write:

$$f(x+y) = f(x) + \sum_{k\geqslant 1} g_k(x)y^k, \tag{1}$$

where g_k are the functions not depending on y.

Since, if $x = 0$ in the previous development, we find f expressed as a power series:

$$f(y) = f(0) + \sum_{k\geqslant 1} g_k(0)y^k, \tag{2}$$

from our modern extensional point of view, we could be tempted to claim that Lagrange is assuming each function f to be analytic at 0 (in the modern sense). But indeed this is not true, since analytic functions cannot be defined without reference to local disks of convergence (again a rigorous theory will be given by Weierstrass who correctly succeeded in founding complex variable theory upon power series, see [5]) and Lagrange does not take into account any disk or domain for his functions, indeed he is not concerned about convergence whatsoever.

Rather, we should consider a function according to Lagrange more as a formal power series, a purely algebraic object: he was not concerned about convergence in general, even if he states time and again that at specific values of their argument, functions may well be not defined. For these reasons, in his 1823 textbook *Résumé des leçons sur le calcul infinitésimal*, Augustin-Louis Cauchy dismissed gracefully but bluntly Lagrange's approach, claiming that [21, p. v],

> I thought I should reject the developments of functions in infinite series, whenever the series obtained are not convergent; and I saw myself forced to return to integral calculus the formula of Taylor, this formula not being able to be admitted as general any longer unless the series it contains is reduced to a finite number of terms and completed by a definite integral. I am aware that the illustrious author of *Analytical Mechanics* took the formula in question as the basis of his theory of *derived functions*. But, despite all the respect commanded by such a great authority, most geometers now agree in recognizing the uncertainty of the results to which one can be led by the use of divergent series.[4]

Cauchy is right of course, his analysis is the 19th century one (for a complete discussion, see [8]): however, our aim is to understand Lagrange viewpoint and stress that, according to his intensional approach, *analytic* means that, as far as y is an indeterminate and to x is not assigned a specific value, equation (1) holds true since $f(x+y)$ is supposed to differ from $f(x)$ by a quantity which is of the form $y \cdot g(x,y)$, where $g(x,y)$ can in turn be separated into the sum of a part depending only on x and a rest, and so on. Thus, the series development is an algebraic recurrent device, not a topological property of the function around a point in the complex plane.

2.2. Lagrange algebraic foundation of differential calculus

For any function Lagrange supposes that $f(x+y)$ can be written as:

$$f(x+y) = f(x) + y \cdot g_1(x,y),$$

where $g_1(x,0)$ is well defined (thus, it is an expression which can be computed to provide a number). Therefore:

$$g_1(x,y) = \frac{f(x+y) - f(x)}{y},$$

is the increment of f at x by y, and for Lagrange $g_1(x,0)$ is the *first derivative* of f at x. For example, let $f(x) = \sqrt{x}$: then from:

$$\sqrt{x+y} = \sqrt{x} + y \cdot g_1(x,y),$$

it follows that:

$$g_1(x,y) = \frac{\sqrt{x+y} - \sqrt{x}}{y} = \frac{x+y-x}{y(\sqrt{x+y}+\sqrt{x})} = \frac{1}{\sqrt{x+y}+\sqrt{x}},$$

so that $g_1(x,0) = 1/(2\sqrt{x})$.

On iterating the same reasoning on $g(x,y)$, writing it as:

$$g_1(x,y) = g_1(x,0) + y \cdot g_2(x,y),$$

one gets the second derivative $(1/2)g_2(x,0)$ and so on. In this way, we can also compute the exact rest of the approximation: for example, if $f(x) = \sqrt{x}$, then on writing:

$$g_1(x,y) = g_1(x,0) + y \cdot g_2(x,y) = \frac{1}{2\sqrt{x}} + y \cdot g_2(x,y),$$

we get:

$$\begin{aligned} g_2(x,y) &= \frac{1}{y}\left(\frac{1}{\sqrt{x+y}+\sqrt{x}} - \frac{1}{2\sqrt{x}}\right) = \frac{\sqrt{x}-\sqrt{x+y}}{2y\sqrt{x}(\sqrt{x+y}+\sqrt{x})} \\ &= -\frac{1}{2\sqrt{x}(\sqrt{x+y}+\sqrt{x})^2}. \end{aligned}$$

Actually Lagrange does not use the full series development of equation (1) but performs this recursive construction until needed, thus up to a power y^k, writing the rest in an exact form. Indeed, while the terms in the series (2) are, up to a $k!$ factor, the derivatives of the function which we compute by limiting processes defined pointwise, Lagrange computes them in terms of finite algebraic operations. In some sense, the idea of Lagrange calculus is a global one (as expressed in intensional form) as opposed to the local one which today we successfully use [7].

Building on these foundations, which are quite unstable from the modern point of view, Lagrange proves several theorems of differential calculus in a purely algebraic way [9]. For example, the identification of the terms $g_1(x,0)$, $(1/2)g_2(x,0)$, $(1/3)g_3(x,0)$... with $f'(x), f''(x), f'''(x)$... is done by a purely algebraic reasoning which involves essentially the principle of identity of two power series (they are equal if their coefficients are: this is obvious by the extensional definition of power series as sequences).

By the same method, Lagrange proves that mixed partial derivatives of a function of several variables are equal and other classic results which are today deduced via limit process (and which may fail if suitable hypotheses are not satisfied): see previous works [5,7]. Most of the geometric applications of calculus to Analytic Geometry (tangent computations, maxima and minima, constrained maxima and minima, etc.) known at the time are deduced by Lagrange by these algebraic means: he was essentially doing algebraic geometry on the real field, in the sense of Walker [22, p. iii].

