Biconic semi-copulas with a given section

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Abstract

Inspired by the notion of biconic semi-copulas, we introduce biconic semi-copulas with a given section. Such semi-copulas are constructed by linear interpolation on segments connecting the graph of a continuous and decreasing function to the points $(0,0)$ and $(1,1)$. Special classes of biconic semi-copulas with a given section such as biconic (quasi-)copulas with a given section are considered. Some examples are also provided.

Keywords: Biconic copula, Conic copula, Quasi-copula, Copula, Linear interpolation

1. Introduction

Semi-copulas have recently gained importance in several areas of research, such as reliability theory, fuzzy set theory and multi-valued logic [6, 10, 12]. Special classes of semi-copulas, such as quasi-copulas and copulas, are widely studied. For instance, quasi-copulas appear in fuzzy set theoretical approaches to preference modeling and similarity measurement [3, 4, 5]. Due to Sklar's theorem [23], copulas have received ample attention from researchers in probability theory and statistics [14].

Recall that a semi-copula [8, 9] is a function $S : [0,1]^2 \to [0,1]$ satisfying the following conditions:

(i) for any $x \in [0,1]$, it holds that

$$S(x,0) = S(0,x) = 0, \quad S(x,1) = S(1,x) = x\,;$$

(ii) for any $x, x', y, y' \in [0,1]$ such that $x \le x'$ and $y \le y'$, it holds that $S(x,y) \le S(x',y')$.

In other words, a semi-copula is nothing else but a binary aggregation function with neutral element 1. The functions $T_{\mathbf{M}}$ and $T_{\mathbf{D}}$ given by $T_{\mathbf{M}}(x,y) = \min(x,y)$ and $T_{\mathbf{D}}(x,y) = \min(x,y)$ whenever $\max(x,y) = 1$, and $T_{\mathbf{D}}(x,y) = 0$ elsewhere, are examples of semi-copulas. Moreover, for any semi-copula S the inequality $T_{\mathbf{D}} \le S \le T_{\mathbf{M}}$ holds.

A semi-copula Q is a quasi-copula [11, 13, 20] if it is 1-Lipschitz continuous, i.e. for any $x, x', y, y' \in [0,1]$, it holds that

$$|Q(x',y') - Q(x,y)| \le |x' - x| + |y' - y|\,.$$

A semi-copula C is a copula [1, 21] if it is 2-increasing, i.e. for any $x, x', y, y' \in [0,1]$ such that $x \le x'$ and $y \le y'$, it holds that $V_C([x,x'] \times [y,y']) :=$

$$C(x',y') + C(x,y) - C(x',y) - C(x,y') \ge 0\,.$$

$V_C([x,x'] \times [y,y'])$ is called the C-volume of the rectangle $[x,x'] \times [y,y']$. The copulas $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$, with $T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$, are respectively the greatest and the smallest copula, i.e. for any copula C, it holds that $T_{\mathbf{L}} \le C \le T_{\mathbf{M}}$. Another important copula is the product copula Π defined by $\Pi(x,y) = xy$.

To increase modelling flexibility, new methods to construct semi-copulas, quasi-copulas and copulas are being proposed continuously in the literature. Several of these methods have been introduced starting from given sections. Such sections can be the diagonal section and/or the opposite diagonal section [2, 15, 17], or a horizontal section and/or a vertical section [7, 19, 22]. All the above methods have used sections that are determined by straight lines in the unit square such as the diagonal, the opposite diagonal, a horizontal line or a vertical line. In the present paper, we consider sections that are determined by a curve in the unit square that represents a continuous and decreasing function.

For any continuous and decreasing $[0,1] \to [0,1]$ function f with $f(0) = 1$ and $f(1) = 0$, the surface of the semi-copula $T_{\mathbf{M}}$ is constituted from (linear) segments connecting the points $(0,0,0)$ and $(a, f(a), f(a))$ as well as segments connecting the points $(a, f(a), f(a))$ and $(1,1,1)$, with $f(a) \le a$, and segments connecting the points $(0,0,0)$ and $(a, f(a), a)$ as well as segments connecting the points $(a, f(a), a)$ and $(1,1,1)$, with $f(a) \ge a$. This observation has motivated the present construction.

