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Binary Extensions and Choice Theory

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Nos choix sont plus nous que nous.

André Suarès

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Contents

CHAPTER 1

General introduction

1.1 Rational choice hypothesis

Rational choice theory postulates that preferences determine choice. For example: if a is preferred to b, then b will not be chosen when a is available[1]. Preferences rank the alternatives and individuals select from the available alternatives the ones that have the highest ranking. We call this the *'Rational choice hypothesis'* (RCH):

RCH: Individuals choose the alternatives that maximize their preferences.

The RCH is composed of two subhypothesises. The first states that choices are optimal with respect to some binary relation, and the second states that this relation equals a preference relation. The restrictions imposed on a preference relation depend to a large extent on the model under consideration, for instance on whether the model is dynamic or static, deterministic or stochastic, etc. But, there are two constraints that are widely, i.e. for almost all models, imposed. These are the properties of *completeness* and *transitivity*. Completeness states that for any two alternatives a and b, either a is at least as good as b or b is at least as good as a. A violation of completeness results in situations where it is infeasible to choose between two alternatives. As an example, we can imagine a shipwrecked person on a desolated island who starves to death because he is not able to choose between eating chicken or coconut. Transitivity states that if a is at least as good as b and if b is at least as good as c, then a is at least as good as c. Transitivity can be supported on the same grounds as completeness: any violation of transitivity can lead to choice situations where an individual is unable to attain a consistent choice.

[1]We say that a is preferred to b if a is at least as good as b while b is not at least as good as a, and we say that a is indifferent to b if a is at least as good as b and b is at least as good as a.

1.2 Revealed preferences

Often, it is the case that preferences are not observable while choices are. A test for the RCH should therefore focus on the constraints that these observations impose on the domain of potential preferences. For example, if a is selected while b was available and b is selected while c was available, one should not observe instances where c is chosen and a is rejected. Otherwise, one could deduce that a is at least as good as b, b is at least as good as c and c is preferred to a. This contradicts transitivity.

If a is selected while b is available, we say that a is *revealed preferred* to b and we write $(a, b) \in R_v$. If $(a, b) \in R_v$ and if, in addition, b is not selected, we say that a is *strictly revealed preferred* to b and write $(a, b) \in P_v$. Evidently $P_v \subseteq R_v$. The relation R_v is derived from observations on choices and is only indirectly (if the RCH holds) deduced from the genuine preference relation.

If the RCH applies and if $(a, b) \in R_v$, it ought to be that a is at least as good as b. Indeed, if on the contrary, b is strictly preferred to a, a should not be chosen while b is available. As such, it is sometimes possible to retrace the entire preference relation from the revealed preference relation. For example, if we observe the choices from each two element choice set, we can specify for each pair $\{a, b\}$ whether $(a, b) \in R_v$ or whether $(b, a) \in P_v$, and verify if the RCH holds.

1.3 Rationalizability

The difficulty arises if it is infeasible to reconstruct the true preference relation from the revealed preference relation. For example: if (a, b) and $(b, c) \in R_v$ but the agent has never been granted the opportunity to choose between a and c, we are inept to verify whether the preferences are transitive, i.e. whether a is at least as good as c. On the other hand, if $(c, a) \in P_v$ we are certain that preferences are not transitive and we may dismiss the RCH. Let us define the concept of *rationalizability*

A collection of observations is rationalizable if there exist preferences which reproduces the observations according to the RCH.

If the observations are not rationalizable, we reject the RCH with certainty. Nevertheless, it is possible that the observations are rationalizable and the RCH is false, i.e. the power of the test is less than one. For example, if a is at least as good as b, b is at least as good as c and c is preferred to a, we must conclude that the preferences are not transitive, hence, the RCH is false. Assume now that the agent has to select between a and b and between b and c. Thus, $(a, b) \in R_v$ and $(b, c) \in R_v$. These observations are rationalizable. Indeed, it is possible to conceive (transitive and complete) preferences for which a is at least as good as b, b is at least as good as c and a is at least as good as c. To conclude: if it is infeasible to derive the entire preference relation from the revealed preference relation, there must exist a relation which induces the same behavior, but is not a preference relation.

An elegant way to measure the reliability of the rationalizability criterion is via the computation of a *power measure* (e.g. [Bronars, 1987]). In order to do this, one defines an alternative hypothesis of how agents make their choices and one constructs (artificially) choices following choice rules that obey this alternative hypothesis. The power measure is computed as the fraction of these choice rules that satisfy the rationalizability criterion. The development of an alternative hypothesis and the computation of the power measure is a very fascinating topic, nevertheless, in this research, we are more concerned with the construction of the rationalizability tests themeselves.

If $(a, b) \in R_v$ and $(b, c) \in R_v$, we say that a is *indirectly revealed preferred* to c. Moreover, we may broaden this definition to permit any finite number of alternatives, i.e. if $(a, b) \in R_v$, $(b, c) \in R_v, \ldots, (p, q) \in R_v$, then a is indirectly revealed preferred to q. Richter [1966] presents following '*congruence axiom*':

For all alternatives a and b, if a is indirectly revealed preferred to b, then $(b, a) \notin P_v$.

Richter shows that a set of observations is rationalizable by a complete and transitive relation if and only if they satisfy this congruence axiom.

It is easy to see why Richter's congruence axiom is necessary. Indeed, if on the contrary, there are a and b such that a is indirectly revealed preferred to b and $(b, a) \in P_v$, then for any transitive preference relation, satisfying the RCH, it must be that a is at least as good as b and b is preferred to a, a contradiction.

1.4 The congruence axiom and binary extensions

The sufficiency part of Richter's rationalizability characterization requires some notation and results from the theory on binary relations. Consider a set of alternatives X and the cartesian product $X \times X = \{(a, b)|a, b \in X\}$. A *binary relation* in X is a subset of $X \times X$. For example, the preference relation, R, where $(a, b) \in R$ if a is at least as good as b and the revealed preference relations R_v and P_v are all examples of binary relations. The pair $\{a, b\}$ belongs to the *asymmetric part* of R if $(a, b) \in R$ and $(b, a) \notin R$. We denote this also by $(a, b) \in P(R)$. For example, the strict preference relation is the asymmetric part of the preference relation. A pair $\{a, b\}$ belongs to the *transitive closure* of the relation R if there exist a number n and elements $a = x_1, x_2, \ldots, x_n = b$ such that $(x_i, x_{i+1}) \in R$ for all $i = 1, ..., n - 1$. We also denote this by $(a, b) \in T(R)$. For example: the transitive closure of the revealed preference relation, R_v is given by the indirect revealed preference relation, $T(R_v)$.

A binary relation R' is an *extension* of a relation R if $R \subseteq R'$ and $P(R) \subseteq P(R')$. A fundamental result in the theory of binary extensions, better known as *Szpilrajn's lemma*, [Szpilrajn, 1930], states that:

Any transitive relation has a complete and transitive extension.

This result is generalized by Suzumura [1976] who demonstrated that:

A binary relation R has a complete and transitive extension if and only if for all a and b, $(a, b) \in T(R)$ implies $(b, a) \notin P(R)$.

Suzumura names this condition *'consistency'*[2].

The indirect revealed preference relation, $T(R_v)$, is transitive, hence, it has, by Szpilrajn's lemma, a complete and transitive extension. The proof of Richter's rationalizability characterization is completed by validating that (by using Richter's congruence axiom) this complete and transitive extension rationalizes the observations.

