NLTS Hamiltonians from good quantum codes

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Abstract

The NLTS (No Low-Energy Trivial State) conjecture of Freedman and Hastings [FH14] posits that there exist families of Hamiltonians with all low energy states of non-trivial complexity (with complexity measured by the quantum circuit depth preparing the state). We prove this conjecture by showing that the recently discovered families of constant-rate and linear-distance QLDPC codes correspond to NLTS local Hamiltonians.

1 Introduction

Ground- and low-energy states of local Hamiltonians are the central objects of study in condensed matter physics. For example, the QMA-complete local Hamiltonian problem is the quantum analog of the NP-complete constraint satisfaction problem (CSP) with ground-states (or low-energy states) of local Hamiltonians corresponding to solutions (or near-optimal solutions) of the problem [KSV02]. A sweeping insight into the computational properties of the low energy spectrum is embodied in the quantum PCP conjecture, which is arguably the most important open question in quantum complexity theory [AAV13]. Just as the classical PCP theorem establishes that CSPs with constant fraction promise gaps remain NP-complete, the quantum PCP conjecture asserts that local Hamiltonians with a constant fraction promise gap remain QMA-complete. Despite numerous results providing evidence both for [AALV09,FH14,NV18] and against [BV05,BH13,AE15] the validity of the quantum PCP conjecture, the problem has remained open for nearly two decades.

The difficulty of the quantum PCP conjecture has motivated a flurry of research beginning with Freedman and Hastings' *No low-energy trivial states (NLTS) conjecture* [FH14]. The NLTS conjecture posits that there exists a fixed constant $\epsilon > 0$ and a family of n qubit local Hamiltonians such that every state of energy $\leq \epsilon n$ requires a quantum circuit of super-constant depth to generate. The NLTS conjecture is a necessary consequence of the quantum PCP conjecture, because QMA-complete problems do not have NP solutions and a constant-depth quantum circuit generating a low-energy state would serve as a NP witness. Thus, this conjecture addresses the inapproximability of local Hamiltonians by classical means.

Previous progress [EH17,NVY18,Eld21,BKKT19,AN22,AB22] provided solutions to weaker versions of the NLTS conjecture, but the complete conjecture had eluded the community.

Theorem 1 (No low-energy trivial states)**.** *There exists a fixed constant $\epsilon > 0$ and an explicit family of $O(1)$-local frustration-free commuting Hamiltonians $\{\mathbf{H}^{(n)}\}_{n=1}^{\infty}$ where $\mathbf{H}^{(n)} = \sum_{i=1}^{m} h_i^{(n)}$ acts on n particles and consists of $m = \Theta(n)$ local terms such that for any family of states $\{\psi_n\}$ satisfying $\operatorname{tr}\big(\mathbf{H}^{(n)}\psi\big) < \epsilon n$, the circuit complexity of the state ψ_n is at least $\Omega(\log n)$.*

The local Hamiltonians for which we can show such robust circuit-lower bounds correspond to constant-rate and linear-distance quantum LDPC error-correcting codes with an additional property related to the clustering of approximate code-words of the underlying classical codes. We show that the property holds for the *quantum Tanner code* construction of Leverrier and Zémor [LZ22] (Section 3). We suspect that the property is true for other constructions of constant-rate and linear-distance QLDPC codes [PK21, BE21, DHLV22], however we do not prove this outright. While we show that the property is sufficient for NLTS, it is an interesting open question if the property is inherently satisfied by all constant-rate and linear-distance constructions.

Quantum code To formalize this property, recall a CSS code with parameters $[[n, k, d]]$. The code is constructed by taking two classical codes C_{x} and C_{z} such that $C_{\mathsf{z}} \supset C_{\mathsf{x}}^{\perp}$. The code C_{z} is the kernel of a row- and column-sparse matrix $H_{\mathsf{z}} \in \mathbb{F}_2^{m_{\mathsf{z}} \times n}$; the same for C_{x} and $H_{\mathsf{x}} \in \mathbb{F}_2^{m_{\mathsf{x}} \times n}$. The rank of H_{z} will be denoted as r_{z} and likewise r_{x} is the rank of H_{x}. Therefore, $n = k + r_{\mathsf{x}} + r_{\mathsf{z}}$. If the code is constant-rate and linear-distance, then $k, d, r_{\mathsf{x}}, r_{\mathsf{z}} = \Omega(n)$. For the codes considered in this work, we also have $m_{\mathsf{z}}, m_{\mathsf{x}} = \Omega(n)$.

For any subset $S \subset \{0,1\}^n$, define a distance measure $|\cdot|_S$ as $|y|_S = \min_{s \in S} |y + s|$ where $|\cdot|$ denoted Hamming weight. We define G_{z}^{δ} as the set of vectors which violate at most a δ-fraction of checks from C_{z}, i.e. $G_{\mathsf{z}}^{\delta} = \{y : |H_{\mathsf{z}} y| \leq \delta m_{\mathsf{z}}\}$. We similarly define G_{x}^{δ}.

Property 1 (Clustering of approximate code-words)**.** *We say that a* $[[n, k, d]]$ *CSS code defined by classical codes* $(C_{\mathsf{x}}, C_{\mathsf{z}})$ clusters approximate code-words *if there exist constants* c_1, c_2, δ_0 *such that for sufficiently small* $0 \leq \delta < \delta_0$ *and every vector* $y \in \{0,1\}^n$*,*

1. *If* $y \in G_{\mathsf{z}}^{\delta}$*, then either* $|y|_{C_{\mathsf{x}}^{\perp}} \leq c_1 \delta n$ *or else* $|y|_{C_{\mathsf{x}}^{\perp}} \geq c_2 n$*.*
2. *If* $y \in G_{\mathsf{x}}^{\delta}$*, then either* $|y|_{C_{\mathsf{z}}^{\perp}} \leq c_1 \delta n$ *or else* $|y|_{C_{\mathsf{z}}^{\perp}} \geq c_2 n$*.*

Note that this property holds for classical Tanner codes with spectral expansion (see [AB22, Theorem 4.3]) and was used to prove the combinatorial NLTS conjecture. In fact, Lemma 9 in the Appendix shows that more general classical codes with small-set expanding interactions graphs satisfy Property 1 with $|\cdot|$ used instead of $|\cdot|_{C_{\mathsf{x}}^{\perp}}$. The quantum analog above is sufficient for proving the full NLTS conjecture.