This paper is organized as follows. In the following section, we introduce biconic functions with a given section. In Section 3, we characterize the classes of biconic semi-copulas with a given section and biconic quasi-copulas with a given section. In Section 4, we also characterize under some additional assumptions the class of biconic copulas with a given section. Finally, some conclusions are stated.

2. Biconic functions with a given section

In this section we introduce the definition of a biconic function with a given section. We denote the *(linear) segment* with endpoints $\mathbf{x}, \mathbf{y} \in [0,1]^2$ as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \{\theta \mathbf{x} + (1-\theta)\mathbf{y} \mid \theta \in [0,1]\}\,.$$

We denote the set of continuous and strictly decreasing functions $f : [0,1] \to [0,1]$ that satisfy $f(x) \leq 1-x$ for any $x \in [0,1]$ as $\mathcal{U}$.
Let $f \in \mathcal{U}$, $f(0) = d'$ and d be the smallest value in $[0,1]$ such that $f(d) = 0$. We introduce the following notations

$$\begin{aligned} S_f &= \{(x,y) \in [0,d] \times [0,1] \mid y < f(x)\} \\ \Delta_{d'} &= \Delta_{\{(0,d'),(0,1),(1,1)\}} \\ \Delta_d &= \Delta_{\{(d,0),(1,1),(1,0)\}} \\ F_f &= [0,1]^2 \setminus (S_f \cup \Delta_{d'} \cup \Delta_d)\,. \end{aligned}$$

The sets S_f and F_f as well as the triangles $\Delta_{d'}$ and Δ_d are depicted in Figure 1. Let C be a semi-copula and $g_C : [0,1] \to [0,1]$ be defined by $g_C(x) = C(x, f(x))$. Then the function $A_{f,g_C} : [0,1]^2 \to [0,1]$ defined by $A_{f,g_C}(x,y) =$

$$\begin{cases} \dfrac{g_C(x_0)}{x_0} x & \text{, if } (x,y) \in S_f \setminus \{(0,0)\}\,, \\ 1 - \dfrac{1-g_C(x_1)}{1-x_1}(1-x) & \text{, if } (x,y) \in F_f \setminus \{(1,1)\}\,, \\ \min(x,y) & \text{, otherwise}\,, \end{cases} \tag{1}$$

where $(x_0, f(x_0))$ (resp. $(x_1, f(x_1))$ is the unique point such that (x,y) is located on the segment $\langle (0,0), (x_0, f(x_0)) \rangle$ (resp. $\langle (x_1, f(x_1)), (1,1) \rangle$), is well defined. The function A_{f,g_C} is called *a biconic function with section* (f, g_C) since $A_{f,g_C}(t, f(t)) = g_C(t)$ for any $t \in [0,1]$, and since it is linear on each segment $\langle (0,0), (t, f(t)) \rangle$ on S_f as well as on each segment $\langle (t, f(t), (1,1) \rangle$ on F_f.

Note that for $g_C(x) = 0$, the class of binary conic functions is retrieved [18]. Note also that for $f(x) = 1-x$, the class of biconic functions with a given opposite diagonal section is retrieved [16].

Let us introduce, for a biconic function A_{f,g_C}, the functions $\varphi_f, \widehat{\varphi}_f, \psi_{g_C}, \widehat{\psi}_{g_C} :]0,d[\to \mathbb{R}$ defined by

$$\varphi_f(x) = \frac{x}{f(x)}\,, \quad \widehat{\varphi}_f(x) = \frac{1-x}{1-f(x)}\,,$$

$$\psi_{g_C}(x) = \frac{g_C(x)}{x}\,, \quad \widehat{\psi}_{g_C}(x) = \frac{1-g_C(x)}{1-x}\,.$$

These functions will be used along the paper.

3. Biconic semi-(resp. quasi-)copulas with a given section

Here, we characterize the class of biconic semi-(resp. quasi-)copulas with a given section.

