On unification and admissible rules in Gabbay–de Jongh logics

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In this paper we study the admissible rules of intermediate logics. We establish some general results on extension of models and sets of formulas. These general results are then employed to provide a basis for the admissible rules of the Gabbay–de Jongh logics and to show that these logics have finitary unification type.

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1 Introduction

The admissible rules of a logic are precisely those rules under which the set of its theorems is closed. These rules arise naturally from a logic, though they may not be plainly visible from its axiomatisation. When A implies B the rule A/B surely is admissible. In classical propositional logic (CPC) all admissible rules are of this form. Here we study intuitionistic propositional logic (IPC) and its axiomatic extensions, for which the situation is a bit more subtle.

Harrop (1960, Theorem 3.1) proved that the rule below is admissible for IPC. This rule is not included in standard axiomatisations of intuitionistic propositional logic, yet it is admissible for all of them: adjoining this rule yields no new theorems. An even stronger result was proven by Prucnal (1979, Theorem 1), who proved this rule to be admissible for all intermediate logics.

$$\frac{\neg C \to (A \vee B)}{(\neg C \to A) \vee (\neg C \to B)}$$

Friedman (1975, Problem 40) conjectured that there is a decision procedure to determine the admissible rules of IPC, which was later confirmed by Rybakov (1984). De Jongh and Visser conjectured that the *Visser rules* form a basis of the admissible rules of IPC, that is to say, all admissible rules of IPC become derivable after adjoining the Visser rules. This was independently proven by Rozière (1992) and Iemhoff (2001b).

The Visser rules come with a natural stratification along the natural numbers, yet they are usually studied as a whole. Although this suffices for IPC and several other intermediate logics, it falls short when considering

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logics that admit the Visser rules only up to a certain stratum. In this paper we attempt to fill that gap. Using an equivalent but ostensibly weaker form of the Visser rules, dubbed the de Jongh rules, we show that the n^{th} Visser rules constitute a basis for the admissible rules of the $(n+1)^{\text{th}}$ Gabbay–de Jongh logic as introduced by Gabbay and de Jongh (1974). The same result holds for any extension in which these rules are admissible. Due to the proof method we can show that these logics enjoy finitary unification as but a simple corollary.

The proof of the main result follows a pattern by now common in the study of admissible rules, such as in the work of Ghilardi (1999), Cintula and Metcalfe (2010), and Iemhoff and Metcalfe (2009). It is shown that the n^{th} de Jongh rule is admissible in the $(n+1)^{\text{th}}$ Gabbay–de Jongh logic, and that because of this every formula can be approximated by finitely many projective formulae, whose disjunction is admissibly equivalent to the original formula. We use a semantic characterisation introduced by Ghilardi (1999) in the context of IPC that easily generalises to the Gabbay–de Jongh logics to prove that the formulae in this admissible approximation are indeed projective. This characterisation states that a formula is projective in the $(n+1)^{\text{th}}$ Gabbay–de Jongh logic if and only if it has the n^{th} extension property. The latter means that every n models of the formula can be extended with a single root in such a way that the result is again a model of the formula.

We identify exactly those sets of formulae that satisfy the n^{th} extension property, in order to later apply the above characterisation. In Section 3 we identify those sets of formulae that give a "best saturated approximation" of a certain Kripke model. Such sets are very convenient in forming extensions, in that they exactly predict what will hold in the resulting extension. Following Iemhoff (2001b) we then seek such sets employing the vacuous implications of a model. This leads to a characterisation of the existence of extensions using a semantic and a syntactic property in Theorem 1.

From there onwards it is an easy task to achieve our goals. In Section 4 we characterise those intermediate logics with the disjunction property that have the n^{th} extension property as exactly those logics in which the n^{th} de Jongh rule is admissible. Based on results in Iemhoff (2005), this implies that the n^{th} de Jongh rule is equivalent to the n^{th} Visser rule. In Section 5 we apply these results to the Gabbay–de Jongh logics. We show that in these logics formulas have admissible approximations. Using this we can prove that the de Jongh rules up to level $n+1$ form a basis of admissibility in the n^{th} Gabbay–de Jongh logic and that these logics have finitary unification.

The phenomenon of admissibility, that is, of the existence of nontrivial admissible rules, is not restricted to intermediate logics. In many transitive modal logics the notion has the same characteristics as in intuitionistic logic and the Gabbay–de Jongh logics: there is an simple basis for the admissible rules and the unification type is finitary Ghilardi (2000), Jeřábek (2005), Rybakov (1997). Temporal and substructural logics have also been studied from this point of view Dzik (2007), Jeřábek (2010), Babenyshev and Rybakov (2011). In this paper our focus is on intermediate logics and therefore the discussion of other contexts will be limited to the few remarks above.

2 Preliminaries

We consider propositional formulae, that is, those expressions built up from a fixed set of (propositional) variables and falsity using the standard logical connectives of conjunction, disjunction and implication. In Backus–Naur form:

$$\mathcal{L}_{\mathsf{prop}} ::= \mathsf{Var} \mid \bot \mid \mathcal{L}_{\mathsf{prop}} \wedge \mathcal{L}_{\mathsf{prop}} \mid \mathcal{L}_{\mathsf{prop}} \vee \mathcal{L}_{\mathsf{prop}} \mid \mathcal{L}_{\mathsf{prop}} \to \mathcal{L}_{\mathsf{prop}}.$$

We will use upper-case latin letters at the beginning of the alphabet to refer to formulae in $\mathcal{L}_{\mathsf{prop}}$. Sets of such are denoted by lower-case latin letters near the end of the alphabet. Upper case Greek letters refer to finite sets of formulae. For aesthetic reasons we occasionally use sequents, denoted $\Gamma \Rightarrow \Delta$. We let S range over sequents, sets of sequents will be denoted by a calligraphic letter and sets of sets of sequents will be given in boldface. For notational convenience we write $\neg A$ to abbreviate $A \to \bot$ and $A \leftrightarrow B$ to abbreviate $(A \to B) \wedge (B \to A)$. We also write $\bigvee \Delta$ and $\bigwedge \Delta$ for respectively the disjunction and conjunction of the formulae in Δ, bracketing and order remain ambiguous as they are irrelevant in the logics we use. We see falsity $\bot$ as the empty disjunction and the empty conjunction as $p \to p$. A sequent $\Gamma \Rightarrow \Delta$ is said to be irreducible when Δ contains only propositional variables and Γ contains only propositional variables or implications between them.

Each formula A corresponds to a sequent, namely $\Rightarrow A$. Conversely, we interpret sequents as formulae in the standard manner as follows.

1 Definition (Associated formulae)
The *associated formula* of a sequent (set of sequents) is defined as follows.

$$\underline{\Gamma \Rightarrow \Delta} := \bigwedge_{A \in \Gamma} A \to \bigvee_{A \in \Delta} A. \qquad \underline{\mathcal{S}} := \bigwedge_{\Gamma \Rightarrow \Delta \in \mathcal{S}} \underline{\Gamma \Rightarrow \Delta}$$

2 Definition (Variables)
The *variables* in a formula (sequent, set of sequents) can be identified recursively as follows, where C abbreviates $\wedge$, $\vee$, and $\to$.

$$\begin{array}{rlrl} \mathsf{vars}\,(p) &:= \{p\} & \mathsf{vars}\,(\bot) &:= \emptyset \\ \mathsf{vars}\,(A \mathrel{\mathsf{C}} B) &:= \mathsf{vars}\,(A) \cup \mathsf{vars}\,(B) & \mathsf{vars}\,(\Gamma) &:= \bigcup_{A \in \Gamma} \mathsf{vars}\,(A) \\ \mathsf{vars}\,(\Gamma \Rightarrow \Delta) &:= \mathsf{vars}\,(\Gamma) \cup \mathsf{vars}\,(\Delta) & \mathsf{vars}\,(\mathcal{S}) &:= \bigcup_{S \in \mathcal{S}} \mathsf{vars}\,(S) \end{array}$$

A substitution maps formulae to formulae in a structure-preserving way. We define substitutions formally as below. For our purposes it is convenient that substitutions touch only finitely many variables, so we impose this non-essential restriction.

3 Definition (Substitution)
A *substitution* is map $\sigma : \mathsf{Var} \to \mathcal{L}_{\mathsf{prop}}$ that equals the identity on a cofinite subset of Var. We call the set of variables $p \in \mathsf{Var}$ such that $\sigma(p) \neq p$ its *domain*, written $\operatorname{dom}\sigma$. We extend substitutions recursively to $\mathcal{L}_{\mathsf{prop}}$ as below.

$$\begin{array}{rcll} \sigma(\bot) &:=& \bot & \\ \sigma(A \mathrel{\mathsf{C}} B) &:=& \sigma(A) \mathrel{\mathsf{C}} \sigma(B) & \text{for any } \mathsf{C} = \wedge, \vee, \to \end{array}$$

Intermediate logics play a central role in this paper, so they merit a proper definition. We will identify intermediate logics with the formulae they derive. In this way we stay clear of the intricate inner workings of the potentially distinct axiomatisations. To stress the relation between derivability and set-membership of the set of theorems we write $x \ni A$ instead of the more common $A \in x$. This has the additional benefit of keeping the position of formulae more stable in the proofs to come, as we will often switch between membership, derivability and validity in a Kripke model. We refer to the properties (i) and (ii) of the definition below as respectively being closed under substitution and satisfying *modus ponens*.

4 Definition (Intermediate logic)
An *intermediate logic* is a proper subset x of $\mathcal{L}_{\mathsf{prop}}$ containing the theorems of IPC, satisfying:

(i) if σ is a substitution and $x \ni A$ then $x \ni \sigma(A)$;

(ii) if $x \ni A \to B$ and $x \ni A$ then $x \ni B$.

A *rule* is an ordered pair of finite sets of formulae, written Γ/Δ. We call such a rule *single-conclusion* whenever $|\Delta| \leq 1$. Consequence-relations, see for instance Wójcicki (1988), provide a convenient language to reason about rules. We will work with multi-conclusion consequence relations as described by Cintula and Metcalfe (2010). The standard notion of a consequence relation is a set of single-conclusion rules satisfying the single-conclusion variants of Definition 5 below, except that there single-conclusion rules may not have empty conclusions. The reason to consider multi-conclusion consequence relations is that they can be used to express the disjunction property, as in Example 1 below.