Local Hamiltonian definition The aforementioned quantum codes lead to a natural commuting frustration-free local Hamiltonian. For every row w_{z} of H_{z} – i.e. a stabilizer term $Z^{w_{\mathsf{z}}}$ of the code, we associate a Hamiltonian term $\frac{1}{2}(\mathbb{I} - Z^{w_{\mathsf{z}}})$. We define $\mathbf{H}_{\mathsf{z}}$ as the sum of all such terms for H_{z}. $\mathbf{H}_{\mathsf{x}}$ is defined analogously and the full Hamiltonian is $\mathbf{H} = \mathbf{H}_{\mathsf{x}} + \mathbf{H}_{\mathsf{z}}$. The number of local terms is $m_{\mathsf{x}} + m_{\mathsf{z}} = \Theta(n)$ and $\mathbf{H}$ has zero ground energy. We refer the reader to the preliminaries of [AN22, Section 2] for more technical definitions and notation.

Open questions There are three questions that we leave unanswered.

- Does Property 1 "morally" hold for all constant-rate and linear-distance quantum codes?
- In the language of chain-complexes, Property 1 seems closely related to the small-set boundary and co-boundary expansion in [HL22, Definition 1.2]. Does this suggest a classical analog of the NLTS property, since [HL22] construct classical Hamiltonians that are hard-to-approximate using sum-of-squares heirarchy?

- Our construction does *not* require quantum local testability. Property 1 is sufficient for clustering of the classical distributions of low-energy states but it is weaker than local testability. [EH17] used local testability to argue clustering for their proof that local testability implies NLTS. What are the implications of codes with Property 1 for the quantum PCP conjecture [AAV13]?
- Can our proof techniques be generalized to prove non-trivial lower bounds for non-commuting Hamiltonians?

2 Proof of the NLTS theorem

The proof, that the local Hamiltonian corresponding to a constant-rate and linear-distance code satisfying Property 1 is NLTS, is divided into a few steps. We first show that the classical distributions generated by measuring any low-energy state in the standard or Hadamard bases are approximately supported on a particular structured subset of vectors. Then, we show that the subsets cluster into a collection of disjoint components which are far in Hamming distance from each other. Finally, we show that the distribution in one of the two bases cannot be too concentrated on any particular cluster. This shows that the distribution is *well-spread* which can be used to prove a circuit depth lower bound.

The supports of the underlying classical distributions Consider a state ψ on n qubits such that $\mathrm{tr}(H\psi) \leq \epsilon n$. Let D_{x} and D_{z} be the distributions generated by measuring the ψ in the (Hadamard) $X-$ and (standard) $Z-$ bases, respectively. We find that D_{z} is largely supported on $G_{\mathsf{z}}^{O(\epsilon)}$. Formally, this is because, by construction,

$$\epsilon n \geq \mathrm{tr}(\mathbf{H}\psi) \geq \mathrm{tr}(\mathbf{H}_{\mathsf{z}}\psi) = \mathop{\mathbf{E}}_{y\sim D_{\mathsf{z}}} |H_{\mathsf{z}}y|.$$

Here, the last equality holds since for a Pauli operator Z^a, $\langle y| \frac{\mathbb{1}-Z^a}{2} |y\rangle = \frac{1-(-1)^{a.y}}{2} = a.y$. Let $q \stackrel{\text{def}}{=} D_{\mathsf{z}}(G_{\mathsf{z}}^{\epsilon_1})$ be the probability mass assigned by D_{z} to $G_{\mathsf{z}}^{\epsilon_1}$. Then,

$$\mathop{\mathbf{E}}_{y\sim D_{\mathsf{z}}} |H_{\mathsf{z}}y| \geq 0 \cdot q + (1-q) \cdot \epsilon_1 m_{\mathsf{z}} = (1-q)\epsilon_1 m_{\mathsf{z}}.$$

Therefore, $D_{\mathsf{z}}(G_{\mathsf{z}}^{\epsilon_1}) \geq 1 - \epsilon n/(\epsilon_1 m_{\mathsf{z}})$. A similar argument shows that $D_{\mathsf{x}}(G_{\mathsf{x}}^{\epsilon_1}) \geq 1 - \epsilon n/(\epsilon_1 m_{\mathsf{x}})$. With the choice $\epsilon_1 = \frac{200n}{\min\{m_{\mathsf{x}}, m_{\mathsf{z}}\}} \cdot \epsilon$, we find

$$D_{\mathsf{z}}(G_{\mathsf{z}}^{\epsilon_1}), D_{\mathsf{x}}(G_{\mathsf{x}}^{\epsilon_1}) \geq \frac{199}{200}$$

for both the bases.

The supports are well clustered Given that D_{z} is well supported on $G_{\mathsf{z}}^{\epsilon_1}$, it is helpful to understand the structure of $G_{\mathsf{z}}^{\epsilon_1}$. For $x, y \in G_{\mathsf{z}}^{\epsilon_1}$, notice that $x \oplus y \in G_{\mathsf{z}}^{2\epsilon_1}$ since $x \oplus y$ satisfies every check that both x and y satisfy. By Property 1 (and assuming $2\epsilon_1 \leq \delta_0$), then either

$$|x \oplus y|_{C_{\mathsf{x}}^{\perp}} \leq 2c_1\epsilon_1 n \quad \text{or else} \quad |x \oplus y|_{C_{\mathsf{x}}^{\perp}} \geq c_2 n.$$

Define a relation '$\sim$' such that for $x, y \in G_{\mathsf{z}}^{\epsilon_1}$, $x \sim y$ iff $|x \oplus y|_{C_{\mathsf{x}}^{\perp}} \leq 2c_1\epsilon_1 n$. To prove that the relation is transitive and therefore an equivalence relation, notice that if $x \sim y$ and $y \sim z$, then

$$|x \oplus z|_{C_{\mathsf{x}}^{\perp}} \leq |x \oplus y|_{C_{\mathsf{x}}^{\perp}} + |y \oplus z|_{C_{\mathsf{x}}^{\perp}} \leq 4c_1\epsilon_1 n.$$

However, $x \oplus z \in G_{\mathsf{z}}^{2\epsilon_1}$ and for sufficiently small ϵ_1 such that $4c_1\epsilon_1 < c_2$, Property 1 implies that $|x \oplus z|_{C_{\mathsf{x}}^\perp} \leq 2c_1\epsilon_1 n$. Thus, $x \sim z$ and hence $\sim$ forms an equivalence relation. We can now divide the set $G_{\mathsf{z}}^{\epsilon_1}$ into clusters $B_{\mathsf{z}}^1, B_{\mathsf{z}}^2, \ldots$, according to the equivalence relation $\sim$. Furthermore, the distance between any two clusters is $\geq c_2 n$, since for x in one cluster and x' in another cluster, we have $|x \oplus x'| \geq |x \oplus x'|_{C_{\mathsf{x}}^\perp} \geq c_2 n$. Lastly, the same argument holds for $G_{\mathsf{x}}^{\epsilon_1}$.

