A time-domain discretisation of Maxwell's equations in nontrivial media using collocated fields

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***Abstract*—In this paper we present an unconditionally stable time-domain discretisation of Maxwell's equations where the discretized fields are collocated. This allows us to combine Maxwell's equations, which are usually discretized on a staggered grid, with constitutive differential equations which may not be well-suited for staggered grids. This approach guarantees that the numerical dispersion relation always closely mimics the exact one.**

I. Introduction

In magnetized plasmas, constitutive differential equations occur which use all three components of the electric field at a certain position. This is difficult to combine with Maxwell's equations, which, for each of their scalar components, involve only derivatives of two electric/magnetic field components at a certain position, and are therefore ideally discretized on Yee-cell-like grids. Conditionally stable solutions to this problem based on interpolation between collocated and staggered positions have been proposed [1]. Here we will present an alternative approach.

FDTD does not work very well on collocated grids, because central-difference approximations of the derivatives never contain a term corresponding to the field at the position where the derivative is calculated. Naively using FDTD on a collocated grid simply results in several independent solutions.

II. A fully implicit approach

Our approach is fully implicit, both in space and in time. The electric fields, magnetic fields, and other degrees of freedom which may be necessary for a time-domain constitutive equation, are discretized on a collocated cartesian grid. The distance between neighbouring points in space is $\Delta_x = \Delta_y = \Delta_z = \Delta$ and in time is Δ_t.

Each cartesian component of Maxwell's equations is discretized independently. Each of these components is a scalar partial differential equation containing one temporal derivative and zero, one or two spatial derivatives (depending on the amount of spatial dimensions used). The terms in the equation are interpolated (in space and in time) to the center of a line/square/cube whose edges are along the directions along which the equation contains derivatives. Now suppose the equation contains the x-derivative $\frac{\partial}{\partial x}$ of a certain field, and other term(s) which do not contain x-derivatives. Then, the x-derivative becomes a central difference and introduces a factor $\sin(k\Delta/2)2/\Delta$ in the discrete dispersion relation. The other term(s) are all interpolated to the same position and thus introduce a factor $\cos(k\Delta/2)$. The total result is that k in the exact dispersion relation is replaced by $\tan(k\Delta/2)2/\Delta$ in the discrete dispersion relation. Similarly, ω in the exact dispersion relation becomes $\tan(\omega\Delta_t/2)2/\Delta_t$. Figure 1 illustrates this for the y-component of Faraday's law in 2D.

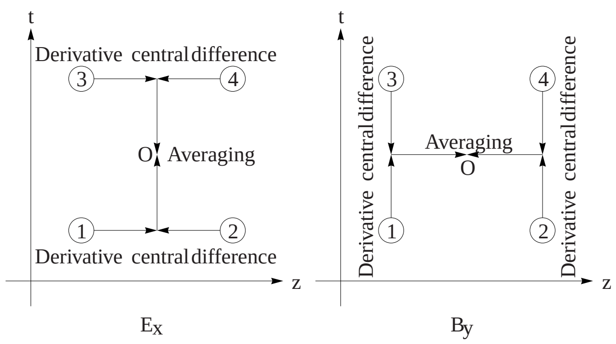

Fig. 1. Discretisation of the 1D equation $\frac{\partial B_y}{\partial t} = \frac{\partial E_x}{\partial z}$. The discrete equation expresses equality between a time-averaged space-derivative and a space-averaged time-derivative.

III. Time stepping

This method is implicit, and thus, every time step a sparse set of equations must be solved. For linear materials, the system to be solved is the same every time step, and a direct solver based on sparse LU decomposition can be used. In our experience, such a direct solver is significantly faster than an iterative solver and the space needed to store the LU factors appears to increase only linearly with the amount of discretisation points.

The system to be solved is always of the form

$$(S_A - S_D\Delta_t/2)V_{t+1} = (S_A + S_D\Delta_t/2)V_t \tag{1}$$

where V_{t+1} are the fields at the next time step, V_t are the fields at this time step, S_A is a spatial interpolation operator, and S_D is a spatial central difference operator.

The eigenvalues of $(S_A - S_D\Delta_t/2)^{-1}(S_A + S_D\Delta_t/2)$ lie on the unit circle for all Δ_t, i.e. our method is unconditionally stable. A detailed stability proof is outside of the scope of this paper.

IV. Cold Magnetized Plasma

The main motivation for this approach is its ability to faithfully reproduce complicated dispersion relations, such as the one for magnetized cold plasma. The equation describing currents in this medium is [1]

$$\frac{\partial \vec{J}_s}{\partial t} = \epsilon_0 \omega_s^2 \vec{E} - \vec{\Omega}_s \times \vec{J}_s \tag{2}$$

where s refers to the different particle species present in the plasma, ω_s is a characteristic plasma frequency related to the density, and $\vec{\Omega}_s = q_s \vec{B}_0 / m_s$ is the cyclotron frequency due to the background magnetic field. Note that the direction of $\vec{\Omega}_s$ is arbitrary: unless the coordinate system is such that $\vec{B}_0$ lies along a coordinate axis, $\frac{\partial \vec{J}_s}{\partial t}$ will depend on all three components of $\vec{J}_s$. This is why this equation is best discretized on a collocated grid.