5 Definition (Finitary multi-conclusion consequence relation)
A *finitary multi-conclusion consequence relation* or *m-logic* is, denoted $\vdash$, is a relation between finite sets of formulae subject to the following axioms, where A is a formula and $\Gamma, \Theta, \Delta, \Pi$ are finite sets of formulae. We call $\vdash$ a derivability relation.

reflexivity	$A \vdash A$;
monotonicity	if $\Gamma \vdash \Delta$ then $\Gamma, \Theta \vdash \Delta, \Pi$;
transitivity	if $\Gamma \vdash \Delta, A$ and $A, \Theta \vdash \Pi$, then $\Gamma, \Theta \vdash \Delta, \Pi$;
structurality	if $\Gamma \vdash \Delta$ then $\sigma(\Gamma) \vdash \sigma(\Delta)$.

We extend the notation to infinite sets on the left by defining $x \vdash \Delta$ to mean that there exists a finite $\Gamma \subseteq x$ such that $\Gamma \vdash \Delta$. A formulae A is said to be a *theorem* of this consequence relation if $\vdash A$. Given a set of rules $\mathcal{R}$ and an *m*-logic $\vdash$ we denote the smallest *m*-logic extending $\vdash$ and $\mathcal{R}$ as $\vdash_{\mathcal{R}}$. An *m*-logic is *consistent* when there exists a formula that is not a theorem.

Any intermediate logic x gives rise to an *m*-logic $\vdash_x$ defined by

$$\Gamma \vdash_x \Delta \text{ if and only if } x \ni \bigwedge \Gamma \to A \text{ for some } \Delta \ni A.$$

One can check that all the axioms indeed are satisfied.[1] Additionally, this *m*-logic satisfies the *deduction theorem*, in that $\Gamma, A \vdash B$ is equivalent to $\Gamma \vdash A \to B$. In the following, every occurrence of $\vdash$ denotes the *m*-logic thusly associated to an intermediate logic. All lemmas are parameterised by the intermediate logic used therein unless otherwise specified.

Given a set of formulae x, a substitution σ is said to be *identity modulo* x when for all $p \in \mathsf{Var}$ we have $x \vdash \sigma(p) \leftrightarrow p$. A *unifier* for a formula A is a substitution σ such that $\sigma(A)$ is a theorem. We say that a formula A is *projective* whenever it has a unifier which is the identity modulo A, and we refer to this unifier as the *projective unifier* of A.

Above we speak of a unifier, this is indeed a unifier in the sense of Baader (1992) as explained by Baader and Ghilardi (2011). The above definition of a projective unifier coincides with that of Ghilardi (1999). One can order substitutions by generality, where σ less general than τ when there exists some substitution ρ such that $\sigma(p) \leftrightarrow \rho\tau(p)$ is a theorem for any $p \in \mathsf{Var}$. A unifier of A is said to be a *most general unifier* when it is the maximum substitution amongst the unifiers of A.

One can easily prove that whenever σ is the identity modulo x, it acts as the identity on all propositional formulae. In symbols, $x \vdash \sigma(A) \leftrightarrow A$ for all formulae A. Moreover, if σ is the identity modulo A and τ is such that $\vdash \tau(A)$, then τ is less general than σ. Indeed, $\tau(A) \vdash \tau(\sigma(p)) \leftrightarrow \tau(p)$ clearly holds for all $p \in \mathsf{Var}$. Therefore projective unifiers are, in particular, most general unifiers.

6 Definition (Admissible rule)
A rule Γ / Δ is *admissible*, written $\Gamma \mathrel{|\!\sim} \Delta$, if every substitution σ that is a unifier of all of Γ is a unifier of some formulae in Δ too.

1 Remark
Note that in the above definition we might as well assume σ to be only defined on $\mathrm{vars}(\Gamma)$. The point is that σ can be split up into two parts; one with domain contained within $\mathrm{vars}(\Gamma)$ and the other outside. Now σ is the composition of these two parts, where the second part is conjugated by a suitably chosen renaming.

It is easy to see that admissibility itself is an *m*-logic, containing $\vdash$. A set of rules $\mathcal{R}$ is a *basis* for $\mathrel{|\!\sim}$ if $\mathrel{|\!\sim} = \vdash_{\mathcal{R}}$.

1 Example (Disjunction Property)
An example of a rule is $p \vee q / \{p, q\}$. An intermediate logic which admits this rule is said to enjoy the *disjunction property* (DP). Concretely, this rule is admissible whenever $A \vee B$ is a theorem if and only if one of A or B is. As is well-known, IPC has DP and CPC does not.

We use Kripke models to reason semantically. A *Kripke structure* is a pair $\langle K, v \rangle$ where K is a partial order and $v : K \to \mathbf{P}(\mathsf{Var})$ is a monotone map, forcing is defined as usual, see for instance Troelstra and Dalen (1988). We speak of a Kripke structure when one would normally speak of a Kripke model (of IPC), and reserve the name Kripke model for a structure which is actually a model of the intermediate logic at hand. We say that a Kripke structure $\langle K, v \rangle$ is a Kripke structure *over* c to mean that $v(k) \subseteq c$ for all $k \in K$. A Kripke structure is said to be *rooted* whenever K has a least element. To each Kripke structure we associate its *theory*, the set of formulae which it forces, as

$$\mathrm{Th}(K) := \left\{ A \mid K \Vdash A \right\}.$$

[1] Note that the notation $\vdash_x$ is a special case of the notation $\vdash_{\mathcal{R}}$, by taking $\mathcal{R}$ to be the appropriate set of empty premise rules. In general though we trust that the appropriate concept will be clear from context.

Given a set of Kripke structures $\mathcal{K}$ one may construct a new Kripke structure denoted $\coprod \mathcal{K}$, the *coproduct* or *disjoint union* of $\mathcal{K}$. Here the underlying order of K is simply the coproduct of the underlying orders of $\mathcal{K}$, and the valuation is extended in the natural manner.

The logics we are concerned with in this paper are the Gabbay–de Jongh logics, which form a countable infinite increasing sequence of logics with the disjunction property introduced by Gabbay and de Jongh (1974).

7 Definition
The n^{th} Gabbay–de Jongh logic $\mathbf{D}_n$ is given as the theory of the class of finite Kripke structures of at most $(n+1)$-fold branching rooted trees. The logic $\mathbf{D}_n$ is the least intermediate logic containing the following axiom.

$$\bigwedge_{i=1}^{n+2} \left(\left(A_i \to \bigvee_{j \neq i} A_j \right) \to \bigvee_{j \neq i} A_j \right) \to \bigvee_{i=1}^{n+2} A_i$$

Given a Kripke structure K we can make a new model by adjoining a least element to K. The valuation of K carries over, and needs to be extended to this new root. There is some choice here, the possible extensions of the valuation correspond precisely with the sets of propositional variables contained within $\mathrm{Th}(K)$.

As explained in the introduction, projective formulas, one of the main ingredients in the proofs of the main theorems, can be characterised via extensions of models, which is defined as follows.

8 Definition (Extension)
Let $K = \langle K, v \rangle$ be a Kripke structure and $x \subset \mathrm{Th}(K)$ be a set of formulae. We define the *extension of* K *with* x, denoted K/x, as the Kripke structure $\langle L, v' \rangle$. Here L is the partial order with underlying set $K + \{x\}$, ordered as before on K but with $x \leq k$ for all $k \in L$. The valuation function v' is an extension of v with $v'(x) = x \cap \mathsf{Var}$. For convenience, we may speak of the extension x whenever we mean K/x. When $\mathcal{K}$ is a set of Kripke structures, we write $\mathcal{K}/x$ to mean $\coprod \mathcal{K}/x$, and we call this the extension of $\mathcal{K}$ with x.

Note that the above definition yields a bonafide Kripke structure. It is by no means necessary that $\mathrm{Th}(K/x)$ bears any meaningful relation to x, other than that they both contain the same propositional variables. In Section 3 we will identify those sets of formulae x for which $\mathrm{Th}(K/x)$ equals x.

Given a set of models $\mathcal{K}$ of a theory x one may wonder whether an extension of $\coprod \mathcal{K}$ forcing x exists. Let us define some properties regarding the existence of extensions.

9 Definition (Extension property)
A class of rooted Kripke structures $\mathcal{M}$ has the n^{th} extension property (EP_n) if for every $\mathcal{K} \subseteq \mathcal{M}$ of size at most n there is a set of formulae y such that $\mathcal{K}/y \in \mathcal{M}$. We say that $\mathcal{M}$ has the *extension property* when it has EP_n for all $n \in \mathbb{N}$. Furthermore, we say that a set of formulae has the (n^{th}) extension property when its class of Kripke structures has it.

3 Extensions

We are interested in finding extensions of (possibly non-rooted) models. In this search we may forget the model, and only look at the associated theory. In Theorem 1, the main theorem of this section, we characterise the existence of extensions in terms of sets of formulas. It will turn out that the theory of a model contains enough information to be able to reflect the existence of extensions.

In this section and the following we use a certain set of formulae c to keep a tight grip on the amount of information present. On first reading it is useful to think of c as simply the set of all formulae $\mathcal{L}_{\mathsf{prop}}$. In Section 5 we instantiate c instead to a subset of Var, in order to prove projectivity of certain formulae. For now though, simply forget c and any restrictions it imposes. This extra flexibility is used later on, yet most results remain interesting when taking c to be all propositional formulae.

The unrelativised version of most of the lemmas in this section (when c is the set of all formulae $\mathcal{L}_{\mathsf{prop}}$) have appeared implicitly in the literature before. The generalisation to arbitrary c is, however, a nontrivial one, as can be seen from the proofs below.

10 Definition (Saturated set)
Let $x \subseteq y$ and c be sets or formulae. We say that x is c-saturated in y, written $x \preccurlyeq_c y$, whenever

$$x \vdash \bigvee \Delta \quad \text{entails} \quad y \cap \Delta \neq \emptyset \quad \text{for all finite } \Delta \subseteq c.$$

A set is simply said to be c-saturated whenever it is c-saturated within itself. When c is the set of all formulae, we omit c from the nomenclature.

