Composition and exponential of compactly supported generalized integral kernel operators

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Abstract

We extend the theory of distributional kernel operators to a framework of generalized functions, in which they are replaced by integral kernel operators. Moreover, in contrast to the distributional case, we show that these generalized integral operators can be composed unrestrictedly. This leads to the definition of the exponential of a subclass of such operators.

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1 Introduction

The theory of nonlinear generalized functions [3, 4, 5, 10, 12], which appears as a natural extension of the theory of distributions, seems to be a suitable framework to overcome the limitations of the classical theory of unbounded operators.

Following a first approach done by D. Scarpalezos in [14], we introduced in [1] a natural concept of integral kernel operators in this setting. In addition, we showed that these operators are characterized by their kernel. Our approach has some relationship with the one of [8], but is less restrictive and uses other technics of proofs. Let us quote that classical operators with smooth or distributional kernel can be canonically extended in the framework of generalized functions, through the sheaf embeddings of $\mathcal{C}^\infty(\cdot)$ or $\mathcal{D}'(\cdot)$ into $\mathcal{G}(\cdot)$, the sheaf of spaces of generalized functions. This shows that our theory is a natural extension of the classical one.

After recalling briefly the mathematical framework, we focus on the case of generalized integral operators with compactly supported kernel. We show that such operators can be composed unrestrictedly and that their composition is still a generalized integral operator with a kernel having a compact support.

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This allows to consider their iterate composition and the question of summation of series of such operators naturally arise. It has been solved for the exponential, with additional assumptions on the growth of the kernel with respect to the scaling parameter, in view of applications to theoretical physics.

Let us mention that the question of composition of generalized integral kernel operators has been investigated in more general cases in [1], namely for integral operators with properly supported kernel and with kernel in the algebra $\mathcal{G}_{L^2}$, constructed from H^∞. In this last case, the exponential of generalized integral kernel operators can also be defined with the above mentioned assumptions on the kernel.

2 Generalized integral operators

2.1 The sheaf of nonlinear generalized functions

In this section, we recall briefly some elements of the theory of generalized numbers and functions. We refer the reader to [2, 3, 4, 5, 10, 11, 12] for more details.

Let $\mathrm{C}^\infty(\cdot)$ be the sheaf of complex valued smooth functions on $\mathbb{R}^d$ ($d \in \mathbb{N}$) endowed with the usual topology of uniform convergence of all the derivatives on compact sets. For every open set Ω of $\mathbb{R}^d$, this topology can be described by the family of semi norms $(p_{K,l}(\cdot))_{K \Subset \Omega, l \in \mathbb{N}}$ with $p_{K,l}(f) = \sup_{x\in K, |\alpha|\leq l} |\partial^\alpha f(x)|$, for all f in $\mathrm{C}^\infty(\Omega)$ (the notation $K \Subset \Omega$ means that the set K is a compact set included in Ω).

Set, with $\mathcal{E}(\Omega) = \mathrm{C}^\infty(\Omega)^{(0,1]}$,

$$\mathcal{E}_M(\Omega) = \left\{ (f_\varepsilon) \in \mathcal{E}(\Omega) \,|\, \forall K \Subset \Omega,\ \forall l \in \mathbb{N},\ \exists q \in \mathbb{N},\ p_{K,l}(f_\varepsilon) = \mathrm{O}\left(\varepsilon^{-q}\right) \text{ as } \varepsilon \to 0 \right\},$$
$$\mathcal{I}(\Omega) = \left\{ (f_\varepsilon) \in \mathcal{E}(\Omega) \,|\, \forall K \Subset \Omega,\ \forall l \in \mathbb{N},\ \forall p \in \mathbb{N},\ p_{K,l}(f_\varepsilon) = \mathrm{O}(\varepsilon^{p}) \text{ as } \varepsilon \to 0 \right\}.$$

The functor $\mathcal{E}_M : \Omega \to \mathcal{E}_M(\Omega)$ (*resp.* $\mathcal{I} : \Omega \to \mathcal{I}(\Omega)$) defines a sheaf of subalgebras of the sheaf $\mathcal{E}(\cdot)$ (*resp.* a sheaf of ideals of the sheaf $\mathcal{E}_M(\cdot)$) [11].