The distributions are not concentrated on any one cluster To apply known circuit-depth lower bounding techniques to D_{z}, it suffices to show that D_{z} is not concentrated on any one cluster B_{z}^i. However, it is not immediate how to show this property for D_{z}. Instead, what we can show is that is impossible for both D_{z} to be concentrated on any one cluster B_{z}^i and D_{x} to be concentrated on any one cluster B_{x}^j.

Lemma 2. *For ϵ_1 such that $2c_1\epsilon_1 \leq \left(\frac{k-1}{4n}\right)^2$, either $\forall\, i$, $D_{\mathsf{z}}(B_{\mathsf{z}}^i) < 99/100$ or else $\forall\, j$, $D_{\mathsf{x}}(B_{\mathsf{x}}^j) < 99/100$.*

Proof. Assume there exists some i such that $D_{\mathsf{z}}(B_{\mathsf{z}}^i) \geq 99/100$. We will employ the following fact that captures the well-known uncertainty of measurements in the standard and Hadamard bases; a proof is provided in the appendix.

Fact 3. *Given a state ψ and corresponding measurement distributions D_{x} and D_{z}, for all subsets $S, T \subset \{0,1\}^n$, $D_{\mathsf{x}}(T) \leq 2\sqrt{1 - D_{\mathsf{z}}(S)} + \sqrt{|S| \cdot |T|/2^n}$.*

For any j, we employ this fact with $S = B_{\mathsf{z}}^i$ and $T = B_{\mathsf{x}}^j$. To bound $|B_{\mathsf{z}}^i|$, fix any string $z \in B_{\mathsf{z}}^i$. Any other string $z' \in B_{\mathsf{z}}^i$ has the property that its Hamming distance from $z \oplus w$ (for some $w \in C_{\mathsf{x}}^\perp$) is at most $2c_1\epsilon_1 n$. Since $\left|C_{\mathsf{x}}^\perp\right| = 2^{\dim C_{\mathsf{x}}^\perp} = 2^{n - \dim C_{\mathsf{x}}} = 2^{r_{\mathsf{x}}}$, the size of the cluster B_{z}^i is at most

$$2^{r_{\mathsf{x}}} \cdot \binom{n}{2c_1\epsilon_1 n} \leq 2^{r_{\mathsf{x}}} \cdot 2^{2\sqrt{2c_1\epsilon_1}n}.$$

A similar bound can be calculated of $\left|B_{\mathsf{x}}^j\right| \leq 2^{r_{\mathsf{z}}} \cdot 2^{2\sqrt{2c_1\epsilon_1}n}$. Then applying Fact 3 with the bound on ϵ_1 as stated in the Lemma,

$$\forall\, j, \quad D_{\mathsf{x}}\left(B_{\mathsf{x}}^j\right) \leq \frac{1}{5} + \sqrt{2^{r_{\mathsf{x}} + r_{\mathsf{z}} - n} \cdot 2^{4\sqrt{2c_1\epsilon_1}n}} = \frac{1}{5} + 2^{\frac{-k}{2} + 2\sqrt{2c_1\epsilon_1}n} \leq \frac{99}{100}.$$

□

A lower bound using the *well-spread* nature of the distribution Assume, without loss of generality, from Lemma 2 that D_{z} is not too concentrated on any cluster B_{z}^i. Recall that $D_{\mathsf{z}}(\bigcup_i B_{\mathsf{z}}^i) \geq 199/200$. Therefore, there exist disjoint sets M and M' such that[1]

$$D_{\mathsf{z}}\left(\bigcup_{i \in M} B_{\mathsf{z}}^i\right) \geq \frac{1}{400} \quad \text{and} \quad D_{\mathsf{z}}\left(\bigcup_{i \in M'} B_{\mathsf{z}}^i\right) \geq \frac{1}{400}.$$

Furthermore, recall that since the distance between any two clusters is at least $c_2 n$, the same distance lower bound holds for the union of clusters over M and M' as well. This proves that the distribution D_{z} is *well-spread* which implies a circuit lower bound due to the following known fact[2] (see Appendix for proof):

[1]Consider building the set M greedily by adding terms until the mass exceeds $1/400$. Upon adding the final term to overcome the threshold, the total mass is at most $397/400$ since no term is larger than $99/100$. Therefore, the remainder of terms not included in M must have a mass of at least $199/200 - 397/400 = 1/400$.

[2]Versions of this lower-bound for well-spread distributions can be found in [AB22], Theorem 4.6], [EH17, Corollary 43], and [AN22, Lemma 13].

Fact 4. *Let D be a probability distribution on n bits generated by measuring the output of a quantum circuit in the standard basis. If two sets $S_1, S_2 \subset \{0,1\}^n$ satisfy $D(S_1), D(S_2) \geq \mu$, then the depth of the circuit is at least*

$$\frac{1}{3}\log\left(\frac{\mathrm{dist}(S_1,S_2)^2}{400n \cdot \log\frac{1}{\mu}}\right).$$

An immediate application of this fact gives a circuit-depth lower bound of $\Omega(\log n)$ for D_{z} since $\mathrm{dist}(S_1, S_2) \geq c_2 n$ and $\mu = \frac{1}{400}$. Since the circuit depth of D_{z} is at most one more than the circuit depth of ψ, the lower bound is proven.

