The "dottrick TDEFIE": a DC stable integral equation for analyzing transient scattering from PEC bodies

Francesco P. Andriulli*[1], Kristof Cools[2], Femke Olyslager[2], and Eric Michielssen[1]

[1] Department of Electrical Engineering and Computer Science
University of Michigan at Ann Arbor
[2] Department of Information Technology (INTEC), Ghent University

Introduction

Marching on in time, time domain integral equation solvers represent an increasingly appealing avenue for analyzing transient electromagnetic interactions with large and complex structures. Compared to their differential equation counterparts, these solvers automatically impose radiation conditions, do not require unknown fields to be discretized throughout homogeneous volumes, and are highly immune to numerical dispersion. Among their many incarnations, marching on in time, time domain electric field integral equation (MOT-TDEFIE) solvers remain the most widely used. Unfortunately MOT-TDEFIE solvers often are plagued by DC instabilities, i.e. constant and linear-in-time solutions of MOT-TDEFIE systems that reside in the null space of the (non-causal) TDEFIE operator. In the past, these instabilities have been partially cured by using loop-tree decompositions [1] and by enforcing boundary conditions on normal magnetic field components [2]. Unfortunately, neither technique completely annihilates the static null space of the TDEFIE operator, nor guarantees that MOT-TDEFIE solutions are free of DC remnants. In this paper, a modified TDEFIE that resolves static and linear-in-time currents is presented. The equation is obtained by leveraging the time domain Calderón identities in conjunction with the "dot-trick", viz. a careful rearrangement of temporal derivative operators appearing in sequences of TDEFIE operators. The effectiveness of the dottrick TDEFIE for both open and closed structures is demonstrated theoretically, proving the resonance free behavior of the dottrick TDEFIE, and numerically.

Formulation

Let Γ and $\hat{\boldsymbol{n}}_{\boldsymbol{r}}$ denote the surface of a perfect electrically conducting smooth object and its outward pointing unit normal at $\boldsymbol{r}$, respectively. Assume that Γ resides in a homogeneous medium with electric permittivity ϵ and magnetic permeability μ, and is illuminated by the electric field $\boldsymbol{E}^i(\boldsymbol{r}, t)$. The current density $\boldsymbol{J}(\boldsymbol{r}, t)$ induced on Γ in response to this excitation satisfies the TDEFIE

$$\frac{\partial \mathcal{T}(\boldsymbol{J})}{\partial t} = \frac{\partial (\mathcal{T}_s + \mathcal{T}_h)(\boldsymbol{J})}{\partial t} = -\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \frac{\partial \boldsymbol{E}^i(\boldsymbol{r}, t)}{\partial t} \tag{1}$$

where

$$\mathcal{T}_h(\boldsymbol{J}) = \left(\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \nabla \mathcal{R} \left(\int_0^t \frac{\nabla_s \cdot \boldsymbol{J}}{\epsilon} \, \mathrm{d}t \right) \right) \tag{2}$$

$$\mathcal{T}_s(\boldsymbol{J}) = -\left(\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \mathcal{R} \left(\mu \frac{\partial \boldsymbol{J}}{\partial t} \right) \right). \tag{3}$$

with $\mathcal{R}(f) = \int_\Gamma \frac{f(\boldsymbol{r}', t - |\boldsymbol{r} - \boldsymbol{r}'|/c)}{4\pi |\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d}\boldsymbol{r}'$. The temporal differentiations in (1) undo the inconvenient temporal integration in (2). To numerically solve (1), Γ is approximated by a mesh of planar triangles, and the current density $\boldsymbol{J}(\boldsymbol{r}, t)$ is approximated as $\boldsymbol{J}(\boldsymbol{r}, t) \approx \sum_{j=1}^{N_t} \sum_{n=1}^{N_s} J_{j,n} \boldsymbol{f}_n(\boldsymbol{r}) T_j(t)$ where $\boldsymbol{f}_n(\boldsymbol{r})$, $n = 1, \ldots, N_s$ are Rao-Wilton-Glisson basis functions defined on the mesh's N_s interior edges and $T_j(t)$, $j = 1, \ldots, N_t$ are higher-order polynomial interpolants satisfying $T_j(t) = T(t - j\Delta t)$ with $T(t) = 0 \ \forall t < \Delta t$; Δt denotes the time step size. Substituting the above expression for $\boldsymbol{J}(\boldsymbol{r}, t)$ in (1) and spatial Galerkin testing the resulting equation at time $t_j = j\Delta t$ yields

