Review of Floyd and Mühlhölzer on Wittgenstein's Annotations þÿto Hardy : Juliet Floyd* and Felix Mühlhölzer.** þÿAnnotations to Hardy s Course of Pure Mathema þÿInvestigation of Wittgenstein s Non-Extensional Understanding of the Real Numbers

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Review of Floyd and Mühlhölzer's *Wittgenstein's Annotations to Hardy's Course of Pure Mathematics: An Investigation of Wittgenstein's Non-Extensionalist Understanding of the Real Numbers*

There are some books that one learns from them, what one expected or hoped to learn; but then there are books that go well beyond the brief laid out in their title, and the book under review is one such. Called *Wittgenstein's Annotations to Hardy's Course of Pure Mathematics*, the book is actually an extensive and deeply informed examination of Wittgenstein's philosophy of mathematics in *all* of its aspects, the pretext here being Wittgenstein's annotations to Hardy's classic text *Course of Pure Mathematics* (CPM henceforth), but in effect drawing on the entirety of Wittgenstein's writings on the foundations of mathematics. As the authors note, the annotations are brief and not very numerous, consisting, if one is lucky, of a few sentences at most. Yet in spite of this scarcity of source material, Floyd and Mühlhölzer's book turns out to be one of the richest and deepest confrontations with Wittgenstein's philosophy of mathematics one has ever come across.

As a genre, annotations are interesting. So just as the genre of artists' drawings reveals the immediacy of the hand-eye connection, destroying the space between the eye and the canvas, so the genre of annotations annihilates the space between the thinking and the page. Annotations in books are the royal road into the thinking of a philosopher—or of anyone else, for that matter.[1]

[1]See for example Gödel's mere underlinings to Husserl's *Ideen I* in the Gödel *Nachlass*,

And here we have Wittgenstein reading Hardy! Nothing short of the fox invading the chicken coop. But rather than laying waste—or *only* laying waste—what Floyd and Mühlhölzer show us is how Wittgenstein pulls the curtain back on the human, step by step of it all underlying the basic theory of the real numbers; how Wittgenstein marks up CPM with his own particular set of philosophical attunements, or rather, marks up its prose, for the *prose* around the theorems of CPM is Wittgenstein's main interest—the prose that explains the theorems or sets the scene for them; the prose that masks all the philosophical and logical evasions mathematicians make in order to get things off the ground; the prose that carries the extensional standpoint which Wittgenstein had always set himself against. It is important that Wittgenstein chooses a textbook to annotate rather than a published paper, as it is in textbooks that one finds such unbuttoned prose, illustrative language directed at the beginner with the intent of helping them—but more often than not, according to Wittgenstein, misleading them instead.

The annotations were written in 1942-43 and they are the basis of part 5 of *Remarks on the Foundations of Mathematics II*, though Wittgenstein was concurrently writing the third version of the *Philosphical Investigations*, the typescript (TS 239) of which was submitted in 1943 but according to Wittgenstein's wishes never published. This crucial period sees Wittgenstein refining and reabsorbing the confrontation with Turing's work in the years 1936-7, a confrontation extensively analysed in the text subsequently and also in brilliant and extensive previous work by both authors. Floyd in particular has argued that it was exactly in the wake of Wittgenstein's confrontation with Turing's 1936 "On computable numbers, with an application to the *Entscheidungsproblem* that

> ... Wittgenstein's mature philosophical method and style of writing solidified, and he turned toward his latest concerted work on logic and the foundations of mathematics.[2]

By 1944 Wittgenstein turned to other areas than foundations of mathematics and logic, and thus these annotations represent one of Wittgenstein's last attempts at clarifying his thinking on mathematics per se. As the authors remark, "Properly speaking, these writings form a kind of "application"

which reveal a great deal about his reception of phenomenology in the 1950s.

[2] p. 10

of his mature thoughts about the nature of the proposition and meaning to the foundations of mathematics."[3]

We consider each chapter of Floyd and Mühlhölzer's book in turn in this review. As we will see, the philosophical interest of the text is enormous. We learn a great deal about Wittgenstein; but we also learn a great deal about how to *read* Wittgenstein, and in so doing we learn a great deal how to read a philosopher full stop.

Floyd and Mühlhölzer begin their chapter one with a brief look at the *Tractatus*, asking the crucial question, "...what kind of *philosophical progress* had been made between the writing of the *Tractatus* and the writing of the *Philosphical Investigations*?"[4] Though as the authors note, already Wittgenstein's middle period 1929-33 marks a significant change in his thinking, surrendering "his ideal of a "final analysis" in which the simplicity of objects would be shown in the structure of the totality of mutually independent elementary sentences" and rejecting "the idea that there *must* be a "general form of proposition"."[5]

Chapter two lays out Wittgenstein's nonextensionalist standpoint, particularly as it relates to the completion of the real line, the terminology of "gaps," and how these are, from the extensionalist point of view, "filled in" by the irrational numbers.

