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Constructive Cut Elimination in Geometric Logic

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Abstract

A constructivisation of the cut-elimination proof for sequent calculi for classical and intuitionistic infinitary logic with geometric rules – given in earlier work by the second author – is presented. This is achieved through a procedure in which the non-constructive transfinite induction on the commutative sum of ordinals is replaced by two instances of Brouwer's Bar Induction. Additionally, a proof of Barr's Theorem for geometric theories that uses only constructively acceptable proof-theoretical tools is obtained.

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1 Introduction

Notable parts of algebra and geometry can be formalised as *coherent theories* over first-order classical or intuitionistic logic. Their axioms are *coherent implications*, i.e., universal closures of implications $D_1 \supset D_2$, where both D_1 and D_2 are built up from atoms using conjunction, disjunction and existential quantification. Examples include all algebraic theories, such as the theory of groups and the theory of rings, all essentially algebraic theories, such as category theory [7], the theory of fields, the theory of local rings, lattice theory [22], projective and affine geometry [22, 17], the theory of separably closed local rings (aka "strictly Henselian local rings") [9, 17, 25].

Although wide, the class of coherent theories leaves out certain axioms used in algebra such as the axioms of torsion Abelian groups or of Archimedean ordered fields, or used in the theory of connected graphs, as well as in the modelling of epistemic social notions such as common knowledge. All the latter examples can however be axiomatised by means of *geometric axioms*, a generalization of coherent axioms that allows infinitary disjunctions.

Geometric implications give a Glivenko class [18], as shown by Barr's Theorem:

▶ **Theorem 1** (Barr's Theorem [3]). *If $\mathcal{T}$ is a coherent (geometric) theory and A is a coherent (geometric) sentence provable from $\mathcal{T}$ in (infinitary) classical logic, then A is provable from $\mathcal{T}$ in (infinitary) intuitionistic logic.*

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Barr's Theorem[1] has its origin, through appropriate completeness results, in the theory of sheaf models, with the following formulation:

▶ **Theorem 2** ([12], Ch.9, Thm.2). *For every Grothendieck topos $\mathcal{E}$ there exists a complete Boolean algebra $\mathbf{B}$ and a surjective geometric morphism $Sh(\mathbf{B}) \longrightarrow \mathcal{E}$.*

An extremely simple and purely syntactic proof of the first-order Barr's Theorem for coherent theories has been given in [14] by means of **G3** sequent calculi: it is shown how to express coherent implications by means of rules that preserve admissibility of the structural rules. As a consequence, Barr's theorem is proved by simply noticing that a proof in **G3C**.**G** – i.e., the calculus for classical logic extended with rules for coherent implications – is also a proof in the intuitionistic multisuccedent calculus **G3I**.**G**.

In [16], this approach to Barr's Theorem has been generalized to (infinitary) geometric theories using **G3**-style calculi for classical and intuitionistic infinitary logic $\mathbf{G3}[\mathbf{CI}]_\omega$ (with finite sequents instead of countably infinite sequents) and their extension with rules expressing geometric implications $\mathbf{G3}[\mathbf{CI}]_\omega.\mathbf{G}$. To illustrate, the geometric axiom of torsion Abelian groups

$$\forall x. \bigvee_{n>0} nx = 0$$

is expressed by the infinitary rule

$$\frac{\{nx = 0, \Gamma \Rightarrow \Delta \mid n > 0\}}{\Gamma \Rightarrow \Delta} \quad .$$

The main results in [16] are that in $\mathbf{G3}[\mathbf{CI}]_\omega.\mathbf{G}$ all rules are height-preserving invertible, the structural rules of weakening and contraction are height-preserving admissible, and cut is admissible. Hence, Barr's Theorem for geometric theories is proved in [16] as it was done in [14] for coherent ones: a proof in $\mathbf{G3C}_\omega.\mathbf{G}$ is also a proof in the intuitionistic multisuccedent calculus $\mathbf{G3I}_\omega.\mathbf{G}$.

We observe that the cut-elimination procedure given in Sect. 4.1 of [16] is not constructive. This is an instance of a typical limitation of cut eliminations in infinitary logics [6, 11, 23] since these proofs use the "natural" (or Hessenberg) commutative sum of ordinals $\alpha \# \beta$:

$$(\omega^{\alpha_m} + \cdots + \omega^{\alpha_0}) \# (\omega^{\beta_n} + \cdots + \omega^{\beta_0}) = (\omega^{\gamma_{m+n+1}} + \cdots + \omega^{\gamma_0})$$

where $\gamma_{m+n+1}, \ldots, \gamma_0$ is a decreasing permutation of $\alpha_m, \ldots, \alpha_0, \beta_n, \ldots, \beta_0$; see [24, 10.1.2B]. The resort to the natural sum is inescapable for proofs using cut-height (i.e., the sum of the derivation-height of the premisses of cut) as inductive parameter: it ensures that we can apply the inductive hypothesis when permuting the cut upwards in the derivation of one premisses. Nevertheless, it makes the proof non-constructive since

> [its] definition utilises the Cantor normal form of ordinals to base ω. This normal form is not available in **CZF** [Constructive Zermelo–Fraenkel set theory] (or **IZF** [Intuitionistic Zermelo–Fraenkel set theory]) and thus a different approach is called for. [20, p. 369]

[1] Barr's theorem is often alleged to achieve more in that it also allows to eliminate uses of the axiom of choice. That such formulations of Barr's theorem should be taken with caution is demonstrated in [20] where the *internal* vs. *external* addition of the the axiom of choice is considered and it is shown that the latter preserves conservativity whereas the former does not.

We constructivise[2] the cut-elimination proof for $\mathbf{G3[CI]}_\omega.\mathbf{G}$ by giving a procedure that replaces the induction on the cut-height with two transfinite inductions on the height of the derivations of the right and the left premiss of cut respectively – see Lemmas 20 and 21 – and it replaces the main induction on the depth of the cut-formula with two instances of Brouwer's principle of Bar Induction – see Theorem 23.[3] As a consequence, we are able to give a proof of Barr's Theorem for geometric theories that uses only constructively acceptable proof-theoretic tools. Moreover, our proof strategy allows to constructivise the cut-elimination procedure for other infinitary calculi.

2 Syntax and sequent calculi for infinitary logics

Let $\mathcal{S}$ be a signature containing, for every $n \in \mathbb{N}$, a countable (i.e., finite, possibly empty, or countably infinite) set $REL^{\mathcal{S}}_n$ of n-ary predicate letters $P^n_1, P^n_2, \ldots$, and a countable set CON of individual constants $c_1, c_2, \ldots$. Let VAR be a denumerable set of variables $x_1, x_2, \ldots$. The language contains the following logical symbols: $=$, $\top$, $\bot$, $\wedge$, $\vee$, $\supset$, $\forall$, $\exists$, as well as countable conjunction $\bigwedge_{n>0}$ and countable disjunction $\bigvee_{n>0}$.