$$\mathsf{T}\mathsf{J} = \mathsf{E} \tag{4}$$

where

$$\mathsf{T} = \begin{pmatrix} \mathsf{T}_0 & & & \\ \mathsf{T}_1 & \mathsf{T}_0 & & \\ \mathsf{T}_2 & \mathsf{T}_1 & \mathsf{T}_0 & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathsf{J} = \begin{pmatrix} \mathsf{J}_0 \\ \mathsf{J}_1 \\ \mathsf{J}_2 \\ \vdots \end{pmatrix}, \quad \mathsf{E} = \begin{pmatrix} \mathsf{E}_0 \\ \mathsf{E}_1 \\ \mathsf{E}_2 \\ \vdots \end{pmatrix}, \quad \{\mathsf{J}_j\}_n = J_{j,n}$$

$$\{\mathsf{T}_k\}_{m,n} = \left\langle \boldsymbol{f}_m(\boldsymbol{r}), \frac{\partial \mathcal{T}(T_k \boldsymbol{f}_n)}{\partial t} \right\rangle \bigg|_{t=0}, \quad \{\mathsf{E}_j\}_n = \left\langle \boldsymbol{f}_n(\boldsymbol{r}), -\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \frac{\partial \boldsymbol{E}^i(\boldsymbol{r}, t)}{\partial t} \right\rangle \bigg|_{t=t_j}$$

with $\langle a(\boldsymbol{r}, t), b(\boldsymbol{r}, t) \rangle = \int_\Gamma a(\boldsymbol{r}, t) b(\boldsymbol{r}, t) \, \mathrm{d}\boldsymbol{r}$. Equation (4) can be cast in Marching-On-in-Time (MOT) form as

$$\mathsf{T}_0 \mathsf{J}_j = \mathsf{E}_j - \sum_{k=0}^{j-1} \mathsf{T}_k \mathsf{J}_{j-k}, \quad j \geq 0. \tag{5}$$

It is easy to see that constant functions and linear in time solenoidal functions reside in the null space of $\partial \mathcal{T}/\partial t$ [3]. In other words, given the physical solution $\boldsymbol{J}(\boldsymbol{r}, t)$ of the scattering problem, every current $\boldsymbol{J}(\boldsymbol{r}, t) + \boldsymbol{J}_0(\boldsymbol{r}, t)$, with $\boldsymbol{J}_0(\boldsymbol{r}, t)$ static or linear in time and solenoidal, will be a valid solution to (1). As a consequence, the numerical solution of (1) will be corrupted by a constant or linear in time spurious offsets [3] further termed DC instabilities. To construct a TDEFIE that upon discretization is immune to DC instabilities, define the operators

$$\tilde{\mathcal{T}}_h(\boldsymbol{J}) = \frac{c}{4\pi} \hat{\boldsymbol{n}} \times \int_\Gamma dS' \nabla \frac{\nabla_s' \cdot \boldsymbol{J}(\boldsymbol{r}', t)}{R}, \quad \tilde{\mathcal{T}}_s(\boldsymbol{J}) = -\frac{1}{4\pi c} \hat{\boldsymbol{n}} \times \int_\Gamma dS' \frac{\boldsymbol{J}(\boldsymbol{r}', t - R/c)}{R}. \tag{6}$$

