A Variation on the Firefighter Problem on Graphs

by

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Abstract

In the classic version of the game of Firefighter, on the first turn a fire breaks out on a vertex in a graph G and then k firefighters protect k vertices. On each subsequent turn, the fire spreads to the collective unburnt neighbourhood of all the burning vertices and the firefighters again protect k vertices. Once a vertex has been burnt or protected it remains that way for the rest of the game. A common objective with respect to some infinite graph G is to determine how many firefighters are necessary to stop the fire from spreading after a finite number of turns, commonly referred to as *containing* the fire. We introduce the concept of *distance-restricted firefighting* where the firefighters' movement is restricted so they can only move up to some fixed distance d per turn rather than being able to move without restriction. We establish some general properties of this new game in contrast to properties of the original game, and we investigate specific cases of the distance-restricted game on the infinite strong, hexagonal, and square grids. We conjecture that two firefighters are insufficient on the square grid when $d = 2$, and we pose some questions about how many firefighters are required in general when $d = 1$.

To my grandfather, Thomas Edgett. I hope I can continue to make you proud.

Lay summary

The Firefighter Problem is a process in which a fire spreads in a network while a team of firefighters seek to defend against this fire. The fire and the firefighters take turns spreading to new parts of the network and defending parts of the network respectively. The firefighters are generally trying to achieve some predetermined goal, usually related to limiting the spread of the fire. These goals can include minimizing how much of the network is burnt, determining if specific parts of the network can be saved, and determining if the fire will burn forever in the case of infinite networks.

A common thread among applications of this model is that there are not any restrictions imposed on how the firefighters move from one turn to the next. This is somewhat unrealistic and in general the model is improved by adding such restrictions. We propose implementing two restrictions. First we restrict the distance that a firefighter can move on any given turn to be less than some fixed value d. We can see that this is very applicable to the idea of fighting an actual fire since the firefighters can't just move instantly between two positions, so there will be some maximum distance they can travel during their turn. Our second restriction is to require that when the firefighters move from one position to the next they do not move through any parts of the network that have been burnt. This restriction is hugely applicable, for example when fighting against a cyber attack sending messages through a compromised node gives the attacker the opportunity to view and potentially manipulate those messages. When these restrictions are in place we refer to the game as Distance-Restricted Firefighting to distinguish it from the original game.

In this thesis we examine the game on certain classes of networks with both restrictions for the most part, but we do also draw some conclusions about how the game behaves with only the first restriction. Our main focus is on determining if the fire will burn forever with these restrictions on some commonly studied infinite grids. We also examine some general properties of the game with these restrictions in contrast to the original game.

Acknowledgements

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I would also like to thank my grandparents Jill, Tom, Jeanine and Ron, my parents Anne and Mark, as well as my siblings Catie, James, Billy, Angus, Grace, and Tom for always supporting me in my academic pursuits and listening to my rambles about math.

Finally I would like to again thank my grandfather Thomas Edgett for always supporting me and sharing stories of your times at school. Your wisdom has always helped to guide my path.

Statement of contribution

Dr. David Pike and Dr. Andrea Burgess suggested the idea of the research and supervised the entirety of the research. All theorems and lemmas were proven by the author unless otherwise indicated. The manuscript was done with the supervision of Dr. Andrea Burgess and Dr. David Pike.

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Chapter 1

Introduction

1.1 Background on Graph Theory

In order to understand the content of this thesis we first review some basic graph theory. For additional information on graph theory we recommend the textbook by West [18]. Firstly, a *graph* G consists of a set called the vertex set (typically denoted $V(G)$) and a set of 2-subsets of the vertex set called the edge set (typically denoted $E(G)$). A pair of vertices $u, v \in V(G)$ are *adjacent* if the edge $\{u, v\} \in E(G)$. Using this terminology we can now define a *subgraph*. A *subgraph* H of a graph G is a graph such that $V(H) \subseteq V(G)$ and if $\{u, v\} \in E(H)$ then $\{u, v\} \in E(G)$. Essentially a subgraph is just a graph contained within a larger graph, which is depicted in Figure 1.1.1.

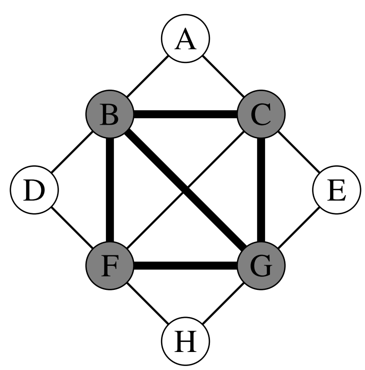

Figure 1.1.1: A graph with a subgraph highlighted within it.

A special case of this concept comes about when discussing *distance* in a graph. In order to define the *distance* between a pair of vertices u and v we first define a *uv-path*. A *uv-path* of *length* n is a sequence of $n+1$ distinct vertices $u_1,\ldots,u_{n+1}$ such that $u=u_1$, $v=u_{n+1}$, and $\{u_i,u_{i+1}\}\in E(G)$ for all $1\leq i\leq n$. The *distance* between two vertices u and v in a graph G is simply defined as the length of a shortest uv-path and is commonly written $d_G(u,v)$ or $d(u,v)$ when it is clear what graph we are referring to.

The *degree* of a vertex u in a graph G is defined to be the number of vertices which are adjacent to u. A *k-regular* graph is a graph where every vertex has degree k. For the introduction of *distance-restricted firefighting* the concept of a *neighbourhood* is important. The *neighbourhood* of a vertex u in a graph G is the set of vertices which are adjacent to u, denoted $N(u)$. Similarly we can describe the neighbourhood of a set $S\subseteq V(G)$ in G as the union of the neighbourhoods of all vertices in S ($\cup_{u\in S}N(u)$), and we write this as $N(S)$.

A *graph isomorphism* is a bijection ϕ between the vertex sets of two graphs G and H such that u is adjacent to v in G if and only if $\phi(u)$ is adjacent to $\phi(v)$ in H. If G and H are the same graph we call this a *graph automorphism* and if for every pair of vertices u,v in G there is an automorphism ϕ such that $\phi(u)=v$ then we say G is *vertex transitive*. A *subdivision* is a map which replaces an edge $\{u,v\}\in E(G)$ by adding a new vertex w to $V(G)$, removing the edge $\{u,v\}$, and adding in the edges $\{u,w\}$ and $\{w,v\}$. Any graph that can be obtained from G by applying zero or more of these maps is referred to as a *subdivision* of G.

The concept of a planar graph is also briefly mentioned in this thesis but it is not the focus of our study, so we will simply say that *planar graphs* are graphs which can be drawn in the plane without any pair of edges crossing over each other. For examples of planar graphs, observe the graphs in Figures 1.1.2 and 1.1.3.

For the purposes of this thesis we will be focusing on three specific graphs with a few exceptions. These graphs are known as the infinite *Strong* ($G_{\boxtimes}$), *Square* ($G_{\square}$), and *Hexagonal* ($G_{\hexagon}$) grids. The latter of these two grids are easily defined as the vertices and edges of a tiling of the plane with squares and hexagons respectively as depicted in Figures 1.1.2 and 1.1.3. For the strong grid we simply take the square grid and add in the 'diagonal' edges as depicted in Figure 1.1.4.

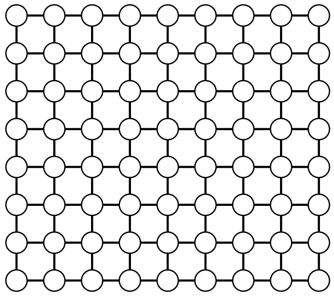

Figure 1.1.2: A 9×8 portion of the square grid.

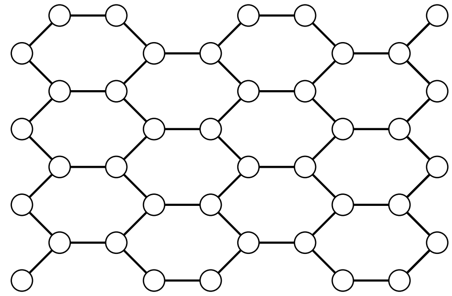

Figure 1.1.3: A portion of the hexagonal grid.

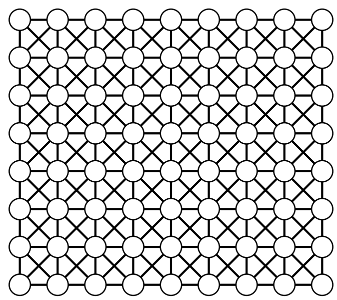

Figure 1.1.4: A 9×8 portion of the strong grid.

As noted in [14] these graphs have a relationship such that $V(G_{\square}) = V(G_{\boxtimes})$ and $E(G_{\square}) \subset E(G_{\boxtimes})$. This nesting can be further extended to the infinite hexagonal ($G_{\hexagon}$) grid by noting $V(G_{\hexagon}) = V(G_{\square})$ and $E(G_{\hexagon}) \subset E(G_{\square})$ as illustrated in Figure 1.1.5.