Note that $\bigvee \emptyset = \bot$ by convention, so the above definition ensures that x must be consistent when $x \preccurlyeq_c y$ for any c and y. Observe that for any rooted Kripke structure K its theory $\mathrm{Th}\,(K)$ is saturated. Indeed, if $\mathrm{Th}\,(K) \vdash \bigvee \Delta$ then the root of K forces $\bigvee \Delta$, hence it also forces some element of Δ. Throughout the literature there are several synonyms in use for saturated sets, amongst them are prime theories and theories with the disjunction property. Here a theory is a set of formulae x where $x \vdash A$ and $x \ni A$ are equivalent, an evident consequence of being saturated.

Let us inspect the relation $\preccurlyeq_c$. First note that provability of formulae in c out of a c-saturated set coincides with membership: if x is c-saturated and $c \ni A$ then $x \vdash A$ holds if and only if $x \ni A$ does. From now on we will freely interchange these two statements. For notational convenience we write $x \subseteq_c y$ to mean that if $x \ni A$ then $y \ni A$ for all $c \ni A$. We extend this notation in the obvious way to write $\subset_c$ and $=_c$. The following properties are all fairly straightforward, but it is convenient to have them in mind for the following.

1 Lemma
Note the following elementary properties of $\preccurlyeq_c$.

(i) $\preccurlyeq_c \subseteq \subseteq_c$.

(ii) $\subseteq \circ \preccurlyeq_c \circ \subseteq = \preccurlyeq_c$;

(iii) $\preccurlyeq_c$ is transitive;

(iv) if x is c-saturated then for all $x \subseteq y$ also $x \preccurlyeq_c y$.

Maximality of saturated sets is a bit subtle. Demanding maximality with respect to the order $\subseteq$ is too stringent whereas maximality with respect to $\subseteq_c$ is too weak. The following definition falls comfortably in the middle. One may think of this definition as demanding that there is no strict extension of x using only elements from c that is c-saturated in y.

11 Definition
Let x, y and c be sets of formulae. We say that x is *c-maximally saturated in y* when $x \preccurlyeq_c y$ and any set $z \supseteq x$ such that $z \preccurlyeq_c y$ satisfies $x =_c z$.

We may then wonder whether there exists sets that are c-maximally saturated in y. This turns out to be the case. Moreover, such sets are always saturated in themselves. These two assertions we will prove in the following two lemmas. They arise as a natural generalisation of a lemma well-established in folklore, see for instance Iemhoff (2001b, Lemma 3.4). The final statement in the Maximal saturation Lemma is void when c is $\mathcal{L}_{\mathsf{prop}}$, yet quite important when it is Var.

2 Lemma (Maximal saturation Lemma)
Let x be c-saturated in y. There exists a set m which is c-maximally saturated in y such that both $x \subseteq m$ and $m - x \subseteq c$.

Proof We will construct a sequence of sets $x =: m_0 \subseteq m_1 \subseteq \ldots$ such that m_n is c-saturated within y. The set $m := \bigcup_{n \in \mathbb{N}} m_n$ will be a c-maximal set c-saturated in y extending x.

The sequence is constructed in the following manner. Consider any enumeration of c, say $\left(C_n\right)_{n \in \mathbb{N}}$. We inductively define the following. Note that this construction clearly ensures that $m - x \subseteq c$.

$$m_{n+1} := \begin{cases} m_n \cup \{C_n\} & \text{whenever } m_n \cup \{C_n\} \preccurlyeq_c y \\ m_n & \text{otherwise.} \end{cases}$$

We first prove that $m \preccurlyeq_c y$. It is clear from the construction that $m \subseteq y$. So take some finite $\Delta \subseteq c$ and assume that $m \vdash \bigvee \Delta$. By definition this ensures the existence of a finite set $\Gamma \subseteq m$ such that $\Gamma \vdash \bigvee \Delta$. There must be some $n \in \mathbb{N}$ such that $\Gamma \subseteq m_n$. As a consequence $m_n \vdash \bigvee \Delta$, and as $m_n \preccurlyeq_c y$ we now know $\Delta \cap y \neq \emptyset$.

We now prove maximality. So assume that $z \supseteq m$ is such that $z \preccurlyeq_c y$. We will show that $z \subseteq_c m$. To this end, take a $z \cap c \ni C$. We know that $C = C_n$ for some $n \in \mathbb{N}$, fix such an n. Note that

$$m_n \cup \{C\} \subseteq m \cup \{C\} \subseteq z \preccurlyeq_c y,$$

so $m_n \cup \{C\} \preccurlyeq_c y$. The construction of $(m_n)_{n \geq 0}$ now ensures that $C \in m_{n+1} \subseteq m$. This proves $z \subseteq_c m$ as desired. ■

3 Lemma
Let m be c-maximally saturated in y. Now m is c-saturated.

Proof Let $\Delta \subseteq c$ be a finite set of formulae. Suppose that $m \vdash \bigvee \Delta$, we need to show that m intersects Δ. When there is some $D \in \Delta$ such that $m + D$ is c-saturated within y we know that $m + D =_c m$ by the maximality of m, proving that Δ intersects m (keeping in mind that $D \in c$). We reason by contradiction, so we assume that this is not the case.

As such, we can pick a set of formulae $\textbf{ctx}\, D \subseteq c$ for each $D \in \Delta$ such that $m + D \vdash \bigvee \textbf{ctx}\, D$ but $\textbf{ctx}\, D \cap y = \emptyset$. This set $\textbf{ctx}\, D$ can be thought of as a *counterexample* to $m + D \preccurlyeq_c y$. We consequently know that

$$m \vdash \bigvee_{D \in \Delta} \bigvee \textbf{ctx}\, D.$$

Because $m \preccurlyeq_c y$, this must entail that $\textbf{ctx}\, D$ intersects y for some $D \in \Delta$. But this is excluded by design, a contradiction. Thus the desired follows. ■

The following Corollary 2 is quite old, proofs of this go back to at least Theorem 3.1 of Aczel (1968). When taking c to be the set of all formulae, this lemma simply states that if $x \nvdash \bigvee \Delta$, there exists a saturated extension of x not intersecting Δ.

1 Corollary (Negative saturation lemma)
Let x and $c \supseteq \Delta$ be sets of formulae such that $x \nvdash \bigvee \Delta$. There exists a c-saturated set $m \supseteq x$ such that $m - x \subseteq c - \Delta$.

Proof Define the set $y := (x \cup c) - \Delta$. We claim that $x \preccurlyeq_c y$. Assuming the claim, the Maximal saturation Lemma and Lemma 3 yield a c-saturated set $m \supseteq x$ such that $m - x \subseteq c$ and $m \preccurlyeq_c y$. We now but need to prove that $m - x \subseteq c - \Delta$. Assuming that there exists a $m \ni A$ with $A \in \Delta$ we get from $m - x \subseteq c$ and $m \preccurlyeq_c y$ that $A \in y$, a clear contradiction.

We now need but to prove the claim. Suppose that $\Pi \subseteq c$ is such that $x \vdash \bigvee \Pi$. If $\Pi \subseteq \Delta$ then $x \vdash \bigvee \Delta$ would yield a contradiction. Whence we know $\Pi \nsubseteq \Delta$, so $\Pi \cap y \neq \emptyset$ as desired. ■

2 Corollary
Let x and c be sets of formulae such that $x \nvdash A \to B$ and $c \ni A, B$. Now there exists a c-saturated set $z \supseteq x$ with $z - x \subseteq c$, such that $z \ni A$ and $z \not\ni B$.

The following lemma is folklore. Its proof is included for completeness' sake.

4 Lemma
Let x be saturated. We define the Kripke model x^* as the partial order of all saturated sets extending x. The valuation function maps a saturated set to the propositional variables it contains. Now $x = \mathrm{Th}\,(x^*)$.

Proof The desired is but an immediate corollary of the equivalence

$$\text{for all } x \leq y, \quad y \vdash C \text{ if and only if } y \Vdash C.$$

which can easily be proven with structural induction along C. For the implication case one uses Corollary 2. ■

Recall that the final aim of this section is to find extensions of models. More precisely, given a set of formulae x we wish to know whether a model K has an extension that is a model of x. In other words, is there a $y \subseteq \mathrm{Th}(K)$ such that $K/y \Vdash x$. We will identify sets of formulae y where K/y forces exactly y. Needless to say, such sets will be saturated and in some sense they lie closely beneath $\mathrm{Th}(K)$. This latter property we now define. As before, one can freely ignore the restrictions imposed upon c at first reading. In short, a tight predecessor of y is a best saturated approximation of y, in that anything larger will be larger than y.

12 Definition (Tight predecessor)
Let x, y and c be sets of formulae. We say that x is a c-tight predecessor of y when x is c-saturated, $x \subseteq y$ and for each c-saturated set $z \supseteq x$ such that $\emptyset \neq z - x \subseteq c$ we have $y \subseteq_c z$.

The relation between tight predecessors of a theory and sets maximally saturated in that theory is not completely straightforward. In Lemma 8 we show under which conditions the latter is an instance of the former. We do not investigate the other direction.

2 Example
Let the ambient intermediate logic be IPC, and consider a two-point Kripke model with nodes $0 \leq 1$ such that $1 \Vdash p$ and $0 \nVdash p$ for one fixed p. It is clear that $\mathrm{Th}(1)$, the theory of 1, is saturated. As a consequence $\mathrm{Th}(1)$ is maximally saturated within $\mathrm{Th}(1)$, and nothing else is. One can also verify that $\mathrm{Th}(1)$ is a tight predecessor of $\mathrm{Th}(1)$, which follows quite trivially from the definition.

We will now argue that $\mathrm{Th}(0)$ is a tight predecessor of $\mathrm{Th}(1)$ as well. Now suppose that $x \supset \mathrm{Th}(0)$ is a saturated set. There must be some formula A such that $x \vdash A$ and $0 \nVdash A$. If $\mathrm{Th}(1) \not\subseteq x$ then there is some formula B such that $1 \Vdash B$ yet $x \nvdash B$. Now see that $0 \Vdash A \to B$ and so $x \vdash A \to B$ holds as well. We already know that $x \vdash A$, so saturation and *modus ponens* yield $x \vdash B$, a contradiction.

