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Performance Analysis of Iterative Decision-Directed Phase Noise Estimation

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Abstract: This contribution deals with estimation and compensation of phase noise in single-carrier digital communications. We present an iterative feedforward decision-directed phase noise estimation algorithm, that is based on approximating the phase noise process by an expansion of DCT basis functions containing only a few terms. An extension to the estimation algorithm is proposed, improving the performance in terms of the mean-square error. We demonstrate that the resulting (linearized) mean-square estimation error consists of two contributions: a contribution from the additive noise, that equals the Cramer-Rao lower bound, and a noise-independent contribution that results from the phase noise modeling error. The phase estimate that yields the lowest possible mean-square error is obtained, assuming knowledge of the phase noise statistics at the receiver.

Keywords: phase noise estimation, decision-directed, MMSE, basis expansion model, DCT

1. Introduction

Digital communication systems make use of carrier oscillators for up-conversion (at the transmitter) and down-conversion (at the receiver). Phase noise refers to random perturbations in the carrier phase, caused by imperfections in both transmitter and receiver oscillators. Compensation of this phase noise is critical since these disturbances can considerably degrade the error performance of the communication system. The phase noise process typically has a low-pass spectrum [1]. A description of the characteristics of oscillator phase noise is given in [2]. Several methods for phase noise estimation exist.

- Phase noise can be estimated by means of a feedback algorithm that operates according to the principle of the PLL. As feedback algorithms give rise to rather long acquisition periods, they are not well suited to systems with burst transmission [3].

- Phase noise is approximated as piecewise constant over the observation interval. In each subinterval over which the phase is assumed to be constant, a conventional feedforward algorithm is used to estimate the local time-average of the phase [3,4]. For strong phase noise, the subintervals have to be small, in which case the phase noise estimate is sensitive to the channel noise.

- Recently, a factor graph approach for the estimation of Markov-type phase noise has been presented in [5], but the algorithm appears rather cumbersome and assumes detailed knowledge at the receiver about the phase noise statistics.

Here, we apply the basis expansion model (BEM) to the problem of phase noise estimation. The considered basis functions are those from the discrete cosine transform (DCT). Basically, from the received signal a few coefficients of the basis expansion are estimated, from which an estimate of the phase noise is computed. In [6] and [7] this technique has been used to obtain a phase noise estimate from pilot symbols only. In [8], the estimate from [7] is iteratively improved by making use of soft decisions of the unknown data symbols. In this contribution, the phase noise estimation algorithm from [8] is modified to obtain an improved phase estimate for low signal-to-noise ratio. We analyze the mean-square error (MSE) of the estimation algorithm for both hard and soft decisions and devise an expression for the linearized minimum mean-square error of the phase.

2. System description

We consider the transmission of a block of K symbols over an AWGN channel that is affected by phase noise. The resulting received signal can be represented as:

$$r(k) = a(k)e^{j\theta(k)} + w(k);\ k = 0, ..., K-1 \tag{1}$$

where the index k refers to the k-th symbol interval of length T, the additive noise $\{w(k)\}$ is a sequence of i.i.d. zero-mean circular symmetric complex-valued Gaussian random variables with $E[|w(k)|^2] = N_0$, and $\theta(k)$ is the sum of a static phase offset θ_0 and a zero-mean phase noise process with KxK correlation matrix $\mathbf{R}_\theta$. The symbol sequence $\{a(k)\}$ contains K_P known pilot symbols at positions $k \in I_P = \{\ k_i,\ i = 0, ..., K_P - 1\}$, with constant magnitude: $|a(k_i)|^2 = E_s$. The remaining $K - K_P$ data symbols are unknown to the receiver; they belong to a constellation A, with $E[|a(k)|^2] = E_s$ for $k \notin I_P$. In order to obtain a decision-directed (DD) estimate of the phase $\theta(k)$, an initial estimate $\hat{\theta}_0(k)$ is calculated using the BEM-based algorithm from [7], that exploits only the pilot symbols. In order to reduce the probability of phase wrapping, we first rotate the observation $r(k)$ over the estimated average angle $\hat{\theta}_{avg}$, which is obtained using the hard/soft decisions $\tilde{a}(k)$ resulting from the initial phase estimate $\hat{\theta}_0(k)$ and is given by

$$\hat{\theta}_{avg} = arg\left(\sum_{k=0}^{K-1} r(k)\tilde{a}^*(k)\right) \tag{2}$$