Theorem 5 (Formal statement of the NLTS theorem)**.** *Consider a $[[n,k,d]]$ CSS code satisfying Property 1 with parameters δ_0, c_1, c_2 as stated. Let $\mathbf{H}$ be the corresponding local Hamiltonian. Then for*

$$\epsilon < \frac{1}{400c_1}\left(\frac{\min\{m_{\mathsf{x}}, m_{\mathsf{z}}\}}{n}\right)\cdot \min\left\{\left(\frac{k-1}{4n}\right)^2, \delta_0, \frac{c_2}{2}\right\},$$

and every state ψ such that $\mathrm{tr}(\mathbf{H}\psi) \leq \epsilon n$, the circuit depth of ψ is at least $\Omega(\log n)$. For constant-rate and linear-distance codes satisfying[3] Property 1, the bound on ϵ is a constant.

3 Proof that Property 1 holds for quantum Tanner codes [LZ22]

Definition of quantum Tanner codes For a group G, consider a right Cayley graph $\mathrm{Cay}^r(G,A)$ and a left Cayley graph $\mathrm{Cay}^\ell(G,B)$ for two generating sets $A, B \subset G$, which are assumed to be symmetric, i.e. $A = A^{-1}$ and $B = B^{-1}$ and of the same cardinality $\Delta = |A| = |B|$. Further, we define the double-covers of $\mathrm{Cay}^r(G,A)$ and $\mathrm{Cay}^\ell(G,B)$ that we will denote $\mathrm{Cay}_2^r(G,A)$ and $\mathrm{Cay}_2^\ell(G,B)$.[4] The vertex sets of $\mathrm{Cay}_2^r(G,A)$ and $\mathrm{Cay}_2^\ell(G,B)$ are $\{\pm\} \times G$ and $G \times \{\pm\}$, respectively. The edges of $\mathrm{Cay}_2^r(G,A)$ are labeled by $A \times G$ and are of the form $(g,+) \sim (ag,-)$. Similarly, the edges of $\mathrm{Cay}_2^\ell(G,B)$ are labeled by $G \times B$ and are of the form $(+,g) \sim (-,gb)$.

Quantum Tanner codes are defined on the balanced product of the two Cayley graphs $X' = \mathrm{Cay}_2^r(G,A) \times_G \mathrm{Cay}_2^\ell(G,B)$, see [BE21, Section IV-B]. It is given by the Cartesian product $\mathrm{Cay}_2^r(G,A) \times \mathrm{Cay}_2^\ell(G,B)$ with the (canonical) anti-diagonal action of G factored out. To understand the set of vertices V' of X', we first note that the vertices of the Cartesian product are labeled by $\{\pm\} \times G \times G \times \{\pm\}$. The group G acts via right-multiplication on the left copy of G and via inverse left-multiplication on the right copy of G. Factoring out this action identifies the vertices $(\pm, a, b, \pm)$ with $(\pm, ag, g^{-1}b, \pm)$ for all $g \in G$. This means that two vertices $(\pm, a, b, \pm)$ and $(\pm, c, d, \pm)$ are identified if and only if $ab = cd$ and the outer signs agree. By passing from these equivalence classes to $ab \in G$, we obtain a unique labeling of the vertices V' of X' by $\{\pm\} \times G \times \{\pm\}$. Thus, V' can be partitioned into the *even-parity vertices* V_0', which are all vertices of the form $(+,g,+)$ and $(-,g,-)$, and the *odd-parity vertices* V_1', which are all vertices of the form $(+,g,-)$ and $(-,g,+)$. The complex X' is called the "quadripartite version" in [LZ22].

Note that besides the natural action of G, there is an addition action of $\mathbb{Z}_2 = \langle\sigma\rangle$ on $\mathrm{Cay}_2^r(G,A)$ and $\mathrm{Cay}_2^\ell(G,B)$, which operates on the labels $\{\pm\}$ via $\sigma(+) = -$ and $\sigma(-) = +$. Hence, there is an

[3] While the distance parameter d does not appear in the bound on ϵ, Property 1 for $\delta = 0$ implies constant distance.

[4] The reason for defining the double-covers is convenience; the covering allows us to label each edge directly by specifying a vertex (group element) and a generator, which is not immediately possible in the original Cayley graphs.

operation of the group $G \times \mathbb{Z}_2$. We can thus analogously define the alternative balanced product complex $X = \mathrm{Cay}_2^r(G, A) \times_{(G\times\mathbb{Z}_2)} \mathrm{Cay}_2^\ell(G, B)$. The complex X is called the "bipartite version" in [LZ22]. Here, we will consider the complex X instead of X'. Using the same arguments as previously for X', we see that the vertices V of X can be labeled by $G \times \{\pm\}$ which fall into the sets V_0, which are all vertices of the form $(g, +)$, and V_1, which are all vertices of the form $(g, -)$.

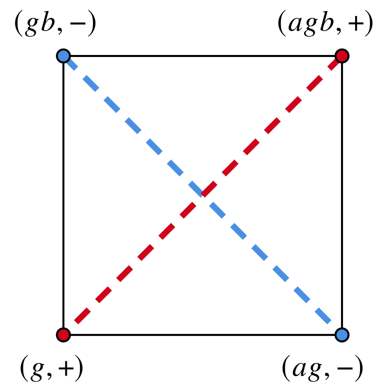

Figure 1: A face of the balanced product complex $X = \mathrm{Cay}_2^r(G, A) \times_{(G\times\mathbb{Z}_2)} \mathrm{Cay}_2^\ell(G, B)$. Each face is incident to two vertices in V_0 (red) and two vertices in V_1 (blue). This fact is used in [LZ22] to define two graphs $\mathcal{G}_0^\square$ and $\mathcal{G}_1^\square$ whose edges connect the vertices in V_0 (red dashed line) and V_1 (blue dashed line), respectively. Importantly, the edge-sets of $\mathcal{G}_0^\square$ and $\mathcal{G}_1^\square$ are both in one-to-one correspondence with the faces of X (and thus with each other).

The quantum Tanner code is now defined as follows. From the balanced product complex X we define two graphs $\mathcal{G}_0^\square$ and $\mathcal{G}_1^\square$. The vertices of $\mathcal{G}_0^\square$ are the vertices in V_0. Note that there are exactly two vertices belonging to V_0 per face in X, see Figure 1. Hence, we connect two vertices by an edge in $\mathcal{G}_0^\square$ if and only if they belong to the same face, or equivalently, all edges in $\mathcal{G}_0^\square$ are of the form $(g, +) \sim (agb, +)$. Similarly, we can define the graph $\mathcal{G}_1^\square$ using the fact that there are exactly two vertices in V_1 per face. Note that both $\mathcal{G}_0^\square$ and $\mathcal{G}_1^\square$ are regular graphs of degree Δ^2, as edges surrounding a vertex are labeled by $A \times B$. Further, $\mathcal{G}_0^\square$ and $\mathcal{G}_1^\square$ are expanders: Let $\lambda(\mathcal{G}) = \max\{|\lambda_2(\mathcal{G})|, |\lambda_n(\mathcal{G})|\}$, where $\lambda_2(\mathcal{G}), \lambda_n(\mathcal{G})$ are the second largest and the smallest eigenvalues of the adjacency matrix of the graph $\mathcal{G}$.

