Convergence analysis of domain decomposition methods

Nonlinear elliptic and linear parabolic equations

Engström, Emil

Convergence analysis of domain decomposition methods

Convergence analysis of domain decomposition methods

Nonlinear elliptic and linear parabolic equations

by Emil Engström

LUND
UNIVERSITY

Thesis for the degree of Licentiate of Philosophy in Numerical Analysis
Thesis advisors: Prof. Eskil Hansen, Dr. Tony Stillfjord
Faculty opponent: Prof. Axel Målqvist

To be presented, with the permission of the Faculty of Engineering of Lund University, for public criticism in the lecture hall MH:309A at the Centre for Mathematical Sciences, Sölvegatan 18A, Lund, on Monday, the 3rd of April 2023 at 13:15.

Organization	Document name
LUND UNIVERSITY Centre for Mathematical Sciences Box 118 SE–221 00 LUND Sweden	**LICENTIATE THESIS** Date of disputation 2023-04-03 Sponsoring organization Swedish Research Council, grant number 2019-05396
Author(s) Emil Engström	

Title and subtitle

Convergence analysis of domain decomposition methods: Nonlinear elliptic and linear parabolic equations

Abstract

Domain decomposition methods are widely used tools for solving partial differential equations in parallel. However, despite their long history, there is a lack of rigorous convergence theory for equations with non-symmetric differential operators. This includes both nonlinear elliptic equations and linear parabolic equations. The aim of this thesis is therefore twofold: First, to construct frameworks, based on new Steklov–Poincaré operators, that allow the study of nonoverlapping domain decomposition methods for nonlinear elliptic and linear parabolic equations. Second, to prove convergence of the Robin–Robin method using these frameworks. For the nonlinear elliptic case, this involves studying L^p-variants of the Lions–Magenes space. In the parabolic case, we use a variational formulation based on fractional time-regularity. The analysis is performed with weak requirements on the spatial domain, where we only assume that the domains have Lipschitz regularity, and for the solutions to the equations, where we assume that their normal derivatives over the interface is in $L^2(\Gamma)$.

Key words

Nonoverlapping domain decomposition, Steklov–Poincaré operator, Robin–Robin method, Convergence, Nonlinear elliptic equation, Space-time

Classification system and/or index terms (if any)

Supplementary bibliographical information	Language
	English
ISSN and key title 1404-028X Licentiate theses in mathematical sciences	ISBN 978-91-8039-579-3 (print) 978-91-8039-578-6 (pdf)
Recipient's notes	Number of pages 77 Price
	Security classification

Convergence analysis of domain decomposition methods

Nonlinear elliptic and linear parabolic equations

by Emil Engström

LUND
UNIVERSITY

Funding information: The thesis work was financially supported by the Swedish Research Council under the grant 2019–05396.

Faculty of Engineering, Centre for Mathematical Sciences

Licentiate theses in mathematical sciences 2023:2
ISSN: 1404-028X

ISBN: 978-91-8039-579-3 (print)
ISBN: 978-91-8039-578-6 (pdf)
LUTFNA-2007-2023

Printed in Sweden by Media-Tryck, Lund University, Lund 2023

Contents

List of publications

This thesis is based on the following publications, referred to by their Roman numerals:

I **Convergence Analysis of the Nonoverlapping Robin–Robin Method for Nonlinear Elliptic Equations**
E. Engström, E. Hansen
SIAM J. Numer. Anal., 60(2):585–605, 2022

II **Time-dependent Steklov–Poincaré operators and space-time Robin–Robin decomposition for the heat equation**
E. Engström, E. Hansen
Submitted to Numerische Mathematik, preprint available on
https://arxiv.org/abs/2210.13868

Acknowledgements

This licentiate thesis is the result of my first three years as a Ph.D. student. As such, I want to thank everyone that has supported and aided me in this time.

First of all, I would like to sincerely thank my supervisor Eskil Hansen for the support, guidance, collaboration, and the numerous insightful discussions, all of which has been crucial for me to reach this point in my studies. I would also like to thank my co-supervisor Tony Stillfjord for his support and feedback.

Finally, I wish to thank all my friends and colleagues at the Centre for Mathematical Sciences that have made this time a great experience.

Populärvetenskaplig sammanfattning på svenska

Många modeller inom naturvetenskap kan utryckas som någon form av partiell differentialekvation. En partiell differentialekvation är en ekvation vars lösning är en funktion i en eller flera rumsdimensioner och möjligen en tidsdimension. De kan beskriva, till exempel, värmefördelning i ett rum vid jämviktsläge eller en vätska som flyter genom ett rör. Dessa ekvationer är svåra att lösa och därför har mycket forskning ägnats åt detta, och mer specifikt, att göra detta på ett effektivt sätt. Komplexa modeller ger vanligtvis differentialekvationer som inte kan lösas analytiskt, det går alltså inte att skriva ner lösningen exakt. Vi vänder oss därför till numeriska metoder, d.v.s., vi approximerar lösningarna med beräkningar utförda på datorer.

Numeriska metoder för partiella differentialekvationer kräver en enorm mängd minne bara för att lagra problemen och snabba processorer för att lösa dem. Utvecklingen av processorhastighet och minne per processor har saktats ned de senaste åren och vi har i stället sett en ökning av distribuerade system. Dessa är stora kluster bestående av flera datorer som arbetar tillsammans. För att utnyttja denna teknologi behöver vi konstruera metoder som kan användas parallellt: Problemet måste kunna delas upp i flera mindre beståndsdelar så att varje del kan lösas oberoende av de andra. Ett sätt att göra detta på är med områdesuppdelande metoder.

Områdesuppdelande metoder fungerar genom att dela upp rumsområdet, d.v.s. rummet och röret i exemplen ovan, i flera mindre områden, eller delar. Varje del ger upphov till ett mindre problem, men problemen är inte oberoende: Temperaturen längst in i rummet beror på temperaturen längs ut, och samma princip gäller röret. Detta beroende löses genom att introducera en iterativ process som överför information mellan delarna.

För numeriska metoder är det inte bara viktigt att de är effektiva, men de måste också vara pålitliga. Helst vill vi veta på förhand hur bra approximation vi kommer att få. En stor del av området numerisk analys är därför att studera konvergens av numeriska metoder. Med konvergens av en metod menar vi att ju mer arbete vi utför, desto närmre kommer approximationen den verkliga lösningen. Trots att områdesuppdelande metoder har använts länge så saknas det fortfarande konergensbevis i många fall.

Målet med denna avhandling är därför att först konstruera ett nytt ramverk för att analysera konvergens av områdesuppdelande metoder i två fall som inte har studerats ordentligt tidigare. Mer specifikt studerar vi fallet då ekvationen är ickelinjär och det tidsberoende fallet. Det andra målet med avhandlingen är att bevisa konvergens av en specifik metod, Robin-Robin metoden, som överför en kombination av randvärden och derivata mellan delarna.

All analys utförs genom att endast undersöka vad som händer på randen mellan delarna. På randen kan områdesuppdelande metoder uttryckas genom Steklov-Poincaré operatorn, som beskriver relationen mellan randvärdena och derivatan. Därför är det nödvändigt att studera egenskaperna hos denna operator. I båda det ickelinjära och det tidsberoende fallet visar vi att operatorn har de egenskaper som krävs för att bevisa konvergens för Robin-Robin metoden.

Chapter 1

Introduction

Approximating solutions to partial differential equations requires a significant amount of computational power, which is typically found on distributed hardware. In order to take advantage of these hardware structures, one requires methods that can be used in parallel. Domain decomposition methods are methods for approximating solutions to partial differential equations that allow such parallel computations.

Despite their long history and the vast amount of methods available, there is no general convergence theory for nonlinear elliptic equations, or for parabolic equations. The purpose of this thesis is therefore to generalize the theory of nonoverlapping domain decomposition methods to nonlinear elliptic equations and linear parabolic equations. In the linear elliptic case, the existing convergence proofs typically exploit symmetry of the partial differential operator, which is not possible for these cases. We will discuss three basic domain decomposition methods, but keep the main focus on the Robin–Robin method since this convergence proof does not require any symmetry assumptions on the differential operator.

The goal of this thesis is thus: First, to construct a general framework for analyzing nonoverlapping domain decomposition methods, for both nonlinear elliptic equations and linear parabolic equations. Second, to prove convergence of the Robin–Robin method in both of these contexts using these new frameworks.

All analysis is performed in the continuous setting, that is, not considering any space or time discretizations of the partial differential equation. This is done in order to study the convergence of the domain decomposition method without the effect of the discretization. Since the continuous case is, in the sense of the Banach spaces involved, more complex than the discrete case, it is also likely that this captures the essential parts of the analysis for the discrete equations. In fact, for the nonlinear

elliptic case it is straightforward to apply the analysis to a finite element discretization, although no convergence order is available. For the parabolic case, the effect of the discretization is not as straightforward and needs to be studied further.

The thesis is structured as follows: In Chapter 2 we introduce the geometrical and analytical tools as well as the notation used in the rest of the thesis. In Chapter 3 we give a review of the theory of domain decomposition methods applied to linear elliptic equations, starting with the theory of Steklov–Poincaré operators and then showing how these operators can be used to prove convergence of some common domain decomposition methods. In Chapter 4 we show how to generalize this theory to nonlinear elliptic equations, which is the subject matter of Paper I. In Chapter 5 we first show how the analysis of Paper I can be applied to a discretized equation and then give some new unpublished numerical results, which verifies our convergence results. In Chapter 6 we generalize the analysis of domain decomposition methods for linear elliptic equations to linear parabolic equations, which is the topic of Paper II. We then provide new numerical results for parabolic equations in Chapter 7. Finally, we give some ideas on future work in Chapter 8.

Chapter 2

Preliminaries

2.1 Geometry

In all applications we consider a nonoverlapping domain decomposition of the spatial domain $\Omega \subset \mathbb{R}^d$, for some $d = 2, 3, \ldots$. For simplicity we only cover the case of two subdomains Ω_1, Ω_2 with the interface Γ, i.e.,

$$\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2, \quad \Omega_1 \cap \Omega_2 = \emptyset \quad \text{and} \quad \Gamma = (\partial\Omega_1 \cap \partial\Omega_2) \setminus \partial\Omega,$$

see Figures 2.1a and 2.1b. This setting also includes decompositions as in Figure 2.1c since we can write

$$\Omega_i = \bigcup_{\ell=1}^{N_i} \Omega_{i\ell}, \quad N_i \in \mathbb{N}, \quad i = 1, 2.$$

Decompositions of this form allow our methods to be used in parallel, since the differential equation on the domain Ω_i can be divided into N_i independent problems on $\Omega_{i\ell}$, $\ell = 1, \ldots, N_i$. Although important for parallelization, we do not consider decompositions of the form Figure 2.1d since they contain crosspoints. That is, points where two parts of the interface intersect. These typically lead to variational formulations with different trial and test spaces, see, e.g., [Paper I, Remark 6.5].

We assume no further regularity of our domains other than that they are Lipschitz, see [38, Section 6.2] for a definition. See also [58] for the definition of a Lipschitz manifold.

Assumption 1. *The domains Ω, Ω_i, $i = 1, 2$, are bounded and Lipschitz. Furthermore the sets Γ, $\partial\Omega_i$, $i = 1, 2$, are $(d-1)$-dimensional Lipschitz manifolds.*

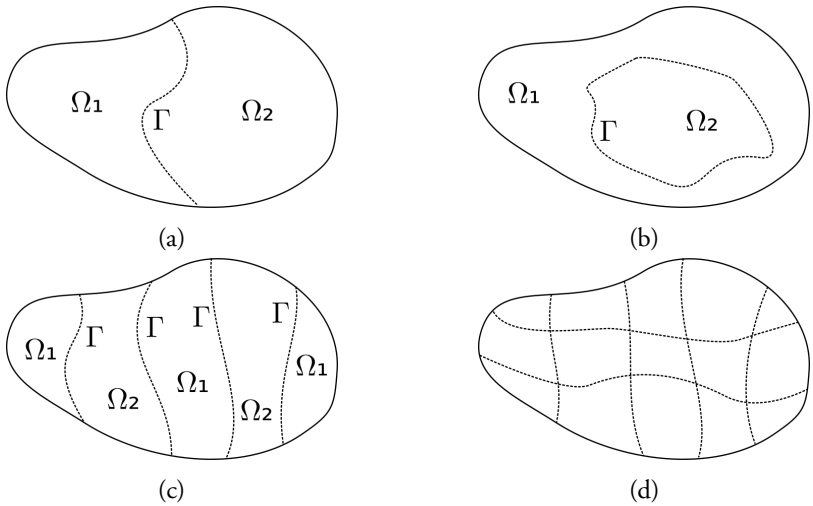


Figure 2.1: Examples of decompositions of Ω: (a) two subdomains with two intersection points; (b) two subdomains without intersection points; (c) multiple subdomains without crosspoints; (d) multiple subdomains with crosspoints.

For some equations we need a further assumption, which is related to whether the equation is naturally coercive only in seminorm or in the full norm. In the former case, we must apply Poincaré's inequality, which requires the following assumption.

Assumption 2. *The sets $\partial\Omega \setminus \partial\Omega_i$, $i = 1, 2$, are (non-empty) $(d-1)$-dimensional Lipschitz manifolds.*

Note that this excludes decompositions of the form Figure 2.1b, since in this case, $\partial\Omega \setminus \partial\Omega_1$ is not $d-1$-dimensional.

Assumption 1 implies that we can define the integrals over Γ, $\partial\Omega_i$, $\partial\Omega \setminus \Gamma$, and $\partial\Omega \setminus \partial\Omega_i$ respectively, and thus also the corresponding L^p spaces, see [58]. See also [38] for a comprehensive overview of integrals and functions spaces on Lipschitz boundaries.

Remark 1. *Note the abuse of notation $\partial\Omega_i \setminus \Gamma$, which is not necessarily an open subset of $\partial\Omega_i$, for example in Figure 2.1a. Therefore when writing $\partial\Omega_i \setminus \Gamma$ we always mean the interior of this set in $\partial\Omega_i$.*

2.2 Function spaces

All analysis is performed in Banach and Hilbert spaces. For a proper introduction we refer to [41]. For a real Banach space X we will denote the dual of X by X^* and the dual pairing by $\langle \cdot, \cdot \rangle_{X^* \times X}$ or by $\langle \cdot, \cdot \rangle$ if the spaces are obvious from the context. Moreover, for two Banach spaces X and Y we define the intersection space $X \cap Y$ with the natural norm

$$\|x\|_{X\cap Y}^2 = \|x\|_X^2 + \|x\|_Y^2,$$

which induces a Banach space structure on $X \cap Y$. We introduce the parameter $1 < p < \infty$, which depends on the context. In particular for the linear cases in Chapters 3, 6 and 7 we are only interested in $p = 2$. We define the Banach spaces

$$V = W_0^{1,p}(\Omega), \quad V_i^0 = W_0^{1,p}(\Omega_i), \quad \text{and}$$
$$V_i = \{v \in W^{1,p}(\Omega_i) : (T_{\partial\Omega_i} v)|_{\partial\Omega_i \setminus \Gamma} = 0\},$$

where $W^{1,p}$ and $W_0^{1,p}$ denotes the standard Sobolev spaces, see [38, Chapter 5]. Moreover we introduce the Lions–Magenes space

$$\Lambda = \{\mu \in L^p(\Gamma) : E_i \mu \in W^{1-1/p,p}(\partial\Omega_i)\}, \quad \text{with}$$
$$\|\mu\|_\Lambda = \|E_i \mu\|_{W^{1-1/p,p}(\partial\Omega_i)}.$$

Here, $E_i : L^p(\Gamma) \to L^p(\partial\Omega_i)$ denotes the extension by 0 and $T_{\partial\Omega_i} : W^{1,p}(\Omega_i) \to W^{1-1/p,p}(\partial\Omega_i)$ denotes the trace operator with the bounded linear right inverse $R_{\partial\Omega_i}$, see [38, Theorems 6.8.13 and 6.9.2]. Note that $R_{\partial\Omega_i}$ is not unique, i.e., there exists many linear extensions from $W^{1-1/p,p}(\partial\Omega_i)$ to $W^{1,p}(\Omega_i)$. For a proper definition of the fractional Sobolev space $W^{1-1/p,p}(\partial\Omega_i)$ and its properties we refer to [38, Chapter 6.8]. For our purposes it is important to note that this is a reflexive Banach space and a Hilbert space for $p = 2$.