Tight predecessors of theories of models are particularly interesting. We can show that a tight predecessor x of (the theory of) a model K is such that the extension K/x has x as its theory. As a consequence, when we seek an extension of K such that y holds, we need but find a tight predecessor of $\mathrm{Th}(K)$ containing y. A crucial part is played by the set of implications that are true in the model K whereas their assumption is not, given in Definition 13. This definition is a generalisation of the set Δ as introduced in Iemhoff (2001a, p. 167). No saturated set can contain both a vacuous implication and its assumption, as is immediate from the definition. It is exactly this property which makes this set a "benign" assumption for a de Jongh rule, as we shall see in Lemma 11.

13 Definition (Vacuous implications)
Let x and c be sets of formulae. We define

$$\mathrm{I}_c(x) := \left\{ A \to B \;\middle|\; c \ni A, B \text{ and } x \ni A \to B \text{ but } x \not\ni A \right\}.$$

When c is the set of all formulae, we simply write $\mathrm{I}(x)$ to mean $\mathrm{I}_c(x)$.

For notational convenience we introduce the following. The lemma below can easily be seen to hold.

14 Definition
Given a set of implications x, we call x^{a} the set of assumptions of x, formally defined as

$$x^{\mathrm{a}} := \{ A \mid x \ni A \to B \}.$$

5 Lemma
Let K be a Kripke structure, let $x \subseteq \mathrm{Th}(K)$ be arbitrary and let $y \subseteq \mathrm{Th}(K)$ be a set of implications. If $y^{\mathrm{a}} \cap \mathrm{Th}(K) = \emptyset$ then $K/x \Vdash y$.

3 Corollary
Let K be a Kripke structure. For any $x \subseteq \mathrm{Th}(K)$ we have $\mathrm{I}(\mathrm{Th}(K)) \subseteq \mathrm{Th}(K/x)$.

We now have the machinery ready to prove that tight predecessors yield extensions, given that they include the vacuous implications of the model they are extending. That is to say, if x is a tight predecessor of the theory of a Kripke model K then the model can be extended so as to force exactly x. Matters become more subtle when x is not a full but merely a c-tight predecessor. At first glance one might expect that this yields only information on the truth of c-formulae in K/x. But we in fact can go one nesting of implications higher, that is to say, if A and B are in c then $x \vdash A \to B$ if and only if $K/x \Vdash A \to B$. We later will apply this technique when c consists of all (relevant) propositional variables.

6 Lemma (Extension lemma)
Let c be a set of formulae closed under taking subformulae, let K be a Kripke model and let x be a c-tight predecessor of $\mathrm{Th}(K)$. Write $d := c \cup c \to c$ where $c \to c := \{A \to B \mid c \ni A, B\}$. Now $\mathrm{I}_c(\mathrm{Th}(K)) \subseteq x$ holds if and only if $\mathrm{Th}(K/x) =_d x$.

Proof The implication from right to left follows immediately from Corollary 3. Suppose $\mathrm{I}_c(\mathrm{Th}(K)) \subseteq x$ to hold. We are done when we can show the following equivalence for all $C \in d$.

$$K/x \Vdash C \quad \text{if and only if} \quad x \vdash C \tag{1}$$

We proceed to prove (1) by induction along the structure of C. If C is a propositional variable we know the equivalence to hold by definition. The disjunctive and conjunctive case follow readily by induction and c-saturation.

Let us now handle the implicative case, so suppose that $C = A \to B$. Bear in mind that $c \ni C$ need not hold, but $c \ni A, B$ is always true. From left to right we reason by contraposition. Suppose that $x \nvdash A \to B$. Extend x to z such that $z \not\ni B$ and $z \ni A$, as per Corollary 2. Do note that $z - x \subseteq c$.

We distinguish two cases, namely $z = x$ and $z \neq x$. In the former case, $x \vdash A$ and $x \nvdash B$, so induction yields $K/x \Vdash A$ and $K/x \nVdash B$. As a consequence we know $K/x \nVdash A \to B$.

In the latter case we know that $\mathrm{Th}(K) \subseteq_c z$. We assume that $K \Vdash A \to B$. Again we distinguish two cases. If $K \Vdash A$ then $K \Vdash B$, so as $\mathrm{Th}(K) \subseteq_c z$ we know $z \ni B$, a contradiction. If $K \nVdash A$ then $A \to B \in \mathrm{I}_c(\mathrm{Th}(K)) \subseteq x \subseteq z$. Again, this ensures us that $z \ni B$, a contradiction. Consequently, $K \Vdash A \to B$ cannot hold, proving $K/x \nVdash A \to B$ as desired.

From right to left we reason by contradiction. Assume that $x \vdash A \to B$ and $K/x \nVdash A \to B$. This gives us some node $k \in K/x$ such that $k \Vdash A$ and $k \nVdash B$. In case that $k \in K$ we get a contradiction, because $x \subseteq \mathrm{Th}(K) \vdash A \to B$. If $k = x$ then $x \vdash A$ and $x \nvdash B$ follow by induction, but $x \vdash B$ holds as well. We have thus shown every case to yield a contradiction, proving the desired. ■

When considering all formulae a tight predecessor always contains the vacuous implications, as explicated by the following lemma. Its proof is similar the implicative case of the above proof.

7 Lemma
Let x be a tight predecessor of y. Now $\mathrm{I}(y) \subseteq x$.

4 Corollary (Bounding of tight predecessors)
Let K be a Kripke model. If x and y are tight predecessors of $\mathrm{Th}(K)$ and they contain the same propositional variables, then they are equal.

The Extension Lemma gives a good reason to search for tight predecessors of the theory of a model above its set of vacuous implications. The following lemma indicates a situation in which a tight predecessor exists, which will be one of the key ingredients of Theorem 1. In the case that c is the set of all formulae, this lemma simply proves that a set m maximally saturated in y is a tight predecessor of y whenever y is closed under deduction and m contains the vacuous implications of y.

8 Lemma (Tight predecessor lemma)
Let y, m and c be sets of formulae such that m is c-maximally saturated in y and y is closed under deduction. If $\mathrm{I}_c(y) \subseteq m$ then m is a c-tight predecessor of y.

Proof Let $z \supseteq m$ be c-saturated and assume that $\emptyset \neq z - m \subseteq c$. We need to show that $y \subseteq_c z$. We proceed by contradiction, so assume that there is some $c \ni B$ with $y - z \ni B$.

If $z \subseteq_c y$ then $z \subseteq y$, because $z - c \subseteq m \subseteq y$. We now know that z is c-saturated within y, so $z \preccurlyeq_c y$. Therefore c-maximality of m yields $z =_c m$, a contradiction.

Consequently, we can safely assume that $z \not\subseteq_c y$, which provides us with a $c \ni A$ such that $z - y \ni A$. Recall that $y \ni B$, so $y \vdash A \to B$ follows. We also know that $y \not\ni A$. These combine to prove that $\mathrm{I}_c(y) \ni A \to B$. But we know that $\mathrm{I}_c(y) \subseteq m \subseteq z$, and $z \ni A$. By *modus ponens* and the c-saturation of z we know that $z \ni B$. Yet we chose B such that $z \not\ni B$, a clear contradiction proving the desired. ■

The following theorem explicates the connection between extensions, vacuous implications and tight predecessors. This theorem is quite general, it works for any intermediate logic.

1 **Theorem**
Let K be a Kripke model, let x be a subset of $\mathrm{Th}(K)$ and let c be a set of formulae closed under subformulae. Write d for $c \cup c \to c$. Each item implies the item below. In case that $x \subseteq d$, all the following are equivalent.

(i) The set $x \cup \mathrm{I}_c(\mathrm{Th}(K))$ is c-saturated within $\mathrm{Th}(K)$;

(ii) There is a c-tight predecessor of $\mathrm{Th}(K)$ containing x and $\mathrm{I}_c(\mathrm{Th}(K))$;

(iii) There is a $y \subseteq \mathrm{Th}(K)$ such that $x \subseteq y$ and $y =_d \mathrm{Th}(K/y)$.

Proof Suppose that (i) holds. The Tight predecessor Lemma and Maximal saturation Lemma now yield the desired c-tight predecessor of $\mathrm{Th}(K)$ containing both x and $\mathrm{I}_c(\mathrm{Th}(K))$, proving (ii).

Now suppose that (ii) holds. Let y be the c-tight predecessor of $\mathrm{Th}(K)$ containing x and $\mathrm{I}_c(\mathrm{Th}(K))$. By the Extension Lemma we know that $y =_d \mathrm{Th}(K/y)$, proving (iii).

Finally, assume that $x \subseteq d$ and that (iii) holds. Observe that $\mathrm{Th}(K/y) \subseteq \mathrm{Th}(K)$. When we can prove that $x \cup \mathrm{I}_c(\mathrm{Th}(K)) \subseteq \mathrm{Th}(K/y)$ we are done, for $\mathrm{Th}(K/y)$ is clearly saturated, being the theory of a rooted model. By (iii) we know that $x = x \cap d \subseteq \mathrm{Th}(K/y)$, and $\mathrm{I}_c(\mathrm{Th}(K)) \subseteq \mathrm{Th}(K/y)$ holds due to Corollary 3. This finishes the proof of (i). ■

2 **Remark**
Suppose that the intermediate logic we work in has the disjunction property and let K be a Kripke model of this logic. Do note that in this case the set of theorems is saturated in itself. By the above theorem the following are equivalent.

(i) The set $\mathrm{I}(\mathrm{Th}(K))$ is saturated in $\mathrm{Th}(K)$;

(ii) There is a tight predecessor of $\mathrm{Th}(K)$;

(iii) There exists an extension of K that is a model of the logic.

4 Admissible Rules

In this section we introduce two rule schemes, the latter a convenient and technical generalisation of the former. The first scheme is a slight adaptation of the well-known Visser Rules that form a basis for the admissible rules of IPC. One can think of the Visser rules as providing saturation in the presence of a predetermined finite amount of information.

We chose to work with this form of the Visser rules as it is slightly more convenient, but this change is not essential. When working with single-conclusion rules subtleties involving the disjunction property come into play, see Citkin (2012) for details concerning this.

