



Zhu, M., Macdonald, J. H. G., Jiang, J. Z., & Neild, S. A. (2022). Modelling a uniaxial inerter in a 2D or 3D environment: Implications of centripetal acceleration. *Journal of Sound and Vibration*, 540, 1-14. [117290]. <https://doi.org/10.1016/j.jsv.2022.117290>

Publisher's PDF, also known as Version of record

License (if available):
CC BY

Link to published version (if available):
[10.1016/j.jsv.2022.117290](https://doi.org/10.1016/j.jsv.2022.117290)

[Link to publication record in Explore Bristol Research](#)
PDF-document

This is the final published version of the article (version of record). It first appeared online via Elsevier at <https://doi.org/10.1016/j.jsv.2022.117290>. Please refer to any applicable terms of use of the publisher.

University of Bristol - Explore Bristol Research

General rights

This document is made available in accordance with publisher policies. Please cite only the published version using the reference above. Full terms of use are available: <http://www.bristol.ac.uk/red/research-policy/pure/user-guides/ebr-terms/>



Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsv

Modelling a uniaxial inerter in a 2D or 3D environment: Implications of centripetal acceleration

Ming Zhu, John H.G. Macdonald[†], Jason Zheng Jiang^{*}, Simon A. Neild

Faculty of Engineering, University of Bristol, Bristol, UK

ARTICLE INFO

Keywords:

Inerter
3D modelling
Multibody model
Centripetal acceleration

ABSTRACT

The inerter completes the force-current analogy between mechanical and electrical components, providing the mechanical equivalent to the capacitor. As such, it is a two-terminal passive element that, when implemented ideally, is normally said to generate a force proportional to the relative acceleration between its two terminals. However, this is applicable only if the inerter does not rotate, so the only relative motion between the device's terminals is axial. In many applications, this restriction is acceptable, such as in car suspension systems. However, in this paper, it is shown that the relationship between the terminal accelerations and the generated force is more complex if the inerter is used in a 2-dimensional (2D) or 3-dimensional (3D) environment, such as within a multi-bar mechanism (e.g., robotic arms or railway pantographs). Specifically, the inerter force is not given by simply the relative acceleration between the two terminals. The centripetal acceleration, resulting from the rotation of the inerter, needs to be accounted for to find the second derivative of the inerter length, which defines the generated force. Two case studies are presented to demonstrate the effects of this normally neglected centripetal acceleration term. It is shown that when an inerter is operating in a 2D or 3D environment, significant errors may occur in evaluating the inerter force and also the system response if the centripetal acceleration term is neglected. Equations are provided for both modelling the inerter in different coordinate systems and for incorporating the inerter in 2D and 3D multibody systems.

1. Introduction

In 2002, Smith [1] introduced a new two-terminal ideal passive mechanical element, termed the inerter, with the property that the force F_I at its two terminals is proportional to the relative acceleration between them (when the inerter does not rotate). The introduction of the inerter achieves a full analogy between mechanical and electrical networks in which force (or velocity) corresponds to current (or voltage) and a fixed point in an inertial reference frame corresponds to electrical ground. In this analogy, a spring (or damper or inerter) corresponds to an inductor (or resistor or capacitor, respectively). Since the concept of the inerter was proposed, many beneficial mechanical configurations with inerters have been explored for a wide range of engineering applications. In particular, an inerter-based device, called the 'J-damper' as a decoy name to keep the technology secret from competitors, was developed by Cambridge University and helped McLaren Racing achieve victory at the 2005 Spanish Grand Prix [2]. A seismic control

^{*} Corresponding author: Jason Zheng Jiang.

E-mail address: z.jiang@bristol.ac.uk (J.Z. Jiang).

[†] This work is dedicated in memory of John Macdonald, who passed away on 5th March 2022.

<https://doi.org/10.1016/j.jsv.2022.117290>

Received 31 December 2021; Received in revised form 20 August 2022; Accepted 5 September 2022

Available online 8 September 2022

0022-460X/© 2022 The Authors. Published by Elsevier Ltd. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

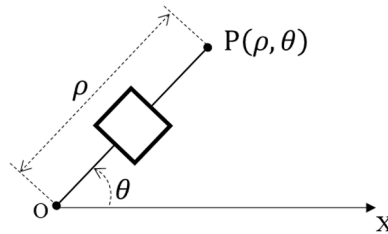


Fig. 1. Two-terminal uniaxial element in 2D polar coordinates.

device, the tuned viscous mass damper (TVMD), has been investigated in structural systems [3,4] and the soil–structure interaction of a structure with an inerter system has been studied [5]. Furthermore, beneficial inerter-based absorbers have been identified for a wide range of mechanical structures, such as buildings [6,7], cables [8,9], automotive vehicles [10–12], railway vehicles [13,14], aircraft landing gear [15], offshore wind turbines [16] and railway pantographs [17]. There has been some recent work come out on the use of a pair of oblique inerters in nonlinear vibration isolators and suppression systems [18, 19]. In addition, inerter-combined active or semiactive vibration control has also drawn interest from researchers, e.g., [20–23].

However, the original model of the inerter has the constraint that only translational movement along the element axis between two terminals of the inerter is permitted, thus it is only suitable to model an inerter in a one-dimensional (1D) environment. For other mechanisms and devices, such as robotic arms, excavator arms or railway pantographs, they may operate in a 2-dimensional (2D) or 3-dimensional (3D) environment, where the motions include both translation and rotation in general. Another example is inerter-based vibration absorbers in earthquake engineering applications. Many researchers have proposed inerter-based devices, such as tuned inerter damper [24], and tuned mass damper inerter [25, 26], which have shown significant effects on the protection of building structures under earthquake conditions. In some of these studies, the inerter-based absorbers are implemented as a ‘cross-bracing’ element [27] where rotational movement will be involved. Meanwhile, the excitations of the earthquake contain components in multiple directions, which can also lead to rotational movement of the inerter-based vibration absorber installed in building structures – this is a field where the present work will have a significant impact.

According to the original definition of the inerter in reference [1], the uniaxial inerter force in a 1D environment can be modelled as *a force that is proportional to the relative acceleration between its two terminals*. However, this simple model is not generally applicable in a 2D or 3D environment. The general inerter force in a 2D or 3D environment should be modelled as *a force proportional to the second time derivative of the distance between its two terminals*. The difference between the simple inerter model and the general inerter model is that the latter takes into account not only the projection of the relative acceleration onto the element axis but also the centripetal acceleration term (which could be neglected intuitively).

In this paper, a model of a uniaxial inerter in a 2D or 3D environment and the methodology of applying a uniaxial inerter in a multibody system is developed. Furthermore, the effect of the centripetal acceleration correction term and the method of implementing a uniaxial inerter in a 2D multibody system are demonstrated via two examples. The sections are arranged as follows. In Section 2, the concept of a uniaxial inerter within a 2D or 3D environment is introduced, initially in a 2D polar coordinate system. Subsequently, a model of a uniaxial inerter in a 2D or 3D environment is derived in general vector form, and its expressions in Cartesian and Spherical coordinates are presented. A 2D inerter-included model is then shown as an example to demonstrate the effect of the centripetal acceleration correction on the inerter force and the system response. In Section 3, the method of adopting a uniaxial inerter in a multibody system is introduced. The proposed modelling methodology is then demonstrated using an inerter-included two-bar mechanism as an example, where the evident effect of the centripetal acceleration correction term on the system responses can be observed. The conclusions are presented in Section 4.

2. Uniaxial inerter in a 2D or 3D environment

In this section, the concept of using a uniaxial inerter in a 2D or 3D environment is first introduced, using a 2D polar coordinate system. It is shown that the centripetal acceleration needs to be accounted for in modelling the inerter force when it rotates. Then the general models of all three uniaxial mechanical elements – springs, dampers and inerters - are derived in a 3D environment in general vector form, showing how inerters need to be treated differently. The general model of the inerter in 2D or 3D is then expressed in Cartesian and spherical coordinates for ease of use. Lastly, a simple 2D 1DOF inerter-included lumped mass system is used as an example to demonstrate the effect of the centripetal acceleration correction on the inerter force and the system response.

2.1. The concept of a uniaxial inerter in a 2D polar coordinate system

Uniaxial elements in this paper are defined as elements whose generated forces, *i.e.*, the forces that the elements apply at the terminals, are only aligned along their axis, *i.e.*, the axis across their two terminals. For example, a hydraulic damper can be modelled as a uniaxial element. A two-terminal uniaxial element, symbolised as a square block, is established in 2D polar coordinates in Fig. 1, where one terminal of the uniaxial element is fixed to the pole, point O, of the coordinate system. Without loss of generality, this symbol can represent a spring, a damper or an inerter. In this polar coordinate system, the position of the free terminal of the uniaxial

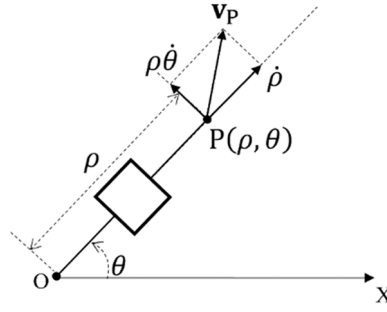


Fig. 2. The velocity of the free terminal, point P, in 2D polar coordinates.