Remark 2. *The Lions–Magenes space satisfies $\Lambda \subset W^{1-1/p,p}(\Gamma)$. At least for $p = 2$, the inclusion is strict and the norm of Λ is stronger than the norm of $W^{1-1/p,p}(\Gamma)$, see [60, Chapter 1.2].*

Remark 3. *Note that it is not obvious from the definition that Λ is independent of $i = 1, 2$. For $p = 2$ we can use [68, Lemma A.8] to identify Λ with the interpolation space $[H_0^{1/2}(\Gamma), L^2(\Gamma)]_{1/2}$. Note that this reference is without proof, but there is a proof for the case of a smooth domain, see [45, Theorem 11.7]. We conjecture that a similar identification can be made for any $1 < p < \infty$ and for the remainder of the thesis we will assume that Λ is independent of $i = 1, 2$ for $p \neq 2$ as well. This is done for the sake of presentation, but it is not a necessary assumption for any of the convergence results stated here, see Paper I for the analysis with i-dependent Λ.*

The standard seminorm and norm on V is denoted by

$$|v|_V = \|\nabla v\|_{L^p(\Omega)^d}, \qquad \|v\|_V = \|v\|_{L^p(\Omega)} + |v|_V,$$

respectively, and similarly for V_i. It follows from the above definitions that the interface trace operator

$$T_i : V_i \to \Lambda : v \mapsto (T_{\partial\Omega_i} v)|_\Gamma$$

is bounded with the bounded linear right inverse $R_i = R_{\partial\Omega_i} E_i$.

To analyze parabolic equations, we require Sobolev–Bochner spaces, see [37, Chapter 2.5.d]. We note that these are equivalent to Hilbert tensor spaces and refer to [69, Chapter 3.4] for a general introduction. Since we are only interested in linear equations, we restrict ourselves to the case $p = 2$ and define the Sobolev–Bochner spaces

$$\begin{aligned} W &= H^{1/2}\big(\mathbb{R}, L^2(\Omega)\big) \cap L^2\big(\mathbb{R}, V\big), \\ W_i^0 &= H^{1/2}\big(\mathbb{R}, L^2(\Omega_i)\big) \cap L^2\big(\mathbb{R}, V_i^0\big), \\ W_i &= H^{1/2}\big(\mathbb{R}, L^2(\Omega_i)\big) \cap L^2\big(\mathbb{R}, V_i\big), \\ Z &= H^{1/4}\big(\mathbb{R}, L^2(\Gamma)\big) \cap L^2\big(\mathbb{R}, \Lambda\big). \end{aligned}$$

The trace operator has an extension to the Sobolev–Bochner space of the form

$$\begin{aligned} T_{\partial\Omega_i} : H^{1/2}\big(\mathbb{R}, L^2(\Omega_i)\big) \cap L^2\big(\mathbb{R}, H^1(\Omega_i)\big) \to \\ H^{1/4}\big(\mathbb{R}, L^2(\partial\Omega_i)\big) \cap L^2\big(\mathbb{R}, H^{1/2}(\partial\Omega_i)\big), \end{aligned}$$

which we refer to as the spatial trace operator. We also formally introduce the temporal Hilbert transform as

$$\mathcal{H}v(t) = \lim_{\epsilon \to 0^+} \frac{1}{\pi} \int_{|s| \geq \epsilon} \frac{1}{s} v(t-s) \mathrm{d}s$$

and note that it extends to an isomorphism on the Sobolev–Bochner spaces in the following ways

$$\mathcal{H}_i : W_i \to W_i \quad \text{and} \quad \mathcal{H}_\Gamma : Z \to Z.$$

Moreover, the spatial interface trace operator of the form

$$T_i : W_i \to Z$$

is bounded, see Paper II for more details.

Remark 4. *The loss of time regularity when applying the spatial trace operator is somewhat surprising. Note that the trace operator is also bounded as an operator*

$$T_{\partial\Omega_i} : L^2\big(\mathbb{R}, H^1(\Omega_i)\big) \to L^2\big(\mathbb{R}, H^{1/2}(\partial\Omega_i)\big),$$

but it is not well defined on the space $H^{1/2}\big(\mathbb{R}, L^2(\Omega_i)\big)$ alone.

Finally, we will denote by c, c_i, C and C_i generic positive constants.

Chapter 3

Domain decomposition for linear elliptic equations

3.1 Introduction to the analysis of domain decomposition methods

For an introduction to linear elliptic equations see [20, 32, 45]. For a general introduction to domain decomposition methods for linear elliptic equations we refer to [60, 68]. See also [17] for a more implementation oriented introduction and the review article [70] for an introduction to the theory of nonoverlapping domain decomposition methods. The first domain decomposition method was introduced in [62] to prove existence of solutions to the Poisson problem for nonrectangular and noncircular domains. This became known as the alternating Schwarz method and was later proposed as a numerical method in [57]. It was analyzed and generalized in the context of numerical methods in [47, 48, 49]. Since then, a large amount of different methods have been introduced and analyzed, see [60, 68] for examples.

Domain decomposition methods are typically divided into overlapping and nonoverlapping methods. The overlapping methods, which are based on overlapping decompositions, transfer information on the interior of one domain to the boundary of another. In contrast, nonoverlapping methods use an information transfer that only considers the boundary information, thus requiring minimal knowledge of the transmitting domain. In this thesis, we only consider nonoverlapping methods, also known as iterative substructuring methods. Furthermore, we limit ourselves to three basic methods which are often used as building blocks for more complex methods: The Dirichlet–Neumann method, the Neumann–Neumann method, and the Robin–

Robin method. The remainder of this chapter is heavily inspired by [60].

We first explain the idea of the analysis in more detail, but formally, using the strong formulations. To keep the notation simple we only consider the Poisson equation with homogeneous Dirichlet boundary conditions. The strong formulation of the Poisson problem is

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Given a decomposition Ω_1, Ω_2 of Ω, the problem can then be rewritten as a transmission problem

$$\begin{cases} -\Delta u_i = f_i & \text{in } \Omega_i, \\ u_i = 0 & \text{on } \partial\Omega_i \setminus \Gamma, \text{ for } i = 1, 2, \\ u_1 = u_2 & \text{on } \Gamma, \\ \nabla u_1 \cdot \nu_1 = -\nabla u_2 \cdot \nu_2 & \text{on } \Gamma, \end{cases} \tag{3.2}$$

where $f_i = f|_{\Omega_i}$ and ν_i is the unit outward normal vector of Ω_i. Note, in particular, that $\nu_1 = -\nu_2$ on Γ. The last two equations here are referred to as the first and second transmission condition, respectively. For some function η on the interface Γ we denote the solution to the nonhomogeneous boundary value problem

$$\begin{cases} -\Delta u_i = f_i & \text{in } \Omega_i, \\ u_i = 0 & \text{on } \partial\Omega_i \setminus \Gamma, \\ u_i = \eta & \text{on } \Gamma, \end{cases} \tag{3.3}$$

by $u_i = F_i\eta + G_i f_i$ for $i = 1, 2$. Here $F_i\eta$ and $G_i f_i$ are the solutions to (3.3) with $f_i = 0$ and $\eta = 0$, respectively. With this notation we can rewrite the entire transmission problem as a problem only on the interface Γ, i.e.,

$$\nabla(F_1\eta + G_1 f_1) \cdot \nu_1 = -\nabla(F_2\eta + G_2 f_2) \cdot \nu_2 \qquad \text{on } \Gamma.$$

This is known as the Steklov–Poincaré equation and the operator

$$S_i : \eta \mapsto \nabla F_i \eta \cdot \nu_i$$

is called the Steklov–Poincaré operator or the Dirichlet-to-Neumann operator. Similarly, we introduce the notation

$$\chi_i = -\nabla G_i f_i \cdot \nu_i.$$

Finally, introducing $S = S_1 + S_2$ and $\chi = \chi_1 + \chi_2$, the Steklov–Poincaré equation can be written as

$$S\eta = \chi \qquad \text{on } \Gamma. \tag{3.4}$$

$$
\begin{array}{ccccc}
\Omega & & \Omega_i & & \Gamma \\
\mathrm{DE} & \Longleftrightarrow & \mathrm{TP} & \Longleftrightarrow & \mathrm{SP} \\
 & & \text{approx.}\downarrow & & \downarrow\text{ approx.} \\
 & & \mathrm{DD} & \Longleftrightarrow & \mathrm{II}
\end{array}
$$

Figure 3.1: The basic idea for the analysis of a nonoverlapping domain decomposition method. The graph shows the connection between the differential equation (DE), the transmission problem (TP), the Steklov–Poincaré equation (SP), the domain decomposition method (DD), and the interface iteration (II).

The basic idea is summarized in the first row in Figure 3.1: We reduce the differential equation to a problem on the interface Γ by showing equivalence of the differential equation, the transmission problem, and the Steklov–Poincaré equation.

The aim is now to approximate the solution to the transmission problem using a domain decomposition method. The Dirichlet–Neumann method is one of the simplest domain decomposition method for nonoverlapping decompositions and has been studied extensively for linear elliptic equations [7, 9, 22, 54, 55]. The idea is to approximate (u_1, u_2), the solution to the transmission problem (3.2), by iterating between a Dirichlet problem and a Neumann problem. The method is to find (u_1^n, u_2^n) for $n = 1, 2, \ldots$, such that

$$
\begin{cases}
-\Delta u_1^{n+1} = f_1 & \text{in } \Omega_1, \\
u_1^{n+1} = 0 & \text{on } \partial\Omega_1 \setminus \Gamma, \\
u_1^{n+1} = \eta^n & \text{on } \Gamma, \\
-\Delta u_2^{n+1} = f_2 & \text{in } \Omega_2, \\
u_2^{n+1} = 0 & \text{on } \partial\Omega_2 \setminus \Gamma, \\
\nabla u_2^{n+1} \cdot \nu_2 = \nabla u_1^{n+1} \cdot \nu_2 & \text{on } \Gamma,
\end{cases}
\tag{3.5}
$$

with $\eta^{n+1} = s_0\, u_2^{n+1}\big|_\Gamma + (1 - s_0)\eta^n$. Here $s_0 > 0$ is a method parameter and η^0 an initial guess.

A similar method is the the Neumann–Neumann method, which was first introduced and analyzed in [8]. It consists of finding $(u_1^n, u_2^n, w_1^n, w_2^n)$ for $n = 1, 2, \ldots$, such

that

$$\begin{cases} -\Delta u_i^{n+1} = f_i & \text{in } \Omega_i, \\ u_i^{n+1} = 0 & \text{on } \partial\Omega_i \setminus \Gamma, \\ u_i^{n+1} = \eta^n & \text{on } \Gamma, \text{ for } i = 1, 2, \\ -\Delta w_i^{n+1} = 0 & \text{in } \Omega_i, \\ w_i^{n+1} = 0 & \text{on } \partial\Omega_i \setminus \Gamma, \\ \nabla w_i^{n+1} \cdot \nu_1 = \nabla u_1^{n+1} \cdot \nu_1 - \nabla u_2^{n+1} \cdot \nu_1 & \text{on } \Gamma, \text{ for } i = 1, 2, \end{cases} \tag{3.6}$$

with $\eta^{n+1} = \eta^n - (s_1\, w_1^{n+1}|_\Gamma - s_2\, w_2^{n+1}|_\Gamma)$. Here $s_1, s_2 > 0$ are method parameters and η^0 an initial guess.

Finally, we have the Robin–Robin method, which was first introduced in [49]. The method is to find (u_1^n, u_2^n), for $n = 1, 2, \ldots$, such that

$$\begin{cases} -\Delta u_1^{n+1} = f_1 & \text{in } \Omega_1, \\ u_1^{n+1} = 0 & \text{on } \partial\Omega_1 \setminus \Gamma, \\ \nabla u_1^{n+1} \cdot \nu_1 + s u_1^{n+1} = \nabla u_2^n \cdot \nu_1 + s u_2^n & \text{on } \Gamma, \\ -\Delta u_2^{n+1} = f_2 & \text{in } \Omega_2, \\ u_2^{n+1} = 0 & \text{on } \partial\Omega_2 \setminus \Gamma, \\ \nabla u_2^{n+1} \cdot \nu_2 + s u_2^{n+1} = \nabla u_1^{n+1} \cdot \nu_2 + s u_1^{n+1} & \text{on } \Gamma. \end{cases} \tag{3.7}$$

Here $s > 0$ is a method parameter and u_2^0 an initial guess. Various modifications, accelerations and generalizations of the Robin–Robin method have been studied in, e.g., [5, 10, 33]. Note, in particular, that the Robin–Robin method is the zeroth order optimized Schwarz method [23, 26]. See also [52, 53, 72] for a modified optimized Schwarz method on arbitrary domains. Although not discussed further here, the Robin–Robin method has independently been introduced to solve the Helmholtz equation [14, 29].

Analogous to the equivalence between the transmission problem and the Steklov–Poincaré equation, the Dirichlet–Neumann and Neumann–Neumann methods are equivalent to the interface iterations

$$\eta^{n+1} = \eta^n + s_0 S_2^{-1}(\chi - S\eta^n)$$

and

$$\eta^{n+1} = \eta^n + (s_1 S_1^{-1} + s_2 S_2^{-1})(\chi - S\eta^n),$$

respectively. These iterations, in turn, approximate the solution to the Steklov–Poincaré equation (3.4). This idea is summarized in Figure 3.1. The Robin–Robin method can

also be interpreted as an interface iteration, but this is nonstandard for the linear elliptic theory and we therefore introduce it in Chapter 4.

3.2 Steklov–Poincaré theory for the Poisson equation

The main ingredient for the analysis of nonoverlapping domain decomposition methods for linear elliptic equations is the corresponding Steklov–Poincaré theory. Here we present the core ideas of the analysis applied to the Poisson equation, which can be found in [60, Chapter 4]. Note that the theory applies more generally to equations with symmetric elliptic operators and, with some conditions on the size of the skew-symmetric part, to equations with non-symmetric linear elliptic operators [60, Chapter 5.1]. The key idea is to consider the weak formulations of (3.1), (3.2) and (3.4) and show that these are equivalent. Then we derive the fundamental properties of the Steklov–Poincaré operators, which are required to study domain decomposition methods.

For the remainder of this chapter we consider the spaces from Section 2.2 with $p = 2$, in which case we have the Hilbert spaces

$$V = H_0^1(\Omega), \quad V_i^0 = H_0^1(\Omega_i), \quad V_i = \{v \in H^1(\Omega_i) : (T_{\partial\Omega_i} v)|_{\partial\Omega_i \setminus \Gamma} = 0\},$$
$$\text{and} \quad \Lambda = \{\mu \in L^2(\Gamma) : E_i \mu \in H^{1/2}(\partial\Omega_i)\}.$$

Let $a : V \times V \to \mathbb{R}$ and $a_i : V_i \times V_i \to \mathbb{R}$ be the bilinear forms given by

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x \quad \text{and} \quad a_i(u, v) = \int_{\Omega_i} \nabla u \cdot \nabla v \, \mathrm{d}x,$$

respectively. It is easy to verify the bounds

$$a(u, v) \leq C \|u\|_V \|v\|_V \qquad \text{for all } u, v \in V \text{ and}$$
$$a(u, u) \geq c \|u\|_V^2 \qquad \text{for all } u \in V,$$

with similar estimates for a_i, $i = 1, 2$. Note that the second estimate, known as the coercivity estimate, requires an application of Poincaré's inequality, which holds for any decomposition such that Assumption 2 is satisfied. Note also that this assumption is redundant if one replaces the Laplace operator Δ by $\Delta + \lambda I$ for some $\lambda > 0$, since then the bilinear forms can be proven to be coercive without having to apply Poincaré's inequality.

We assume that the source term $f \in L^2(\Omega)$, which implies that the decomposed sources $f_i = f|_{\Omega_i}$ satisfy $f_i \in L^2(\Omega_i)$. Now we can introduce the weak formulation

of the Poisson problem (3.1), which is to find $u \in V$ such that

$$a(u, v) = (f, v)_{L^2(\Omega)} \qquad \text{for all } v \in V. \tag{3.8}$$

The bounds above imply that the weak problem (3.8) is uniquely solvable by the Lax–Milgram lemma [41, Chapter 6, Theorem 6].