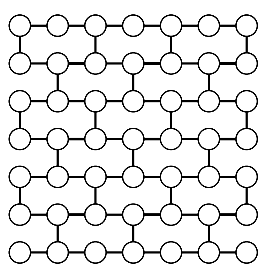

Figure 1.1.5: 7×7 Portion of the hexagonal grid as a subgraph of the square grid.

We define a *cycle* on n vertices as the graph with vertex set $v_1, \ldots, v_n$ where v_1 and v_n are adjacent and v_i is adjacent to v_{i+1} for $1 \leq i \leq n-1$ and we denote this graph as C_n. By adding one extra vertex to C_n and making it adjacent to every vertex we obtain a *wheel* graph. A *complete rooted n-ary tree* of height m is best defined by its construction. The construction starts with a single vertex known as the *root* and then n vertices which are all adjacent to the initial vertex are added and no other vertices. Then for each vertex of degree one add n more vertices and make them all adjacent to that degree one vertex and no other vertices. Repeat this process until there are n^{m-1} vertices of degree 1. Figure 1.1.6 depicts a 3-ary tree of height 3 to better illustrate the concept. We can also define a complete rooted n-ary tree of infinite height as the graph obtained from repeating this process indefinitely rather than stopping once a certain number of vertices has been reached.

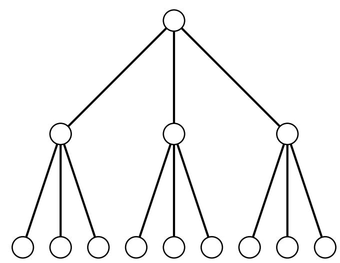

Figure 1.1.6: A 3-ary tree of height 3.

We now define two infinite graphs, the *infinite ray* $P_{\mathbb{N}}$ and the *infinite path* $P_{\mathbb{Z}}$. The graph $P_{\mathbb{N}}$ has vertex set $\mathbb{N} = \{0, 1, 2, \ldots\}$ where i is adjacent to $i+1$ for all $i \in \mathbb{N}$. Similarly, we define $P_{\mathbb{Z}}$ to be the graph with vertex set $\mathbb{Z}$ where i is adjacent to $i+1$ for all $i \in \mathbb{Z}$.

The *Cartesian product* of two graphs G, H, denoted $G \square H$, is a graph with vertex set $V(G) \times V(H)$. Two vertices (u, v) and (x, y) are adjacent if $u = x$ and v is adjacent to y in H, or if u is adjacent to x in G and $v = y$. We can consider the infinite square grid as the Cartesian product of $P_{\mathbb{Z}}$ with itself.

1.2 Background on Firefighting

The game of firefighter was first introduced by Hartnell in 1995 [11] as a model for combatting the spread of a fire. In his version, the game starts with a fire breaking out on a single vertex in a graph G. The game then alternates between having a single firefighter defend a vertex and the fire spreading to all undefended vertices which have a burning neighbour. Once a vertex is burning or defended it stays that way for the remainder of the game. This process continues until the fire can no longer spread. This problem has since been generalized to allow for multiple firefighters as well as multiple initial burning vertices [8].

There are a number of objectives that can be pursued with respect to firefighting. Some common objectives that have been studied are minimizing the expected number of vertices that will burn given a random initial burning vertex [17], saving the maximum number of vertices [12], and determining if a specific subset of vertices can be saved [12]. The related decision problem for determining the maximum number of vertices that can be saved has been shown to be NP-complete even when restricted to trees of maximum degree three [7]. Furthermore the decision problem of determining if a specific subset of vertices can be saved has been shown to be NP-complete even when restricted to trees of maximum degree three with the leaves of the tree being the set of vertices to be saved [12]. For further information on firefighting on finite graphs we recommend the survey paper by Finbow and MacGillivray [8] or the recent thesis by Wagner [16].

Another well studied goal of firefighting is determining whether or not the fire will continue burning forever in an infinite graph. This is referred to as containment and generally the objective is to determine the minimum number of firefighters needed to contain a fire for some fixed graph G. We define $f(G,u)$ to be the minimum number of firefighters needed to contain a fire that breaks out at vertex u in graph G and we refer to this quantity as the *firefighter number* of G for a fire at u. If the graph is vertex transitive, like the grids we will be focusing on, we can simply write $f(G)$ since the choice of u does not matter and we refer to this simply as the firefighter number of G.

One of the first major results on containment came from the 2002 paper by Wang and Moeller [17] where it was shown that the square grid requires exactly two firefighters. Shortly afterwards Fogarty proved Theorem 1.2.1 in her MSc thesis [9], which has been a very good tool for establishing lower bounds on the number of firefighters required for a given graph.

Theorem 1.2.1 ([9]). *Let D_k denote the set of vertices at distance k from the original burning vertex and let $B_k \subseteq D_k$ denote the set of burned vertices after k time intervals that are distance k from the original burning vertex. Also let f denote the number of firefighters available at each iteration and let r_k denote the number of firefighters in $D_{k+1} \cup D_{k+2} \cup \cdots$ after k time intervals. Finally for $A \subseteq D_k$ let $N^+(A) = N(A) \cap D_{k+1}$.*

If G is a graph such that for all k, every $A \subseteq D_k$ satisfies $|N^+(A)| \geq |A| + f$, then

$|B_n| \geq 1 + r_n$ *for all n.*

Theorem 1.2.1, and the stronger version of the theorem described in [6] have been the main tools for proving lower bounds in firefighting since their conception. Fogarty also gave lower bounds for the number of firefighters required for the triangular grid and proved that two firefighters suffice to contain any finite source fire on the square grid. In her MSc thesis [13], Messinger introduced the concept of fractional firefighting, which was later renamed to be average firefighting, where the number of firefighters available is no longer constant and is computed as the average number of firefighters available. In her MSc thesis and in a 2008 paper on average firefighting [14], Messinger showed that for any positive integer T, $\frac{3}{2} + \frac{1}{3T+2}$ firefighters suffice on the square grid, $3 + \frac{1}{T}$ firefighters suffice on the strong grid, and $2 + \frac{1}{T + \frac{1}{2T+4}}$ firefighters suffice on the triangular grid.

Messinger also conjectured that one firefighter does not suffice to contain a fire on the hexagonal grid [13], which has come to be known as Messinger's Conjecture. This conjecture is still unproven but it has been shown that small relaxations to the conjecture do allow for the fire to be contained. In 2014 Gavenčiak et al. [10] showed that if there are two (possibly equal) integers t_1, t_2 such that there is an extra firefighter available at time t_1 and another extra firefighter available at time t_2 then the fire can be contained. This has recently been improved by Dean et al. [5] to only need an extra firefighter at a single time t_1.

The remainder of this thesis focuses on our new variant of the game which we introduce in Chapter 2. In Chapters 3 and 4 we examine how this new variant plays out on the square, strong and hexagonal grids. In Chapter 4 we also outline a special case of Messinger's conjecture and give a brief overview of how one might attempt to prove this special case and why it is likely easier to prove than Messinger's conjecture. In the final chapter we pose some questions about some general properties of the game as well as some questions about a specific case of our variant. Several of the results in this thesis are also included in the manuscript [2] created by the author which has been submitted for publication.

Chapter 2

Distance-Restricted Firefighting

2.1 A New Variant of Firefighting

In the original game there is no relationship between the firefighters' positions at time t and time $t+1$, which is not always a good representation of the real world since firefighters are often restricted by how fast they can move. This concept has been previously studied in [3, 4] but only for the case where firefighters are restricted to moving to a neighbour of their previous position. Our variation of the rules, which we have dubbed *distance-restricted* or *distance d* firefighting, seeks to make the game more realistic by forcing each firefighter to only move a maximum of some specified distance d between time t and time $t+1$. This condition can be explicitly defined by saying that given a fixed, positive integer d and k firefighters labelled as $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_k$, then for every $t \in \mathbb{N}$, there must exist a bijection g_t between the sets POS_t and POS_{t+1} where $POS_t = \{(\mathscr{F}_i, u_i) \mid \mathscr{F}_i \text{ occupies vertex } u_i \text{ at time } t\}$ and $g_t(\mathscr{F}_i, u) = (\mathscr{F}_j, v)$ implies that the distance between u and v is at most d and that $i = j$. Notice that there is an ambiguity in whether or not the distance between u and v is determined with respect to the entire graph or just the subgraph induced by the unburnt vertices. We will be using the distance in the induced subgraph since it is a more realistic model, but will occasionally make note of how things may be different if the firefighters are allowed to move through the fire.

