Characterizing Coherence, Correcting Incoherence

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to crank out COHERENCE CHARACTERIZATIONS

1. Context

Basic setup:

- Finite possibility space Ω
- Finite set of gambles $\mathcal{K}$ on Ω
- Lower previsions $\underline{P}$ on $\mathcal{K}$

Matrix notation:

- $|\Omega|$-by-$|\mathcal{K}|$ matrix K with gambles as columns
- the rows of K (columns of $K^\top$) are the degenerate previsions
- the set $\mathcal{S}$ of matrices S obtained from the identity matrix $\mathbb{I}$ by changing at most one 1 to -1
- all-one (zero) column vector 1 (0)

2. Goals

Given K, find a non-redundant H-representations for the set of all $\underline{P}$

A. that avoid sure loss ($[\Lambda_A|\alpha_A]$),

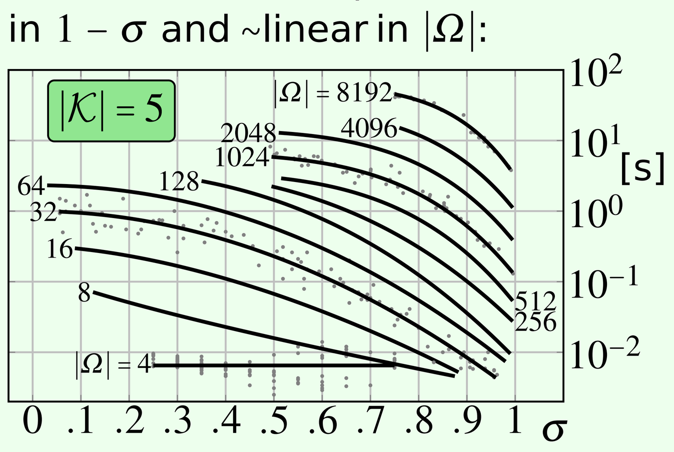
B. that avoid sure loss and for which $\underline{P} \geq \min$ ($[\Lambda_B|\alpha_B]$),

C. that are coherent ($[\Lambda_C|\alpha_C]$).

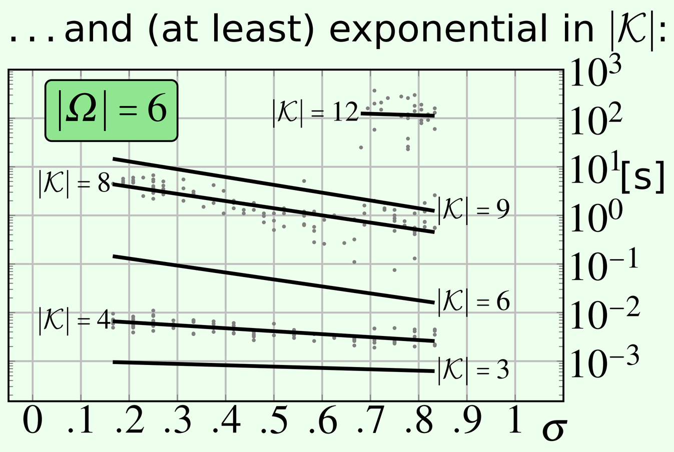
7. Experiments

The sparsity σ is the fraction of zero components in K.

Procedure C1 is exponential in $1-\sigma$ and ~linear in $|\Omega|$:



... and (at least) exponential in $|\mathcal{K}|$:



3. Goal A: Characterizing ASL

Based on the existence of a dominating linear prevision:

A1. $\exists \mu_{\mathbb{I}}, \nu_{\mathbb{I}} \geq 0 : \underline{P} = K^\top\mu_{\mathbb{I}} - \mathbb{I}\nu_{\mathbb{I}} \wedge 1^\top\mu_{\mathbb{I}} = 1 \longrightarrow \begin{bmatrix} K^\top & -\mathbb{I} \\ 1^\top & 0^\top \end{bmatrix} \xrightarrow{\text{EN, RR}} [\Lambda_A|\alpha_A]$