2.3. Lagrange attempt to an algebraic foundation of variational calculus

It is well known [23, p. 110] that the term *calculus of variations* was coined by Euler after that Lagrange in 1755, at the age of 19 years, communicated to the great Swiss mathematician his own algebraic approach to the problem of maxima and minima of integrals [16]: Euler had developed geometric and numerical methods while Lagrange found a symbolic device which is essentially still used in the books of Analytical Mechanics. Euler liked the idea so much that he (already a renowned scientist) wrote to the unknown Turin teenager that [24, p. 144],

> After reading your last letter, which saw you to erect the theory of maximum and minimum to the highest level of perfection, I can't stop admiring the distinguished sagacity of your ingenuity. [...] I immediately looked through your analysis, which enabled me to formulate solutions to such problems through just analysis to a much wider range than my method based on geometrical ideas.[5]

The young Lagrange used formal expressions in his calculus of variations, not concerning about convergence and always considering integrals as indefinite (i.e., antiderivatives) rather than definite integrals as Cauchy will do later: thus, for Lagrange, the result of the process of integration is always a function, never a number.

It is worth to notice that Lagrange infers by his methods that each function of a single variable is integrable; thus it has an antiderivative, while he claims that this is not true in the case of several variables [19, p. 401]. The integrability condition he found for the existence of the primitive in the general case appears also to be the necessary condition for the stationarity of an integral in the Calculus of Variations: for this reason, Lagrange, in his later works, changes his approach to Calculus of Variations, which Euler liked so much, and uses his formal methods in the *Théorie* [18, p. 273ff], and in the *Leçons* [19, p. 441ff], with no reference to infinitesimal reasonings, which were used in his letters to Euler and early papers, see Goldstine [23]: for an account of the classical exposition of Lagrange's Calculus of Variations [25]; for an account of the exposition that Lagrange gave of the Calculus of Variations using his analytical function calculus [26]; for an analysis of these two different foundations that Lagrange gave during the years.

Suppose $f(x, y, y', y'', \ldots)$ be a function where y is supposed to be a function of x; one cannot know the primitive function of f unless y is known in terms of x. The idea of Lagrange is, as usual, to develop $f(x, y, y', y'', \ldots)$ in series after adding an increment to its arguments as:

$$f(x, y+\omega, y'+\omega', y''+\omega'', \ldots),$$

being ω a function of x (which is nothing else than the first variation δy that Lagrange does not want to use in this context to avoid any reference to infinitesimal reasonings). By algebraic manipulation of this development, Lagrange shows that $f(x, y, y', y'', \ldots)$ has a primitive if and only if the following condition holds true, independently on any relation between x and y:

$$f'(y) - [f'(y')]' + [f'(y'')]'' - \cdots = 0.$$

In modern notation, this equation reads:

$$\frac{df}{dy} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} - \cdots = 0, \tag{3}$$

i.e., the Lagrange equation.

The interesting fact is that this integrability condition is also the necessary condition on a curve $y(x)$ to be the maximum or minimum of the antiderivative of $f(x, y, y', y'', \ldots)$ in a given interval $a \leqslant x \leqslant b$. Namely, one looks for the value of $y(x)$ such that $y(x) = 0$ for $x = a$ and $y(x)$ is the maximum or minimum of the primitive of $f(x, y, y', y'', \ldots)$ for $x = b$.

To illustrate that, Lagrange puts forth the following example [18, p. 281]:

$$f(x, y, y') = y'^2 + 2myy' + ny^2.$$

Then, equation (3) becomes:

$$2my' + 2ny - 2y'' - 2my' = 0,$$

thus $ny = y''$, a differential equation which Lagrange solves easily by his algebraic methods as $y = g e^{x\sqrt{n}} + h e^{-x\sqrt{n}}$ if $n \geqslant 0$ or as $y = g \sin(kx + h)$ if $n = -k^2$. This is the primitive of f up to arbitrary constants g, h such that, viewed as a curve, on its points, the original expression has maximum or minimum value: Lagrange shows also how to determine such constants once values of x are bound to a certain interval.

However, the proof that Lagrange provides for the necessary condition of stationarity of the integral of $f(x, y, y', \ldots)$ is incorrect: indeed he does assume that a single primitive exists for the first variation of this integral, which is in general false and invalidates his algebraic reasoning (for a historical and critical account of Lagrange error when applying his formal computations to Calculus of Variations, see previous works [8,9,26]).

While algebraic computations performed by Lagrange are not quite satisfactory today, since they leave aside existence, convergence, and uniqueness questions (in particular, the reasonings of Lagrange are by no means correct to determine primitive and maxima and minima according to modern standards), nevertheless the unification of integrability conditions and stationarity necessary conditions for the Calculus of Variations is noteworthy.

3. Formal Lagrangian mechanics

Hereinafter, we will try to recover some analytical mechanical concepts using Lagrange viewpoint: of course we are aware, as Lagrange did not, that the formal approach can only address a restricted class of differentiable functions and that our expressions may well not converge anywhere. However, we think it is worth to experiment with this Lagrangian approach to the calculus behind Lagrangian Mechanics, and the ease with which we will recover some results is, we believe, interesting to say the least.

It is worth to remind that some attempts at an algebraic description of Analytical Mechanics have been done time and again (e.g., previous works [27–31]) but, we believe, with different spirit, methods, and purposes w.r.t. those we are presenting here.

Let us fix our notations for differential calculus in the formal power series ring over a field K with zero characteristic (we will not use any result from the theory of formal power series other than well-known facts which are collected in standard references such as [32, iv.4]). We stick to the $n = 1$ degree of freedom case in the following, therefore working in the algebra $K[[x, \dot{x}]]$: in future works, we will extend our discussion to the general case.