1.5 This research

In the consecutive chapters, we develop rationalizability tests for preferences that satisfy, besides transitivity and completeness, other appealing properties. Let us look at the example of monotonicity. Let X be a subset from a vector space where each vector represents a commodity bundle. For two bundles a and b, we denote $a \geq b$ if the bundle a contains at least as much of every good as the bundle b. A relation, R, is *monotonic* if for all a and b in X, $a \geq b$ implies $(a, b) \in R$.

When are observations rationalizable by a complete, transitive and monotonic relation? Following Richter, we solve this problem in two steps. In a first step, we identify the conditions for a relation to have a complete, transitive and monotonic extension and in a second step, we apply this outcome to the framework of revealed preferences. Let us commence with the first step.

1.5.1 Generalizing Suzumura's consistency condition

The relation R is consistent if and only if $T(R)$ extends R. Indeed, if R is consistent it is impossible that $(a, b) \in P(R)$ and $(b, a) \in T(R)$. Together with $R \subseteq T(R)$, this implies that $T(R)$ extends R. On the other hand, if consistency is violated, i.e. $(a, b) \in T(R)$ and $(b, a) \in P(R)$, it is impossible that $T(R)$ extends R.

It is easy to validate that this statement is true for any increasing function[3] (see also lemma 2.1 in section 2.2):

Let us construct the *transitive and monotonic closure* in two steps. First, we consider the relation $\overline{R}$ by adding to R all the elements (a, b) for which $a \geq b$ and second, we take the transitive closure of $\overline{R}$, $T(\overline{R})$. This closure is increasing, hence, we can write:

The relation $T(\overline{R})$ extends R if and only if for all $(a, b) \in T(\overline{R})$, $(b, a) \notin P(R)$.

The transitive closure of R is the smallest (with respect to set inclusion) transitive relation that contains R. Therefore, if R has a transitive extension, it is consistent. Indeed, if R is not consistent, then there are alternatives a and b such that $(a, b) \in T(R)$ and $(b, a) \in P(R)$. Any transitive extension R' of R should therefore obey $(a, b) \in R' \supseteq T(R)$ and $(b, a) \in P(R')$, a contradiction. Besides this, a relation R is transitive if and only if $R = T(R)$ (see also section 2.3.1.i). Suzumura's theorem therefore implies that:

[2] For further generalizations of Szpilrajn's lemma, see Duggan [1999] and Donaldson and Weymark [1998] among others.

[3] A function F is increasing if for all relations R, $R \subseteq F(R)$.

A relation, R, has an extension $R' = T(R')$ if and only if it has a complete extension $R^* = T(R^*)$.

In other words: the existence of a complete and transitive extension does not impose additional restraints beyond the existence of a transitive extension. Can we find additional functions for which this is true? In section 2.3.2 we show that:

A relation R has a complete and monotonic extension R' if and only if for all $(a, b) \in T(\overline{R})$, $(b, a) \notin P(R)$, i.e. $T(\overline{R})$ is an extension of R.

Let us now continue with the second step of our rationalizability result.

1.5.2 Rationalizability once again

The proof of this step is almost identical to Richter's original proof but rather than Szpilrajn's lemma, we use the above given variant of Suzumura's theorem. This provides following outcome for the property of monotonicity (see also section 2.4):

A collection of observations is rationalizable by a monotonic complete and transitive relation if and only if for all $(a, b) \in T(\overline{R_v})$, $(b, a) \notin P_v$.

In this research we discuss besides monotonicity (chapter 2) also the properties of convexity and homotheticity (chapter 2), absolute (relative) time-consistency and impatience (chapter 3) and independence (chapter 4)

1.6 Outline

Before we close this general introduction, we briefly summarize the coming chapters.

In chapter 2, we present a general extension result and we apply it to the properties of convexity, homotheticity and monotonicity. As mentioned above, this general extension result generalizes Suzumura's theorem. If we substitute in Suzumura's theorem the transitive closure with a general function F we obtain that:

A relation R has a complete extension $R^* = F(R^*)$ if and only if for all $(a, b) \in F(R)$, $(b, a) \notin P(R)$.

The initial part of chapter 2 (section 2.2) derives a class of functions for which this is true. In the second part of this chapter (section 2.3), we apply this result to the properties of convexity, homotheticity and monotonicity. For example, we develop a function C such that a relation R has a complete, transitive and convex extension if and only if for all $(a, b) \in C(R)$, $(b, a) \notin P(R)$. In the closing section of the chapter (section 2.4), we apply these extension results to a choice theoretic framework. We extend Richter's congruence axiom and characterize the set of choice functions that are rationalizable by a complete extension, R^*, that satisfies $R^* = F(R^*)$. For example, we find that a choice function is rationalizable by a complete, convex and transitive relation if and only if for all alternatives a and b, if $(a, b) \in C(R_v)$, then $(b, a) \notin P_v$.

In chapter 3, we focus on dynamic models and we determine the class of choice functions that are rationalizable by a complete, transitive, absolute (relative) time-consistent and impatient relation. We construct an absolute (relative) time consistent, transitive and impatient closure and we confirm that the following congruence condition characterizes the set of choice functions that are rationalizable.

For all a, b, if (a,b) belongs to the absolute (relative) time-consistent, transitive and impatient closure of R_v then $(b,a) \notin P_v$.

Chapter 4 is joined work with Luc Lauwers. In this chapter, we focus on models with choice under uncertainty. Lotteries are probability distributions over the set of alternatives and preferences are defined over the set of lotteries. Besides completeness and transitivity, we impose that preferences are independent, i.e. if lottery a is preferred to lottery b, then any mixture of a with a third lottery c is preferred to the same mixture of b and c. We develop a transitive and independent closure and consider following congruence condition:

If (a,b) belongs to the transitive and independent closure of R_v, then $(b,a) \notin P_v$.

We prove that this congruence condition characterizes the set of choice functions that are rationalizable by a transitive, complete and independent relation (section 4.3). Then, we extend this framework to multiple and interacting individuals. In this setting, each individual is confronted with a set of actions from which he has to make a choice. Following Sprumont [2000], we can define a joint choice function to be Nash rationalizable if there exist a profile of preference relations over the set of strategy profiles such that the observed outcomes coincide with the Nash equilibria based upon these preferences. We extend this framework to allow for lotteries over action-profiles and define Nash rationalizability when individuals have a menu of mixtures at their disposal. We characterize Nash rationalizability by two conditions (section 4.4):

A choice function is Nash rationalizable if it satisfies a modified version of Richter's congruence condition and the axiom of non-cooperation.

The first condition (modified congruence condition) is a congruence condition which is similar to the one for the single person choice framework (see section 4.3). The second condition (non-cooperation) connects the individual behavior to the collective behavior: if a strategy profile belongs to the collective choice, it should also be selected when the actions of all but one individual are kept fixed. We finalize the chapter by linking our result to the analysis of Sprumont [2000] (section 4.5).

In the last chapter, chapter 5, we address some generalizations (section 5.1), discuss for each property some implementation issues (section 5.2) and we give concluding remarks (section 5.3).

CHAPTER 2

A general extension method with applications to convexity, homotheticity and monotonicity

2.1 Introduction

Consider a set of alternatives, X, and a binary relation, R, on X, with asymmetric part $P(R)$. The main objective of this chapter is to characterize the set of relations R for which there exists a relation, R', satisfying:

i R' is an extension of R, i.e. $R \subseteq R'$ and $P(R) \subseteq P(R')$,

ii R' is complete and transitive, and

iii R' satisfies some additional properties like convexity, homotheticity or monotonicity.