Lemma 6 ([LZ22, Lemma 4]). *If* $\mathrm{Cay}^r(G, A)$ *and* $\mathrm{Cay}^\ell(G, B)$ *are Ramanujan graphs, then*

$$\lambda(\mathcal{G}_0^\square), \lambda(\mathcal{G}_1^\square) \leq 4\Delta.$$

Taking two suitable local codes $C_A, C_B \subset \mathbb{F}_2^\Delta$, we define $C_0 = C_A \otimes C_B$ and $C_1 = C_A^\perp \otimes C_B^\perp$. Finally, we define Tanner codes $C_{\mathsf{z}} = C(\mathcal{G}_0^\square, C_0^\perp)$ and $C_{\mathsf{x}} = C(\mathcal{G}_1^\square, C_1^\perp)$ [Tan81, SS96]. It can be shown [LZ22] that $C_{\mathsf{z}} \supset C_{\mathsf{x}}^\perp$, so that we obtain a well-defined CSS code.

For these codes to have constant-rate and linear-distance, the graphs and local codes need to fulfill certain conditions: The Cayley graphs are required to be Ramanujan expanders [LPS88, Mar88]. Further, the local codes are required to be *robust* and *resistant to puncturing*. More precisely, we call $C_1^\perp = (C_A^\perp \otimes C_B^\perp)^\perp = C_A \otimes \mathbb{F}_2^B + \mathbb{F}_2^A \otimes C_B$ *w-robust* if any code word $|x|$ of Hamming weight bounded as $|x| \leq w$ has its support included in $|x|/d_A$ columns and $|x|/d_B$ rows, where d_A and d_B are the minimum distances of C_A and C_B, respectively. Further, $C_1^\perp$ has w-robustness with *resistance to puncturing p* if for any $A' \subset A$, $B' \subset B$ with $|A'|, |B'| \geq \Delta - w'$ with $w' \leq p$ the code $C_1^\perp$ remains w-robust when punctured outside of $A' \times B'$.

Clustering of code-words We will now show that the quantum Tanner Codes defined above satisfy Property 1. We start with the following claim which is stated along the same lines as [LZ22, Theorem 1], and proved in the appendix. We have changed some constants, for consistency purposes.

Claim 7. *Fix $\lambda \in (0, \frac{1}{2})$, $\gamma \in (\frac{1}{2}+\lambda, 1)$ and $\kappa > 0$. Suppose C_A, C_B have distance at least $\kappa\Delta$ and $C_0^\perp, C_1^\perp$ are $\Delta^{\frac{3}{2}-\lambda}$-robust with Δ^γ resistance to puncturing. Then there exist constants c_1, c_2, δ_0 such that the following holds when $\delta \leq \delta_0$.*

1. *For any $x \in G_{\mathsf{x}}^\delta$ with $c_1\delta m_{\mathsf{x}} \leq |x| \leq c_2 n$, there is a $y \in C_{\mathsf{z}}^\perp$ satisfying $|x \oplus y| < |x|$.*
2. *For any $z \in G_{\mathsf{z}}^\delta$ with $c_1\delta m_{\mathsf{z}} \leq |z| \leq c_2 n$, there is a $w \in C_{\mathsf{x}}^\perp$ satisfying $|z \oplus w| < |z|$.*

Note that δ_0 is chosen simply to ensure that $c_1\delta_0 m_{\mathsf{x}} \leq c_2 n$ and $c_1\delta_0 m_{\mathsf{z}} \leq c_2 n$.

We will now establish Property 1 using this claim. For $x \in G_{\mathsf{x}}^\delta$, if $c_1\delta m_{\mathsf{x}} \leq |x|_{C_{\mathsf{z}}^\perp} \leq c_2 n$, then there is a $y' \in C_{\mathsf{z}}^\perp$ such that $c_1\delta m_{\mathsf{x}} \leq |x \oplus y'| \leq c_2 n$. Note that $x \oplus y' \in G_{\mathsf{x}}^\delta$, since $H_{\mathsf{x}} y' = 0$. Thus, we can invoke Claim 7 (many times) to conclude that there is a $y \in C_{\mathsf{z}}^\perp$ such that $|x \oplus y \oplus y'| < c_1\delta m_{\mathsf{x}}$. But $|x|_{C_{\mathsf{z}}^\perp} \leq |x \oplus y \oplus y'| < c_1\delta m_{\mathsf{x}}$, leading to a contradiction. Thus, either $|x|_{C_{\mathsf{z}}^\perp} \geq c_2 n$ or $|x|_{C_{\mathsf{z}}^\perp} \leq c_1\delta m_{\mathsf{x}} = c_1\delta\frac{m_{\mathsf{x}}}{n} \cdot n$. We can argue similarly for G_{z}^δ. Thus, Property 1 is satisfied with modified constant $\delta_0 \to \delta_0 \cdot \frac{\min\{m_{\mathsf{x}}, m_{\mathsf{z}}\}}{n}$.