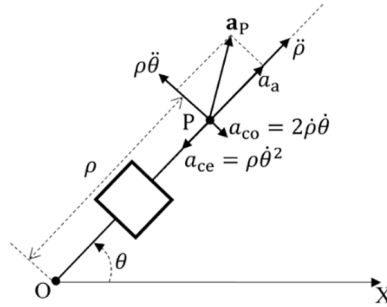


Fig. 3. Acceleration of the free terminal, point P, in 2D polar coordinates.

element, point P, is determined by the radius ρ , i.e., a distance from the pole, and polar angle θ , i.e., the angle from the polar axis X. The motion of the free terminal, point P, is given by translation along the axis OP and rotation of the axis OP about point O.

For spring with stiffness k , the force is proportional to the change in distance between its two terminals compared with its undeformed length ρ_0 . In this 2D polar coordinate system, if tension forces are defined as positive, the spring force is given by

$$F_S = k(\rho - \rho_0) \tag{1}$$

In this 2D polar coordinate system, the relative velocity between the element's two terminals, v_P , is a superposition of the radial velocity $\dot{\rho}$ along the element axis and the circumferential velocity $\rho\dot{\theta}$ orthogonal to the element axis, as shown in Fig. 2. Note that the overhead ‘ \cdot ’ denotes the time derivative. As the circumferential velocity $\rho\dot{\theta}$ is orthogonal to the element axis, it has no effect on the damper force. Hence, for a damper with damping coefficient c , the force is proportional to the component of relative velocity between the element's two terminals along its axis. That is the damper force is given by

$$F_D = c\dot{\rho}. \tag{2}$$

The original inerter force, defined in [1] for a 1D environment, is proportional to the relative acceleration between the inerter's two terminals. In the 2D polar coordinate system, let the relative acceleration between the element's two terminals, points O and P, be \mathbf{a}_P . Based on the definition of the inerter in [1], if there is no rotation of the inerter, i.e., the movement is restricted to be along the element axis, so $\dot{\theta} = 0$, the formula for the inerter force is

$$F_{I_no_rotation} = ba_a \tag{3}$$

where b is the inertance and a_a is the component of \mathbf{a}_P along the element axis (the only component in this case).

However, this definition is not applicable to the case where $\dot{\theta} \neq 0$, i.e., when the element is allowed to move in a 2D environment. This is because the relative acceleration between the element's two terminals, \mathbf{a}_P , consists of four components: the radial acceleration $\ddot{\rho}$ along the element axis and the circumferential acceleration $\rho\ddot{\theta}$ orthogonal to the element axis, but also the centripetal acceleration $a_{ce} = \rho\dot{\theta}^2$ along the element axis towards the pole, and the Coriolis acceleration $a_{co} = 2\dot{\rho}\dot{\theta}$ orthogonal to the element axis (see Fig. 3). Obviously, the circumferential acceleration $\rho\ddot{\theta}$ and the Coriolis acceleration a_{co} have no effect on the inerter force, being orthogonal to the element axis. In the component of acceleration along the element axis, $a_a = \ddot{\rho} - a_{ce}$, the centripetal acceleration a_{ce} also has no effect on the inerter force. This is because only the second time derivative of the distance between the inerter's two terminals, $\ddot{\rho}$, can generate an inerter force. For example, imagine that an inerter experiences pure rotation with a constant length between its two terminals – the inerter force is zero in this scenario as there is no change in the distance between the two terminals. Hence, when the inerter force is modelled in a 2D (or 3D) environment, the centripetal acceleration must be removed from the relative acceleration between the terminals to find the inerter force. In this way, the correct inerter force within this 2D polar coordinate system is

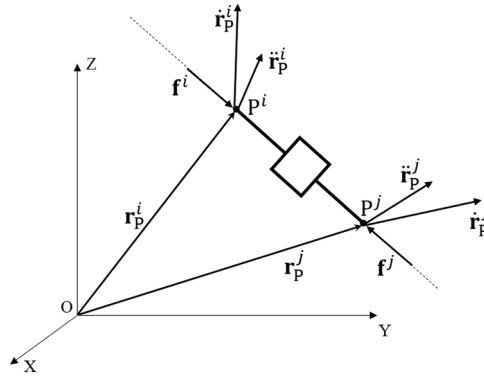


Fig. 4. Two-terminal uniaxial element in a 3D environment.

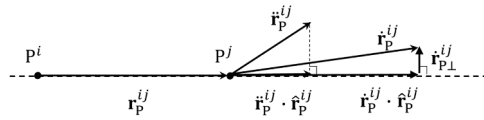


Fig. 5. Diagram of the relative displacement, velocity and acceleration vectors between the two terminals, points Pⁱ and P^j.

$$F_I = b(a_a - (-a_{cc})) = b\ddot{\rho}. \tag{4}$$

2.2. Model of a uniaxial inerter in general vector form

In the previous subsection, the concept of the inerter force was introduced within a polar coordinate system. The discussion shows that the centripetal acceleration has to be removed from the relative acceleration between the two terminals to find the inerter force in a 2D (or 3D) environment. In this subsection, models of uniaxial elements, including inerters, in a 3D (or 2D) environment are derived in general vector form.

Let points Pⁱ and P^j be the two terminals of a uniaxial element as shown in Fig. 4. Note that the superscript of Pⁱ (or P^j) indicates the body to which the point is connected (as in Section 3). These two terminals are assumed to be connected by pinned-pinned joints but constraining the rotational degree of freedom (DOF) about the PⁱP^j axis. For point Pⁱ (or P^j), its absolute position, velocity and acceleration with respect to the global coordinates X-Y-Z are denoted as 3-element vectors r_Pⁱ, r_Pⁱ and r_Pⁱ (or r_P^j, r_P^j and r_P^j), respectively.

For a spring, the force along its axis can be expressed in this 3D environment as

$$F_S = k \left((r_P^{ij} \cdot r_P^{ij})^{\frac{1}{2}} - L_0 \right) \tag{5}$$

where r_P^{ij} = r_P^j - r_Pⁱ is the relative position vector from point Pⁱ to point P^j within the global coordinates X-Y-Z, and L₀ is the undeformed length between the uniaxial element's terminals. Note that, in Eq. (5), the physical meaning of (r_P^{ij} · r_P^{ij})^{1/2} is the instantaneous distance between uniaxial element's terminals.

For a damper, by differentiating the distance between the uniaxial element's terminals, (r_P^{ij} · r_P^{ij})^{1/2}, the damper force along its axis can be expressed in this 3D environment as

$$F_D = c \left((r_P^{ij} \cdot r_P^{ij})^{-\frac{1}{2}} (\dot{r}_P^{ij} \cdot r_P^{ij}) \right) \tag{6}$$

where r_P^{ij} = d(r_P^{ij})/dt. Note that Eq. (6) can be rewritten as

$$F_D = c (r_P^{ij} \cdot \hat{r}_P^{ij}), \tag{7}$$

where r_P^{ij} = r_P^{ij} / ((r_P^{ij} · r_P^{ij})^{1/2}) is the unit vector along PⁱP^j and the physical meaning of r_P^{ij} · r_P^{ij} can be understood as the projection of r_P^{ij} onto the element axis across the two terminals (see Fig. 5).