If $F = \sum_{r,s \geqslant 0} \varphi_{r,s} x^r \dot{x}^s$, its partial derivative w.r.t. variable x is the derivation:

$$\frac{\partial F}{\partial x} = \sum_{r,s \geqslant 0} r \cdot \varphi_{r,s} x^{r-1} \dot{x}^s = \sum_{r,s \geqslant 0} (r+1) \varphi_{r+1,s} x^r \dot{x}^s,$$

and similarly:

$$\frac{\partial F}{\partial \dot{x}} = \sum_{r,s \geqslant 0} s \cdot \varphi_{r,s} x^r \dot{x}^{s-1} = \sum_{r,s \geqslant 0} (s+1) \varphi_{r,s+1} x^r \dot{x}^s.$$

The maps $\partial/\partial x, \partial/\partial \dot{x} : K[[x, \dot{x}]] \to K[[x, \dot{x}]]$ are K-linear and surjective, being $K[[\dot{x}]]$ and $K[[x]]$ their kernels; thus, we have the exact sequences:

$$0 \to K[[\dot{x}]] \to K[[x, \dot{x}]] \xrightarrow{\partial/\partial x} K[[x, \dot{x}]]/K[[\dot{x}]] \to 0,$$

$$0 \to K[[x]] \to K[[x, \dot{x}]] \xrightarrow{\partial/\partial \dot{x}} K[[x, \dot{x}]]/K[[x]] \to 0,$$

which induce isomorphisms:

$$\frac{K[[x, \dot{x}]]}{K[[\dot{x}]]} \to \frac{K[[x, \dot{x}]]}{K[[\dot{x}]]} \quad \text{and} \quad \frac{K[[x, \dot{x}]]}{K[[x]]} \to \frac{K[[x, \dot{x}]]}{K[[x]]},$$

whose inverses we denote as:

$$\int F \, dx \in \frac{K[[x, \dot{x}]]}{K[[\dot{x}]]}, \qquad \int F \, d\dot{x} \in \frac{K[[x, \dot{x}]]}{K[[x]]}.$$

Of course, we have (modulo $K[[\dot{x}]]$ and $K[[x]]$, respectively):

$$\int F\,dx = \sum_{r,s\geqslant 0} \frac{1}{r+1}\,\varphi_{r,s}x^{r+1}x^{s}, \qquad \int F\,d\dot{x} = \sum_{r,s\geqslant 0} \frac{1}{s+1}\,\varphi_{r,s}x^{r}x^{s+1},$$

so that we have:

$$\frac{\partial}{\partial x}\int F\,dx = F, \qquad \frac{\partial}{\partial \dot{x}}\int F\,d\dot{x} = F.$$

3.1. Helmholtz conditions and Lagrange equations

We take seriously the idea of Lagrange that the fundamental equations of Calculus of Variations can be understood as integrability conditions: therefore, we try to formulate in our setting this idea, which was pursued by outstanding scientists after Lagrange, using his analytical methods in our modern notation.

Let us consider the ordinary differential equation:

$$F(x,\dot{x})\ddot{x} + G(x,\dot{x}) = 0, \tag{4}$$

where $\dot{x}$ represents the derivative with respect to the time t, and suppose F and G to be defined as formal power series in the indeterminates x, $\dot{x}$:

$$F = \sum_{r,s\geqslant 0} \varphi_{r,s}x^{r}\dot{x}^{s}, \qquad G = \sum_{r,s\geqslant 0} \gamma_{r,s}x^{r}\dot{x}^{s},$$

with coefficients $\varphi_{r,s} = \varphi_{r,s}(t)$ and $\gamma_{r,s} = \gamma_{r,s}(t)$ functions of the variable t (possibly formal power series in turn). From now on, we will leave the dependence of the coefficients on t out.

Let us not think to F and G as extensional functions but as formal power series, and to $x, \dot{x}, \ddot{x}$ as generators of our algebra subject to the relations:

$$\frac{dx}{dt} = \dot{x}, \qquad \frac{d\dot{x}}{dt} = \ddot{x},$$

which extend the derivation on power series coefficients to indeterminates.

Next, consider a power series:

$$L = \sum_{r,s\geqslant 0} \lambda_{r,s}x^{r}\dot{x}^{s}.$$

We will compute its *Lagrangian derivative*:

$$\frac{\delta L}{\delta x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x}.$$

Let us do that explicitly:

$$\begin{aligned}
\frac{\delta L}{\delta x} &= \frac{d}{dt}\sum_{r,s}(s+1)\lambda_{r,s+1}x^{r}\dot{x}^{s} - \sum_{r,s}(r+1)\lambda_{r+1,s}x^{r}\dot{x}^{s} \\
&= \sum_{r,s}(s+1)\frac{d}{dt}\lambda_{r,s+1}x^{r}\dot{x}^{s} + \sum_{r,s} r(s+1)\lambda_{r,s+1}x^{r-1}\dot{x}^{s+1} \\
&\qquad + \sum_{r,s} s(s+1)\lambda_{r,s+1}x^{r}\dot{x}^{s-1}\ddot{x} - \sum_{r,s}(r+1)\lambda_{r+1,s}x^{r}\dot{x}^{s}.
\end{aligned}$$

Now let us observe that:

$$\sum_{r,s\geqslant 0} r(s+1)\lambda_{r,s+1}x^{r-1}\dot{x}^{s+1} = \sum_{r\geqslant 0}\sum_{s\geqslant 0}(r+1)\,s\lambda_{r+1,s}x^r\dot{x}^s.$$

Hence, we obtain:

$$\begin{aligned}\frac{\delta L}{\delta x} &= \sum_{r,s}(s+1)\frac{d}{dt}\lambda_{r,s+1}x^r\dot{x}^s + \sum_{r,s}(r+1)s\lambda_{r+1,s}x^r\dot{x}^s\\ &\quad+\sum_{r,s}(s+1)(s+2)\lambda_{r,s+2}x^r\dot{x}^s\ddot{x} - \sum_{r,s}(r+1)\lambda_{r+1,s}x^r\dot{x}^s\\ &= \left[\sum_{r,s}(s+1)(s+2)\lambda_{r,s+2}x^r\dot{x}^s\right]\ddot{x}\\ &\quad+\sum_{r,s}\left[(s+1)\frac{d}{dt}\lambda_{r,s+1} + (r+1)(s-1)\lambda_{r+1,s}\right]x^r\dot{x}^s.\end{aligned}$$

Notice that we have deduced an expression of the form $F\ddot{x}+G$: on supposing the coefficients of F and G to be given, let us determine, by means of the identity principle for formal power series, the $\lambda_{r,s}$ which fulfill the identity $(\delta L/\delta x) = F\ddot{x}+G$.