Definition 1 *The sheaf of factor algebras $\mathcal{G}(\cdot) = \mathcal{E}_M(\cdot)/\mathcal{I}(\cdot)$ is called the sheaf of* Colombeau type algebras.

The sheaf $\mathcal{G}$ turns to be a sheaf of differential algebras and a sheaf of modules on the factor ring $\overline{\mathbb{C}} = \mathcal{X}(\mathbb{C})/\mathcal{N}(\mathbb{C})$ with

$$\mathcal{X}(\mathbb{K}) = \left\{ (r_\varepsilon) \in \mathbb{K}^{(0,1]} \,|\, \exists q \in \mathbb{N},\ |r_\varepsilon| = \mathrm{O}\left(\varepsilon^{-q}\right) \text{ as } \varepsilon \to 0 \right\},$$
$$\mathcal{N}(\mathbb{K}) = \left\{ (r_\varepsilon) \in \mathbb{K}^{(0,1]} \,|\, \forall p \in \mathbb{N},\ |r_\varepsilon| = \mathrm{O}(\varepsilon^{p}) \text{ as } \varepsilon \to 0 \right\},$$

where $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$.

Notation 2 *For $(f_\varepsilon) \in \mathcal{E}_M(\Omega)$, $Cl(f_\varepsilon)$ will denote its class in $\mathcal{G}(\Omega)$.*

As $\mathcal{G}$ is a sheaf, the notion of support of a section $f \in \mathcal{G}(\Omega)$ (Ω open subset of $\mathbb{R}^d$) makes sense. Thus the *support* of a generalized function $f \in \mathcal{G}(\Omega)$, denoted by *supp* f, is the complement in Ω of the largest open subset of Ω where f is null. We denote by $\mathcal{G}_C(\Omega)$ the subset of elements of $\mathcal{G}(\Omega)$ with compact supports. In particular, such compactly supported generalized functions have the following property: Every $f \in \mathcal{G}_C$ has a representative $(f_\varepsilon) \in \mathcal{E}_M(\Omega)$ such that each f_ε has the same compact support. We say that such a representative has a *global compact support.*

2.2 Definitions and first properties

Let X (*resp.* Y) be an open subset of $\mathbb{R}^m$ (*resp.* $\mathbb{R}^n$). We denote by $\mathcal{G}_{ps}(X \times Y)$ the set of generalized functions g of $\mathcal{G}(X \times Y)$ properly supported in the following sense:

$$\begin{array}{r} \forall\ O_1 \subset X \text{ relatively compact open subset,} \\ \exists K_2 \Subset Y \ / \ supp\, g \cap (O_1 \times Y) \ \subset O_1 \times K_2. \end{array} \tag{1}$$

The set $\mathcal{G}_{ps}(X \times Y)$ is clearly a subalgebra of $\mathcal{G}(X \times Y)$.

Proposition 3 *[1] For g in $\mathcal{G}_{ps}(X \times Y)$, there exists $G \in \mathcal{G}(X)$ such that, for all relatively compact open subset O_1 of X,*

$$G_{|O_1} = Cl\left(\left(x \mapsto \int_{K_2} g_\varepsilon(x,y)\,\mathrm{d}y\right)_{|O_1}\right)$$

where (g_ε) is a representative of g and $K_2 \Subset Y$ is such that $supp\, g \cap (O_1 \times Y) \ \subset O_1 \times K_2$.

Notation 4 *With a slight abuse, we shall denote $G = \int g(\cdot, y)\,\mathrm{d}y$ or $G(\cdot_1) = \int g(\cdot_1, y)\,\mathrm{d}y$.*

Definition 5 *Let H be in $\mathcal{G}_{ps}(X \times Y)$. We call* generalized integral operator *the map*

$$\begin{array}{cccl} \widehat{H} & : & \mathcal{G}(Y) & \rightarrow & \mathcal{G}(X) \\ & & f & \mapsto & \widehat{H}(f) = \int H(\cdot, y) f(y)\,\mathrm{d}y. \end{array}$$

We say that H is the kernel *of the generalized integral operator $\widehat{H}$.*

This map is well defined, due to proposition 3 since the application $g = H(\cdot_1, \cdot_2) f(\cdot_2)$ is in $\mathcal{G}_{ps}(X \times Y)$, for all $f \in \mathcal{G}(Y)$. Moreover, it is linear.