For an inerter, based on the definition of inerter in a 1D environment, i.e., only movement along its axis is allowed, F_{I_no-rotation} is defined as

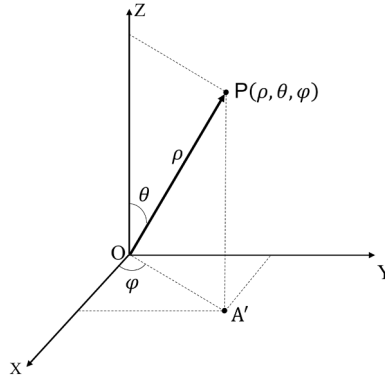


Fig. 6. Diagram of the spherical coordinate system.

$$F_{I_no_rotation} = b \left(\ddot{\mathbf{r}}_p^{ij} \cdot \hat{\mathbf{r}}_p^{ij} \right) \quad (8)$$

where $\ddot{\mathbf{r}}_p^{ij} = \frac{d(\dot{\mathbf{r}}_p^{ij})}{dt}$, and $\ddot{\mathbf{r}}_p^{ij} \cdot \hat{\mathbf{r}}_p^{ij}$ is the projection of the relative acceleration between the two terminals onto the element axis. Based on the discussion in SubSection 2.1, in a 3D environment, the centripetal acceleration has to be taken into account in modelling the inerter force. F_I in a 3D environment is defined as a force proportional to the second time derivative of the relative distance between the inerter's two terminals as

$$F_I = b \frac{d^2}{dt^2} \left(\left(\mathbf{r}_p^{ij} \cdot \mathbf{r}_p^{ij} \right)^{\frac{1}{2}} \right) \quad (9)$$

$$= b \left\{ \ddot{\mathbf{r}}_p^{ij} \cdot \hat{\mathbf{r}}_p^{ij} + \frac{1}{\left(\mathbf{r}_p^{ij} \cdot \mathbf{r}_p^{ij} \right)^{\frac{3}{2}}} \left[\dot{\mathbf{r}}_p^{ij} \cdot \dot{\mathbf{r}}_p^{ij} - \left(\dot{\mathbf{r}}_p^{ij} \cdot \hat{\mathbf{r}}_p^{ij} \right)^2 \right] \right\}.$$

In the curly braces of Eq. (9), the first term, $\ddot{\mathbf{r}}_p^{ij} \cdot \hat{\mathbf{r}}_p^{ij}$, is the projection of the relative acceleration onto the element axis across the two terminals (as in Eq. (8)), and the second term, $\frac{1}{\left(\mathbf{r}_p^{ij} \cdot \mathbf{r}_p^{ij} \right)^{\frac{3}{2}}} \left[\dot{\mathbf{r}}_p^{ij} \cdot \dot{\mathbf{r}}_p^{ij} - \left(\dot{\mathbf{r}}_p^{ij} \cdot \hat{\mathbf{r}}_p^{ij} \right)^2 \right]$, is the centripetal acceleration resulting from the rotation of the element. Specifically, the physical meaning of $\dot{\mathbf{r}}_p^{ij} \cdot \dot{\mathbf{r}}_p^{ij} - \left(\dot{\mathbf{r}}_p^{ij} \cdot \hat{\mathbf{r}}_p^{ij} \right)^2$ is the square of the circumferential velocity, which is denoted as $|\dot{\mathbf{r}}_{p\perp}^{ij}|^2$ (see Fig. 5).

The uniaxial inerter force model in Eq. (9), expressed in terms of vectors, is a general model which is valid in both 2D and 3D environments. This general inerter force model can be applied to different types of models, for example, multibody models or finite element models, and can be expressed in different coordinate systems, as in the following subsection.

2.3. Inerter force in cartesian and spherical coordinate systems

The general formula for the uniaxial inerter force, Eq. (9), can be applied in different 2D or 3D coordinate systems. The expressions for Cartesian and spherical coordinates are presented in this subsection.

In 3D Cartesian coordinates, defining the relative displacement vector between the inerter's two terminals, points P^j and P^i , as $\mathbf{r}_p^{ij} = [x \ y \ z]^T$ (where superscript "T" indicates transpose), using Eq. (9) the inerter force can be expressed as

$$F_I = \frac{b}{\sqrt{x^2 + y^2 + z^2}} \left\{ (\dot{x}x + \dot{y}y + \dot{z}z) + (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{(\dot{x}x + \dot{y}y + \dot{z}z)^2}{x^2 + y^2 + z^2} \right\}. \quad (10)$$

In 2D Cartesian coordinates, the same equation can be used, simply dropping the terms in z and its derivatives.

The spherical coordinate system (ρ, θ, φ) is another common coordinate system for a 3D environment, as shown in Fig. 6.

In these spherical coordinates, using Eq. (9), with the inerter terminals at points O and P, the inerter force can be expressed simply as

$$F_I = b\ddot{\rho}. \quad (11)$$

i.e. the same as for 2D polar coordinates, as in Eq. (4).

2.4. Example: a 2D 1DOF inerter-included lumped mass system

The model of a uniaxial inerter in a 2D or 3D environment was introduced in the previous subsections. In this subsection, a simple

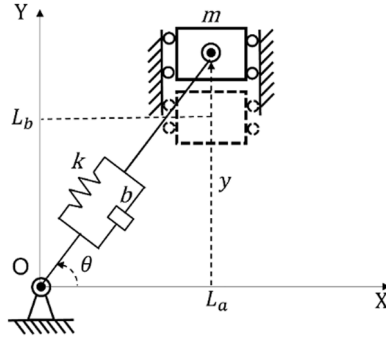


Fig. 7. An example 1DOF inerter-included lumped mass system.

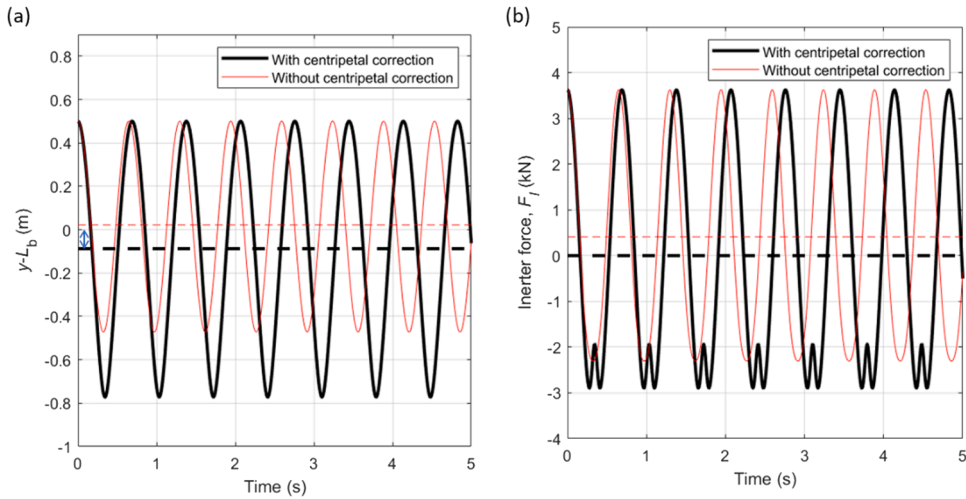


Fig. 8. Numerical time histories of (a) displacement responses relative to the equilibrium position (with the correction included, there is a mean displacement offset as indicated by the blue double-headed arrow) and (b) inerter forces, with and without the centripetal acceleration correction term. The dashed lines show the corresponding mean values.

2D 1DOF inerter-included lumped mass system is developed as an example to demonstrate the effect of the centripetal acceleration correction term on the inerter force and response of the system.

A 1DOF inerter-included lumped mass system in a 2D environment in the horizontal plane is defined as shown in Fig. 7. The lumped mass, m , is constrained to travel only in the Y direction. A uniaxial spring and a uniaxial inerter, in parallel, link the mass and the origin of the system, point O , with a revolute joint. Friction and damping are neglected. The offset of the line of motion of the mass from the origin, point O , in the X direction, is L_a . The static equilibrium position of the mass in the Y direction is L_b , that is, the undeformed length of the spring is $L_0 = \sqrt{L_a^2 + L_b^2}$. The position of the mass in the XY coordinate system can be expressed as a vector $\mathbf{r} = \begin{bmatrix} L_a \\ y \end{bmatrix}$. If the initial position of the mass in the Y direction is set to be y_0 ($y_0 \neq L_b$), when released the mass will oscillate about the equilibrium position in the Y direction. Using Eq. (10), the inerter force can be expressed as

$$F_i = \frac{by}{\sqrt{L_a^2 + y^2}}\ddot{y} + \frac{bL_a^2}{(\sqrt{L_a^2 + y^2})^3}\dot{y}^2. \tag{12}$$

Note that the second term on the right-hand side of Eq. (12), $\frac{bL_a^2}{(\sqrt{L_a^2 + y^2})^3}\dot{y}^2$, is the centripetal acceleration correction term. Meanwhile, using Eq. (1) or Eq. (5), the spring force in this example is $F_s = k(\sqrt{L_a^2 + y^2} - L_0)$. Thus, the equation of motion of this 1DOF inerter-included lumped mass system is

$$m\ddot{y} = - \left(\frac{by}{\sqrt{L_a^2 + y^2}}\ddot{y} + \frac{bL_a^2}{(\sqrt{L_a^2 + y^2})^3}\dot{y}^2 + k(\sqrt{L_a^2 + y^2} - L_0) \right) \frac{y}{\sqrt{L_a^2 + y^2}} \tag{13}$$

