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Deferred correction with mono-implicit Runge–Kutta methods for first-order IVPs

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Abstract

To reach a high order of accuracy for numerical solutions of IVPs with mono-implicit Runge–Kutta (MIRK) methods, the technique of deferred correction is used. Special attention is paid to the possible increase of the order and the stability of such schemes. Several schemes are given. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

For the numerical solution of first-order IVPs

$$y' = f(x, y), \quad y(x_0) = y_0, \quad y \in \mathbb{R}^{\mathrm{d}} \quad \text{and} \quad \mathrm{f}: \mathbb{R} \times \mathbb{R}^{\mathrm{d}} \to \mathbb{R}^{\mathrm{d}}, \tag{1.1}$$

the following representation of s-stage implicit Runge–Kutta methods (IRK), known as *parameterized* IRK methods, was presented by Burrage et al. [1]:

$$y_{n+1} = y_n + h \sum_{i=1}^{s} b_i f(x_n + c_i h, Y_i),$$

$$Y_i = (1 - v_i) y_n + v_i\, y_{n+1} + h \sum_{j=1}^{s} x_{ij} f(x_n + c_j h, Y_j), \quad i = 1, \ldots, s.$$

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Hence, a s-stage parameterized IRK method is completely determined by the tableau

$$\begin{array}{c|c|cccc} c_1 & v_1 & x_{11} & x_{12} & \dots & x_{1s} \\ c_2 & v_2 & x_{21} & x_{22} & \dots & x_{2s} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_s & v_s & x_{s1} & x_{s2} & \dots & x_{ss} \\ \hline & & b_1 & b_2 & \dots & b_s \end{array} \tag{1.2}$$

Comparing this representation with the description of a general IRK method by means of its Butcher tableau (c,A,b) [2], it is easy to verify that the relationship $A = X + v.b^{\mathrm{T}}$ holds. For all methods considered, we will assume that the row-sum condition holds, i.e. $A.e = c$ where e is the s-vector with unit entries. By imposing that X (or X after a rearrangement of its rows and columns according to a same permutation) is a strictly lower triangular matrix one obtains mono-implicit Runge–Kutta (MIRK) methods [1].

Several results concerning MIRK methods have been established. Well-known are the following bounds: the order $p \leqslant s+1$ and the stage order is 3 at most. Also, in [1] a complete characterization is given of methods of order $p \leqslant 6$ with $s \leqslant p$ stages. Another family of MIRK methods is given in [7]: here $s = p$ and $c_i = 0, 1, \dots, s-1$.

Also, there is no problem to find stable MIRK methods: when a MIRK method is applied to the test problem $y' = \lambda y$, $y(x_0) = y_0$ with fixed stepsize h, one obtains $y_n = R^n(\lambda h) y_0$ where $R(z) = P(e-v,z)/P(-v,z)$ with $P(w,z) = 1 + \sum_{i=1}^{s} z^i b^{\mathrm{T}}.X^{i-1}.w$. This reveals one of the main problems one is confronted with when using MIRK methods: the Jacobian of the implicit system to be solved (which is of dimension d), is in practice approximated by the following non-linear expression in $J = \partial f/\partial y$:

$$I - \sum_{i=1}^{s} h^i J^i b^{\mathrm{T}}.X^{i-1}.v.$$

This requires the computation of powers of J (an operation with complexity $\mathcal{O}(d^3)$). To avoid the computation of high powers of J, we propose to use the technique of *deferred correction* (DC). While Cash [3,4] used this technique for BVPs, we will apply it to IVPs.

2. The DC algorithm

Suppose we want to approximate the solution of IVP (1.1) on the mesh $x_0 < x_2 < x_2 < \cdots$ and let $h = \max_i h_i$ where $h_i{:=}x_{i+1} - x_i$. Let Δy be the restriction of the continuous solution $y(x)$ to the grid and let η and η^* be approximations to Δy.

We rely on a theorem proven by Skeel [6], which we reformulate in a slightly modified form.