The sets TER of *terms* is the union of VAR and CON. The set of *formulas* of the language $\mathcal{L}^{\mathcal{S}}_\omega$ is generated by:

$$A ::= P^n_i t_1, \ldots, t_n \mid t_1 = t_2 \mid \top \mid \bot \mid A \wedge A \mid A \vee A \mid A \supset A \mid \forall x A \mid \exists x A \mid \bigwedge_{n>0} A_n \mid \bigvee_{n>0} A_n$$

where $t_i \in TER$, $P^n_i \in REL^{\mathcal{S}}_n$, and $x \in VAR$.
We use the following metavariables:

- x, y, z for variables and $\vec{x}, \vec{y}, \vec{z}$ for lists thereof;
- t, s, r for terms;
- P, Q, R for atomic formulas;
- A, B, C for formulas.

We use $A(\vec{x})$ to say that the variables having free occurrences in A are included in $\vec{x}$. We follow the standard conventions for parentheses. The formulas $\top$, $\neg A$ and $A \supset\subset B$ are defined as expected. When considering (infinitary) classical logic we can shrink the set of primitive logical symbols by means of the well-known De Morgan's dualities (including $\bigvee_{n>0} A_n \supset\subset \neg \bigwedge_{n>0} \neg A$), however also in the classical case we consider a language where all operators (excluding $\neg$ and $\supset\subset$) are taken as primitive. This is not just useful but even necessary since our purpose is to extract the constructive content of classical proofs and many of the interdefinabilities do not hold in intuitionistic logic.

The notions of *free* and *bound occurrences* of a variable in a formula are the usual ones. We posit that no formula may have infinitely many free variables. A *sentence* is a formula without free occurrences of variables. Given a formula A, we use $A(t/x)$ to denote the formula obtained by replacing each free occurrence of x in A with an occurrence of t, provided that t is free for x in A – i.e., no new occurrence of t is bound by a quantifier.

Each formula A has a countable ordinal $\mathtt{d}(A)$ as its *depth* (the successor of the supremum of the depths of its immediate subformulas). More precisely

[2] By "constructive" here we mean not relying on classical logical principles such as excluded middle or linearity of ordinals but we do not mean acceptable in all schools of constructive mathematics.

[3] See [20, §7] for a different proof, based on constructive ordinals, of cut elimination in infinitary logic. The proof in [2] does not use ordinals, but it is inherently classical in that it uses a one-sided calculus based on De Morgan's dualities.

▶ **Definition 3** (Depth of A). $\mathtt{d}(A) = sup\{\mathtt{d}(B) \mid B \text{ is an immediate subformula of } A\} + 1$.

For example, $\bot$ and atoms P have depth 1, since they have no immediate subformulas and the supremum of an empty family of ordinals is 0. The definition of depth implies that, if A' is a proper subformula of A, then $\mathtt{d}(A') < \mathtt{d}(A)$.

Sequents $\Gamma \Rightarrow \Delta$ have a finite multiset of formulas on each side. The inference rules for $\bigvee_{n>0}$ are thus:

$$\frac{\{\Gamma, A_n \Rightarrow \Delta \mid n > 0\}}{\Gamma, \bigvee_{n>0} A_n \Rightarrow \Delta}\ L\bigvee \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n, A_k}{\Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n}\ R\bigvee_k.$$

Observe that $L\bigvee_{n>0}$ has countably many premisses, one for each $n > 0$. The rules for $\bigwedge_{n>0}$ are dual to the above ones.

Derivations built using these rules are thus (in general) infinite trees, with countable branching but where (as may be proved by induction on the definition of derivation) each branch has finite length. The *leaves* of the trees are those where the two sides have an atomic formula in common, and also instances of rules $L\bot$, $R\top$. To make this precise, we give a formal definition of the notion of *derivation* $\mathcal{D}$ and the associated notions of its *height* $\mathtt{ht}(\mathcal{D})$ and its *end-sequent*.

▶ **Definition 4** (Derivations, their height and their end-sequent).

1. *Any sequent* $\Gamma \Rightarrow \Delta$*, where some atomic formula occurs in both* Γ *and* Δ*, is a derivation, of* height 0 *and with* end-sequent $\Gamma \Rightarrow \Delta$.
2. *Let* $\beta \leq \omega$*. If each* $\mathcal{D}_n$*, for* $0 < n < \beta$*, is a derivation of height* α_n *and with end-sequent* $\Gamma_n \Rightarrow \Delta_n$*, and*

$$\frac{\ldots \quad \Gamma_n \Rightarrow \Delta_n \quad \ldots}{\Gamma \Rightarrow \Delta}\ R$$

is an instance of a rule with β *premisses, then*

$$\frac{\ldots \quad \begin{array}{c}\vdots\,\mathcal{D}_n\\ \Gamma_n \Rightarrow \Delta_n\end{array} \quad \ldots}{\Gamma \Rightarrow \Delta}\ R$$

is a derivation, of height *the countable ordinal* $sup_n(\alpha_n)+1$ *and with* end-sequent $\Gamma \Rightarrow \Delta$.[4]

If $\mathbf{X}$ *is a calculus, we use* $\mathbf{X} \vdash \Gamma \Rightarrow \Delta$ *to say that* $\Gamma \Rightarrow \Delta$ *is derivable in the calculus* $\mathbf{X}$.

By this definition each derivation has a countable ordinal *height* (the successor of the supremum of the heights of its immediate subderivations). Thus, if Γ and Δ have an atomic formula in common, then $\Gamma \Rightarrow \Delta$ has a derivation $\mathcal{D}$ of height $\mathtt{ht}(\mathcal{D}) = 0$. The sequent $\bot, \Gamma \Rightarrow \Delta$ (regarded as a zero-premiss rule) has a derivation of height 1. Observe that the definitions of depth of a formula and of height of a derivation differ from those in [6]: we use the successor of a supremum rather than the supremum of the successors (note that $sup_{n>0}(n+1) = \omega \neq \omega + 1 = (sup_{n>0}(n)) + 1$). It follows that, if $\mathcal{D}'$ is a sub-derivation of $\mathcal{D}$, then $\mathtt{ht}(\mathcal{D}') < \mathtt{ht}(\mathcal{D})$. If a sequent has a derivation of height α we say it is *α-derivable* and write $\vdash^\alpha \Gamma \Rightarrow \Delta$.

[4] Derivations can thus be represented as (infinite) trees, where the nodes are the sequents in the derivation, and a node that corresponds to a premiss of a rule is an immediate successor of the node that corresponds to the conclusion of such rule. Therefore, a node that corresponds to the conclusion of a rule with β premisses has β immediate successors.

▶ **Definition 5** (Sequent calculi for infinitary logics with equality).

1. $\mathbf{G3C}_\omega$ *is defined by the rules in Tables 1 and 3;*
2. $\mathbf{G3I}_\omega$ *is defined as* $\mathbf{G3C}_\omega$ *with the exception of rules* $L\supset$, $R\supset$, $R\forall$, *and* $R\bigwedge$ *that are defined as in Table 2.*

By $\mathbf{G3[CI]}_\omega$ we denote any one of the two calculi above. Observe that a multi-succedent intuitionistic calculus as the one we use is closer to a classical calculus than the usual calculus with the restriction that the succedent of sequents should consist of at most one formula (used, for example in [20]). As in the finitary case such a multi-succedent choice is particularly useful for proving Glivenko-style results [15].