Following the literature on firefighting, it is natural to introduce the concept of *average distance-restricted firefighting*. Average firefighting, which is introduced in [14], simply allows the number of firefighters available on any given turn to fluctuate in a cyclic manner with respect to some finite length sequence. That is to say, if the sequence is of length n then the number of firefighters available at time t is determined by $t \bmod n$. Here, as the name might suggest, the number of firefighters is counted as the average of this sequence. Note that our definition of the distance-restricted game requires some refinement to make it compatible with instances where the number of firefighters available varies from turn to turn. If the number of firefighters increases by ℓ on any given turn, then before the firefighters move, the ℓ additional firefighters are placed on vertices that currently have at least one firefighter on them, and then the firefighters move as normal. If the number of firefighters decreases by ℓ on any given turn, then before the firefighters move, ℓ of them are chosen and removed from play, and then the remaining firefighters move as normal. So the concept of *average firefighting* along with this refinement of the rules for *distance-restricted firefighting* yields the idea of *average distance-restricted firefighting*.

We can now define $f_d(G,u)$ analogously to $f(G,u)$ by considering the distance d firefighting game and refer to $f_d(G,u)$ and $f_d(G)$ as the distance-restricted firefighter number. The notation $f_d^*(G,u)$ is defined analogously to $f_d(G,u)$, except that the firefighters are allowed to move through the fire.

2.2 General Properties of Distance-Restricted Firefighting

There are some nice properties of the original game that do not hold in the distance-restricted game. For example, in the original game, if H is a subgraph of G with $u \in V(H)$ then $f(G,u) \geq f(H,u)$. This is apparent from the fact that a strategy that contains the fire with k firefighters on G can be slightly modified and then played on H as well. For any placement of the firefighters there are two cases to consider. At time t if all the vertices that would have been protected in the strategy on G exist in H and are unprotected, then the firefighters protect those vertices. Otherwise, any missing vertices can be considered as protected since the fire cannot spread to them and so rather than protecting those missing vertices, an arbitrary set of vertices can be protected without hindering the strategy. Since the fire cannot spread to any additional vertices due to the fact that all the same vertices from G are either protected or missing, the fire cannot spread on H more than it did on G and thus the fire is contained in H as it is contained in G. However, this property does not hold in the distance-restricted firefighting game. This fact will become apparent shortly after observing some properties of certain infinite grids.

Observe that the graph in Figure 2.2.1 is both a subdivision and a subgraph of the hexagonal grid. This subgraph will be denoted as $sub(G_{\hexagon})$. The paths of length 3 formed by the degree 2 vertices and their neighbours can be replaced with single edges in order to recover the hexagonal grid.

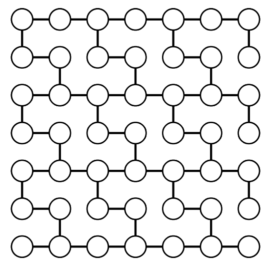

Figure 2.2.1: 7×7 Portion of the subdivided hexagonal grid as a subgraph of the hexagonal grid.

If d is sufficiently large then $sub(G_{\hexagon})$ only requires one firefighter when $d \geq 11$ as depicted[1] in Figure 2.2.2, but $sub(G_{\hexagon})$ contains an infinite path[2] which stretches out in two directions and requires two firefighters to contain the fire for any value of d. Thus the firefighter number of a subgraph is not bounded by the firefighter number of the original graph in the distance-restricted game. In fact, Theorem 2.2.1 states that the ratio and difference of $f_d(G,u)$ and $f_d(H,u)$ for H a subgraph of G and $u \in V(H)$ can in fact both be unbounded.

[1]There are many diagrams in this thesis that all follow the same labelling conventions, which we outline here. The firefighters are usually labelled so that the even numbered turns have black vertices and the odd numbered turns have gray vertices. The fire follows a similar convention with the even turns being red and the odd turns being orange. We also adopt the convention that the first turn is turn zero and the vertices where the fire and firefighters start are diamond shaped. There are also diagrams where there is no fire, and in those cases the first firefighters may be diamond shaped if necessary for clarity. We also occasionally have a firefighter who is distinguished from the other firefighters in some way, and in those cases the firefighter may be coloured blue instead of black or gray.

[2]Simply consider any set of vertices $\{(x,y) \mid x \in \mathbb{Z}\}$ for some fixed value of y.

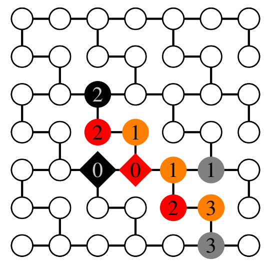

Figure 2.2.2: A strategy to contain the fire in 4 turns on the subdivided hexagonal grid when $d = 11$. The case where the fire begins on a degree 2 vertex is trivial and thus omitted.

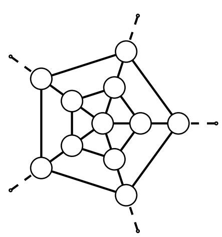

Figure 2.2.3: The first 3 layers of $P_{\mathbb{N}} \Box C_5$ modified as described in the proof of Theorem 2.2.1.

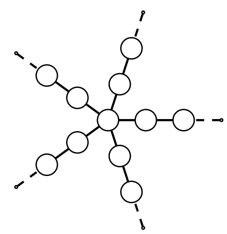

Figure 2.2.4: An infinite subgraph of the graph in Figure 2.2.3 with distance-restricted firefighter number 5.

Theorem 2.2.1. *There exist graphs G, H such that H is a subgraph of G with $u \in V(H) \subseteq V(G)$ such that the value of $\left(f_d(H,u) - f_d(G,u)\right)$ as well as the value of $\left(\frac{f_d(H,u)}{f_d(G,u)}\right)$ can be arbitrarily large for any value of d.*

Proof. The graph in Figure 2.2.3 can be thought of as taking $P_{\mathbb{N}} \Box C_m$ (the Cartesian product of an infinite ray and a cycle) and replacing the cycle corresponding to the end vertex of the ray with a wheel on $m+1$ vertices. This graph has distance-restricted firefighter number 1 for a fire breaking out at the center of the wheel for any value of d since if we start our firefighter in the $(m+1)^{th}$ cycle then the firefighter can just walk around the cycle and form a barrier. However if we consider the subgraph depicted in Figure 2.2.4, then any fire breaking out at the center of the wheel will require m firefighters to contain for any value of d. Thus both the difference and the ratio are unbounded. $\square$

In fact since in the proof of Theorem 2.2.1 $f_d(G,u) = 1$ and $f_d(H,u)$ could take on any positive integer value, it is also true that the ratio and difference of these values can take on any positive integer value.

This leads us to the question of how large the difference between the firefighter number and the distance-restricted firefighter number can be.

Theorem 2.2.2. *There exists a graph G such that the value of* $\left(f_d(G,u) - f(G,u)\right)$ *as well as the value of* $\left(\frac{f_d(G,u)}{f(G,u)}\right)$ *can be arbitrarily large for any value of* d*.*

Proof. Simply observe the subgraph from the proof of Theorem 2.2.1. This graph clearly only requires one firefighter in the original game, but requires m firefighters in the distance-restricted game. Thus both the ratio and the difference are unbounded. □

We can also consider taking a rooted complete n-ary tree of infinite height and replacing each edge with a path of length $n+1$. If the fire breaks out at the root of the tree then in the original game a single firefighter can contain the fire by defending the vertices of degree $n+1$ which were initially adjacent to the root. However in the distance-restricted game, if we have fewer than n firefighters then, since the fire burning the root splits the graph into n subgraphs that the firefighters cannot move between, there will be a subgraph which is undefended. Since all these subgraphs will burn forever if left undefended, there must be n firefighters in order to contain the fire. Thus we have a second example of a graph where the firefighter number and the distance-restricted firefighter number have an arbitrarily large ratio and difference.

Observe that Theorem 2.2.2 does not extend to when the firefighters are allowed to pass through the fire (i.e. it is false that there exists a graph where $\left(f_d^*(G,u) - f(G,u)\right)$ can be arbitrarily large for every value of d). If G is a connected graph where f firefighters suffice in the original game, then there is a finite distance d where for all t the firefighters at time t and the firefighters at time $t+1$ are within distance d. The existence of d is apparent from the fact that the firefighters only move for a finite number of turns so it is guaranteed that a maximum distance moved by the firefighters exists. This is due to the fact that the firefighters can move through the fire so there is always a set of paths from the positions at time t to the positions at time $t+1$. Thus there is always a value of d where the values of $f(G,u)$ and $f_\delta^*(G,u)$ are the same for all $\delta \geq d$.

It is also true that if we fix d then $f_d^*(G,u) - f(G,u)$ and $\frac{f_d^*(G,u)}{f(G,u)}$ can be arbitrarily large. Consider the modified n-ary trees discussed after the proof of Theorem 2.2.2 and take n to be much larger than d. Containing a fire only requires a single firefighter in the original game as before. However, since n is much larger than d the firefighters will not be able to defend more than d of the subtrees since after time d the fire has spread far enough down the tree that the smallest distance between an unburnt vertex in one undefended subtree and an unburnt vertex in another undefended subtree is at least d. A single firefighter can now save no more than d subtrees. Now if $n > \ell d$ from some $\ell \in \mathbb{Z}^+$ then clearly at least ℓ firefighters are required to contain the fire. Thus the difference $f_d^*(G,u) - f(G,u)$ and the ratio $\frac{f_d^*(G,u)}{f(G,u)}$ can both be unbounded if we fix d.