A first step toward a solution is given by Suzumura [1976] who showed that a relation, R, has a complete and transitive extension if and only if for all (a, b) in the transitive closure of R, $(b, a) \notin P(R)$. He calls this condition *consistency*. The proof of this outcome can be split up in two steps. In the first step, one shows that a binary relation, R, has a transitive extension if and only if R is consistent. In the second step, one uses Szpilrajn's lemma [Szpilrajn, 1930] which states that any transitive relation has a complete and transitive extension.

Suzumura's consistency condition characterizes the set of relations that satisfy requirements (i) and (ii). Therefore, any relation that fulfills (i), (ii) and (iii) must also be consistent. On the other hand, consistency it is not sufficient. Consider for example commodity bundles, a, b, c and d, and assume that $(a, b) \in R, (c, b) \in R, (b, d) \in P(R)$ and that there exist an $\alpha \in]0, 1[$ such that $d = \alpha a + (1 - \alpha)c$. The relation R meets consistency, and has thus a complete and transitive extension. Even so, it has no complete, transitive and convex extension. Indeed, if R'

is a complete transitive and convex extension of R, then it follows that $(c, b) \in R', (a, b) \in R'$ and, by convexity, $(d, b) \in R'$. This contradicts that $(b, d) \in P(R')$.

The idea is to strengthen the requirement of consistency by replacing the transitive closure with another function F. The adapted 'consistency' requirement reads: for all a and b, if $(a, b) \in F(R)$, then $(b, a) \notin P(R)$. We develop a class of functions for which this adjusted 'consistency' condition characterizes the set of relations that have a complete extension, $R^* = F(R^*)$. The remaining problem is to show that a complete relation R^* equals $F(R^*)$ if and only if R^* is transitive and fulfills requirement (iii).

In section 2.2, we introduce notation and basic definitions and deduce the main extension method. As mentioned above, we construct a class of functions, F such that a relation R has a complete extension $R^* = F(R^*)$ if and only if for all $(a, b) \in F(R)$, $(b, a) \notin P(R)$.

In section 2.3, we apply this result to specific properties, i.e. convexity, homotheticity and monotonicity. We define the corresponding functions C and H and we show that a relation R has a complete transitive and (homothetic and monotonic) convex extension if and only if $((a, b) \in H(R))$ $(a, b) \in C(R)$ implies that $(b, a) \notin P(R)$. We derive akin results for monotonicity and strict monotonicity.

In section 2.4, we build a bridge between the revealed preference literature and our result. By generalizing the congruence axiom of Richter [1966], we characterize the set of choice functions which are rationalizable by a complete and transitive relation, $R^* = F(R^*)$. By substituting the functions C and H for F, we derive rationalizability results for the properties of convexity and homotheticity.

Before we begin the next section, let us provide an example. Consider an individual with a preference relation over three goods. We present his preferences as a binary relation over a convex subset of $\mathbb{R}^3$. Consider six bundles in this set:

$$\begin{aligned}
x_1 &= (10, 9, 3) \\
x_2 &= (9, 7, 8) \\
x_3 &= (12, 4, 8) \\
x_4 &= (4, 8, 12) \\
x_5 &= (8, 9, 7) \\
x_6 &= (8, 12, 4)
\end{aligned}$$

Assume that we have observations such that x_1 is revealed preferred to x_2 and x_5, x_3 is revealed preferred to x_4, x_6 is revealed preferred to x_4 and x_4 is strictly revealed preferred to x_1. These observations satisfy the conditions of Richter's congruence condition[1], hence, there does exist a complete and transitive relation that rationalizes these observations. However, at the end of section 2.4 we will reconsider this example and show that these observations are not rationalizable by a complete, transitive and convex relation.

[1]Recall from section 1.3 that Richter's congruence condition states that, if a is indirectly revealed preferred to b then b is not strictly revealed preferred to a. See also section 2.4 for a formal definition

2.2 A general extension method

Consider a set X of alternatives. A set $R \subseteq X \times X$ is called a *binary relation* on X. We denote the set of all binary relations on X by $\mathcal{R}$. Given a relation R, we define its *inverse* R^{-1} by $(a,b) \in R^{-1}$ if and only if $(b,a) \in R$. The *symmetric part* of R is given by $R \cap R^{-1}$ and is denoted by $I(R)$, the *asymmetric part* $R - I(R)$ is denoted by $P(R)$ and the *non-comparable* part $X \times X - (R \cup R^{-1})$ is denoted by $N(R)$.

A binary relation R is *complete* if for all $a, b \in X : (a,b) \in R$ or $(b,a) \in R$. A binary relation R is *transitive* if for all $a,b,c \in X$: if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$.

Definition 2.1 (Extension). *A relation R' is an extension of the relation R, denoted $R \preceq R'$, if $R \subseteq R'$ and $P(R) \subseteq P(R')$.*

The relation $\preceq$ is reflexive: for all $R, R \preceq R$, and transitive: if $R \preceq R'$ and $R' \preceq R''$, then $R \preceq R''$.

Consider a function $F : \mathcal{R} \to \mathcal{R}$ and let $\mathcal{R}^* = \{R \in \mathcal{R} | R \preceq F(R)\}$. In the ensuing section, we provide several examples for $F(.)$. Until then, it may be usefull to think of the transitive closure[2] $T(.)$ as an example.

The following result characterizes the set $\mathcal{R}^*$.

Lemma 2.1. *Let $R \subseteq F(R)$. The relation R belongs to $\mathcal{R}^*$ if and only if $F(R) \cap P(R)^{-1} = \emptyset$.*

Proof. ($\rightarrow$) Let $R \preceq F(R)$. If $(a,b) \in P(R)$, we derive that $(a,b) \in P(F(R))$, hence $(b,a) \notin F(R)$. Conclude that $F(R) \cap P(R)^{-1} = \emptyset$.

($\leftarrow$) Let $F(R) \cap P(R)^{-1} = \emptyset$. If $(a,b) \in P(R)$, then $(b,a) \notin F(R)$. From $R \subseteq F(R)$ and $P(R) \subseteq R$, we derive that $(a,b) \in P(F(R))$. Conclude that $R \preceq F(R)$. □

Recall, from the introduction (section 2.1), that Suzumura [1976] characterized the set of relations $R \in \mathcal{R}$ that have a complete and transitive extension, by the requirement that $T(R) \cap P(R)^{-1} = \emptyset$.

Likewise, we would like to characterize the relations $R \in \mathcal{R}$ that have a complete extension $R' \in \mathcal{R}^*$ (or equivalently $R' = F(R')$), by the condition $F(R) \cap P(R)^{-1} = \emptyset$.

Lemma 2.2. *Let $F : \mathcal{R} \to \mathcal{R}$ and let $\mathcal{R}^* = \{R \in \mathcal{R} | R \preceq F(R)\}$. If F satisfies the following conditions:*

C1: for each well-ordered chain $R_0 \subset R_1 \subset \ldots \subset R_\alpha \subset \ldots$ of relations in $\mathcal{R}^*$, the union $\bigcup_{0 \leq \alpha} R_\alpha$ is also in $\mathcal{R}^*$, and,

C2: for each relation $R \in \mathcal{R}^*$ such that $N(R) \neq \emptyset$, there exists a non-empty subset T of $N(R)$ such that $R \cup T \in \mathcal{R}^*$,

[2] For a relation $R \in \mathcal{R}$, we say that $(a,b) \in T(R)$ if there exists a natural number n and elements $x_1, \ldots, x_n$ in X such that $x_1 = a$, $x_n = b$ and $(x_i, x_{i+1}) \in R$ for all $i = 1, \ldots, n-1$.

then in order for a relation $R \in \mathcal{R}$ with $R \subseteq F(R)$, to have a complete extension $R^* = F(R^*)$ it is sufficient that $F(R) \cap P(R)^{-1} = \emptyset$.