As the phase fluctuation $\phi(k) = \theta(k) - \hat{\theta}_{avg}$ is essentially a lowpass process, it can be well approximated by the weighed sum of a ***limited number*** N ($<< K$) of suitable basis functions:

$$\phi(k) \approx \sum_{n=0}^{N-1} x_n \psi_n(k),\ k = 0, ..., K-1 \tag{3}$$

Next, the ML-estimate $\{\hat{x}_n, n = 0, ..., N-1\}$ of the expansion coefficients $\{x_n, n = 0, ..., N-1\}$ is calculated. Finally, we obtain the phase estimate $\hat{\theta}(k) = \hat{\theta}_{avg} + \sum_{n=0}^{N-1} \hat{x}_n \psi_n(k)$, $k = 0, ..., K-1$

3. Phase noise estimation algorithm

In this contribution we make use of the orthonormal discrete cosine transform (DCT) basis functions, that are defined as

$$\psi_n(k) = \begin{cases} \sqrt{\frac{1}{K}} & n = 0 \\ \sqrt{\frac{2}{K}} cos\left(\frac{\pi n}{K}\left(k + \frac{1}{2}\right)\right) & n > 0 \end{cases}$$

for $k = 0, ..., K-1$. Hence, x_n is the n-th DCT-coefficient of $\phi(k)$. From the observation (1) we will produce an estimate $\{\hat{x}_n, n = 0, ..., N-1\}$ of the N DCT-coefficients $\{x_n, n = 0, ..., N-1\}$, using the phase model (3) with equality. The corresponding phase estimate $\hat{\theta}(k)$ is obtained by computing the inverse DCT of $\{\hat{x}_n\}$ and adding the constant phase $\hat{\theta}_{avg}$. The decision-directed estimate $\hat{\theta}(k)$ is presented in [8] and is given by

$$\hat{\theta}(k) \quad = \quad \hat{\theta}_{avg} \; + \sum_{k'=0}^{K-1} (\mathbf{\Psi_K} \mathbf{\Psi_K}^T)_{k,k'} r'(k')$$

with

$$\begin{aligned} r'(k) &= arg(r(k)\tilde{a}^*(k)e^{-j\hat{\theta}_{avg}}) && (4) \\ &\approx \theta(k) - \hat{\theta}_{avg} + n(k) && (5) \end{aligned}$$

and $(\mathbf{\Psi_K})_{k,n} = \psi_n(k)$, for $k = 0, ..., K-1; n = 0, ..., N-1$, and where $n(k)$ is real-valued white gaussian noise, $n(k) \sim N(0, N_0/2)$. The approximation in (5) is based on the linearization of the argument function, assuming the phase noise fluctuations are small and a sufficiently large E_s/N_0. This linear model, however, is no longer valid for low E_s/N_0, where the channel noise variance becomes too large. Therefore, we propose the following modification to our phase noise estimation algorithm. Assume we transmit a block of symbols with length $K = BM$ symbols, where B and M are integers and $M \geq N$. In order to reduce the non-linear effect, we now produce an estimate of the coefficients $\{x_n, n = 0, ..., N-1\}$ by averaging the observations over B successive samples in order to reduce the channel noise variance. The estimation is now based on $r'_B(m)$ for $m = 0, ..., M-1$:

$$\begin{aligned} r'_B(m) &= arg\left(\sum_{b=0}^{B-1} r(b+mB)e^{-j\hat{\theta}_{avg}}\tilde{a}^*(b+mB)\right) && (6) \\ &\approx \frac{1}{B}\sum_{b=0}^{B-1} \phi(b+mB) + u(m) \\ &\approx \sum_{n=0}^{N-1} x_n \psi_n^{avg}(m) + u(m) \end{aligned}$$

where the basis functions $\psi_n^{avg}(m)$ are given by

$$\psi_n^{avg}(m) = \frac{1}{B}\sum_{b=0}^{B-1} \psi_n(b+mB)$$

and $u(m)$ is real-valued white gaussian noise with variance $E[u^2(m)] = N_0/(2B)$. Performing ML-estimation of x_n yields the following phase estimate:

$$\hat{\theta}_B(k) = \hat{\theta}_{avg} + \sum_{m=0}^{M-1} \left(\mathbf{\Psi_K}(\mathbf{\Psi_{avg}}^T \mathbf{\Psi_{avg}})^{-1}\mathbf{\Psi_{avg}}\right)_{k,m} r'_B(m)$$

where $(\mathbf{\Psi}_{avg})_{m,n} = \Psi_n^{avg}(m)$ for $m = 0, ..., M-1$; $n = 0, ..., N-1$. It is clear that the channel noise variance is reduced by a factor B, which, if B is chosen sufficiently large, allows the use of the linearized model (5). Averaging the observations over a group of B symbols however, also reduces the phase noise estimation accuracy, since the fluctuation of the phase noise over this group of symbols is approximated by a constant phase. The larger the number of symbols B in a group, the lower the estimation accuracy becomes.

4. Performance analysis

The performance of the phase noise estimation algorithm is assessed by considering the means-square error (MSE) of the phase, which is defined as

$$MSE = \frac{1}{K} E\left[\sum_{k=0}^{K-1} (\theta(k) - \hat{\theta}(k))^2\right] \tag{7}$$

We assume the linearized observation model (5) is valid and calculate the MSE. Assuming that after convergence the hard/soft decisions can be approximated by the true symbol values, the corresponding MSE is given by

$$MSE \;=\; \frac{N}{K}\frac{N_0}{2E_s} + MSE_\infty \tag{8}$$

with

$$MSE_\infty = \frac{1}{K} trace\left((\mathbf{\Psi_K}\mathbf{\Psi_K}^T - \mathbf{I_K})\mathbf{R}_\theta(\mathbf{\Psi_K}\mathbf{\Psi_K}^T - \mathbf{I_K})^T\right)$$

where $\mathbf{I_K}$ is the $K \times K$ identity matrix [8]. The Cramer-Rao lower bound (CRB) on the MSE (7), based on the observation (1), the phase noise model (3) with equality and assuming all symbols are known, is given by

$$MSE \geq \frac{N}{K}\frac{N_0}{2E_s} \tag{9}$$

The first term in (8) denotes the contribution from the additive noise and is equal to the CRB (9), whereas the second term constitutes a MSE floor, caused by the phase noise modeling error. We note that the CRB is proportional to N. The influence of the number of estimated coefficients N on the MSE floor can be better understood by taking into account that the phase $\theta(k)$ can be written as follows:

$$\theta(k) \;=\; \sum_{n=0}^{N-1} x_n \psi_n(k) + \theta_e(k) \tag{10}$$

where $\theta_e(k) = \theta(k) - \sum_{n=0}^{N-1} x_n \psi_n(k)$ is the phase noise modeling error. Substituting (10) into (7) leads to the following expression for the MSE floor:

$$MSE_\infty = \frac{1}{K} trace\left((\mathbf{\Psi_K}\mathbf{\Psi_K}^T - \mathbf{I_K})\mathbf{R}_{\theta_e}(\mathbf{\Psi_K}\mathbf{\Psi_K}^T - \mathbf{I_K})^T\right)$$

where $\mathbf{R}_{\theta_e}$ is the correlation matrix of the phase noise modeling error. This expression shows that the MSE floor decreases when N increases, since $\mathbf{R}_{\theta_e}$ approaches zero when N increases. For a detailed analysis of the MSE (8), we refer to [8].

When the phase noise statistics are known at the receiver, we can calculate the minimum mean-square error (MMSE) estimate of the phase. Assuming the linearized observation model (5), the MMSE phase estimate is given by

$$\hat{\theta}_{MMSE}(k) = \sum_{k'=0}^{K-1} (\mathbf{M})_{k,k'} r'(k'),\ k = 0, ..., K-1$$

where the $K \times K$ matrix $\mathbf{M}$ is chosen such that the MSE is minimized. Assuming the phase noise process can be approximated by a Gaussian process with $K \times K$ covariance matrix $\mathbf{R}_\theta$, the MSE is minimized for $\mathbf{M} = \mathbf{R}_\theta(\mathbf{R}_\theta + \sigma^2 \mathbf{I_K})^{-1}$, where $\sigma^2 = N_0/2$. The MMSE is given by