The equalities $\mathcal{T}_h \mathcal{T}_s = \tilde{\mathcal{T}}_h \tilde{\mathcal{T}}_s$, $\mathcal{T}_s \mathcal{T}_h = \tilde{\mathcal{T}}_s \tilde{\mathcal{T}}_h$ hold as spatial integrations and temporal differentiations commute. This together with the fact that $\mathcal{T}_h^2 = 0$ allow us to express $\mathcal{T}^2$ as $\mathcal{T}^2 = \mathcal{T}_s^2 + \mathcal{T}_h \mathcal{T}_s + \mathcal{T}_s \mathcal{T}_h + \mathcal{T}_h^2 = \mathcal{T}_s^2 + \tilde{\mathcal{T}}_h \tilde{\mathcal{T}}_s + \tilde{\mathcal{T}}_s \tilde{\mathcal{T}}_h$. Thus (1) can be modified into

$$\left(\mathcal{T}_s^2 + \tilde{\mathcal{T}}_h \tilde{\mathcal{T}}_s + \tilde{\mathcal{T}}_s \tilde{\mathcal{T}}_h \right) [\boldsymbol{J}](\boldsymbol{r}, t) = -\mathcal{T}[\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{E}^i](\boldsymbol{r}, t) \tag{7}$$

henceforth termed the "dottrick TDEFIE". The dottrick TDEFIE can be discretized as

$$\left(\mathsf{T}_s \mathsf{G}^{-1} \mathsf{T}_s + \tilde{\mathsf{T}}_h \mathsf{G}^{-1} \tilde{\mathsf{T}}_s + \tilde{\mathsf{T}}_s \mathsf{G}^{-1} \tilde{\mathsf{T}}_h \right) \mathsf{J} = \mathsf{T}\mathsf{E}, \tag{8}$$

where the matrices T_s, $\tilde{\mathsf{T}}_h$, $\tilde{\mathsf{T}}_s$, and G are detailed in [3], where it is also shown that (8) can be cast in a MOT form similar to (5). Equation (8) has several advantages over (1). First, it can be solved rapidly by iterative solvers [4]. Second, it contains no temporal integral and therefore can be conveniently implemented. And third, as

will be shown next, it is immune to DC instabilities: given a static or linear-in-time $\boldsymbol{J}(\boldsymbol{r}, t)$ satisfying

$$\left(\mathcal{T}_s^2 + \tilde{\mathcal{T}}_h\tilde{\mathcal{T}}_s + \tilde{\mathcal{T}}_s\tilde{\mathcal{T}}_h\right)[\boldsymbol{J}](\boldsymbol{r}, t) = 0 \quad \Rightarrow \quad \boldsymbol{J}(\boldsymbol{r}, t) = 0. \tag{9}$$

In other words, the static and linear-in-time kernels of $\mathcal{T}$ and $\dot{\mathcal{T}}$ are not present in $\mathcal{T}_s^2 + \tilde{\mathcal{T}}_h\tilde{\mathcal{T}}_s + \tilde{\mathcal{T}}_s\tilde{\mathcal{T}}_h$, and (7) is a DC-stable equation.

It is sufficient to prove (9) only for static currents $\boldsymbol{J}(\boldsymbol{r}, t) = \boldsymbol{J}(\boldsymbol{r})$. Indeed, if $\boldsymbol{J}(\boldsymbol{r}, t)$ is linear-in-time then

$$\left(\mathcal{T}_s^2 + \tilde{\mathcal{T}}_h\tilde{\mathcal{T}}_s + \tilde{\mathcal{T}}_s\tilde{\mathcal{T}}_h\right)[\boldsymbol{J}](\boldsymbol{r}, t) = 0 \tag{10}$$

implies

$$\partial_t \left(\mathcal{T}_s^2 + \tilde{\mathcal{T}}_h\tilde{\mathcal{T}}_s + \tilde{\mathcal{T}}_s\tilde{\mathcal{T}}_h\right)[\boldsymbol{J}](\boldsymbol{r}, t) = \left(\mathcal{T}_s^2 + \tilde{\mathcal{T}}_h\tilde{\mathcal{T}}_s + \tilde{\mathcal{T}}_s\tilde{\mathcal{T}}_h\right)[\partial_t\boldsymbol{J}](\boldsymbol{r}, t) = 0. \tag{11}$$