Similarly, if d is fixed then Theorem 2.2.1 extends to f_d^* which is apparent from the same reasoning as used in the previous paragraph, but using the graph from the proof of Theorem 2.2.1. This result clearly does not extend if d is not fixed since for a large enough value of d, $f_d^*(G,u) = f(G,u) \geq f(H,u) = f_d^*(H,u)$.

Chapter 3

Square Grid

Our consideration of $G_{\square}$ can be divided into two parts: the case of $d = 2$ and the case of $d \neq 2$. The case of $d = 2$ has shown to be much more complex than the case of $d \neq 2$ so the majority of this chapter focuses on this more complex case.

3.1 Square Grid when d $\neq$ 2

The case of $d = 1$ on the square grid was shown to require four firefighters in [3, 4]. The idea in this case is that in order to stop the fire from burning along the four infinite straight paths that start at the origin, there must be four firefighters, and four firefighters are obviously sufficient because the fire can be trivially surrounded on turn zero.

The case of $d = 3$ also has an easy solution. Figure 3.1.1 portrays a strategy that works with two firefighters. The two firefighters form a barrier by spiraling around the fire until they are able to contain the fire.

Lemma 3.1.1. *Two firefighters are necessary and sufficient to contain the fire on the square grid in the game with $d = 3$.*

Proof. Figure 3.1.1 illustrates a strategy with two firefighters. Thus two firefighters suffice.

One firefighter does not suffice since if one firefighter was sufficient for $d = 3$ then one firefighter would be sufficient for the original game which is not true [17]. $\square$

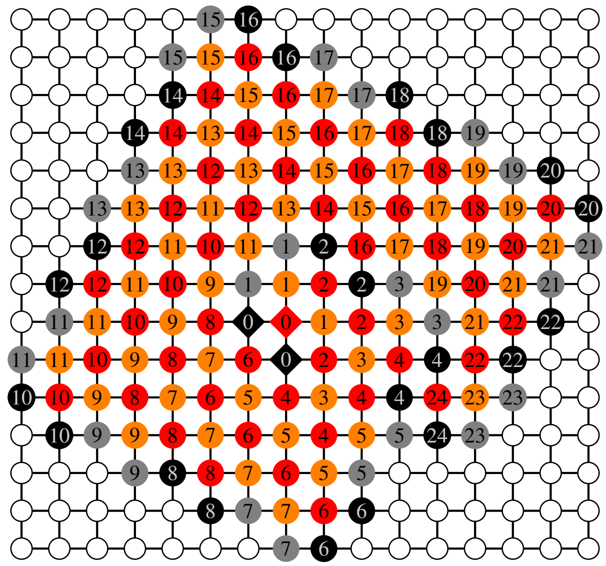

Figure 3.1.1: A strategy for containment when $d = 3$. For extra clarity, this figure is augmented so that the turn numbers are visible on the vertices.

For any distance greater than three the same strategy portrayed in Figure 3.1.1 can be played so for any $d > 3$ two firefighters are also necessary and sufficient.

The more interesting case when $d = 3$ is if we allow the fire to burn for one or more turns before the two firefighters start defending vertices. If we attempt to use the same strategy of spiraling around the fire, we can equate the problem of whether or not the fire will be contained with a problem related to the growth of a recurrence relation that can be derived from the strategy. In order to set up this recurrence we first have to derive the number of turns it takes for two firefighters to cover the first k corners[1] when $d = 3$, denoted as t_k for all $k \geq 1$. We can derive this as a recurrence relation by considering the number of turns it takes to go from the $(k-1)^{th}$ corner to the k^{th} corner, represented as $t_k - t_{k-1}$. This is done indirectly by instead finding the absolute value of the difference in the x-coordinate of the corners on the turn when the $(k-1)^{th}$ corner is covered since this difference is in fact the number of turns it takes to go from the $(k-1)^{th}$ corner to the k^{th} corner. First we will demonstrate that it takes $3 + 3t_{k-6} + 2(k-6)$ turns for the fire to reach the $(k-6)^{th}$ corner. In order to understand this first observe Figure 3.1.2.

[1] A corner is the vertex where the firefighters change direction in their strategy.

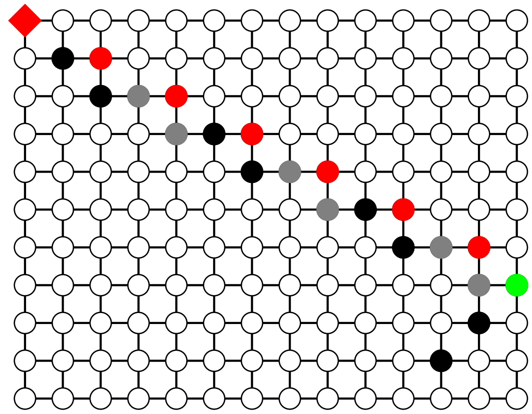

Figure 3.1.2: An illustration of how the fire moves along the outside of the firefighters placed in previous turns.

Figure 3.1.2 represents a situation where in previous rounds the firefighters, while spiraling around the fire, have created a barrier (alternating black and gray vertices) and fire has just moved along the top of the barrier. For each firefighter turn there is a red vertex that is in the same position relative to the firefighters' positions. These red vertices are all distance 3 apart, so the fire will take 3 turns per firefighter turn to move along this barrier. The fire has moved beyond the barrier when it reaches the green vertex, which is in the same position as the initial red vertex but relative to the next barrier the firefighters have created. As a result it takes the fire 2 extra turns for every corner that it passes. From here we combine these two pieces to see that the fire will take 3 turns per firefighter turn ($3t_{k-6}$), plus 2 turns per corner ($2(k-6)$) to reach the initial vertex, plus however many turns it takes the fire to get to the first initial vertex relative to the first barrier. Figure 3.1.3 represents an initial configuration for the game when the fire spreads twice before any firefighters are deployed, and demonstrates that it takes 3 turns for the fire to get to the initial vertex, represented by the green vertex, relative to the first barrier. This gives $+3$ on the end of the expression. Thus to reach the $(k-6)^{th}$ corner it takes the fire $3t_{k-6}+2(k-6)+3$ turns as desired.

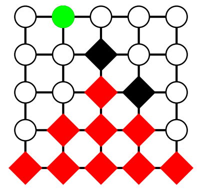

Figure 3.1.3: An illustration of how the fire gets to the desired initial position.

The fire will then have to burn for $2(t_{k-5}-t_{k-6})+1$ turns to have a clear path straight towards the initial vertex next to the $(k-5)^{th}$ corner. This is due to the fact that the firefighters move 2 towards the $(k-5)^{th}$ corner each turn in the desired direction. The firefighters need to defend one past this position, which gives us $2(t_{k-5}-t_{k-6})+1$ turns for the fire to reach the next inital position. This is also along the line towards the $(k-1)^{th}$ corner, so the downward[2] growth of the fire from here is exactly how far away the furthest downward direction vertex on fire will be when the firefighters reach the $(k-1)^{th}$ corner.

Now until turn t_{k-1} the fire is burning in this downward direction, enlarging the distance between the position where the firefighters will reach corner $k-1$ and the furthest downward vertex on fire until the firefighters reach the position for corner $k-1$. So we take the two quantities, $3t_{k-6}+2(k-6)+3$ and $2(t_{k-5}-t_{k-6})+1$, and subtract them from t_{k-1} to get an equation for t_k-t_{k-1} which gives:

$$t_k = 2t_{k-1}-2t_{k-5}-t_{k-6}-2k+8$$

A reference image is shown in Figure 3.1.4 to aid in understanding all of this.

This recurrence along with the initial conditions $t_1=2$, $t_2=5$, $t_3=11$, $t_4=22$, $t_5=41$, $t_6=74$ counts the number of turns if the fire is only allowed to burn for one turn before the firefighters play. By varying these initial conditions we can have the recurrence model what happens for different numbers of turns that the firefighters don't initially play. If we can show that $t_k \geq 3+3t_{k-3}+2(k-3)$ then the firefighters can never catch up to the leading edge of the fire since it takes more turns for the firefighters to reach corner k than it does for the fire to burn past corner $k-3$ (and thus guarantees the existence of corner $k+1$). We also conjecture that if the sequence ever begins to decrease or becomes negative then the fire is contained using this strategy.

[2]By downward here we mean in reference to the rotation of the game that gives the same concept as in Figure 3.1.4.