Recall the operator $R_i : \Lambda \to V_i$, defined in Section 2.2. The weak formulation of the transmission problem (3.2) is to find $(u_1, u_2) \in V_1 \times V_2$ such that

$$\begin{cases} a_i(u_i, v_i) = (f_i, v_i)_{L^2(\Omega_i)} & \text{for all } v_i \in V_i^0, i = 1, 2, \\ \quad T_1 u_1 = T_2 u_2, & \\ \sum_{i=1}^2 a_i(u_i, R_i\mu) - (f_i, R_i\mu)_{L^2(\Omega_i)} = 0 & \text{for all } \mu \in \Lambda. \end{cases} \tag{3.9}$$

The first two equations are straightforward to derive from the strong formulation of the transmission problem (3.2). The third equation can be derived by multiplying the second transmission condition with $\mu \in \Lambda$, integrating, and applying Green's formula as follows

$$\begin{aligned} 0 &= \sum_{i=1}^2 \int_{\partial\Omega_i} \nabla u_i \cdot \nu_i \, \mu \, \mathrm{d}S = \sum_{i=1}^2 a(u_i, R_i\mu) + \int_{\Omega_i} \Delta u_i R_i \mu \, \mathrm{d}x \\ &= \sum_{i=1}^2 a(u_i, R_i\mu) - (f_i, R_i\mu)_{L^2(\Omega_i)}. \end{aligned}$$

The weak problem (3.8) and the weak transmission problem are equivalent, see [60, Lemma 1.2.1]. As in Section 3.1, let $F_i : \Lambda \to V_i$ denote the harmonic extension operator, i.e., mapping η to $u_i = F_i\eta$ solving

$$\begin{cases} a_i(u_i, v) = 0 & \text{for all } v \in V_i^0, \\ \quad T_i u_i = \eta & \end{cases}$$

and $G_i : (V_i^0)^* \to V_i^0$ be the operator mapping f_i to $u_i = G_i f_i$ solving

$$a_i(u_i, v) = (f_i, v)_{L^2(\Omega_i)} \qquad \text{for all } v \in V_i^0. \tag{3.10}$$

Note that $T_i F_i \eta = \eta$ and therefore F_i is another right inverse to T_i. Moreover, F_i and G_i are bounded operators. With these operators, the weak forms of the Steklov–Poincaré operators $S_i : \Lambda \to \Lambda^*$ and the functionals $\chi_i \in \Lambda^*$ can be defined as

$$\langle S_i\eta, \mu\rangle = a_i(F_i\eta, R_i\mu) \quad \text{and} \quad \langle \chi_i, \mu\rangle = (f_i, R_i\mu)_{L^2(\Omega_i)} - a_i(G_i f_i, R_i\mu),$$

respectively. Introducing also $S = S_1 + S_2$ and $\chi = \chi_1 + \chi_2$ we have the weak Steklov–Poincaré equation, which is to find $\eta \in \Lambda$ such that $S\eta = \chi$, or equivalently,

$$\sum_{i=1}^{2} \langle S_i \eta, \mu \rangle = \sum_{i=1}^{2} \langle \chi_i, \mu \rangle \quad \text{for all } \mu \in \Lambda. \tag{3.11}$$

The weak Steklov–Poincaré equation is equivalent to the weak transmission problem, see [60, Remark 1.2.2].

Remark 5. *Some authors define the Steklov–Poincaré operators as*

$$\langle S_i \eta, \mu \rangle = a_i(F_i \eta, F_i \mu).$$

We note that this definition is equivalent since $T_i(F_i\mu - R_i\mu) = 0$ *implies that* $F_i\mu - R_i\mu \in V_i^0$. *By definition of* F_i *this gives that*

$$a_i(F_i\eta, F_i\mu) = a_i(F_i\eta, F_i\mu - R_i\mu) + a_i(F_i\eta, R_i\mu) = a_i(F_i\eta, R_i\mu).$$

This also applies to the weak formulation of the transmission problem, which means that we can replace R_i *by* F_i *in (3.9).*

For the Poisson equation, we have the following properties of the Steklov–Poincaré operators.

Theorem 1. *Suppose that Assumptions 1 and 2 hold. The Steklov–Poincaré operators* $S_i, S : \Lambda \to \Lambda^*$ *are well defined and bounded, i.e.,*

$$\langle S_i \eta, \mu \rangle \leq C \|\eta\|_\Lambda \|\mu\|_\Lambda, \qquad \langle S\eta, \mu \rangle \leq C \|\eta\|_\Lambda \|\mu\|_\Lambda \quad \textit{for all } \eta, \mu \in \Lambda, i = 1, 2.$$

Moreover, they are coercive, i.e.,

$$\langle S_i \eta, \eta \rangle \geq c \|\eta\|_\Lambda^2, \qquad \langle S\eta, \eta \rangle \geq c \|\eta\|_\Lambda^2 \quad \textit{for all } \eta \in \Lambda, i = 1, 2.$$

Finally, they are symmetric, i.e.,

$$\langle S_i \eta, \mu \rangle = \langle S_i \mu, \eta \rangle, \qquad \langle S\eta, \mu \rangle = \langle S\mu, \eta \rangle \quad \textit{for all } \eta, \mu \in \Lambda, i = 1, 2.$$

For a proof of Theorem 1, see [60, Chapter 1.2]. Note that Theorem 1 together with the Lax–Milgram lemma implies that the Steklov–Poincaré equation (3.11) is uniquely solvable, although this can also be seen as a consequence of the equivalence to the weak problem (3.8).

3.3 Convergence of the Dirichlet–Neumann and Neumann–Neumann methods

The goal of this section is to show that the Dirichlet–Neumann and Neumann–Neumann methods converge. For reference, we introduce the weak formulations of the Dirichlet–Neumann and Neumann–Neumann methods. The weak form of the Dirichlet–Neumann method is to find $(u_1^n, u_2^n) \in V_1 \times V_2$ for $n = 1, 2, \ldots$, such that

$$\begin{cases} a_1(u_1^{n+1}, v_1) = (f_1, v_1)_{L^2(\Omega_1)} & \text{for all } v_1 \in V_1^0, \\ T_1 u_1^{n+1} = \eta^n, & \\ a_2(u_2^{n+1}, v_2) = (f_2, v_2)_{L^2(\Omega_2)} & \text{for all } v_2 \in V_2^0, \\ a_2(u_2^{n+1}, R_2\mu) - (f_2, R_2\mu)_{L^2(\Omega_2)} + a_1(u_1^{n+1}, R_1\mu) & \\ \quad - (f_1, R_1\mu)_{L^2(\Omega_1)} = 0 & \text{for all } \mu \in \Lambda, \end{cases} \tag{3.12}$$

with $\eta^{n+1} = s_0 T_2 u_2^{n+1} + (1 - s_0)\eta^n$. Similarly the weak form of the Neumann–Neumann method is to find $(u_1^n, u_2^n, w_1^n, w_2^n) \in V_1 \times V_2 \times V_1 \times V_2$ for $n = 1, 2, \ldots$, such that

$$\begin{cases} a_i(u_i^{n+1}, v_i) = (f_i, v_i)_{L^2(\Omega_i)} & \text{for all } v_i \in V_i^0, \\ T_i u_i^{n+1} = \eta^n & \text{for } i = 1, 2, \\ a_1(w_1^{n+1}, v_1) = 0 & \text{for all } v_1 \in V_1^0, \\ a_1(w_1^{n+1}, R_1\mu) = a_1(u_1^{n+1}, R_1\mu) - (f_1, R_1\mu)_{L^2(\Omega_1)} & \\ \quad + a_2(u_2^{n+1}, R_2\mu) - (f_2, R_2\mu)_{L^2(\Omega_2)} & \text{for all } \mu \in \Lambda \\ a_2(w_2^{n+1}, v_2) = 0 & \text{for all } v_2 \in V_2^0, \\ a_2(w_2^{n+1}, R_2\mu) = -a_1(u_1^{n+1}, R_1\mu) & \\ \quad + (f_1, R_1\mu)_{L^2(\Omega_1)} & \\ \quad - a_2(u_2^{n+1}, R_2\mu) + (f_2, R_2\mu)_{L^2(\Omega_2)} & \text{for all } \mu \in \Lambda, \end{cases} \tag{3.13}$$

with $\eta^{n+1} = \eta^n - (s_1 T_1 w_1^{n+1} - s_2 T_2 w_2^{n+1})$. It can be shown that the weak form of the Dirichlet–Neumann method is equivalent to the interface iteration given by finding $\eta^n \in \Lambda$, for $n = 1, 2, \ldots$, such that

$$\langle S_2 \eta^{n+1}, \mu \rangle = \langle S_2 \eta^n, \mu \rangle + s_0 \langle -S\eta^n + \chi, \mu \rangle \quad \text{for all } \mu \in \Lambda. \tag{3.14}$$

Similarly, the weak form of the Neumann–Neumann method is equivalent to the iteration given by finding $(\lambda_1^n, \lambda_2^n, \eta^n) \in \Lambda \times \Lambda \times \Lambda$, for $n = 1, 2, \ldots$, such that

$$\begin{cases} \langle S_i \lambda_i^{n+1}, \mu \rangle = \langle -S\eta^n + \chi, \mu \rangle & \text{for all } \mu \in \Lambda,\ i = 1, 2, \\ \eta^{n+1} = \eta^n + s_1 \lambda_1^{n+1} + s_2 \lambda_2^{n+1}, \end{cases} \tag{3.15}$$

see [60, Chapter 1.3]. It follows immediately from Theorem 1 and the Lax–Milgram lemma that both of these iterations are well defined, i.e., that each step in the iteration is uniquely solvable. Moreover, we have the following geometric convergence result for the Dirichlet–Neumann method, which is a consequence of the abstract result [60, Theorem 4.2.2] together with the properties in Theorem 1. We include a proof for comparison with the techniques used in Section 3.4, Paper I and Paper II.

Theorem 2. *Suppose that Assumptions 1 and 2 hold. Let u_i^n be the iterates of the Dirichlet–Neumann method (3.12) with initial guess $\eta^0 \in \Lambda$ and some parameter $s_0 > 0$ small enough. Moreover, let (u_1, u_2) denote the solution to (3.9) and η the solution to (3.11). Then there exists constants $L < 1$ and $C > 0$ such that*

$$\|u_i^n - u_i\|_{V_i} \leq C L^n \|\eta^0 - \eta\|_\Lambda, \quad i = 1, 2.$$

Proof. We can write the error iteration on Λ as

$$\begin{aligned} \eta^{n+1} - \eta &= \eta^n - \eta + s_0 S_2^{-1}(\chi - S\eta^n) \\ &= \eta^n - \eta + s_0 S_2^{-1}\big(\chi - S\eta - S(\eta^n - \eta)\big) \\ &= \eta^n - \eta - s_0 S_2^{-1} S(\eta^n - \eta). \end{aligned}$$

By Theorem 1 we have that

$$(\mu, \lambda)_{S_2} = \langle S_2 \mu, \lambda \rangle$$

defines an inner product, which induces a norm $\|\cdot\|_{S_2}$ that is equivalent to $\|\cdot\|_\Lambda$. The properties of Theorem 1 now yield

$$\begin{aligned} \|\eta^{n+1} - \eta\|_{S_2}^2 &= \langle S_2(\eta^{n+1} - \eta), \eta^{n+1} - \eta \rangle \\ &= \big\langle S_2\big(\eta^n - \eta - s_0 S_2^{-1} S(\eta^n - \eta)\big), \eta^n - \eta - s_0 S_2^{-1} S(\eta^n - \eta) \big\rangle \\ &= \big\langle S_2(\eta^n - \eta), \eta^n - \eta \big\rangle + s_0^2 \big\langle S(\eta^n - \eta), S_2^{-1} S(\eta^n - \eta) \big\rangle \\ &\quad - s_0 \big(\langle S(\eta^n - \eta), \eta^n - \eta \rangle + \langle S_2(\eta^n - \eta), S_2^{-1} S(\eta^n - \eta) \rangle \big) \\ &= \langle S_2(\eta^n - \eta), \eta^n - \eta \rangle + s_0^2 \langle S(\eta^n - \eta), S_2^{-1} S(\eta^n - \eta) \rangle \\ &\quad - 2 s_0 \langle S(\eta^n - \eta), \eta^n - \eta \rangle \\ &\leq (1 + C s_0^2 - c s_0) \|\eta^n - \eta\|_{S_2}^2. \end{aligned}$$

If $s_0 > 0$ is small enough then $1 + Cs_0^2 - cs_0 < 1$. By using the norm equivalence again we have the convergence

$$\|\eta^n - \eta\|_\Lambda \leq CL^n \|\eta^0 - \eta\|_\Lambda.$$

with $L = 1 + Cs_0^2 - cs_0$. The result for $i = 1$ now follows from the fact that

$$\begin{aligned}\|u_1^{n+1} - u_1\|_{V_1} &= \|(F_1\eta^n + G_1 f_1) - (F_1\eta + G_1 f_1)\|_{V_1} = \|F_1(\eta^n - \eta)\|_{V_1} \\ &\leq C\|\eta^n - \eta\|_\Lambda \leq CL^n \|\eta^0 - \eta\|_\Lambda.\end{aligned}$$

A similar computation together with the observation

$$u_2^{n+1} = \frac{1}{s_0}\left(F_2\eta^{n+1} - (1 - s_0)F_2\eta^n\right)$$

gives the convergence result for $i = 2$. □

Remark 6. *For convenience the convergence result of the Dirichlet–Neumann method, as well as all other methods in this thesis, is stated in the norms of V_i, $i = 1, 2$. This can be reformulated as convergence in the norm of $V_1 \times V_2$. This norm is in general weaker than the norm of V, but is equivalent for functions $v \in V$. If we introduce the element $v^n = (F_1\eta^n, F_2\eta^n) \in V$, we have convergence of v^n in the norm of V. Note however, that the function $F_2\eta^n$ is not present as an iterate in the Dirichlet–Neumann method.*

The constant L can be completely characterized by the boundedness and coercivity constants in Theorem 1, see [60, Remark 4.2.3]. In fact, we have that Theorem 2 holds with

$$L = 1 + s_0^2 \frac{(c_1 + c_2)^2}{C_2^2} - 2s_0 \frac{C_1 + C_2}{c_2},$$

where C_i, c_i denote the boundedness and coercivity constants of S_i, respectively. The proof also holds for some elliptic equations with non-symmetric operators, if the non-symmetric part is small enough, see [60, Chapter 5.1]. Moreover, a similar proof shows that the same result holds for the Neumann–Neumann method, see [60, Theorem 4.2.5].

Theorem 3. *Suppose that Assumptions 1 and 2 hold. Let u_i^n be the iterates of the Neumann–Neumann method (3.13) with initial guess $\eta^0 \in \Lambda$ and some parameters $s_1, s_2 > 0$ small enough. Moreover, let (u_1, u_2) denote the solution to (3.9) and η the solution to (3.11). Then there exists constants $L < 1$ and $C > 0$ such that*

$$\|u_i^n - u_i\|_{V_i} \leq CL^n \|\eta^0 - \eta\|_\Lambda, \quad i = 1, 2.$$

3.4 Convergence of the Robin–Robin method

The Robin–Robin method was first introduced and analyzed in [49] with a convergence proof showing weak convergence. Furthermore, the method has been shown to converge (strongly) in $V_1 \times V_2$ as well [60], although this proof is missing a regularity requirement which is not obvious for Lipschitz domains. Geometric convergence rate has since been derived in various contexts for the discretized version in [51, 71]. We then get a geometric convergence rate with an error reduction constant of the form $L = 1 - \mathcal{O}(\sqrt{h})$ where h denotes the mesh width, showing that the convergence rate deteriorates to one as h tends to zero. In the continuous case, geometric convergence rate is typically not achieved, note that [71] even gives a counterexample showing that geometric convergence is impossible for the continuous Robin–Robin method.

However, there are several advantages of the Robin–Robin method over the Dirichlet–Neumann and Neumann–Neumann methods. First, it will converge regardless of the choice of parameter. Second, the method can be implemented without calculating a derivative, see [17, Chapter 2.3.1]. Third, as shown in Paper I and Paper II, it is possible to prove convergence of the Robin–Robin method without relying on the symmetry of the differential operator, which is a necessary assumption to prove convergence for both the Dirichlet–Neumann and Neumann–Neumann methods using the standard analysis presented in Section 3.3.