V. Numerical Examples

A. Mode conversion

As a first example, we simulate 1D wave propagation in a cold plasma with position-dependent background magnetic field. The dispersion relation is shown in figure 2. Note how the long-wavelength fast wave connects to the much shorter ion-cyclotron wave (ICW). In figure 3, we see the appearance of this short wave in time-domain.

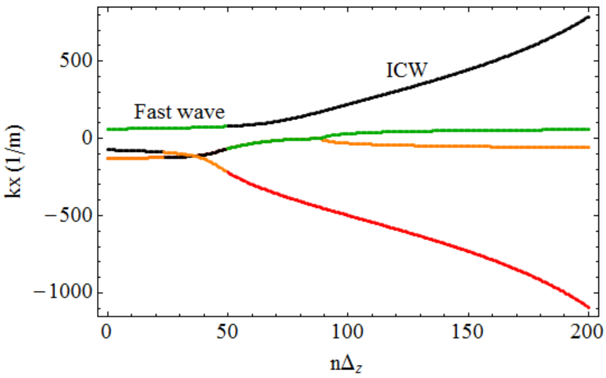

Fig. 2. Space-dependent dispersion relation describing mode conversion in a cold magnetized plasma.

B. Numerical reproduction of the exact dispersion relation of uniform cold plasma

In Fig. 4, the numerical dispersion relation for a plasma like that to the left of Fig. 2 is shown. We used periodic boundary conditions and calculated all eigenvectors and eigenvalues of the time-stepping operator. Thanks to uniformity and periodicity, the eigenvectors are pure complex exponentials, all with real k_x (imaginary or complex k_x cannot obey the periodic boundary conditions). The real roots of the exact continuous dispersion relation in infinite space are also shown, as well as the modified roots obtained by replacing k_x by $\tan(k_x \Delta_x/2)2/\Delta_x$ and ω by $\tan(\omega \Delta_t/2)2/\Delta_t$.

This shows quite convincingly that this method is in fact capable of faithfully reproducing rather complex dispersion relations.

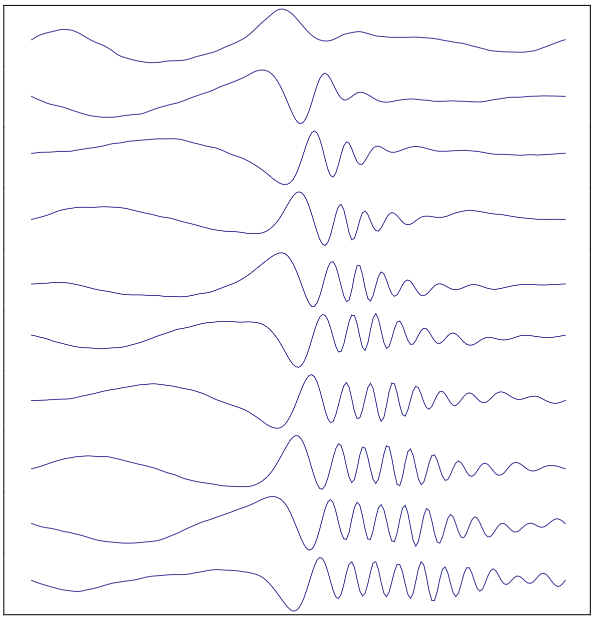

Fig. 3. E_x after 1000 (top),2000, . . . , 10000 (bottom) time steps, at $c\Delta_t = 100\Delta_x$.

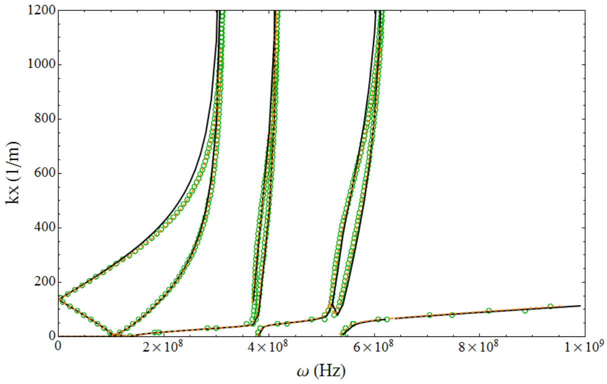

Fig. 4. Green : Numerical value of $|k_x|$ versus ω obtained by calculating the eigenvectors of the system matrix and Fourier transforming them in space. The obtained k_x is always real. Black: Analytically determined $|k_x|$ for propagating (real k_x) solutions of the exact dispersion relation. Orange: Analytically predicted discrete dispersion relation by replacing ω and k_x by the relevant tangent functions in the continuous dispersion relation.

VI. Conclusion

We have described an unconditionally stable discretisation of Maxwell's equations on collocated grids. This discretisation is especially usefull for the time-domain simulation of magnetized cold plasmas, whose constitutive equations are not well-suited for staggered grids. In the future, we will attempt to simulate far larger problems and/or different materials (such as "warm" plasmas).

References

[1] D. N. Smithe, "Finite-difference time-domain simulation of fusion plasmas at radiofrequency time scales," *Physics of Plasmas*, vol. 14, no. 14, pp. 2537–2549, Apr. 2007.