Onwards we will conflate sequents and the formulae they abbreviate. That is to say, we will speak of unifiers and derivability of sequents, where we actually mean unifiers and derivability of their associated formulae. Take care to note that when the ambient intermediate logic x has DP then $\vdash_x \Delta$ and $\vdash_x \bigvee \Delta$ are equivalent for all finite Δ, and as such $\Gamma \mathrel{|\!\sim} \Delta$ is equivalent to $\Gamma \mathrel{|\!\sim} \bigvee \Delta$.

15 Definition (Visser rule)
Let n be a natural number. The n^{th} *Visser rule* determined by finite sets of formulae Γ and Δ is given below, under the condition that Γ contains only implications and is of size at most n. We call such a rule an *irreducible Visser rule* when $\Gamma \Rightarrow \Delta$ is an irreducible sequent.

$$\frac{\Gamma \Rightarrow \Delta}{\left\{\, \Gamma \Rightarrow A \;\middle|\; A \in \Delta \cup \Gamma^{\mathrm{a}} \,\right\}}\ \mathsf{V}_n$$

Below a small example of an instance of a Visser rule with $n = 1$. The Visser rules at $n = 0$ express exactly DP. See Iemhoff (2006) for more examples of instances of the Visser rules and note that admissibility of V_n coincides with the property P_n of Iemhoff (2001a).

3 Example
Consider Harrop's rule as given in the introduction. We can naively translate this rule into the notation of Definition 6 and arrive at the rule below on the left. It is clear that admissibility of the left-hand rule yields admissibility of the right-hand rule whenever the ambient intermediate logic enjoys DP. The other direction requires the observation that if $\vdash \neg C \to C$ then $\vdash \neg C \Rightarrow \bot$.

$$\frac{\{\Rightarrow \neg C \to A \vee B\}}{\{\, \Rightarrow (\neg C \to A) \vee (\neg C \to B)\}} \qquad \frac{\{\neg C \Rightarrow A, B\}}{\{\neg C \Rightarrow A, \neg C \Rightarrow B, \neg C \Rightarrow C\}}$$

Iemhoff (2001b) proved that the rules V_n for all $n \in \mathbb{N}$ together form a basis of the admissible rules of any intermediate logic in which they are admissible. In particular, they are a basis for the admissible rules of IPC. Also, Iemhoff (2001a, Lemma 4.5) showed that any intermediate logic with the disjunction property admits V_n if and only if it has EP_n. We will use our machinery to prove a similar result for the rule (scheme) J_n as defined below.

The rule scheme J_n, the n^{th} de Jongh rules, are an ostensible generalisation of the Visser rules. In the Visser rules we restrict the amount of assumptions, the size of Γ, and as such indirectly influence the number of conclusions the rule has. In the de Jongh rules we take a more direct approach, and divide the assumptions of Γ into a specified number of groups, each of which can serve as a conclusion. That is to say, if $\Gamma \Rightarrow \Delta$ is provable, we do not demand that $\Gamma \Rightarrow A$ is derivable for some $A \in \Delta$ or $A \in \Gamma^{\mathrm{a}}$ as with the Visser rules. Instead we guarantee that $\Gamma \Rightarrow \Pi$ is derivable for some $\Pi \in \Delta$ or $\Pi \in \mathcal{U}$, where $\mathcal{U}$ is a predetermined subdivision of Γ^{a} of a bounded size. Let us give a definition.

16 Definition
Given a set X, we say that $\mathcal{U}$ is a n-cover of X if $|\mathcal{U}| \leq n$ and $\bigcup \mathcal{U} = X$.

17 Definition (de Jongh rule)
Let n be a natural number. The n^{th} *de Jongh rule* determined by finite sets of formulae Γ, Δ and an n-cover $\mathcal{U}$ of Γ^{a} is given below, under the condition that Γ contains only implications. We call such a rule an *irreducible de Jongh rule* when $\Gamma \Rightarrow \Delta$ is an irreducible sequent.

$$\frac{\Gamma \Rightarrow \Delta}{\left\{\, \Gamma \Rightarrow \Pi \;\middle|\; \Pi \in \Delta \cup \mathcal{U} \,\right\}}\ \mathsf{J}_n$$

In the definition above we write $\Pi \in \Delta \cup \mathcal{U}$, here Δ is to be interpreted as a set of singletons. This slight abuse of notation allows us to define the rules far more succinctly. Colloquially we speak of the n^{th} de Jongh rule when we actually mean all the possible instances of an n^{th} de Jongh rule.

The n^{th} de Jongh rule is, in case that $n \geq 2$, a generalised disjunction property. As such one would want the admissibility of J_n to entail DP. To show the plausibility of this we compare the semantic properties associated to these syntactic properties. The semantic counterpart of DP will be described below, and it is easy to see that this is weaker than EP_n. The following lemma is a direct consequence of Maksimova (1986, Theorem 1) and the duality between finite Heyting algebras and finite Kripke models well-established in folklore.

9 Lemma
Let x be an intermediate logic. Now x has DP if and only if to every pair of rooted Kripke models $K_1, K_2 \Vdash x$ there is another rooted Kripke model $K \Vdash x$ such that K_1 and K_2 are generated submodels.

5 Corollary
Let $n \geq 2$ be arbitrary. If an intermediate logic has EP_n, then it has DP.

Let us first prove that the n^{th} de Jongh rule (scheme) indeed holds in a logic which admits EP_n. The converse also holds, we eventually show this in Theorem 2.

10 Lemma (Covers via extensions)
Let $n \geq 2$ be arbitrary and let x be an intermediate logic with EP_n. The associated *m*-logic $\vdash$ admits J_n.

Proof Suppose that x has EP_n. Let Γ and Δ be finite sets of formulae with Γ containing only implications and let $\mathcal{U}$ be a n-cover of Γ^{a}. It suffices to prove that if $\vdash \Gamma \Rightarrow \Delta$ then $\vdash \Gamma \Rightarrow \Pi$ for some $\Pi \in \mathcal{U} \cup \Delta$.

We proceed by contradiction, so assume that $\vdash \Gamma \Rightarrow \Delta$ yet $\nvdash \Gamma \Rightarrow \Pi$ for all $\Pi \in \mathcal{U} \cup \Delta$. By the completeness of x with respect to rooted Kripke models we now know of $K_\Pi \Vdash x$ such that $K_\Pi \Vdash \bigwedge \Gamma$ and $K_\Pi \nVdash \bigvee \Pi$ for all $\Pi \in \mathcal{U} \cup \Delta$.

For convenience we index the elements of Δ as $\{D_1, \ldots, D_m\}$. We will construct rooted Kripke models $K_0, \ldots, K_m$ such that:

(i) K_i is a model of x for all $i \leq m$;

(ii) $K_i \nVdash D_i$ for all $1 \leq i \leq m$;

(iii) $\mathsf{Th}\,(K_{i+1}) \subseteq \mathsf{Th}\,(K_i)$ for all $i \leq m$;

(iv) $\mathsf{Th}\,(K_i) \cap \Gamma^{\mathrm{a}} = \emptyset$ and $\mathsf{Th}\,(K_i) \Vdash \bigwedge \Gamma$.

Assume for a moment that we have such models. By (iii), (iv) and Lemma 5 we know that $K_m \Vdash \bigwedge \Gamma$. Also see that $K_m \nVdash \bigvee \Delta$. Indeed, the contrary would entail $K_m \Vdash D_i$ for some $1 \leq i \leq m$ by rootedness, and (iii) now shows $K_i \Vdash D_i$, contradicting (ii). These facts combine to prove that $K_m \nVdash \Gamma \Rightarrow \Delta$. But from (i) and $\vdash \Gamma \Rightarrow \Delta$ we now derive a contradiction, proving the desired.

Let us now give this construction. Note that it will suffice to prove (iv) for $i = 0$ by (iii). We simultaneously construct the chain and prove these properties by induction along i. First consider the Kripke model $\coprod_{\Pi \in \mathcal{U}} K_\Pi$, which is a model of x. There must be an extension of this model as x has EP_n, call this extension K_0. Note that $K_0 \nVdash A$ for all $A \in \Gamma^{\mathrm{a}}$. Indeed, to every $A \in \Gamma^{\mathrm{a}}$ there is a $\Pi \in \mathcal{U}$ such that $A \in \Pi$, so $K_0 \Vdash A$ would entail $K_\Pi \Vdash \bigvee \Pi$, a clear contradiction. This proves (iv). Note that (ii) is vacuous for $i = 0$.

Now assume we have constructed the chain and proven all properties up to i. Define K_{i+1} to be any extension of K_i and K_{D_i} that is a model of x, guaranteed to exist by EP_n and (i). Properties (i)-(iv) clearly hold, finishing the argument. ■

3 Remark
The situation for $n \leq 1$ is quite different from that of $n \geq 2$. By Corollary 5 we know that EP_n entails DP in the latter case. The former is, however, not true in general. Take for instance the intermediate logic CPC, which does not have DP, but which does have EP_1.

We say that a set of set of formulae x is *c-closed under a rule* Γ/Δ if for all substitutions σ which map variables to elements of c we have that

$$\text{if } x \vdash \sigma(A) \text{ for all } A \in \Gamma \text{ then } x \ni \sigma(A) \text{ for some } A \in \Delta.$$

This definition is meant to include being closed under deduction. Observe that, when taking c to be the set of variables, a set is c-closed under J_n precisely if it is closed under irreducible n^{th} de Jongh rules. The converse of the above Lemma 10 can easily be proven using the now-available machinery. We need but show that underneath the theories of n rooted models one can find a saturated and appropriately negatively closed set. The following lemma ensures this. Its proof is quite similar to that of Iemhoff (2001b, p. 288).

11 Lemma (Saturation lemma)
Let n be a natural number, c a set of formulae, x a set of formulae c-closed under J_n and Y a set of n c-saturated sets extending x, and write $u := \bigcap Y$. Then $x \cup \mathrm{I}_c(u)$ is also c-saturated in u.

Proof Let $\Delta \subseteq c$ be some finite set of formulae. Suppose that $x \cup \mathrm{I}_c(u) \vdash \bigvee \Delta$. This yields some finite $\Gamma \subseteq \mathrm{I}_c(u)$ such that $x \vdash \Gamma \Rightarrow \Delta$. From this we will prove that $\Delta \cap u \neq \emptyset$.