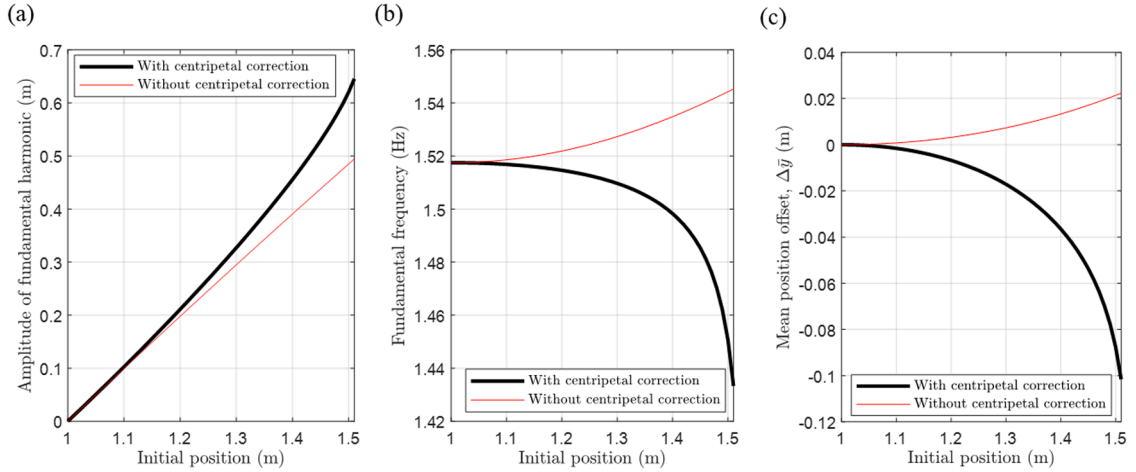


Fig. 9. Comparison of responses between the models with and without the centripetal acceleration correction term in the inerter force, for different initial positions: (a) amplitude of fundamental harmonic, (b) fundamental frequency and (c) mean position offset.

which can be rewritten as

$$\left(m + \frac{by^2}{L_a^2 + y^2}\right)\ddot{y} + \frac{bL_a^2 y \dot{y}^2}{(L_a^2 + y^2)^2} + \frac{k(\sqrt{L_a^2 + y^2} - L_0)y}{\sqrt{L_a^2 + y^2}} = 0. \quad (14)$$

The second term, involving \dot{y}^2 , is the centripetal acceleration correction term.

As an example, for a given set of parameter values, $m = 5 \text{ kg}$, $b = 100 \text{ kg}$, $k = 10,000 \text{ Nm}^{-1}$, $L_a = 1 \text{ m}$ and $L_b = 1 \text{ m}$ and initial position $y_0 = 1.5 \text{ m}$, the numerical time-history responses, with and without the inclusion of the centripetal acceleration correction term, are shown in Fig. 8(a). It can be observed that the correction term affects both the magnitude and frequency of the response. With the correction term, the magnitude of the oscillations is larger and the fundamental frequency is lower than the case where the term is neglected. Also, defining the mean position offset relative to the equilibrium position as $\Delta \bar{y} = \bar{y} - L_b$ where \bar{y} is the mean value of y , it can be observed that there is a negative mean position offset when the correction is included, in contrast with a small positive mean offset if it is neglected. This can be explained as follows: considering only the mean value and fundamental harmonic of the response, it can be approximated as $y \approx \bar{y} + A \cos(\omega t)$, where A is the amplitude and ω the angular frequency of the fluctuating component. The centripetal correction term in the inerter force (Eq. (13)), involving $y^2 = \frac{A^2 \omega^2}{2} \{1 - \cos(2\omega t)\}$, leads to a constant term proportional to A^2 in the equation of motion, which causes a negative mean offset of the displacement relative to the static equilibrium position. The inerter forces with and without the centripetal acceleration correction term are also compared in Fig. 8(b). The amplitude of the fluctuating component of the inerter force including the centripetal acceleration correction term is larger and its frequency is lower than without the correction term. Meanwhile, the mean inerter force including the centripetal acceleration correction term is smaller than without the correction term.

To explore the behaviour of the example system more broadly, numerical time-history responses of Eq. (14) were obtained using a series of different initial positions, with and without the inclusion of the centripetal acceleration correction term. For each initial position, the periodic response can be decomposed into a Fourier series. The amplitude and frequency of the fundamental harmonic, and the mean position offset in relation to the initial position, are shown in Fig. 9. For the response with the centripetal acceleration correction term, the fundamental frequency decreases from the natural frequency (1.518 Hz) as the initial position increases, while the mean position offset also decreases. (The natural frequency is found by linearising the equation of motion, Eq. (14), as shown in Appendix A). However, if the centripetal acceleration correction term is neglected, the fundamental frequency and mean position offset both show opposite trends (i.e. both increase) with increasing initial position (Fig. 9 (b&c)). It is noted that a pair of oblique inerters in nonlinear vibration isolators [18] and suppression systems [19] has been studied recently. It can be seen that Eq. (12) is identical to the model of the oblique inerter derived in [18, 19], which is a special case of Eq. (10), the derived general inerter model expressed in Cartesian coordinate. The case study analysed here shown in Fig. 7 is a simplified version of the nonlinear inerter mechanism (NIM) quasi-zero-stiffness (QZS) isolator studied in [18]. The results obtained in this case study can be validated and further explained with comparisons to [18, 19]. For example, the observed influence on the system's fundamental frequency resulting from the centripetal acceleration term is in line with the analysis in [18].

This example has demonstrated the importance of correctly modelling the inerter force in a 2D (or 3D) environment. If the centripetal acceleration correction term is neglected, as the response departs from the very low amplitude linearised behaviour, the calculated amplitude, frequency and mean position offset of the system depart significantly from the correct response, and indeed the trends of the frequency and mean position are in the opposite direction. Therefore, the correction term must be included when using a uniaxial inerter in a 2D or 3D environment when the rotation of the inerter is significant.

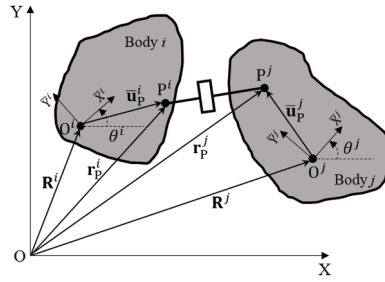


Fig. 10. A uniaxial inerter with its two terminals P^i and P^j respectively on body i and body j in a 2D multibody system.

3. Uniaxial inerter in a multibody system

The general equations for a uniaxial inerter in a 2D or 3D environment have been derived in Section 2. However, when using an inerter in a multibody system, the point where a terminal is connected to a body does not necessarily correspond to the body's reference point. In order to incorporate the inerter in a multibody system, the inerter force needs to be expressed in terms of the generalised coordinates of the bodies to which it is connected, where, for a rigid body, the generalised coordinates are the coordinates of the body reference point and the orientation of the body. In this section, the force of the uniaxial inerter is expressed in the multibody system framework, and the methodology for implementing it in a multibody model is discussed in detail.

3.1. Inerter force in the framework of a 2D multibody system

In a 2D multibody system, motions of each rigid body are described using three coordinates, i.e., two coordinates describing the translational motions of the reference point O^i of body i , $\mathbf{R}^i = [x^i \ y^i]^T$, and one coordinate defining its orientation, θ^i . Hence, the generalised coordinates of body i can be denoted as a 3-element vector $\mathbf{q}^i = [\mathbf{R}^{iT} \ \theta^i]^T$. Assume that the two terminals of a uniaxial inerter are point P^i and point P^j , which are respectively located on body i and body j , as shown in Fig. 10. Note that the terminal point P^i (or P^j) does not generally coincide with the body's reference point O^i (or O^j). The local positions of points P^i and P^j with respect to the bodies' local coordinate frames $\bar{X}^i\text{-}\bar{Y}^i$ and $\bar{X}^j\text{-}\bar{Y}^j$ are $\bar{\mathbf{u}}_p^i = [\bar{x}_p^i \ \bar{y}_p^i]^T$ and $\bar{\mathbf{u}}_p^j = [\bar{x}_p^j \ \bar{y}_p^j]^T$. Note that the overbar in \bar{X}^i, \bar{Y}^i (or \bar{X}^j, \bar{Y}^j) indicates local coordinates. In this work, as all bodies are considered rigid, $\bar{\mathbf{u}}_p^i$ and $\bar{\mathbf{u}}_p^j$ are constant vectors. In this 2D multibody system, the relative position, velocity and acceleration vectors between the two terminals of the inerter can be written as [29, pp. 107–110]

$$\mathbf{r}_p^{ij} = (\mathbf{R}^j + \mathbf{A}^j \bar{\mathbf{u}}_p^j) - (\mathbf{R}^i + \mathbf{A}^i \bar{\mathbf{u}}_p^i), \tag{15}$$

$$\dot{\mathbf{r}}_p^{ij} = (\dot{\mathbf{R}}^j + \dot{\theta}^j \mathbf{A}_\theta^j \bar{\mathbf{u}}_p^j) - (\dot{\mathbf{R}}^i + \dot{\theta}^i \mathbf{A}_\theta^i \bar{\mathbf{u}}_p^i), \tag{16}$$