First, we get:

$$\lambda_{r,s+2} = \frac{1}{(s+1)(s+2)}\varphi_{r,s}. \tag{5}$$

In this way, all $\lambda_{r,s}$ for each $r\geqslant 0$ and each $s\geqslant 2$ are uniquely determined.

Next, we have:

$$(s+1)\frac{d}{dt}\lambda_{r,s+1} + (r+1)(s-1)\lambda_{r+1,s} = \gamma_{r,s}. \tag{6}$$

We use this condition to recover $\lambda_{r,1}$ and to find out a necessary relation between F and G.

In the first place, on putting $s=0$ in the previous identity, we get:

$$\frac{d}{dt}\lambda_{r,1} - (r+1)\lambda_{r+1,0} = \gamma_{r,0},$$

so that $\lambda_{r,1}$ are determined up to $\lambda_{r,0}$, which we may choose arbitrarily.

For $s=1$, we get a compatibility condition between F and G, since:

$$2\frac{d}{dt}\lambda_{r,2} = 2\frac{d}{dt}\frac{1}{2}\varphi_{r,0} = \gamma_{r,1}\ ,$$

thus:

$$\frac{d}{dt}\varphi_{r,0} = \gamma_{r,1}\ .$$

Moreover:

$$\begin{aligned}&(s+3)\frac{d}{dt}\lambda_{r,s+3} + (r+1)(s+1)\lambda_{r+1,s+2},\\ &= (s+3)\frac{d}{dt}\frac{1}{(s+2)(s+3)}\varphi_{r,s+1} + (r+1)(s+1)\frac{1}{(s+1)(s+2)}\varphi_{r+1,s},\\ &= \frac{1}{s+2}\frac{d}{dt}\varphi_{r,s+1} + \frac{r+1}{s+2}\varphi_{r+1,s}.\end{aligned}$$

Therefore, from equation (6), the following integrability conditions, which correspond to the classical *Helmholtz conditions* [1], have to hold for the differential equation (4):

$$\frac{d}{dt}\varphi_{r,s+1} + (r+1)\varphi_{r+1,s} = (s+2)\gamma_{r,s+2}.$$

We resume our findings in the following.

Theorem 1. Given $F = \sum_{r,s} \varphi_{r,s} x^r \dot{x}^s$ and $G = \sum_{r,s} \gamma_{r,s} x^r \dot{x}^s$, then, if the Helmholtz conditions:

$$\begin{cases} \dot{\varphi}_{r,0} = \gamma_{r,1}, \\ \dot{\varphi}_{r,s+1} + (r+1)\varphi_{r+1,s} = (s+2)\gamma_{r,s+2}, \end{cases}$$

are fulfilled for each $r, s \geqslant 0$, then, on putting:

$$\lambda_{r,1} = \int \left(\gamma_{r,0} + (r+1)\lambda_{r+1,0}\right) dt, \quad \lambda_{r,s+2} = \frac{1}{(s+1)(s+2)}\varphi_{r,s},$$

we determine $L = \sum_{r,s} \lambda_{r,s} x^r \dot{x}^s$ up to $\lambda_{r,0} = (1/r!)(\partial^r L/\partial x^r)(0,0)$, so that:

$$\frac{\delta L}{\delta x} = F\ddot{x} + G\ .$$

Remark 1. Helmholtz conditions in Theorem 1 may be written in terms of F and G as follows:

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\dot{x} = \frac{\partial G}{\partial \dot{x}}\ . \tag{7}$$

Indeed,

$$\begin{aligned} \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\dot{x} &= \sum_{r,s} \dot{\varphi}_{r,s} x^r \dot{x}^s + \sum_{r,s} (r+1)\varphi_{r+1,s} x^r \dot{x}^{s+1}, \\ &= \sum_{r} \dot{\varphi}_{r,0} x^r + \sum_{r \geqslant 0}\sum_{s \geqslant 1} \left[\dot{\varphi}_{r,s} + (r+1)\varphi_{r+1,s-1}\right] x^r \dot{x}^s, \end{aligned}$$

and

$$\frac{\partial G}{\partial \dot{x}} = \sum_{r,s} (s+1)\gamma_{r,s+1} x^r \dot{x}^s = \sum_{r} \gamma_{r,1} x^r + \sum_{r \geqslant 0}\sum_{s \geqslant 1} (s+1)\gamma_{r,s+1} x^r \dot{x}^s,$$

whose equality let us recover, term-wise, Helmholtz conditions as we stated in Theorem 1.

Notice that Helmholtz conditions require:

$$\frac{\partial G}{\partial \dot{x}}(x,0) = \frac{\partial F}{\partial t}(x,0).$$

This is surely fulfilled if G does not depend on $\dot{x}$.

In particular, if we require the coefficients $\lambda_{r,s}$ to be constant with respect to t, the conditions (5) and (6) on L simplify to:

$$\lambda_{r,s+2} = \frac{1}{(s+1)(s+2)}\varphi_{r,s}, \quad \gamma_{r,1} = 0, \quad \lambda_{r+1,s} = \frac{1}{(r+1)(s-1)}\gamma_{r,s}\ (s \neq \lambda),$$

while

Corollary 1. Given $F = \sum_{r,s} \varphi_{r,s} x^r \dot{x}^s$ and $G = \sum_{r,s} \gamma_{r,s} x^r \dot{x}^s$, with $\varphi_{r,s}$ and $\gamma_{r,s}$ constant with respect t, then, if the Helmholtz conditions:

$$\begin{cases} \gamma_{r,1} = 0 \\ (r+1)\varphi_{r+1,s} = (s+2)\gamma_{r,s+2} \end{cases},$$

are fulfilled for each $r, s \geqslant 0$, then on putting:

$$\lambda_{r+1,s} = \frac{1}{(r+1)(s-1)}\gamma_{r,s}, \quad \lambda_{r,s+2} = \frac{1}{(s+1)(s+2)}\varphi_{r,s},$$

we determine $L = \sum_{r,s} \lambda_{r,s} x^r \dot{x}^s$ up to $\lambda_{0,0} = L(0,0)$ and $\lambda_{0,1} = (\partial L/\partial \dot{x})(0,0)$, so that:

$$\frac{\delta L}{\delta x} = F\ddot{x} + G .$$

3.2. Simple examples

Using theorems deduced in the previous section, let us determine in some examples whether a Lagrange function for a Newtonian dynamical system $F\ddot{x} + G = 0$ does exist, and in this case to build it. Notice how easy is to prove or disprove Helmholtz conditions and to build a Lagrangian when it is possible by this purely algebraic method.