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A Omitted Proofs

Proof of Fact 3: Consider a purification of the state ψ as $|\psi\rangle$ on a potentially larger Hilbert space. Write $|\psi\rangle$ as $\sum_{z\in\{0,1\}^n} |\psi_z\rangle \otimes |z\rangle$ where the second register is the original n qubit code-space. Define $C \stackrel{\text{def}}{=} \sum_{z\in S} \| \, |\psi_z\rangle \, \|^2$ and

$$|\psi'\rangle = \frac{1}{\sqrt{C}} \sum_{z\in S} |\psi_z\rangle \otimes |z\rangle \stackrel{\text{def}}{=} \sum_{z\in S} |\psi'_z\rangle \otimes |z\rangle \, .$$

Since $C = D_{\mathsf{z}}(S) \stackrel{\text{def}}{=} 1-\eta$, by the gentle measurement lemma [Win99] we have $\frac{1}{2}\| \, |\psi\rangle\langle\psi| - |\psi'\rangle\langle\psi'| \, \|_1 \leq 2\sqrt{\eta}$. Measuring $|\psi'\rangle$ in the computational basis, we obtain a string $z \in S$ with probability $\| \, |\psi'_z\rangle \, \|^2$. Measuring $|\psi'\rangle$ in the Hadamard basis, we obtain a string x with probability

$$p(x) \stackrel{\text{def}}{=} \frac{1}{2^n} \left\| \sum_z (-1)^{x.z} |\psi'_z\rangle \right\|^2 = \frac{1}{2^n} \left(\sum_{z,w} (-1)^{x.(z\oplus w)} \langle \psi'_w | \psi'_z \rangle \right).$$

Then we can compute the collision probability of $p(x)$:

$$
\begin{aligned}
\sum_x p(x)^2 &= \frac{1}{2^{2n}} \sum_x \left(\sum_{z,w} (-1)^{x.(z\oplus w)} \left\langle \psi'_w \middle| \psi'_z \right\rangle \right)^2 \\
&= \frac{1}{2^{2n}} \left(\sum_x \sum_{s,t,z,w} (-1)^{x.(z\oplus w\oplus s\oplus t)} \left\langle \psi'_s \middle| \psi'_t \right\rangle \left\langle \psi'_w \middle| \psi'_z \right\rangle \right) \\
&= \frac{1}{2^n} \left(\sum_{s,t,z,w: z\oplus w\oplus s\oplus t=0} \left\langle \psi'_s \middle| \psi'_t \right\rangle \left\langle \psi'_w \middle| \psi'_z \right\rangle \right) \\
&= \frac{1}{2^n} \left(\sum_{s,t,w} \left\langle \psi'_s \middle| \psi'_t \right\rangle \left\langle \psi'_w \middle| \psi'_{s\oplus t\oplus w} \right\rangle \right) \\
&\le \frac{1}{2^n} \left(\sum_{s,t} \| |\psi'_s\rangle \| \, \| |\psi'_t\rangle \| \cdot \left(\sum_w \| |\psi'_w\rangle \| \, \| |\psi'_{s\oplus t\oplus w}\rangle \| \right) \right) \\
&\le \frac{1}{2^n} \left(\sum_{s,t} \| |\psi'_s\rangle \| \, \| |\psi'_t\rangle \| \cdot \left(\sqrt{\sum_w \| |\psi'_w\rangle \|^2} \sqrt{\sum_w \| |\psi'_{s\oplus t\oplus w}\rangle \|^2} \right) \right) \\
&= \frac{1}{2^n} \left(\sum_{s,t} \| |\psi'_s\rangle \| \, \| |\psi'_t\rangle \| \right) = \frac{1}{2^n} \left(\sum_{s\in S} \| |\psi'_s\rangle \| \right)^2 \le \frac{1}{2^n} \cdot |S| \cdot \left(\sum_s \| |\psi'_s\rangle \|^2 \right) = \frac{|S|}{2^n}.
\end{aligned}
$$

The previous line follows by an application of the Cauchy-Schwarz inequality. Apply it again to calculate that

$$
\sum_{x\in T} p(x) \le \sqrt{|T| \sum_x p(x)^2} \le \sqrt{\frac{|S| \cdot |T|}{2^n}}.
$$

Since $\frac{1}{2} \| |\psi\rangle\langle\psi| - |\psi'\rangle\langle\psi'| \|_1 \le 2\sqrt{\eta}$, we conclude that $D_{\mathsf{x}}(T) \le 2\sqrt{\eta} + \sqrt{\frac{|S|\cdot|T|}{2^n}}$. □

Proof of Fact 4: Let $|\rho\rangle = U\,|0\rangle^{\otimes m}$ on $m \ge n$ qubits, where U is a depth t quantum circuit such that when $|\rho\rangle$ is measured in the standard basis, the resulting distribution is p. Note that $m \le 2^t n$ without loss of generality (see [AN22, Section 2.3] for a justification based on the light cone argument). The Hamiltonian

$$
G = \mathop{\mathbf{E}}_{i=1}^{m} U\,|1\rangle\langle 1|_i\, U^\dagger
$$

has $|\rho\rangle$ as its unique ground-state, is commuting, has locality 2^t, and has eigenvalues $0, 1/m, 2/m, \ldots 1$. There exists a polynomial P of degree f, built from Chebyshev polynomials, such that

$$
P(0) = 1, \qquad |P(i/m)| \le \exp\left(-\frac{f^2}{100m}\right) \le \exp\left(-\frac{f^2}{100\cdot 2^t n}\right) \text{ for } i = 1, 2, \ldots, m.
$$

See [AAG22, Theorem 3.1] (or [KLS96, BCDZ99]) for details on the construction of P. Applying the polynomial P to the Hamiltonian G results in an *approximate ground-state projector*, $P(G)$, such that

$$
\| |\rho\rangle\langle\rho| - P(G) \|_\infty \le \exp\left(-\frac{f^2}{100\cdot 2^t n}\right)
$$

Furthermore, $P(G)$ is a $f \cdot 2^t$ local operator. Setting $u \stackrel{\text{def}}{=} \text{dist}(S_1, S_2)$ and choosing $f \stackrel{\text{def}}{=} \frac{u}{2^{t+1}}$, we obtain

$$\| \,|\rho\rangle\langle\rho| - P(G)\|_\infty \leq \exp\left(-\frac{u^2}{400 \cdot 2^{3t} n}\right).$$

Let Π_{S_1}, Π_{S_2} be projections onto the strings in sets S_1, S_2 respectively. Note that $\Pi_{S_1} P(G) \Pi_{S_2} = 0$, which implies

$$\|\Pi_{S_1} \,|\rho\rangle\langle\rho|\, \Pi_{S_2}\|_\infty \leq \exp\left(-\frac{u^2}{400 \cdot 2^{3t} \cdot n}\right).$$

However

$$\|\Pi_{S_1} \,|\rho\rangle\langle\rho|\, \Pi_{S_2}\|_\infty = \sqrt{\langle\rho|\, \Pi_{S_1} \,|\rho\rangle \cdot \langle\rho|\, \Pi_{S_2} \,|\rho\rangle} = \sqrt{p(S_1)p(S_2)} \geq \mu.$$

Thus, $2^{3t} \geq \frac{u^2}{400 \cdot \log \frac{1}{\mu} \cdot n}$, which rearranges into the fact statement.