Theorem 2.1. *Consider the DC scheme*:

$$\begin{aligned} \phi(\eta) &= 0, \\ \phi(\eta^*) &= \psi(\eta). \end{aligned} \tag{2.3}$$

Suppose (i) $\eta = \Delta y + \mathcal{O}(h^p)$, (ii) $\psi(\Delta y) = \phi(\Delta y) + \mathcal{O}(h^{p^*})$, *and* (iii) $\psi(\Delta w) = \mathcal{O}(h^r)$ *for arbitrary functions w having at least r continuous derivatives*, *then*

$$\eta^* = \Delta y + \mathcal{O}(h^{\min(p^*, p+r)}). \tag{2.4}$$

As already mentioned, in our case ϕ will correspond to a MIRK method of order p and $\psi := \phi - \phi^*$ where ϕ^* corresponds to a MIRK method of order $p^* > p$ (we will systematically denote the quantities that relate to ϕ^* with a *-superscript: s^*, a^*_{ij}, b^*_i, $c^*_i, \ldots$). The interesting thing about ϕ^* being a MIRK method is that $\psi(\eta) = -\phi^*(\eta)$ can be computed directly. Although one could argue that ϕ can be any RK method, we also choose it to be a MIRK method since in that case all systems to be solved have dimension d.

For ϕ we have $\phi(\Delta y)_n := (y_{n+1} - y_n)/h_n - \sum_{i=1}^{s} b_i f(x_n + c_i h_n, Y_i)$, with $y_i := y(x_i)$ and

$$Y_i = (1 - v_i) y_n + v_i y_{n+1} + h_n \sum_{j=1}^{i-1} x_{ij} f(x_n + c_j h_n, Y_j)$$

$$= y_n + h_n \sum_{j=1}^{s} a_{ij} f(x_n + c_j h_n, Y_j) + \mathcal{O}(h_n^p).$$

Assumption (i) is a representation of the global error of the method ϕ with p the order of the method. If $y'(x) = f(x, y(x))$, then a Taylor series expansion gives

$$\phi(\Delta y)_n = (1 - b^{\mathrm{T}}.e) f_n + (\tfrac{1}{2}(1 - 2b^{\mathrm{T}}.A.e) f_n\, {}^y f_n + (1 - 2b^{\mathrm{T}}.c)\, {}^x f_n) h_n + \mathcal{O}(h_n^2),$$

whereby the superscript denotes the derivation and the subscript n means that all evaluations are taken in $x = x_n$. One notices that, if the series expansion is carried out as far as $\mathcal{O}(h_n^p)$, in this way all the order conditions to achieve order p can be recognised. It thus becomes clear that the term in h_n^i, $0 \leqslant i \leqslant p-1$ becomes zero when the method is of order p. We thus have $\phi(\Delta y) = \mathcal{O}(h_n^p)$. In the same way condition (ii) of Theorem 2.1 expresses the order of the residual with the higher-order method ϕ^*. Analogous to the previous derivation, $\phi^*(\Delta y)_n = \mathcal{O}(h_n^{p^*})$ can be deduced. The value r from assumption (iii) follows from the expansion of

$$\psi(\Delta w)_n = \sum_{i=1}^{s} b_i f(x_n + c_i h_n, Y_i) - \sum_{i=1}^{s^*} b_i^* f(x_n + c_i^* h_n, Y_i^*).$$

One finds

$$Y_i = w_n + (v_i w_n' + (X.e)_i f_n) h_n$$
$$+ (\tfrac{1}{2} v_i w_n'' + (X.c)_i\, {}^x f_n + [(X.v)_i w_n' + (X.X.e)_i f_n]\, {}^y f_n) h_n^2 + \mathcal{O}(h_n^3)$$

and

$$f(x_n + c_i h_n, Y_i)$$
$$= f_n + (c_i\, {}^x f_n + (v_i w_n' + (X.e)_i f_n)\, {}^y f_n) h_n$$
$$+ (\tfrac{1}{2} c_i^2\, {}^{xx} f_n + c_i (v_i w_n' + (X.e)_i f_n)^{xy} f_n + \tfrac{1}{2}(v_i w_n' + (X.e)_i f_n)^2\, {}^{yy} f_n$$
$$+ \tfrac{1}{2} v_i w_n''\, {}^y f_n + (X.c)_i\, {}^x f_n\, {}^y f_n + (X.v)_i w_n' ({}^y f_n)^2 + (X.X.e)_i f_n ({}^y f_n)^2)\, h_n^2$$
$$+ \mathcal{O}(h_n^3),$$