Example 1. Consider the simplest Newtonian system: $m\ddot{x} = 0$. This is just a uniform motion on a line. We have $F = m$ and $G = 0$, so that Corollary 1 applies and provides $\lambda_{r,s} = 0$ but for $\lambda_{0,2} = \varphi_{0,0}/2 = m/2$, hence:

$$L = \frac{m}{2}\dot{x}^2 .$$

Example 2. Now consider the equation $m\ddot{x} + a + bx = 0$, thus F is constant and G is linear in x. This is a harmonic motion on a line. Again it is easy to check that Helmholtz conditions do hold, since they reduce to $0 = 0$! The Lagrangian is built as:

$$\lambda_{r+1,s} = \begin{cases} -a & \text{if } r = s = 0 \\ \dfrac{-b}{2} & \text{if } r = 1,\ s = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Moreover, $\lambda_{0,2} = \varphi_{0,0}/2 = m/2$. Therefore:

$$L = -ax - \frac{b}{2}x^2 + \frac{m}{2}\dot{x}^2 .$$

Example 3. Let us consider the so-called damped harmonic oscillator, whose equation is $m\ddot{x} + ax + b\dot{x} = 0$ [1]. Here still $F = m$ but $G = ax + b\dot{x}$, so that the only non-zero coefficients are $\varphi_{0,0} = m$, $\gamma_{1,0} = a$ and $\gamma_{0,1} = b$. This last identity violates Helmholtz conditions! Therefore, there is no Lagrangian L for this dynamical system.

The Lagrangian in the latter case would be:

$$\begin{cases} \lambda_{2,0} = -\dfrac{a}{2} \\ \lambda_{0,2} = \dfrac{m}{2} \end{cases},$$

with $\lambda_{0,0}$ and $\lambda_{0,1}$ arbitrary, thus:

$$L = \lambda_{0,0} + \lambda_{0,1}\dot{x} - \frac{a}{2}x^2 + \frac{m}{2}\dot{x}^2 .$$

But, of course,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m\ddot{x} + ax + \lambda_{0,1} \neq m\ddot{x} + ax + b\dot{x} .$$

Example 4. Another variant of the harmonic motion equation is:

$$\ddot{x} - x^3 + x = 0 .$$

In this case $\varphi_{0,0} = 1$, $\gamma_{1,0} = 1$, $\gamma_{3,0} = -1$, therefore:

$$L = \frac{1}{2}\dot{x}^2 + \frac{1}{4}x^4 - \frac{1}{2}x^2 .$$

Example 5. More generally, we can also recover the Lagrangian for the pendulum equation:

$$\ddot{x} + a\sin x = 0.$$

Since $F = 1$ and $G = a\sin x$ in this case, we have $\varphi_{rs} = 0$ but for $\varphi_{00} = 1$. Moreover:

$$G = \sum_{r \geqslant 0} a\frac{(-1)^r}{(2r+1)!}x^{2r+1},$$

thus, for $r \geqslant 0$, $\gamma_{2r,0} = 0$ and $\gamma_{2r+1,0} = a(-1)^r/(2r+1)!$; moreover $\gamma_{r,s} = 0$ if $s > 0$. Combining these relations with our definition for λ in terms of φ, we get:

$$\begin{cases} \lambda_{2r+1,0} = 0 \\ \lambda_{2r+2,0} = \dfrac{1}{2r+2}a\dfrac{(-1)^r}{-(2r+1)!} = -a\dfrac{(-1)^r}{(2r+2)!} \\ \lambda_{r+1,s} = 0 \qquad \text{if } s > 0 . \end{cases}$$

Furthermore, since the only non-zero φ_{rs} is $\varphi_{00} = 1$, we have:

$$\begin{cases} \lambda_{0,2} = \dfrac{1}{2} \\ \lambda_{r,s+2} = 0 \quad \text{if } r > 0 \text{ or } s > 0, \end{cases}$$

so that, on choosing $\lambda_{00} = a$, we get:

$$L = \lambda_{00} - \sum_{r \geqslant 0} a\frac{(-1)^r}{(2r+2)!}x^{2r+2} + \frac{1}{2}\dot{x}^2 = \frac{1}{2}\dot{x}^2 - a\cos x .$$

Example 6. Another classical example is the equation of motion of a point along a parabola rotating around its vertical axis (cf. [33, p. II.5.2]):

$$\left(1 + \lambda x^2\right)\ddot{x} + \left(a + \lambda\dot{x}^2\right)x = 0.$$

In this case: $\varphi_{0,0}=1$, $\varphi_{2,0}=\lambda$, $\gamma_{1,0}=a$, $\gamma_{1,2}=\lambda$, so that Helmholtz conditions reduce to $0=0$ and $2\lambda=2\lambda$, and the Lagrangian is:

$$L=-\frac{a}{2}x^2+\frac{1}{2}\dot{x}^2+\frac{\lambda}{2}x^2\dot{x}^2 .$$

3.3. Dissipative systems à la Rayleigh

In some cases, dissipative systems may be described via Lagrangian formalism as well: this kind of systems stems from the classical Rayleigh dissipative systems and have been studied still in recent times (for a review, see Bersani and Caressa [1]).