> Mathematicians, of course, have perfectly good reasons to go beyond the domain of rational numbers. For Wittgenstein this presents no problem at all in cases of *individual* numbers like $\sqrt{2}$ and π and so on when these are determined by definite techniques of producing their decimal expansion or by similar means of identification. He balks, however, at the transition from the rational to "all" real numbers at one stroke, so to speak, which in CPM is accomplished by means of the general notion of a cut ("section" in Hardy's terminology) of the rational numbers.[6]

As the authors point out, Wittgenstein is not advocating that we reject irrational numbers, or, related to this, impredicativity, or indeed classical disjunction. He is instead asking us to attend to what may be "potentially

[3]op cit
[4]p. 5
[5]p. 6
[6]p. 29

confusing metaphors involved in their heuristics and motivations";[7] to examine in earnest the extensionalist point of view:

> Wittgenstein's criticism is directed at the common view of modern mathematicians that mathematics is about extensions. We call this the *extensionalist point of view* and the contrary view proffered by Wittgenstein the *non-extensionalist* one. We do not use the term "intensionalist" because it would suggest that words get their meaning via certain entities individually attached to them, entities called "intensions", and this is a conception totally contradicting Wittgenstein's stance. According to him, words (linguistic expressions, signs) get their meaning—their "life", as he sometimes says—through the uses we make of them within language, and for him it is a sort of category mistake to base or ground this so-called meaning in immediate relations to certain entities accompanying the words.[8]

Wittgenstein's critique of extensionalism might easily lead one to think that he advocated revising classical mathematical methods along the lines of what had been suggested by Weyl with predicativity, or Brouwer, with the rejection of classical negation. And indeed a student of Wittgenstein's drew this very conclusion, pegging Wittgenstein as a Brouwerian intuitionist, and was reprimanded for it.[9] For Floyd and Mühlhölzer this thinking of Wittgenstein in reformist terms misprizes Wittgenstein's philosophical project, what he was up to *philosophically* with respect to foundational questions. And indeed clearing up the question of Wittgenstein's possible predicativist or constructivist commitments—he doesn't seem to have any, it turns out—is an important achievement of Floyd and Mühlhölzer's text. So what is Wittgenstein's non-extensionalist point of view?

> With the term "extension" Wittgenstein has two things in mind. First, he will strictly distinguish between sequences of numbers that the extensionalist considers to be, in Cantor's sense, "finished" [fertig] entities or sets—these are the "extensions"—from

[7] p. 29

[8] p. 30

[9] The episode is recounted in Cheryl Misak's biography *Frank Ramsey: A Sheer Excess of Powers*.

> the techniques or rules by means of which such entities may be produced, assessed, or accessed. If there are such techniques, the extensionalist's interest is ultimately only in their results, the produced sequences, and not the possible processes or conceptual motifs or definitions leading to them. For the non-extensionalist, on the other hand, it is the processes and structured conceptual motifs, the grammar or logic of the notions, we should be concerned with. The most important cases discussed in Wittgenstein's texts are given by the conception of a real number as a rule-governed calculational procedure for which we can see that for any given n it will generate n digits: for example, a recipe for generating more and more successive digits of $\sqrt{2}$. About the number $\sqrt{2}$ Wittgenstein explicitly says that "we [as extensionalists] have a tendency to think that there is one result produced by $\sqrt{2}$, viz., an infinite decimal fraction." But on his view $\sqrt{2}$ produces a series of results, but no single result: "$\sqrt{2}$ is a rule for producing a fraction, not an extension".[10]

The authors note that regarding infinite decimals as finished entities also enables the metaphor of a gap free real line—to the mathematician, of course. As for the philosopher, to accept metaphors like gap-freeness is to give in to a wrong picture:

> Wittgenstein's investigation of the limitations of the extensionalist viewpoint is motivated by a diagnosis: that it all too easily leads to illusory thinking. It should therefore be refused as anything more than metaphor. In this sense he does not reject the extensionalist viewpoint altogether. Instead, he accepts it as a sort of specific phraseology, but one that should be assigned a relatively peripheral place, or rather, one that actually has a peripheral place already within our mathematical practice.

Wittgenstein is correct in the sense that metaphors do not appear in proofs. This is an important fact and Wittgenstein is right that we do need to develop a sense of its philosophical meaning. Viewed from within the practice, metaphors are, of course, all over the place. There is, for example, *compression of information*: "Collapsing cardinals," for example, a metaphor

[10] p. 31

common to set theory, compresses the complex process of changing cardinalities by forcing, into a single, powerful thought-image—and the theory of forcing would suffer greatly for the lack of it.

Robert Frost spoke about the value of metaphor in his "Education by Poetry," an address delivered at Amherst College in 1931. He spoke of the "lost soul," the person "who gets lost in the material without a gathering metaphor to throw it into shape and order"; the idea that metaphor is all of thinking; the idea that, sans metaphor, we are *unsafe*:

> What I am pointing out is that unless you are at home in the metaphor, unless you have had your proper poetical education in the metaphor, you are not safe anywhere. Because you are not at ease with figurative values: you don't know the metaphor in its strength and its weakness. You don't know how far you may expect to ride it and when it may break down with you. You are not safe with science; you are not safe in history.