The analysis of [49] differs from that of the Dirichlet–Neumann and Neumann–Neumann methods since no Steklov–Poincaré theory is employed. For reference, we include a result showing strong convergence in $V_1 \times V_2$. This proof closely follows [60, Chapter 4.5], but it is unclear if this is the original reference. Since the proof does not use the weak formulation of the Robin–Robin method we leave this for Chapter 4.

Theorem 4. *Suppose that Assumptions 1 and 2 hold. Let u_i^n be the iterates of the Robin–Robin method (3.7) for some $s > 0$ and let (u_1, u_2) denote the solution to (3.2). Moreover, assume that*

$$\nabla(u_i^n - u_i) \cdot \nu_i \in L^2(\Gamma) \quad \text{for } n = 1, 2, \ldots. \tag{3.16}$$

Then

$$\|u_i^n - u_i\|_{V_i} \to 0, \quad i = 1, 2,$$

as n tends to infinity.

Proof. Note first that by (3.16) we have

$$\langle \nabla(u_i^{n+1} - u_i) \cdot \nu_i, u_i^{n+1} - u_i \rangle_{\Lambda^* \times \Lambda} = \int_\Gamma (u_i^{n+1} - u_i) \nabla(u_i^{n+1} - u_i) \cdot \nu_i \, \mathrm{d}S.$$

Thus using the $H(\mathrm{div}, \Omega_i)$-version of Green's formula [31, Chapter 1, Corollary 2.1] yields

$$|u_i^{n+1} - u_i|_{V_i}^2 = \int_{\Omega_i} |\nabla(u_i^{n+1} - u_i)|^2 \,\mathrm{d}x = \langle \nabla(u_i^{n+1} - u_i) \cdot \nu_i, u_i^{n+1} - u_i \rangle_{\Lambda^* \times \Lambda}$$
$$= \int_\Gamma (u_i^{n+1} - u_i) \nabla(u_i^{n+1} - u_i) \cdot \nu_i \,\mathrm{d}S.$$

We write $e_i^n = u_i^n - u_i$ and use the identity

$$e_i^{n+1} \nabla e_i^{n+1} \cdot \nu_i = \frac{1}{4s} \big((\nabla e_i^{n+1} \cdot \nu_i + s e_i^{n+1})^2 - (\nabla e_i^{n+1} \cdot \nu_i - s e_i^{n+1})^2 \big)$$

to get

$$|e_i^{n+1}|_{V_i}^2 = \frac{1}{4s} \int_\Gamma (\nabla e_i^{n+1} \cdot \nu_i + s e_i^{n+1})^2 \,\mathrm{d}S - \frac{1}{4s} \int_\Gamma (\nabla e_i^{n+1} \cdot \nu_i - s e_i^{n+1})^2 \,\mathrm{d}S.$$

From the transmission conditions

$$\nabla e_2^{n+1} \cdot \nu_2 + s e_2^{n+1} = -\nabla e_1^{n+1} \cdot \nu_1 + s e_1^{n+1} \quad \text{and}$$
$$\nabla e_1^{n+1} \cdot \nu_1 + s e_1^{n+1} = -\nabla e_2^n \cdot \nu_2 + s e_2^n$$

on $L^2(\Gamma)$ we have

$$\begin{aligned} |e_1^{n+1}|_{V_1}^2 + |e_2^{n+1}|_{V_2}^2 &= \frac{1}{4s} \int_\Gamma (\nabla e_2^n \cdot \nu_2 - s e_2^n)^2 \,\mathrm{d}S \\ &\quad - \frac{1}{4s} \int_\Gamma (\nabla e_1^{n+1} \cdot \nu_1 - s e_1^{n+1})^2 \,\mathrm{d}S \\ &\quad + \frac{1}{4s} \int_\Gamma (\nabla e_1^{n+1} \cdot \nu_1 - s e_1^{n+1})^2 \,\mathrm{d}S \\ &\quad - \frac{1}{4s} \int_\Gamma (\nabla e_2^{n+1} \cdot \nu_2 - s e_2^{n+1})^2 \,\mathrm{d}S \\ &= \frac{1}{4s} \int_\Gamma (\nabla e_2^n \cdot \nu_2 - s e_2^n)^2 \,\mathrm{d}S \\ &\quad - \frac{1}{4s} \int_\Gamma (\nabla e_2^{n+1} \cdot \nu_2 - s e_2^{n+1})^2 \,\mathrm{d}S. \end{aligned}$$

Summing up yields

$$\sum_{n=0}^{N} \Big(|e_1^{n+1}|_{V_1}^2 + |e_2^{n+1}|_{V_2}^2 \Big) = \frac{1}{4s} \int_\Gamma (\nabla e_2^0 \cdot \nu_2 - s e_2^0)^2 \,\mathrm{d}S$$
$$- \frac{1}{4s} \int_\Gamma (\nabla e_2^{N+1} \cdot \nu_2 - s e_2^{N+1})^2 \,\mathrm{d}S,$$

which shows that the sum is bounded. Therefore

$$|e_1^{n+1}|_{V_1}^2 + |e_2^{n+1}|_{V_2}^2 \to 0,$$

as n tends to infinity. □

It is still possible to formulate the Robin–Robin method as an interface iteration and prove convergence this way, but we leave this for Chapter 4.

3.5 Domain decomposition for discretized linear elliptic equations

Since our results are based on the variational formulation of the equation we restrict ourselves to finite element methods. For the theory of finite element methods applied to linear elliptic equations we refer to [12, 65]. One can formulate the Dirichlet–Neumann, Neumann–Neumann, and Robin–Robin methods on finite element spaces and show that these also converges. For the Dirichlet–Neumann and Neumann–Neumann methods the convergence results hold with the same constant L when considering a finite element approximation, see [60, Chapters 4.3-4.4]. For the Robin–Robin method, see Section 5.1.

Chapter 4

Domain decomposition for nonlinear elliptic equations (Paper I)

In this chapter we study domain decomposition applied to nonlinear elliptic equations. For the theory of nonlinear elliptic equations that is employed here we refer to [61, 73]. While the theory of domain decomposition methods for linear elliptic equations is well understood, the corresponding theory for nonlinear equations is by no means fully developed. For overlapping domain decomposition methods there are convergence results, see for example [18, 48, 66, 67], but for nonoverlapping methods the results are very few. Some such results for a specific equation in one dimension can be found in [5, 6], but this approach does not fit the equations considered here. The aim of Paper I is therefore to generalize parts of the theory from Chapter 3 to nonlinear elliptic equations. In particular, we introduce new nonlinear Steklov–Poincaré operators and prove convergence of the Robin–Robin method for a large class of nonlinear equations. Note that we do not attempt to prove convergence of the Dirichlet–Neumann and Neumann–Neumann methods, since the theories for both of these methods require the partial differential operator to be linear and symmetric.

The equations under consideration are of the form

$$\begin{cases} -\nabla \cdot \alpha(\nabla u) + g(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

for some functions $\alpha : \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R} \to \mathbb{R}$. As in the linear case we consider

the transmission problem of (4.1), which is

$$\begin{cases} -\nabla \cdot \alpha(\nabla u_i) + g(u_i) = f_i & \text{in } \Omega_i, \\ u_i = 0 & \text{on } \partial\Omega_i \setminus \Gamma, \text{ for } i = 1, 2, \\ u_1 = u_2 & \text{on } \Gamma, \\ \alpha(\nabla u_1) \cdot \nu_1 = -\alpha(\nabla u_2) \cdot \nu_2 & \text{on } \Gamma. \end{cases} \tag{4.2}$$

Note that the second transmission condition is a nonlinear generalization of the Neumann condition from (3.2). It is the natural condition in the sense that it appears when deriving the weak formulation from the strong formulation. The transmission problem can then be approximated by the nonlinear Robin–Robin method, which is to find (u_1^n, u_2^n) for $n = 1, 2, \ldots$, such that

$$\begin{cases} -\nabla \cdot \alpha(\nabla u_1^{n+1}) + g(u_1^{n+1}) = f_1 & \text{in } \Omega_1, \\ u_1^{n+1} = 0 & \text{on } \partial\Omega_1 \setminus \Gamma, \\ \alpha(\nabla u_1^{n+1}) \cdot \nu_1 + s u_1^{n+1} = \alpha(\nabla u_2^n) \cdot \nu_1 + s u_2^n & \text{on } \Gamma, \\ -\nabla \cdot \alpha(\nabla u_2^{n+1}) + g(u_2^{n+1}) = f_2 & \text{in } \Omega_2, \\ u_2^{n+1} = 0 & \text{on } \partial\Omega_2 \setminus \Gamma, \\ \alpha(\nabla u_2^{n+1}) \cdot \nu_2 + s u_2^{n+1} = \alpha(\nabla u_1^{n+1}) \cdot \nu_2 + s u_1^{n+1} & \text{on } \Gamma. \end{cases} \tag{4.3}$$

The idea is to mimic the analysis of Chapter 3, but with added complexities. The first obvious issue is that we need an interface interpretation of the Robin–Robin method. This turns out to be the Peaceman–Rachford iteration,

$$\eta_2^{n+1} = (sI + S_2)^{-1}(sI - S_1)(sI + S_1)^{-1}(sI - S_2)\eta_2^n,$$

first introduced in [59] to approximate solutions to elliptic and parabolic differential equations. In the linear case the connection between the Robin–Robin method and Peaceman–Rachford iteration has been studied in, for example [1, 16], and has been utilized to prove convergence of the Robin–Robin method applied to a discretized equation in [51, 71]. The second problem is that the equation, and in extension the Steklov–Poincaré operators, are nonlinear. This means that it is not possible to apply the Lax–Milgram lemma. Instead, we will use the nonlinear generalization, the Browder–Minty theorem, see [73, Theorem 26.A]. After establishing the proper definitions, which differ slightly from the linear elliptic case, the difficulty lies in proving the properties required to apply the Browder–Minty theorem. The third issue is that we are not aware of a variational proof of the convergence of the Peaceman–Rachford iteration. To remedy this, we will restrict our operators to unbounded operators on $L^2(\Gamma)$ and then apply the abstract result on the convergence of the Peaceman–Rachford iteration given in [50].

We first make the following assumptions on the structure of the equation.

Assumption 3. *The functions $\alpha : \mathbb{R}^d \to \mathbb{R}^d, g : \mathbb{R} \to \mathbb{R}$ and the parameters (p, r) satisfy*

- *The parameters p and r satisfy $p \in [2, \infty)$ and $r \in (1, \infty)$. If $p < d$ then $r \leq dp/\big(2(d-p)\big) + 1$.*
- *The functions α and g are continuous and satisfy the growth conditions*

$$|\alpha(z)| \leq C|z|^{p-1} \quad \textit{and} \quad |g(x)| \leq C|x|^{r-1} \quad \textit{for all } z \in \mathbb{R}^d, x \in \mathbb{R}.$$

- *The function α is strictly monotone, i.e.,*

$$\big(\alpha(z) - \alpha(\tilde{z})\big) \cdot (z - \tilde{z}) \geq c|z - \tilde{z}|^p \quad \textit{for all } z, \tilde{z} \in \mathbb{R}^d.$$

- *The function α is coercive, i.e.,*

$$\alpha(z) \cdot z \geq c|z|^p \quad \textit{for all } z \in \mathbb{R}^d.$$

- *The function g is strictly monotone and coercive, i.e.,*

$$\big(g(x) - g(\tilde{x})\big)(x - \tilde{x}) \geq c|x - \tilde{x}|^r \quad \textit{and} \quad g(x)x \geq c|x|^r \quad \textit{for all } x, \tilde{x} \in \mathbb{R}.$$

Recall the spaces V, V_i, V_i^0, Λ defined in Section 2.2. We let $p \in [2, \infty)$ and introduce the nonlinear forms $a : V \times V \to \mathbb{R}$ and $a_i : V_i \times V_i \to \mathbb{R}$ by

$$a(u, v) = \int_\Omega \alpha(\nabla u) \cdot \nabla v + g(u)v \,\mathrm{d}x \quad \text{and}$$
$$a_i(u, v) = \int_{\Omega_i} \alpha(\nabla u) \cdot \nabla v + g(u)v \,\mathrm{d}x,$$

respectively. Note that the forms are nonlinear in the first argument, but linear in the second argument, which is fundamental for the analysis. Consider the weak formulation of (4.1), given by finding $u \in V$ such that

$$a(u, v) = (f, v)_{L^2(\Omega)} \quad \text{for all } v \in V. \tag{4.4}$$

Moreover, as in the linear case, we consider the weak formulation of the transmission problem, which is to find $(u_1, u_2) \in V_1 \times V_2$ such that

$$\begin{cases} a_i(u_i, v_i) = (f_i, v_i)_{L^2(\Omega_i)} & \text{for all } v_i \in V_i^0, i = 1, 2, \\ \quad T_1 u_1 = T_2 u_2, & \\ \sum_{i=1}^2 a_i(u_i, R_i\mu) - (f_i, R_i\mu)_{L^2(\Omega_i)} = 0 & \text{for all } \mu \in \Lambda. \end{cases} \tag{4.5}$$

The weak formulations of the equation (4.4) and the transmission problem (4.5) are equivalent, see [Paper I, Theorem 5.2]. We also introduce the weak formulation of the Robin–Robin method, which is to find $(u_1^n, u_2^n) \in V_1 \times V_2$ for $n = 1, 2, \ldots,$ such that

$$\begin{cases} a_1(u_1^{n+1}, v_1) = (f_1, v_1)_{L^2(\Omega_1)}, & \text{for all } v_1 \in V_1^0, \\ a_1(u_1^{n+1}, R_1\mu) - (f_1, R_1\mu)_{L^2(\Omega_1)} + a_2(u_2^n, R_2\mu) \\ \quad - (f_2, R_2\mu)_{L^2(\Omega_2)} = s(T_2u_2^n - T_1u_1^{n+1}, \mu)_{L^2(\Gamma)}, & \text{for all } \mu \in \Lambda, \\ a_2(u_2^{n+1}, v_2) = (f_2, v_2)_{L^2(\Omega_2)}, & \text{for all } v_2 \in V_2^0, \\ a_2(u_2^{n+1}, R_2\mu) - (f_2, R_2\mu)_{L^2(\Omega_2)} + a_1(u_1^{n+1}, R_1\mu) \\ \quad - (f_1, R_1\mu)_{L^2(\Omega_1)} = s(T_1u_1^{n+1} - T_2u_2^{n+1}, \mu)_{L^2(\Gamma)}, & \text{for all } \mu \in \Lambda. \end{cases} \tag{4.6}$$

We introduce the nonlinear solution operators $F_i : \Lambda \to V_i$ as the solution to

$$\begin{cases} a_i(u_i, v) = (f_i, v)_{L^2(\Omega_i)} & \text{for all } v \in V_i^0, \\ T_iu_i = \eta. \end{cases}$$

Note that F_i is now a nonlinear right inverse to the linear operator T_i. The nonlinear Steklov–Poincaré operators can then be defined as

$$\langle S_i\eta, \mu\rangle = a_i(F_i\eta, R_i\mu) - (f_i, R_i\mu)_{L^2(\Omega_i)} \qquad \text{and}$$
$$\langle S\eta, \mu\rangle = \sum_{i=1}^{2} \Big(a_i(F_i\eta, R_i\mu) - (f_i, R_i\mu)_{L^2(\Omega_i)} \Big).$$

Note that the definitions differ slightly from the standard definitions of the Steklov–Poincaré operators in Chapter 3. In particular, we include the influence of f_i, both by the term $(f_i, R_i\mu)_{L^2(\Omega_i)}$ and by using the full extension F_i instead of the solution operator corresponding to the homogeneous problem. This makes the notation simpler, but introduces nonlinearity to the Steklov–Poincaré operators. However, this is not an issue since the operators are already nonlinear.

We now wish to apply the nonlinear generalization of the Lax–Milgram lemma, namely the Browder–Minty theorem, see [73, Theorem 26.A]. For reference we include the statement here. We say that an operator $A : X \to X^*$ on a Banach space X is monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0$$

for all $x, y \in X$ and coercive if

$$\lim_{\|x\|_X \to \infty} \frac{\langle Ax, x\rangle}{\|x\|_X} = \infty.$$

Finally, the operator A is said to be demicontinuous if

$$\langle Ax^k - Ax, y\rangle \to 0$$

as $x^k \to x$ in X for all $y \in X$.