A2. $\exists \mu_{\mathbb{I}} \geq 0 : \underline{P} \leq K^\top\mu_{\mathbb{I}} \wedge 1^\top\mu_{\mathbb{I}} = 1 \longrightarrow \begin{bmatrix} \mathbb{I} & -K^\top & 0 \\ & -\mathbb{I} & 0 \\ & 1^\top & 1 \\ & -1^\top & -1 \end{bmatrix} \xrightarrow{\text{PJ}_{\underline{P}}\text{, RR}} [\Lambda_A|\alpha_A]$

4. Goal B: Characterizing ASL ≥ min

B1. Starting from $[\Lambda_A|\alpha_A]$: $\begin{bmatrix} \Lambda_A & \alpha_A \\ -\mathbb{I} & -\min \end{bmatrix} \xrightarrow{\text{RR}} [\Lambda_B|\alpha_B]$

5. Goal C: Characterizing coherence

Based on the existence of S-dominating linear previsions:

C1. Analogous to A1 & intersection over all S in $\mathcal{S}$:

$\forall S \in \mathcal{S} : \exists \mu_S, \nu_S \geq 0 : \underline{P} = K^\top\mu_S - S\nu_S \wedge 1^\top\mu_S = 1 \longrightarrow \begin{bmatrix} K^\top & -S \\ 1^\top & 0^\top \end{bmatrix} \xrightarrow{\text{EN, IS}_{S\in\mathcal{S}}\text{, RR}} [\Lambda_C|\alpha_C]$

C2. Analogous to A2 & intersection over all S in $\mathcal{S}$:

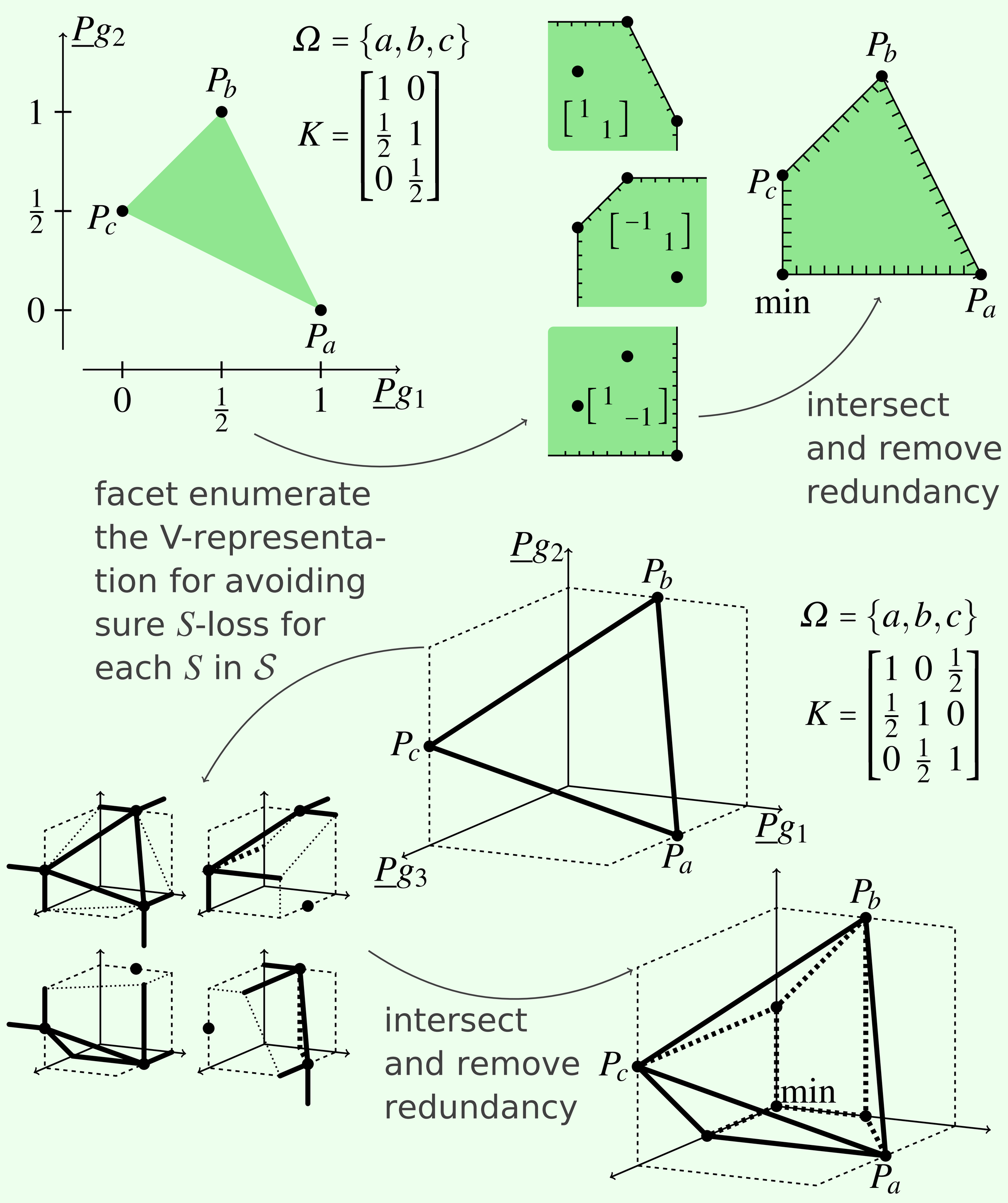
$\forall S \in \mathcal{S} : \exists \mu_S \geq 0 : S\underline{P} \leq SK^\top\mu_S \wedge 1^\top\mu_S = 1 \longrightarrow \begin{bmatrix} S & -SK^\top & 0 \\ & -\mathbb{I} & 0 \\ & 1^\top & 1 \\ & -1^\top & -1 \end{bmatrix} \xrightarrow{\text{PJ}_{\underline{P}}\text{, IS}_{S\in\mathcal{S}}\text{, RR}} [\Lambda_C|\alpha_C]$