$$\ddot{\mathbf{r}}_p^{ij} = (\ddot{\mathbf{R}}^j + \ddot{\theta}^j \mathbf{A}_\theta^j \bar{\mathbf{u}}_p^j - (\dot{\theta}^j)^2 \mathbf{A}^j \bar{\mathbf{u}}_p^j) - (\ddot{\mathbf{R}}^i + \ddot{\theta}^i \mathbf{A}_\theta^i \bar{\mathbf{u}}_p^i - (\dot{\theta}^i)^2 \mathbf{A}^i \bar{\mathbf{u}}_p^i), \tag{17}$$

where the 2D transformation matrix $\mathbf{A}^i = \begin{bmatrix} \cos\theta^i & -\sin\theta^i \\ \sin\theta^i & \cos\theta^i \end{bmatrix}$ and $\mathbf{A}_\theta^i = \frac{d\mathbf{A}^i}{d\theta^i} = \begin{bmatrix} -\sin\theta^i & -\cos\theta^i \\ \cos\theta^i & -\sin\theta^i \end{bmatrix}$, and similarly for \mathbf{A}^j and \mathbf{A}_θ^j . Eqs. (16&17) can be rewritten in compact matrix form as

$$\dot{\mathbf{r}}_p^{ij} = \mathbf{H}^{ij} \dot{\mathbf{q}}^{ij}, \tag{18}$$

$$\ddot{\mathbf{r}}_p^{ij} = \mathbf{H}^{ij} \ddot{\mathbf{q}}^{ij} + \mathbf{a}_n^{ij}, \tag{19}$$

where $\mathbf{H}^{ij} = \begin{bmatrix} -\mathbf{I}_{2 \times 2} & -\mathbf{A}_\theta^i \bar{\mathbf{u}}_p^i & \mathbf{I}_{2 \times 2} & \mathbf{A}_\theta^j \bar{\mathbf{u}}_p^j \end{bmatrix}$, $\mathbf{a}_n^{ij} = (\dot{\theta}^j)^2 \mathbf{A}^j \bar{\mathbf{u}}_p^j - (\dot{\theta}^i)^2 \mathbf{A}^i \bar{\mathbf{u}}_p^i$, $\dot{\mathbf{q}}^{ij} = [\dot{\mathbf{R}}^{iT} \ \dot{\theta}^i \ \dot{\mathbf{R}}^{jT} \ \dot{\theta}^j]^T$, $\ddot{\mathbf{q}}^{ij} = [\ddot{\mathbf{R}}^{iT} \ \ddot{\theta}^i \ \ddot{\mathbf{R}}^{jT} \ \ddot{\theta}^j]^T$, and $\mathbf{I}_{2 \times 2}$ is the 2×2 identity matrix (similarly $\mathbf{I}_{3 \times 3}$ is the 3×3 identity matrix in SubSection 3.2). Substituting Eqs. (18&19) into Eq. (9) yields the equation for the uniaxial inerter force expressed in terms of the generalised coordinates of the 2D multibody system, as

$$F_1 = b \left\{ [(\mathbf{H}^{ij} \ddot{\mathbf{q}}^{ij} + \mathbf{a}_n^{ij}) \cdot \hat{\mathbf{r}}_p^{ij}] + \frac{1}{(\mathbf{r}_p^{ij} \cdot \mathbf{r}_p^{ij})^{\frac{3}{2}}} [(\mathbf{H}^{ij} \dot{\mathbf{q}}^{ij}) \cdot (\mathbf{H}^{ij} \dot{\mathbf{q}}^{ij}) - ((\mathbf{H}^{ij} \dot{\mathbf{q}}^{ij}) \cdot \hat{\mathbf{r}}_p^{ij})^2] \right\}. \tag{20}$$

Similarly to Eq. (9), the first term in the curly braces of Eq. (20), $(\mathbf{H}^{ij} \ddot{\mathbf{q}}^{ij} + \mathbf{a}_n^{ij}) \cdot \hat{\mathbf{r}}_p^{ij}$, corresponds to the projection of the relative acceleration between the terminals onto the element axis, and the second term in the curly braces, $\frac{1}{(\mathbf{r}_p^{ij} \cdot \mathbf{r}_p^{ij})^{\frac{3}{2}}} [(\mathbf{H}^{ij} \dot{\mathbf{q}}^{ij}) \cdot (\mathbf{H}^{ij} \dot{\mathbf{q}}^{ij}) - ((\mathbf{H}^{ij} \dot{\mathbf{q}}^{ij}) \cdot \hat{\mathbf{r}}_p^{ij})^2]$, corresponds to the term due to the centripetal acceleration.

3.2. Inerter force in the framework of a 3D multibody system

The equation of the uniaxial inerter force expressed in terms of the generalised coordinates of a 2D multibody system has been derived in Eq. (20). This equation is also valid for a 3D multibody system. Compared with the 2D environment, the only difference is that the vectors and matrices used in Eq. (20), i.e., \mathbf{q}^{ij} , \mathbf{H}^{ij} , \mathbf{a}_n^{ij} and \mathbf{r}_p^{ij} , have different expressions for a 3D multibody system.

In a 3D multibody system, motions of a rigid body are described using six coordinates, i.e., three coordinates describing the translational motions of the body and three coordinates defining the body's orientation. The translational motions of body i can be defined using the position of a reference point fixed on the body, which can be denoted using three-dimensional Cartesian coordinates $\mathbf{R}^i = [x^i \ y^i \ z^i]^T$. In this work, Euler angles $\theta^i = [\phi^i \ \theta^i \ \psi^i]^T$ are used to describe the body's orientation. Initially, the body's local coordinate system, $\bar{X}^i\text{-}\bar{Y}^i\text{-}\bar{Z}^i$, coincides with the global coordinate system X-Y-Z. ϕ^i , θ^i and ψ^i represent three successive rotations which are performed about the current \bar{Z}^i axis, \bar{X}^i axis and \bar{Z}^i axis in turn. Hence, the generalised coordinates of body i in this 3D multibody system can be represented by a 6-element vector $\mathbf{q}^i = \begin{bmatrix} \mathbf{R}^i \\ \theta^i \end{bmatrix}$

As in SubSection 3.1, let a uniaxial inerter be installed between point P^i on body i and point P^j on body j . The local positions of points P^i and P^j with respect to the bodies' local coordinate frames $\bar{X}^i\text{-}\bar{Y}^i\text{-}\bar{Z}^i$ and $\bar{X}^j\text{-}\bar{Y}^j\text{-}\bar{Z}^j$ are $\bar{\mathbf{u}}_p^i$ and $\bar{\mathbf{u}}_p^j$. In a 3D multibody system, Eq. (15) is still valid but the transformation matrix for body i , \mathbf{A}^i , (and similarly for body j throughout this subsection) is a function of the Euler angles and its formulation in detail is shown in Appendix B. Note that $\mathbf{A}^i\bar{\mathbf{u}}_p^i$ can be denoted as \mathbf{u}_p^i , representing the local position vector of point P^i relative to reference point O^i , expressed in global coordinates. For the velocity and acceleration vectors, i.e., Eqs. (18&19), they are still valid but with modified \mathbf{q}^{ij} , \mathbf{H}^{ij} and \mathbf{a}_n^{ij} for the 3D multibody system. The generalised coordinates of the body i and j where the inerter is connected in this 3D multibody system are modified as

$$\mathbf{q}^{ij} = \begin{bmatrix} \mathbf{R}^i \\ \theta^i \\ \mathbf{R}^j \\ \theta^j \end{bmatrix}. \tag{21}$$

The modified matrix \mathbf{H}^{ij} in the 3D multibody system is [29, pp. 397–398]

$$\mathbf{H}^{ij} \begin{bmatrix} -I_{3 \times 3} & \underline{\mathbf{u}}_p^i \mathbf{G}^i & I_{3 \times 3} & \underline{\mathbf{u}}_p^j \mathbf{G}^j \end{bmatrix} \tag{22}$$

where

$$\mathbf{G}^i = \begin{bmatrix} 0 & \cos\theta^i & \sin\theta^i \sin\phi^i \\ 0 & \sin\theta^i & -\sin\theta^i \cos\phi^i \\ 1 & 0 & \cos\theta^i \end{bmatrix}, \tag{23}$$

and the underbar denotes the skew-symmetric matrix of the corresponding vector. For example, the skew-symmetric matrix of the vector $\mathbf{u}_p^i = [x_p^i \ y_p^i \ z_p^i]^T$ is

$$\underline{\mathbf{u}}_p^i = \begin{bmatrix} 0 & -z_p^i & -y_p^i \\ z_p^i & 0 & -x_p^i \\ -y_p^i & x_p^i & 0 \end{bmatrix}. \tag{24}$$

For the 3D multibody system, \mathbf{a}_n^{ij} also needs to be modified as [29, pp398]

$$\mathbf{a}_n^{ij} = \left((\mathbf{G}^j \dot{\theta}^j) (\mathbf{G}^i \dot{\theta}^i) \mathbf{u}_p^i - \underline{\mathbf{u}}_p^i \dot{\mathbf{G}}^i \dot{\theta}^i \right) - \left((\mathbf{G}^i \dot{\theta}^i) (\mathbf{G}^j \dot{\theta}^j) \mathbf{u}_p^j - \underline{\mathbf{u}}_p^j \dot{\mathbf{G}}^j \dot{\theta}^j \right). \tag{25}$$

With the modified \mathbf{r}_p^{ij} , \mathbf{q}^{ij} , \mathbf{H}^{ij} and \mathbf{a}_n^{ij} for the 3D multibody system, as in Eqs. (15, 21, 22&25), the force of the uniaxial inerter, expressed in terms of the generalised coordinates of the 3D multibody system, can be obtained using Eq. (20).