such that

$$\sum_{i=1}^{s} b_i f(x_n + c_i h_n, Y_i) = b^{\mathrm{T}}.ef_n + (b^{\mathrm{T}}.c\,{}^{x}f_n + (b^{\mathrm{T}}.vw_n' + b^{\mathrm{T}}X.ef_n)\,{}^{y}f_n)h_n$$
$$+(\tfrac{1}{2}b^{\mathrm{T}}.c^2\,{}^{xx}f_n + (b^{\mathrm{T}}.(cv)w_n' + b^{\mathrm{T}}.(cX.e)f_n)\,{}^{xy}f_n$$
$$+\tfrac{1}{2}b^{\mathrm{T}}.(vw_n' + X.ef_n)^2\,{}^{yy}f_n + \tfrac{1}{2}b^{\mathrm{T}}.vw_n''\,{}^{y}f_n$$
$$+b^{\mathrm{T}}X.c\,{}^{x}f_n\,{}^{y}f_n + b^{\mathrm{T}}X.vw_n'({}^{y}f_n)^2 + b^{\mathrm{T}}XX.ef_n({}^{y}f_n)^2)h_n^2 + \mathcal{O}(h_n^3).$$

Doing the same kinds of operations for the higher-order method and substracting, one finds that $\psi(\Delta w) = \phi(\Delta w) - \phi^*(\Delta w)$ is $\mathcal{O}(h_n^r)$ where $r = \min(p,q)$ and

$$q = \begin{cases} 1 & \text{if } b^{\mathrm{T}}.v \neq b^{*\mathrm{T}}.v^* \\ 2 & \text{if } b^{\mathrm{T}}.v = b^{*\mathrm{T}}.v^* \\ & \text{but } |b^{\mathrm{T}}.(cv) - b^{*\mathrm{T}}.(c^*v^*)| + |b^{\mathrm{T}}X.v - b^{*\mathrm{T}}X^*.v^*| + |b^{\mathrm{T}}.v^2 - b^{*\mathrm{T}}.v^{*2}| \neq 0 \\ 3 & \text{if } b^{\mathrm{T}}.v = b^{*\mathrm{T}}.v^* \\ & \text{and } |b^{\mathrm{T}}.(cv) - b^{*\mathrm{T}}.(c^*v^*)| + |b^{\mathrm{T}}X.v - b^{*\mathrm{T}}X^*.v^*| + |b^{\mathrm{T}}.v^2 - b^{*^{\mathrm{T}}}.v^{*2}| = 0 \\ & \text{but } \ldots \\ \ldots . & \end{cases}$$

We thus find that, while the value r in condition (iii) is 1 in general, it can be raised to 2 or even higher. In [3,4], where symmetric methods are used, the value r=2 is obtained since for all symmetric methods $b^{\mathrm{T}}.v = \frac{1}{2}$. Combining the three conditions of Theorem 2.1, is it clear that there will be a gain $\mathcal{O}(h^g)$ with the DC technique based on ϕ and ϕ^*, where $g = \min(r, p^* - p) = \min(p, q, p^* - p)$. Since one may expect that, if $p = q = p^* - q$, the ratio *accuracy/computational cost* is optimal, we will call these schemes optimal.

The basis of coupling two methods by DC, can be enlarged to several methods. The general scheme of DC by coupling m methods is of the following form:

$$\begin{aligned} &\phi_1(\eta_1) = 0, \\ &\phi_1(\eta_i) = \phi_1(\eta_{i-1}) - \phi_i(\eta_{i-1}), \quad i = 2, \ldots, m. \end{aligned} \tag{2.5}$$

We will call ϕ_1 the basic method while ϕ_i, $i = 2, \ldots, m$ are called the composing methods. Adding an extra method can, despite the extra computational cost, be interesting for reasons of accuracy and/or stability. In this paper we will restrict ourselves to schemes for which each of the composing methods raises the accuracy. For such schemes, we will analyze the stability.

3. Linear stability of DC-schemes

To analyze the linear stability properties of the method obtained, we introduce some new notations. Let $R_i(z) = N_i(z)/D_i(z)$ whereby $N_i(0) = D_i(0) = 1$ denote the linear stability function associated to method ϕ_i, then the linear stability function Z_m associated to scheme (2.5) is recursively defined by

$$Z_1(z) := R_1(z),$$

$$Z_i(z) := \frac{(D_1(z) - D_i(z))Z_{i-1}(z) + N_i(z)}{D_1(z)}, \quad i = 2, \dots, m.$$

In this way, it is clear that the denominator of $Z_m(z)$ is $D_1^m(z)$.