As usual, we consider only derivations of *pure sequents* – i.e., sequents where no variable has both free and bound occurrences. We say that $\Gamma \Rightarrow \Delta$ is $\mathbf{G3[CI]}_\omega$*-derivable* (with height α), and we write $\mathbf{G3[CI]}_\omega \vdash^{(\alpha)} \Gamma \Rightarrow \Delta$, if there is a $\mathbf{G3[CI]}_\omega$-derivation (of height at most α) of $\Gamma \Rightarrow \Delta$ or of an alphabetic variant of $\Gamma \Rightarrow \Delta$. A rule is said to be *(height-preserving) admissible* in $\mathbf{G3[CI]}_\omega$, if, whenever its premisses are $\mathbf{G3[CI]}_\omega$-derivable (with height at most α), also its conclusion is $\mathbf{G3[CI]}_\omega$-derivable (with height at most α). A rule is said to be *(height-preserving) invertible* in $\mathbf{G3[CI]}_\omega$, if, whenever its conclusion is $\mathbf{G3[CI]}_\omega$-derivable (with height at most α), also its premisses are $\mathbf{G3[CI]}_\omega$-derivable (with height at most α). In each rule depicted in Tables 1, 2, and 3 the multisets Γ and Δ are called *contexts*, the formulas occurring in the conclusion are called *principal*, and the formulas occurring in the premiss(es) only are called *active*.

3 From geometric implications to geometric rules

By a *geometric implication* we mean the universal closure of an implicative formula whose antecedent and consequent are formulas constructed from atomic formulas and $\bot, \top$ using only $\wedge$, $\vee$, $\exists$, and $\bigvee_{n>0}$. More precisely:

▶ **Definition 6** (Geometric implication).

- *A formula is* Horn *iff it is built from atoms and* $\top$ *using only* $\wedge$*;*
- *A formula is* geometric *iff it is built from atoms and* $\top, \bot$ *using only* $\wedge, \vee, \exists$, *and* $\bigvee_{n>0}$*;*
- *A sentence is a* geometric implication *iff it is of the form* $\forall\vec{x}(A\supset B)$ *where* A *and* B *are geometric formulas.*

By a *coherent implication* we mean a geometric implication without occurrences of $\bigvee_{n>0}$.

As is well known, for geometric implications we have a normal form theorem.

▶ **Theorem 7** (Geometric normal form (GNF)). *Any geometric implication is equivalent to a possibly infinite conjunction of sentences of the form*

$$\forall\vec{x}(A\supset B)$$

where A *is Horn and* B *is a possibly infinite disjunction of existentially quantified Horn formulas.*

This normal form theorem is important because, as shown in [14] for coherent implications and in [16] for geometric ones, we can extract from a sentence G in GNF a *geometric rule* L_G (where the name L_G indicates that it is a *left rule*) that can be added to a sequent calculus without altering its structural properties. To be more precise, let us consider the following sentence G in GNF:

$$\forall\vec{x}(P_1(\vec{x}) \wedge \cdots \wedge P_k(\vec{x}) \supset \bigvee_{n>0} \exists\vec{y}(Q_{n_1}(\vec{x},\vec{y}) \wedge \cdots \wedge Q_{n_m}(\vec{x},\vec{y}))) \qquad (G)$$

Table 1 The calculus $\mathbf{G3C}_\omega$ (z fresh in $L\exists$ and $R\forall$).

Initial sequents:
$P, \Gamma \Rightarrow \Delta, P$

$$\frac{}{\Gamma \Rightarrow \Delta, \top}\ R\top \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ R\wedge \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ R\vee$$

$$\frac{}{\bot, \Gamma \Rightarrow \Delta}\ L\bot \qquad \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ L\vee$$

$$\frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \supset B}\ R\supset \qquad \frac{\Gamma \Rightarrow \Delta, A(y/x), \exists x A}{\Gamma \Rightarrow \Delta, \exists x A}\ R\exists \qquad \frac{\Gamma \Rightarrow \Delta, A(z/x)}{\Gamma \Rightarrow \Delta, \forall x A}\ R\forall$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta}\ L\supset \qquad \frac{A(z/x), \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta}\ L\exists \qquad \frac{A(y/x), \forall x A, \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\ L\forall$$

$$\frac{\{\Gamma \Rightarrow \Delta, A_i \mid i > 0\}}{\Gamma \Rightarrow \Delta, \bigwedge_{n>0} A_n}\ R\bigwedge \qquad \frac{\Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n, A_k}{\Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n}\ R\bigvee$$

$$\frac{A_k, \bigwedge_{n>0} A_n, \Gamma \Rightarrow \Delta}{\bigwedge_{n>0} A_n, \Gamma \Rightarrow \Delta}\ L\bigwedge \qquad \frac{\{A_i, \Gamma \Rightarrow \Delta \mid i > 0\}}{\bigvee_{n>0} A_n, \Gamma \Rightarrow \Delta}\ L\bigvee$$

Table 2 Non-classical rules for $\mathbf{G3I}_\omega$ (z fresh in $R\forall$).

$$\frac{A \supset B, \Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta}\ L\supset \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow \Delta, A \supset B}\ R\supset \qquad \frac{\Gamma \Rightarrow A(z/x)}{\Gamma \Rightarrow \Delta, \forall x A}\ R\forall \qquad \frac{\{\Gamma \Rightarrow A_i \mid i > 0\}}{\Gamma \Rightarrow \Delta, \bigwedge A_n}\ R\bigwedge$$

Table 3 Rules for equality in $\mathbf{G3[CI]}_\omega$.

$$\frac{s = s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Ref} \qquad \frac{P(t/x), s = t, P(s/x), \Gamma \Rightarrow \Delta}{s = t, P(s/x), \Gamma \Rightarrow \Delta}\ \mathit{Repl}$$

Table 4 Geometric rule L_G expressing the geometric sentence G.

$$\frac{\ldots \qquad Q_{n_1}(\vec{x}, \vec{y}_n), \ldots, Q_{n_m}(\vec{x}, \vec{y}_n), P_1(\vec{x}), \ldots, P_k(\vec{x}), \Gamma \Rightarrow \Delta \qquad \ldots}{P_1(\vec{x}), \ldots, P_k(\vec{x}), \Gamma \Rightarrow \Delta}\ L_G$$

Such a sentence G determines the (finitary or infinitary) *geometric rule* given in Table 4 with one premiss for each of the countably many disjuncts in $\bigvee_{n>0}(Q_{n_1}(\vec{x},\vec{y}) \wedge \cdots \wedge Q_{n_m}(\vec{x},\vec{y}))$. The variables in $\vec{y}_n$ are chosen to be *fresh*, i.e. are not in the conclusion; and without loss of generality they are all distinct. The list $\vec{y}_n$ of variables may vary as n varies, and maybe no finite list suffices for all the countably many cases. The variables $\vec{x}$ (finite in number) may be instantiated with arbitrary terms. Henceforth we shall normally omit mention of the variables.