3.3. Implementation of a uniaxial inerter in a multibody model

The equation for the uniaxial inerter force expressed in terms of generalised coordinates of 2D or 3D multibody systems has been derived in SubSections 3.1 and 3.2. In this subsection, the methodology of implementing the uniaxial inerter in a multibody system is introduced.

In a centroidal coordinate system where the origin of each body's local coordinate frame is rigidly attached to its centre of mass, the equations of motion for a multibody system can be written as [29, pp. 430]

$$\begin{bmatrix} \mathbf{M} & \mathbf{C}_q^T \\ \mathbf{C}_q & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_s \\ \dot{\lambda}_s \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_c \\ \mathbf{Q}_d \end{bmatrix} \tag{26}$$

where \mathbf{M} is the system mass matrix, \mathbf{C}_q is the constraint Jacobian matrix, \mathbf{q}_s is the system generalised coordinate vector, \mathbf{Q}_c is the generalised external force vector, λ_s is the vector of Lagrange multipliers which are used to calculate the reaction forces resulting from the constraints and \mathbf{Q}_d is a vector that absorbs terms that are quadratic in the velocities of the second-time differentiation of the constraint equation $\mathbf{C}(\mathbf{q}_s, t) = 0$ [29, pp. 428]. Assume that this model includes n_b bodies with e DOFs for each ($e = 6$ for a 3D model and $e = 3$ for a 2D model) and total n_c constrained DOFs due to the mechanical joints. The definitions and formulations of the above matrices and vectors are standard and can be found in textbooks on multibody system modelling, e.g., [28, 29].

An inerter can be added to the multibody system by applying two equal and opposite external forces to the relevant bodies. These forces need to be implemented into the original equations of motion (Eq. (26)) in the form of the resulting generalised external force vector associated with the generalised coordinates of body i and body j on which the inerter force acts, \mathbf{Q}_i^{ij} , as

$$\begin{bmatrix} \mathbf{M} & \mathbf{C}_q^T \\ \mathbf{C}_q & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_s \\ \dot{\lambda}_s \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_c + \mathbf{L}^{ij} \mathbf{Q}_i^{ij} \\ \mathbf{Q}_d \end{bmatrix} \tag{27}$$

where \mathbf{L}^{ij} is the incidence matrix of the inerter indicating the connection topology between the original multibody system and the inerter. The incidence matrix \mathbf{L}^{ij} is comprised of two columns, each column consisting of n_b $e \times e$ blocks. Apart from two $e \times e$ identity matrices located at the i th row 1st column and j th row 2nd column corresponding to body i and body j connected to the two terminals of the inerter, the remaining blocks in \mathbf{L}^{ij} are filled with $e \times e$ null matrices, giving

$$\mathbf{L}^{ij} = \begin{bmatrix} \mathbf{0}_{e \times e}^1 & \mathbf{0}_{e \times e}^1 \\ \vdots & \vdots \\ \mathbf{I}_{e \times e}^i & \mathbf{0}_{e \times e}^j \\ \vdots & \vdots \\ \mathbf{0}_{e \times e}^j & \mathbf{I}_{e \times e}^i \\ \vdots & \vdots \\ \mathbf{0}_{e \times e}^{n_b} & \mathbf{0}_{e \times e}^{n_b} \end{bmatrix} \quad (i, j \in \{1, \dots, n_b\} \text{ and } i \neq j). \tag{28}$$

The generalised external force vector of F_i associated with the generalised coordinates of body i and body j in the multibody system [29, pp. 243] is

$$\mathbf{Q}_i^{ij} = -F_i \mathbf{H}^{ijT} \hat{\mathbf{r}}_p^{ij}. \tag{29}$$

Substituting the equation for the inerter force, i.e., Eq. (20), into Eq. (29), the full generalised external force vector of the uniaxial inerter can be written as

$$\mathbf{Q}_i^{ij} = -b \mathbf{H}^{ijT} \hat{\mathbf{r}}_p^{ij} \left\{ [(\mathbf{H}^{ij} \dot{\mathbf{q}}^{ij} + \mathbf{a}_n^{ij}) \cdot \hat{\mathbf{r}}_p^{ij}] + (\mathbf{r}_p^{ij} \cdot \mathbf{r}_p^{ij})^{-\frac{1}{2}} [(\mathbf{H}^{ij} \dot{\mathbf{q}}^{ij}) \cdot (\mathbf{H}^{ij} \dot{\mathbf{q}}^{ij}) - ((\mathbf{H}^{ij} \dot{\mathbf{q}}^{ij}) \cdot \hat{\mathbf{r}}_p^{ij})^2] \right\}. \tag{30}$$

Eq. (30) can be rewritten in a compact format as

$$\mathbf{Q}_i^{ij} = -b \mathbf{H}^{ijT} \hat{\mathbf{r}}_p^{ij} \hat{\mathbf{r}}_p^{ijT} \mathbf{H}^{ij} \dot{\mathbf{q}}^{ij} + \mathbf{Q}_{iv}^{ij} \tag{31}$$

where $\mathbf{Q}_{iv}^{ij} = -b \mathbf{H}^{ijT} \hat{\mathbf{r}}_p^{ij} \hat{\mathbf{r}}_p^{ijT} \mathbf{a}_n^{ij} - b (\mathbf{r}_p^{ij} \cdot \mathbf{r}_p^{ij})^{-\frac{1}{2}} \mathbf{H}^{ijT} \hat{\mathbf{r}}_p^{ij} [(\mathbf{H}^{ij} \dot{\mathbf{q}}^{ij}) \cdot (\mathbf{H}^{ij} \dot{\mathbf{q}}^{ij}) - ((\mathbf{H}^{ij} \dot{\mathbf{q}}^{ij}) \cdot \hat{\mathbf{r}}_p^{ij})^2]$ which absorbs the terms related to the relative velocities. Hence, substituting Eq. (31) into Eq. (27), the full multibody model, including the added uniaxial inerter is

$$\begin{bmatrix} \mathbf{M} & \mathbf{C}_q^T \\ \mathbf{C}_q & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_s \\ \dot{\lambda}_s \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_c + \mathbf{L}^{ij} \left(-b \mathbf{H}^{ijT} \hat{\mathbf{r}}_p^{ij} \hat{\mathbf{r}}_p^{ijT} \mathbf{H}^{ij} \dot{\mathbf{q}}^{ij} + \mathbf{Q}_{iv}^{ij} \right) \\ \mathbf{Q}_d \end{bmatrix}. \tag{32}$$

This matrix equation represents a set of differential-algebraic equations. For normal differential-algebraic equations of a multibody system without an inerter, numerical solutions can be obtained using direct time integration. However, including the inerter into the system, the \mathbf{Q}_i^{ij} in the right-hand side of Eq. (27) or (32) is a function of the current accelerations of the generalised coordinates, $\ddot{\mathbf{q}}^{ij}$, which makes Eq. (32) implicit. For implicit equations, direct time integration cannot be used to obtain numerical solutions. Although some integration algorithms, for example, the predictor-corrector method [30], can be applied to solve implicit equations numerically, they can be much more complicated and time-consuming. Moreover, it is inevitable that numerical errors will be introduced by using an implicit integration algorithms. In order to use a direct time integration method to solve the multibody system of equations including an inerters, a modification of Eq. (32) is introduced in the following.

The generalised inerter force vector \mathbf{Q}_i^{ij} consists of two parts, as in Eq. (31), i.e., the term involving the acceleration of the

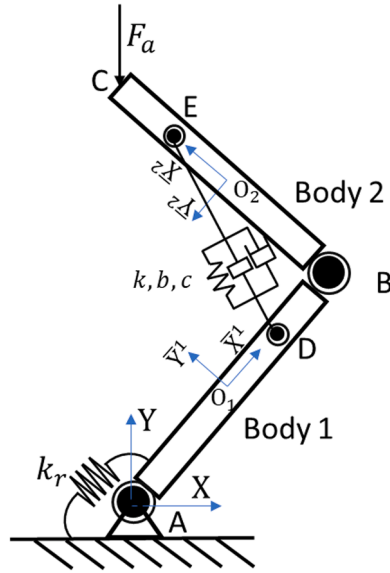


Fig. 11. Example 2D inerter-included two-bar mechanism.