One way of describing what Wittgenstein is doing in the annotations is that he is probing the breakdown of the mathematical metaphor, the touch and go of it, in Frost's terms:

> All metaphor breaks down somewhere. That is the beauty of it. It is touch and go with the metaphor, and until you have lived with it long enough you don't know when it is going. You don't know how much you can get out of it and when it will cease to yield. It is a very living thing. It is as life itself.

Hardy's metaphors irritate Wittgenstein. When Hardy speaks of figures that "look and behave like a real line," Wittgenstein's marks such passages with, simply, "Unsinn." "Deny yourself the metaphor and then you'll see what's really going on with the figure," Wittgenstein's seems to be saying. "See things as they are!"

With predicativity or intutionism there is another coping mechanism on the table for the non-extensionalist, and this is the axiomatic approach to the extensionalist view, advocated by Hilbert and others. But Wittgenstein is not at all concerned with axiomatics in these annotations, and indeed, to be fair, neither is Hardy in CPM—or at least not to the extent that he should be, perhaps. The authors view the omission of axiomatics as a "severe defect" of Wittgenstein's considerations here. In an insightful passage the content of which is also fleshed out later in the book, the authors suggest that non-extensionalism and axiomatics in Hilbert's sense, somehow "speak past eachother":

> By 1942-1943 Wittgenstein was aware of the analysis that Turing gave of the general notion of a formal system in his famous paper "On Computable Numbers, with an Application to the Entscheidungsproblem" (Turing 1936). In analyzing the notion of taking a step in a formal system, Turing adopted in this paper a non-extensionalist perspective, treating "computable" real numbers on the basis of a "comparison" between a human calculating with pencil and paper and the step-by-step processes of a machine... These discussions [with Watson and Turing JK] had an impact on Wittgenstein's thinking between 1937 and 1942-1943, among the most important of which was Wittgenstein's drawing in the notions of form of life [Lebensform] and "technique" [Technik] into his later philosophy, and applying these to topics in the foundations of mathematics, ultimately the problem of what it is to follow a rule. Wittgenstein may have assumed that he could treat the issue of axiomatization under the broad heading of "following a rule", or "taking a step in a formal system" conceived of as computation, where the notion of computation, as analyzed by Turing, would be founded upon a "comparison" between a human being following a circumscribed recipe with symbols and a "step" in a formal system of logic. If so, then the importance of the everyday fact that we must take up or accept or take in or understand a rule would be deeply embedded in his approach, at the foundation, so to speak, lying closer to the heart of the matter than broader and more mathematical questions about the axiomatic method.

The authors close the chapter with the tantalizing remark that "As his philosophy matured, moreover, Wittgenstein came to emphasize the lack of any sharp conceptual dichotomy between step-by-step proceduralizing in formal and informal languages."

Chapter three treats the first block of annotations, contextualising and connecting them to passages in his other major texts, and also to material in the Nachlass. The problem of applications, the question of sense and nonsense, the idea of the workings of language; the "beautiful question," they cite Wittgenstein asking, of "How does the working wheel pass into an idle one?""[11] are put to use in laying out what the annotations give us, but also what they don't. For if attending to the uses we put language is the philosophical move we *must* make, then how to account for pure mathematics, mathematics out of "mufti," out of its civilian guise? The authors see Wittgenstein regarding not only set theory but the analysis of CPM, and indeed logicist foundations as having the "character of a collection of idle wheels"—and they take him to task over it.

Section 3.4 takes up Wittgenstein's emphatically critical remarks on Hardy's proof of the irrationality of the square root of two, or, again, the prose around the actual proof. As an aside, throughout the text the authors point out a critical distinction for Wittgenstein, between "prose" and *calculus*, the latter being, for Wittgenstein, the true locus of meaning—in a special, wholly contextualized sense of "meaning." As Mühlhölzer points out later in the text:

> Rather, he wants to point out that what we consider to be "the usual meanings" relies on meaning-determinations based on deliberately chosen methods, and not on entities given in advance which we approach and grasp by means of our methods.[12]

Returning to section 3.4, needless to say all the talk of "invincible feelings," of "common sense," and even of "supposing, if possible" are "cowardly evasions," for Wittgenstein. (Instead of "supposing, if possible," one should instead use "try," or to "check.") Such language works toward solidifying a conception Wittgenstein wants to push against, namely the idea that one establishes, with a proof, a static statement, a single fact, with a unitary meaning. Instead, what Wittgenstein wants to say is that different proofs, different paths to a proof, yield different senses of e.g. the statement that $\sqrt{2}$ is irrational. From the *Nachlass* manuscript MS 126:

> What troubles me in an exposition like, e.g., Hardy's is the seemingly useless variety of proofs of one and the same sentence. I

[11] p. 47
[12] p. 141

> want to say: each of these proofs belongs to a particular occasion where it has to be precisely applied.[13]

Stepping outside of the deep and specifically Wittgensteinian concerns of Floyd and Mühlhölzer's text for a moment, Wittgenstein's criticism of Hardy here is puzzling. It is an essential feature of mathematical "occasions," or contexts, that they are *porous*, they leak into eachother. A proof in set theory, for example, that a partial order has the countable chain condition, may well draw on the context of topology. The proof of the Prime Number Theorem involved complex integration, originally, but then Erdös and Selberg gave a proof of it in the context of elementary number theory. A notable example is Fermat's Last Theorem, which emerges in the context of elementary number theory, but whose proof draws on many different sophisticated techniques from the contexts of algebraic geometry. As for the idea that statements have different senses, this would introduce chaos into mathematics—a brutal simplification perhaps, but one which gestures at the shared conviction among mathematicians that mathematics is a descriptive science—difficult as this conviction is to make any sense of *philosophically.*