We need also a further condition for height-preserving admissibility of contraction to hold:

▶ **Definition 8** (Closure condition). *Given a calculus with geometric rules, if it has a rule with an instance with repetition of some principal formula such as:*

$$\frac{\ldots \qquad Q_1,\ldots,Q_n,P_1,\ldots,P_{k-2},P,P,\Gamma \Rightarrow \Delta \qquad \ldots}{P_1,\ldots,P_{k-2},P,P,\Gamma \Rightarrow \Delta}\ L^c_G$$

then also the contracted instance

$$\frac{\ldots \qquad Q_1,\ldots,Q_m,P_1,\ldots,P_{k-2},P,\Gamma \Rightarrow \Delta \qquad \ldots}{P_1,\ldots,P_{k-2},P,\Gamma \Rightarrow \Delta}\ L^c_G$$

has to be included in the calculus.

As for the finitary case [14], also in the infinitary case the condition is unproblematic, since each atomic formula contains only a finite number of variables and therefore so are the instances; it follows that, for each geometric rule, the number of rules that have to be added is finite. Moreover, in many cases contracted instances need not be added since they are admissible in the calculus without them. To illustrate, we consider the coherent rule *Repl* for equality given in Table 3:

$$\frac{P(t/x), s=t, P(s/x), \Gamma \Rightarrow \Delta}{s=t, P(s/x), \Gamma \Rightarrow \Delta}\ Repl$$

This rule generates contracted instances only when its two principal formulas (as well as its active formula) are copies of the same reflexivity atom $t=t$. In this case, after having applied contraction, we can replace the instance of *Repl* with an instance of *Ref* (instead of *Repl*^{c}). That is, we can transform:

$$\frac{t=t, t=t, t=t, \Gamma \Rightarrow \Delta}{t=t, t=t, \Gamma \Rightarrow \Delta}\ Repl \qquad \text{into} \qquad \cfrac{\cfrac{t=t, t=t, t=t, \Gamma \Rightarrow \Delta}{t=t, t=t, \Gamma \Rightarrow \Delta}\ LC}{t=t, \Gamma \Rightarrow \Delta}\ Ref$$

But this does not hold in general. For example, if $<$ is an Euclidean relation, we must add both of the following rules:

$$\frac{s<r, t<s, t<r, \Gamma \Rightarrow \Delta}{t<s, t<r, \Gamma \Rightarrow \Delta}\ Euc \qquad \text{and} \qquad \frac{s<s, t<s, \Gamma \Rightarrow \Delta}{t<s, \Gamma \Rightarrow \Delta}\ Euc^c$$

otherwise the valid sequent $t<s \Rightarrow s<s$ would not be contraction-free derivable. In presence of *Ref*, no added rule is needed.

▶ **Theorem 9** ([16]). *If we add to the calculus* $\mathbf{G3[CI]}_\omega$ *a finite or infinite family of geometric rules* L_G*, then we can prove all of the geometric sentences* G *from which they were determined.*

In the following, we shall denote with $\mathbf{G3[CI]}_\omega.\mathbf{G}$ any extension of $\mathbf{G3[CI]}_\omega$ with a finite or infinite family of geometric rules L_G (together with all needed contracted instances thereof).

Before proceeding with the structural properties, we give some examples of geometric axioms and their corresponding rules.

▶ **Example 10** (Geometric axioms and rules).

1. The axiom of **torsion Abelian groups**, $\forall x. \bigvee_{n>1}(nx = 0)$, becomes the rule

$$\frac{\dots \quad nx = 0, \Gamma \Rightarrow \Delta \quad \dots}{\Gamma \Rightarrow \Delta} R_{Tor}$$

2. The axiom of **Archimedean ordered fields**, $\forall x. \bigvee_{n\geq 1}(x < n)$, becomes the rule

$$\frac{\dots \quad x < n, \Gamma \Rightarrow \Delta \quad \dots}{\Gamma \Rightarrow \Delta} R_{Arc}$$

3. The axiom of **connected graphs**,

$$\forall xy. x = y \vee xRy \vee \bigvee_{n>1} \exists z_0 \dots \exists z_n (x = z_0 \,\&\, y = z_n \,\&\, z_0 R z_1 \,\&\, \dots \,\&\, z_{n-1} R z_n)$$

becomes the rule

$$\frac{x = y, \Gamma \Rightarrow \Delta \quad xRy, \Gamma \Rightarrow \Delta \quad \dots \quad x = z_0, y = z_n, z_0 R z_1, \dots, z_{n-1} R z_n, \Gamma \Rightarrow \Delta \quad \dots}{\Gamma \Rightarrow \Delta} R_{Conn}$$

3.1 Structural rules

We present here the results concerning the admissibility of the structural rules, cut excluded, in the calculi $\mathbf{G3[CI]}_\omega.\mathbf{G}$. All these results have been proved in Sect. 4 of [16] by simple transfinite induction on ordinals, either on the depth of a formula or on the height of a derivation.

▶ **Lemma 11** (Generalised initial sequents). *The sequent $A, \Gamma \Rightarrow \Delta, A$ is $\mathbf{G3[CI]}_\omega.\mathbf{G}$-derivable, for A arbitrary formula.*

▶ **Lemma 12** (α-conversion). *If $\mathbf{G3[CI]}_\omega.\mathbf{G} \vdash^\alpha \Gamma \Rightarrow \Delta$ then $\mathbf{G3[CI]}_\omega.\mathbf{G} \vdash^\alpha \Gamma' \Rightarrow \Delta'$, for $\Gamma' \Rightarrow \Delta'$ a bound alphabetic variant of $\Gamma \Rightarrow \Delta$.*

▶ **Lemma 13** (Substitution). *If $\mathbf{G3[CI]}_\omega.\mathbf{G} \vdash^\alpha \Gamma \Rightarrow \Delta$ and t is free for x in $\Gamma \Rightarrow \Delta$ then $\mathbf{G3[CI]}_\omega.\mathbf{G} \vdash^\alpha \Gamma(t/x) \Rightarrow \Delta(t/x)$.*

▶ **Theorem 14** (Weakening). *The left and right rules of weakening:*

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} RW$$

are height-preserving admissible (hp-admissible, for short) in $\mathbf{G3[CI]}_\omega.\mathbf{G}$.

▶ **Lemma 15** (Invertibility).

1. *Each rule of $\mathbf{G3C}_\omega.\mathbf{G}$ is hp-invertible.*

2. *Each rule of $\mathbf{G3I}_\omega.\mathbf{G}$ except $R\supset$, $R\forall$, and $R\bigwedge$ is hp-invertible.*

▶ **Theorem 16** (Contraction). *The rules of left and right contraction:*

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ LC \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ RC$$

are hp-admissible in $\mathbf{G3[CI]}_\omega.\mathbf{G}$.

4 Constructive cut elimination

We are now ready to prove that the following context-sharing rule of cut:

$$\frac{\Gamma \Rightarrow \Delta, C \quad C, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Cut$$

is eliminable in the calculus $\mathbf{G3[CI]}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$ obtained by extending $\mathbf{G3[CI]}_\omega.\mathbf{G}$ with *Cut*. In order to give a proof of cut elimination that uses only constructively admissible proof-theoretical tools we must avoid the "natural" (or Hessenberg) commutative sum of ordinals: we cannot use cut-height as inductive parameter as done in the Gentzen- and Dragalin-style proofs. In order to avoid it, we make use of a proof strategy introduced in [13] for fuzzy logics that has been extensively used in the context of hypersequent calculi; see [4, 8, 10]. This proof strategy can be seen as a simplified and local version of the proof given by H.B. Curry in [5]. The proof is based on two main lemmas (Lemmas 20 and 21 below) that are proved by induction on the height of the derivation of the right and of the left premiss of cut, respectively. Moreover, (almost) all non-principal instances of cut are taken care of by a separate lemma (Lemmas 18 and 19) which shows that cut can be permuted upwards with respect to rule instances not having the cut formula among their principal formulas.