The idea of Rayleigh [11] was to generalize Lagrangian potentials, which depend only upon x, to potentials which also take $\dot{x}$ into account, as follows: let us suppose that on the Lagrangian system with potential energy $U(x)$ acts also an external generalized potential $R(x,\dot{x})=\sum_{r,s}\rho_{r,s}x^r\dot{x}^s$, so that motion equations are:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}-\frac{\partial L}{\partial x}-\frac{\partial R}{\partial \dot{x}}=0 .$$

If we try to ask whether:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}-\frac{\partial L}{\partial x}-\frac{\partial R}{\partial \dot{x}}=F\ddot{x}+G,$$

then, by computations which slightly extend the ones we did before in the case $R=0$, we get the following conditions:

$$\begin{cases}\lambda_{r,s+2}=\dfrac{1}{(s+1)(s+2)}\varphi_{r,s}\\ (s+1)\dot{\lambda}_{r,s+1}+(r+1)(s-1)\lambda_{r+1,s}-(s+1)\rho_{r,s+1}=\gamma_{r,s}\end{cases},$$

which, if we look for Lagrangians not explicitly depending on t, reduce to:

$$\begin{cases}\lambda_{r,s+2}=\frac{1}{(s+1)(s+2)}\varphi_{r,s}\\ (r+1)(s-1)\lambda_{r+1,s}-(s+1)\rho_{r,s+1}=\gamma_{r,s}\end{cases}.$$

Next, as we did before, we may compute special cases of the differential condition (for $s=0,1,2$) to get:

$$\begin{aligned}&\dot{\lambda}_{r,1}-(r+1)\lambda_{r+1,0}-\rho_{r,1}=\gamma_{r,0},\\ &\rho_{r,2}=\frac{1}{2}\left(\dot{\varphi}_{r,0}-\gamma_{r,1}\right),\\ &\rho_{r,s+3}=\frac{1}{s+3}\left(\frac{\dot{\varphi}_{r,s+1}}{s+2}+\frac{r+1}{s+2}\varphi_{r+1,s}-\gamma_{r,s+2}\right).\end{aligned}$$

Theorem 1 and its corollary may be formulated as follows.

Theorem 2. Given $F=\sum_{r,s}\varphi_{r,s}x^r\dot{x}^s$ and $G=\sum_{r,s}\gamma_{r,s}x^r\dot{x}^s$, then on putting:

$$\rho_{r,2}=\frac{1}{2}\left(\dot{\varphi}_{r,0}-\gamma_{r,1}\right),\quad \rho_{r,s+3}=\frac{1}{s+3}\left(\frac{\dot{\varphi}_{r,s+1}}{s+2}+\frac{r+1}{s+2}\varphi_{r+1,s}-\gamma_{r,s+2}\right),$$

we determine $R = \sum_{r,s} \rho_{r,s} x^r \dot{x}^s$ up to $\rho_{r,0} = (1/r!)(\partial^r R/\partial x^r)(0,0)$, and on putting:

$$\lambda_{r,1} = \int \big(\gamma_{r,0} + \rho_{r,1} + (r+1)\lambda_{r+1,0}\big)\, dt, \quad \lambda_{r,s+2} = \frac{1}{(s+1)(s+2)} \varphi_{r,s},$$

we determine $L = \sum_{r,s} \lambda_{r,s} x^r \dot{x}^s$ up to $\lambda_{r,0} = (1/r!)(\partial^r L/\partial x^r)(0,0)$, so that:

$$\frac{\delta L}{\delta x} - \frac{\partial R}{\partial \dot{x}} = F\ddot{x} + G.$$

In particular, if we require the coefficients of L and R to be constant with respect to t, the conditions simplify to the following corollary.

Corollary 2. Given $F = \sum_{r,s} \varphi_{r,s} x^r \dot{x}^s$ and $G = \sum_{r,s} \gamma_{r,s} x^r \dot{x}^s$, then on putting:

$$\rho_{r,2} = -\frac{1}{2}\gamma_{r,1}, \quad \rho_{r,s+3} = \frac{1}{s+3}\left(\frac{r+1}{s+2}\varphi_{r+1,s} - \gamma_{r,s+2}\right),$$

we determine $R = \sum_{r,s} \rho_{r,s} x^r \dot{x}^s$ up to $\rho_{r,0} = (1/r!)(\partial^r R/\partial x^r)(0,0)$, and on putting:

$$\lambda_{r+1,s} = \frac{\gamma_{r,s} + (s+1)\rho_{r,s+1}}{(r+1)(s-1)}, \quad \lambda_{r,s+2} = \frac{\varphi_{r,s}}{(s+1)(s+2)},$$

we determine $L = \sum_{r,s} \lambda_{r,s} x^r \dot{x}^s$ up to $\lambda_{r,0} = (1/r!)(\partial^r L/\partial x^r)(0,0)$, so that:

$$\frac{\delta L}{\delta x} - \frac{\partial R}{\partial \dot{x}} = F\ddot{x} + G.$$

Conditions expressed in the previous theorem and corollary almost automatically imply the form of the L and R series, if they exist.

Example 7. Let us again consider the damped pendulum equation:

$$m\ddot{x} + ax + b\dot{x} = 0.$$

We have $F = \varphi_{0,0} = m$, $G = \gamma_{1,0}x + \gamma_{0,1}\dot{x} = ax + b\dot{x}$ as only non-zero terms. From the previous corollary, it follows immediately that $\rho_{0,2} = -\gamma_{0,1}/2 = -b/2$, while $\rho_{r,s} = 0$ for $s \geqslant 3$. Next, we find $\lambda_{r,s} = 0$ for $s \geqslant 2$ but for $\lambda_{0,2} = m/2$, and $\lambda_{2,0} = -(\gamma_{1,0} + \rho_{0,1})/2 = -a/2$. Therefore, we have:

$$L = \frac{m}{2}\dot{x}^2 - \frac{a}{2}x^2, \quad R = -\frac{b}{2}\dot{x}^2.$$

Example 8. A signum variation in Example 6 leads to a non-conservative equation:

$$\big(1 + \lambda x^2\big)\ddot{x} + \big(a - \lambda \dot{x}^2\big)x = 0.$$

Indeed, $\varphi_{0,0} = 1$, $\varphi_{2,0} = \lambda$, $\gamma_{1,0} = a$, $\gamma_{1,2} = -\lambda$, so that Helmholtz conditions are $0 = 0$ and $2\lambda = -2\lambda$, so that they holds if and only if $\lambda = 0$. However, the recipe in the previous corollary provides us with L and R as follows:

$$L = -\frac{a}{2}x^2 + \frac{1}{2}\dot{x}^2 + \frac{\lambda}{2}x^2\dot{x}^2, \quad R = \frac{2}{3}\lambda x \dot{x}^3.$$