Before we provide the proof, let us first outline the intuition behind $C1$ and $C2$. Remember, from lemma 2.1, that for $R \subseteq F(R)$, $F(R) \cap P(R)^{-1} = \emptyset$ if and only if $R \in \mathcal{R}^*$. The idea is to enlarge R by repeatedly adding elements of $N(R)$, such that these enlarged relations remain in $\mathcal{R}^*$. This is exactly what condition $C2$ allows to do. If X is finite, $C2$ is sufficient to end up with a complete extension. This is no longer true if X is infinite. For these cases we added the, rather technical, condition $C1$. Notice that this condition requires the index α to be an index over ordinal numbers.

Proof of lemma 2.2. Let $\Omega = \{R' \in \mathcal{R}^* | R \preceq R'\}$ collect all the extensions of R in R^* including R. Consider a well-ordered chain $R_0 \subseteq R_1 \subseteq ... \subseteq R_\alpha \subseteq ...$ in Ω and define $B = \bigcup_{0 \leq \alpha} R_\alpha$. Let us verify that Ω contains B. By $C1$, $B \preceq F(B)$. We are left to affirm that $R \preceq B$. Clearly, $R \subseteq B$. If, on the contrary, $(a,b) \in P(R)$ and $(b,a) \in B$, there must exist an R_α in the well ordering that contains (b,a). This violates $R_\alpha \in \Omega$.

We apply Zorn's lemma and conclude that the set Ω has a maximal element. Let R^* be such an element.

We verify that R^* is complete. If, on the contrary, $N(R^*) \neq \emptyset$, there exists, by $C2$ a relation $T \subseteq N(R^*)$ such that $R^* \cup T \in \mathcal{R}^*$. Let us verify that $R^* \cup T \in \Omega$ by showing that $R \preceq R^* \cup T$. The set Ω contains R^*, thus, $R \preceq R^*$. Let $(a,b) \in P(R^*)$ and assume on the contrary that $(b,a) \in N(R^*)$. Clearly $R^* \preceq R^* \cup T$, hence, by transitivity of $\preceq$ we derive that $R \preceq R^* \cup T$.

We must conclude that R' is not maximal, a contradiction. Hence, R^* is complete.

We complete the proof by demonstrating that $R^* = F(R^*)$. As $R^* \preceq F(R^*)$, we immediately deduce that $R^* \subseteq F(R^*)$. To verify the reverse, assume that $(a,b) \in F(R^*)$. If $(b,a) \in P(R^*)$, we would derive that $(b,a) \in P(F(R^*))$, a contradiction. Therefore, it must be that $(b,a) \notin P(R^*)$. From completeness of R^*, we conclude that $(a,b) \in R^*$. Hence, $F(R^*) \subseteq R^*$. □

Lemma 2.3. *Let $F : \mathcal{R} \to \mathcal{R}$ and let $\mathcal{R}^* = \{R \in \mathcal{R} | R \preceq F(R)\}$. Let F satisfy the following condition:*

C3: for all R and $R' \in \mathcal{R}$, if $R \subseteq R'$, then $F(R) \subseteq F(R')$.

Then, if a relation $R \in \mathcal{R}$ with $R \subseteq F(R)$, has a complete extension $R^* = F(R^*)$, then $F(R) \cap P(R)^{-1} = \emptyset$.

Proof. Let $R \subseteq F(R)$ extend to a complete relation $R^* = F(R^*)$. Assume, on the contrary, that $(a,b) \in F(R)$ and $(b,a) \in P(R)$. Deduce that $(b,a) \in P(R^*)$ and (by $C3$) $(a,b) \in F(R^*) = R^*$, a contradiction □

The combination of lemma 2.1, 2.2 and 2.3 leads to the following result.

Theorem 2.1. *Let $F : \mathcal{R} \to \mathcal{R}$ satisfy the conditions $C1$, $C2$, $C3$. Then in order that a relation $R \in \mathcal{R}$ with $R \subseteq F(R)$ has a complete extension $R^* = F(R^*)$ it is necessary and sufficient that $F(R) \cap P(R)^{-1} = \emptyset$.*

In the remaining part of this section, we will impose restrictions on the function F beyond $C1$, $C2$ and $C3$. There are multiple reasons for this. First of all, it allows us to impose a more familiar structure on F: although the conditions $C1$, $C2$ and $C3$ are fairly general, they do not correspond to a particular known class of functions. Second, we have been unable to find any economically interesting applications for which the function F satisfies conditions $C1$, $C2$ and $C3$, but not these extra conditions. Finally, imposing these additional restrictions here allows us to simplify and shorten the proofs in the next section.

Definition 2.2 (Closure operator)**.** *The function F is a closure operator if it satisfies condition $C3$,*

C4: for all $R \in \mathcal{R} : R \subseteq F(R)$, and,

C5: for all $R \in \mathcal{R} : F(F(R)) \subseteq F(R)$.

A closure operator F is algebraic if

C6: for all $R \in \mathcal{R}$ and all $(a, b) \in F(R)$, there is a finite relation $R' \subseteq R$ such that $(a, b) \in F(R')$.

Let us show how algebraic closure operators relate to the conditions $C1$ and $C2$.

Lemma 2.4. *Let $F : \mathcal{R} \to \mathcal{R}$ be an algebraic closure operator and let $\mathcal{R}^* = \{R \in \mathcal{R} | R \preceq F(R)\}$. If F satisfies,*

C7: for all $R \in \mathcal{R}$, if $R = F(R)$ and $N(R) \neq \emptyset$, there exists a non-empty subset T of $N(R)$ such that $R \cup T \in \mathcal{R}^*$

then F satisfies $C1$ and $C2$.

Proof. Let us start by verifying that F satisfies $C2$. First, notice that $C4$ and $C5$ together imply $F(F(R)) = F(R)$ for all $R \in \mathcal{R}$. Take any relation $R \in \mathcal{R}^*$ for which $N(R) \neq \emptyset$.
If $R = F(R)$, we have from $F(F(R)) = F(R)$, that $F(R) \in \mathcal{R}^*$. In this case, condition $C2$ is equivalent to condition $C7$.
If $R \subset F(R)$, we consider the set $T = F(R) - R$. Again, by $F(F(R)) = F(R)$, we derive that $F(R) = R \cup T \in \mathcal{R}^*$. The proof is complete if we can demonstrate that $T \subseteq N(R)$. Assume, on the contrary, that $(a, b) \in T$ and $(a, b) \notin N(R)$. There are two options: $(a, b) \in R$ or $(b, a) \in P(R)$. Let us show that both lead to a contradiction.

(i) $(a, b) \in R$. From the construction of T, we deduce that $(a, b) \notin T$, a contradiction.

(ii) $(b, a) \in P(R)$. From $T \subseteq F(R)$, we deduce that $(a, b) \in F(R)$. This contradicts with $R \preceq F(R)$.