□

Proof of Claim 7: We prove the first part of the claim. The second part follows along the same lines. Following [LZ22], we define $\mathcal{G}^{\square}_{1,x}$ as the sub-graph of $\mathcal{G}^{\square}_1$ that is induced by $x \in G^{\delta}_{\mathsf{x}}$ (in other words, we only consider those edges of $\mathcal{G}^{\square}_1$ for which the corresponding squares have a '1' assigned by x). Let $S \subset V_1$ be the set of vertices in $\mathcal{G}^{\square}_{1,x}$. Most vertices v in S have their local view according to $C_1^{\perp}$. But, x is an approximate code-word from G^{δ}_{x}. So there are no restrictions on the local views of at most δm_{x} vertices in S. We now modify the definition of 'exceptional vertices' from [LZ22]. Let $S_e \subset S$ be the set of vertices v which satisfy one of the two conditions:

- The degree is at least $\Delta^{\frac{3}{2}-\lambda}$ in $\mathcal{G}^{\square}_{1,x}$.
- The local view of x at v violates a check in $C_1^{\perp}$.

Since $|S| \geq \frac{2|x|}{\Delta^2}$, we choose $c_1 \stackrel{\text{def}}{=} \frac{\Delta^{3-2\lambda}}{256}$ to conclude that $|S| \geq \frac{\Delta^{1-2\lambda}}{128} \delta m_{\mathsf{x}}$. Now, we establish the following bound on $|S_e|$, which modifies [LZ22, Claim 9].

$$|S_e| \leq \frac{256|S|}{\Delta^{1-2\lambda}} + 2\delta m_{\mathsf{x}} \leq \frac{512|S|}{\Delta^{1-2\lambda}}. \tag{1}$$

To establish this bound, we proceed the same as [LZ22]. Note that all the vertices in S that are not 'violated' by x have degree at least $\kappa\Delta$ (distance of the local code $C_1^{\perp}$). Thus, setting $c_2 \stackrel{\text{def}}{=} \frac{\kappa \Delta^{\frac{1}{2}-\lambda}}{16} \cdot \frac{|V_1|}{n}$ and noting that $|S| \geq 2\delta m_{\mathsf{x}}$ for large constant Δ, we obtain

$$\frac{|S|}{2} \leq (|S| - \delta m_{\mathsf{x}}) \leq \frac{2|x|}{\kappa\Delta} \implies |S| \leq \frac{4|x|}{\kappa\Delta} \leq \frac{|V_1|}{4\Delta^{\frac{1}{2}+\lambda}}.$$

If $|S_e| \leq 2\delta m_{\mathsf{x}}$, Equation (1) is verified. Otherwise, by using the expander mixing lemma and Lemma 6, we have

$$\frac{\Delta^{\frac{3}{2}-\lambda}}{2}|S_e| \leq \Delta^{\frac{3}{2}-\lambda}(|S_e| - \delta m_{\mathsf{x}}) \leq E(S_e, S) \leq \frac{\Delta^2 |S_e||S|}{|V_1|} + 4\Delta\sqrt{|S_e||S|} \leq \frac{\Delta^{\frac{3}{2}-\lambda}}{4}|S_e| + 4\Delta\sqrt{|S_e||S|},$$

which implies $|S_e| \leq \frac{256|S|}{\Delta^{1-2\lambda}}$.

Having established Equation (1), which modifies a similar expression in [LZ22] by a constant factor of 8, we proceed further in a very similar manner. We define the normal vertices ($S \setminus S_e$), heavy edges and the set T in the same manner. The upper bound on $|T|$ in [LZ22, Claim 11] remains unchanged. To arrive at [LZ22, Claim 12], the definition of α is slightly modified according to Equation (1). We need a vertex in T that is not adjacent to large number of vertices in S_e. For this, [LZ22] upper bound $|E(S_e, T)|$ using the expander mixing lemma. The modified constants lead to a new upper bound

$$|E(S_e, T)| \leq \frac{256}{\Delta^{\frac{1}{2}-\lambda}}|T| + 128\Delta^{\lambda}\sqrt{|S||T|} \stackrel{\text{def}}{=} \beta\Delta^{\frac{1}{2}+\lambda}|T|, \quad \beta = 256 + \frac{512}{\Delta}.$$

The rest of the argument remains unchanged with the modified constants α, β. □

Small-set expansion

Definition 8. *Let G be a d-left-regular bipartite graph between vertex sets L and R. A subset $A \subset L$ is said to be γ-expanding if $|\Gamma(A)| \geq (1-\gamma)d|A|$ where $\Gamma(A) \subset R$ is the set of neighbors of A. We say that G is (γ, α)-small set expanding if every set A of size $\leq \alpha|L|$ is γ-expanding.*

Lemma 9. *For a classical error correcting code with check matrix $H \in \mathbb{F}_2^{m\times n}$, draw the interaction graph G between the set of vertices, $V = [n]$, and the set of checks, $C = [m]$, with an edge $v \sim c$ if v participates in the check c. If G is (γ, α)-small set expanding for $\gamma < \frac{1}{2}$, then the code satisfies the classical version of Property 1.*

Proof. Consider any $y \in \{0,1\}^n$. If $|y| < \alpha n$, then y is the indicator vector for a small subset $A \subset V$, and $|\Gamma(A)| \geq (1-\gamma)d|A|$. Let $\Gamma^+(A)$ be the subset of $\Gamma(A)$ with a unique neighbor in A. Since the number of edges between A and $\Gamma(A)$ is $d|A|$, then

$$\begin{aligned} d|A| &\geq \left|\Gamma^+(A)\right| + 2\cdot\left(|\Gamma(A)| - \left|\Gamma^+(A)\right|\right) \\ &= -\left|\Gamma^+(A)\right| + 2(1-\gamma)d|A| \end{aligned}$$

Therefore, $|\Gamma^+(A)| \geq (1-2\gamma)d|A|$. Since every check in $\Gamma(A)$ is adjacent to a unique vertex in A, $\Gamma^+(A)$ is a subset of the checks that will be violated by y. Set $c_2 \stackrel{\text{def}}{=} \alpha$ and $c_1 \stackrel{\text{def}}{=} \frac{m}{(1-2\gamma)dn}$. If $|y| < \alpha n$, then

$$\delta m \geq |Hy| \geq \left|\Gamma^+(A)\right| \geq (1-2\gamma)d|A| = (1-2\gamma)d|y|.$$

This shows that, in fact, $|y| < c_1\delta n$.