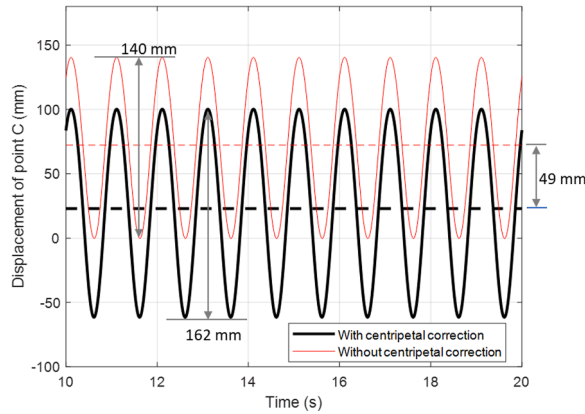


Fig. 12. Comparison of the displacements, relative to the equilibrium position, of point C in the Y direction, between the models with and without the centripetal correction term for the example two-bar mechanism (dashed lines show the corresponding mean values).

generalised coordinates, $-b\mathbf{H}^{ijT}\hat{\mathbf{r}}_p^{ij}\hat{\mathbf{r}}_p^{ijT}\mathbf{H}^{ij}\dot{\mathbf{q}}^{ij}$, and the term that absorbs the velocities of the generalised coordinates, \mathbf{Q}_{iv}^{ij} . Noting that $\ddot{\mathbf{q}}^{ij} = \mathbf{L}^{ijT}\ddot{\mathbf{q}}_s$, the acceleration term can be moved to the left-hand side of Eq. (32) and be combined with the system mass matrix \mathbf{M} , giving

$$\begin{bmatrix} \mathbf{M} + \mathbf{M}_I & \mathbf{C}_q^T \\ \mathbf{C}_q & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_s \\ \dot{\lambda}_s \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_e + \mathbf{Q}_{iv} \\ \mathbf{Q}_d \end{bmatrix} \tag{33}$$

where $\mathbf{M}_I = b\mathbf{L}^{ij}\mathbf{H}^{ijT}\hat{\mathbf{r}}_p^{ij}\hat{\mathbf{r}}_p^{ijT}\mathbf{H}^{ij}\mathbf{L}^{ijT}$ and $\mathbf{Q}_{iv} = \mathbf{L}^{ij}\mathbf{Q}_{iv}^{ij}$.

In this way, the right-hand side of Eq. (33) is a function of the positions and velocities of the generalised coordinates, *i.e.*, \mathbf{q}_s and $\dot{\mathbf{q}}_s$, but not of any accelerations, which makes it suitable for the use of direct time integration to obtain numerical solutions.

3.4. Example: a 2D inerter-included two-bar mechanism system

The method of implementing a uniaxial inerter in a multibody system has been introduced in previous subsections. In this subsection, a 2D inerter-included two-bar mechanism is used as an example, and the effect of the centripetal acceleration correction term on the system response is demonstrated.

The example 2D inerter-included two-bar mechanism, in the horizontal plane, is defined as shown in Fig. 11. Body 1 and body 2 have masses of M_1 and M_2 and rotational inertias of J_1 and J_2 , respectively. Both bodies are 1 m long. One end of body 1 is constrained

by a revolute joint with the ground at point A, via a rotational spring k_r , with 45° undeformed reference angle relative to the X axis positive direction. The other end of body 1 is linked with one end of body 2 by a revolute joint at point B. A spring, a damper and an inerter, labelled by k , c and b , are connected in parallel between points D and E, located on body 1 and body 2 respectively. With the spring undeformed, the initial orientation of body 2 relative to the X axis positive direction is 135° . Friction forces are neglected. An excitation force F_a is applied on the top of body 2, point C. In this example, the excitation force is set to be a sine wave with an amplitude of 8 N at 1 Hz, i.e., $F_a = 8\sin(2\pi t)$ N, with t in s.

For this 2D model, the origin of the global X-Y coordinate system is selected as point A. Local coordinate frames $\bar{X}^1-\bar{Y}^1$ and $\bar{X}^2-\bar{Y}^2$ are established for body 1 and body 2 respectively, with the origins at their centres of mass, i.e., O_1 and O_2 , and the \bar{X}^1 and \bar{X}^2 axes orientated along the bodies' lengths, as shown in Fig. 11. Other data of the model are given in detail in Appendix C. In the equation of motion, Eq. (33), for this example, $\ddot{\mathbf{q}}_s = \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}$ and $\lambda_s = [\lambda_1 \ \lambda_2 \ \lambda_3 \ \lambda_4]^T$, corresponding to the 4 DOFs constrained by revolute joints A and B. Matrix \mathbf{M}_i and vector \mathbf{Q}_{iv} , which are introduced by the uniaxial inerter, can be obtained following the method explained in SubSection 3.3. The remaining matrices and vectors in Eq. (33), \mathbf{M} , \mathbf{C}_q , \mathbf{Q}_e and \mathbf{Q}_d , can be found using the standard multibody modelling method [28, 29].

Using direct time integration, the steady-state displacement, relative to the equilibrium position, of point C in the Y direction is shown in Fig. 12, with and without the inclusion of the centripetal correction term. It can be observed that the peak-peak response of the model without the centripetal acceleration correction is 140 mm, compared with 162 mm for the model including the centripetal acceleration correction. Meanwhile, the mean displacement offset is increased from 23 mm to 72 mm for the model without the centripetal acceleration correction compared with that of the model including the centripetal acceleration correction.

4. Conclusions

In this paper, we consider the implications of using an idealised uniaxial inerter in a 2D or 3D environment. Specifically, the equations for modelling a uniaxial inerter in these environments and the methodology of its implementation in a multibody system are developed. Firstly, the concept of a uniaxial inerter in a 2D polar coordinate system is discussed, and then the model of an inerter is derived in general vector form in a 2D or 3D environment. It is shown that the centripetal acceleration due to rotation has to be accounted for in modelling the uniaxial inerter in a 2D or 3D environment. Expressions for the inerter force in Cartesian and spherical coordinates are also provided for ease of use. A simple 2D 1DOF inerter-included lumped mass system is analysed as an example to show that the centripetal acceleration correction term affects the system responses, in terms of the oscillation amplitude, the fundamental frequency of free vibrations and the mean position offset. If the centripetal acceleration correction term in the inerter force is omitted, significant errors can occur in evaluating the inerter forces and system responses when operating in a 2D or 3D environment. A methodology for applying the uniaxial inerter to a multibody system is also introduced. Then, using an example of a two-bar mechanism containing an inerter, it is shown that the effect of the centripetal acceleration correction term can again be significant. It is noted that, in line with much of the literature, we treat the inerter as an idealised modelling element. When the device is implemented physically, in addition to the centripetal acceleration effects highlighted here, further device-dependant considerations may need to be addressed, including the mass moment of inertia and possibly gyroscopic effects, as well as friction and compliance.

CRedit authorship contribution statement

Ming Zhu: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Data curation, Writing – original draft, Writing – review & editing, Visualization. **John H.G. Macdonald:** Supervision, Writing – review & editing, Investigation, Validation. **Jason Zheng Jiang:** Supervision, Writing – review & editing, Validation, Funding acquisition. **Simon A. Neild:** Supervision, Writing – review & editing, Validation.

Declaration of Competing Interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests.

Data Availability

No data was used for the research described in the article.

Acknowledgement

Ming Zhu is supported by a China Scholarship Council - University of Bristol joint-funded PhD scholarship. Jason Zheng Jiang is supported by an EPSRC Fellowship (EP/T016485/1).

Appendix A. Linearisation of Eq. (14)

If a small perturbation is applied to the equilibrium point of the system in the 2D inerter-included lumped mass system, the position of the lumped mass can be expressed as

$$y = L_b + y_d \tag{A.1}$$

where y_d is the dynamic component of the response. Differentiating both sides of Eq. (A.1) yields

$$\dot{y} = \dot{y}_d, \tag{A.2}$$

$$\ddot{y} = \ddot{y}_d. \tag{A.3}$$

Substituting Eq. (A.1) into the length between the two terminals, $\sqrt{L_a^2 + y^2}$, yields

$$\sqrt{L_a^2 + y^2} = L_0 \sqrt{1 + \frac{2L_b y_d + y_d^2}{L_0^2}} \approx L_0 \left(1 + \frac{2L_b y_d + y_d^2}{2L_0^2} \right). \tag{A.4}$$

If only the first-order term of y_d is retained, $\sqrt{L_a^2 + y^2}$ can be approximated as

$$\sqrt{L_a^2 + y^2} \approx L_0 + \frac{L_b}{L_0} y_d. \tag{A.5}$$

Using Eq. (A.5), Eq. (14) can be linearised as

$$\left(m + \left(\frac{L_b}{L_0} \right)^2 b \right) \ddot{y}_d + k \left(\frac{L_b}{L_0} \right)^2 y_d = 0. \tag{A.6}$$

In the linearised system, the equivalent system mass, m_e , is $m + \left(\frac{L_b}{L_0} \right)^2 b$ and the equivalent system stiffness, k_e , is $k \left(\frac{L_b}{L_0} \right)^2$. So, the natural frequency of the linearised system is $f_e = \frac{1}{2\pi} \sqrt{\frac{k_e}{m_e}}$ which, for the given parameter values, is 1.518Hz.