$\longrightarrow [A_{S,\underline{P}} \; A_{S,\mu_S} | b_0]$

C3. Block matrix form of C2:

$$[A_{\underline{P}} \; A_\mu | b] := \begin{bmatrix} A_{\mathbb{I},\underline{P}} & A_{\mathbb{I},\mu_{\mathbb{I}}} & & & b_0 \\ \vdots & & \ddots & & \vdots \\ A_{S,\underline{P}} & & & A_{S,\mu_S} & b_0 \\ \vdots & & & \ddots & \vdots \end{bmatrix} \xrightarrow{\text{PJ}_{\underline{P}}\text{, RR}} [\Lambda_C|\alpha_C]$$

6. Illustrations of Procedure C1



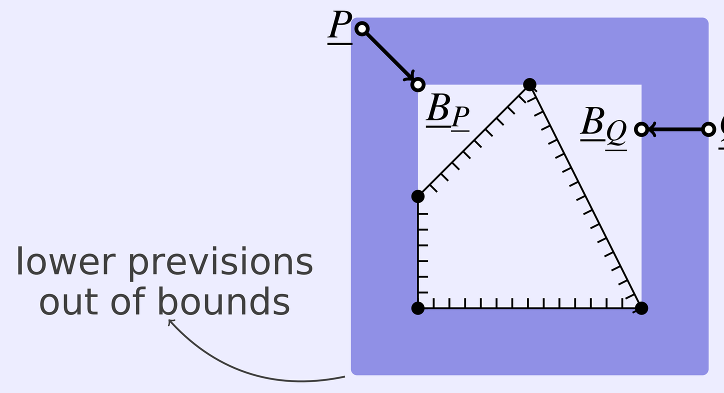
1. Context & Goal

Given: incoherent lower prevision $\underline{P}$.
Goal: Find a coherent correction to it.