Finally, we need to show that F satisfies $C1$. Consider a well-ordered chain $R_0 \subseteq R_1 \subseteq \ldots \subseteq R_\alpha \subseteq \ldots$ in $\mathcal{R}^*$ and let $B = \bigcup_{\alpha \geq 0} R_\alpha$. We have to verify that $B \in \mathcal{R}^*$. Applying condition $C4$, we derive that $B \subseteq F(B)$. Hence, from lemma 2.1, we only need to affirm that $F(B) \cap P(B)^{-1} = \emptyset$. Assume, on the contrary, that $(a, b) \in F(B)$ and $(b, a) \in P(B)$. From $C6$, there exists a finite subset $B' \subseteq B$ for which $(a, b) \in F(B')$. Consider a relation R_α in the well-ordered chain for which $B' \subseteq R_\alpha$. The existence of such relation is guaranteed by finiteness of B'. From $C3$, we derive that $(a, b) \in F(R_\beta)$ for all $\beta \geq \alpha$. Also, from the construction of B, we deduce that there is an $\alpha' \geq 0$ such that $(b, a) \in P(R_{\beta'})$ for all $\beta' \geq \alpha'$. Conclude that $(a, b) \in F(R_{\alpha''}) \cap P(R_{\alpha''})^{-1}$ for all $\alpha'' \geq \max\{\alpha, \alpha'\}$. This violates $R_{\alpha''} \in \mathcal{R}^*$. □

Theorem 2.1, together with lemma 2.1 and lemma 2.4, gives the following result:

Theorem 2.2. *Consider an algebraic closure operator $F : \mathcal{R} \to \mathcal{R}$ that satisfies $C7$. Then, a relation $R \in \mathcal{R}$ has a complete extension $R^* = F(R^*)$ if and only if $F(R) \cap P(R)^{-1} = \emptyset$.*

We finish this section by providing a characterization for closure operators which will be usefull in the next section. This characterization is well-known (e.g. [Cohn, 1965]), but we prove it for completeness.

Lemma 2.5. *Assume that F satisfies $F(X \times X) = X \times X$. Then F is a closure operator if and only if for all $R \in \mathcal{R}$:*

$$F(R) = \bigcap \{Q \supseteq R | Q = F(Q)\}.$$

Proof. ($\leftarrow$) Let $F(R) = \bigcap\{Q \supseteq R | Q = F(Q)\}$ for all $R \in \mathcal{R}$. Let us demonstrate that F satisfies $C4$, $C3$ and $C5$

(i) $C4$. If $(a, b) \in R$, then, $(a, b) \in Q$ for all $Q \supseteq R$, hence, also for those relations Q that satisfy $Q = F(Q)$. Therefore $(a, b) \in F(R)$.

(ii) $C3$. Let $R \subseteq R'$ and assume that $(a, b) \in F(R)$. Then $(a, b) \in Q$ for all $Q \supseteq R$ that satisfy $Q = F(Q)$. As $R' \supseteq R$, we must have that $(a, b) \in Q'$ for all $Q' \supseteq R'$ that satisfy $Q' = F(Q')$. Hence, $(a, b) \in F(R')$.

(iii) $C5$. If $(a, b) \in F(F(R))$, then $(a, b) \in Q$ for all $Q \supseteq F(R)$ that satisfy $Q = F(Q)$. If, on the contrary $(a, b) \notin F(R)$, there must be a $Q' \supseteq R$ for which $(a, b) \notin Q'$ and $Q' = F(Q')$. As, $Q' \supseteq R$, we derive from $C3$ that $F(Q') \supseteq F(R)$. Together with $Q' = F(Q')$, we deduce that $Q' \supseteq F(R)$. This violates the assumption that $(a, b) \in F(F(R))$.

($\rightarrow$) Let F satisfy conditions $C3$, $C4$ and $C5$. From $C3$ and $C5$, we derive that $F(R) = F(F(R))$ for all $R \in \mathcal{R}$. Hence, if $(a, b) \in Q$ for all $Q \supseteq R$ that satisfy $Q = F(Q)$, we must also have that $(a, b) \in F(R)$. Conclude that $\bigcap\{Q \supseteq R | Q = F(Q)\} \subseteq F(R)$. To see the reverse, let $(a, b) \in F(R)$. By $C3$, we find that $(a, b) \in F(Q)$ for all $Q \supseteq R$. In particular, this must also hold for all Q that satisfy $Q = F(Q)$. Conclude that $F(R) \subseteq \bigcap\{Q \supseteq R | Q = F(Q)\}$. □

2.3 Transitive, convex, monotonic and homothetic extensions

This section applies theorem 2.2 to several properties. The procedure that we will adopt for each of these properties takes the following steps:

i. Define a function F.

ii. Demonstrate that a (complete) relation R^* equals $F(R^*)$ if and only if R^* satisfies transitivity and the desired properties

iii. Verify that F is an algebraic closure operator that satisfies condition $C7$. We do this in three steps. :

  iii.1 Proof that F is a closure operator, i.e. $F(R) = \bigcap\{Q \supseteq R | Q = F(Q)\}$,

  iii.2 show that the closure operator F is algebraic, i.e. F satisfies $C6$, and

  iii.3 confirm that F satisfies $C7$.

iv. Use theorem 2.2 to conclude that a relation, R, has a complete extension $R^* = F(R^*)$ if and only if $F(R) \cap P(R)^{-1} = \emptyset$.

2.3.1 Transitive extensions

In this section, we reproduce the result of Suzumura [1976] that a relation has a complete and transitive extension if and only if it is consistent[3].

We start by introducing some notation and definitions.

Definition 2.3 (Sequence). *A finite sequence s in X of length $n_s \in \mathbb{N}$ is a function*

$$s : \{1, \ldots, n_s\} \to X : i \to s(i).$$

Let S collect all the finite sequences in X. Sometimes, it will be convenient to define the sequence $s \in S$ by the enumeration of its image: $s = s(1), \ldots, s(n_s)$.

For two sequences s_1 and $s_2 \in S$ we may construct the compound sequence $s' \in S$ of length $(n_{s_1} + n_{s_2})$, given by $s' = s_1(1), ..., s_1(n_{s_1}), s_2(1), ..., s_2(n_{s_2})$. We denote this sequence by $s' = s_1 \oplus s_2$.

Definition 2.4 (Transitivity). *A relation R in X is transitive if for all a, b and $c \in X$:*

$$\text{if} \quad (a, b) \in R \quad \text{and} \quad (b, c) \in R \quad \text{then} \quad (a, c) \in R.$$

[3] A relation R is consistent if for each natural number n and each sequence $x_1 \ldots, x_n$ of elements in X, if $x_1 = a$, $x_n = b$ and $(x_i, x_{i+1}) \in R$ for all $i = 1, \ldots, n-1$, then $(b, a) \notin P(R)$.

Now, we are set up to apply steps (i)-(iv) mentioned in the introductory paragraph of section 2.3 We start by introducing the function T.

i. Define the function T

The function $T : \mathcal{R} \to \mathcal{R}$ is given by $(a, b) \in T(R)$ if and only if there is a sequence $s \in S$ such that $s(1) = a$, $s(n_s) = b$ and for all $i = 1, \ldots, n_s - 1$:

$$(s(i), s(i + 1)) \in R.$$

In step (ii), we relate the function T to the property of transitivity.

ii. For all $R \in \mathcal{R} : R = T(R) \leftrightarrow R$ **is transitive.**

Proof. ($\to$) If $(a, b) \in R$ and $(b, c) \in R$, we can construct the sequence $s = a, b, c$, and derive from the definition of T that $(a, c) \in T(R) = R$. Conclude that R is transitive.