Appendix B. Transformation matrix for body i expressed with defined Euler angles

$$A^i(\theta^i) = \begin{bmatrix} \cos\psi^i \cos\phi^i - \cos\theta^i \sin\phi^i \sin\psi^i & -\sin\psi^i \cos\phi^i - \cos\theta^i \sin\phi^i \cos\psi^i & \sin\theta^i \sin\phi^i \\ \cos\psi^i \sin\phi^i + \cos\theta^i \cos\phi^i \sin\psi^i & -\sin\psi^i \sin\phi^i + \cos\theta^i \cos\phi^i \cos\psi^i & -\sin\theta^i \cos\phi^i \\ \sin\theta^i \sin\psi^i & \sin\theta^i \cos\psi^i & \cos\theta^i \end{bmatrix} \tag{B.1}$$

Appendix C. Parameters of the example 2D inerter-included two-bar mechanism

See Appendix Table C1 and Table C2

Table C1
Local coordinates of each point of the two-bar mechanism.

Body	Point	Local coordinates (m) (\bar{X}^i, \bar{Y}^i)
Body 1	A	(-0.5, 0)
	B	(0.5, 0)
	D	(0.25, 0)
Body 2	B	(-0.5, 0)
	C	(0.5, 0)
	E	(0.25, 0)

Table C2
Parameters of the two-bar mechanism.

Parameter	Value	Units
k_r	1000	Nmrad ⁻¹
k	100	Nm ⁻¹
c	100	Nsm ⁻¹
b	10	kg
M_1	1	kg
M_2	1	kg
J_1	0.083	kgm ²
J_2	0.083	kgm ²

References

- [1] M.C. Smith, Synthesis of mechanical networks: the inerter, *IEEE Trans. Autom. Control.* 47 (10) (2002) 1648–1662.
- [2] C. Scarborough, Technical insight: renault's J-damper, *Autosport* (2008). <https://www.autosport.com/general/news/technical-insight-renaults-j-damper-5078923/5078923/> (accessed 20 Nov. 2021).
- [3] K. Ikago, K. Saito, N. Inoue, Seismic control of single-degree-of-freedom structure using tuned viscous mass damper, *Earthq. Eng. Struct. Dyn.* 41 (3) (2012) 453–474.
- [4] K. Ikago, Y. Sugimura, K. Saito, N. Inoue, Modal response characteristics of a multiple-degree-of-freedom structure incorporated with tuned viscous mass dampers, *J. Asian Archit. Build. Eng.* 11 (2) (2012) 375–382.
- [5] Q. Chen, Z. Zhao, R. Zhang, C. Pan, Impact of soil–structure interaction on structures with inerter system, *J. Sound Vib.* 433 (2018) 1–15.
- [6] I.F. Lazar, S.A. Neild, D.J. Wagg, Using an inerter-based device for structural vibration suppression, *Earthq. Eng. Struct. Dyn.* 43 (8) (2014) 1129–1147.
- [7] S.Y. Zhang, J.Z. Jiang, S. Neild, Optimal configurations for a linear vibration suppression device in a multi-storey building, *Struct. Control Health Monit.* 24 (3) (2017) e1887.
- [8] X. Shi, & S. Zhu, Dynamic characteristics of stay cables with inerter dampers, *J. Sound Vib.* 423 (2018) 287–305.
- [9] J. Luo, J.Z. Jiang, J.H. Macdonald, Cable vibration suppression with inerter-based absorbers, *J. Eng. Mech.* 145 (2) (2019), 04018134.
- [10] M.C. Smith, F.C. Wang, Performance benefits in passive vehicle suspensions employing inerters, *Veh. Syst. Dyn.* 42 (4) (2004) 235–257.
- [11] Y. Shen, L. Chen, X. Yang, D. Shi, J. Yang, Improved design of dynamic vibration absorber by using the inerter and its application in vehicle suspension, *J. Sound Vib.* 361 (2016) 148–158.
- [12] S.Y. Zhang, M. Zhu, Y. Li, J.Z. Jiang, R. Ficca, M. Czechowicz, R. Neilson, S.A. Neild, G. Herrmann, Ride comfort enhancement for passenger vehicles using the structure-immittance approach, *Veh. Syst. Dyn.* 59 (4) (2021) 504–525.
- [13] F.C. Wang, M.K. Liao, B.H. Liao, W.J. Su, H.A. Chan, The performance improvements of train suspension systems with mechanical networks employing inerters, *Veh. Syst. Dyn.* 47 (7) (2009) 805–830.
- [14] J.Z. Jiang, A.Z. Matamoros-Sanchez, R.M. Goodall, M.C. Smith, Passive suspensions incorporating inerters for railway vehicles, *Veh. Syst. Dyn.* 50 (sup 1) (2012) 263–276.
- [15] Y. Li, J.Z. Jiang, S. Neild, Inerter-based configurations for main-landing-gear shimmy suppression, *J. Aircr.* 54 (2) (2016) 684–693.
- [16] Y.Y. Li, S. Park, J.Z. Jiang, M. Lackner, S. Neild, I. Ward, Vibration suppression for monopile and spar-buoy offshore wind turbines using the structure-immittance approach, *Wind Energy* 23 (10) (2020) 1966–1985.
- [17] M. Zhu, S.Y. Zhang, J.Z. Jiang, J. Macdonald, S. Neild, P. Antunes, P. João, S. Cullingford, M. Askill, S. Fielder, Enhancing pantograph-catenary dynamic performance using an inertance-integrated damping system, *Veh. Syst. Dyn.* 60 (6) (2021) 1909–1932.
- [18] J. Yang, J.Z. Jiang, S.A. Neild, Dynamic analysis and performance evaluation of nonlinear inerter-based vibration isolators, *Nonlinear Dyn* 99 (3) (2020) 1823–1839.
- [19] Z. Dong, B. Shi, J. Yang, T. Li, Suppression of vibration transmission in coupled systems with an inerter-based nonlinear joint, *Nonlinear Dyn* (2021) 1–26.
- [20] P. Brzeski, T. Kapitaniak, P. Perlikowski, Novel type of tuned mass damper with inerter which enables changes of inertance, *J. Sound Vib.* 349 (2015) 56–66.
- [21] M. Zilletti, Feedback control unit with an inerter proof-mass electrodynamic actuator, *J. Sound Vib.* 369 (2016) 16–28.
- [22] N. Alujević, D. Čakmak, H. Wolf, M. Jokić, Passive and active vibration isolation systems using inerter, *J. Sound Vib.* 418 (2018) 163–183.
- [23] H. He, Y. Li, J.Z. Jiang, S. Burrow, S. Neild, A. Conn, Using an inerter to enhance an active-passive-combined vehicle suspension system, *Int. J. Mech. Sci.* 204 (2021), 106535.
- [24] I.F. Lazar, S.A. Neild, D.J. Wagg, Using an inerter-based device for structural vibration suppression, *Earthq. Eng. & Struct. Dyn.* 43 (8) (2014) 1129–1147.
- [25] L. Marian, & A. Giaralis, Optimal design of a novel tuned mass-damper–inerter (TMDI) passive vibration control configuration for stochastically support-excited structural systems, *Probab. Eng. Mech.* 38 (2014) 156–164.
- [26] D. De Domenico, & G. Ricciardi, An enhanced base isolation system equipped with optimal tuned mass damper inerter (TMDI), *Earthq. Eng. & Struct. Dyn.* 47 (5) (2018) 1169–1192.
- [27] Y. Li, L. Lombardi, F. De Luca, Y. Farbiarz, J.J. Blandon, L.A. Lara, J.F. Rendon, J.Z. Jiang, S. Neild, Optimal design of inerter-integrated vibration absorbers for seismic retrofitting of a high-rise building in Colombia, *J. Phys.* 1264 (1) (2019).
- [28] P.E. Nikravesh, *Computer-aided Analysis of Mechanical Systems*, 1st Ed., Prentice-Hall, New Jersey, 1988.
- [29] A.A. Shabana, *Computational Dynamics*, 2nd Ed., John Wiley & Sons, New York, 2009.
- [30] J.C. Butcher, *Numerical Methods For Ordinary Differential Equations*, 1st Ed., John Wiley & Sons, New York, 2003.