Theorem 5. *Let X be a reflexive Banach space and let $A : X \to X^*$ be an operator that is monotone, coercive and demicontinuous. Then for any $y \in X^*$ there exists a unique solution $x \in X$ to the equation*

$$Ax = y.$$

Demicontinuity is actually a stronger assumption than the standard Browder–Minty theorem, but it is sufficient for our purposes. It is clear that a coercive and bounded linear operator $A : X \to X^*$ satisfies the assumptions of Theorem 5 and therefore, the Browder–Minty theorem is a generalization of the Lax–Milgram lemma. We summarize the main results on the Steklov–Poincaré operators from Paper I that are necessary to apply the Browder–Minty theorem. In fact, we even have a slightly stronger monotonicity condition.

Theorem 6. *Suppose that Assumptions 1 and 3 hold. Then $S_i, S : \Lambda \to \Lambda^*$ are well defined and satisfy the monotonicity condition*

$$\langle S_i\eta - S_i\mu, \eta - \mu\rangle \geq c_i\big(\|\nabla(F_i\eta - F_i\mu)\|^p_{L^p(\Omega_i)^d} + \|F_i\eta - F_i\mu\|^r_{L^r(\Omega_i)}\big),$$

$$\langle S\eta - S\mu, \eta - \mu\rangle \geq c\sum_{i=1}^{2}\big(\|\nabla(F_i\eta - F_i\mu)\|^p_{L^p(\Omega_i)^d} + \|F_i\eta - F_i\mu\|^r_{L^r(\Omega_i)}\big)$$

for all $\eta, \mu \in \Lambda$ and $i = 1, 2$. Moreover, S_i, S are coercive and demicontinuous.

For the proof, see [Paper I, Chapter 7].

Using the Steklov–Poincaré operators we can express the Robin–Robin method as the interface iteration to find $\eta^n \in \Lambda$ for $n = 1, 2, \ldots,$ such that

$$\begin{cases} \langle (sJ + S_1)\eta_1^{n+1}, \mu\rangle = \langle (sJ - S_2)\eta_2^n, \mu\rangle & \text{for all } \mu \in \Lambda, \\ \langle (sJ + S_2)\eta_2^{n+1}, \mu\rangle = \langle (sJ - S_1)\eta_1^{n+1}, \mu\rangle & \text{for all } \mu \in \Lambda. \end{cases} \tag{4.7}$$

As already observed, this iteration is known as the Peaceman–Rachford iteration. Here,

$$J : L^2(\Gamma) \to L^2(\Gamma)^* : \eta \mapsto (\eta, \cdot)_{L^2(\Gamma)}$$

denotes the Riesz isomorphism. A more detailed explanation with a proof can be found in [Paper I, Lemma 6.3]. As observed before, the remaining issue is that even

with the properties of Theorem 6 there is no variational convergence proof of the Peaceman–Rachford iteration. We must therefore restrict the Steklov–Poincaré operators to unbounded operators on $L^2(\Gamma)$ before we can apply the abstract result [50] on the convergence of the Peaceman–Rachford iteration. A detailed proof can be found in [Paper I, Theorem 8.9]. Note that we require the same type of regularity assumption as in the linear case, see Theorem 4.

Assumption 4. *Let $u \in V$ denote the solution to (4.4). Then the functionals*

$$\mu \mapsto a_i(u|_{\Omega_i}, R_i\mu), \quad i = 1, 2,$$

are elements in $L^2(\Gamma)^*$.

For examples that satisfy both Assumptions 3 and 4, see [Paper I, Example 2.7].

Theorem 7. *Suppose that Assumptions 1, 3 and 4 hold. Let u_i^n be the iterates of the nonlinear Robin–Robin method (4.6) for some $s > 0$ and let $(u_1, u_2) \in V_1 \times V_2$ denote the solution to the weak transmission problem (4.4). Then*

$$\|u_i^n - u_i\|_{V_i} \to 0, \quad i = 1, 2,$$

as n tends to infinity.

Chapter 5

Numerical results for nonlinear elliptic equations

5.1 Discretization of nonlinear elliptic domain decomposition methods

For convergence of finite element methods for nonlinear equations, see for example [3, 4, 11, 15, 19]. In this section we consider some Galerkin approximation of (4.4). That is, a finite dimensional subspace $V_h \subset V$ and the weak discrete equation to find $u^h \in V_h$ such that

$$a(u^h, v^h) = \langle f, v^h \rangle \quad \text{for all } v^h \in V_h. \tag{5.1}$$

Notice that we do not assume that V_h arises from a finite element approximation, although the notation suggests it, with h denoting the mesh width. We introduce the decomposed finite dimensional spaces V_h^1, V_h^2 and assume that $V_h \subset V_h^1 \times V_h^2$. Furthermore, we assume that there is a common finite dimensional trace space, which we denote by

$$\Lambda_h = T_1(V_h^1) = T_2(V_h^2).$$

Note that all of these assumptions hold for a finite element discretization where the interface is contained in the edges of the mesh, see for example Figure 5.1. Moreover, let $F_i^h : \Lambda_h \to V_i^h$ be the discrete solution operator. Then we can introduce the discrete Steklov–Poincaré operators $S_i^h : \Lambda_h \to \Lambda_h^*$ as

$$\langle S_i^h \eta^h, \mu^h \rangle = a_i(F_i^h \eta^h, R_i^h \mu^h) - (f_i, R_i^h \mu^h)_{L^2(\Omega_i)},$$

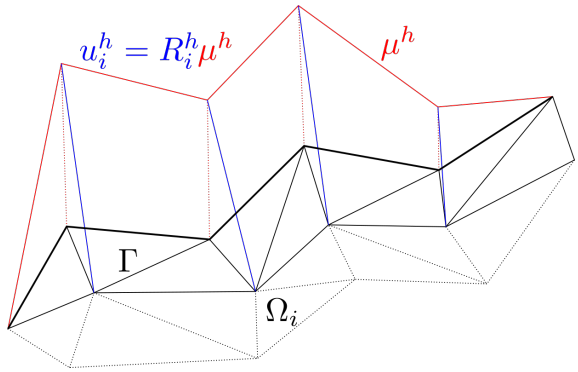


Figure 5.1: The natural extension $u_i^h = R_i^h \mu^h$ on Ω_i of the discrete function μ^h on Γ. The function μ^h is shown in red and the extension u_i^h in red and blue.

and $S^h = S_1^h + S_2^h$. Here, $R_i^h : \Lambda_h \to V_i^h$ can be taken to be any linear extension operator, but in an implementation it is convenient to use the extension such that $R_i^h \mu^h = 0$ on the interior degrees of freedom, see Figure 5.1.

The operators S_i^h have similar properties to the continuous Steklov–Poincaré operators.

Theorem 8. *Suppose that Assumptions 1 and 3 hold. Then the operators $S_i^h, S^h : \Lambda_h \to \Lambda_h^*$ are well defined and satisfy the same monotonicity condition, coercivity condition and continuity condition as S_i, S in Theorem 6.*

Although not specified here, the discrete nonlinear Robin–Robin method can be defined by replacing the test and trial spaces in (4.6) with the discrete variants. The same argument as in Paper I now gives the following convergence result.

Theorem 9. *Suppose that Assumptions 1 and 3 hold and that Assumption 4 holds with u replaced by $u^h \in V^h$, the solution to the discrete weak equation (5.1). Moreover, let $u_i^h = u^h|_{\Omega_i} \in V_i^h$, $i = 1, 2$, and $u_i^{h,n}$ be the iterates of the discrete nonlinear Robin–Robin method. Then $(u_1^{h,n}, u_2^{h,n})$ converges to (u_1^h, u_2^h) in $V_1^h \times V_2^h$, i.e.,*

$$\|u_i^{h,n} - u_i^h\|_{V_i} \to 0, \qquad i = 1, 2,$$

as n tends to infinity.

Remark 7. *It is evident that if one chooses a piecewise linear finite element discretization and the function g in (4.1) is continuous, then the generalized normal derivative is piecewise continuous and therefore Assumption 4 is satisfied.*

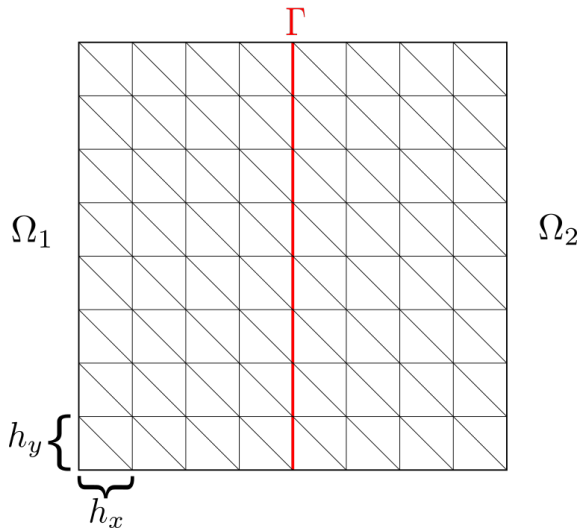


Figure 5.2: Domain decomposition applied to a uniform spatial finite element grid.

5.2 Numerical results

In order to illustrate the convergence results of Chapter 4 and Section 5.1, we test the Robin–Robin method applied to the problem

$$\begin{cases} -\nabla \cdot (|\nabla u|^{p-2}\nabla u) + u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

where f is chosen such that

$$u(x, y) = (x - x^2)(y - y^2)$$

solves (5.2). We use the domain $\Omega = (0, 1)^2$ and the decomposition

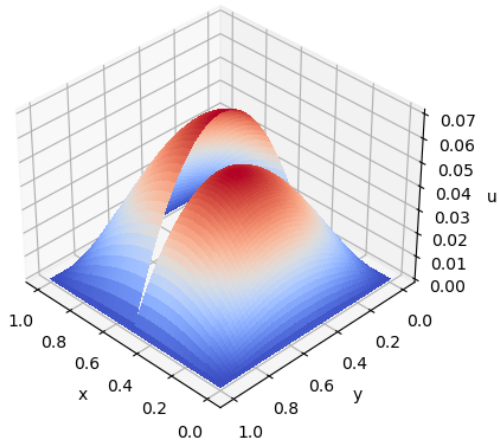
$$\Omega_1 = (0, 1/2) \times (0, 1), \quad \Omega_2 = (1/2, 1) \times (0, 1), \quad \text{and} \quad \Gamma = \{1/2\} \times (0, 1).$$

We discretize using piecewise linear finite elements using a uniform mesh of the form in Figure 5.2, with mesh size $h_x = h_y = 1/512$. We use three different values of $p = 2, 3, 6$ with the parameters $s = 4.84, 1.12, 0.006$, respectively. The parameters have been chosen by trial and error to achieve near optimal convergence rates. In Figure 5.3 we plot the first two iterations of the Robin–Robin method for $p = 3$. We also plot the relative error

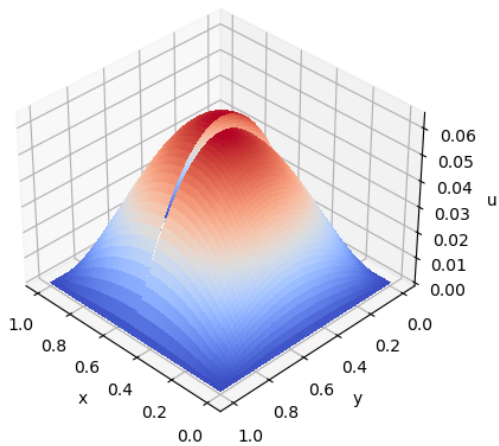
$$e_{\text{rel}} = \frac{\|u_1^n - u\|_{W^{1,p}(\Omega_1)} + \|u_2^n - u\|_{W^{1,p}(\Omega_2)}}{\|u\|_{W^{1,p}(\Omega_1)} + \|u\|_{W^{1,p}(\Omega_2)}}$$

at each iteration in Figure 5.4. In Figure 5.4a we plot the relative error with u being the solution to the discrete problem (5.1) and in Figure 5.4b we plot the relative error with u being the exact solution interpolated to a finer mesh with $h_x = h_y = 1/2048$. For reference we also plot three horizontal lines corresponding to the errors of the finite element solutions on the whole of Ω.

From Figure 5.4a we see that the discrete Robin–Robin method seems to converge to the solution to (5.1). From Figure 5.4b we see that after a certain amount of iterations the error is dominated by the error of the spatial discretization and therefore we do not see any error reduction after that point. Moreover, we find from both Figures 5.4a and 5.4b that the speed of convergence degrades as the nonlinearity becomes stronger, i.e., as the parameter p increases.

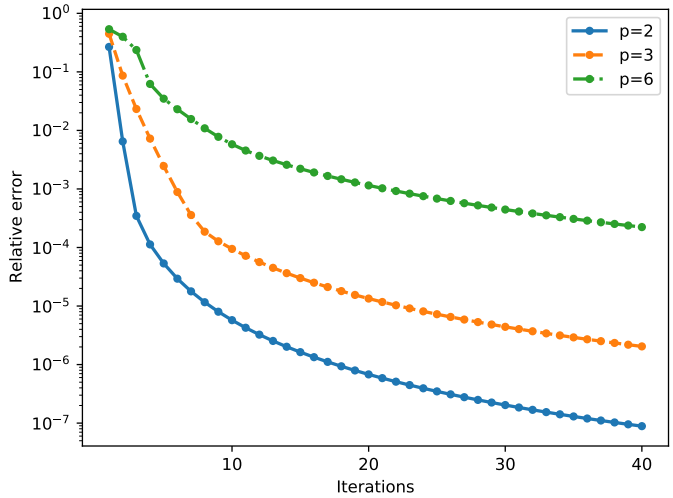


(a)

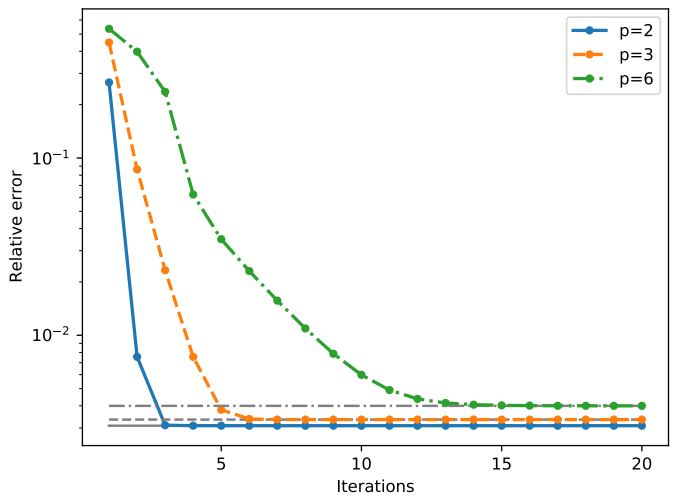


(b)

Figure 5.3: The first (a) and second (b) iterations of the nonlinear Robin–Robin method with $p = 3$.



(a)



(b)

Figure 5.4: The relative errors of the nonlinear Robin–Robin method compared to: (a) the finite element solution; (b) the exact solution. The dotted lines in (b) indicates the spatial discretization errors.

Chapter 6

Domain decomposition for linear parabolic equations (Paper II)

Domain decomposition methods have a long history of being applied to parabolic equation. These space-time decomposition methods have been studied both in the context of parallel time integrators, see [24, Chapter 3] for an overview, and for domains with different material properties in the different subdomains, see [2, 35]. However, there has hardly been any general convergence results. Instead, the standard proofs rely on the Fourier transform, but this method has only been applied in rectangular domains [25, 27, 28, 30, 43, 44]. One exception is [34], which uses a method-specific analysis similar to [49], see also Section 3.4. Notice that if an implicit Euler time-stepping scheme is used for the parabolic equation then one attains an elliptic equation at each time-step and the convergence results of Chapter 3 apply. However, this only gives pointwise convergence in time. The aim of this chapter, which is a summary of Paper II is therefore to provide a Steklov–Poincaré theory for the heat equation and prove convergence of the Robin–Robin method in this context.