2. Bring within bounds

If $\underline{P}f \notin [\min f, \max f]$ for some f in $\mathcal{K}$, it is out of bounds. To bring it within bounds:

$$\underline{B}_{\underline{P}}f := \begin{cases} \min f & \underline{P}f \leq \min f, \\ \max f & \underline{P}f \geq \max f, \\ \underline{P}f & \text{otherwise.} \end{cases}$$



3. Downward correction

As the downward correction of $\underline{P}$ we take the lower envelope of the maximal coherent dominated lower previsions (proposed earlier by Pelessoni & Vicig, following Weichselberger), so the nadir point $\underline{D}_{\underline{P}}$ of the MOLP (cf. C)

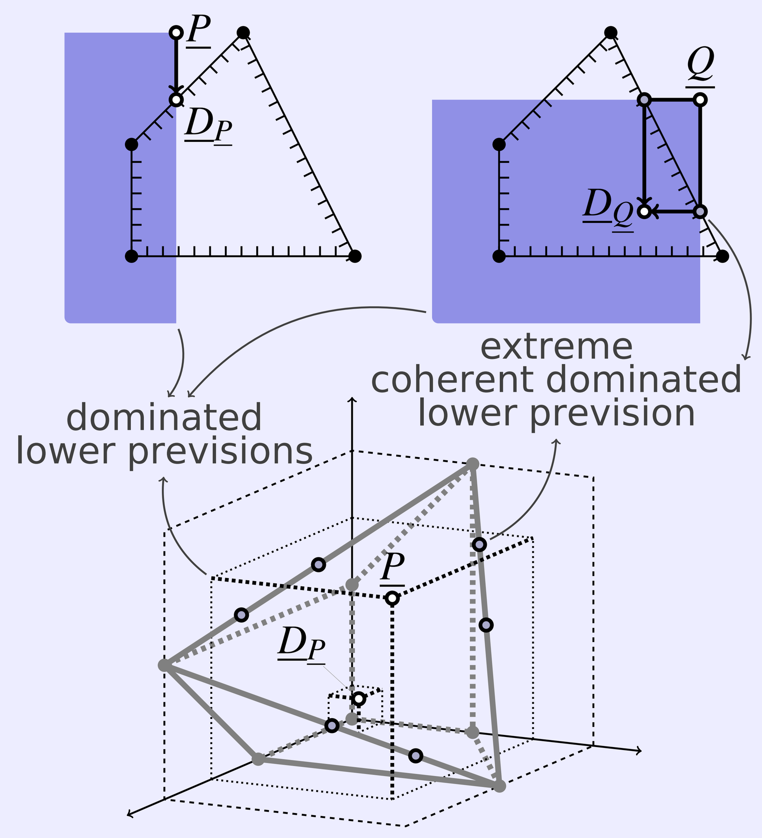
(†) maximize $\underline{Q}$, subject to $\Lambda_C\underline{Q} \leq \alpha_C$, $\underline{Q} \leq \underline{P}$

or the MOLP (cf. C3)

(‡) maximize $\underline{Q}$, subject to $A_{\underline{Q}}\underline{Q} + A_\mu\mu \leq b$, $\underline{Q} \leq \underline{P}$.

Some desirable properties:

- It is the maximal neutral correction ('no component tradeoffs').
- The imprecision of the correction is nondecreasing with incoherence.



For the future: *Can the computation be simplified for special classes of $\underline{P}$?*

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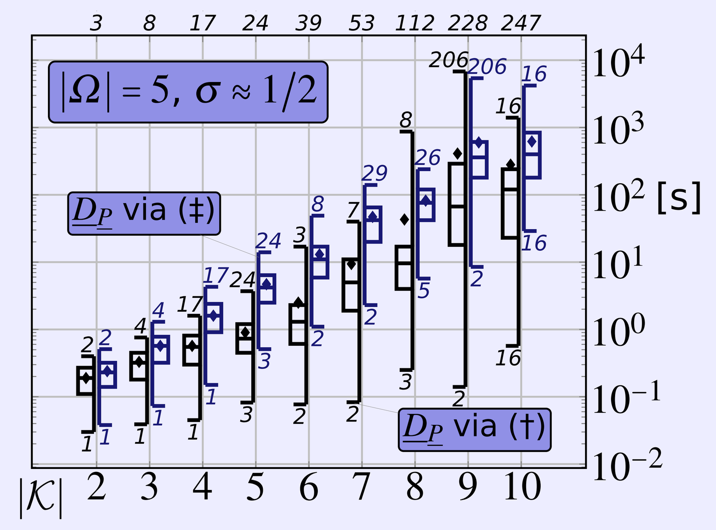
to ERADICATE INCOHERENCE utterly

4. Experiments

With the M3-solver we used, computation appears exponential in $|\mathcal{K}|$; using pre-computed constraints (†) is more efficient than not (‡):



We expect other solvers and certainly direct M2-solvers to perform more efficiently, but could not test any yet.

