Some results on PA-provably recursive functions (Draft)

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Abstract

We provide some results which emerged from joint research carried out at CRM. The theorems are inspired by analogy with situations related to forcing.

1 Vague analogies with forcing

In set theory one has a "ground model" with a given set of functions from ω to ω. Then three things can happen when passing to a larger model, as exemplified by:

1. Sacks forcing: One adds new functions but any new function is dominated by an old (ground model) function.

2. Cohen forcing: One adds a new function that cannot be dominated by a ground model function but no single function which dominates (mod finite) all ground model functions. If one adds two such functions f,g using "Cohen times Cohen" forcing, then in addition any function added by both f and g is in the ground model.

3. Hechler forcing: One adds a new function that dominates (mod finite) all ground model functions. If one adds two such functions f,g using "Hechler times Hechler" forcing, then again any function added by both f and g is in the ground model.

The analogy in proof theory is the following: Fix a theory T like PA. Let us take the "T-provably recursive" functions to be those with primitive recursive graph (the honest functions) such that for some choice of primitive recursive representation of that graph, totality of the function is T-provable.

Our Theorem 1 says the following: There is a "natural" total recursive function f with primitive recursive graph which is not PA-provably recursive (via any primitive recursive representation of its graph) and such that no provably recursive function of $PA+$"f is total " (expressed using any primitive recursive graph representation) dominates (mod finite) all provably recursive function of PA. This is the proof-theoretic analogue of Cohen forcing.

The situation is similar for Theorem 2. It says that there are "natural" functions f_0, f_1 with primitive recursive graph which are not provably recursive in PA (via any primitive recursive graph representation), yet any function which is provably recursive in both $PA + Total(f_0)$ and $PA + Total(f_1)$ (where the latter are expressed using primitive recursive graph representations) is in fact provably recursive in PA. This is the analogue of Cohen times Cohen forcing.

2 The results

Theorem 1 *There exists an honest number-theoretic function f such that f is not provably recursive in PA and such that any g which is provably total in $PA + Tot(f)$ does not eventually dominate every PA-recursive function.*

Proof. (The proof is inspired by [2].) We construct f in stages. $d_0 := 0$. Assume we are at stage $s = 2i$ and that d_s is defined. Assume that $f(x)$ is defined for $x < d_s$. $d_{s+1} := H_{\varepsilon_0}(d_s)$. We extend f by $f(x) := H_{\varepsilon_0}(x)$ for $d_s \leq x < d_{s+1}$.

Assume that $s = 2i + 1$. Let $d'_{s+1} := H_{\varepsilon_0}(d_s)$ and $d_{s+1} := H_{\varepsilon_0}(d'_{s+1})$. We extend f by $f(x) := d'_{s+1} + x$ for $d_s \leq x < d_{s+1}$. Moreover let $\overline{f}_s(x) := f(x)$ for $x < d_s$ and $\overline{f}_s(x) := d'_{s+1} + x$ for $d_s \leq x$.

Since $f(x) = H_{\varepsilon_0}(x)$ for infinitely many x we see that f is not provably recursive in PA. Assume now that $PA + Tot(f)$ proves $Tot(g)$. Then there exists an $\alpha < \varepsilon_0$ such that for all x we have $g(x) < f^\alpha(x)$ where

$$f^\alpha(x) := \max(\{f(x)\} \cup \{f^\beta(f^\beta(x)) : \beta < \alpha \wedge N\beta \leq f(N\alpha + x)\}).$$

Here $N\alpha$ is defined via $N0 := 0$ and $N\alpha := N\beta + N\gamma + 1$ if α has the normal form $\omega^\beta + \gamma$.

By construction we know that for each odd s we have $\overline{f}_s(x) \leq H_{d'_{s+1}}(x)$. Choose an odd stage s with $2 \cdot N\alpha + 12 \leq d'_{s+1}$. Then $\overline{f}_s^\alpha(d'_{s+1}) \leq H_{\varepsilon_0}(d'_{s+1})$.