The situation of having many proofs of a theorem lies in constrast with theorems in which a variety of proofs are *not* admitted; occasions in which the particulars of the given context *matter too much.* A detail, for example, slight changes in which will cause the proof to fail, so that other avenues to the proof are closed off; other ways of looking at things are shut down. The set-up, in short, is off. In such a situation one is tempted to speak of instability, or a lack of robustness; whereas the existence of a variety of proofs points toward stability, towards robustness and invariance—towards the mathematically *substantive.*

For Wittgenstein one senses that this is pulling the camera too far back ("invariance"?). What Wittgenstein's wants us to see is the *point* of the proof; to extract from it a *procedure.* So if instead of letters we substitute numbers in Hardy's proof of the irrationality of $\sqrt{2}$, which Wittgenstein actually does in the annotations, and what Floyd and Mühlhölzer call "a strange transformation of Hardy's proof," we will see that wherever we start we will be led to a series of inferences involving smaller and smaller numbers until we are finally confronted with an impossible situation. Alternative proofs bring a different series of inferences to the fore, leading to different

[13] p. 56

impossibilities—or to no impossibilities, in the case that a contradiction is not what is sought in that particular path to a proof.

Hardy simply calls an alternative proof "interesting," but doesn't say why. Wittgenstein is bothered: "But in what way *is* it interesting?"[14] The suggestion here is that Hardy is relying on a shared set of convictions regarding the value of alternative proofs. Proofs do have a point, and Wittgenstein is right that we should live them and live through them, instead of wrapping them in possibly misleading prose. But there is also the larger picture; the invincible feeling, to use Hardy's odious—for Wittgenstein—terminology, which is offstage and slightly out of view, and decidedly *not* part of the practice, but at the same time essentially guiding it, that something like invariant mathematical content exists, in Gödel's words, independently of our definitions and constructions; independently of what *we* do. The incorrigible feeling mathematicians have about their work, that mathematical truths are not occasion sensitive, or perspective dependent, or true in a framework, or whatnot. The profundity of Wittgenstein's point of view is that he does not see coming at invariance directly, or truth, as profitable for the philosopher, as the proper subject for philosophy. In fact, one senses that for Wittgenstein it would almost be in poor taste to bring this up, and this is not a completely unsupportable way of looking at things.

Floyd and Mühlhölzer do not defend Wittgenstein where he cannot be defended, and indeed the complete lack of hagiography is one of the virtues of their text. Floyd and Mühlhölzer's insight here is to see this attention to the life of a proof, if one may put it this way, as laying down a path to Wittgenstein's later work, in particular to the project of "ushering Russell's notion of acquaintance back to the everyday."

> There are a variety of contexts in which mathematicians put words, proofs, particular numbers and theorems to work. This is what is at issue in Wittgenstein's annotations to CPM.
>
> As we have repeatedly emphasized, it is central for Wittgenstein to carefully keep track of the difference between words, sentences, signs and pictures that are being used to a purpose of some kind or other, and those that we—in our treatment and conceptions of them—have made non-useful... Given this point, we can see why a great deal of philosophical argumentation in the later Wittgen-

[14] p. 56

> stein is devoted to criticizing the idea that words, symbols, actions, thoughts or signs must be seen to be immediately and intrinsically meaningful, sans context.[15]

Floyd and Mühlhölzer's chapter four analyzes the possible alignment of Wittgenstein and Brouwer on the issue of the acceptance of the law of the excluded middle (LEM), and the subtle but nevertheless insurmountable differences between them on this. This is a delicate issue for the logician, who may easily read Wittgenstein as advocating the outright denial of LEM as a normative principle. For if mathematics is to be seen as a set of concrete calculi, calculi based on the moves human beings make, mathematically—a collection of everyday *activities*, if you like—then what place can there possibly be for the idea that every proposition is either true or false? It would seem that validity should hang wholly on proof, rather than any commitment to what appears to be a rather heavy, *philosophical* conception of truth, namely the bivalent conception. In other words, constructive mathematics would seem to be the perfect implementation of Wittgenstein's world view.

Wittgenstein finesses the issue: the acceptance of the LEM takes us *outside* of mathematics, meaning, there is no sense of construing the law as valid *mathematically*:

> But does this mean that there is no such problem as: "Does the pattern ϕ occur in this expansion?"?—To ask this is to ask for a rule regarding the occurrence of ϕ. And the alternative of the existence or non-existence of such a rule is at any rate not a mathematical one.[16]

One might infer that staying *inside*, staying within the domain of mathematics, entails adopting constructive mathematics in some form. But this also takes us outside, away from focussing on what it is we want to do with our mathematics in the first place—once again, a *philosophical* distraction. As Floyd and Mühlhölzer remind us, Wittgenstein is not an adherent of any foundational school or ism, and the conviction was emphatically held. In his 1939 Cambridge lectures on the foundations of mathematics he explicitly said: "[It] will be most important not to interfere with the mathematicians"[17]

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[15] p. 65

[16] *Remarks on the Foundations of Mathematics* V, 20. Quoted on p. 73 of the text.