Note that $\Gamma \Rightarrow \Delta$ is of the appropriate form to apply J_n for c-formulae, as Γ can only contain implications. The trick now is to construct an n-cover $\mathcal{U}$ of Γ^{a} such that if $\Pi \in \mathcal{U}$ then $\Pi \cap y$ is empty for some $y \in Y$. This is not all that hard; simply set $\mathcal{U} := \{U_y := \Gamma^{\mathrm{a}} - y \mid y \in Y\}$.

First we show that this indeed is indeed an n-cover. The size of $\mathcal{U}$ is not in question, but does it really cover Γ? Indeed it does, for if $c \ni A$ would be excluded from each U_y then this means that $y \vdash A$ for each $y \in Y$, contradicting $\Gamma \subseteq \mathrm{I}_c(u)$. In symbols:

$$\bigcup_{z \in Y} (\Gamma^{\mathrm{a}} - z) = \Gamma^{\mathrm{a}} - \bigcap Y = \Gamma^{\mathrm{a}} - u = \Gamma^{\mathrm{a}}.$$

We can now conclude that $x \ni \underline{\Gamma \Rightarrow \Pi}$ for some $\Pi \in \mathcal{U} \cup \Delta$ through the c-closedness of x under J_n. We claim that $\Pi \in \Delta$ has to hold. From this claim we can conclude the desired. Write χ for the formula such that $\Pi = \{\chi\}$ and assume the above claim. Now as $x \vdash \Gamma \Rightarrow \chi$ we know that $x \cup \mathrm{I}_c(u) \vdash \chi$. Consider any $y \in Y$, and observe that $x \cup \mathrm{I}_c(u) \subseteq y$, whence $y \vdash \chi$. Due to the c-saturation of y we can thus derive that $y \ni \chi$. Consequently we know the intersection of u and Δ to be inhabited, which is what we had to prove.

We now but need to prove our claim. So suppose that $x \vdash \Gamma \Rightarrow \Pi$ for some $\Pi \in \mathcal{U}$. This provides us with a $y \in Y$ such that $\Pi \cap y = \emptyset$, as per the construction of the n-cover $\mathcal{U}$. But do note that y contains both x and $\mathrm{I}_c(u)$, which ensures us that y proves both $\Gamma \Rightarrow \Pi$ and $\bigwedge \Gamma$. As a consequence, $y \vdash \bigvee \Pi$. The c-saturation of y now ensures a non-empty intersection between y and Π, a clear contradiction. ■

6 Corollary (Stratified extension lemma)
Let x be a saturated set closed under J_n. Then x has the n^{th} extension property.

Proof Consider n Kripke models $K_1, \ldots, K_n \Vdash x$. We need to find an extension of $K := \coprod_{i=1}^{n} K_i \Vdash x$. By the above Lemma 11 we know that $x \cup \mathrm{I}(\mathrm{Th}(K))$ is saturated in $\mathrm{Th}(K)$. By Theorem 1 we now obtain an extension y of K such that $K/y \Vdash x$ as desired. ■

For the main results in the next section we need to apply the Saturation Lemma to c-closed sets which elements have a specific form relative to the set c. The result is the following lemma. The reason for considering this particular form of c-closed sets will become clear in the next section.

12 Lemma
Let n be a natural number, c a set of formulae, x a set of formulae c-closed under J_n and such that every formula in x is of the form $\underline{\Gamma \Rightarrow \Delta}$ for some $\Delta \subseteq c$ and $\Gamma \subseteq c \cup c \to c$. Then x has the n^{th} extension property.

Proof Let $\mathcal{K}$ be a set of at most n rooted models of x. Our task is to find an extension z of the model $K := \coprod \mathcal{K}$ such that $K/z \Vdash x$.

We can now apply Lemma 11 to this situation, by letting Y be $\{\ \mathrm{Th}(M) \mid M \in \mathcal{K}\ \}$. Clearly, Y is a set of saturated sets extending x. Theorem 1 now provides us with a set of formulae $z \supseteq x$ such that $K/z =_d z$ for $d := c \cup c \to c$.

We but need to prove that K/z is a model of x. To this end fix $\underline{\Gamma \Rightarrow \Delta}$ in x and take some node $k \in K/z$ such that $k \Vdash \bigwedge \Gamma$. If $k \in K$ then $k \Vdash \Gamma \Rightarrow \Delta$ holds as $K \Vdash x$. Now consider the remaining case, where $k = z$, the root of K/z. It follows that $z \Vdash A$ for all $A \in \Gamma$, so $z \vdash A$ holds for all $A \in \Gamma$ because $\Gamma \subseteq c \cup c \to c$. Now as $z \supseteq x$ we know $z \vdash \bigvee \Delta$. By c-saturation of z we obtain a $B \in \Delta$ such that $z \vdash B$, because $\Delta \subseteq c$. For these reasons we know $z \Vdash B \in \Delta$, which in turn proves $z \Vdash \bigvee \Delta$. ■

2 Theorem
Let x be an intermediate logic with the disjunction property and write $\vdash$ for the associated m-logic. Now $\vdash$ admits J_n if and only if x has EP_n.

Proof The implication from right to left is given by Lemma 10. We need but prove the other direction, so assume that $\vdash$ admits J_n. It is clear that x is a saturated set. Note that this set is closed under all admissible rules of $\vdash$, so in particular, it is closed under J_n. Now apply the above Corollary 6, whence the desired is immediate. ■

5 Characterising Admissibility

In this section we prove the main theorems of this paper. They provide a basis for the admissible rules of the Gabbay–de Jongh logics and establish that their unification type is finitary. Since we are only concerned with logics that have the disjunction property, throughout the following we tacitly assume the logic to be such.

The key point in the proofs of the main theorems is that in the Gabbay–de Jongh logics every formula has an admissible approximation. Admissible approximations, to be defined below, are closely related to projective approximations as introduced by Ghilardi (1999, 2000, 2002). The projective approximation of a formula A is a set of projective formulae Π that each prove A which is minimal in a certain sense. The crucial link to admissible rules in the case of IPC, as pointed out by Ghilardi (1999), is that each unifier of a A is also a unifier of some $D \in \Pi$. It however takes considerable effort to derive this property from the definition of a projective approximation. As suggested by Iemhoff (2003, 2005) we sidestep the projective approximation altogether, and immediately skip to this more desirable property in Definition 19.

As we will see, from the existence of admissible approximations the main theorems easily follow. The rest of the section is devoted to showing that admissible approximations exist. We thereby use a set of rewriting steps that can be successively applied to a formula. It is shown that the finitely many normal forms of this rewriting process are closed under the n^{th} de Jongh rule, which, using the results in the previous section, implies that they have the n^{th} extension property. This implies that these formulas are projective in the $(n-1)^{\text{th}}$ Gabbay–de Jongh logic. The other requirements of an admissible approximation can be easily inferred from this.

There is a tiny technical detail that has to be addressed before we proceed. Unlike in projective approximations, the formulas in an admissible approximation in general contain propositional variables that do not occur in the formula of which they form an approximation. For this reason we have to slightly broaden the notions of derivability and admissibility.

18 Definition
A rule Γ/Δ is said to be *broadly derivable with respect to the rules* $\mathcal{R}$, written $\Gamma \Vdash_{\mathcal{R}} \Delta$, whenever for each substitution σ there is a substitution τ agreeing with σ on $\mathrm{vars}(\Gamma)$ such that $\tau(\Gamma) \vdash_{\mathcal{R}} \bigvee \tau(\Delta)$.

We say that such a rule is *broadly admissible*, written $\Gamma \Vdash\!\sim \Delta$, when for each unifier σ of Γ there is a unifier τ of some formula form Δ such that τ agrees with σ on $\mathrm{vars}(\Gamma)$.

19 Definition (Admissible approximation)
A finite set of projective formulae Π is said to be an *admissible approximation* of A if

$$\bigvee \Pi \vdash A \quad \text{and} \quad A \Vdash\!\sim \Pi \tag{2}$$

We say that an admissible approximation Π of A is *anchored by* $\mathcal{R}$ if $A \Vdash_{\mathcal{R}} \bigvee \Pi$. When one wants to characterise admissibility, it suffices to know the anchors of admissible approximations. This is explicated by the following lemma.

13 Lemma
Let A, B be formulae and let Π be an admissible approximation of A. Assume that $\mathrm{vars}(\Pi) \cap \mathrm{vars}(B) \subseteq \mathrm{vars}(A)$. Now (i) and (ii) are equivalent. Moreover, if Π is anchored by $\mathcal{R}$ and $\mathcal{R} \subseteq {\mid\!\sim}$ then all of the following are equivalent.

(i) $A \mid\!\sim B$;

(ii) $\bigvee \Pi \vdash B$;

(iii) $A \vdash_{\mathcal{R}} B$.

Proof Suppose that (i) holds. To prove (ii) it suffices to prove $D \vdash B$ for all $D \in \Pi$. So let $D \in \Pi$ be arbitrary. As D is projective we know it to have a projective unifier, σ say. We know that $\vdash \sigma(D)$, so it follows that $\vdash \sigma(\bigvee \Pi)$. As Π is an admissible approximation of A we get $\vdash \sigma(A)$. From (i) it now follows that $\vdash \sigma(B)$. This, together with the observation that σ is identity modulo D, yields $D \vdash B$ as desired. Consequently, (ii) follows.

Suppose that (ii) holds. Let σ be a substitution such that $\sigma(A)$ is a theorem. By Definition 19 we now know that $\bigvee \tau(\Pi)$ is a theorem for a given τ such that τ coincides with σ on $\mathrm{vars}(A)$. For convenience we assume $\mathrm{dom}\,\tau \subseteq \mathrm{vars}(\Pi)$. When we can prove that τ coincides with σ on $\mathrm{vars}(B)$ we are done. Consider any $p \in \mathrm{vars}(B) - \mathrm{vars}(A)$ and note that p must not be contained within $\mathrm{vars}(\Pi)$. Consequently, both σ and τ act as the identity on p. This shows that $\tau(B) = \sigma(B)$, proving (i).