Observe that, differently from [4, 13, 10], we will not consider an arbitrary instance of *Cut* of maximal rank (i.e., such that its cut formula has maximal depth among the cut formulas occurring in the derivation), but we will always consider an uppermost instance of *Cut*, i.e. a cut the premisses of which are cut-free derivations. Otherwise, in Lemmas 20 and 21 as well as in Theorem 23, we would have to assume that ordinals are linearly/totally ordered; but in a constructive setting this assumption implies the law of excluded middle [1]. In Theorem 23 we will proceed, instead, by using two instances of Brouwer's principle of Bar Induction: the first will be used to show that an uppermost instance of *Cut* is eliminable and the second to show that all instances of *Cut* are eliminable. Note that although it is a constructively admissible principle, Bar Induction increases the proof-theoretic strength of **CZF**, cf. [19].

▶ **Definition 17** (Cut-substitutive rule). *A sequent rule* Rule *is* cut-substitutive *if each instance of cut with cut formula not principal in the last rule instance* Rule *of one of the premisses of cut can be permuted upwards w.r.t.* Rule *as in the following example:*

$$\frac{\dfrac{A, \Gamma \Rightarrow \Delta, B, C}{\Gamma \Rightarrow \Delta, A \supset B, C}\ R\supset \quad C, \Gamma \Rightarrow \Delta, A \supset B}{\Gamma \Rightarrow \Delta, A \supset B}\ Cut \quad \rightsquigarrow \quad \frac{\dfrac{A, \Gamma \Rightarrow \Delta, B, C \quad \dfrac{C, \Gamma \Rightarrow \Delta, A \supset B}{A, C, \Gamma \Rightarrow \Delta, B}\ \textit{hp-inv}}{A, \Gamma \Rightarrow \Delta, B}\ Cut}{\Gamma \Rightarrow \Delta, A \supset B}\ R\supset$$

▶ **Lemma 18.** *Each rule of* $\mathbf{G3C}_\omega.\mathbf{G}$ *is cut-substitutive.*

Proof. By inspecting the rules in Tables 1 and 3 it is immediate to realise that each of them is cut-substitutive because they are all hp-invertible (using Lemma 13 for rules $L\exists$, $R\forall$, and for geometric rules with a variable condition). ◀

▶ **Lemma 19.** *Each rule of* $\mathbf{G3I}_\omega.\mathbf{G}$ *except* $R\supset$, $R\forall$ *and* $R\bigwedge$ *is cut-substitutive.*

Proof. Same as for $\mathbf{G3C}_\omega.\mathbf{G}$. ◀

▶ **Lemma 20** (Right reduction)**.** *If we are in* $\mathbf{G3}[\mathbf{CI}]_\omega.\mathbf{G}$ *and all of the following hold:*

1. $\mathcal{D}_1 \vdash \Gamma \Rightarrow \Delta, A$
2. $\mathcal{D}_2 \vdash A, \Gamma \Rightarrow \Delta$
3. *A is principal in the last rule instance applied in $\mathcal{D}_1$*
4. *A is not of shape $\exists x B$ or $\bigvee_{n>0} B_n$.*

Then there is a $\mathbf{G3}[\mathbf{CI}]_\omega.\mathbf{G} + \{\mathbf{Cut}\}$*-derivation $\mathcal{D}$ concluding $\Gamma \Rightarrow \Delta$ containing only cuts on proper subformulas of A.*

Proof. By transfinite induction on $\mathtt{ht}(\mathcal{D}_2)$.

If $\mathcal{D}_2$ is a one node tree, the lemma obviously holds.

Else, we have two cases depending on whether A is principal in the last rule instance *Rule* applied in $\mathcal{D}_2$ or not.

In the latter case, if we are in $\mathbf{G3C}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$, the lemma holds thanks to Lemma 18: we permute the cut upwards in $\mathcal{D}_2$ and then we apply the inductive hypothesis and an instance of *Rule*. If we are in $\mathbf{G3I}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$ and the last step of $\mathcal{D}_2$ is not by one of $R\supset$, $R\forall$, and $R\bigwedge$ then it holds by Lemma 19. In the remaining three cases, we have two subcases according to whether $\mathcal{D}_1$ ends with a step by an invertible rule or not. In the latter subcase, $\mathcal{D}_1$ ends with one of $R\supset$, $R\forall$, and $R\bigwedge$. We permute the cut upwards in the right premiss. To illustrate, we consider the case of $R\bigwedge$. We transform

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{11}\\ \Gamma \Rightarrow B(y/x)\end{array}}{\Gamma \Rightarrow \Delta', \bigwedge A_n, \forall x B}\,R\forall \qquad \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{2i}\\ \{\forall x B, \Gamma \Rightarrow A_i \mid i>0\}\end{array}}{\forall x B, \Gamma \Rightarrow \Delta', \bigwedge A_n}\,R\bigwedge}{\Gamma \Rightarrow \Delta', \bigwedge A_n}\,Cut$$

into

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{11}\\ \Gamma \Rightarrow B(y/x)\end{array}}{\Gamma \Rightarrow \forall x B}\,R\forall \qquad \begin{array}{c}\vdots\,\mathcal{D}_{2i}\\ \{\forall x B, \Gamma \Rightarrow A_i \mid i>0\}\end{array}}{\{\Gamma \Rightarrow A_i \mid i>0\}}\,IH_i, i>0}{\Gamma \Rightarrow \Delta', \bigwedge A_n}\,R\bigwedge$$

If, instead, $\mathcal{D}_1$ ends by an invertible rule then we apply invertibility, thus transforming the derivation into one having only cuts on proper subformulas of A. To illustrate, if $\mathcal{D}_1$ ends with a step by $R\wedge$, we transform

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{11}\\ \Gamma \Rightarrow \Delta', \bigwedge A_n, B\end{array} \quad \begin{array}{c}\vdots\,\mathcal{D}_{12}\\ \Gamma \Rightarrow \Delta', \bigwedge A_n, C\end{array}}{\Gamma \Rightarrow \Delta', \bigwedge A_n, B \wedge C}\,R\wedge \qquad \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{2i}\\ \{B \wedge C, \Gamma \Rightarrow A_i \mid i>0\}\end{array}}{B \wedge C, \Gamma \Rightarrow \Delta', \bigwedge A_n}\,R\bigwedge}{\Gamma \Rightarrow \Delta', \bigwedge A_n}\,Cut$$

into

$$\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{12}\\ \Gamma \Rightarrow \Delta', \bigwedge A_n C\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{11}\\ \Gamma \Rightarrow \Delta', \bigwedge A_n, B\end{array}}{C, \Gamma \Rightarrow \Delta', \bigwedge A_n, B}\,LW \qquad \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{2}\\ B \wedge C, \Gamma \Rightarrow \Delta', \bigwedge A_n\end{array}}{B, C, \Gamma \Rightarrow \Delta', \bigwedge A_n}\,Lem.\ 15}{C, \Gamma \Rightarrow \Delta', \bigwedge A_n}\,Cut}{\Gamma \Rightarrow \Delta', \bigwedge A_n}\,Cut$$

Next, we consider the case with A principal in the last rule instance applied in $\mathcal{D}_2$. We have cases according to the shape of A.