[17] *Lectures on the Foundations of Mathematics*, quoted on p. 63 of the text.

There is more to be said here, about the terms "inside" and "outside." Floyd and Mühlhölzer map "inside" and "outside" onto Wittgenstein's distinction between the material of an edifice, and its scaffolding:

> This logical structure of the "law of excluded middle" is then, as Wittgenstein often put it throughout his life, a kind of scaffolding for the erection of an edifice. It is not part of the building blocks, elements, and materials that will be needed to structure the deciding of the problem and the assertion of a mathematical result, but something more modular, which we will bring to bear on the construction to help build it as we wish.
>
> As philosophers we should be sensitive to this: in mathematics one must wait until a new mathematical edifice is erected in order to make such a decision, assertion, rule, and so on . . . The picture of the law of excluded middle alone is not enough to get at our working decisions and proofs, or to clarify them very far.[18]

This undermines the idea of LEM as a rule one should adopt, or not, as settings in which the rule may have obtained can collapse into situations in which a decision can be made after all. In other words, one has obtained a concrete proof of p, or a concrete proof of its negation. In that case, Wittgenstein will say that the rule has changed its status, and we are no longer *outside* mathematics, but wholly within.

Chapter five treats the next block of annotations, and on their basis stages a confrontation between Wittgenstein's non-extensionalism and the Dedekind construction of the real numbers, through the notion of a cut. At issue is "Hardy's seemingly unruffled transition from cuts that are defined by specifically given singular numbers—unproblematic from the non-extensionalist point of view—to a general notion of "all cuts", imagined as utterly uncoupled from all specific ways of producing them."[19]

As the authors note, Hardy does gesture at the transition at the beginning of the section in question, but this falls short of what needs to be be said, for Wittgenstein. The issue turns on the notion of variable, as opposed to the objects in the variable's domain of quantification. And indeed Hardy's exposition here is a bit odd, including as it does phrases like "policeman x,

[18] p. 73
[19] p. 89

the driver of cab x, the year x, the xth day of the week"[20] together with variables x ranging over the real or natural numbers. In fact it is something of an achievement—for Wittgenstein, and through him, the authors'—that such locutions as "policeman x" begin to sound odd to the reader. These examples from "ordinary life"—another phrase which starts to induce vertigo in the reader—were expunged from later versions of Hardy's text. This is an important fact in itself as well as evidence of Floyd and Mühlhölzer's amazing attention to detail.

In any case it is here with the Dedekind construction of the real numbers—an impredicative construction, as Weyl pointed out—that the rubber meets the road; that the hard surface of extensionalism comes up against Wittgenstein's philosophical sensibility. Again, the problem for Wittgenstein is not the impredicativity of the construction, the problem rather lies in the fact that "Hardy's geometrical imagery lets the extensionalist standpoint appear all too natural."[21] The authors criticize Hardy for this, and more generally for his "all too unreflective step into extensionalism." There is some tension here: classical analysis as it was practiced then and is practiced now is entirely and unapologetically extensional, modulo certain hesitations coming the fringes, e.g. from people like Kroneker and others—so how could Hardy have done otherwise in his presentation of the real numbers? One doesn't pick up CPM in order to learn about impredicative definitions. The tension is resolved by reminding oneself of what the authors urge throughout, that Hardy's move into extensionalism is a move the *philosopher* must notice; that is, the remarks in the annotations are for *philosophers*—if indeed the annotations are meant for anyone but himself.

It is also important to say at this point that the authors are of course aware of the importance of extensionalism in mathematics. As Floyd writes in her chapter eight:

> It is crucial to understand that we have the notion of extension in mathematics for a reason: to suppress the diversity of actual human techniques and procedures of proof used in informal mathematics of the real numbers. We do this in order to handle the infinite, as well as to separate the idea of order from that of collection.[22]

[20] p. 90
[21] p. 93
[22] p. 206

In a deeply profound and illuminating passage, Floyd and Mühlhölzer instruct us in the depth of Wittgenstein's qualms here:

> Here Wittgenstein sees danger in the idea that grammatical form *per se* can determine a certain sort of content, and danger also in the idea that a certain content determines altogether the form that is relevant to individuating it. He also worries about presuming that we are in final possession of all the forms we will ever need. He is wondering—here in connection with Dedekind—about the *grammatical* (in his broad sense, *logical*) significance of the fact that a mathematical procedure has been set up and works smoothly. The parallel he draws, and worries about, is this. Aristotelian logic's "subject-predicate" structure works smoothly with its procedures. But what about the further idea that its syllogistic forms are empty "forms", unprejudicial and available for extension into every possible context? This is truly a dangerous assumption. . . one could also equally well wonder whether his [i.e. Frege's JK] notion of "concept"—or indeed Dedekind?s notions of "system" and "cut"— will serve foundationally, so to speak, all possible extensions into the future. Frege's unbridled notion of concept got him into trouble, after all, with his Axiom V: there are dangers of contradictions. More generally, there is something portentous and perhaps ill-omened about the intrusion of logic into mathematics: the specific differences and similarities among the variety of mathematical techniques may be obscured or mis-categorized or glossed over.[23]