For a proof we apply Theorem 4 from [1] (and tacitly Lemma 9 from the same article). With this we obtain from $\overline{f}_s(x) \leq H_{d'_{s+1}}(x)$ (which holds for all x) that

$$\begin{aligned}
& \overline{f_s}^\alpha(d'_{s+1}) \\
\leq\ & H_{\omega^{\alpha+d'_{s+1}+1}+8}(d'_{s+1}) \\
\leq\ & H_{\omega^{\alpha+d'_{s+1}+1}+8}(d'_{s+1} + N\alpha + 10) \\
\leq\ & H_{\omega^{\alpha+\omega}}(d'_{s+1} + N\alpha + 10) \\
\leq\ & H_{\omega^{\alpha+\omega}}(H_{\alpha+10}(d'_{s+1})) \\
\leq\ & H_{\omega^{\alpha+\omega}+\alpha+10}(d'_{s+1}) \\
\leq\ & H_{\varepsilon_0}(d'_{s+1}).
\end{aligned}$$

Now we prove: $(*)$ $\quad \overline{f_s}^{\alpha}(y) \leq H_{\varepsilon_0}(d'_{s+1}) \Rightarrow f^{\alpha}(y) = \overline{f_s}^{\alpha}(y)$ by induction on $\alpha < \varepsilon_0$. Assume that $f^{\alpha}(y) = f^{\beta}(f^{\beta}(y))$ for some $\beta < \alpha$ with $N\beta \leq f(N\alpha + y)$. We know that $\overline{f_s}(N\alpha + y) \leq H_{\varepsilon_0}(d'_{s+1})$ hence $\overline{f_s}(N\alpha + y) = f(N\alpha + y) \geq N\beta$. Thus $\overline{f_s}^{\beta}(\overline{f_s}^{\beta}(y)) \leq \overline{f_s}^{\alpha}(y) \leq H_{\varepsilon_0}(d'_{s+1})$.

So $\overline{f_s}^{\beta}(y) \leq H_{\varepsilon_0}(d'_{s+1})$ and the i.h. yields $\overline{f_s}^{\beta}(y) = f^{\beta}(y)$ and hence $\overline{f_s}^{\beta}(\overline{f_s}^{\beta}(y)) = f^{\beta}(\overline{f_s}^{\beta})(y)) = f^{\beta}(f^{\beta}(y))$.

Since $f^{\alpha}(y) = f^{\beta}(f^{\beta}(y)) = \overline{f_s}^{\beta}(\overline{f_s}^{\beta}(y)) \leq \overline{f_s}^{\alpha}(y)$ we are done with the proof of $(*)$.

Putting things together we obtain for large enough s that

$$g(d'_{s+1}) < f^{\alpha}(d'_{s+1}) = \overline{f_s}^{\alpha}(d'_{s+1}) \leq H_{\omega^{\alpha \# \omega}+\alpha+10}(d'_{s+1}).$$

So the function $H_{\omega^{\alpha \# \omega}+\alpha+10}$ is not eventually dominated by g.

Theorem 2 *There are two honest recursive functions f_0, f_1 which are not provably recursive in PA such that if a function g is provably recursive in $PA + Tot(f_0)$ and $PA + Tot(f_1)$ then g is provably recursive in PA.*

Proof. (The proof is inspired by [2].) We construct f_0, f_1 in stages. $d_0 := 0$. Assume we are at stage $s = 2i$ and that d_s is defined. Assume that $f_i(x)$ are defined for $x < d_s$. Put $\hat{d}_{s+1} := H_{\varepsilon_0}(d_s)$. Define $\overline{f_{1,s}}$ by $\overline{f_{1,s}}(x) := f_1(x)$ for $x < d_s$ and $\overline{f_{1,s}}(x) := f_1(d_s - 1) + x$ for $x \geq d_s$.

Put

$$d'_{s+1} := \mu n : \exists x < n[x \geq \hat{d}_{s+1} \wedge \overline{f_{1,s}}^{\omega_i}(x) < H_{\varepsilon_0}(x) \leq n]$$

and $d_{s+1} := H_{\varepsilon_0}(d'_{s+1})$.

Extend f_0 by $f_0(x) := f_0(x)$ for $x < d_s$, $f_0(x) := \hat{d}_{s+1} + x$ for $\hat{d}_{s+1} > x \geq d_s$ and $f_0(x) := H_{\varepsilon_0}(x)$ for $d'_{s+1} > x \geq \hat{d}_{s+1}$ and $f_0(x) := d_{s+1} + x$ for $d_{s+1} > x \geq d'_{s+1}$. Define $\overline{f_{0,s}}$ by $\overline{f_{0,s}}(x) := f_0(x)$ for $x < d'_{s+1}$ and $\overline{f_{0,s}}(x) := d_{s+1} + x$ for $x \geq d'_{s+1}$. Extend f_1 by $f_1(x) := f_1(x)$ for $x < d_s$ and $f_1(x) := \overline{f_{1,s}}(x)$ for $d_{s+1} > x \geq d_s$. Assume we are at stage $s = 2i + 1$ and that d_s is defined. We interchange the roles of f_0 and f_1. That means: Assume that $f_i(x)$ are defined for $x < d_s$. Put $\hat{d}_{s+1} := H_{\varepsilon_0}(d_s)$. Define $\overline{f_{0,s}}$ by $\overline{f_{0,s}}(x) := f_0(x)$ for $x < d_s$ and $\overline{f_{0,s}}(x) := f_0(d_s - 1) + x$ for $x \geq d_s$.