It is interesting to notice that the equation in the previous example stems from field-theoretical works in chiral Lagrangian theories of pion interactions [34] and are studied in the classical setting in Mathews and Lakshmanan [35]: therein, a Lagrangian is exhibited for the equation, and built by means of an integrating factor, a well-known procedure related to the inverse problem of variational calculus [1]. More precisely, a Lagrangian for the equation:

$$\ddot{x} + x\frac{a - \lambda \dot{x}^2}{(1 + \lambda x^2)} = 0,$$

is

$$L = \frac{1}{2}\left(\frac{\dot{x}^2 - ax^2}{1 + \lambda x^2}\right).$$

This Lagrangian is not a classical $L = T - U$ Lagrangian, but nevertheless it provides the motion equation by the classical procedure.

Example 9. Van der Pol equation:

$$\ddot{x} + \mu\left(x^2 - 1\right)\dot{x} + x = 0,$$

is another important dissipative generalization of the harmonic motion: indeed, Helmholtz conditions are violated, as a simple check by means of Corollary 1 shows, but we may write down Lagrange and Rayleigh functions by noticing that $\varphi_{0,0} = 1$, $\gamma_{1,0} = 1$, $\gamma_{0,1} = -\mu$, and $\gamma_{2,1} = \mu$, getting:

$$L = \frac{\dot{x}^2}{2} - \frac{x^2}{2}, \quad R = \frac{\mu}{2}\dot{x}^2\left(1 - x^2\right).$$

It is interesting to notice that Corollary 2 does not impose any condition on $\rho_{0,1}$ but for the relation $1 + \rho_{0,1} = -2\lambda_{2,0}$: we find it a posteriori to match the original equation.

Example 10. More generally, let us consider Liénard's equation:

$$\ddot{x} + f(x)\dot{x} + g(x) = 0.$$

In this case, with the previous notations, $f = \sum_r \gamma_{r,1}x^r$ and $g = \sum_r \gamma_{r,0}x^r$. Therefore, we find:

$$\begin{cases} \rho_{r,2} = -\dfrac{1}{2}\gamma_{r,1} \\ \rho_{r,s+3} = -\dfrac{1}{s+3}\gamma_{r,s+2} \\ \lambda_{0,2} = \dfrac{1}{2} \\ \lambda_{r+1,0} = -\dfrac{\gamma_{r,0} + \rho_{r,1}}{r+1} = -\dfrac{\gamma_{r,0}}{r+1} \\ \lambda_{r+1,1} = -\dfrac{\gamma_{r,1} + 2\rho_{r,2}}{r+1} = -\dfrac{\gamma_{r,1} - \gamma_{r,1}}{r+1} = 0 \end{cases}.$$

These relations may be expressed as:

$$R = -\frac{\dot{x}^2}{2}f(x), \quad L = \frac{\dot{x}^2}{2} - \int g(x)dx.$$

Acknowledgements

The first author is member of the Gruppo Nazionale per la Fisica Matematica (GNFM) of the Istituto Nazionale di Alta Matematica (INdAM, Italy).

Funding

The author(s) received no financial support for the research, authorship, and/or publication of this article.

ORCID iD

Alberto Maria Bersani https://orcid.org/0000-0001-8525-7363

Notes

1. We will quote from the most recent editions of Lagrange's books published when he was still alive.
2. *Le calcul des fonctions a le même objet que le calcul différentiel pris dans le sens plus étendu, mais il n'est point sujet aux difficultés qui se rencontrent dans les principes et dans la marche ordinaire de ce calcul: il sert de plus à lier le calcul différentiel immédiatement à l'algèbre, dont on peut dire qu'il fait jusq'à présent une science séparée.*
3. *l'analyse ne doir avoir d'autre métaphysique que celle qui consiste dans les premier principes et dans ler premières opérations fondamental du calcul.*
4. *J'ai cru devoir rejeter les développemens des fonctions en série infinies, toutes les fois que les séries obtenues ne sont pas convergentes; et je me suis vu forcé de renvoyer au calcul intégral la formule de* TAYLOR, *cette formule ne pouvant plus être admise comme générale qu'autant que la série qu'elle renferme se trouve réduite à un nombre fini de termes, et complétée par un intégrale définie. Je n'ignore pas que l'illustre auteur de la Mécanique analytique a pris la formule dont il s'agit pour base de sa théorie des fonctions dérivées. Mais, malgré tout le respecte que commande une si grande autorité, la plupart des géomètres s'accordent maintenant à reconna tre l'incertitude des résultats auxquels ou peut être conduit par l'emploi des séries divergentes.*
5. *Perlectis tui postremi litteris, quibus Theoriam maximorum ac minimorum ad summum fere perfectionis fastigium erexisse videris, eximiam ingenii tui sagacitatem satis admirari non possum. [...] Statim autem perspexi analysin tuam, qua meas hujusmodi problematum solutiones per sola analyseo præ cepta elicuisse multo latius patere mea methodo ideis geometricis innixa.*