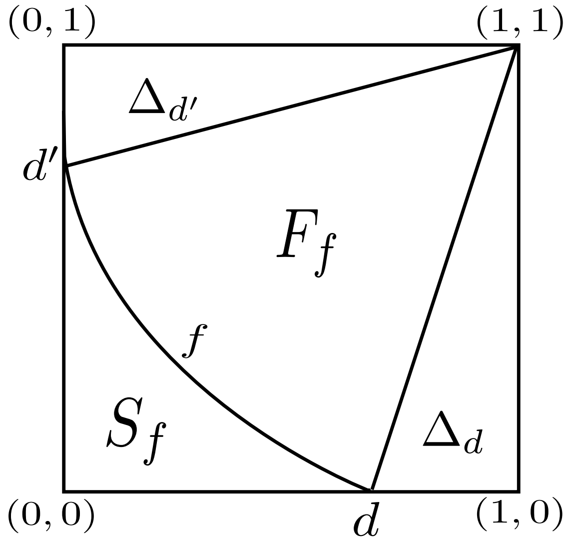

Figure 1: Illustration of the sets F_f and S_f as well as the triangles Δ_d and $\Delta_{d'}$.

Proposition 1 *Let $f \in \mathcal{U}$ and C be a semi-copula. The function A_{f,g_C} defined in (1) is a semi-copula if and only if*

(i) *the functions $\varphi_f \psi_{g_C}$ and ψ_{g_C} are increasing and decreasing, respectively;*
(ii) *the functions $\widehat{\varphi}_f \widehat{\psi}_{g_C}$ and $\widehat{\psi}_{g_C}$ are decreasing and increasing, respectively.*

Proposition 2 *Let $f \in \mathcal{U}$ and C be a quasi-copula. The function A_{f,g_C} defined in (1) is a quasi-copula if and only if*

(i) *the conditions of Proposition 1 are satisfied;*
(ii) *the functions $\frac{1}{\varphi_f} - \psi_{g_C}$ and $\varphi_f(1 - \psi_{g_C})$ are decreasing and increasing, respectively;*
(iii) *the functions $\frac{1}{\widehat{\varphi}_f} - \widehat{\psi}_{g_C}$ and $\widehat{\varphi}_f(1 - \widehat{\psi}_{g_C})$ are increasing and decreasing, respectively.*

Example 1 *Let $f \in \mathcal{U}$ and $C = T_{\mathbf{L}}$. One easily verifies that the conditions of Proposition 1 are satisfied and the corresponding biconic function A_{f,g_C} is a semi-copula. On the other hand, A_{f,g_C} is a quasi-copula if and only if the functions $\frac{f(x)}{1-x}$ and $\frac{x}{1-f(x)}$ are decreasing and increasing on the interval $]0,d[$, respectively.*

Example 2 *Let $f \in \mathcal{U}$ and C be the product copula, i.e. $C = \Pi$. One easily verifies that the conditions of Proposition 2 are satisfied and the corresponding biconic function A_{f,g_C} is a quasi-copula, and hence a semi-copula. Consequently, when the considered semi-copula is Π, the class of biconic semi-copulas with a given section and the class of biconic quasi-copulas with a given section coincide.*

4. Biconic copulas with a given section

Next, we characterize for specific cases the class of biconic copulas with a given section. A function

$f : [0,1] \to [0,1]$ is called piecewise linear if its graph is constituted of segments. Next, we restrict our attention to the case when $C = \Pi$ and hence, $g_C(x) = xf(x)$ for any $x \in [0,1]$.

Proposition 3 *Let $f \in \mathcal{U}$ such that f is piecewise linear and let $C = \Pi$. The function A_{f,g_C} defined in (1) is a copula if and only if the functions φ_f and $\widehat{\varphi}_f$ are convex.*

Example 3 *Let $f : [0,1] \to [0,1]$ be defined by $f(x) = 1 - x$ for any $x \in [0,1]$, and let $C = \Pi$. One easily verifies that the functions φ_f and $\widehat{\varphi}_f$ are convex and the corresponding biconic function A_{f,g_C} is a copula, and is given by*

$$A_{f,g_C}(x,y) = \begin{cases} \dfrac{xy}{x+y} & , \text{ if } y \le 1-x\,, \\ \dfrac{xy-(x+y-1)^2}{2-x-y} & , \text{ otherwise.} \end{cases}$$

Using the same technique as in [18], Proposition 3 can be generalized for any element from $\mathcal{U}$.