Conjecture 3.1.2. *If, for some set of initial conditions and some value $k \in \mathbb{N}$, $t_k > t_{k+1}$, then two firefighters can contain the fire in the game corresponding to the given initial conditions using the spiraling strategy.*

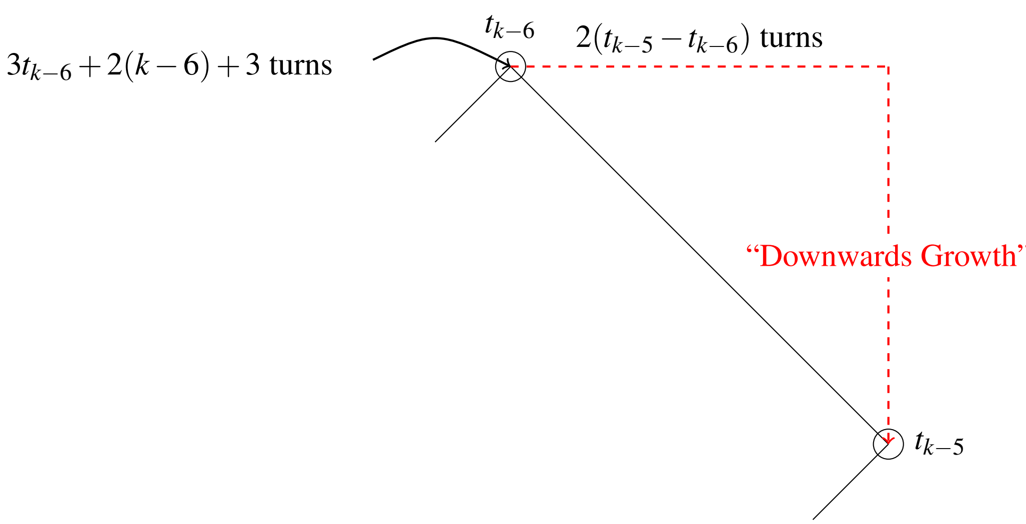

Figure 3.1.4: A sketch of how the recurrence relation is derived.

It is also worth noting that if we change our initial conditions to $t_1 = 1$, $t_2 = 2$, $t_3 = 4$, $t_4 = 7$, $t_5 = 11$, $t_6 = 16$, which matches up with the case of a single source fire, then the sequence stops growing after turn 8 which corresponds to when the fire is contained. This is the only piece of evidence we have about how the recurrence behaves in a situation where the fire is contained.

We made attempts to solve the recurrence in order to see how it grows, but solving it by hand would be a serious undertaking of its own, and when we gave it to the Sympy **rsolve** recurrence solver [15] to solve, it ran for 90 days without coming to a solution. It did, however, produce the general form of the solution as displayed in Equation 3.1:

$$t_k = A(i)^k + A(2k+4) + B(-i)^k + \frac{C(A+Bk)}{A}\left(\frac{1-\sqrt{5}}{2}\right)^k + \frac{D(A+Bk)}{A}\left(\frac{1+\sqrt{5}}{2}\right)^k. \tag{3.1}$$

3.2 Square Grid when d = 2

As previously mentioned, the case of $d = 2$ has proven to be much more challenging than the case of $d \neq 2$, so we need to introduce more complex strategies. Observe that in Theorem 3.2.1 we refer to *corralling* the fire to a column, which simply means that the two firefighters are moving in the same direction in two parallel lines and if they continued like this then only vertices between those two lines would burn (See Figure 3.2.1). We also require that this column of fire is only burning in one direction, rather than in two directions.

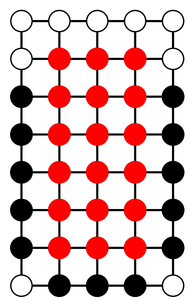

Figure 3.2.1: Two firefighters corralling the fire to a column.

Theorem 3.2.1. *Let $d = 2$. If two firefighters can corral the fire to a column of finite width in the infinite square grid, then they can contain the fire.*

Proof. The proof of this theorem relies on the fact that two firefighters can contain any finite source fire in the half grid when $d = 2$. To see this consider a half diamond of fire along the border of the half grid such that the diamond of fire extends out some distance k from the vertices on the border of the grid (see Figure 3.2.2). Initially, place the firefighters at the edge of the fire with one firefighter on the border and the other on the first firefighter's neighbour that is not on the border (see Figure 3.2.2). From here the firefighters move upwards until they are $2k$ vertices above the peak of the fire, and they then move over the fire and are high enough above the fire that they will have a clear path across. They can then follow the same strategy but rotated ninety degrees to get far enough past the far edge of the fire that they can then move downwards and contain the fire. Since this initial diamond can be arbitrarily large it can cover all sources of fire, and thus any finite source fire can be contained as well.

So now suppose the fire is contained to a column in the full grid. Both firefighters can continue building the protective barriers on either side of the column for an arbitrarily large number of turns, thus these protective barriers can be arbitrarily long. One of the firefighters continues to move up the side of the fire while the other begins to move two spaces at a time in order to move up above the fire, across the top, and down to where the other firefighter is. At this point the firefighters can build a protective barrier up ahead of the fire on that side and then build a protective barrier across so they are now back in the column where the firefighter that was skipping vertices was. From here observe that there is a finite number of vertices burning on the outer side of the protective barriers that form the column, and the length of these protective barriers is arbitrarily large. Thus the firefighters can contain the fire if they treat the remainder of the game as if the protective barrier of the column is the border of a half grid since both firefighters are on the same side of that border as the fire. □

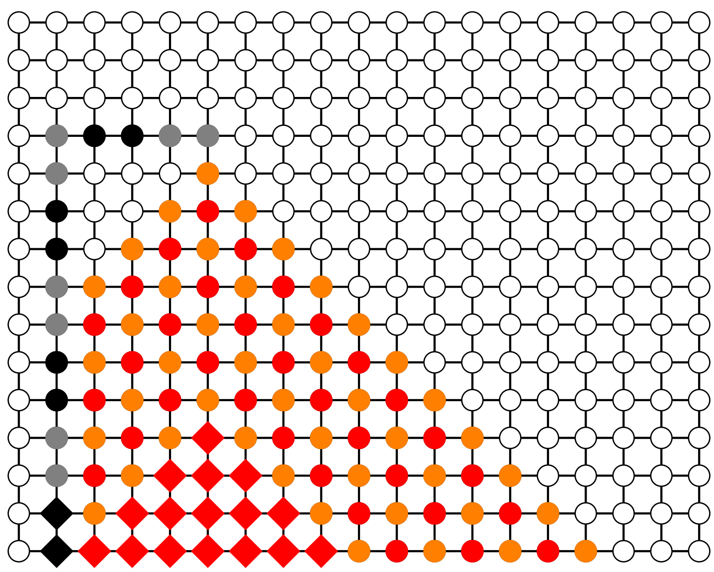

Figure 3.2.2: The beginning of the strategy described in the proof of Theorem 3.2.1 with $k = 3$. Note that this is a half grid for which the border is along the bottom of the diagram.

We conjecture that the converse of Theorem 3.2.1 also holds.

Conjecture 3.2.2. *Let $d = 2$. Two firefighters can contain the fire in the square grid if and only if they can contain the fire to a column.*

Determining $f_2(G_\Box)$ has proven to be a very stubborn problem. We can easily establish that $f_2(G_\Box) \geq 2$ since two firefighters are optimal in the original game. With three firefighters the fire is contained in seven turns as illustrated in Figure 3.2.3, and thus $2 \leq f_2(G_\square) \leq 3$.

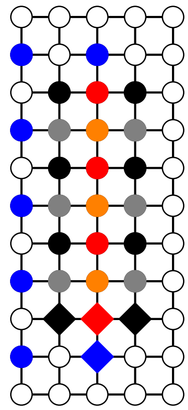

Figure 3.2.3: A strategy for containment with three firefighters when $d = 2$.

However a successful strategy with two firefighters remains elusive. It appears that two firefighters can only contain the fire to a quarter plane in the case of $d = 2$. On the other hand, observe that the square grid without the vertices of the form $(0,n)$, for all $n \in \mathbb{Z}^-$ is containable with two firefighters when $d = 2$.

Lemma 3.2.3. *Let $d = 2$. Then a fire in the infinite square grid, minus the vertices of the form $(0,t)$ for all $t \in \mathbb{Z}^-$, can be contained by two firefighters.*

Proof. First, if the fire starts on a vertex $(\pm 1,t)$ for some $t \in \mathbb{Z}^-$ then the first firefighter starts directly above the fire and the second starts right of the fire if the fire is at $(1,t)$ and left of the fire if the fire is at $(-1,t)$. The firefighters then build a wall downwards through $(\pm 2,t)$ until they are ahead of the fire, which is guaranteed because the firefighters are building the wall two vertices at a time and the fire is moving one vertex at a time. Then the firefighters turn in towards the y-axis and have thus surrounded the fire.