Put

$$d'_{s+1} := \mu n : \exists x < n[x \geq \hat{d}_{s+1} \wedge \overline{f_{0,s}}^{\omega_i}(x) < H_{\varepsilon_0}(x) \leq n]$$

and $d_{s+1} := H_{\varepsilon_0}(d'_{s+1})$. Extend f_1 by $f_1(x) := f_1(x)$ for $x < \hat{d}_{s+1}$ and $f_1(x) := H_{\varepsilon_0}(x)$ for $d'_{s+1} > x \geq \hat{d}_{s+1}$ and $f_1(x) := d_{s+1} + x$ for $d_{s+1} > x \geq d'_{s+1}$.

Define $\overline{f_{1,s}}$ by $\overline{f_1}(x) := f_1(x)$ for $x < d'_{s+1}$ and $\overline{f_{1,s}}(x) := d_{s+1} + x$ for $x \geq d'_{s+1}$. Extend f_0 by $f_0(x) := f_0(x)$ for $x < d_s$ and $f_0(x) := \overline{f_{0,s}}(x)$ for $d_{s+1} > x \geq d_s$.

We write $f_0 \wedge f_1$ for $x \mapsto \min\{f_0(x), f_1(x)\}$. Then $(f_0 \wedge f_1)(x) \leq 2 \cdot x$.

Assume that $PA + Tot(f_1) \vdash Tot(f_0)$. Then, by [1], there is an $\alpha < \varepsilon_0$ such that for all x we have $f_0(x) < f_1^{\alpha}(x)$. Choose i_1 such that $\alpha < \omega_{i_1}$. Then for $x \geq N\alpha$ we have $(**)$ $f_0(x) < f_1^{\omega_{i_1}}(x)$. Assume that s is $2i+1$ with $i > \max\{N\alpha, i_1\}$. Then there is an $x < d'_{s+1}$ with $x \geq d_s$ such that $\overline{f_1}^{\omega_i}(x) < H_{\varepsilon_0}(x) \leq d'_{s+1}$. Since f_1 and $\overline{f_1}$ agree on $x < d_{s+1}$ we obtain $f_1^{\omega_i}(x) = \overline{f_1}^{\omega_i}(x) < H_{\varepsilon_0}(x) = f_0(x)$. Contradiction with $(**)$. By a symmetric argument $PA + Tot(f_0)$ does not prove $Tot(f_1)$.

Now assume that $PA + Tot(f_0) \vdash Tot(g)$ and $PA + Tot(f_1) \vdash Tot(g)$. Then there exist α_i such that for all x we have $g(x) < f_i^{\alpha_i}(x)$. Our idea is now to show that $(f_0^{\alpha_0} \wedge f_1^{\alpha_1})(x) \leq (f_0 \wedge f_1)^{\alpha_0 \# \alpha_1}(x)$. Since $(f_0 \wedge f_1)(x) \leq 2 \cdot x$ this would yield the claim. But the obvious verification doesn't seem to work.

For this purpose let us introduce another iteration hierarchy.

$$f_\alpha(x) := \max(\{f(x)\} \cup \{f_\beta(f_\beta(x)) : \beta < \alpha \wedge N\beta \leq N\alpha + x\}).$$

This hierarchy behaves better with respect to the $\wedge$ operator.