Several stability properties can be proven. A property which is useful in the construction of optimal DC schemes is given in the following theorem:

Theorem 3.1. *If* $R_i(z) = \exp(z) + \mathcal{O}(z^{g\,i+1})$, $i = 1, \dots, m$ *then* $Z_m(z) = \exp(z) + \mathcal{O}(z^{g\,m+1})$ *if and only if* $D_1(z) - D_i(z) = \mathcal{O}(z^g)$, $i = 1, \dots, m$.

Proof. Let $R_i(z) = N_i(z)/D_i(z) = \exp(z) + E_i(z)$ and $Z_i(z) = \tilde{N}_i(z)/\tilde{D}_i(z) = \exp(z) + \tilde{E}_i(z)$ where $E_i(z) = \mathcal{O}(z^{g\,i+1})$ and $\tilde{E}_i(z) = \mathcal{O}(z^{g\,i+1})$. Then

$$\frac{\tilde{N}_i(z)}{\tilde{D}_i(z)} = \frac{(D_1(z) - D_i(z))(\exp(z) + \tilde{E}_{i-1}(z)) + D_i(z)(\exp(z) + E_i(z))}{D_1(z)}$$

$$= \exp(z) + \frac{D_1(z) - D_i(z)}{D_1(z)}\tilde{E}_{i-1}(z) + \mathcal{O}(z^{g\,i+1})$$

from which is follows that $(D_1(z) - D_i(z))/D_1(z) = \mathcal{O}(z^g)$. □

If a DC scheme is set up consisting of m MIRK methods this condition means that, for $i = 0, 1, \dots, g-1$, $b^{\mathrm{T}}.X^i.v$ has the same value for all m methods.

We recall that our first aim is to reduce the computational work associated to the computation of high powers of J. Since the number of powers is determined by the degree of $D_1(z)$, we may want to choose a method ϕ_1 for which $D_1(z)$ is linear. In this respect, the trapezoidal rule looks very interesting since it is the only A-stable MIRK method for which $D_1(z)$ is linear which allows $g = 2$. Unfortunately, we have the following result:

Theorem 3.2. *The DC scheme* (2.3) *where* ϕ *is based on the trapezoidal rule and* ϕ^* *is a Runge–Kutta method* M *of order* $p \geqslant 3$, *cannot be A-stable.*

Proof. Let $N(z)/D(z)$ where $N(0) = D(0) = 1$ be the stability function of M. Then the stability function Z_2 of the DC scheme is

$$Z_2(z) = \frac{1 - z^2/4 + [(1 - z/2)N(z) - (1 + z/2)D(z)]}{(1 - z/2)^2}.$$

Since $(1 + z/2)/(1 - z/2) = \exp(z) + \mathcal{O}(z^3)$ and $N(z)/D(z) = \exp(z) + \mathcal{O}(z^{p+1})$, the term between brackets in $Z_2(z)$ is $\mathcal{O}(z^3)$, hence the resulting method is not A-stable. □

From the above result, it follows that if D_1 is linear, ϕ_1 can only be of first order if A-stability is required and thus only $g = 1$ is possible. If one looks for accurate A-stable schemes, it is thus necessary to consider schemes for which the denominator of the basic method is quadratic at least. In

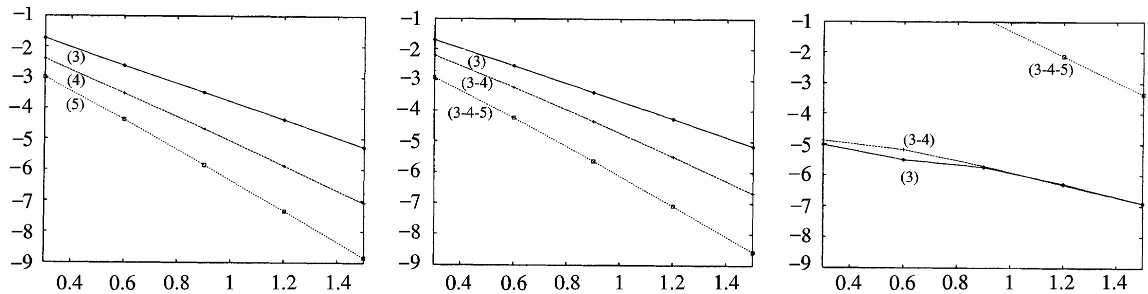

Fig. 1. $-\log_{10} h$ vs. $\log_{10}$ of the global error in $x=1$ with (a) $\lambda=0$ (left), (b) $\lambda=-1$ (middle) and (c) $\lambda=-1000$ (right) for the methods of case A.

this case, it is still possible to avoid the computation of J^2 if D_1 is factorizable in linear terms. Then several systems (for which the iteration matrices are linear in J) have to be solved consecutively.