Now suppose the fire starts on a vertex that is not of the form $(\pm 1,t)$ for $t \in \mathbb{Z}^-$. Then the firefighters can corral the fire to a quarter plane within the modified square grid. This quarter grid can always be set up so that one firefighter ($\mathscr{F}_1$) is moving in the negative y-direction and the other ($\mathscr{F}_2$) is moving away from the missing vertices in the x-direction. If $\mathscr{F}_1$ is between the fire and the missing vertices, then $\mathscr{F}_1$ turns and moves towards the missing vertices until reaching the missing vertices. Otherwise it must be that $\mathscr{F}_1$ is on the line $x = 0$ and so will continue moving in the negative y-direction until reaching the missing vertices. From here it is easy to observe that $\mathscr{F}_1$ can retrace steps and then follow the steps of $\mathscr{F}_2$ at twice the speed as before and meet up with $\mathscr{F}_2$. At this point the firefighters can clearly contain the fire. $\square$

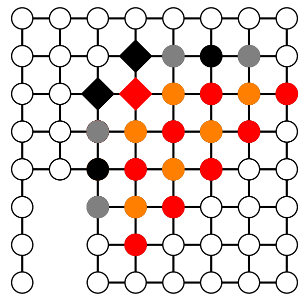

Figure 3.2.4: The beginning of the firefighters' strategy for the grid with a straight path removed.

It is difficult to remove a simpler infinite connected structure from the grid, so we can see that containing the fire here with two firefighters is very close to being able to contain the fire in the original grid. Based on this it appears that two firefighters are nearly sufficient in the square grid when $d = 2$. This notion of being nearly sufficient is further strengthened by Lemma 3.2.4.

Lemma 3.2.4. *In the firefighter game with two firefighters on the infinite square grid with* $d = 2$*, if there is one extra firefighter on an unspecified turn, then the firefighters can contain the fire.*

Proof. Without loss of generality, suppose the fire starts at $(1,1)$ since $G_{\square}$ is vertex transitive. Observe that if the fire is initially corralled by moving along the positive x- and y-axes, then when the extra firefighter is available one of the firefighters can turn ninety degrees and corral the fire to a column (see Figures 3.2.5, 3.2.6, and 3.2.7). Theorem 3.2.1 then tells us the fire can be contained. $\square$

Figure 3.2.5 depicts examples of how some strategies work when deploying the extra firefighter on different turns. The fire starts in the bottom left of the column in all four diagrams on the left with the extra firefighter being deployed on the fourth, third, second, and first turns respectively. Figure 3.2.6 shows how the right firefighter moves over to meet the left firefighter, and Figure 3.2.7 shows an example of how the two firefighters subsequently move over to the top of the fire.

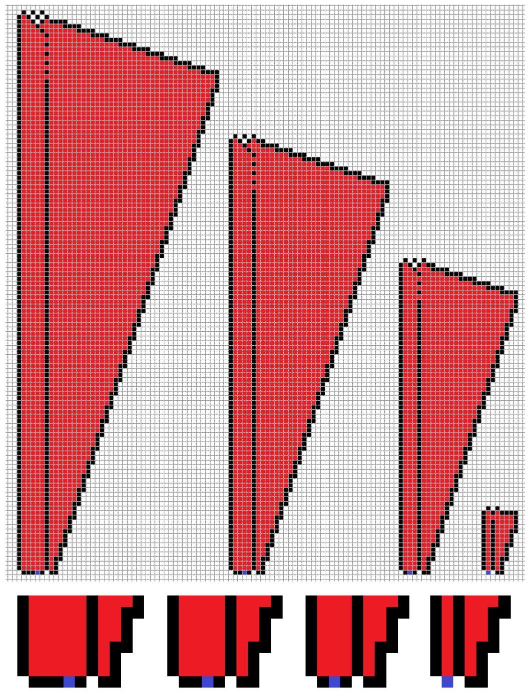

Figure 3.2.5: Some examples of the strategy with 2,2,. . . ,2,3,2,2,. . . firefighters. The bottom image is zoomed in to make the extra firefighter (blue) more visible.

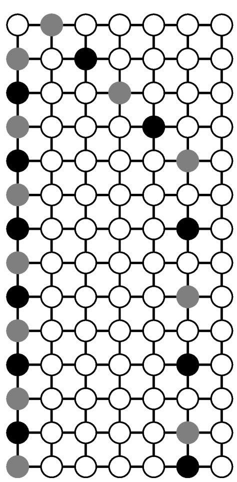

Figure 3.2.6: How the right firefighter moves to join the left firefighter.

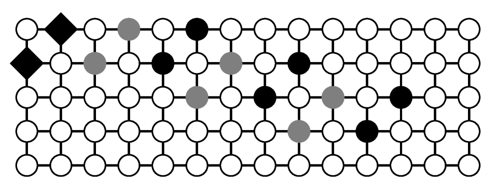

Figure 3.2.7: How the two firefighters move to be above the fire.

We can also allow for a slightly different variation on our rules to further solidify that two firefighters are almost sufficient in the case of $d = 2$. If we require that the firefighters' average distance moved on each turn is at most two, then the firefighters can very nearly contain the fire. In fact the firefighters can contain the fire if they are permitted to move through the fire once. We refer to this modified game as *sum-distance* firefighting and we can see the aforementioned containment with two firefighters in Figure 3.2.8. Observe that the firefighters have the fire corralled to a column at the last turn in the figure, so by Theorem 3.2.1 the firefighters can contain the fire. Notice that both relaxations of the rules are necessary for this strategy to work. From turn 0 to turn 1 the top firefighter both moves through the fire and moves a distance of 3, and from turn 1 to turn 2 the left firefighter also moves a distance of 3.

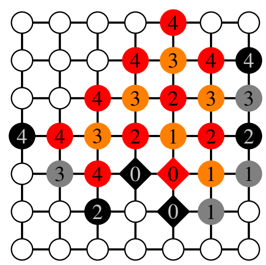

Figure 3.2.8: The initial part of the strategy for containing the fire in the sum-distance game with two firefighters when $d = 2$.

From Theorem 3.2.1, Lemma 3.2.3, Lemma 3.2.4, and Figure 3.2.8 it is clear that very slight deviations from the problem of containing the fire with two firefighters when $d = 2$ results in the firefighters being able to contain the fire. This is despite the fact that it appears to be impossible for two firefighters to contain the fire on the infinite square grid when $d = 2$, which we formalize in Conjecture 3.2.5.

Conjecture 3.2.5. *Two firefighters do not suffice to contain the fire on the infinite square grid when $d = 2$.*

We further expand upon Conjecture 3.2.5 below in Conjecture 3.2.6 using a notion from [10] with different notation. The notion of saving some portion $\rho \in [0,1]$ of the vertices with some predetermined strategy is defined as $\liminf_{n\to\infty} \frac{|B_n|}{|D_n|} = \rho$. Here D_n is the set of vertices at distance n or less from where the fire broke out and B_n is the set of vertices at distance n or less from where the fire broke out that will eventually burn.

Conjecture 3.2.6. *Two firefighters cannot save more than $\frac{3}{4}$ of the vertices on the square grid when $d = 2$.*

If Conjecture 3.2.6 is true then it represents a tight bound since the firefighters can easily save $\frac{3}{4}$ of the grid by initially defending the vertices left of the fire and below the fire and then moving along the edge of the fire as it spreads.

We can also note that if two firefighters were to suffice to contain the fire, the strategy used would have to be somewhat clever since as we will see in the proof of Theorem 3.2.7 the firefighters cannot simply defend vertices which are about to burn.

Theorem 3.2.7. *A fire on $G_\Box$ cannot be contained by two firefighters when $d = 2$ if the firefighters only play in the set of vertices that are about to burn.*

Proof. This proof can be split into three cases.

Case 1: The firefighters initially protect two vertices that are diagonal from one another.

In this case we can see that the firefighters will only ever have one choice for where to go and that the resulting strategy does not contain the fire. Figure 3.2.9 below represents burned vertices in red, protected vertices in black, and vertices that are about to be burned and can be protected in green. Clearly the firefighters will only be able to move along the sides of the fire and thus will never contain it.

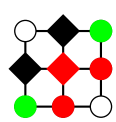

Figure 3.2.9: First strategy when $d = 2$.

Case 2: The firefighters initially protect two vertices on opposite sides of the fire.

In this case the firefighters initially have two choices as to where they can move, but once they have picked their initial move the rest of their moves are determined and will not contain the fire, as seen in Figure 3.2.10.

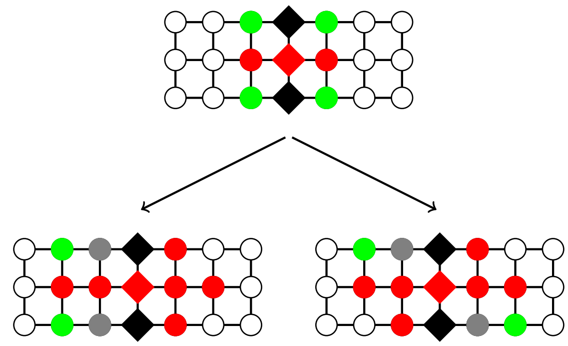

Figure 3.2.10: Second strategy when $d = 2$.