Once (9) is established for static currents and since $\partial_t\boldsymbol{J}(\boldsymbol{r}, t)$ is static

$$\tilde{\mathcal{T}}_h\tilde{\mathcal{T}}_s[\partial_t\boldsymbol{J}](\boldsymbol{r}, t) = 0 \Rightarrow \partial_t\boldsymbol{J}(\boldsymbol{r}, t) = 0 \Rightarrow \boldsymbol{J}(\boldsymbol{r}, t) \text{ is static.} \tag{12}$$

To demonstrate (9) for static $\boldsymbol{J}(\boldsymbol{r})$, note that for such current $\mathcal{T}_s^2[\boldsymbol{J}](\boldsymbol{r}) = 0$. Condition $\left(\tilde{\mathcal{T}}_h\tilde{\mathcal{T}}_s + \tilde{\mathcal{T}}_s\tilde{\mathcal{T}}_h\right)[\boldsymbol{J}](\boldsymbol{r}) = 0$ implies

$$\nabla_s \cdot \left(\tilde{\mathcal{T}}_h\tilde{\mathcal{T}}_s + \tilde{\mathcal{T}}_s\tilde{\mathcal{T}}_h\right)[\boldsymbol{J}](\boldsymbol{r}) = 0 \tag{13}$$

and (since $\nabla_s \cdot \tilde{\mathcal{T}}_h\tilde{\mathcal{T}}_s[\boldsymbol{J}](\boldsymbol{r}) = 0$)

$$\nabla_s \cdot \tilde{\mathcal{T}}_s\tilde{\mathcal{T}}_h[\boldsymbol{J}](\boldsymbol{r}) = 0. \tag{14}$$

To proceed, the following lemma, proven in [3], is needed:

Lemma: Given a simply connected surface Γ, the operator $\tilde{\mathcal{T}}_s$ defined on Γ in (8), and a static tangential vector field $\boldsymbol{f}(\boldsymbol{r})$ with $\nabla_s \cdot \boldsymbol{f}(\boldsymbol{r}) = 0$, it follows that if $\nabla_s \cdot \tilde{\mathcal{T}}_s[\boldsymbol{f}](\boldsymbol{r}) = 0$ then $\boldsymbol{f}(\boldsymbol{r}) = 0$.

Since $\nabla_s \cdot \tilde{\mathcal{T}}_h[\boldsymbol{J}](\boldsymbol{r}) = 0$, the lemma can be applied to (14) with $\boldsymbol{f}(\boldsymbol{r}) = \tilde{\mathcal{T}}_h[\boldsymbol{J}](\boldsymbol{r})$, yielding

$$\boldsymbol{f}(\boldsymbol{r}) = \tilde{\mathcal{T}}_h[\boldsymbol{J}](\boldsymbol{r}) = \frac{c}{4\pi}\hat{\boldsymbol{n}} \times \int_\Gamma dS' \nabla \frac{\nabla_s' \cdot \boldsymbol{J}(\boldsymbol{r}')}{R} = 0 \Rightarrow \int_\Gamma dS' \frac{\nabla_s' \cdot \boldsymbol{J}(\boldsymbol{r}')}{R} = const. \tag{15}$$

Note that

$$const = \int_\Gamma dS' \frac{\nabla_s' \cdot \boldsymbol{J}(\boldsymbol{r}')}{R} \propto \frac{1}{C} \int_\Gamma \nabla_s' \cdot \boldsymbol{J}(\boldsymbol{r}') dS' = 0 \tag{16}$$