($\leftarrow$) Assume that R is transitive. Notice that T satisfies $C4$, i.e. $R \subseteq T(R)$. In order to verify that $T(R) \subseteq R$, assume that $(a, b) \in T(R)$, i.e. there exist a sequence $s \in S$ such that $s(1) = a, s(n) = b$ and for all $i = 1, \ldots, n - 1$, $(s(i), s(i + 1)) \in R$. We have to proof that $(a, b) \in R$. We proceed by induction on n.

For $n = 1$, we immediately derive that $(a, b) = R$.

Suppose the result holds up to $n = \ell$. Consider a sequence, s of length $\ell + 1$ such that $s(1) = a, s(\ell + 1) = b$ and for all $i = 1, ..., \ell$, $(s(i), s(i + 1)) \in R$. From the induction basis, we obtain $(a, s(\ell)) \in R$. From transitivity, and $(s(\ell), b) \in R$, we deduce that $(a, b) \in R$. □

In step (iii), we show that T satisfies the requirements from theorem 2.2.

iii. The function T **is an algebraic closure operator which satisfies property** $C7$**.**

First we show that T is a closure operator.

iii.1 For all $R \in \mathcal{R} : T(R) = \bigcap\{Q \supseteq R | Q = T(Q)\}$**.**

Proof. ($\subseteq$) Let $(a, b) \in T(R)$ and assume that $R \subseteq Q = T(Q)$. Then, there exist a sequence $s \in S$ such that $s(1) = a, s(n_s) = b$ and for all $i = 1, \ldots, n_s - 1$, $(s(i), s(i + 1)) \in R \subseteq Q$. Conclude that $(a, b) \in T(Q) = Q$. Hence, $(a, b) \in \bigcap\{Q \supseteq R | Q = T(Q)\}$.

($\supseteq$) First, we verify that $T(R)$ is transitive. Therefore, assume that (a, b) and $(b, c) \in T(R)$. Then, there are sequences $s, s' \in S$ such that $s(1) = a, s(n_s) = b = s'(1), s'(n_{s'}) = c$ and for all $i = 1, \ldots, n_s - 1$ and $j = 1, \ldots, n_{s'} - 1$, $(s(i), s(i + 1)$ and $(s(j), s(j + 1)) \in R$. If we use the sequence $s'' = s(1), \ldots, s(n_s), s'(2), \ldots, s'(n_{s'})$ in the definition of $T(.)$, we derive that $(a, b) \in T(R)$. Infer from section 2.3.1.ii that $T(T(R)) = T(R)$ and that $T(R) \in \{Q \supseteq R | Q = T(Q)\}$. □

Next, we show that T is algebraic.

iii.2 The function T satisfies condition $C6$.

Proof. Let $(a, b) \in T(R)$. Then, there is a sequence $s \in S$ for which $s(1) = 1, s(n_s) = b$ and for all $i = 1, \ldots, n_s - 1$, $(s(i), s(i+1)) \in R$. Construct the finite relation $R \cap D \times D$ with $D = \{s(1), \ldots, s(n_s)\}$ and observe that $(a, b) \in T(R \cap D \times D)$. Conclude that $T(.)$ satisfies $C6$. □

Finally, we show that T satisfies $C7$.

iii.3 The function T satisfies condition $C7$.

Proof. Let $R = T(R)$ and $N(R) \neq \emptyset$. Let us show that for $(a, b) \in N(R)$, the relation $F(R \cup \{(a, b)\})$ extends $R \cup \{(a, b)\}$. Define $R' = R \cup \{(a, b)\}$.

By $C4$ and lemma 2.1, we only need to verify that $T(R') \cap P(R')^{-1} = \emptyset$. Assume, on the contrary that $(c, d) \in T(R')$ and $(d, c) \in P(R')$ and consider first the case where $(c, d) \neq (a, b)$. Then there exist a sequence $s \in S$ with $s(1) = c, s(n_s) = d$ and for all $i = 1, \ldots, n_s - 1$, $(s(i), s(i+1)) \in R'$. Clearly, there is an i for which $(s(i), s(i+1)) = (a, b)$. Else, we would deduce that $(c, d) \in T(R) = R$, a contradiction.

Let ℓ be the highest interger such that $(s(\ell - 1), s(\ell)) = (a, b)$ and let f be the smallest integer such that $(s(f), s(f+1)) = (a, b)$. Apply the definition of T to the sequence $s' = s(\ell), \ldots, s(n_s), s(1), \ldots, s(f)$ and conclude that $(b, a) \in T(R) = R$, a contradiction.

If $(c, d) = (a, b)$, we apply the definition of T to the sequence $s' = s(\ell), ..., s(n_s)$ (if there is no i for which $(s(i), s(i+1)) = (a, b)$, we put $\ell = 1$) and conclude again that $(b, a) \in T(R) = R$, a contradiction . □

iv. Conclusion

The function T is an algebraic closure operator that satisfies $C7$. We can use theorem 2.2 and proof that R has a complete and transitive extension if and only if $T(R) \cap P(R)^{-1} = \emptyset$.

2.3.2 Convex extensions

In this section, we look for the existence of complete, transitive and convex extensions (see also Bossert and Sprumont [2001] and Scapparone [1999]). We assume that X is a convex[4] subset of $\mathbb{R}^n$.

[4] It is possible to reproduce the results of this section without this condition. However, this would drastically increase the notational complexity without really adding something fundamental to the analysis.

For each finite set $A \subseteq X$, we denote by $V(A)$ the interior of the convex hull spanned by the elements of A:

$$V(A) = \left\{ a \in X \middle| a = \sum_{x_i \in A} \alpha_i x_i \text{ where for all i, } \alpha_{\text{i}} > 0 \text{ and } \sum_{\text{i}} \alpha_{\text{i}} = 1 \right\}.$$

The definition of convexity has many variants, depending on the additional requirements imposed on the relation under consideration[5] (e.g. completeness). We will use the following:

Definition 2.5 (Convexity). *A relation R is convex if for all finite sets $A \subseteq X$ and all $b \in X$:*

- *if $(x_i, b) \in R$ for all $x_i \in A$, then for all $a \in V(A) : (a, b) \in R$, and*
- *if $(x_i, b) \in R$ for all $x_i \in A$ and there is an $x_j \in A$ for which $(x_j, b) \in P(R)$, then for all $a \in V(A) : (a, b) \in P(R)$.*

Let us determine the condition for which a relation has a complete, transitive and convex extension by going through steps (i)-(iv) of the introductory paragraph of this section. The most intuitive way to develop the function F (step (i)) is to work in a sequential manner. Indeed, it is possible to define an algebraic closure operator $C^*(.)$ such that (for all complete relations R^*) $R^* = C^*(R^*)$ if and only if R^* is convex. Then, one may define the function $F(R) = T(C^*(R))$ and verify that $F(R^*) = R^*$ if and only if R^* is transitive and R^* is convex (this is step (ii)). Unfortunately, the function $F(.)$ is not a closure operator, hence, it does not satisfy step (iii). Therefore, it is necessary to impose both conditions (transitivity and convexity) simultaneously in the same function. Let us define the function C.

i. Define the function C

Consider a finite number of sequences $s_1, ..., s_m \in S$. For an element $s_j(i)$, $i < n_{s_j}$, we say that the set A is compatible with $s_j(i)$ if

- $A \subseteq \{s_k(v) | k \in \{1, ..., m\}, v \in \{1, ..., n_{s_k}\}\}$ and,
- $s_j(i + 1) \in A$.