Case 3: The firefighters initially both protect the same vertex.

This case is essentially the same as the previous case. The firefighters each initially have one choice and can then only move along the edge of the fire. This is demonstrated below in Figure 3.2.11.

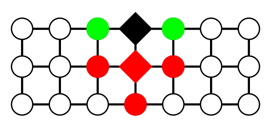

Figure 3.2.11: Third strategy when $d = 2$.

Thus since these three cases represent all possible starting configurations (up to isomorphism) we have proven the Theorem. $\square$

Chapter 4

Strong and Hexagonal Grids

4.1 Strong Grid

One of the main results from [4] shows that eight firefighters are necessary on the strong grid when $d = 1$ (i.e. $f_1(G_\boxtimes) = 8$), and the strategy illustrated in Figure 4.1.1 shows that $f_2(G_\boxtimes) \leq 4$. It is also known that in the original game four firefighters is the minimum number of firefighters where the fire can be contained on the strong grid [13]. Thus the firefighters cannot contain the fire with three firefighters at any distance. If the distance is increased, the same strategy can be used to contain the fire, so all distances greater than two also require four firefighters.

Lemma 4.1.1. *Four firefighters are necessary and sufficient to contain the fire on the strong grid when $d \geq 2$.*

Proof. Refer to Figure 4.1.1 to see that four firefighters are sufficient[1]. To confirm that four firefighters are necessary, see Theorem 22 from [13] which states that three firefighters do not suffice in the original game and therefore they do not suffice for the distance-restricted game as well. Thus four firefighters are both necessary and sufficient to contain the fire on the strong grid when $d = 2$. $\square$

[1]There are instances in Figure 4.1.1 where a vertex will have a firefighter on that vertex on multiple turns. In these cases we colour the inside of the vertex the colour the second firefighter would have, and the outside of the vertex is coloured the colour of the initial firefighter. This could cause issues if both firefighters are there on even turns or they are both there on odd turns. However, for us this does not happen so we can disregard it.

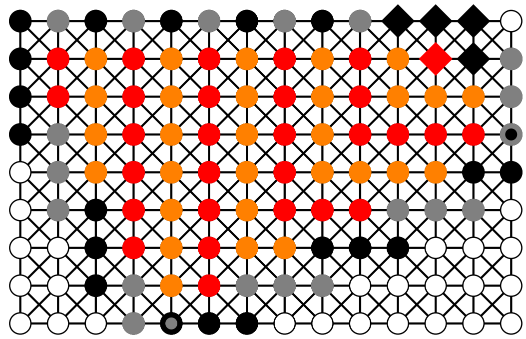

Figure 4.1.1: A strategy for containment with four firefighters on the strong grid when $d = 2$. One firefighter moves across the top while the other three spiral around from the other side.

In terms of average firefighting, note that a strategy is given for the strong grid which uses $3 + \frac{1}{T}$ firefighters for any $T \in \mathbb{Z}^+$ in [14]. Specifically, there is a strategy where whenever the turn number is 0 (mod T) there are four firefighters available, and the rest of the time there are three firefighters available. This strategy almost satisfies the restrictions of the distance-restricted game when $d = 2$ and only needs slight modification to satisfy the restrictions.

The initial part of the strategy has the firefighters contain the fire to a quarter plane, but then has one of the firefighters jump from being on the right side to being on the left side. We simply modify this so that instead of maintaining the right wall and moving around the fire from top to bottom on the left, we maintain the top wall and move around the fire from right to left along the bottom. The first four moves of this strategy are drawn below in Figure 4.1.2, when $T = 2$.

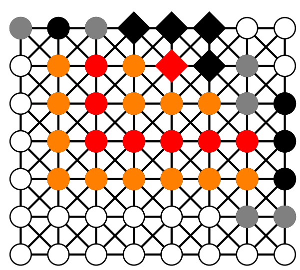

Figure 4.1.2: The start of a strategy for containment for a sequence with an average of $3 + \frac{1}{T}$ firefighters on the strong grid when $d = 2$ and $T = 2$.

4.2 Hexagonal Grid

The hexagonal grid represents an interesting challenge as the minimum number of firefighters required in the original firefighting game is the subject of Messinger's Conjecture [13]. Messinger's Conjecture states that on the hexagonal grid, one firefighter is insufficient to contain the fire. It was shown in [10] that a fire can be contained on the hexagonal grid if there is always one firefighter available except on turns t_1, t_2 where there is one additional firefighter. The bound was further improved to only require one extra firefighter on a single turn t_1 in [5]. This presents a new challenge for us as unlike previous cases we do not inherit a good lower bound from the original game.

First we will observe another interesting difference about the distance-restricted game in comparison to the original game. If we consider the strategies for containing the fire with one firefighter plus an extra firefighter on an extra turn or two and try to find a distance that will always allow the strategy to be transferred, then we will quickly see that a problem arises. Namely, the distance becomes a monotonic increasing function of the turn number on which the extra firefighter is available. Thus no matter what distance we pick, there will be a turn number t_1 where if the extra firefighter comes on or after t_1 then the strategy given in [5] will violate our distance restriction. So we can see that in this case, there may be no strategy for which one firefighter with one extra firefighter or even two extra firefighters can always contain the fire for some fixed distance, although given a large enough distance there will be small values of t_1 where the firefighters can contain the fire.

For the case of $d = 1$ observe in Figure 4.2.1 that every vertex has three paths that can be drawn out in a similar way as the four paths from the case of $d = 1$ on the square grid. That is to say that defending one of these paths stops that firefighter from defending either of the other two paths. This can be formalized by saying that the i^{th} vertices (assuming the initial shared vertex is the 0^{th} vertex) in any pair of these paths are at a minimum distance of i from each other. Thus when $d = 1$ three firefighters are required and clearly three firefighters suffice.

Lemma 4.2.1. *Three firefighters are necessary and sufficient to contain the fire on the hexagonal grid when $d = 1$.*

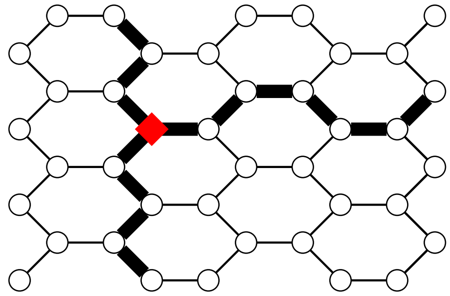

Figure 4.2.1: The three paths used to show that two firefighters do not suffice on the hexagonal grid when $d = 1$.

Now observe that in the case of $d = 2$ the fire can be contained in five turns as illustrated in Figure 4.2.2. For any distance greater than 2 the same strategy can be applied and thus two firefighters are also sufficient when $d > 2$.

Lemma 4.2.2. *Two firefighters suffice to contain the fire on the hexagonal grid when $d = 2$.*

Proof. See Figure 4.2.2. □

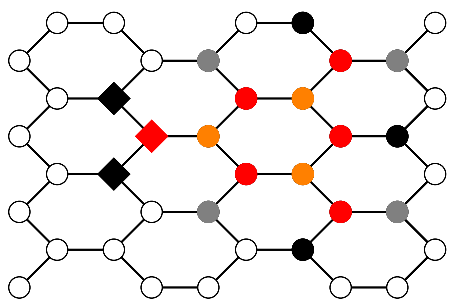

Figure 4.2.2: A strategy for containment with two firefighters on the hexagonal grid when $d = 2$.

We also conjecture that two firefighters are necessary to contain the fire in the case of $d = 2$ which we formalize in Conjecture 4.2.3. This conjecture is a weaker version of Messinger's Conjecture, since if one firefighter is insufficient for the original game then it is certainly the case that one firefighter is insufficient for the distance-restricted game. Thus at least two firefighters would be necessary as stated in Conjecture 4.2.3.

Conjecture 4.2.3. *One firefighter does not suffice to contain the fire in the hexagonal grid when $d = 2$.*

This special case of Messinger's Conjecture has a nice property which should make a proof more attainable. Namely, whenever the firefighter moves a distance of 2 its start and end point have exactly one neighbour in common. This common neighbour vertex now has all but one of its neighbours protected, and so this vertex is essentially protected since the fire reaching this vertex does not change anything in terms of how much the fire will spread as it cannot spread from this vertex to any new vertices. So in essence, any strategy that the firefighter employs is equivalent to protecting the vertices of some walk in the grid. This could be used to show that any strategy containing the fire would imply that a spiral strategy would work. In Figure 4.2.3 the start of the spiral strategy on the hexagonal grid is demonstrated.