□

References

[AAG22] Anurag Anshu, Itai Arad, and David Gosset. An area law for 2d frustration-free spin systems. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing*, STOC 2022, page 12–18, New York, NY, USA, 2022. Association for Computing Machinery. 8

[AALV09] Dorit Aharonov, Itai Arad, Zeph Landau, and Umesh Vazirani. The detectability lemma and quantum gap amplification. In *Proceedings of the Forty-First Annual ACM Symposium on Theory of Computing*, STOC '09, page 417–426, New York, NY, USA, 2009. Association for Computing Machinery. 1

[AAV13] Dorit Aharonov, Itai Arad, and Thomas Vidick. Guest Column: The Quantum PCP Conjecture. *SIGACT News*, 44(2):47–79, June 2013. 1, 3

[AB22] Anurag Anshu and Nikolas P. Breuckmann. A construction of Combinatorial NLTS, 2022. 1, 2, 4

[AE15] Dorit Aharonov and Lior Eldar. The Commuting Local Hamiltonian Problem on Locally Expanding Graphs is Approximable in NP. *Quantum Information Processing*, 14(1):83–101, January 2015. 1

[AN22] Anurag Anshu and Chinmay Nirkhe. Circuit Lower Bounds for Low-Energy States of Quantum Code Hamiltonians. In Mark Braverman, editor, *13th Innovations in Theoretical Computer Science Conference (ITCS 2022)*, volume 215 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 6:1–6:22, Dagstuhl, Germany, 2022. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. 1, 2, 4, 8

[BCDZ99] H. Buhrman, R. Cleve, R. De Wolf, and C. Zalka. Bounds for small-error and zero-error quantum algorithms. pages 358–368, 1999. 8

[BE21] Nikolas P. Breuckmann and Jens N. Eberhardt. Balanced Product Quantum Codes. *IEEE Transactions on Information Theory*, 67(10):6653–6674, 2021. 2, 5

[BH13] Fernando G.S.L. Brandao and Aram W. Harrow. Product-state approximations to quantum ground states. In *Proceedings of the Forty-Fifth Annual ACM Symposium on Theory of Computing*, STOC ’13, page 871–880, New York, NY, USA, 2013. Association for Computing Machinery. 1

[BKKT19] Sergey Bravyi, Alexander Kliesch, Robert Koenig, and Eugene Tang. Obstacles to state preparation and variational optimization from symmetry protection. *https://arxiv.org/abs/1910.08980*, 2019. 1

[BV05] Sergey Bravyi and Mikhail Vyalyi. Commutative version of the local hamiltonian problem and common eigenspace problem. *Quantum Info. Comput.*, 5(3):187–215, May 2005. 1

[DHLV22] Irit Dinur, Min-Hsiu Hsieh, Ting-Chun Lin, and Thomas Vidick. Good Quantum LDPC Codes with Linear Time Decoders, 2022. 2

[EH17] L. Eldar and A. W. Harrow. Local hamiltonians whose ground states are hard to approximate. In *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 427–438, 2017. 1, 3, 4

[Eld21] Lior Eldar. Robust Quantum Entanglement at (Nearly) Room Temperature. In James R. Lee, editor, *12th Innovations in Theoretical Computer Science Conference (ITCS 2021)*, volume 185 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 49:1–49:20, Dagstuhl, Germany, 2021. Schloss Dagstuhl–Leibniz-Zentrum für Informatik. 1

[FH14] Michael H. Freedman and Matthew B. Hastings. Quantum systems on non-k-hyperfinite complexes: A generalization of classical statistical mechanics on expander graphs. *Quantum Info. Comput.*, 14(1–2):144–180, January 2014. 1

[HL22] Max Hopkins and Ting-Chun Lin. Explicit lower bounds against omega(n)-rounds of sum-of-squares, 2022. 2

[KLS96] Jeff Kahn, Nathan Linial, and Alex Samorodnitsky. Inclusion-exclusion: Exact and approximate. *Combinatorica*, 16(4):465–477, Dec 1996. 8

[KSV02] A. Yu. Kitaev, A. H. Shen, and M. N. Vyalyi. *Classical and Quantum Computation*. American Mathematical Society, USA, 2002. 1

[LPS88] Alexander Lubotzky, Ralph Phillips, and Peter Sarnak. Ramanujan graphs. *Combinatorica*, 8(3):261–277, 1988. 6

[LZ22] Anthony Leverrier and Gilles Zémor. Quantum Tanner codes, 2022. 2, 5, 6, 7, 9, 10

[Mar88] Grigorii Aleksandrovich Margulis. Explicit group-theoretical constructions of combinatorial schemes and their application to the design of expanders and concentrators. *Problemy peredachi informatsii*, 24(1):51–60, 1988. 6

[NV18] Anand Natarajan and Thomas Vidick. Low-degree testing for quantum states, and a quantum entangled games PCP for QMA. In Mikkel Thorup, editor, *59th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2018, Paris, France, October 7-9, 2018*, pages 731–742. IEEE Computer Society, 2018. 1

[NVY18] Chinmay Nirkhe, Umesh Vazirani, and Henry Yuen. Approximate Low-Weight Check Codes and Circuit Lower Bounds for Noisy Ground States. In Ioannis Chatzigiannakis, Christos Kaklamanis, Dániel Marx, and Donald Sannella, editors, *45th International Colloquium on Automata, Languages, and Programming (ICALP 2018)*, volume 107 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 91:1–91:11, Dagstuhl, Germany, 2018. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. 1

[PK21] Pavel Panteleev and Gleb Kalachev. Asymptotically Good Quantum and Locally Testable Classical LDPC Codes, 2021. 2

[SS96] Michael Sipser and Daniel A Spielman. Expander codes. *IEEE transactions on Information Theory*, 42(6):1710–1722, 1996. 6

[Tan81] R Tanner. A recursive approach to low complexity codes. *IEEE Transactions on information theory*, 27(5):533–547, 1981. 6

[Win99] A. Winter. Coding theorem and strong converse for quantum channels. *IEEE Transactions on Information Theory*, 45(7):2481–2485, 1999. 7