$$MMSE \;=\; \frac{1}{K} trace\left(\mathbf{R}_\theta - \mathbf{R}_\theta(\mathbf{R}_\theta + \sigma^2 \mathbf{I_K})^{-1}\mathbf{R}_\theta\right) \tag{11}$$

and the corresponding MMSE estimate is

$$\hat{\theta}_{MMSE}(k) = \sum_{k'=0}^{K-1} \left(\mathbf{R}_\theta(\mathbf{R}_\theta + \sigma^2 \mathbf{I_K})^{-1}\right)_{k,k'} r'(k') \tag{12}$$

We now apply the BEM to represent the phase noise process and produce the MMSE estimate of N BEM coefficients. The resulting phase estimate is found to be

$$\begin{aligned} \hat{\theta}_{MMSE,BEM}(k) &= \sum_{k'=0}^{K-1} \left(\mathbf{\Psi_K}\mathbf{\Psi_K^T}\mathbf{R}_\theta(\mathbf{R}_\theta + \sigma^2 \mathbf{I_K})^{-1}\right)_{k,k'} r'(k') \\ &= \sum_{n=0}^{N-1} \Psi_n(k) \sum_{k'=0}^{K-1} \Psi_n(k')\hat{\theta}_{MMSE}(k') \end{aligned} \tag{13}$$

Comparing equations (12) and (13), we observe that using the BEM and performing MMSE estimation of the basis expansion coefficients leads to a phase estimate that is worse than the MMSE phase estimate. From (13) it is clear that the phase estimate $\hat{\theta}_{MMSE,BEM}(k)$ is the same as the phase obtained when selecting only the N lower-order expansion coefficients of $\hat{\theta}_{MMSE}(k)$. Hence, the $K - N$ higher-order coefficients are neglected, which will inevitably lead to a worse performance. This is confirmed by the corresponding MSE:

$$MSE_{MMSE,BEM} \;=\; \frac{1}{K} trace\left(\mathbf{R}_\theta\right) - \frac{1}{K} trace\left(\mathbf{\Psi_K}^T\mathbf{R}_\theta(\mathbf{R}_\theta + \sigma^2 \mathbf{I_K})^{-1}\mathbf{R}_\theta\mathbf{\Psi_K}\right)$$

5. Numerical results

In this section we assess the performance of the proposed technique in terms of the MSE of the phase estimate by means of computer simulations. We assume transmission of uncoded QPSK symbols over an AWGN channel in the presence of Wiener phase noise $\theta(k)$, which is described by the following equation:

$$\theta(k+1) = \theta(k) + \Delta(k),\ k = 0, ..., K-2$$

where the initial phase noise value $\theta(0)$ is uniformly distributed in $[-\pi, \pi]$ and $\Delta(k)$ is a sequence of i.i.d. zero-mean Gaussian random variables with variance σ_Δ^2. In the following, we assume the transmission of a block of $K = 100$ symbols containing