Remark 6 *Any $H \in \mathcal{G}(X \times Y)$ compactly supported satisfies (1) and $\widehat{H}$ is well defined. Moreover, a straightforward calculation shows that the image of $\widehat{H}$ is included in $\mathcal{G}_C(X)$. Furthermore, the definition of $\widehat{H}$ does not need to refer to proposition 3 in this case. Indeed, if H is in $\mathcal{G}_C(X \times Y)$ with $supp\, H \subset K_1 \times K_2$ ($K_1 \Subset X$, $K_2 \Subset Y$) and f in $\mathcal{G}(Y)$, we have*

$$\widehat{H}(f) = Cl\left(x \mapsto \int_{K_2} H_\varepsilon(x,y) f_\varepsilon(y)\,\mathrm{d}y\right)$$

where (H_ε) *(resp.* (f_ε)*) is any representative of* H *(resp.* f*).*

Remark 7 *If* H *is in* $\mathcal{G}(X \times Y)$ *without hypothesis on the support, we can define a map* $\widehat{H} : \mathcal{G}_C(Y) \to \mathcal{G}(X)$ *in the same way, since for all* f *in* $\mathcal{G}_C(Y)$*, the function* $H(\cdot_1, \cdot_2) f(\cdot_2)$ *is in* $\mathcal{G}_{ps}(X \times Y)$*. In this case, the generalized integral operator could also be defined globally since* f *has a representative with a global compact support, as quoted above.*

This case leads us to make the link between the classical theory of integral operators acting on $\mathcal{D}(Y)$ and the generalized one. Indeed, if h belongs to $\mathcal{D}'(X \times Y)$ and $\widehat{h}$ is the classical operator of kernel h, then the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{D}(Y) & \overset{\widehat{h}}{\to} & \mathcal{D}'(X) \\ \downarrow \sigma & & \downarrow i_S \\ \mathcal{G}_C(Y) & \overset{\widehat{i'_S(h)}}{\to} & \mathcal{G}(X), \end{array}$$

where σ (*resp.* i_S, i'_S) is the usual embedding of $\mathcal{D}(Y)$ into $\mathcal{G}_C(Y)$ (*resp.* $\mathcal{D}'(X)$ into $\mathcal{G}(X)$, $\mathcal{D}'(X \times Y)$ into $\mathcal{G}(X \times Y)$). This shows that our theory extends "canonically" the classical one. We refer to [1] for more details on the relationship with classical cases and to [7, 12] for the definition of the sheaves embeddings of $\mathcal{D}(\cdot)$ and $\mathcal{D}'(\cdot)$ into $\mathcal{G}(\cdot)$.

Remark 8 *The map*$\widehat{\ }$: $\mathcal{G}_{ps}(X \times Y) \to \mathcal{L}(\mathcal{G}(Y), \mathcal{G}(X))$ *is a linear map of* $\overline{\mathbb{C}}$*-modules. Moreover,* $\widehat{H}$ *is continuous for the sharp topologies [14].*
Conversely, the third author showed in [6] that any continuous linear map from $\mathcal{G}_C(Y)$ *to* $\mathcal{G}(X)$*, satisfying appropriate growth hypothesis with respect to the regularizing parameter* ε*, can be written as a generalized integral kernel operator.*

The following result shows that the map$\widehat{\ }$, defined in remark 8, is injective.

Theorem 9 *[1, 9] Characterization of generalized integral operators by their kernel: One has* $\widehat{H} = 0$ *if and only if* $H = 0$*.*

2.3 Composition of generalized integral operators

For this topic, we only consider in this paper generalized integral operators with compactly supported kernel.