References

[1] Bersani, AM, and Caressa, P. Lagrangian descriptions of dissipative systems: a review. *Math Mech Solids* 2020; 26: 785–803.
[2] Abraham, R, Marsden, JE, Ratiu, T, et al. *Foundations of mechanics*. 2nd ed. Madison, WI: AMS Chelsea Publishing, 2008.
[3] Demidov, SS. Leonhard Euler, treatise on the differential calculus, (1755). In: Grattan-Guinness I (ed.) *Landmark writings in Western mathematics 1640–1940*. Amsterdam: Elsevier, 2005, pp. 191–198.
[4] Reich, K. Leonhard Euler, introduction to analysis, (1748). In: Grattan-Guinness I (ed.) *Landmark writings in Western mathematics 1640–1940*. Amsterdam: Elsevier, 2005, pp. 181–190.
[5] Bottazzini, U. *Il calcolo sublime: storia dell'analisi matematica da Euler a Weierstrass. Lezioni e seminari*. Turin: Boringhieri, 1981.
[6] Fraser, C. Joseph Louis Lagrange's algebraic vision of the calculus. *Hist Math* 1987; 14: 38–53.
[7] Fraser, C. Joseph Louis Lagrange, Théorie des fonctions analytiques, first edition (1797). 2005, pp. 258–276, http://homes.chass.utoronto.ca/~cfraser/lagrangetheorie.pdf
[8] Grabiner, JV. *The origins of Cauchy's rigorous calculus (Dover books on mathematics)*. Mineola, NY: Dover Publications, 2005.
[9] Grabiner, JV. *Historian looks back the calculus as algebra and selected writings: spectrum*. Washington, DC: Mathematical Association of America, 2014.
[10] Pepe, L. Il giovane Lagrange e i fondamenti dell'analisi. In: *Sfogliando la Méchanique analytique: Giornata di studio su Louis Lagrange*. Milano, 19 Ottobre 2006, pp. 37–49, Milano: Istituto Lombardo. Accademia Di Scienze E Lettere.
[11] Rayleigh, JWS. *The theory of sound*, 2 Vols in one. Mineola, NY: Dover, 1945.
[12] Dell'Isola, F, Andreaus, U, Cazzani, A, et al. On a debated principle of Lagrange analytical mechanics and on its multiple applications. In: *The complete works of Gabrio Piola: volume I, volume 38 of advanced structured materials*. Berlin: Springer, 2014, https://hal.archives-ouvertes.fr/hal-00991089
[13] Dell'Isola, F, Laudato, M, and Scerrato, D. On the applications of principles of analytical mechanics by Lagrange to the principal problems. In: *The complete works of Gabrio Piola: volume II, volume 97 of advanced structured materials*. Berlin: Springer, 2019, https://www.springerprofessional.de/en/on-the-applications-of-principles-of-analytical-mechanics-by-lag/16180256
[14] Capecchi, D. *History of virtual work laws: a history of mechanics prospective—science networks: historical studies no.42*. Cham: Springer, 2012.
[15] Yourgrau, W, and Mandelstam, S. *Variational principles in dynamics and quantum theory*. 3rd ed. London: Pitman, 1968.
[16] Caressa, P. *Piccola storia della matematica*, vol. 2. Milan: Alphatest, 2010.

[17] Fraser, C. Lagrange's analytical mathematics, its cartesian origins and reception in Comte's positive philosophy. *Stud Hist Philos Sci* 1990; 21: 243–256.
[18] Lagrange, JL. *Théorie des fonctions analytiques*. 2nd ed. Morton, IL: Courcier, 1813.
[19] Lagrange, JL. *Leçons sur le calcul des fonctions*. 2nd ed. Morton, IL: Courcier, 1806.
[20] Church, A. *The calculi of lambda-conversion: annals of mathematical studies*. Princeton, NJ: Princeton University Press, 1941.
[21] Cauchy, AL. *Résumé des leçons données à l'École royale polytechnique sur le calcul infinitésimal*, vol. I. Paris: Debure, 1823.
[22] Walker, RJ. *Algebraic curves*. Princeton, NJ: Princeton University Press, 1950.
[23] Goldstine, HH. *A history of the calculus of variations from the 17th through the 19th century (Studies in the history of mathematics and physical sciences no.5)*. Cham: Springer, 1980.
[24] Lagrange, JL. *Oeuvres de Lagrange, publiées par les soins de M. J.-A Serret et de Gaston Darboux*, vol. 14. Paris: Gauthier-Villars et Fils, 1892.
[25] Todhunter, I. *A history of the progress of the calculus of variations during the nineteenth century*. New York: Macmillan, 1861.
[26] Fraser, C. JL Lagrange's changing approach to the foundations of the calculus of variations. *Arch Hist Exact Sci* 1985; 32: 151–191.
[27] Gel'fand, IM, and Dikii, LA. Asymptotic behaviour of the resolvent of Sturm–Liouville equations and the algebra of the Korteweg–de Vries equations. *Russian Math Surv* 1975; 30: 77–113.
[28] Gel'fand, IM, and Dorfman, IY. Hamiltonian operators and algebraic structures related to them. *Funct Anal Appl* 1980; 13: 248–262.
[29] Vinogradov, AM, and Krasil'shchik, IS. What is the Hamiltonian formalism? *Russian Math Surv* 1975; 30: 177–202.
[30] Caressa, P. Examples of Poisson modules, I. *Rend Circ Mate Paler* 2003; 52: 419–452.
[31] Caressa, P. Examples of Poisson modules, II. *Rend Circ Mate Paler* 2004; 53: 23–60.
[32] Bourbaki, N. *Algèbre* (chapitres 4 à 7), volume II. Cham: Springer, 2007.
[33] Andronov, AA, Vitt, AA, and Khaikin, SE. *Theory of oscillators, volume 4 of international series of monograph in physics*. Oxford: Pergamon Press, 1966.
[34] Delbourgo, R, Salam, A, and Strathdee, J. Infinities of nonlinear and Lagrangian theories. *Phys Rev* 1969; 187: 1999–2007.
[35] Mathews, PM, and Lakshmanan, M. On a unique nonlinear oscillator. *Quart Appl Math* 1974; 32: 7.