where C is the (always positive) capacitance of Γ. Equations (16) and (15) imply

$$\nabla_s \cdot \boldsymbol{J}(\boldsymbol{r}) = 0 \tag{17}$$

from which it follows that $\tilde{\mathcal{T}}_s\tilde{\mathcal{T}}_h[\boldsymbol{J}](\boldsymbol{r}) = 0$. It only remains to be shown that $\tilde{\mathcal{T}}_h\tilde{\mathcal{T}}_s[\boldsymbol{J}](\boldsymbol{r}) = 0$ (with $\nabla_s \cdot \boldsymbol{J}(\boldsymbol{r}) = 0$) implies $\boldsymbol{J}(\boldsymbol{r}) = 0$. To this end, note that

$$\tilde{\mathcal{T}}_h\tilde{\mathcal{T}}_s[\boldsymbol{J}](\boldsymbol{r}) = \hat{\boldsymbol{n}} \times \int_\Gamma dS' \nabla \frac{\nabla_s' \cdot \tilde{\mathcal{T}}_s[\boldsymbol{J}](\boldsymbol{r}')}{R} = 0 \Rightarrow \int_\Gamma dS' \frac{\nabla_s' \cdot \tilde{\mathcal{T}}_s[\boldsymbol{J}](\boldsymbol{r}')}{R} = const. \tag{18}$$
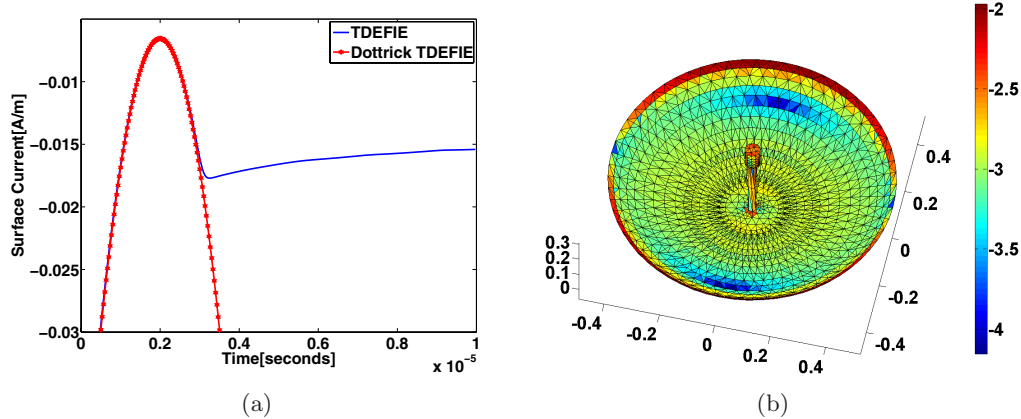
Figure 1: (a) Magnitude of the surface current on the radar antenna tip as a function of the time step Δt obtained using the TDEFIE and the dottrick TDEFIE; (b) Magnitude of the surface current on the radar dish at $t = 0.2e - 5$ seconds

A line of reasoning similar to that in (16) implies

$$\nabla_s \cdot \tilde{\mathcal{T}}_s[\boldsymbol{J}](\boldsymbol{r}) = 0. \tag{19}$$

It is now sufficient to reapply the lemma with $\boldsymbol{f}(\boldsymbol{r}) = \boldsymbol{J}(\boldsymbol{r})$ (recall that $\nabla_s \cdot \boldsymbol{J}(\boldsymbol{r}) = 0$) to obtain $\boldsymbol{J}(\boldsymbol{r}) = 0$. This proves that (7) is immune to DC instabilities.

The dottrick TDEFIE has been tested on a radar dish of diameter 1m residing in the xy-plane (Fig. 1(b)), and discretized using 5008 unknowns. The incident wave is a Gaussian $\boldsymbol{E}^i(\boldsymbol{r}, t) = 4\hat{\boldsymbol{x}}e^{-\gamma^2} / (T\sqrt{\pi})$ with $\gamma = 4\left(ct - ct_0 - \hat{\boldsymbol{z}} \cdot \boldsymbol{r}\right)/T$, $T = 200$ meter, and $t_0 = 300$ seconds. Fig. 1(a) compares the surface currents obtained using the TDEFIE and the dottrick TDEFIE; from the figure it is evident that the dottrick TDEFIE is completely DC stable.

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