From now on we assume Π to be anchored by a set of admissible rules $\mathcal{R}$. Suppose that (ii) holds, we wish to prove (iii). We know that $A \Vdash_{\mathcal{R}} \bigvee \Pi$ by anchoring. Now consider the identity substitution, and note that the former ensures us a substitution τ such that $\tau(A) \vdash_{\mathcal{R}} \bigvee \tau(\Pi)$ and $\tau(p) \neq p$ entails $p \in \mathrm{vars}(\Pi) - \mathrm{vars}(A)$. Structurality and transitivity now prove that $\tau(A) \vdash_{\mathcal{R}} \tau(B)$. As τ acts as the identity on both $\mathrm{vars}(A)$ and $\mathrm{vars}(B)$, (iii) immediately follows. This finishes the proof, for the implication from (iii) to (i) is clear. ■

7 **Corollary**
Let $\mathcal{R}$ be a set of admissible rules. Suppose that for each pair of formulae A and B there is an admissible approximation Π anchored by $\mathcal{R}$ with $\mathrm{vars}(\Pi) \cap \mathrm{vars}(B) \subseteq \mathrm{vars}(A)$. Then $\mathcal{R}$ is a basis of admissibility.

Admissible approximations are not only useful in finding bases of admissible rules in a logic, they can also be used to establish the unification type of a logic as defined in Baader and Ghilardi (2011). Namely, in the situation that each formula in the intermediate logic has an admissible approximation, the logic has finitary unification type. That is to say, each formula A has a finite set of unifiers such that any unifier is less general than one of those chosen few.

14 **Lemma**
Let x be an intermediate logic with DP. If each formula has an admissible approximation then x has finitary unification type.

Proof We need to prove that for every formula A there is a finite set of substitutions Σ such that if σ unifies A then σ is less general than a substitution in Σ. To prove just this, let A be any formula and take Π to be an admissible approximation of A, guaranteed to exist by assumption. We know of a projective unifier σ_D per $D \in \Pi$. Define $\Sigma := \{ \sigma_D \restriction \mathrm{vars}(A) \mid D \in \Pi \}$, where $\restriction$ denotes the restriction of a function to a (potentially smaller) domain.

Now take τ to be any unifier of A. We know of some ρ that coincides with τ on $\mathrm{vars}(A)$ such that $\vdash \rho(D)$ for some $D \in \Pi$ because Π is a admissible approximation of A. Because σ_D is a projective unifier of D we know that there is some substitution ν such that $\nu\sigma_D = \rho$. We compute

$$\nu\left(\sigma_D \restriction \mathrm{vars}(A)\right) = \left(\nu\sigma_D\right) \restriction \mathrm{vars}(A) = \rho \restriction \mathrm{vars}(A) = \tau$$

proving the desired. ■

In the following we will use the language of rewrite systems. For details on this we refer to Terese (2003) and Baader and Nipkow (1999). We can characterise irreducible sequents as normal forms of a rewrite system. This rewrite system deconstructs the formula along its syntactical structure, maintaining provability from right to left and broad provability in the other direction. The rewrite relation will be well-founded (or terminating), hence we can easily obtain a finite set of irreducible sequents "approximating" another set of sequents in a sense to be made precise in Lemma 17. Recall that $\mathcal{S}$ and $\mathcal{T}$ stand for finite sets of sequents. In the definition below we write $\mathcal{S}[\mathcal{T}]$ to mean $\mathcal{S} \cup \mathcal{T}$ where $\mathcal{S}$ and $\mathcal{T}$ do *not* overlap. The elements of $\mathcal{S}$ and $\mathcal{T}$ are divided by semi-columns rather than commas, so as not to confuse them with the commas that occur in the sequents. In our context, a normal form will be a set of sequents $\mathcal{S}$ such that if $\mathcal{S} \rightarrowtail \mathcal{T}$ then $\mathcal{T} = \mathcal{S}$.

20 Definition (Syntactic deconstruction)
Syntactic deconstruction is defined as the following rewrite system on $\mathbf{P}(\mathcal{L}_{\mathsf{prop}})$. In the rules $(\to\Rightarrow)$ and $(\Rightarrow\to)$ the variables p and q are understood to be fresh in the usual sense, and in the rule $(\to\Rightarrow)$ it is assumed that $A \to B$ is not an implication between propositional variables.

$$\begin{array}{lrcl}
(\bot\Rightarrow) & \mathcal{S}[\,\bot,\Gamma\Rightarrow\Delta\,] & \rightarrowtail & \mathcal{S} \\
(\Rightarrow\bot) & \mathcal{S}[\,\Gamma\Rightarrow\bot,\Delta\,] & \rightarrowtail & \mathcal{S}[\,\Gamma\Rightarrow\Delta\,] \\
(\wedge\Rightarrow) & \mathcal{S}[\,A\wedge B,\Gamma\Rightarrow\Delta\,] & \rightarrowtail & \mathcal{S}[\,A,B,\Gamma\Rightarrow\Delta\,] \\
(\Rightarrow\wedge) & \mathcal{S}[\,\Gamma\Rightarrow\Delta,A\wedge B\,] & \rightarrowtail & \mathcal{S}[\,\Gamma\Rightarrow\Delta,A\,;\,\Gamma\Rightarrow\Delta,B\,] \\
(\vee\Rightarrow) & \mathcal{S}[\,A\vee B,\Gamma\Rightarrow\Delta\,] & \rightarrowtail & \mathcal{S}[\,A,\Gamma\Rightarrow\Delta\,;\,B,\Gamma\Rightarrow\Delta\,] \\
(\Rightarrow\vee) & \mathcal{S}[\,\Gamma\Rightarrow\Delta,A\vee B\,] & \rightarrowtail & \mathcal{S}[\,\Gamma\Rightarrow\Delta,A,B\,] \\
(\to\Rightarrow) & \mathcal{S}[\,A\to B,\Gamma\Rightarrow\Delta\,] & \rightarrowtail & \mathcal{S}[\,p\Rightarrow A\,;\,p\to q,\Gamma\Rightarrow\Delta\,;\,B\Rightarrow q\,] \\
(\Rightarrow\to) & \mathcal{S}[\,\Gamma\Rightarrow\Delta,A\to B\,] & \rightarrowtail & \mathcal{S}[\,\Gamma\Rightarrow\Delta,p\,;\,p,A\Rightarrow B\,]
\end{array}$$

The fresh variables p and q are such that have an empty intersection with the sets $\mathrm{vars}(\mathcal{S})$, $\mathrm{vars}(\Gamma\Rightarrow\Delta)$, $\mathrm{vars}(A)$ and $\mathrm{vars}(B)$. We can extend the set of variables which p and q are not supposed to hit by any finite amount of variables, later on we will do so to accommodate Corollary 7.

15 Lemma
A set of sequents is a set of irreducible sequents if and only if it is in normal form with respect to $\rightarrowtail$.

Proof If $\mathcal{S}$ consist only of irreducible sequents, it is immediately clear that only $(\to\Rightarrow)$ stands a chance of being applicable. But as the left-hand side of $(\to\Rightarrow)$ must contain a non-atomic implication on the left, this rule too is not applicable. Hence $\mathcal{S}$ must be in normal form. The converse is equally clear. ■

21 Definition (Degree of a formula)
The *degree* of a formula is given by the following map $|\cdot|:\mathcal{L}_{\mathsf{prop}}\to\mathbb{N}$ defined via structural recursion as below.

$$\begin{array}{rcll}
|p| & := & 0 & \\
|\bot| & := & 1 & \\
|A \mathbin{\mathsf{C}} B| & := & 1+\max(\,|A|\,,|B|\,) & \qquad \mathsf{C}=\wedge,\vee,\to
\end{array}$$

16 Lemma
The relation $\rightarrowtail$ is well-founded.

Proof The map $|\cdot|$ can be extended to sets of formulae and sequents as follows:

$$|\Gamma| := \sum_{A\in\Gamma}|A| \qquad\qquad |\Gamma\Rightarrow\Delta| := |\Gamma|+|\Delta|$$

We now order sets of sequents by $\succ$, the multi-set order of Dershowitz and Manna (1979) on the multi-sets of their degrees. This is a well-founded order. We are done when we can prove that $\rightarrowtail$ is a refinement of $\succ$. This is a matter of verifying inequalities for the rules in Definition 20. We treat the implication cases as examples, all other inequalities are at most as hard to check as these. This means that we must show the following.

$$\mathcal{S}[\,A\to B,\Gamma\Rightarrow\Delta\,]\succ\mathcal{S}[\,p\Rightarrow A\,;\,p\to q,\Gamma\Rightarrow\Delta\,;\,B\Rightarrow q\,] \tag{3}$$
$$\mathcal{S}[\,\Gamma\Rightarrow\Delta,A\to B\,]\succ\mathcal{S}[\,\Gamma\Rightarrow\Delta,p\,;\,p,A\Rightarrow B\,] \tag{4}$$

To prove (3), note that $A\to B,\Gamma\Rightarrow\Delta$ clearly dominates $p\Rightarrow A$ and $B\Rightarrow q$. To see that it is larger than $p\to q,\Gamma\Rightarrow\Delta$ we use the fact that at least one of A and B has positive degree, whereas p and q do not. This proves the desired inequality. From here (4) is also fairly obvious, as the left-hand sequents dominates all sequents on the right. ■

17 Lemma
If $\mathcal{S}\rightarrowtail\mathcal{T}$ then the following hold:

(i) $\mathcal{S}$ is derivable from $\mathcal{T}$;

(ii) $\mathcal{T}$ is broadly derivable from $\mathcal{S}$;

(iii) $\mathcal{T}$ is broadly admissible from $\mathcal{S}$;

Proof Note that (iii) is but a weaker form of (ii), so we need but focus on the first two statements. We proceed by case analysis; we ought to cover all rules of Definition 20. In each case, (i) can easily be seen to hold. Indeed, all but the rules for implication correspond quite directly to rules of the well-known Gentzen proof system for intuitionistic logic **G3i** as given by Troelstra and Schwichtenberg (1996). The implication rules deviate a bit more, yet the first statement still clearly holds.

One can prove (ii) in a manner analogous to the inversion lemma of Troelstra and Schwichtenberg (ibid., page 67), for all but the implication cases. We will treat these in some detail.