In a footnote Floyd and Mühlhölzer establish a link between Wittgenstein's thought here with that of his notorious remark in *Remarks on the Foundations of Mathematics* concerning the ""disastrous invasion" of mathematics by logic." This is Wittgenstein's abhorrence of the idea that a single conceptual/logical scheme ought to be superimposed on a dynamic practice, one that is above all else continually *made* and *remade*, revisited, recontextualized, reconceptualized, remodeled, endlessly torn down and rebuilt—mathematics, in other words:

> . . . when a totalizing perspective sets itself up, one had better be careful about assuming that one is in possession of one, overar-

[23]p. 94

> ching way of thinking that is now necessarily fundamental. One must take into account the uses of one's own words when enunciating a logical distinction and make sure the perspective is thoroughgoing.[24]

This block of notations exhibits Wittgenstein at his most biting, with talk of Hardy's "crudeness," or of the "prudishness" of a line of proof. "A line of proof is prudish," Wittgenstein writes, "if one anxiously avoids the least logical ambiguity, but tolerates crude nonsense."[25] Floyd and Mühlhölzer cash out "nonsense" here in terms of impredicativity, and indeed this is an interesting way to think about impredicativity, as a form of nonsense. But they suggest that Wittgenstein's own gloss on the term "prudishness" be replaced by their own:

> ... the word "prudish" appears quite appropriate if it is understood from Wittgenstein's own, non-extensionalist point of view. And in this sense it is a better word for him to have used about Hardy's proof than what is said in his own gloss on the term itself... The proof of Dedekind's theorem, in whatever form, has without doubt the characteristic abstractness expressed in Dedekind's phraseology which, as we have seen, is uncoupled from the familiar invigorated and enriching life of individual numbers like 2 and π in a variety of mathematical areas. In this sense it can be called "prudish", if one conceives of it as lending rigor and precision to an otherwise flabby or unrigorous mode of procedure.[26]

Floyd and Mühlhölzer's comparatively brief Chapter six, treating the last block of annotations, deals with Hardy's presentation of functions, limits and continuity. As one might expect, Hardy's talk of our "common-sense idea of continuity,"[27] which is cashed out in terms of the standard $\epsilon - \delta$ definition of continuity, is looked at by Wittgenstein from the non-extensionalist perspective. As they remark,

> Here we see that Wittgenstein's criticism is the same as in the previous annotation: Hardy has a tendency to skip the non-extensional aspects of our mathematical practice which involve

[24]pp. 95-96
[25]p. 107
[26]p. 108
[27]p. 117

> the methods we use in order prove the statements at hand (for example the statement 'n^2 is large when n is large').[28]

The authors are right to see Wittgenstein reacting to the oddness of capturing something so concrete as "line without breaks" with a definition as odd, at least for the student, as the $\epsilon - \delta$ definition of continuity. (One wonders what Wittgenstein would have had to say about non-standard analysis, presumably it would have been more to his liking.)

The final two chapters consist of separate papers by Mühlhölzer (chapter 7) and by Floyd (chapter 8), leaving Wittgenstein's annotations to CPM behind. Mühlhölzer's paper is primarily concerned with the issue of uncountability in connection with Cantor's diagonal proof as taken up by Wittgenstein in *Remarks on the Foundations of Mathemtics II*, in particular Wittgenstein's provocative beginning: "It means nothing to say: 'Therefore the X numbers are not countable'".

> Here Wittgenstein talks about someone who has understood Cantor's diagonal method, seeing that to any list of decimal expansions of real numbers a further expansion of a real number exists, the "diagonal number", that is not in the list. But when then saying: "Therefore the real numbers are not countable", one simply repeats what was already clear: that to any list of decimal expansions of real numbers a further expansion of a real number exists, viz. the diagonal number, that is not in the list. Consequently, the "Therefore" means nothing. Obviously, Wittgenstein here accepts the notion of uncountability, and he in particular accepts the uncountability of the real numbers. However, he doesn't understand this notion simply in the sense of the negation of the well-known notion of countability, because this alone would only be a meagre formal scaffold that lets us down completely if we look for real mathematical substance. So, the concept of uncountability should be understood with explicit reference to the diagonal method... But then the essence of this method does not consist in letting us discover new mathematical facts but merely in the invention of a conceptual novelty.[29]

[28]p. 121

[29]p. 125

Here we learn about Wittgenstein's distinction between essential and non-essential numbers,[30] about Wittgenstein's notion of discovery; and again, about the paradoxical effect of Cantor's proof of the uncountabilty of the reals, or more precisely a weird asymmetry of it: on the one hand, the proof turns on such a simple trick—and yet at the same time it yields a deep result on the other.