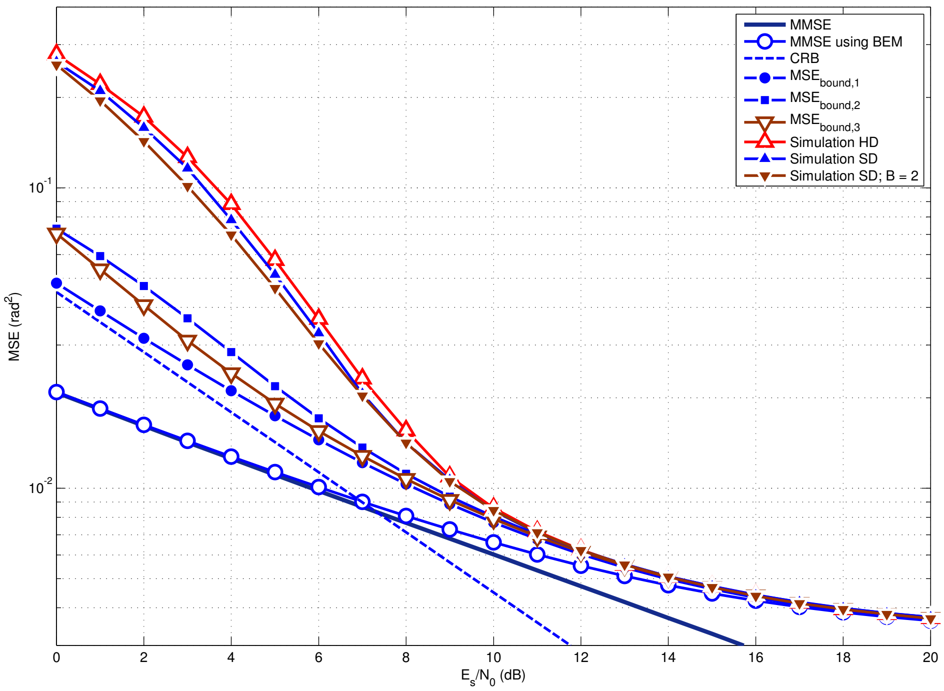

Figure 1: *MSE as function of* $\frac{E_s}{N_0}$ *for* $K = 100$ *and* $N = 9$

$K_P = 10$ pilot symbols and "strong" Wiener phase noise with $\sigma_\Delta = 3°$. An initial pilot-based phase estimate is obtained using the phase noise estimation algorithm from [7], where $N_0 = 3$ DCT-coefficients are estimated. Next, the decision-directed estimate is obtained for $N = 9$ DCT-coefficients. Figure 1 shows the following MSE curves as a function of E_s/N_0:

- $MSE_{bound,1}$ corresponds to the analytically computed MSE (8). As expected, the simulated MSE curve obtained when using soft decisions is closer to $MSE_{bound,1}$ than the corresponding MSE curve when using hard decisions. Both curves, however, deviate significantly from $MSE_{bound,1}$ at low E_s/N_0. This can be attributed to the fact that the linearized model (5) is no longer valid here and to the fact that the hard/soft decisions become unreliable here, so that the phase estimation error $((\hat{\theta}(k) - \theta(k))\, mod\, 2\pi) - \pi$ becomes uniformly distributed in $[-\pi, \pi]$. Hence the MSE approaches $\pi^2/3$ rad^2 for very low E_s/N_0.

- In order to study the impact of the assumption of non-linearity on the MSE, we use the correct data symbols $a(k)$ instead of the hard/soft decisions $\tilde{a}(k)$ in (4) and simulate the MSE. The resulting MSE is represented by $MSE_{bound,2}$. The difference at low E_s/N_0 between $MSE_{bound,1}$ and $MSE_{bound,2}$ in figure 1 is caused by the assumption of the linearized model which is no longer valid for low E_s/N_0.

- Figure 1 also shows the MSE curve obtained via simulations, when $B = 2$ consecutive samples are grouped according to (6). $MSE_{bound,3}$ shows the corresponding

MSE assuming perfect hard/soft decisions ($\tilde{a}(k) = a(k)$). We observe an improvement in the MSE as compared to $MSE_{bound,2}$ for low E_s/N_0. Simulations for $B = 5$ and $B = 10$ have shown that the MSE increases, especially at high E_s/N_0. This is in accordance with the fact that increasing the number of successive samples B per group reduces the non-linear effect, but also reduces the estimation accuracy.

- For high E_s/N_0, all curves lead to a MSE floor, except for the MMSE (11) where the estimate does not make use of the BEM. This MSE floor is caused by the phase noise modeling error due to neglecting $K - N$ higher-order DCT-coefficients.

6. Conclusions

In this contribution we have considered a feedforward iterative decision-directed phase noise estimation algorithm for single-carrier transmission. Linearization of the observation model has indicated that the MSE of the resulting estimate consists of an additive noise contribution (that increases with N) and a MSE floor caused by the phase noise modeling error (that decreases with N). The noise contribution coincides with the Cramer-Rao lower bound. We have proposed a modification to the phase noise estimation algorithm that is able to reduce the degradation in the MSE caused by assuming the linear observation model. Here, the number B of successive symbols that are grouped should be chosen such that the estimation accuracy remains acceptable. An analytical expression has been obtained for both the MMSE of the phase, and the MMSE of the phase when the BEM is used. The MMSE of the phase has been shown to have a floor when the BEM is used, while it is confirmed that the MMSE is the lowest bound on the MSE.

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