Remark 8. *The space-time decomposition methods are related to the waveform relaxation methods originating in circuit simulations, see [24, 42]. These are methods to solve ordinary differential equations, but when applied to a spatially discretized partial differential equation they can be interpreted as space-time decomposition methods. Note that many authors still use the term waveform relaxation synonymously to space-time decomposition.*

We consider a decomposition of the spatial domain Ω as in Section 2.1. This induces a space-time decomposition of the form

$$\overline{\Omega} \times \mathbb{R} = (\overline{\Omega}_1 \times \mathbb{R}) \cup (\overline{\Omega}_2 \times \mathbb{R}).$$

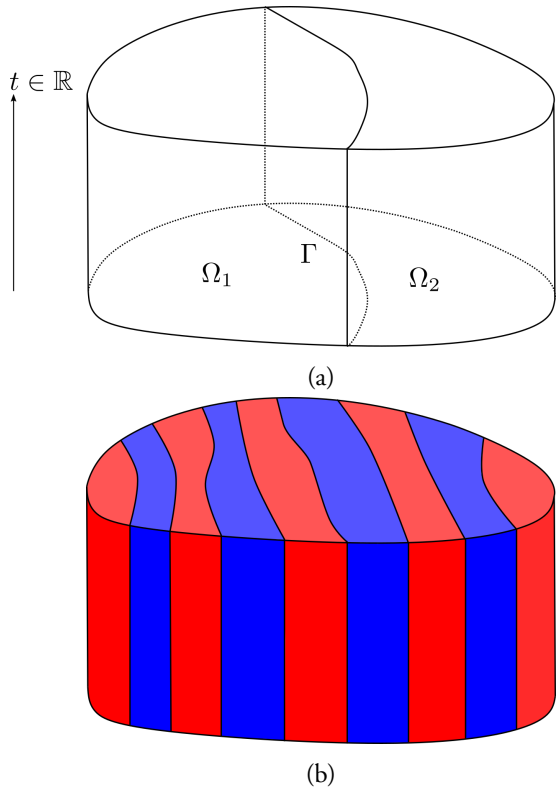


Figure 6.1: Examples of space-time decompositions of $\Omega \times \mathbb{R}$ with: (a) two subdomains; (b) multiple subdomains without crosspoints.

In Figure 6.1 we see two examples of such space-time decompositions. For notational purposes, we focus on decompositions of the form in Figure 6.1a, but as for the elliptic case, the analysis applies to decompositions of the form in Figure 6.1b as well, compare with Section 2.1. In the same way as for elliptic equations, this enables parallel implementations of the space-time domain decomposition methods.

For a general introduction to analysis of parabolic equations, see [20, 40]. In this thesis we consider variational space-time methods for parabolic equations, see [46] for an overview. The theory developed in [13] is also applied heavily in Paper II.

We restrict our analysis to the heat equation with homogeneous Dirichlet boundary conditions

$$\begin{cases} (\partial_t - \Delta)u = f & \text{in } \Omega \times \mathbb{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}. \end{cases} \tag{6.1}$$

Note that we consider the equation on the entire space-time cylinder $\Omega \times \mathbb{R}$. However,

if one chooses a source term f that is zero for times $t < 0$, then this corresponds to a homogeneous initial condition at $t = 0$. As for elliptic equations we can rewrite this as a transmission problem

$$\begin{cases} (\partial_t - \Delta)u_i = f_i & \text{in } \Omega_i \times \mathbb{R} \quad \text{for } i = 1, 2, \\ u_i = 0 & \text{on } (\partial\Omega_i \setminus \Gamma) \times \mathbb{R} \quad \text{for } i = 1, 2, \\ u_1 = u_2 & \text{on } \Gamma \times \mathbb{R}, \\ \nabla u_1 \cdot \nu_1 = -\nabla u_2 \cdot \nu_2 & \text{on } \Gamma \times \mathbb{R}, \end{cases} \tag{6.2}$$

and use this as a basis for constructing domain decomposition methods. In order to approximate the solution to (6.2) we consider the same three domain decomposition methods as for the elliptic equations in Chapter 3. Since the space-time methods differ in the equation being solved they are described here. The space-time Dirichlet–Neumann method is to find (u_1^n, u_2^n) for $n = 1, 2, \ldots$, such that

$$\begin{cases} (\partial_t - \Delta)u_1^{n+1} = f_1 & \text{in } \Omega_1 \times \mathbb{R}, \\ u_1^{n+1} = 0 & \text{on } (\partial\Omega_1 \setminus \Gamma) \times \mathbb{R}, \\ u_1^{n+1} = \eta^n & \text{on } \Gamma \times \mathbb{R}, \\ (\partial_t - \Delta)u_2^{n+1} = f_2 & \text{in } \Omega_2 \times \mathbb{R}, \\ u_2^{n+1} = 0 & \text{on } (\partial\Omega_2 \setminus \Gamma) \times \mathbb{R}, \\ \nabla u_2^{n+1} \cdot \nu_2 = -\nabla u_1^{n+1} \cdot \nu_1 & \text{on } \Gamma \times \mathbb{R}, \end{cases} \tag{6.3}$$

with $\eta^{n+1} = s_0 \, u_2^{n+1}\big|_{\Gamma\times\mathbb{R}} + (1 - s_0)\eta^n$. Here $s_0 > 0$ is a method parameter and η^0 an initial guess. Similarly, the space-time Neumann–Neumann method is to find $(u_1^n, u_2^n, w_1^n, w_2^n)$ for $n = 1, 2, \ldots$, such that

$$\begin{cases} (\partial_t - \Delta)u_i^{n+1} = f_i & \text{in } \Omega_i \times \mathbb{R}, \\ u_i^{n+1} = 0 & \text{on } (\partial\Omega_i \setminus \Gamma) \times \mathbb{R}, \\ u_i^{n+1} = \eta^n & \text{on } \Gamma \times \mathbb{R}, \text{ for } i = 1, 2, \\ (\partial_t - \Delta)w_i^{n+1} = f_i & \text{in } \Omega_i \times \mathbb{R}, \\ w_i^{n+1} = 0 & \text{on } (\partial\Omega_i \setminus \Gamma) \times \mathbb{R}, \\ \nabla w_i^{n+1} \cdot \nu_1 = \nabla u_1^{n+1} \cdot \nu_1 - \nabla u_2^{n+1} \cdot \nu_1 & \text{on } \Gamma \times \mathbb{R}, \end{cases} \tag{6.4}$$

with $\eta^{n+1} = \eta^n - (s_1 \, w_1^{n+1}\big|_{\Gamma\times\mathbb{R}} - s_2 \, w_2^{n+1}\big|_{\Gamma\times\mathbb{R}})$. Here $s_1, s_2 > 0$ are method parameters and η^0 an initial guess. Finally we have the space-time Robin–Robin

method, which is to find (u_1^n, u_2^n) for $n = 1, 2, \ldots$, such that such that

$$\begin{cases} (\partial_t - \Delta)u_1^{n+1} = f_1 & \text{in } \Omega_1 \times \mathbb{R}, \\ u_1^{n+1} = 0 & \text{on } (\partial\Omega_1 \setminus \Gamma) \times \mathbb{R}, \\ \nabla u_1^{n+1} \cdot \nu_1 + s u_1^{n+1} = \nabla u_2^n \cdot \nu_1 + s u_2^n & \text{on } \Gamma \times \mathbb{R}, \\ (\partial_t - \Delta)u_2^{n+1} = f_2 & \text{in } \Omega_2 \times \mathbb{R}, \\ u_2^{n+1} = 0 & \text{on } (\partial\Omega_2 \setminus \Gamma) \times \mathbb{R}, \\ \nabla u_2^{n+1} \cdot \nu_2 + s u_2^{n+1} = \nabla u_1^{n+1} \cdot \nu_2 + s u_1^{n+1} & \text{on } \Gamma \times \mathbb{R}. \end{cases} \tag{6.5}$$

In Paper II we show that these three methods are well defined, i.e., that each step in the iteration is uniquely solvable. We also show that the Robin–Robin method converges to the solution to the transmission problem (6.2) and therefore also to the solution to the heat equation (6.1). Here we state the main results and give a brief overview of the analysis. The basic idea is the same as in the linear elliptic case, but there are a few difficulties. First, we must have a variational formulation of the heat equation (6.1). The standard approach for variational methods is to use the solution space

$$H^1(\mathbb{R}, H^{-1}(\Omega)) \cap L^2(\mathbb{R}, H^1(\Omega)),$$

but this space is problematic since it is not possible to "glue" such functions together, see [13, Example 2.14]. This is solved by instead employing the framework with solutions in

$$H^{1/2}(\mathbb{R}, L^2(\Omega)) \cap L^2(\mathbb{R}, H^1(\Omega)),$$

which is weaker than the standard framework, see [13, Equation (2.3)]. This setting was, e.g., introduced in [46] and has been applied for both boundary element methods [13] and space-time finite elements [21, 39, 64]. The second problem that arises is that the bilinear forms are not coercive in the full norm, but only in the $L^2(\mathbb{R}, H^1(\Omega))$-norm. By extension, the Steklov–Poincaré operators are only coercive in $L^2(\mathbb{R}, \Lambda)$. To solve this, we use an idea that originates from [21], which is to apply a transformation on the test space that turns the bilinear forms coercive. We apply this idea not only on the bilinear form, but also on the Steklov–Poincaré operators, which has not been done before. Note that the space-time Steklov–Poincaré operators have been introduced without analysis in [34] and for more general parabolic problems in [36]. The third problem is that the Steklov–Poincaré operators are not symmetric due to the skew-symmetry of ∂_t. Therefore, we put the emphasis on the Robin–Robin method, since we have seen that this convergence theory does not require any symmetry assumptions. However, note that even without symmetry we can show that the Dirichlet–Neumann and the Neumann–Neumann methods are well defined.

Recall the spaces W, W_i, W_i^0 and Z defined in Section 2.2. For the analysis we introduce the bilinear forms $a : W \times W \to \mathbb{R}$ and $a_i : W_i \times W_i \to \mathbb{R}$ as

$$a(u,v) = \int_{\mathbb{R}} \int_{\Omega} \partial_t uv + \nabla u \cdot \nabla v \,\mathrm{d}x \,\mathrm{d}t \qquad \text{and}$$
$$a_i(u_i,v_i) = \int_{\mathbb{R}} \int_{\Omega_i} \partial_t u_i v_i + \nabla u_i \cdot \nabla v_i \,\mathrm{d}x \,\mathrm{d}t,$$

respectively. Note that the above definition for a is only valid for

$$u \in H^1\big(\mathbb{R}, L^2(\Omega)\big) \cap L^2\big(\mathbb{R}, H^1(\Omega)\big)$$

and similarly for a_i, but these bilinear forms can be extended to $u \in W$ and $u_i \in W_i$ respectively [13, Lemma 2.6]. Alternatively, they can be defined equivalently using fractional derivatives as in [Paper II, Section 4]. The bilinear forms a_i satisfy the following bounds

$$a_i(u_i,v_i) \leq C\|u_i\|_{W_i}\|v_i\|_{W_i} \qquad \text{for all } u_i, v_i \in W_i, \text{ and}$$
$$a(u_i,u_i) \geq c\|u_i\|^2_{L^2(\mathbb{R},H^1(\Omega_i))} \qquad \text{for all } u_i \in W_i,$$

see [Paper II, Lemma 8] for the proof. Notice that the coercivity bound is in a weaker norm and therefore it is not possible to apply the Lax–Milgram lemma as in Chapter 3. This can be resolved by applying a Hilbert transform, see [21]. We leave out this step here since we will demonstrate this technique for the Steklov–Poincaré operator later. Let $f \in L^2(\Omega \times \mathbb{R})$ and define $f_i = f|_{\Omega_i \times \mathbb{R}} \in L^2(\Omega_i \times \mathbb{R})$, $i = 1, 2$. The weak formulation of the heat equation (6.1) is to find $u \in W$ such that

$$a(u,v) = (f,v)_{L^2(\Omega\times\mathbb{R})} \qquad \text{for all } v \in W \tag{6.6}$$

and the weak formulation of the transmission problem is to find $(u_1, u_2) \in W_1 \times W_2$ such that

$$\begin{cases} a_i(u_i,v_i) = (f_i,v_i)_{L^2(\Omega_i\times\mathbb{R})} & \text{for all } v_i \in W_i^0,\ i = 1,2, \\ \quad T_1u_1 = T_2u_2, & \\ \sum_{i=1}^2 a_i(u_i,F_i\mu) - (f_i,F_i\mu)_{L^2(\Omega_i\times\mathbb{R})} = 0 & \text{for all } \mu \in Z. \end{cases} \tag{6.7}$$

Here $F_i : Z \to W_i$ is the solution operator of the homogeneous equation, i.e., $u_i = F_i\eta$ solves

$$\begin{cases} a_i(u_i,v) = 0 & \text{for all } v \in W_i^0, \\ \quad T_iu_i = \eta. & \end{cases}$$

Similarly, let $G_i : (W_i^0)^* \to W_i^0$ be the operator mapping f_i to $u_i = G_if_i$ solving

$$a_i(u_i,v) = (f_i,v)_{L^2(\Omega_i\times\mathbb{R})} \qquad \text{for all } v \in W_i^0. \tag{6.8}$$

As in the linear case the weak formulations of the heat equation (6.6) and the transmission problem (6.7) are equivalent, see [Paper II, Lemma 11].

Remark 9. *In the weak formulation of the transmission problem, and the following weak formulations, we use the extension F_i in the second argument of the bilinear form a_i instead of the extension R_i, which was used in Chapters 3 and 4. We remark that the definitions do not depend on the extension, see [Paper II, Remark 3]. Also, compare with Remark 5.*

The weak form of the space-time Dirichlet–Neumann method is to find $(u_1^n, u_2^n) \in W_1 \times W_2$ for $n = 1, 2, \ldots$, such that

$$\begin{cases} a_1(u_1^{n+1}, v_1) = (f_1, v_1)_{L^2(\Omega_1 \times \mathbb{R})} & \text{for all } v_1 \in W_1^0, \\ T_1 u_1^{n+1} = \eta^n, & \\ a_2(u_2^{n+1}, v_2) = (f_2, v_2)_{L^2(\Omega_2 \times \mathbb{R})} & \text{for all } v_2 \in W_2^0, \\ a_2(u_2^{n+1}, F_2\mu) - (f_2, F_2\mu)_{L^2(\Omega_2 \times \mathbb{R})} & \\ \quad + a_1(u_1^{n+1}, F_1\mu) - (f_1, F_1\mu)_{L^2(\Omega_1 \times \mathbb{R})} = 0 & \text{for all } \mu \in Z, \end{cases} \tag{6.9}$$

with $\eta^{n+1} = s_0 T_2 u_2^{n+1} + (1 - s_0)\eta^n$. Similarly, the weak form of the space-time Neumann–Neumann method is to find $(u_1^n, u_2^n, w_1^n, w_2^n) \in W_1 \times W_2 \times W_1 \times W_2$ for $n = 1, 2, \ldots$, such that

$$\begin{cases} a_i(u_i^{n+1}, v_i) = (f_i, v_i)_{L^2(\Omega_i \times \mathbb{R})} & \text{for all } v_i \in W_i^0, \\ T_i u_i^{n+1} = \eta^n & \text{for } i = 1, 2, \\ a_1(w_1^{n+1}, v_1) = 0 & \text{for all } v_1 \in W_1^0, \\ a_1(w_1^{n+1}, F_1\mu) = a_1(u_1^{n+1}, F_1\mu) & \\ \quad - (f_1, F_1\mu)_{L^2(\Omega_1 \times \mathbb{R})} & \\ \quad + a_2(u_2^{n+1}, F_2\mu) - (f_2, F_2\mu)_{L^2(\Omega_2 \times \mathbb{R})} & \text{for all } \mu \in Z \\ a_2(w_2^{n+1}, v_2) = 0 & \text{for all } v_2 \in W_2^0, \\ a_2(w_2^{n+1}, F_2\mu) = -a_1(u_1^{n+1}, F_1\mu) & \\ \quad + (f_1, F_1\mu)_{L^2(\Omega_1 \times \mathbb{R})} & \\ \quad - a_2(u_2^{n+1}, F_2\mu) + (f_2, F_2\mu)_{L^2(\Omega_2 \times \mathbb{R})} & \text{for all } \mu \in Z, \end{cases} \tag{6.10}$$

with $\eta^{n+1} = \eta^n - (s_1 T_1 w_1^{n+1} - s_2 T_2 w_2^{n+1})$. Finally, the weak form of the space-time Robin–Robin method is to find $(u_1^n, u_2^n) \in W_1 \times W_2$ for $n = 1, 2, \ldots$, such

that

$$\begin{cases} a_1(u_1^{n+1}, v_1) = (f_1, v_1)_{L^2(\Omega_1\times\mathbb{R})} & \text{for all } v_1 \in W_1^0, \\ a_1(u_1^{n+1}, F_1\mu) - (f_1, F_1\mu)_{L^2(\Omega_1\times\mathbb{R})} & \\ \quad + a_2(u_2^n, F_2\mu) - (f_2, F_2\mu)_{L^2(\Omega_2\times\mathbb{R})} & \\ \quad = s(T_2u_2^n - T_1u_1^{n+1}, \mu)_{L^2(\Gamma\times\mathbb{R})} & \text{for all } \mu \in Z, \\ a_2(u_2^{n+1}, v_2) = (f_2, v_2)_{L^2(\Omega_2\times\mathbb{R})} & \text{for all } v_2 \in W_2^0, \\ a_2(u_2^{n+1}, F_2\mu) - (f_2, F_2\mu)_{L^2(\Omega_2\times\mathbb{R})} & \\ \quad + a_1(u_1^{n+1}, F_1\mu) - (f_1, F_1\mu)_{L^2(\Omega_1\times\mathbb{R})} & \\ \quad = s(T_1u_1^{n+1} - T_2u_2^{n+1}, \mu)_{L^2(\Gamma\times\mathbb{R})} & \text{for all } \mu \in Z. \end{cases} \tag{6.11}$$