There is much to consider in this chapter, but due to limitations of space we concentrate on the set-theoretic part of Mühlhölzer's chapter. To this end, we take Wittgenstein's definitions of countability and uncountability, respectively (C) and (U), to be the standard ones. Thus a set M is countable if there is a surjection of the natural numbers onto it; M is uncountable if no such surjection exists. Mühlhölzer sees Wittgenstein's objections to the proof having to do with the fact that misleads us about what in mathematics has substance, and what doesn't:

> The notion of *countability* has mathematical substance far beyond its definition in the meager set theoretical language as presented in (C), because there are so many cases of countable sets in which the relevant functions f from $\mathbb{N}$ onto M can be given by easily usable algorithms, for example, if M is the set of the rational or of the algebraic numbers. In these cases there are numerous methods of enumeration that show a pleasant sort of mathematical concreteness. This, however, is totally different in the case of (U), if one doesn't already know Cantor's diagonal method. In this case, we merely have the set theoretic definition, which is only a meager formal scaffold that, considered as such, lets us down completely if we look for real mathematical substance, for example, when we aim at a proof of the uncountability of the set of real numbers.[31]

Mühlhölzer's point, and Wittgenstein's, is that prior to the emergence of Cantor's diagonal method, the concept of uncountability is *empty*:

> According to Wittgenstein, before Cantor's invention of the diagonal method the concept of uncountability, in the plain sense of "not countable", was not a *general* concept but an *empty* one.

[30] p. 237
[31] p. 156

> Conceptual emptiness is not the same as conceptual generality. And even by Cantor's method this concept does not gain real "generality"—one would need *more special* methods than only this one—but it gains substance.[32]

Constructive mathematics looms large here. The uncountability of the real numbers is a constructive truth, however in constructive mathematics there is no theory of cardinality and therefore the conclusions drawn will differ sharply from those drawn in the classical case. But constructive mathematics, the great "road not taken" by Wittgenstein, is a non-starter here.

Lurking in the background of Wittgenstein's objections to Cantor's proof, and brought out cogently by Mühlhölzer, is Wittgenstein's general suspicion of set theory. This is Wittgenstein's view that the "calculus of sets" is too far removed from the symbolic practices of everyday mathematicians, their actual ways of devising proofs; the idea that set theory misses "the *life* of our signs in mathematics,"[33] if not the life of the subject altogether. Some of this is tied up with his notion of generality, of what it must mean for a concept to be a general concept; the prime place of *procedures* in Wittgenstein's view of mathematics, and the idea that a concept is in some sense "idle" if it doesn't give rise to the articulation of possible procedures that put the concept to work—that activate the concept as a working wheel.

Definition (U) of uncountability, for example, "completely leaves us at a loss as to possible proofs of uncountability," as Mühlhölzer remarks, interpreting this passage of Wittgenstein:

> What does the 'generality' of the calculus of sets ["des Mengenkalküls"] consist of? Not in the fact that it is a simple picture of a class of other ('special') calculi? Is it also general without a relation to them?
>
> . . . [a JK] so-called general mathematical investigation brings to light are truths that are then carried out by the more special investigations only in more detail? Whereas the general calculus is 'general' only by referring to special calculi. Because in mathematics nothing is in the *word* but everything in the calculus. That is, because the sign brings no other meaning into pure mathematics than what the calculus itself bestows on it.

[32] p. 162
[33] p. 160

> Mathematics doesn't consist of considerations. And a part of mathematics can be called 'abstract' only insofar as its application, albeit hinted at, is nevertheless left vague.[34]

There is also the problem, for Wittgenstein, of set theory imposing a kind of uniformity on mathematics, the problem we referred to earlier of imposing a logico-conceptual grid on a dynamic and living subject.

Wittgenstein, of course, was reacting to the set theory of his day. But even then—after all, constructions like Gödel's L-hierarchy were already known, as Mühlhölzer notes—it could perhaps have been seen that the importance of set theory does not lie in the specific *ways* it represents mathematical language, e.g. ordered pairs being represented by the Kuratowski notation, rather the importance of its modelling aspect lies in the simple fact that mathematical language *can* be so represented, and extremely economically, i.e. by the addition of a single non-logical constant to the first order predicate calculus. In fact, the modelling aspect of set theory functions with the same amount of robustness one looks for in so-called ordinary mathematics, in that a concept such as the ordered pair, or of real number, admits many different modelings, and set theory is indifferent to a choice of any one of them. (In fact this is pointed out by the authors on p. 86.)

As for Wittgenstein's second qualm, it has become clear that since the time of Wittgenstein's writings that the importance of set theory less in its foundational aspect, perhaps, than in its being a, or one might say, the, correct mathematical theory of the infinite. The sophisticated methodologies which have been developed since Wittgenstein's time, of forcing, of inner models, of infinitary combinatorics, all show us that the higher infinite evinces a beautiful and coherent structure—evinces, in a word, *substance.*

As for generality, one cannot blame Wittgenstein for not foreseeing how much there is to be mined, mathematically, from the general perspective, untethered from "procedures"; he is not to be blamed for missing the virtues of the general standpoint, and the windfall of mathematical opportunities offered therein. Shelah, for example, takes what seems to be an implausibly general standpoint in his *Main Gap Theorem*, but is nonetheless able to provide a powerful classification of *all* first order theories based simply on the number of (non-isomorphic) models of those theories, in different cardinalities.

[34] Wittgenstein, pp. 15-18 of *Nachlass* MS 162a, quoted on p. 157 of the text.