Given the list of sequences $s_1, ..., s_m$, we denote by $\mathcal{A}(s_j(i); s_1, .., s_m)$ the collection of all sets A which are compatible with $s_j(i)$.

To simplify notation, we also write $\mathcal{A}(s_j(i))$ instead of $\mathcal{A}(s_j(i); s_1, ..., s_m)$.

The function $C : \mathcal{R} \to \mathcal{R}$ is defined in the following way: for a relation $R \in \mathcal{R}$ we write $(a, b) \in C(R)$ if there exists a finite number of sequences $s_1, ..., s_m \in S$ such that for all $j = 1, \ldots, m$: $s_j(1) = a$, $s_j(n_{s_j}) = b$ and for all $j = 1, \ldots, m$ and $i = 1, \ldots, n_{s_j} - 1$:

- $(s_j(i), s_j(i + 1)) \in R$ or

[5] See Mas-Colell et al. [1995, p.44] for the intuition behind the idea that preferences should be convex.

- there is a set $A \in \mathcal{A}(s_j(i))$ such that $s_j(i) \in V(A)$.

We will validate that C is an algebraic closure operator which satisfies $C7$, but let us begin by indicating how C relates to the the property of convexity.

ii. If R is complete, then $R = C(R)$ if and only if R is transitive and convex.

Proof. Necessity is straightforward, hence we only show sufficiency.

Assume that R is complete, transitive and convex. From convexity, we deduce that for all finite sets $A \subseteq X$ and all $c \in V(A)$, it is not the case that:

- $(x_i, c) \in R$ for all $x_i \in A$ and $(x_j, c) \in P(R)$ for at least one element $x_j \in A$.

Otherwise, we would deduce that $(c, c) \in P(R)$, a contradiction. Completeness of R, allows us to rewrite this conditions in the following way:
For all $A \subseteq X$, if $c \in V(A)$ then:

- there is an $x_j \in A$ for which $(c, x_j) \in P(R)$ or (1)
- for all $x_i \in A : (c, x_i) \in R$. (2)

Assume that $(a, b) \in C(R)$. Then, there are a finite number of sequences $s_1, \ldots, s_m$ such that for each sequence $j = 1, \ldots, m$: $s_j(1) = a$, $s_j(n_{s_j}) = b$ and for each $i = 1, \ldots, n_{s_j} - 1$ either $(s_j(i), s_j(i+1)) \in R$ or $s_j(i) \in V(A)$ for some $A \in \mathcal{A}(s_j(i))$. We must demonstrate that $(a, b) \in R$.

We proceed by constructing a sequence $s' = s'(1), \ldots, s'(n_{s'})$ such that $s'(1) = a$, $s'(n_{s'}) = b$ and for all $i = 1, \ldots, n_{s'} - 1$: $(s'(i), s'(i+1)) \in R$. The result follows from transitivity of R and section 2.3.1.ii. Consider the following algorithm:

1. Initiate $s'(1) = s_1(1)$ and set $k = 1$. To to step 2.
2. if $s'(k) = b$, we stop. Otherwise, we increase k by one, i.e. $k := k + 1$. Go to step 3.
3. for $s'(k-1) = s_j(i)$, if $(s_j(i), s_j(i+1)) \in R$, we set $s'(k) = s_j(i+1)$ and return to step 2, otherwise, go to step 3.
4. for $s'(k-1) = s_j(i)$, if $s_j(i) \in V(A)$ for some $A \in \mathcal{A}(s_j(i))$, we know from the first part of the proof that there are two cases to consider:
   (a) if (2) holds, we derive that $(s_j(i), s_j(i+1)) \in R$. Then we put $s'(k) = s_j(i+1)$, and we return to step 2,
   (b) if (1) holds, we derive that $(s_j(i), s_w(v)) \in P(R)$, for some element $s_w(v)$ in some sequence s_w. Then we put $s'(k) = s_w(v)$ and we return to step 2,

The algorithm terminates only at step 2, i.e. at the value b. Therefore, the algorithm is well-defined if it reaches this step after a finite number of steps. If, on the contrary, the algorithm does not terminate after a finite number of steps, then, by finiteness of the sequences $s_1, \ldots, s_m$, there must be a loop in the sequence $s'(1), s'(2), \ldots, s'(f), \ldots, s'(\ell), \ldots$ Suppose that $s'(f)$ and $s'(l)$ correspond to the same element in the same sequence. This only occurs if the algorithm passes through step 4.b. Agree, that there must be a strict relation involved, for example $(s'(v), s'(v+1)) \in P(R)$ (with $f \leq v \leq \ell$). Transitivity of R, and the result from section 2.3.1.ii, establishes that $(s'(v+1), s'(v)) \in R$, a contradiction. Hence, the algorithm terminates after a finite number of steps. Conclude that $(a, b) \in R$. □

iii. The function C is an algebraic closure operator which satisfies condition $C7$

We start by showing that C is a closure operator.

iii.1 For all $R \in \mathcal{R}$: $C(R) = \bigcap\{Q \supseteq R | Q = C(Q)\}$.

Proof. It is straightforward to verify that $C(X \times X) = X \times X$. Hence, for all $R \in \mathcal{R}$, the set $\{Q \supseteq R | Q = C(Q)\}$ is non-empty.

($\subseteq$) Let $(a, b) \in C(R)$ and $Q \in \{Q' \supseteq R | Q' = C(Q')\}$. Then, there exists a finite number of sequences $s_1, \ldots, s_m$ where for all $j = 1, \ldots, m$, $s_j(i) = a, s_j(n_{s_j}) = b$ and for all $i = 1, \ldots, n_{s_j} - 1$, $(s_j(i), s_j(i+1)) \in R \subseteq Q$ or $s_j(i) \in V(A)$ for some $A \in \mathcal{A}(s_j(i))$. Hence, $(a, b) \in C(Q) = Q$. Conclude that $(a, b) \in \bigcap\{Q \supseteq R | Q = C(Q)\}$.

($\supseteq$) Let us begin by showing that $C(C(R)) = C(R)$. Evidently $C(R) \subseteq C(C(R))$. To see the reverse, consider elements a and $b \in X$ and assume that $(a, b) \in C(C(R))$. From the definition of C, there must be a finite number of sequences $s_1, \ldots, s_m$ in S such that for all $j = 1, \ldots, m$: $s_j(1) = a$, $s_j(n_{s^j}) = b$, and for all $i = 1, \ldots, n_{s_j} - 1$ either $(s_j(i), s_j(i+1)) \in C(R)$ or $s_j(i) \in V(A)$, where $A \in \mathcal{A}(s_j(i))$. For each $j = 1, \ldots, m$ and $i = 1, \ldots, n_s - 1$, there are two cases:

(i) $(s_j(i), s_j(i+1)) \in R$. Then there are sequences $s_1^{(j,i)}, \ldots, s_{m^{(j,i)}}^{(j,i)}$ where for all $v = 1, \ldots, m^{(j,i)}$, $s_v^{(j,i)}(1) = s_j(i)$, $s_v^{(j,i)}(n_{s^{(j,i)}}) = s_j(i+1)$ and for all $w = 1, \ldots, n_{s_v^{(j,i)}} - 1$, $(s_v^{(j,i)}(w), s_v^{(j,i)}(w+1)) \in R$ or $s_v^{(j,i)}(w) \in V(A)$ with $A \in \mathcal{A}(s_v^{(j,i)}(w))$. Let $q_v^{(j,i)}$ be the sequence $s_v^{(j,i)}$ without its last element.