4. An example

Case A: We select MIRK methods for which $c_i = i - 1$, $i = 1, 2, \ldots, s$. These methods, which still contain some parameters, are described in Section 3 of Van Daele [7] and used in a code in Van Hecke [8]. Since D_1 has to be quadratic at least, we look for a method ϕ_1 which is already of third order. It turns out that within the family considered it is possible to construct a L-stable fifth-order method M_{345}, based on three methods of orders 3, 4, and 5, respectively, for which D_1 is factorizable and, if we call M_3 (resp. M_{34}) the method based on the third-order (resp. third and fourth order) method alone, M_3 and M_{34} are A-stable. The values of the parameters to obtain this are $t=2(\sqrt{3}+1)$ for $m=3$, $t=0$ and $s=2(\sqrt{3}+1)$ for $m=4$ and $s=-2-4\sqrt{3}/19$ and $t=7/2+2\sqrt{3}$ for $m=5$ (with stage order 3).

As it is the case with RK methods in general, one can expect a possible order reduction when applying the method to stiff problems. Therefore, we apply the method to the Prothero–Robinson test problem [5]:

$$y'(x) = \lambda(y(x) - g(x)) + g'(x), \qquad y(0) = g(0) \tag{4.6}$$

with $g(x) = 10 - (10 + x)\exp(-x)$. We integrate this problem whose solution is $y(x) = g(x)$, up to $x = 1$ and we consider the global error for different values of the stiffness parameter λ and different values of the constant stepsize h. For $\lambda = 0$ the problem becomes explicit and the results obtained with deferred correction are those obtained with the last method used. The slopes of the lines in Fig. 1(a) confirm the theoretical order of the methods M_3, M_4, M_5.

For $\lambda \approx 0$, the problem is non-stiff and from Fig. 1(b) one can easily deduce the expected order behaviour of the three methods M_3, M_{34} and M_{345}. However, as λ decreases, the behaviour changes. In Fig. 1(c) we show the case where $\lambda=-1000$, in which case the problem is moderately stiff. One notices that M_{34} does not perform better than M_3, while M_{345} performs very badly. To understand the behaviour of the different schemes, we consider the LTEs and we look at the behaviour in the case $z = -\lambda h \to \infty$ and $h \to 0$ (this is what Prothero et al. call the stiff order).

5. The stiff order of DC schemes

When a parameterized RK method is applied to (4.6) with steplength h one obtains,

$$y_1 = \frac{(1+\hat{h}B^{\mathrm{T}}.(e-v))y + B^{\mathrm{T}}.(hG'(0)-\hat{h}G(0))}{1-\hat{h}B^{\mathrm{T}}.v}, \tag{5.7}$$

where $\hat{h}{:=}\lambda h$, $B^{\mathrm{T}}{:=}b^{\mathrm{T}}.(I-\hat{h}X)^{-1}$ and $G(x)$ and $G'(x)$ are the s-vectors with entries $g(c_i x)$ and $g'(c_i x)$.

Theorem 5.1. *If a parameterized RK method of order p with stage order $q \leqslant p$ is applied to* (4.6), *then*

$$y(h) - y_1 = \frac{h^{q+1}}{(q+1)!} C_{q+1}(\hat{h}) y^{(q+1)}(0) + \mathcal{O}(h^{q+2}), \tag{5.8}$$

where

$$C_{q+1}(\hat{h}) = 1 - (q+1)b^{\mathrm{T}}.c^q + \frac{\hat{h}B^{\mathrm{T}}.(c^{q+1}-(q+1)A.c^q)}{1-\hat{h}B^{\mathrm{T}}.v}.$$

Proof. Developing (5.7) in a Taylor series for h, one obtains on account of $y(x)=g(x)$

$$y_2 = y + \frac{1}{1-\hat{h}B^{\mathrm{T}}.v} \sum_{j=1}^{\infty} B^{\mathrm{T}}.\left(c^{j-1} - \hat{h}\frac{c^j}{j}\right) y^{(j)}(0) \frac{h^j}{(j-1)!}.$$