Theorem 10 *For* H *in* $\mathcal{G}_C(X \times \Xi)$ *and* K *in* $\mathcal{G}_C(\Xi \times Y)$*,* $\widehat{H} \circ \widehat{K} : \mathcal{G}(Y) \to \mathcal{G}_C(X)$ *is a generalized integral operator whose kernel* L *is an element of* $\mathcal{G}_C(X \times Y)$ *defined globally by* $L(\cdot_1, \cdot_2) = \int_\Xi H(\cdot_1, \xi) K(\xi, \cdot_2)\, \mathrm{d}\xi$*.*
Moreover, there exists K_1 *(resp.* K_2 *,* K_3*) a compact set of* X *(resp.* Ξ *,*Y*) such that the support of* H *(resp.* K*) is contained in the interior of* $K_1 \times K_2$ *(resp.* $K_2 \times K_3$*). In this case, the support of* L *is contained in* $K_1 \times K_3$*.*

Proof. For all f in $\mathcal{G}(Y)$, $\widehat{K}\,(f)$ is well defined and belongs to $\mathcal{G}_C(\Xi)$, according to remark 6. This allows the definition of the composition $\widehat{H} \circ \widehat{K}$. Let us verify now the assertion concerning the support of L. Since H (*resp.* K) is in $\mathcal{G}_C(X\times\Xi)$ (*resp.* $\mathcal{G}_C(\Xi\times Y)$), we can find K_1 (*resp.* K_2 , K_3) satisfying the second assertion of the theorem. Then,

$$L(\cdot_1, \cdot_2) = \int_{K_2} H(\cdot_1, \xi) K(\xi, \cdot_2)\, \mathrm{d}\xi$$

is a well defined generalized function, according to the theory of integration of generalized functions on compact sets [3, 5]. Denote by (H_ε) (*resp.* (K_ε)) a representative of H (*resp.* K) and set $O_1 = X \setminus K_1$, $O_3 = Y \setminus K_3$. The map $L_\varepsilon(\cdot_1, \cdot_2) = \int_{K_2} H_\varepsilon(\cdot_1, \xi) K_\varepsilon(\xi, \cdot_2)\, \mathrm{d}\xi$ is a representative of L. For $U \Subset X$ and $V \Subset Y$ such that $U \times V \subset O_1 \times O_3$, we have either $U \subset O_1$ or $V \subset O_3$. We shall suppose, for example, that $U \subset O_1$. For $(x, y) \in U \times V$, we have

$$|L_\varepsilon(x, y)| = \left| \int_{K_2} H_\varepsilon(x, \xi) K_\varepsilon(\xi, y)\, \mathrm{d}\xi \right| \le Vol(K_2) p_{U\times K_2,0}\,(H_\varepsilon) p_{K_2\times V,0}\,(K_\varepsilon),$$

where $Vol(K_2)$ denotes the volume of K_2. Therefore

$$p_{U\times V,0}\,(L_\varepsilon) \le Vol(K_2) p_{U\times K_2,0}\,(H_\varepsilon) p_{K_2\times V,0}\,(K_\varepsilon). \tag{2}$$

As $(H_{\varepsilon|O_1\times\Xi})$ is in $\mathcal{I}(O_1\times\Xi)$ and $U\cap K_2 \subset O_1\times\Xi$, it follows that $p_{U\times K_2,0}(H_\varepsilon) = O\,(\varepsilon^m)$ as $\varepsilon \to 0$, for all $m \in \mathbb{N}$. Moreover, (K_ε) is in $\mathcal{E}_M\,(\Xi \times Y)$. Thus, relation (2) implies that $p_{U\times V,0}\,(L_\varepsilon) = O\,(\varepsilon^m)$ as $\varepsilon \to 0$, for all $m \in \mathbb{N}$. Finally, (L_ε) satisfies the null estimate of order 0 for all compact sets included in $O_1 \times O_3$. Using theorem 1.2.3 of [10], we can conclude, without estimates on the derivatives, that $L_{|O_1\times O_3} = 0$. Therefore, the support of L is contained in $K_1 \times K_3$. From this, a straightforward verification, using once more the integration on compact sets of generalized functions, shows that $\widehat{H} \circ \widehat{K} = \widehat{L}$. ■

Repeted applications of theorem 10 show the following:

Corollary 11 *For H in $\mathcal{G}_C(X^2)$ and all $n \ge 2$, the composition n times of $\widehat{H}$ is a well defined operator $\widehat{H}^n$ with image in $\mathcal{G}_C(X)$. Moreover, $\widehat{H}^n$ admits as kernel $L_n \in \mathcal{G}_C(X^2)$ defined by*

$$L_n(\cdot_1, \cdot_2) = \int_{X^{n-1}} H(\cdot_1, \xi_1) H(\xi_1, \xi_2) \cdots H(\xi_{n-1}, \cdot_2)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \cdots \mathrm{d}\xi_{n-1}.$$

Furthermore, for all $n \ge 2$, the support of L_n is contained in the one of H.