If $A \equiv P$ for some atomic formula P, then the last rule instance in $\mathcal{D}_2$ is by a geometric rule (rules for equality included) L_G concluding $P_1, \ldots, P, \ldots, P_k, \Gamma'' \Rightarrow \Delta', P$ and $\mathcal{D}_1$ is the one node tree $P, \Gamma' \Rightarrow \Delta', P$. The conclusion of cut is the initial sequent $P, \Gamma' \Rightarrow \Delta', P$ which is cut-free derivable.

The cases with $A \equiv \bot$ or $A \equiv B \circ C$, for $\circ \in \{\top, \wedge, \vee, \supset\}$, are left to the reader.

If $A \equiv \forall x B$ we transform (if we are in $\mathbf{G3I}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$, Δ is not in the premiss of $R\forall$)

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{11}\\ \Gamma \Rightarrow \Delta, B(y/x)\end{array}}{\Gamma \Rightarrow \Delta, \forall x B}\,R\forall \qquad \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{21}\\ B(t/x), \forall x B, \Gamma \Rightarrow \Delta\end{array}}{\forall x B, \Gamma \Rightarrow \Delta}\,L\forall}{\Gamma \Rightarrow \Delta}\,Cut$$

into the following derivation having only cuts on proper subformulas of A (if we are in $\mathbf{G3I}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$ then Δ is introduced in $\mathcal{D}_{11}$ by height-preserving weakenings, which can be done since $\mathcal{D}_{11}$ is in $\mathbf{G3I}_\omega.\mathbf{G}$):

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{11}\\ \Gamma \Rightarrow \Delta, B(y/x)\end{array}}{\Gamma \Rightarrow \Delta, B(t/x)}\,Lem.\ 13 \qquad \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{1}\\ \Gamma \Rightarrow \Delta, \forall x B\end{array} \quad \begin{array}{c}\vdots\,\mathcal{D}_{21}\\ \forall x B, B(t/x), \Gamma \Rightarrow \Delta\end{array}}{B(t/x), \Gamma \Rightarrow \Delta}\,IH}{\Gamma \Rightarrow \Delta}\,Cut$$

If $A \equiv \bigwedge B_i$ we transform (Δ not in the premisses of $R\bigwedge$ if we are in $\mathbf{G3I}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$)

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{1i}\\ \{\Gamma \Rightarrow \Delta, B_i \,|\, i > 0\}\end{array}}{\Gamma \Rightarrow \Delta, \bigwedge B_n}\,R\bigwedge \qquad \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{21}\\ B_k, \bigwedge B_n, \Gamma \Rightarrow \Delta\end{array}}{\bigwedge B_n, \Gamma \Rightarrow \Delta}\,L\bigwedge}{\Gamma \Rightarrow \Delta}\,Cut$$

into the following derivation having only cuts on proper subformulas of A (if we are in $\mathbf{G3I}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$ then Δ is introduced in $\mathcal{D}_{1k}$ by height-preserving weakenings):

$$\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{1k}\\ \Gamma \Rightarrow \Delta, B_k\end{array} \qquad \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{1}\\ \Gamma \Rightarrow \Delta, \bigwedge B_n\end{array} \quad \begin{array}{c}\vdots\,\mathcal{D}_{21}\\ \bigwedge B_n, B_k, \Gamma \Rightarrow \Delta\end{array}}{B_k, \Gamma \Rightarrow \Delta}\,IH}{\Gamma \Rightarrow \Delta}\,Cut$$

◀

▶ **Lemma 21** (Left reduction). *If we are in* $\mathbf{G3[CI]}_\omega.\mathbf{G}$ *and all of the following hold:*

1. $\mathcal{D}_1 \vdash \Gamma \Rightarrow \Delta, A$
2. $\mathcal{D}_2 \vdash A, \Gamma \Rightarrow \Delta$

Then there is a $\mathbf{G3[CI]}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$*-derivation* $\mathcal{D}$ *concluding* $\Gamma \Rightarrow \Delta$ *containing only cuts on proper subformulas of* A*.*

Proof. By transfinite induction on $\mathtt{ht}(\mathcal{D}_1)$.

If $\mathcal{D}_1$ is a one node tree, the lemma obviously holds.

Else, we have two cases depending on whether A is principal in the last rule instance applied in $\mathcal{D}_1$ or not. In the latter case, the lemma holds thanks to Lemma 18 or 19 (if the last step of $\mathcal{D}_1$ is by an intuitionistic non-invertible rule we proceed as in the analogous case of Lemma 20).

When A is principal in the last rule inference in $\mathcal{D}_1$, we have cases according to the shape of A. If A is an atomic formula, or $\top$, or $\bot$, or $B \circ C$ ($\circ \in \{\wedge, \vee \supset\}$), or $\forall x B$, or $\bigwedge B_n$, the lemma holds thanks to Lemma 20.

If $A \equiv \exists x B$, we transform:

$$\dfrac{\dfrac{\vdots\, \mathcal{D}_{11} \atop \Gamma \Rightarrow \Delta, \exists x B, B(t/x)}{\Gamma \Rightarrow \Delta, \exists x B}\, R\exists \qquad \vdots\, \mathcal{D}_2 \atop \exists x B, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, Cut$$

into the following derivation having only cuts on proper subformulas of A (Lemma 15 can be applied since $\mathcal{D}_2$ is in $\mathbf{G3}[\mathbf{CI}]_\omega.\mathbf{G}$):

$$\dfrac{\dfrac{\vdots\, \mathcal{D}_{11} \atop \Gamma \Rightarrow \Delta, B(t/x), \exists x B \qquad \vdots\, \mathcal{D}_2 \atop \exists x B, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, B(t/x)}\, IH \qquad \dfrac{\vdots\, \mathcal{D}_2 \atop \exists x B, \Gamma \Rightarrow \Delta}{B(t/x), \Gamma \Rightarrow \Delta}\, Lem.\ 15}{\Gamma \Rightarrow \Delta}\, Cut$$

If $A \equiv \bigvee_{n>0} B_n$, we transform:

$$\dfrac{\dfrac{\vdots\, \mathcal{D}_{11} \atop \Gamma \Rightarrow \Delta, \bigvee_{n>0} B_n, B_k}{\Gamma \Rightarrow \Delta, \bigvee_{n>0} B_n}\, R\bigvee \qquad \vdots\, \mathcal{D}_2 \atop \bigvee_{n>0} B_n, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, Cut$$

into the following derivation:

$$\dfrac{\dfrac{\vdots\, \mathcal{D}_{11} \atop \Gamma \Rightarrow \Delta, B_k, \bigvee_{n>0} B_n, \qquad \vdots\, \mathcal{D}_2 \atop \bigvee_{n>0} B_n, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, B_k}\, IH \qquad \dfrac{\vdots\, \mathcal{D}_2 \atop \bigvee_{n>0} B_n, \Gamma \Rightarrow \Delta}{B_k, \Gamma \Rightarrow \Delta}\, Lem.\ 15}{\Gamma \Rightarrow \Delta}\, Cut$$

◀

In order to prove Cut elimination in a constructive way we use Bar Induction as done in [21, p. 18] for ω-arithmetic. This strategy avoids the assumption of total ordering of ordinal numbers. Before proving the theorem we introduce Brouwer's principle of (decidable) Bar Induction.