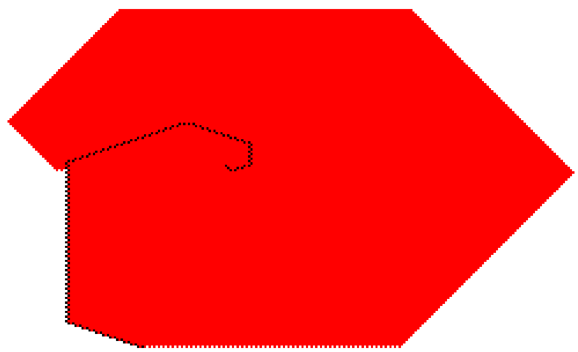

Figure 4.2.3: The spiral strategy on the hexagonal grid. Here the hexagonal grid is represented as a subgraph of the square grid.

We initially planned to tackle this problem similarly to how we approached the problem of a radius k fire on the square grid with two firefighters when $d = 3$ as seen in Section 3.1. The problem with building a recurrence relation for this spiral is that, unlike with the square grid, when the firefighters turn a corner they do not follow the same pattern to build the next segment of the barrier. Thus it becomes much more complicated to build a recurrence since the length of each segment is not only dependent on how many times the firefighter has turned, but also on the parity of the number of times the firefighter has turned.

Graphs of maximum degree 3 have previously been examined using a similar idea. In [7] it is shown that any graph with maximum degree at most 3 that has a vertex r of degree at most 2 permits a polynomial time solution to the firefighter decision problem when the fire breaks out at r. This proof relies on the fact that using a single firefighter the fire can be forced to follow a path of the firefighter's choosing and so the solution only requires finding the shortest path that is either about to self intersect or ends at a vertex of degree less than 3. However our work differs from this as the firefighter is the one forming a walk in our case, rather than the fire.

Chapter 5

Conclusions

Throughout this thesis we have seen many upper bounds on $f_d(G,u)$ for fixed G and d, and progress has been made towards establishing some lower bounds to complement them. Lower bounds are of particular interest in general, and especially for this game. The proofs of these lower bounds cannot make use of widely used tools like Fogarty's Hall-type condition [9] as the firefighters' positions are no longer a sequence of arbitrary moves. The way that every move affects all moves that follow makes the problem of distance-restricted firefighting more complicated than the original game.

Alongside these questions about lower bounds for particular graphs, there are also questions about the game in general. For example we initially wondered if every infinite, planar, k-regular, vertex-transitive graph would require k firefighters to contain the fire when $d = 1$. If we consider $P_{\mathbb{Z}} \square C_n$ for any $n \in \{3,4,5,\ldots\}$, observe that two firefighters can contain the fire by starting far enough away from the fire along $P_{\mathbb{Z}}$ in either direction and then forming a protective barrier in the form of an n-cycle. Moreover, for any vertex in the graph, the subgraph induced by the set of vertices within distance $\lfloor \frac{n-2}{2} \rfloor$ is isomorphic to the subgraph induced by the same process for any vertex in the square grid.

The embedding we have for this graph places the cycles corresponding to the nonnegative integer vertices on the circles of radius $1,2,3,\ldots$ centered around the origin in $\mathbb{R}^2$ and the cycles corresponding to the negative integers on the circles of radius $\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots$ centered around the origin in $\mathbb{R}^2$. However, in [1] infinite planar graphs are required to be embeddable with a *proper embedding*. A *proper embedding* requires the embedding to not have an accumulation point, which our embedding clearly does at $(0,0)$. It is still open as to whether or not every infinite, planar, k-regular, vertex-transitive graph G which also admits a proper embedding in $\mathbb{R}^2$ has $f_1(G) = k$.

Another question we have is under which conditions $f_d(H,u) \leq f_d(G,u)$ for H a subgraph of G and $u \in V(H)$. We have seen that this inequality does not hold in general, but we have seen that the strong, triangular, square, hexagonal and our subdivided hexagonal grid all have this property. All the examples we have of the condition not holding have to do with the fact that, unlike in the original model, the set of vertices the firefighters can eventually reach is reduced if the set of burnt vertices is a vertex cut. Due to this fact, the condition likely involves the connectedness of both G and H, potentially in relation to the degree of their vertices. For example, all of our grids except the subdivided hexagonal grid are k-regular and k-connected, but the subdivided hexagonal grid is not regular. So it is possible that G and H being r-regular and k-connected (for possibly different values of k) could imply that $f_d(H,u) \leq f_d(G,u)$, but a different condition than regularity would be needed for a full characterization.

We of course also have the conjectures we made through the thesis which we reiterate here.

Conjecture 3.2.2. *Let $d = 2$. Two firefighters can contain the fire in the square grid if and only if they can contain the fire to a column.*

Conjecture 3.2.5. *Two firefighters do not suffice to contain the fire on the infinite square grid when $d = 2$.*

Conjecture 3.2.6. *Two firefighters cannot save more than $\frac{3}{4}$ of the vertices on the square grid when $d = 2$.*

Conjecture 4.2.3. *One firefighter does not suffice to contain the fire in the hexagonal grid when $d = 2$.*

We consider Conjectures 3.2.5 and 3.2.6 to be the more interesting conjectures as Conjecture 3.2.2 is something that would likely be proven in order to then prove the other two conjectures from Chapter 3. Even if Conjecture 3.2.6 is unsolved, Conjecture 3.2.5 would still be an interesting result as it would represent a new proof of a lower bound in firefighting that is non-trivial and does not use Fogarty's Hall-type condition.

Conjecture 4.2.3 would also be an interesting result, especially if Messinger's conjecture was shown to be false. If Messinger's conjecture were false, Conjecture 4.2.3 would remain open and would open up a new question of which values of d permit a strategy where one firefighter can contain the fire on the hexagonal grid. Alternatively, proving Conjecture 4.2.3 would imply that any counterexample to Messinger's conjecture would require the firefighter to move a distance greater than 2 at least once. In any case, Conjecture 4.2.3 could lead to some interesting explorations into how the game behaves when the firefighters' strategy can be considered as a walk.

Bibliography

[1] E. W. Aurand. Infinite planar graphs. Phd. thesis, University of North Texas, 2000.

[2] A. Burgess, J. Marcoux, and D. Pike. Firefighting with a distance-based restriction. *arXiv preprint arXiv:2204.01908*, 2022.

[3] X. Chen, X. Hu, C. Wang, and Y. Zhang. Continuous firefighting on infinite square grids. In *Theory and applications of models of computation*, volume 10185 of *Lecture Notes in Comput. Sci.*, pages 158–171. Springer, Cham, 2017.

[4] S. Days-Merrill. Firefighter problem played on infinite graphs. Honors thesis, Bridgewater State University, 2019.

[5] A. Dean, S. English, T. Huang, R. A. Krueger, A. Lee, M. Mizrahi, and C. Wheaton-Werle. Firefighting on the hexagonal grid. *Discrete Appl. Math.*, 305:16–22, 2021.

[6] M. Develin and S. G. Hartke. Fire containment in grids of dimension three and higher. *Discrete Appl. Math.*, 155(17):2257–2268, 2007.

[7] S. Finbow, A. King, G. MacGillivray, and R. Rizzi. The firefighter problem for graphs of maximum degree three. *Discrete Math.*, 307(16):2094–2105, 2007.

[8] S. Finbow and G. MacGillivray. The firefighter problem: a survey of results, directions and questions. *Australas. J. Combin.*, 43:57–77, 2009.

[9] P. Fogarty. Catching the fire on grids. Masters thesis, University of Vermont, 2003.

[10] T. Gavenčiak, J. Kratochvíl, and P. Prałat. Firefighting on square, hexagonal, and triangular grids. *Discrete Math.*, 337:142–155, 2014.

[11] B. Hartnell. Firefighter! An application of domination. In *25th Manitoba Conference on Combinatorial Mathematics and Computing*, 1995.

[12] A. King and G. MacGillivray. The firefighter problem for cubic graphs. *Discrete Math.*, 310(3):614–621, 2010.

[13] M.-E. Messinger. Firefighting on infinite grids. Masters thesis, Dalhousie University, 2005.

[14] M.-E. Messinger. Average firefighting on infinite grids. *Australas. J. Combin.*, 41:15–28, 2008.

[15] A. Meurer, C. P. Smith, M. Paprocki, O. Čertík, S. B. Kirpichev, M. Rocklin, A. Kumar, S. Ivanov, J. K. Moore, S. Singh, T. Rathnayake, S. Vig, B. E. Granger, R. P. Muller, F. Bonazzi, H. Gupta, S. Vats, F. Johansson, F. Pedregosa, M. J. Curry, A. R. Terrel, Š. Roučka, A. Saboo, I. Fernando, S. Kulal, R. Cimrman, and A. Scopatz. SymPy: symbolic computing in python. *PeerJ Comput. Sci.*, 3:e103, Jan. 2017.

[16] C. Wagner. A new survey on the firefighter problem. Masters thesis, University of Victoria, 2021.

[17] P. Wang and S. A. Moeller. Fire control on graphs. *J. Combin. Math. Combin. Comput.*, 41:19–34, 2002.

[18] D. B. West. *Introduction to Graph Theory*. Prentice Hall, 2 edition, September 2000.