Proposition 4 *Let $f \in \mathcal{U}$ and $C = \Pi$. The function A_{f,g_C} defined in (1) is a copula if and only if the functions φ_f and $\widehat{\varphi}_f$ are convex.*

Example 4 *Let $f : [0,1] \to [0,1]$ be defined by*

$$f(x) = \begin{cases} (1-2x)^2 & , \text{ if } x \le 1/2\,, \\ 0 & , \text{ otherwise,} \end{cases}$$

and let $C = \Pi$. One easily verifies that the functions φ_f and $\widehat{\varphi}_f$ are convex and the corresponding biconic function A_{f,g_C} is a copula.

Rather than using the product copula, we consider now any copula C but we will suppose that f is piecewise linear.

Proposition 5 *Let $f \in \mathcal{U}$ such that f is piecewise linear, and let C be a copula. Let $a \in [0,d]$ be the unique value such that $f(a) = a$. The function A_{f,g_C} defined in (1) is a copula if and only if*

(i) *for any $x_1, x_2, x_3 \in [0,d]$ such that $x_1 < x_2 < x_3$, it holds that*

$$\begin{vmatrix} 1-x_1 & 1-y_1 & 1-z_1 \\ 1-x_2 & 1-y_2 & 1-z_2 \\ 1-x_3 & 1-y_3 & 1-z_3 \end{vmatrix} \ge 0 \qquad (2)$$

and

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \ge 0\,, \qquad (3)$$

where $f(x_i) = y_i$ and $g_C(x_i) = z_i$ for any $i \in \{1,2,3\}$;

(ii) *the function $\xi : [0,a[\cup]a,d] \to \mathbb{R}$ defined by $\xi(x) = \frac{x-g_C(x)}{x-f(x)}$ is decreasing on the interval $[0,a[$ as well as on the interval $]a,d]$.*

In order to retrieve the class of conic copulas and the class of biconic copulas with a given opposite diagonal section, we need the following two lemmas.

Lemma 1 *Let f be a real-valued function defined on the interval $[a,b]$. Then it holds that f is convex if and only if*

$$\begin{vmatrix} 1-x_1 & 1-f(x_1) & 1 \\ 1-x_2 & 1-f(x_2) & 1 \\ 1-x_3 & 1-f(x_3) & 1 \end{vmatrix} \ge 0 \qquad (4)$$

holds for any $x_1, x_2, x_3 \in [a,b]$ such that $x_1 < x_2 < x_3$.

Lemma 2 *Let g be a real-valued function defined on the interval $[a,b]$. Then it holds that g is concave if and only if*

$$\begin{vmatrix} 1-x_1 & x_1 & 1-g(x_1) \\ 1-x_2 & x_2 & 1-g(x_2) \\ 1-x_3 & x_3 & 1-g(x_3) \end{vmatrix} \ge 0\,. \qquad (5)$$

holds for any $x_1, x_2, x_3 \in [a,b]$ such that $x_1 < x_2 < x_3$.

Remark 1

(i) *For $g_C(x) = 0$, inequality (2) is equivalent to the convexity of f (see Lemma 1) and hence, the class of conic copulas (when the upper boundary curve of the zero-set is piecewise linear) is retrieved [18].*

(ii) *For $f(x) = 1-x$, inequality (2) is equivalent to the concavity of g_C (see Lemma 2) and hence, the class of biconic copulas with a given opposite diagonal section (when the opposite diagonal section is piecewise linear) is retrieved [16].*

5. Conclusions

We have introduced biconic semi-copulas with a given section. We have also characterized the class of biconic quasi-copulas with a given section. Under some assumptions, we have characterized biconic copulas with a given section. Some known classes of semi-copulas, such as binary conic semi-copulas and biconic semi-copulas with a given opposite diagonal section, turn out to be special cases of biconic semi-copulas with a given section.