Order p and stage order q imply $F{:=}(q+1)A.c^q - c^{q+1} \neq 0$ and $f{:=}(q+1)b^{\mathrm{T}}.c^q - 1$ and

$$X.c^{j-1} = \frac{c^j - v}{j}, \quad j=1,\ldots,q,$$

$$X.c^q = \frac{c^{q+1}-v}{q+1} + \frac{F-fv}{q+1}$$

such that

$$c^{j-1} - \hat{h}\frac{c^j}{j} = (I-\hat{h}X).c^{j-1} - \hat{h}\frac{v}{j}, \quad j=1,\ldots,q,$$

$$c^q - \hat{h}\frac{c^{q+1}}{q+1} = (I-\hat{h}X).c^q - \hat{h}\frac{v}{q+1} + \hat{h}\frac{F-fv}{q+1}.$$

Multiplying each equation in both sides with B^{T} gives

$$B^{\mathrm{T}}.\left(c^{j-1} - \hat{h}\frac{c^j}{j}\right) = \frac{1}{j}(1-\hat{h}B^{\mathrm{T}}.v), \quad j=1,\ldots,q,$$

$$B^{\mathrm{T}}.\left(c^q - \hat{h}\frac{c^{q+1}}{q+1}\right) = \frac{1}{q+1}(1-\hat{h}B^{\mathrm{T}}.v) + \frac{1-\hat{h}B^{\mathrm{T}}.v}{q+1}f + \hat{h}\frac{B^{\mathrm{T}}.F}{q+1}.$$

The result now follows. □

Remark. It may happen that, if $q < p$, $C_{q+1}(\hat{h}) = 0$.

If a method is fitted to solve stiff problems, the rational function $C(z) \sim z^{-p_z}$ with $p_z \geqslant 0$ as $z \to \infty$. For DC-schemes we need to know how the corresponding expression grows out of the expressions for the composing methods. Therefore, we define for each method in the scheme a function $S(h,\hat{h}):=B^{\mathrm{T}}.(hG'(0)-\hat{h}G(0))$, such that we obtain from (5.7) that $y_1=[N(\hat{h})y+S(h,\hat{h})]/D(\hat{h})$. When the scheme (2.5) is applied to problem (4.6), one obtains the approximations $\tilde{y}_{1,i}=Z_i(\hat{h})y+W_i(h,\hat{h})$, $i=1,2,\ldots,m$, where

$$Z_i(z):=\frac{(D_1(z)-D_i(z))Z_{i-1}(z)+N_i(z)}{D_1(z)}, \qquad Z_1(z):=\frac{N_1(z)}{D_1(z)},$$

$$W_i(z,\hat{z}):=\frac{(D_1(z)-D_i(z))W_{i-1}(z,\hat{z})+S_i(z,\hat{z})}{D_1(z)}, \qquad W_1(z,\hat{z}):=\frac{S_1(z,\hat{z})}{D_1(z)}.$$

If we now consider the case where $h \to 0$ and $\hat{h} \to \infty$ and we define $\tilde{q}_m:=\min_{1\leqslant i\leqslant m}\{q_i \mid C_{q_i+1}(\hat{h}) \neq 0\}$ where q_i and $C_{i,q_i+1}(\hat{h})$ follow from (5.8) for method ϕ_i, then

$$y(h)-\tilde{y}_{1,m}=\frac{h^{\tilde{q}_m+1}}{(\tilde{q}_m+1)!}\,y^{(\tilde{q}_m+1)}(0)\tilde{C}_{m,\tilde{q}_m+1}(\hat{h})+\mathcal{O}(h^{\tilde{q}_m+2}),$$

where $\tilde{C}_{1,\tilde{q}_m+1}(z):=C_{1,\tilde{q}_m+1}(z)$ and

$$\tilde{C}_{i,\tilde{q}_m+1}(z):=\frac{(D_1(z)-D_i(z))\tilde{C}_{i-1,\tilde{q}_m+1}(z)+D_i(z)C_{i,\tilde{q}_m+1}(z)}{D_1(z)}.$$

If we now return to Case A, we find that $\tilde{q}_3=2$ since $q_1=2$ and $q_1=q_3=3$ and

$$C_{1,3}(z)=\frac{\sqrt{3}z}{6-2(3+\sqrt{3})z+(2+\sqrt{3})z^2}, \qquad C_{2,3}(z)=0, \qquad C_{3,3}(z)=0,$$

$$\tilde{C}_{2,3}(z):=\frac{\sqrt{3}(3+2\sqrt{3})(z-2)z^2}{2(6-6z-2\sqrt{3}z+2z^2+\sqrt{3}z^2)^2},$$

$$\tilde{C}_{3,3}(z):=(3+2\sqrt{3})(-2+z)z^3\,\frac{(-312-84\sqrt{3}+(624+312\sqrt{3})z-(488+272\sqrt{3})z^2+(157+90\sqrt{3})z^3)}{20\sqrt{3}(6-(6+2\sqrt{3})z+(2+\sqrt{3})z^2)^3}$$

from which one can conclude that for $z \to \infty$ $\tilde{C}_{1,3} \sim z^{-1}$ and $\tilde{C}_{2,3} \sim z^{-1}$ but $\tilde{C}_{3,3} \sim z^1$, which can be verified from Fig. 2.