Suppose that $\mathcal{S} \rightarrowtail \mathcal{T}$ is an instance of $(\to\Rightarrow)$. This means that there is some set of sequents $\mathcal{G}$ such that

$$\mathcal{S} = \mathcal{G} \cup \{A \to B, \Gamma \Rightarrow \Delta\} \quad \text{and} \quad \mathcal{T} = \mathcal{G} \cup \{p \Rightarrow A \; ; \; p \to q, \Gamma \Rightarrow \Delta \; ; \; B \to q\}.$$

Let σ be any substitution with $\operatorname{dom}\sigma \subseteq \operatorname{vars}(\mathcal{S})$. Note that p and q are fresh, so they cannot be contained within $\operatorname{dom}\sigma$. Now define the substitution τ by setting $\tau(p) := \sigma(A)$ and $\tau(q) := \sigma(B)$, and everywhere else we take τ to equal σ. Note that $\underline{\tau(\mathcal{T})}$ is logically equivalent to $\underline{\sigma(\mathcal{S})}$, proving that $\mathcal{S} \Vdash \mathcal{T}$.

We now treat the other implication case, so assume that $\mathcal{S} \rightarrowtail \mathcal{T}$ is an instance of the rule $(\Rightarrow\to)$. Consequently we know of some set of sequents $\mathcal{G}$ such that

$$\mathcal{S} = \mathcal{G} \cup \{\Gamma \Rightarrow \Delta, A \to B\} \quad \text{and} \quad \mathcal{T} = \mathcal{G} \cup \{\Gamma \Rightarrow \Delta, p \; ; \; p, A \Rightarrow B\}.$$

Consider any substitution σ with $\operatorname{dom}\sigma \subseteq \operatorname{vars}(\mathcal{S})$. As above, p cannot be within the domain of σ. We define a substitution τ by setting $\tau(p) := \sigma(A \to B)$ and letting τ equal σ everywhere else. As before, it is easy to see that $\underline{\tau(\mathcal{T})}$ is logically equivalent to $\underline{\tau(\mathcal{S})}$, proving the desired. ∎

We cite the following as a modest generalisation of Ghilardi (1999, Theorem 5). Where he uses IPC and the extension property we use the n^{th} Gabbay–de Jongh logic and the $(n+1)^{\text{th}}$ extension property respectively. The proof is equal up to re-evaluating the definitions used in this different context.

18 Lemma
Let n be a natural number, A be a formula and let the intermediate logic at hand be the n^{th} Gabbay–de Jongh logic $\mathbf{D}_n$. The following are equivalent:

(i) the union of the set of theorems of $\mathbf{D}_n$ with $\{A\}$ has EP_{n+1};

(ii) the formula A is projective.

19 Lemma
Let $\mathcal{S}$ be a set of irreducible sequents and write $c := \operatorname{vars}(\mathcal{S})$. Assume that $\mathcal{S}$ is c-closed under the irreducible $(n+1)^{\text{th}}$ de Jongh rules. Then the class of models of $\mathcal{S}$ over c has EP_{n+1}. Furthermore, $\underline{\mathcal{S}}$ is projective in $\mathbf{D}_n$.

Proof The first part follows immediately from Lemma 12, taking $\underline{\mathcal{S}}$ for x. The second part follows from Lemma 18 once we have shown that the set of theorems of $\mathbf{D}_n$ adjoined with $\underline{\mathcal{S}}$ (call this set y) has EP_{n+1}. Therefore consider $(n+1)$ models of y. As $\mathcal{S}$ has EP_{n+1} there is an extension that satisfies $\underline{\mathcal{S}}$. This extension automatically is a model of $\mathbf{D}_n$, and therefore of y. ∎

22 Definition (Splitting)
Let n be a natural number and let c be a set of propositional variables. We define a relation $\overset{n}{\rightsquigarrow}$ on sets of irreducible sequents over c in the following manner. For any irreducible sequent $\Gamma \Rightarrow \Delta$ over c with Γ implication-only, every set of irreducible sequents $\mathcal{S}$ and every n-cover $\mathcal{U}$ of Γ^{a} such that $\mathcal{S} \vdash \Gamma \Rightarrow \Delta$ and $\Pi \in \mathcal{U} \cup \Delta$ we set

$$\mathcal{S} \overset{n}{\rightsquigarrow} \left(\mathcal{S} \cup \{ \Gamma \Rightarrow \Delta ; \Gamma \Rightarrow \Pi \} \right),$$

and we say that this step is *associated* to the de Jongh rule determined by Γ, Δ and $\mathcal{U}$.

It is not hard to see that the above definition is actually sensible, in that the right-hand side is a set of sets of irreducible sequents. After all, the only "new" sequents that get added have a left-hand side which is equal to an "old" left-hand side, and the right-hand side consists of assumptions of irreducible implications or "old" right-hand sides. Furthermore, every $\overset{n}{\rightsquigarrow}$-chain eventually stabilises, because the size of the related sets is bounded.

Note that $\overset{n}{\rightsquigarrow}$ is neither deterministic nor confluent. It does have the nice property that for any set of irreducible sequents $\mathcal{S}$ we know $\mathcal{S} \Vdash_{\mathcal{R}} \underline{\{ \mathcal{T} \mid \mathcal{S} \overset{n}{\rightsquigarrow} \mathcal{T} \}}$, where $\mathcal{R}$ is a finite set of instances of the n^{th} de Jongh rule. This is easy to see, intuitively because $\mathcal{T}$ is related to $\mathcal{S}$ by $\overset{n}{\rightsquigarrow}$ precisely if the former equals the latter adjoined with one of the conclusions of n^{th} de Jongh rule. Consequently, the set of all normal forms $\mathbf{T}$ of a given set $\mathcal{S}$ is broadly derivable from $\mathcal{S}$ on a finite set of formulae associated to the n^{th} de Jongh rule.

20 Lemma
Let n be a natural number, let $\mathcal{S}$ be a set of irreducible sequents and let $\mathcal{S} \overset{n}{\rightsquigarrow} \mathcal{T}$ be a normal form. The following hold:

(i) Each sequent in $\mathcal{S}$ is provable from $\mathcal{T}$, that is, $\mathcal{T} \vdash S$ for all $S \in \mathcal{S}$.

(ii) The set $\mathcal{T}$ is $\mathrm{vars}\,(\mathcal{S})$-closed under irreducible J_n.

(iii) The formula $\bigwedge \underline{\mathcal{T}}$ is projective in $\mathbf{D}_{n-1}$.

Proof To prove (i), simply note that $\mathcal{S} \subseteq \mathcal{T}$ follows directly from the definition of $\overset{n}{\rightsquigarrow}$. Let us now prove that $\mathcal{T}$ is $\mathrm{vars}\,(\mathcal{S})$-closed under irreducible J_n. Suppose that $\mathcal{T} \vdash \Gamma \Rightarrow \Delta$ for some irreducible sequent over $\mathrm{vars}\,(\mathcal{S})$ with Γ consisting only of implications and let $\mathcal{U}$ be an n-cover of Γ^{a}. As $\mathcal{T}$ is in normal form this implies that $\mathcal{T} \ni \Gamma \Rightarrow \Pi$ for some $\Pi \in \Gamma^{\mathrm{a}} \cup \Delta$, proving the desired. We arrive at (iii) from (ii) and Lemma 19. ∎

3 Theorem
Let A be any formula, let $n \geq 2$ be a natural number and let V be a set of variables. In the $(n-1)^{\text{th}}$ de Jongh logic there is an admissible approximation of A anchored solely by the n^{th} de Jongh rules, whose variables do not overlap $V - \mathrm{vars}\,(\phi)$.

Proof Let A be any formula, and consider the set $\{A\}$. Take any normal form $\mathcal{S}$ under the relation $\rightarrowtail$ satisfying the variable condition, as guaranteed to exist by Lemma 16. We know that $A \Vdash \bigwedge \underline{\mathcal{S}}$ and $\underline{\mathcal{S}} \vdash A$ by Lemma 17. Moreover, $\mathcal{S}$ is a set of irreducible sequents by Lemma 15.

Let $\mathbf{T}$ be the set of normal forms of $\mathcal{S}$ under $\overset{n}{\rightsquigarrow}$. We claim that the set

$$\Pi := \left\{ \bigwedge \underline{\mathcal{T}} \,\middle|\, \mathbf{T} \ni \mathcal{T} \right\}$$

is an admissible approximation of A anchored by some set $\mathcal{R}$, where $\mathcal{R}$ is the sensible finite set of n^{th} de Jongh rules used to arrive at $\mathbf{T}$. First of all, this set Π is clearly finite as $\overset{n}{\rightsquigarrow}$ is well-founded and finitely branching. Moreover, all elements of Π are projective by Lemma 20.

All that is left to do is prove that $\bigvee \Pi \vdash A$, $A \mathrel{|\!\!\sim} \Pi$ and $A \vdash_{\mathcal{R}} \bigvee \Pi$. The second fact follows directly from the third. Observe that

$$\bigvee \Pi \vdash \underline{\mathcal{S}} \vdash A \quad \text{and} \quad A \Vdash \bigwedge \underline{\mathcal{S}} \Vdash_{\mathcal{R}} \underline{\mathbf{T}} = \Pi,$$

whence the desired is immediate. ∎

The now-available machinery immediately entails that the de Jongh rules characterise admissibility in the Gabbay–de Jongh logics, and this result is stratified in the obvious manner. The following theorems are direct corollaries of Theorem 3 adjoined with Corollary 7 and Lemma 14.

4 Theorem
The $(n+1)^{\text{th}}$ de Jongh rules form a basis for the admissible rules of the n^{th} Gabbay–de Jongh logic for all positive n.

5 Theorem
The n^{th} Gabbay–de Jongh logic has finitary unification for all positive n.

6 Theorem
In any extension of the n^{th} Gabbay-de Jongh logic the $(n+1)^{\text{th}}$ de Jongh rules form a basis for the admissible rules whenever they are admissible. In this case the logic has finitary unification.

As can be seen from their semantical characterisation, the intersection of all Gabbay-de Jongh logics is equal to IPC. Also, the collection of all de Jongh rules is a basis for the admissible rules of any intermediate logic in which they are admissible, see Iemhoff (2001a). Theorem 6 therefore shows that in this respect there is a great similarity between IPC and the Gabbay-de Jongh logics.

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