5. Upward correction

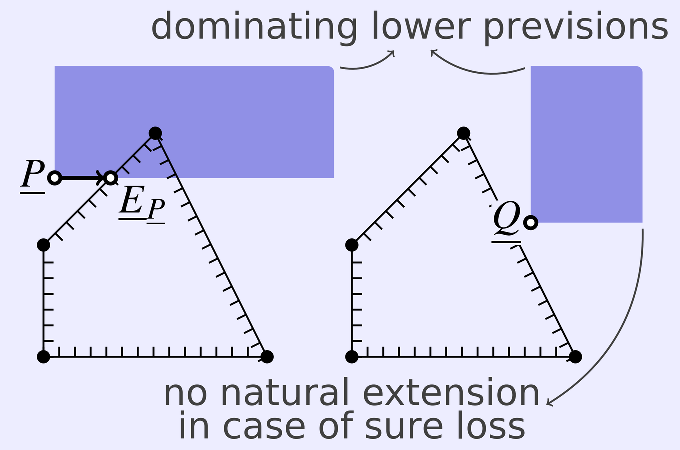
The standard upward correction of $\underline{P}$ is its natural extension $\underline{E}_{\underline{P}}$, the unique minimal pointwise dominating coherent lower prevision, so the the solution to the MOLP (cf. C)

minimize $\underline{E}_{\underline{P}}$, subject to $\Lambda_C\underline{E}_{\underline{P}} \leq \alpha_C$, $\underline{E}_{\underline{P}} \geq \underline{P}$

or the MOLP (cf. C3)

(*) minimize $\underline{E}_{\underline{P}}$, subject to $A_{\underline{E}_{\underline{P}}}\underline{E}_{\underline{P}} + A_\mu\mu \leq b$, $\underline{E}_{\underline{P}} \geq \underline{P}$.

- The problem becomes a plain LP by using the objective $\sum_{g\in\mathcal{K}} \underline{E}_{\underline{P}}g$.
- (*) decomposes into a classical formulation of natural extension.



1. Representations

Any convex polyhedron in $\mathbb{R}^n$ can be described in two ways:

H-representation (intersection of half-spaces)

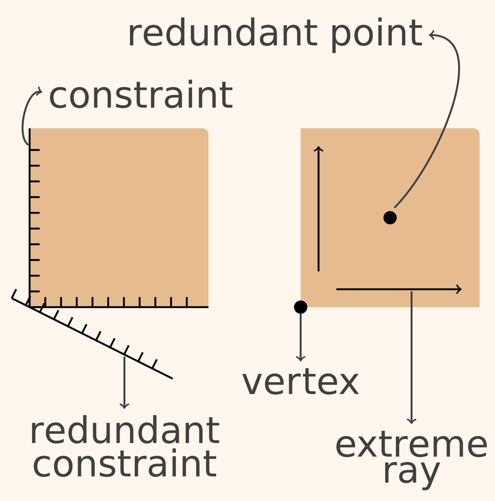
$[A|b] := \{x \in \mathbb{R}^n : Ax \leq b\}$, with constraint matrix in $\mathbb{R}^{k\times n}$ and constraint vector in $\mathbb{R}^k$

V-representation (convex hull of points and rays)

$\begin{bmatrix} V \\ w \end{bmatrix} := \{x \in \mathbb{R}^n : x = V\mu \wedge \mu \geq 0 \wedge w^\top\mu = 1\}$, with vector matrix in $\mathbb{R}^{n\times\ell}$, vector in $\mathbb{R}^\ell$, and vector in $(\mathbb{R}^\ell)_{\geq 0}$ with components defining points ($\neq 0$) and rays ($= 0$)

2. Illustration

Here $n = 2$, $k = 3$, and $\ell = 4$.