3 Application: exponential of generalized integral operators

In this section, we define the exponential of generalized integral operators in a particular case and study some of their properties. We need before to introduce a convenient subsheaf of $\mathcal{G}(\cdot)$. For Ω open set of $\mathbb{R}^d$ ($d \in \mathbb{N}$), set

$$\mathcal{H}_{ln}(\Omega) = \{(u_\varepsilon) \in \mathcal{E}(\Omega) \ / \ \forall K \Subset \Omega,\ \forall l \in \mathbb{N},\ \ p_{K,l}(u_\varepsilon) = O(|\ln \varepsilon|) \text{ as } \varepsilon \to 0\}\,.$$

The set $\mathcal{H}_{ln}(\Omega)$ is a linear subspace of $\mathcal{E}_M(\Omega)$ (but not a subalgebra). Define

$$\mathcal{G}_{ln}(\Omega) = \frac{\mathcal{H}_{ln}(\Omega)}{\mathcal{I}(\Omega)} \text{ and } \mathcal{G}_{C\,ln}(\Omega) = \mathcal{G}_{ln}(\Omega) \cap \mathcal{G}_C(\Omega).$$

Theorem 12 *Let H be in $\mathcal{G}_{C\,ln}(X^2)$. Denote by L_n the kernel of $\widehat{H}^n : \mathcal{G}(X) \to \mathcal{G}_C(X)$ defined as in corollary 11 and $(L_{n,\varepsilon})$ a representative of L_n. For all $\varepsilon \in (0,1]$, the series $\sum_{n\geq 1} \frac{L_{n,\varepsilon}}{n!}$ (by setting $L_1 = H$) normally converges, for the usual topology of uniform convergence on compact subsets of X^2. Denote by S_ε its sum. The net (S_ε) belongs to $\mathcal{E}_M(X^2)$. Furthermore, $S = Cl\,(S_\varepsilon)$ defines a compactly supported element of $\mathcal{G}(X^2)$ only depending on H.*
The well defined operator $\mathrm{e}^{\widehat{H}} = \widehat{S} + Id$ (where Id is the operator identity) will be called the exponential *of $\widehat{H}$.*

The **proof** of this theorem can be divided in three parts. The first part contains the estimates of $\sum_{n\geq 1} \frac{L_{n,\varepsilon}}{n!}$ for a particular representative of L_n, given by a fixed representative of H. The second part deals with the independence of $Cl\,(S_\varepsilon)$ with respect to the chosen representative of L_n, that is of H. The third part shows that S is compactly supported.

We shall give here mainly the first part of this proof and refer the reader to [1] for other parts. Let H be in $\mathcal{G}_{C\,ln}(X^2)$ and (H_ε) one of its representative. According to corollary 11, we have $\widehat{H}^n = \widehat{L_n} : \mathcal{G}(X) \to \mathcal{G}_C(X)$ and $L_n \in \mathcal{G}_C(X^2)$ admits as representative $(L_{n,\varepsilon})$ with

$$L_{n,\varepsilon}(\cdot_1, \cdot_2) = \int_{K^{n-1}} H_\varepsilon(\cdot_1, \xi_1) H_\varepsilon(\xi_1, \xi_2) \cdots H_\varepsilon(\xi_{n-1}, \cdot_2)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \cdots \mathrm{d}\xi_{n-1},$$

where K is a compact set of X such that the support of H is contained in the interior of K^2.