We now define the space-time Steklov–Poincaré operators $S_i : Z \to Z^*$ and the functionals $\chi_i \in Z^*$ as

$$\langle S_i\eta, \mu\rangle = a_i(F_i\eta, F_i\mu) \quad \text{and} \quad \langle \chi_i, \mu\rangle = (f_i, F_i\mu)_{L^2(\Omega_i\times\mathbb{R})} - a_i(G_if_i, F_i\mu).$$

Moreover we introduce $S = S_1 + S_2$ and $\chi = \chi_1 + \chi_2$. The space-time Steklov–Poincaré equations is then to find $\eta \in Z$ such that

$$\langle S\eta, \mu\rangle = \langle \chi, \mu\rangle \quad \text{for all } \mu \in Z, \tag{6.12}$$

compare with (3.11). As in the elliptic case, the weak formulation of the transmission problem (6.7) and the Steklov–Poincaré equation (6.12) are equivalent, see [Paper II, Lemma 13]. Moreover, the space-time domain decomposition methods can also be reformulated entirely on the space-time interface $\Gamma \times \mathbb{R}$. The Dirichlet–Neumann method becomes the preconditioned iteration to find $\eta^n \in Z$, for $n = 1, 2, \ldots$, such that

$$\langle S_2\eta^{n+1}, \mu\rangle = \langle S_2\eta^n, \mu\rangle + s_0\langle -S\eta^n + \chi, \mu\rangle \quad \text{for all } \mu \in Z. \tag{6.13}$$

Similarly, the Neumann–Neumann method is equivalent to the iteration given by finding $(\lambda_1^n, \lambda_2^n, \eta^n) \in Z \times Z \times Z$, for $n = 1, 2, \ldots$, such that

$$\begin{cases} \langle S_i\lambda_i^{n+1}, \mu\rangle = \langle -S\eta^n + \chi, \mu\rangle & \text{for all } \mu \in Z,\ i = 1, 2, \\ \eta^{n+1} = \eta^n + s_1\lambda_1^{n+1} + s_2\lambda_2^{n+1}. & \end{cases} \tag{6.14}$$

The Dirichlet–Neumann and Neumann–Neumann methods should be compared with the linear elliptic case in Section 3.3. Finally, the Robin–Robin method becomes the Peaceman–Rachford iteration, compare with (4.7). That is, to find $(\eta_1^n, \eta_2^n) \in Z \times Z$, for $n = 1, 2, \ldots$, such that

$$\begin{cases} \langle (sJ + S_1)\eta_1^{n+1} - \chi_1, \mu\rangle = \langle (sJ - S_2)\eta_2^n + \chi_2, \mu\rangle & \text{for all } \mu \in Z, \\ \langle (sJ + S_2)\eta_2^{n+1} - \chi_2, \mu\rangle = \langle (sJ - S_1)\eta_1^{n+1} + \chi_1, \mu\rangle & \text{for all } \mu \in Z, \end{cases} \tag{6.15}$$

with $J : L^2(\Gamma \times \mathbb{R}) \to L^2(\Gamma \times \mathbb{R})^*$ denoting the Riesz isomorphism. We now state the main result on the properties of the space-time Steklov–Poincaré operators. Note that they are coercive only in the norm $L^2(\mathbb{R}, \Lambda)$, which does not allow the usage of the Lax–Milgram lemma. Recall, however, that the Hilbert transform defined in Section 2.2 is an isomorphism $\mathcal{H}_\Gamma : Z \to Z$. By defining

$$\mathcal{H}_\Gamma^\varphi = \cos(\varphi) I - \sin(\varphi) \mathcal{H}_\Gamma,$$

we get a new bilinear form

$$(\eta, \mu) \mapsto \langle S_i \eta, \mathcal{H}_\Gamma^\varphi \mu \rangle$$

that is coercive in Z. This is one of the key ideas of Paper II, as it allows the usage of the Lax–Milgram lemma.

Theorem 10. *Suppose that Assumptions 1 and 2 hold. The Steklov–Poincaré operators $S_i, S : Z \to Z^*$ are well defined and bounded, i.e.,*

$$\langle S_i \eta, \mu \rangle \leq C \|\eta\|_Z \|\mu\|_Z, \quad \langle S\eta, \mu \rangle \leq C \|\eta\|_Z \|\mu\|_Z \quad \text{for all } \eta, \mu \in Z, \quad i = 1, 2.$$

Moreover, they are coercive in the norm $L^2(\mathbb{R}, \Lambda)$, i.e.,

$$\langle S_i \eta, \eta \rangle \geq c \|\eta\|^2_{L^2(\mathbb{R}, \Lambda)}, \quad \langle S\eta, \eta \rangle \geq c \|\eta\|^2_{L^2(\mathbb{R}, \Lambda)} \quad \text{for all } \eta \in Z, \quad i = 1, 2.$$

Finally, they are coercive-equivalent in Z, i.e.,

$$\langle S_i \eta, \mathcal{H}_\Gamma^\varphi \eta \rangle \geq c \|\eta\|^2_Z, \quad \langle S\eta, \mathcal{H}_\Gamma^\varphi \eta \rangle \geq c \|\eta\|^2_Z \quad \text{for all } \eta \in Z, \quad i = 1, 2$$

for $\varphi > 0$ small enough.

The proof can be found in [Paper II, Lemma 12]. The next result follows directly from Theorem 10, see [Paper II, Corollary 2].

Theorem 11. *The Dirichlet–Neumann, Neumann–Neumann, and Robin–Robin methods are well defined, in the sense that for each $n = 1, 2, \ldots$, the iterations (6.9) to (6.11) have unique solutions.*

To prove convergence of the Robin–Robin method we restrict the Steklov–Poincaré operators to unbounded operators on $L^2(\Gamma \times \mathbb{R})$. Then we can apply the abstract theorem of [50] on the convergence of the Peaceman–Rachford iteration to get the following result, see [Paper II, Theorem 2] for the full proof. As in the linear and nonlinear elliptic cases we require the following assumption to be able to perform the analysis in $L^2(\Gamma \times \mathbb{R})$.

Assumption 5. *Let $u \in W$ denote the solution to (6.6). Then the functionals*

$$\mu \mapsto a_i(u|_{\Omega_i \times \mathbb{R}}, F_i\mu) - (f_i, F_i\mu)_{L^2(\Omega_i \times \mathbb{R})}, \quad i = 1, 2,$$

are elements in $L^2(\Gamma \times \mathbb{R})^*$.

Theorem 12. *Suppose that Assumptions 1, 2 and 5 hold. Let u_i^n be the iterates of the Robin–Robin method (6.11) for some $s > 0$ and let u_i denote the solution to (6.7). Then*

$$\|u_i^n - u_i\|_{L^2(\mathbb{R}, H^1(\Omega_i))} \to 0, \quad i = 1, 2,$$

as n tends to infinity.

Chapter 7

Numerical results for linear parabolic equations

There is a vast amount of methods for discretizing (6.1). One standard approach is to use a finite element method in space and a time-stepping scheme, for an overview see [40]. We illustrate Theorem 12 by considering the space-time Robin–Robin method applied to the heat equation (6.1) with $d = 2$ and f chosen such that

$$\begin{cases} u(t,x,y) = 0 & \text{for } t \leq 0, \\ u(t,x,y) = \sin(\pi t/2)\sin(\pi x)\sin(\pi y) & \text{for } 0 < t < 1, \\ u(t,x,y) = e^{2(1-t)\pi^2}\sin(\pi x)\sin(\pi y) & \text{for } 1 \leq t \end{cases} \tag{7.1}$$

is the solution. Since u is smooth in space and exponentially decreasing in time, it is clear that u satisfies Assumption 5. We use the spatial domain $\Omega = (0,1)^2$ and the decomposition

$$\Omega_1 = (0,1/2) \times (0,1), \quad \Omega_2 = (1/2,1) \times (0,1), \quad \text{and} \quad \Gamma = \{1/2\} \times (0,1).$$

We discretize using a piecewise linear finite elements in space and the implicit Euler method in time with final time $T = 1$. The spatial mesh is of the form given in Figure 5.2 with mesh size $h_x = h_y = 1/128$ and the time step size is $h_t = 1/128$. We compare the Robin–Robin method with the Dirichlet–Neumann and Neumann–Neumann methods. The parameters are chosen as $s_0 = 0.5, s_1 = 0.25, s_2 = 0.25$. For the Robin–Robin method we use $s = 0.25$ and $s = 0.03$ depending on the test case explained below. All parameters have been found to be near optimal. For all

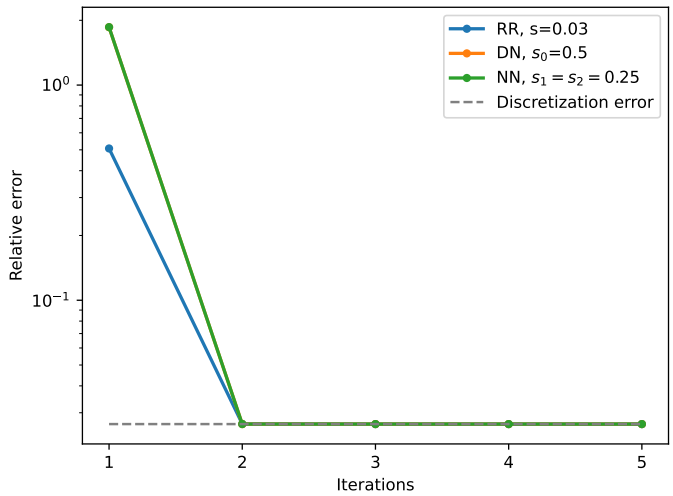
three methods we compute the relative errors

$$e_{H^1} = \frac{\|u_1^n - u\|_{L^2((0,1),H^1(\Omega_1))} + \|u_2^n - u\|_{L^2((0,1),H^1(\Omega_2))}}{\|u\|_{L^2((0,1),H^1(\Omega_1))} + \|u\|_{L^2((0,1),H^1(\Omega_2))}}$$
$$e_{L^2} = \frac{\|u_1^n - u\|_{L^2((0,1),L^2(\Omega_1))} + \|u_2^n - u\|_{L^2((0,1),L^2(\Omega_2))}}{\|u\|_{L^2((0,1),L^2(\Omega_1))} + \|u\|_{L^2((0,1),L^2(\Omega_2))}}$$

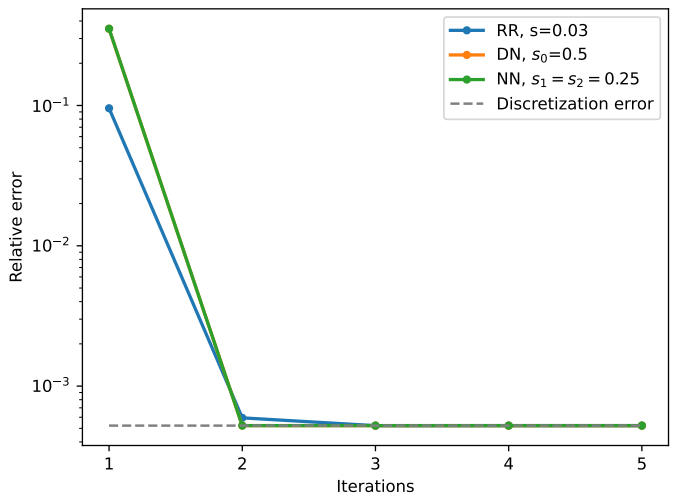
at each iteration. In Figure 7.1a we plot the relative error e_{H^1} with u being the exact solution interpolated to a finer mesh with $h_x = h_y = 1/512$ together with a reference line corresponding to the error of the space-time discretization on the whole of $\Omega \times \mathbb{R}$. In Figure 7.1b we plot the same for the relative error e_{L^2}. Finally, we plot the relative error e_{H^1} with u being the solution to the discrete problem on the whole of $\Omega \times \mathbb{R}$ in Figure 7.2

Remark 10. *Note that the theoretical result gives convergence in $L^2\big(\mathbb{R}, H^1(\Omega_i)\big)$, but we are measuring in the weaker norms $L^2\big((0,1), H^1(\Omega_i)\big)$ and $L^2\big((0,1), L^2(\Omega_i)\big)$ since we are only interested in the solution on a finite time interval.*

From Figure 7.1 we see that the three methods all perform similarly. Note that after a certain amount of iterations the error is dominated by the error of the space-time discretization, which is why we see that the methods stall. After reaching this barrier there is really no practical reason to continue the iteration, but it is still interesting when studying convergence. From Figure 7.2 we see that the discrete Robin–Robin method converges to the solution to the discrete problem, but not nearly as fast as the Dirichlet–Neumann and Neumann–Neumann methods. Still, the convergence of the Robin–Robin method appears to be geometric, which would agree with the discrete linear elliptic case [71].



(a)



(b)

Figure 7.1: The relative errors of the space-time domain decomposition methods compared to the solution to the discrete problem measured in the space-time norm with spatial H^1-norm (a) and L^2-norm (b). Note that the Dirichlet–Neumann behaves very similar to the Neumann–Neumann method and is therefore difficult to see.

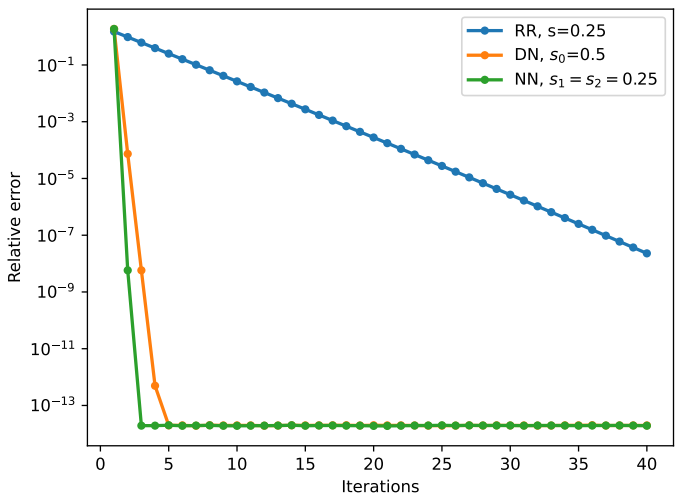


Figure 7.2: The relative errors of the space-time domain decomposition methods compared to the exact solution interpolated to a fine grid.