Floyd in chapter eight weaves together her work on aspect perception in Wittgenstein, on Wittgenstein's notion of "technique," of her examination of the Turing-Wittgenstein nexus and the cross fertilization between them; and on Wittgenstein's Diagonal Argument, so-called, which she, more than anyone, has brought into view in a series of incisive papers in the last decades. Writ large, her assessment of what Wittgenstein is up to (philosophically) in the annotations, is this:

> Wittgenstein's major objection to Hardy's presentation in CPM is that it fails to clearly convey the importance of the distinction between the non-extensionalist and the extensionalist perspectives. Hardy makes it seem as if the transition in perspectives is smooth, requiring no discussion of the prices paid for each or the subtlety of the transitions between them.

Floyd's achievement here is to show how Wittgenstein's preoccupations in the annotations flower into (something like a) resolution in Wittgenstein's mature philosophy; how all of this sets the stage for the emergence of Wittgenstein's all important concept of "forms of life," of his concept of "the everyday," how the Diagonal Argument in Wittgenstein's hands throws the concept of "following a rule" into relief; and how all of this is meant to sit apart from metaphysics as it is traditionally pursued, as well as other a priori styles of philosophizing:

> The narrative in what follows will offer a selective, targeted, historical reconstruction, designed to show that there is a sense of *necessity* in Wittgenstein's turn toward ordinary mathematical practice with real numbers, toward rules and techniques utilized in everyday life. The stress on the objectivity located in these—as opposed to the objecthood per se of the entities discussed—is a theme of the whole later Wittgenstein.[35]

Floyd's remarks here on Wittgenstein's notion of "technique," in contrast to "opinion" or "aspect," and the necessity of Wittgenstein having to "explore the sometimes murky area where technique begins, and where it ends,"[36] are especially interesting.

[35] p. 196
[36] p. 203

As Floyd writes, "[t]he diagonal procedure instantiates, in his view, such a murky place." Due to the lack of space we only consider her analysis of Wittgenstein's Diagonal Argument, the detailed consideration of which she regards, indeed correctly, as the capstone of the authors reading of Wittgenstein's non-extensionalist approach to the construction of the real numbers.

Wittgenstein's 1947 Diagonal Argument[37] is Wittgenstein's gloss on Turing's "Do What You Do" argument, as Floyd aptly calls it, with Turing reacting to E.W. Hobson, and all of them reacting to Cantor. Turing's "Do What You Do" argument given in his 1936 paper reveals the hazards of blithely adapting Cantor's diagonal argument to the setting of computability. For trying to do this delivers us to a moment in the argument in which one must "do what you are doing," and one is now flummoxed. As Floyd writes, ""Do What I Do" . . . is an empty command, or idling wheel, for [the machine JK] H. It is not of course contradictory, but it cannot be followed with any particular step."[38] Turing's is an ingenious piece of analysis and it leads to a correct proof of the negative solution of the *Entscheidungsproblem*.

Wittgenstein extracts philosophical gold from the "Do What You Do" argument, drawing, for Floyd, the ur-Wittgensteinian moral that "an order makes sense only in certain positions [*Stellen*]."[39] As Floyd writes:

> A way to put this in Wittgensteinian terms is that it is part of our concept of a *rule* that we know what in particular to do with it in a particular given situation, we know how to fashion a technique for embedding a rule, word or routine in life. . . his 1939 Cambridge lectures on the foundations of mathematics that Turing attended [LFM], Wittgenstein stressed this point, hammering home the importance of our fashioning of techniques over and over again, not as a psychological or mathematical, but rather as a logical fact.[40]

Floyd and Mühlhölzer's final Chapter 9 reproduces the annotations, so that we can see all the remarks, the underlinings, the essential "Unsinn" on p. 4 of CPM, for ourselves—an appropriate way to end the book.

[37] The argument is given in Wittgenstein's *Remarks on the Philosophy of Psychology* and earlier in MS 135. Quoted in the text on p. 257.

[38] p. 235

[39] Quotation on p. 256.

[40] p. 240

Floyd and Mühlhölzer's text is philosophically powerful. As for those mathematicians who are often confused by Wittgenstein's cryptic or seemingly nonsensical remarks on mathematics, Floyd and Mühlhölzer's text goes a long way toward clearing up these confusions. Mathematicians may also be taken aback by the seemingly ad hominem character of some of Wittgenstein's remarks about Hardy in these annotations.[41] But this is outweighed by the respect for the subject Wittgenstein displayed e.g. in his 1939 Cambridge lectures, with his admonition to philosophers, unheeded by all too many, "not to interfere with the mathematicians."

Hardy, of course, hardly needs defending. One must also keep in mind the problem in historical work generally, in philosophy, of how to handle philosophical esoterica—whether it be that of Leibniz's, or Husserl's, or, and this is an especially difficult case, Gödel's so-called *Max Phil* notebooks. One of the most difficult cases of all has been that of the Wittgenstein *Nachlass*. We see, in the able hands of Floyd and Mühlhölzer, just how to negotiate our way around Wittgenstein's *Nachlass*—just how it is, that one can *find one's way about*.

[41]See p. 127.