Case B: Taking into account the results concerning stiff order, another choice of the parameters, for which M_3 and M_{34} are A-stable M_{345} is quasi L-stable (there is a very small area in the complex plane near the imaginary unit where the method is unstable) is made: $t=0$ for $m=3$, $s=t=0$ for $m=4$, $t=0$ and $s=\frac{4}{57}$ for $m=5$. In this case D_1 is quadratic but no longer factorizable. This different behaviour of the global error (see Fig. 3) compared to Case A is due solely to the choice of the parameters, which causes LTE $\sim h^4\hat{h}^{-1}$ for all methods.

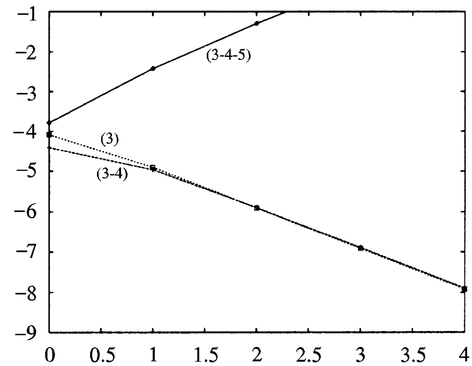

Fig. 2. $\log_{10} z$ vs. $\log_{10}$ of the global error in $x=1$ with $h=\frac{1}{10}$ for the methods of case A.

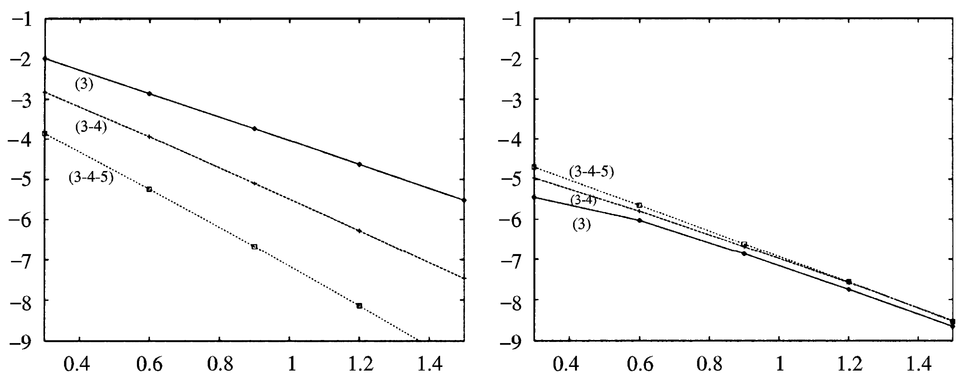

Fig. 3. $-\log_{10} h$ vs. $\log_{10}$ of the global error in $x=1$ with (a) $\lambda=-1$ (left) and (b) $\lambda=-1000$ (right) for the methods of case B.

Case C: A third and last example illustrates the possibility to have a stable DC-scheme with gain $g=3$ with a stable s_1-stage method of order 3 and a s_2-stage method of order 6 both having the maximum stage-order 3. To construct this scheme, we first examined the cases where the total number of stages s_1+s_2 is minimal, taking into account that $s_2 \geqslant 5$ to obtain order 6 and $s_1 \geqslant 3$ to obtain order 3 and stage order 3 and we made use of the fact that expressions of the form $b^{\mathrm{T}}.X^i.v$ and $b^{\mathrm{T}}.X^i.e$ are connected to each other by the order equations. This technique showed that it was impossible to have A-stability for $s_1=3$ and $s_2=5$ or $s_2=6$. We thus chose $s_1=4$ and $s_2=5$. For the sixth-order method, we used the family in [1]. This family contains two parameters $c_3^{(6)}$ and $c_4^{(6)}$. A family of third-order methods with four stages which has stage order 3 and for which the denominator of the stability function has fixed linear and quadratic coefficients also contains two parameters $c_3^{(3)}$ and $c_4^{(3)}$, whereby stability requires that $|(c_3^{(3)}-1)/c_3^{(3)}|<1$. There is one possibility, $c_4^{(6)}=1-c_3^{(6)}$, to make the DC scheme A-stable and L-stability can be obtained for $c_3^{(3)}=\sqrt{2}$. Considering the $\tilde{C}$-expressions reveals that both the basic method and the DC-scheme are $\sim z^{-1}$ irrespective of the choice made for $c_3^{(3)}$. The two remaining conditions, which express that $b^{\mathrm{T}}.(cv)$ and $b^{\mathrm{T}}.(v^2)$ should have a fixed value for both methods, are then used to determine $c_3^{(6)}$ and $c_4^{(3)}$.