For all compact subset of X^2 of the form $K_1 \times K_2$, $(\alpha, \beta) \in \mathbb{N}^d \times \mathbb{N}^d$ and $(x,y) \in K_1 \times K_2$, one has

$$\begin{aligned}\left| \frac{\partial^{\alpha+\beta} L_{2,\varepsilon}}{\partial x^\alpha \partial y^\beta}(x,y) \right| &= \left| \int_K \frac{\partial^\alpha H_\varepsilon}{\partial x^\alpha}(x,\xi) \frac{\partial^\beta H_\varepsilon}{\partial y^\beta}(\xi, y)\, \mathrm{d}\xi \right| \\ &\leq \int_K p_{K_1\times K,|\alpha|}(H_\varepsilon) p_{K\times K_2,|\beta|}(H_\varepsilon)\, \mathrm{d}\xi.\end{aligned}$$

It follows that

$$p_{K_1\times K_2,|(\alpha,\beta)|}(L_{2,\varepsilon}) \leq Vol(K) p^2_{V^2,|(\alpha,\beta)|}(H_\varepsilon),$$

where V is a compact set of X containing K,K_1 and K_2. By an iterative method, we show that, for all $n \geq 2$,

$$p_{K_1\times K_2,|(\alpha,\beta)|}(L_{n,\varepsilon}) \leq Vol(K)^{n-1} p^n_{V^2,|(\alpha,\beta)|}(H_\varepsilon).$$

This last inequality implies that the series $\sum_{n\geq 1} \frac{L_{n,\varepsilon}}{n!}$ normally converges, for the usual topology of uniform convergence of all the derivatives on any compact subset of X^2. Set

$$S_\varepsilon = \sum_{n=1}^{+\infty} \frac{L_{n,\varepsilon}}{n!}.$$

As L_n is in $\mathcal{G}_C(X^2)$ and since the convergence is uniform, S_ε belongs to $\mathcal{C}^\infty(X^2)$, for all $\varepsilon \in (0,1]$. Furthermore, for all compact subset of X^2 of the form $K_1 \times K_2$ and $(\alpha,\beta) \in \mathbb{N}^d \times \mathbb{N}^d$, one has

$$\begin{aligned} p_{K_1\times K_2,|(\alpha,\beta)|}(S_\varepsilon) &\leq \sum_{n=1}^{+\infty} \frac{1}{n!} p_{K_1\times K_2,|(\alpha,\beta)|}(L_{n,\varepsilon}) \\ &\leq \sum_{n=1}^{+\infty} \frac{1}{n!} Vol(K)^{n-1} p^n_{V^2,|(\alpha,\beta)|}(H_\varepsilon) \\ &\leq \frac{1}{Vol(K)} \left[e^{Vol(K) p_{V^2,|(\alpha,\beta)|}(H_\varepsilon)} - 1 \right]. \end{aligned}$$

Since H is in $\mathcal{G}_{C\,ln}(X^2)$, $p_{V^2,|(\alpha,\beta)|}(H_\varepsilon) = O(|\ln \varepsilon|)$ as $\varepsilon \to 0$, that is there exists $k \in \mathbb{N}$ such that $p_{V^2,|(\alpha,\beta)|}(H_\varepsilon) \leq \ln\left(\frac{1}{\varepsilon^k}\right)$, so

$$p_{K_1\times K_2,|(\alpha,\beta)|}(S_\varepsilon) \leq C_K \varepsilon^{-kVol(K)},$$

where C_K is a constant depending only on K and not on the representative of H. Consequently, (S_ε) is in $\mathcal{E}_M(X^2)$ and we denote by S its class in $\mathcal{G}(X^2)$.

The independence of S with respect to the representatives is classically proved by taking two representatives of H, which gives two sums, $\left(S^1_\varepsilon\right)$ and $\left(S^2_\varepsilon\right)$, obtained by the process described above, and by estimating the difference $\left(S^1_\varepsilon - S^2_\varepsilon\right)$. This uses similar estimates as above. Finally, the assertion concerning the support is proved with similar arguments as the ones of the proof of theorem 10.

In [1], it is shown that the exponential defined by theorem 12 inherits the main expected functional properties.

Proposition 13 *If H is in $\mathcal{G}_{Cln}(X^2)$ then*

$$\widehat{H}\circ e^{\widehat{H}} = e^{\widehat{H}}\circ\widehat{H}\ ; \quad e^{a\widehat{H}}\circ e^{b\widehat{H}} = e^{(a+b)\widehat{H}},\ \textit{for all}\ (a,b)\in\mathbb{R}^2\ ; \quad \frac{d}{dt}\ e^{t\widehat{H}} = \widehat{H}\circ e^{t\widehat{H}}.$$

By applying theorem 9 concerning the characterization of generalized integral operators by their kernel, these properties are proved by using the associated kernels.

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