Indeed for two increasing functions f, h we have

$$(f_\alpha \wedge h_\beta)(x) \leq (f \wedge h)_{\alpha \# \beta}(x).$$

This is proved by induction on $\alpha \# \beta$. Assume first that $\beta = 0$. Then $h_\beta(x) = h(x)$. If $\alpha = 0$ then $f_\alpha(x) = f(x)$ and the assertion is clear. If $\alpha > 0$ then $f_\alpha(x) = f_\gamma(f_\gamma(x))$ for some $\gamma < \alpha$ with $N\gamma \leq N\alpha + x$. We have $(f \wedge h)_\alpha(x) \geq (f \wedge h)_\gamma((f \wedge h)_\gamma(x))$. The induction hypothesis yields $(f \wedge h)_\gamma(x) \geq (f_\gamma \wedge h)(x)$. If $(f_\gamma \wedge h)(x) = h(x)$ then $(f \wedge h)_\alpha(x) \geq (f \wedge h)_\gamma(h(x)) \geq h(x)$ and the assertion follows. So assume $(f_\gamma \wedge h)(x) = f_\gamma(x)$. We then have $(f_\gamma \wedge h)(f_\gamma \wedge h) = (f_\gamma \wedge h)(f_\gamma(x))$. If $(f_\gamma \wedge h)(f_\gamma(x)) = h(f_\gamma(x))$ then the assertion follows from $h(f_\gamma(x)) \geq h(x)$. We may assume that $(f_\gamma \wedge h)(f_\gamma(x)) = f_\gamma(f_\gamma(x)) = f_\alpha(x)$. Since $f_\alpha(x) \geq h(x)$ the assertion also holds in this case.

Assume (by symmetry) for the induction step that $(f_\alpha \wedge h_\beta)(x) = h_\beta(x) = h_\gamma(h_\gamma(x))$ for some $\gamma < \beta$ with $N\gamma \leq N\beta + x$. The inequality $f_\alpha(x) \geq h_\beta(x)$ yields $h_\gamma(x) \leq (f_\alpha \wedge h_\gamma)(x) \leq (f \wedge h)_{\alpha \# \gamma}(x)$ by the induction hypothesis. The inequality $f_\alpha(h_\gamma(x)) \geq f_\alpha(x) \geq h_\beta(x)$ yields $h_\gamma(h_\gamma(x)) \leq f_\alpha(h_\gamma(x)) \leq (f_\alpha \wedge h_\gamma)(h_\gamma(x)) \leq (f \wedge h)_{\alpha \# \gamma}(h_\gamma(x))$ by the induction hypothesis. Putting things together we obtain $h_\gamma(h_\gamma(x)) \leq (f \wedge h)_{\alpha \# \gamma}((f \wedge h)_{\alpha \# \gamma}(x))$. Now we have $N(\alpha \# \gamma) \leq N(\alpha \# \beta) + x$. Hence $(f \wedge h)_{\alpha \# \gamma}((f \wedge h)_{\alpha \# \gamma}(x)) \leq (f \wedge h)_{\alpha \# \beta}(x)$ and we are done.

To prove the theorem it suffices to show that $f^\alpha(x) \leq f_{\omega^\alpha + 1}(x)$. This is proved by induction on α. Assume that $f^\alpha(x) = f^\beta(f^\beta(x))$ with $\beta < \alpha$ and $N\beta \leq f(N\alpha + x)$. The induction hypothesis yields $f^\beta(f^\beta(x)) \leq f_{\omega^\beta+1}(f_{\omega^\beta+1}(x))$. Then

$$\begin{aligned} f^\beta(f^\beta(x)) &\leq f_{\omega^\beta+1}(f_{\omega^\beta+1}(x)) \\ &\leq f_{\omega^\beta+1}(f_{\omega^\beta+1}(f(N\alpha + x))) \end{aligned}$$

$$
\begin{aligned}
&\leq f_{\omega^\alpha}(f(N\alpha + x)) \\
&\leq f_{\omega^\alpha}(f_{\omega^\alpha}(x)) \\
&\leq f_{\omega^{\alpha+1}}(x)
\end{aligned}
$$

where we made use of $f(N\alpha + x) \leq f_\alpha(x)$. The last claim is again proved by induction on α. For the induction step for proving this claim note that $f_\alpha(x) = f_\beta(f_\beta(x))$ for some $\beta < \alpha$ with $N\beta = N\alpha + x$. (It is easily seen that here $=$ has to hold). Then $f_\beta(f_\beta(x)) \geq f_\beta(x) \geq f(N\beta + x) \geq f(N\alpha + x + x) \geq f(N\alpha + x)$.

References

[1] Andreas Weiermann: Classifying the provably total functions of PA. Bulletin of Symbolic Logic, 12 (2), (2006), 177-90.

[2] Lars Kristiansen: Subrecursive Degrees and fragments of Peano arithmetic. Archive for Mathematical Logic 40, (2001), 365-397.