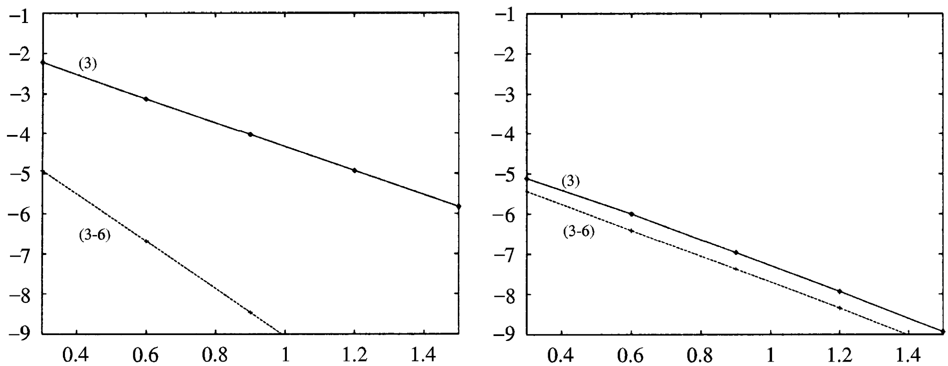

Fig. 4. $-\log_{10} h$ vs. $\log_{10}$ of the global error in $x = 1$ with (a) $\lambda = -1$ (left) and (b) $\lambda = -1000$ (right) for the methods of case C.

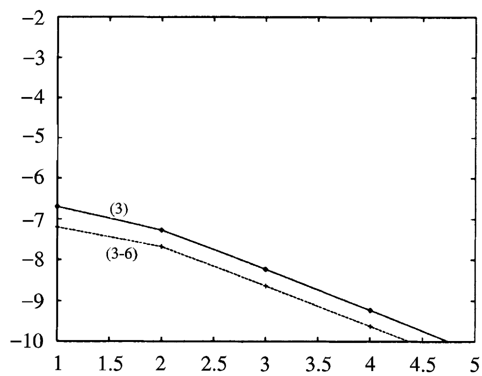

Fig. 5. $\log_{10} z$ vs. $\log_{10}$ of the global error in $x = 1$ with $h = 1/10$ for the methods of case C.

We mention the following solution:

0	0				
1	1	0			
.7071067812	.7928932188	.0606601718	−.1464466094		
−.2670411948	−.2606042131	−.2932210260	−.1257432186	.4125272629	
		1.0863664648	.3492484895	.0353379297	−.4709528840

0	0					
1	1	0				
$c_3^{(6)}$	$v_3^{(6)}$	−1.249716874	−.4341220130			
$1 - c_3^{(6)}$	$1 - v_3^{(6)}$	.4341220130	1.249716874	0		
.5	.5	.1410807817	−.1410807817	−.0077889892	.0077889892	
		.1871016491	.1871016491	−.0047942664	−.0047942664	.6353852345

where $c_3^{(6)} = -0.5322765429$ and $v_3^{(6)} = 1.151562344$.

A final analysis shows that, apart from small regions of instability along the imaginary axis, the basic method of order three is A-stable and the DC-scheme itself is L-stable. The results are shown in Figs. 4 and 5.

6. Conclusion

In this paper we considered the construction of DC schemes out of MIRK methods for the numerical solution of IVPs. It is shown that high-order schemes can be constructed, but that it is insufficient to consider only linear stability. One can make sure that the stability of the DC scheme is ensured also for non-linear systems of equations, but a new problem, which we did not consider so far, is present: for the non-stiff case, there is a natural mechanism present in the DC scheme to perform error control and stepsize selection. For the stiff case, this mechanism is no longer present due to order reduction. This problem will be considered in future work.

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