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References

[1] C. Alsina, M.J. Frank and B. Schweizer, *Associative Functions: Triangular Norms and Copulas*, World Scientific, Singapore, 2006.

[2] B. De Baets, H. De Meyer and M. Úbeda-Flores, Opposite diagonal sections of quasi-copulas and copulas, *Internat. J. Uncertainty, Fuzziness and Knowledge-Based Systems*, 17:481–490, 2009.

[3] B. De Baets and J. Fodor, *Additive fuzzy preference structures: the next generation*, in: Principles of Fuzzy Preference Modelling and Decision Making (B. De Baets and J. Fodor, eds.), Academia Press, 2003, pp. 15–25.

[4] B. De Baets, S. Janssens and H. De Meyer, On the transitivity of a parametric family of cardinality-based similarity measures, *Internat. J. Approximate Reasoning*, 50:104–116, 2009.

[5] S. Díaz, S. Montes and B. De Baets, Transitivity bounds in additive fuzzy preference structures, *IEEE Trans. Fuzzy Systems*, 15:275–286, 2007.

[6] F. Durante, E.P. Klement, R. Mesiar and C. Sempi, Conjunctors and their residual implicators: characterizations and construction methods, *Mediterranean Journal of Mathematics* 4:343–356, 2007.

[7] F. Durante, A. Kolesárová, R. Mesiar and C. Sempi, Copulas with given values on a horizontal and a vertical section, *Kybernetika*, 43:209–220, 2008.

[8] F. Durante, J.J. Quesada Molina and C. Sempi, Semicopulas: characterizations and applicability, *Kybernetika*, 42:287–302, 2006.

[9] F. Durante and C. Sempi, Semicopulae, *Kybernetika*, 41:315–328, 2005.

[10] F. Durante and F. Spizzichino, Semi-copulas, capacities and families of level sets, *Fuzzy Sets and Systems*, 161:269–276, 2010.

[11] C. Genest, J.J. Quesada Molina, J.A. Rodríguez Lallena and C. Sempi, A characterization of quasi-copulas, *Journal of Multivariate Analysis*, 69:193–205, 1999.

[12] P. Hájek and R. Mesiar, On copulas, quasicopulas and fuzzy logic, *Soft Computing*, 12:1239–1243, 2008.

[13] S. Janssens, B. De Baets and H. De Meyer, Bell-type inequalities for quasi-copulas, *Fuzzy Sets and Systems*, 148:263–278, 2004.

[14] H. Joe, *Multivariate Models and Dependence Concepts*, Chapman & Hall, London, 1997.

[15] T. Jwaid, B. De Baets and H. De Meyer, Orbital semilinear copulas, *Kybernetika*, 45:1012–1029, 2009.

[16] T. Jwaid, B. De Baets and H. De Meyer, Biconic aggregation functions, *Information Sciences*, 187:129–150, 2012.

[17] T. Jwaid, B. De Baets and H. De Meyer, Lower semiquadratic copulas with a given diagonal section, *Journal of Statistical Planning and Inference*, 143:1355–1370, 2013.

[18] T. Jwaid, B. De Baets, J. Kalická and R. Mesiar, Conic aggregation functions, *Fuzzy Sets and Systems*, 167:3–20, 2011.

[19] E. Klement, A. Kolesárová and R. Mesiar, Copulas constructed from horizontal sections, *Communications in Statistics: Theory and Methods*, 36:2901–2911, 2007.

[20] A. Kolesárová, 1-Lipschitz aggregation operators and quasi-copulas, *Kybernetika*, 39:615–629, 2003.

[21] R. Nelsen, *An Introduction to Copulas*, Springer, New York, 2006.

[22] J. Rodríguez-Lallena, A class of copulas with piecewise linear horizontal sections, *Journal of Statistal Planning and Inference*, 139:3908–3920, 2009.

[23] A. Sklar, Fonctions de répartition à n dimensions et leurs marges, *Publ. Inst. Statist. Univ. Paris*, 8:229–231, 1959.