▶ **Definition 22** (Bar Induction). *Let B and I be unary predicates (the so-called "base predicate" and "inductive predicate", respectively) of finite lists of natural numbers (to be denoted by $u, v, \dots$). If:*

1. *B is decidable;*
2. *Every infinite sequence of natural numbers has a finite initial segment satisfying B;*
3. *$B(u)$ implies $I(u)$ for every finite list u;*
4. *If $I(u * n)$ holds for all $n \in \mathbb{N}$ then $I(u)$ holds;*

Then I holds for the empty list of natural numbers.

▶ **Theorem 23** (Cut elimination). *Cut is admissible in $\mathbf{G3}[\mathbf{CI}]_\omega$.*

Proof. Throughout this proof, we use finite lists of natural numbers to index (partial) branches of trees, i.e. directed paths from the root to a node, possibly a leaf. Consider a tree such that each node has immediate successors either indexed by ω or else by some $k < \omega$, and such that each branch has finite length, then:

- The empty list $\{\}$ indexes the root of the tree.
- Suppose that u indexes a partial branch $\mathcal{R}$ of the tree and that the last node a has immediate successor nodes indexed by $k < \omega$, and let a natural number n be given. Let $m = n \mod k$: that is, m is the remainder of n after division by k. Then $u * n$ indexes $\mathcal{R}$ extended with the m^{th} immediate successor node of a. For example, in the case of a 2-premiss rule, odd numbers index the left premiss, even numbers the right premiss.

Notice that the above gives a partial surjective map, with decidable domain, from sequences of natural numbers to branches in the given tree. Moreover, this ensures that every infinite sequence has an initial segment that indexes a branch of the tree.[5]

Let $\mathcal{D}$ be a derivation in the calculus $\mathbf{G3[CI]}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$. The proof consists of two parts, each building on an appropriate Bar Induction.

- **Part 1.** We use Bar Induction to show that an uppermost instance of *Cut* with cut-formula C occurring in $\mathcal{D}$ is admissible. We use the method defined above to index the branches of the formation tree of the formula C – where C is the root of the tree and atomic formulas or $\top$ or $\bot$ are its leaves. Let $B(u)$ hold if u indexes a branch whose last element is an atom or $\bot$ or $\top$; let $I(u)$ hold if u indexes a partial branch whose last element is a formula D such that an uppermost cut on D or on some proper subformula thereof in $\mathbf{G3[CI]}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$ is eliminable.
  The following hold:
  1. $B(u)$ is decidable by simply comparing the list with the formation tree;
  2. By definition of the indexing, the n^{th} element of the sequence identifies the n^{th} node in a branch of the formation tree of a formula. After a finite number of steps from the root we find an atom or $\bot$ or $\top$ since all branches of the tree are finite and this identifies an initial segment of the infinite sequence that satisfies B.
  3. $B(u)$ implies $I(u)$ since cuts on atomic formulas, $\top$, or $\bot$ are admissible;
  4. $I(u * n)$ for all n implies $I(u)$: by Lemma 21 an uppermost cut on some formula E can be reduced to cuts on proper subformulas of E.

  By Bar Induction we conclude that the uppermost cut with cut-formula C is eliminable from $\mathbf{G3[CI]}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$.
- **Part 2.** We show that all cuts can be eliminated from $\mathcal{D}$. We consider a derivation $\mathcal{D}$ in $\mathbf{G3[CI]}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$ and, as above, we use lists of natural numbers to index branches of $\mathcal{D}$. Let $B(u)$ hold if u indexes a branch ending in a leaf of $\mathcal{D}$; let $I(u)$ hold if u indexes a partial branch whose last element has a cut-free derivation (i.e., it is $\mathbf{G3[CI]}_\omega.\mathbf{G}$-derivable). All conditions of Bar Induction are satisfied by this choice of B and I:
  1. $B(u)$ is decidable;
  2. Given any infinite sequence of numbers, we have $B(u)$ for every finite initial segment u that represents a full branch $\mathcal{R}$ of the tree, i.e., a root-to-leaf path; and by construction of the representation there are such u.
  3. $B(u)$ implies $I(u)$ since the leaves of $\mathcal{D}$ trivially have a cut-free derivation;
  4. $I(u * n)$ for all n implies $I(u)$: having shown in part 1 that uppermost instances of Cut are admissible, if all the premisses of a rule instance in $\mathcal{D}$ have a cut-free derivation, then also its conclusion has a cut-free derivation.

  By Bar Induction we conclude that the conclusion of $\mathcal{D}$ has a cut-free derivation. ◀

[5] Since the number of nodes of the tree is at most countable, one may also define an encoding such that the correspondence is unique. This however would require more effort and we would lose the property that every infinite sequence has an initial segment that indexes a branch of the tree.

▶ **Corollary 24.** *The rule of context-free cut:*

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Sigma}{\Pi, \Gamma \Rightarrow \Delta, \Sigma}\ Cut_{cf}$$

is admissible in $\mathbf{G3[CI]}_\omega.\mathbf{G}$.

Proof. An immediate consequence of Theorem 23 since rules *Cut* and Cut_{cf} are equivalent when weakening and contraction are admissible. ◀

5 A proof of the infinitary Barr theorem

Barr's theorem is a fundamental result in geometric logic: it guarantees that for geometric theories classical derivability of geometric implications entails their intuitionistic derivability. As recalled in the Introduction (Theorem 2), the result has its origin, through appropriate completeness results, in the theory of sheaf models. The most general form of Barr's theorem [3, 26, 20] is higher-order and includes the axiom of choice, and stated as eliminating not just classical reasoning but also the axiom of choice[6].

If one is interested solely in derivability in geometric logic (finitary or infinitary, but without the axiom of choice), Barr's theorem can be regarded as identifying a Glivenko class, i.e., a class of sequents for which classical derivability entails intuitionistic derivability and a proof entirely internal to proof theory, without any detour through completeness with respect to topos-theoretic models, can be obtained.

Consider now a classical theory axiomatised by coherent or geometric implications. Extending the conversion into rules of [14] to cover the case of infinitary disjunctions and using the results detailed above, we transform the classical geometric theory $\mathbf{G}$ into a contraction- and cut-free sequent calculus $\mathbf{G3C}_\omega.\mathbf{G}$. We shall denote by $\mathbf{G3I}_\omega.\mathbf{G}$ the corresponding intuitionistic extension of $\mathbf{G3I}_\omega$. The following holds:

▶ **Theorem 25** (Barr's theorem). *If a coherent or geometric implication is derivable in* $\mathbf{G3C}_\omega.\mathbf{G}$*, it is derivable in* $\mathbf{G3I}_\omega.\mathbf{G}$.