Chapter 8

Outlook

8.1 Domain decomposition methods for nonlinear parabolic equations

The most obvious next step is to combine the theories of Papers I and II. The difficulty here is not with the Steklov–Poincaré theory, but rather that there is a lack of theoretical results for nonlinear parabolic equations on Lipschitz domains and with temporal $H^{1/2}$-regularity. Such a variational framework for analyzing nonlinear parabolic equations has been developed in [21], but to our knowledge, there is no corresponding trace theory for Lipschitz domains. Another difficulty is that it is not possible to perform the analysis in Hilbert spaces and therefore we cannot rely on the identification between Sobolev–Bochner spaces and Hilbert tensor spaces, which was used in Paper II.

8.2 Convergence of the space-time Dirichlet–Neumann method

Waveform relaxation methods are known to converge for systems of ordinary differential equations [24, 56]. However, to our knowledge there is no convergence result for the space-time Dirichlet–Neumann method applied to partial differential equations without discretization. Although the framework developed in Paper II results in Steklov–Poincaré operators that are not symmetric, there is still cause to study the convergence of Dirichlet–Neumann in the parabolic context. Either by adapting the analytical methods used for ordinary differential equations, or modifying the Dirichlet–Neumann method to get a symmetric preconditioner, i.e., study methods

of the form

$$\eta^{n+1} = \eta^n + s_0 B^{-1}(\chi - S\eta^n)$$

with B chosen to be a coercive and symmetric operator.

8.3 Domain decomposition for discretized parabolic equations

While the nonlinear elliptic result can be applied directly to the discretized equation, it is unclear how to do this for the space-time decomposition methods since standard Galerkin methods require different test and trial spaces, see [63]. This is due to the fact that in order to get a well posed initial value problem we must replace $H^{1/2}(\mathbb{R})$ in the trial space by the Lions–Magenes space

$$H_{00}^{1/2}(\mathbb{R}^+) = \{u \in H^{1/2}(\mathbb{R}^+) : Eu \in H^{1/2}(\mathbb{R})\},$$

where Eu denotes the extension by 0 to $\mathbb{R}$. A promising alternative discretization that fits our framework better can be found in [64].

References

[1] V. I. Agoshkov and V. I. Lebedev. Variational algorithms of the domain decomposition method [translation of Preprint 54, Akad. Nauk SSSR, Otdel. Vychisl. Mat., Moscow, 1983]. volume 5, pages 27–46. 1990. Soviet Journal of Numerical Analysis and Mathematical Modelling.

[2] E. Ahmed, C. Japhet, and M. Kern. Space-time domain decomposition for two-phase flow between different rock types. *Comput. Methods Appl. Mech. Engrg.*, 371:113294, 30, 2020.

[3] J. W. Barrett and W. B. Liu. Finite element approximation of the p-Laplacian. *Math. Comp.*, 61(204):523–537, 1993.

[4] J. W. Barrett and W. B. Liu. Quasi-norm error bounds for the finite element approximation of a non-Newtonian flow. *Numer. Math.*, 68(4):437–456, 1994.

[5] H. Berninger. *Domain decomposition methods for elliptic problems with jumping nonlinearities and application to the Richards equation*. 2009. Thesis–Freie Universität Berlin (Germany).

[6] H. Berninger, R. Kornhuber, and O. Sander. Convergence behaviour of Dirichlet-Neumann and Robin methods for a nonlinear transmission problem. In *Domain decomposition methods in science and engineering XIX*, volume 78 of *Lect. Notes Comput. Sci. Eng.*, pages 87–98. Springer, Heidelberg, 2011.

[7] P. E. Bjørstad and O. B. Widlund. Iterative methods for the solution of elliptic problems on regions partitioned into substructures. *SIAM J. Numer. Anal.*, 23(6):1097–1120, 1986.

[8] J.-F. Bourgat, R. Glowinski, P. Le Tallec, and M. Vidrascu. Variational formulation and algorithm for trace operator in domain decomposition calculations. In *Domain decomposition methods (Los Angeles, CA, 1988)*, pages 3–16. SIAM, Philadelphia, PA, 1989.

[9] J. H. Bramble, J. E. Pasciak, and A. H. Schatz. An iterative method for elliptic problems on regions partitioned into substructures. *Math. Comp.*, 46(174):361–369, 1986.

[10] W. Chen, X. Xu, and S. Zhang. On the optimal convergence rate of a Robin-Robin domain decomposition method. *J. Comput. Math.*, 32(4):456–475, 2014.

[11] S.-S. Chow. Finite element error estimates for nonlinear elliptic equations of monotone type. *Numer. Math.*, 54(4):373–393, 1989.

[12] P. G. Ciarlet. *The finite element method for elliptic problems*. Studies in Mathematics and its Applications, Vol. 4. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1978.

[13] M. Costabel. Boundary integral operators for the heat equation. *Integral Equations Operator Theory*, 13(4):498–552, 1990.

[14] B. Després, P. Joly, and J. E. Roberts. A domain decomposition method for the harmonic Maxwell equations. In *Iterative methods in linear algebra (Brussels, 1991)*, pages 475–484. North-Holland, Amsterdam, 1992.

[15] L. Diening, C. Ebmeyer, and M. Ružička. Optimal convergence for the implicit space-time discretization of parabolic systems with p-structure. *SIAM J. Numer. Anal.*, 45(2):457–472, 2007.

[16] M. Discacciati, A. Quarteroni, and A. Valli. Robin-Robin domain decomposition methods for the Stokes-Darcy coupling. *SIAM J. Numer. Anal.*, 45(3):1246–1268, 2007.

[17] V. Dolean, P. Jolivet, and F. Nataf. *An introduction to domain decomposition methods*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2015. Algorithms, theory, and parallel implementation.

[18] M. Dryja and W. Hackbusch. On the nonlinear domain decomposition method. *BIT*, 37(2):296–311, 1997.

[19] C. Ebmeyer and W. B. Liu. Quasi-norm interpolation error estimates for the piecewise linear finite element approximation of p-Laplacian problems. *Numer. Math.*, 100(2):233–258, 2005.

[20] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[21] M. Fontes. *Parabolic equations with low regularity*. 1996. Thesis–Lunds Universitet (Sweden).

[22] D. Funaro, A. Quarteroni, and P. Zanolli. An iterative procedure with interface relaxation for domain decomposition methods. *SIAM J. Numer. Anal.*, 25(6):1213–1236, 1988.

[23] M. J. Gander. Optimized Schwarz methods. *SIAM J. Numer. Anal.*, 44(2):699–731, 2006.

[24] M. J. Gander. 50 years of time parallel time integration. In *Multiple shooting and time domain decomposition methods*, volume 9 of *Contrib. Math. Comput. Sci.*, pages 69–113. Springer, Cham, 2015.

[25] M. J. Gander and L. Halpern. Optimized Schwarz waveform relaxation methods for advection reaction diffusion problems. *SIAM J. Numer. Anal.*, 45(2):666–697, 2007.

[26] M. J. Gander, L. Halpern, and F. Nataf. Optimized Schwarz methods. In *Domain decomposition methods in sciences and engineering (Chiba, 1999)*, pages 15–27. DDM.org, Augsburg, 2001.

[27] M. J. Gander, F. Kwok, and B. C. Mandal. Dirichlet-Neumann and Neumann-Neumann waveform relaxation algorithms for parabolic problems. *Electron. Trans. Numer. Anal.*, 45:424–456, 2016.

[28] M. J. Gander, F. Kwok, and B. C. Mandal. Dirichlet-Neumann waveform relaxation methods for parabolic and hyperbolic problems in multiple subdomains. *BIT*, 61(1):173–207, 2021.

[29] M. J. Gander, F. Magoulès, and F. Nataf. Optimized Schwarz methods without overlap for the Helmholtz equation. *SIAM J. Sci. Comput.*, 24(1):38–60, 2002.

[30] E. Giladi and H. B. Keller. Space-time domain decomposition for parabolic problems. *Numer. Math.*, 93(2):279–313, 2002.

[31] V. Girault and P.-A. Raviart. *Finite element approximation of the Navier-Stokes equations*, volume 749 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1979.

[32] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[33] W. Guo and L. S. Hou. Generalizations and accelerations of Lions' nonoverlapping domain decomposition method for linear elliptic PDE. *SIAM J. Numer. Anal.*, 41(6):2056–2080, 2003.

[34] T.-T.-P. Hoang, J. Jaffré, C. Japhet, M. Kern, and J. E. Roberts. Space-time domain decomposition methods for diffusion problems in mixed formulations. *SIAM J. Numer. Anal.*, 51(6):3532–3559, 2013.

[35] T.-T.-P. Hoang, C. Japhet, M. Kern, and J. E. Roberts. Space-time domain decomposition for reduced fracture models in mixed formulation. *SIAM J. Numer. Anal.*, 54(1):288–316, 2016.

[36] T.-T.-P. Hoang, C. Japhet, M. Kern, and J. E. Roberts. Space-time domain decomposition for advection-diffusion problems in mixed formulations. *Math. Comput. Simulation*, 137:366–389, 2017.

[37] T. Hytönen, J. van Neerven, M. Veraar, and L. Weis. *Analysis in Banach spaces. Vol. I. Martingales and Littlewood-Paley theory*, volume 63 of *Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Cham, 2016.

[38] A. Kufner, O. John, and S. Fučík. *Function spaces*. Monographs and Textbooks on Mechanics of Solids and Fluids, Mechanics: Analysis. Noordhoff International Publishing, Leyden; Academia, Prague, 1977.

[39] S. Larsson and C. Schwab. Compressive space-time galerkin discretizations of parabolic partial differential equations. Technical Report 2015-04, Seminar for Applied Mathematics, ETH Zürich, Switzerland, 2015.

[40] S. Larsson and V. Thomée. *Partial differential equations with numerical methods*, volume 45 of *Texts in Applied Mathematics*. Springer-Verlag, Berlin, 2003.

[41] P. D. Lax. *Functional analysis*. Pure and Applied Mathematics (New York). Wiley-Interscience, New York, 2002.

[42] E. Lelarasmee, A.E. Ruehli, and A.L. Sangiovanni-Vincentelli. The waveform relaxation method for time-domain analysis of large scale integrated circuits. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, 1(3):131–145, 1982.

[43] F. Lemarié, L. Debreu, and E. Blayo. Toward an optimized global-in-time Schwarz algorithm for diffusion equations with discontinuous and spatially variable coefficients. Part 1: The constant coefficients case. *Electron. Trans. Numer. Anal.*, 40:148–169, 2013.

[44] F. Lemarié, L. Debreu, and E. Blayo. Toward an optimized global-in-time Schwarz algorithm for diffusion equations with discontinuous and spatially variable coefficients. Part 2: The variable coefficients case. *Electron. Trans. Numer. Anal.*, 40:170–186, 2013.

[45] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I*. Die Grundlehren der mathematischen Wissenschaften, Band 181. Springer-Verlag, New York-Heidelberg, 1972.

[46] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. II*. Die Grundlehren der mathematischen Wissenschaften, Band 182. Springer-Verlag, New York-Heidelberg, 1972.

[47] P.-L. Lions. On the Schwarz alternating method. I. In *First International Symposium on Domain Decomposition Methods for Partial Differential Equations (Paris, 1987)*, pages 1–42. SIAM, Philadelphia, PA, 1988.

[48] P.-L. Lions. On the Schwarz alternating method. II. Stochastic interpretation and order properties. In *Domain decomposition methods (Los Angeles, CA, 1988)*, pages 47–70. SIAM, Philadelphia, PA, 1989.

[49] P.-L. Lions. On the Schwarz alternating method. III. A variant for nonoverlapping subdomains. In *Third International Symposium on Domain Decomposition Methods for Partial Differential Equations (Houston, TX, 1989)*, pages 202–223. SIAM, Philadelphia, PA, 1990.

[50] P.-L. Lions and B. Mercier. Splitting algorithms for the sum of two nonlinear operators. *SIAM J. Numer. Anal.*, 16(6):964–979, 1979.

[51] S. H. Lui. A Lions non-overlapping domain decomposition method for domains with an arbitrary interface. *IMA J. Numer. Anal.*, 29(2):332–349, 2009.

[52] S. H. Lui. Convergence estimates for an higher order optimized Schwarz method for domains with an arbitrary interface. *J. Comput. Appl. Math.*, 235(1):301–314, 2010.

[53] S. H. Lui. Optimized Schwarz methods for domains with an arbitrary interface. In *Domain decomposition methods in science and engineering XIX*, volume 78 of *Lect. Notes Comput. Sci. Eng.*, pages 109–116. Springer, Heidelberg, 2011.

[54] L. D. Marini and A. Quarteroni. An iterative procedure for domain decomposition methods: a finite element approach. In *First International Symposium on Domain Decomposition Methods for Partial Differential Equations (Paris, 1987)*, pages 129–143. SIAM, Philadelphia, PA, 1988.

[55] L. D. Marini and A. Quarteroni. A relaxation procedure for domain decomposition methods using finite elements. *Numer. Math.*, 55(5):575–598, 1989.

[56] U. Miekkala and O. Nevanlinna. Convergence of dynamic iteration methods for initial value problem. *SIAM J. Sci. Statist. Comput.*, 8(4):459–482, 1987.

[57] K. Miller. Numerical analogs to the Schwarz alternating procedure. *Numer. Math.*, 7:91–103, 1965.

[58] J. Naumann and C. G. Simader. *Measure and Integration on Lipschitz-Manifolds*. Humboldt-Universität zu Berlin, 2011. Lecture notes.

[59] D. W. Peaceman and H. H. Rachford, Jr. The numerical solution of parabolic and elliptic differential equations. *J. Soc. Indust. Appl. Math.*, 3:28–41, 1955.

[60] A. Quarteroni and A. Valli. *Domain decomposition methods for partial differential equations*. Numerical Mathematics and Scientific Computation. The Clarendon Press, Oxford University Press, New York, 1999.

[61] T. Roubíček. *Nonlinear partial differential equations with applications*, volume 153 of *International Series of Numerical Mathematics*. Birkhäuser/Springer Basel AG, Basel, second edition, 2013.

[62] H.A. Schwarz. Über einen Grenzübergang durch alternierendes Verfahren. *Vierteljahrsschr. Nat.forsch. Ges. Zür.*, 15:272–286, 1870.

[63] O. Steinbach and H. Yang. Space-time finite element methods for parabolic evolution equations: discretization, a posteriori error estimation, adaptivity and solution. In *Space-time methods—applications to partial differential equations*, volume 25 of *Radon Ser. Comput. Appl. Math.*, pages 207–248. De Gruyter, Berlin, 2019.

[64] O. Steinbach and M. Zank. Coercive space-time finite element methods for initial boundary value problems. *Electron. Trans. Numer. Anal.*, 52:154–194, 2020.

[65] G. Strang and G. J. Fix. *An analysis of the finite element method*. Prentice-Hall Series in Automatic Computation. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1973.

[66] X.-C. Tai and M. Espedal. Rate of convergence of some space decomposition methods for linear and nonlinear problems. *SIAM J. Numer. Anal.*, 35(4):1558–1570, 1998.

[67] X.-C. Tai and J. Xu. Global and uniform convergence of subspace correction methods for some convex optimization problems. *Math. Comp.*, 71(237):105–124, 2002.

[68] A. Toselli and O. Widlund. *Domain decomposition methods—algorithms and theory*, volume 34 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2005.

[69] J. Weidmann. *Linear operators in Hilbert spaces*, volume 68 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1980.

[70] J. Xu and J. Zou. Some nonoverlapping domain decomposition methods. *SIAM Rev.*, 40(4):857–914, 1998.

[71] X. Xu and L. Qin. Spectral analysis of Dirichlet-Neumann operators and optimized Schwarz methods with Robin transmission conditions. *SIAM J. Numer. Anal.*, 47(6):4540–4568, 2010.

[72] L. Yin, J. Wu, and Z. Dai. Convergence analysis for a second order Schwarz method for nonoverlapping subdomains. *J. Math. Anal. Appl.*, 436(1):467–477, 2016.

[73] E. Zeidler. *Nonlinear functional analysis and its applications. II/B*. Springer-Verlag, New York, 1990.

Scientific publications

Author contributions

Paper I: Convergence Analysis of the Nonoverlapping Robin–Robin Method for Nonlinear Elliptic Equations

Esil Hansen and I did the analysis together and I wrote most of section 3.

Paper II: Time-dependent Steklov–Poincaré operators and space-time Robin–Robin decomposition for the heat equation

Eskil Hansen and I did the analysis and wrote the paper together.