3. Tasks

RR. Removing redundancy: if j is the number of non-redundant constraints (or vectors), this requires solving k (or ℓ) linear programming problems of size $n \times j$

EN. Moving between H- and V-representations: done using vertex/facet enumeration algorithms; polynomial in n, k, and ℓ.

PJ. Projection on a lower-dimensional space: easy with V-representations, hard with H-representations.

IS. Intersection: easy with H-representations, hard with V-representations.

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to juggle POLYHEDRA like there's no tomorrow

1. Formalization

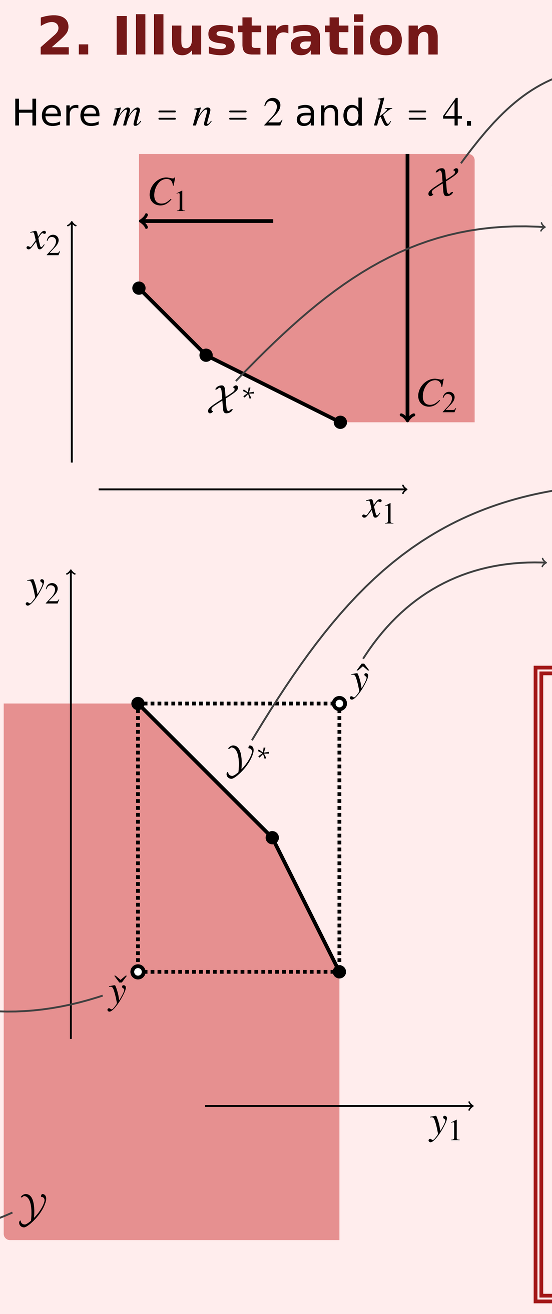
Any multi-objective linear program (MOLP) can be put in the following form:

maximize $y = Cx$, subject to $Ax \leq b$ and $x \geq 0$

(objective matrix in $\mathbb{R}^{m\times n}$; objective vector in $\mathbb{R}^m$; optimization vector in $\mathbb{R}^n$; constraint matrix in $\mathbb{R}^{k\times n}$; constraint vector in $\mathbb{R}^k$)

2. Illustration

Here $m = n = 2$ and $k = 4$.



3. Tasks

Main computational tasks in non-decreasing order of complexity:

M1. Finding $\hat{y}$.
M2. Finding $\check{y}$.
M3. Finding ext $\mathcal{Y}^*$ and characterizing $\mathcal{Y}^*$.
M4. Finding ext $\mathcal{X}^*$.
M5. Characterizing $\mathcal{X}^*$.

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to grok MULTI-OBJECTIVE LINEAR PROGRAMMING

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