Proof. Any derivation in $\mathbf{G3C}_\omega.\mathbf{G}$ uses only rules that follow the (infinitary) geometric rule scheme and logical rules. Observe that geometric implications contain no $\supset$, nor $\forall$, nor $\bigwedge$ in the scope of $\vee$, which means that no instance of the rules that violates the intuitionistic restrictions is used, so the derivation directly gives (through the addition, where needed, of the missing implications in steps of $L\supset$) a derivation in $\mathbf{G3I}_\omega.\mathbf{G}$ of the same conclusion. ◀

A proof of Barr's theorem for *finitary* geometric theories was given in [14] through a cut-free presentation of finitary geometric theories and the choice of an appropriate sequent calculus that, in effect, trivialises the result. By the results above, the same trivialization works for infinitary logic: a classical proof *already* is an intuitionistic proof.

6 Conclusion

This paper has shown how it is possible to constructivise the cut elimination procedure given in [16] for infinitary geometric theories and how, as a consequence, it is possible to obtain a constructive proof of Barr's theorem. The proof used here avoids the use of the natural sum

[6] Cf. Footnote 1.

of ordinals which made non-constructive most cut-elimination procedures for infinitary logics, but it does not avoid the use of transfinite induction on ordinals since all proofs of the results in Section 3.1, as well as the proofs of Lemmas 20 and 21, are by induction on ordinals. We observe, however, that the alternative route of resorting to constructive ordinals has been pursued in [20] to obtain a proof of cut elimination for infinitary logic and of Barr's theorem.

In the future, we plan to get rid of ordinals altogether by introducing a new well-founded inductive parameter that can supplant ordinals. Another open line of research is to extend the purely logical proof of Barr's theorem given here and in [16] to other infinitary Glivenko sequent classes, as it has been done in [15] for the finitary ones.

References

1. Peter Aczel and Michael Rathjen. Notes on constructive set theory. Technical report, Institut Mittag-Leffler (The Royal Swedish Academy of Sciences), 2001. URL: `https://ncatlab.org/nlab/files/AczelRathjenCST.pdf`.
2. Ryota Akiyoshi. An ordinal-free proof of the complete cut-elimination theorem for $\mathbf{\Pi^1_1}$-**CA** + **BI** with the ω-rule. *IfCoLog J. of Logics and their Applications*, 4:867–884, January 2017.
3. Michael Barr. Toposes without points. *J. Pure Appl. Algebra*, 5(3):265–280, 1974. `doi:10.1016/0022-4049(74)90037-1`.
4. Agata Ciabattoni, George Metcalfe, and Franco Montagna. Algebraic and proof-theoretic characterizations of truth stressers for mtl and its extensions. *Fuzzy Sets and Systems*, 161(3):369–389, 2010. Fuzzy Logics and Related Structures. `doi:10.1016/j.fss.2009.09.001`.
5. Haskell B. Curry. *Foundations of Mathematical Logic*. Dover Books on Mathematics Series. Dover Publications, 1977.
6. Solomon Feferman. Lectures on Proof Theory. In *Proceedings of the Summer School in Logic (Leeds, 1967)*, pages 1–107. Springer, Berlin, 1968.
7. Peter Freyd. Aspects of topoi. *Bull. Austral. Math. Soc.*, 7(1):1–76, 1972. `doi:10.1017/S0004972700044828`.
8. Andrzej Indrzejczak. Eliminability of cut in hypersequent calculi for some modal logics of linear frames. *Information Processing Letters*, 115(2):75–81, 2015. `doi:10.1016/j.ipl.2014.07.002`.
9. Peter T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium. Vol. 1 & 2*. Oxford University Press, New York, 2002.
10. Hidenori Kurokawa. Hypersequent calculi for modal logics extending **S4**. In Yukiko Nakano, Ken Satoh, and Daisuke Bekki, editors, *New Frontiers in Artificial Intelligence*, pages 51–68, Cham, 2014. Springer International Publishing. `doi:10.1007/978-3-319-10061-6_4`.
11. E. G. K. Lopez-Escobar. An interpolation theorem for denumerably long formulas. *Fund. Math.*, 57:253–272, 1965. `doi:10.4064/fm-57-3-253-272`.
12. Saunders Mac Lane and Ieke Moerdijk. *Sheaves in Geometry and Logic. A First Introduction to Topos Theory*. Springer-Verlag, 1994. `doi:10.1007/978-1-4612-0927-0`.
13. George Metcalfe, Nicola Olivetti, and Dov Gabbay. *Proof Theory for Fuzzy Logics*. Springer, 2008. `doi:10.1007/978-1-4020-9409-5`.
14. Sara Negri. Contraction-free sequent calculi for geometric theories with an application to Barr's theorem. *Arch. Math. Logic*, 42(4):389–401, 2003. `doi:10.1007/s001530100124`.
15. Sara Negri. Glivenko sequent classes in the light of structural proof theory. *Arch. Math. Logic*, 55(3-4):461–473, 2016. `doi:10.1007/s00153-016-0474-y`.
16. Sara Negri. Geometric rules in infinitary logic. In Ofer Arieli and Anna Zamansky, editors, *Arnon Avron on Semantics and Proof Theory of Non-Classical Logics*, pages 265–293. Springer, 2021. `doi:10.1007/978-3-030-71258-7_12`.
17. Sara Negri and Jan von Plato. *Proof Analysis. A Contribution to Hilbert's Last Problem*. Cambridge University Press, Cambridge, 2011.
18. V.P. Orevkov. Glivenko's sequence classes. In V.P. Orevkov, editor, *Logical and logico-mathematical calculi. Part 1*, pages 131–154. Inst. Steklov, Leningrad, 1968.

19 Michael Rathjen. A note on Bar Induction in constructive set theory. *Mathematical Logic Quarterly*, 52(3):253–258, 2006. doi:10.1002/malq.200510030.

20 Michael Rathjen. Remarks on Barr's theorem. Proofs in geometric theories. In Dieter Probst and Peter Schuster, editors, *Concepts of Proof in Mathematics, Philosophy, and Computer Science*, pages 347–374. de Gruyter, 2016. doi:10.1515/9781501502620-019.

21 Annika Siders. From Stenius' consistency proof to Schütte's cut elimination for ω-arithmetic. *Rev. Symb. Log.*, 9(1):1–22, 2016. doi:10.1017/S1755020315000337.

22 Göran Sundholm. Proof theory, a survey of the omega-rule. *Videnskapsselskapets skrifter, 1. Mat.-naturv. klasse*, 4, April 1983.

23 Gaisi Takeuti. *Proof Theory*. North-Holland, 1987 (second edition).

24 Anne S. Troelstra and Helmut Schwichtenberg. *Basic Proof Theory*. Cambridge University Press, Cambridge, 2nd edition, 2000. doi:10.1017/CBO9781139168717.

25 Gavin C. Wraith. Generic Galois theory of local rings. In Michael P. Fourman et al., editor, *Applications of Sheaves*, pages 739–767. Springer-Verlag, 1979. doi:10.1007/BFb0061844.

26 Gavin C. Wraith. Intuitionistic algebra: some recent developments in topos theory. In *Proceedings of the International Congress of Mathematicians (Helsinki, 1978)*, pages 331–337. Acad. Sci. Fennica, 1980.