(ii) $s_j(i) \in V(A)$. Let $s_1^{(j,i)} = s_1^{(j,i)}(1), s_1^{(j,i)}(2)$, where $s_1^{(j,i)}(1) = s_j(i)$ and $s_1^{(j,i)}(2) = s_j(i+1)$. Let $q_1^{(j,i)}$ be the sequence $s_1^{(j,i)}$ without its last element, i.e. the single element sequence $q_1^{(j,i)} = s_1^{(j,i)}(1)$.

For each $j = 1, \ldots, m$, each $i = 1, \ldots, n_{s_j} - 1$ and each $v = 1, \ldots, m^{(j,i)}$, consider the following compound sequence:

$$s^{(j,i,v)} = q_1^{(j,1)} \oplus q_1^{(j,2)} \oplus \ldots \oplus q_1^{(j,i-1)} \oplus q_v^{(j,i)} \oplus q_1^{(j,i+1)} \oplus \ldots \oplus q_1^{(j,n_{s^j}-2)} \oplus s_1^{(j,n_{s_j}-1)}.$$

Using these sequences in the definition of C, we deduce that $(a, b) \in C(R)$. Hence $C(C(R)) = C(R)$. Conclude that $C(R) \in \{Q \supseteq R | Q = C(Q)\}$. □

Now we demonstrate that the closure operator C is algebraic.

iii.2 The function C satisfies condition $C6$.

Proof. Consider a relation R and assume that $(a, b) \in C(R)$. Then, there exists a finite number of sequences $s_1, ..., s_m$ where for all $j = 1, \dots, m$, $s_j(1) = a$, $s_j(n_{s_j}) = b$ and for all $i = 1, \dots, n_{s_j} - 1$, $(s_j(i), s_j(i+1)) \in R$ or $s_j(i) \in V(A)$ for some $A \in \mathcal{A}(s_j(i))$. Then, $R \cap D \times D$ with, $D = \{s_j(i) | j = 1, \dots, m; i = 1, \dots, n_{s_j}\}$, is a finite relation and $(a, b) \in C(R \cap D \times D)$. Conclude that $C(.)$ satisfies $C6$. □

Finally we verify condition $C7$.

iii.3 The function C satisfies condition $C7$.

Proof. Let $R = C(R)$ and assume that $N(R) \neq \emptyset$. We need to find a nonempty subset T of $N(R)$ such that $R \cup T \in \mathcal{R}^*$. Let $(a, b) \in N(R)$ and consider the relation $R' = R \cup \{(a, b)\}$. We will show that $R' \preceq C(R')$.

By $C4$ and lemma 2.1, we know that we can finish the proof if we demonstrate that $P(R')^{-1} \cap C(R') = \emptyset$. Assume, on the contrary, that there are elements c and $d \in X$ for which $(c, d) \in P(R')$ and $(d, c) \in C(R')$ and consider first the case where $(c, d) \neq (a, b)$.
From the definition of C, we know that there exists a finite number m of sequences $s_1, \dots, s_m \in S$ such that for all $j = 1, \dots, m$, $s_j(1) = d$, $s_j(n_{s_j}) = c$, and for all $i = 1, \dots, n_{s_j} - 1$, $(s_j(i), s_j(i+1)) \in R'$ or $s_j(i) \in V(A)$ for some $A \in \mathcal{A}(s_j(i))$. If for all $s_j(i)$ with $(s_j(i), s_j(i+1)) \in R'$ also $(s_j(i), s_j(i+1)) \in R$, then $(d, c) \in C(R) = R$, a contradiction. Hence, there must be at least one $s_j(i)$ for which $(s_j(i), s_j(i+1)) = (a, b)$.

For any sequence s_j ($j = 1, ..., m$) there are two cases to consider.

1 There is an $i = 1, ..., n_{s_j} - 1$ for which $(s_j(i), s_j(i+1)) = (a, b)$.

2 There is no $i = 1, ..., n_{s_j} - 1$ for which $(s_j(i), s_j(i+1)) = (a, b)$.

As argued above, the set of sequences that fall under case 1 is not empty. Furthermore, for all sequences s_j that fall under case 1 and for all $i = 1, \dots, n_{s_j} - 1$ for which $(s_j(i), s_j(i+1)) = (a, b)$ there are again two cases to consider:

1.1 There is a $v > i$ such that $(s_j(v), s_j(v+1)) = (a, b)$.

1.2 There is no $v > i$ such that $(s_j(v), s_j(v+1)) = (a, b)$.

For each sequence s_j under case 1 and for each $i \in \{1, \dots, n_{s_j} - 1\}$ under case 1.1, we consider the smallest integer $w > i$ such that $(s_j(w), s_j(w+1)) = (a, b)$ and we construct the sequence:

$$s_j(i+1), s_j(i+2), ..., s_j(w-1), s_j(w). \tag{2.1}$$

For each sequence s_j under case 1 and for each $i \in \{1, \ldots, n_{s_j} - 1\}$ under case 1.2, we consider the smallest integer $w > 0$ such that $(s_j(w), s_j(w+1)) = (a, b)$ and we construct the sequence:

$$s_j(i+1), ..., s_j(n_{s_j}), s_j(1), ..., s_j(w-1), s_j(w). \tag{2.2}$$

Consider a sequence s_k that falls under case 1 and Assume that ℓ is the largest integer for which $(s_k(\ell), s_k(\ell)) = (a, b)$. Assume that f is the smallest integer such that $(s_k(f), s_k(f+1)) = (a, b)$.

For each sequence s_j that falls under case 2, we construct the sequence:

$$s_k(\ell+1), ..., s_k(n_{s_k}), s_j(1), s_j(2), ..., s_j(n_{s_j}), s_k(1), s_k(2), ..., s_k(f). \tag{2.3}$$

Applying the definition of C to the finite number of sequences that are constructed by (2.1), (2.2) and (2.3), we establish that $(b, a) \in C(R) = R$, a contradiction. The proof for the case where $(c, d) = (a, b)$ is very similar and is left to the reader. □

iv Conclusion.

We see that the function C is an algebraic closure operator which satisfies $C7$. If we apply theorem 2.2 to the function C, we can conclude that a relation R has a convex, transitive and complete extension if and only if $C(R) \cap P(R)^{-1} = \emptyset$.

2.3.3 Monotonic extensions

The third part of this section focusses on the properties of monotonicity and strict monotonicity[6]. We assume that X is a subset of $\mathbb{R}^m$ and for two elements a and b in X, we say that $a \geq b$ if each component of a is greater than the corresponding component of b. Further, $a > b$ if $a \geq b$ and $a \neq b$.

Definition 2.6 (Monotonicity). *A relation R is monotonic if for all $a, b \in X$:*

$$\text{if } \ a \geq b \ \text{ then } \ (a, b) \in R.$$

Definition 2.7 (Strict monotonicity). *A relation R on X is strict monotonic if R is monotonic and for all $a, b \in X$:*

$$\text{if } \ a > b \ \text{ then } \ (a, b) \in P(R).$$

Given a relation R, we define the relation $\overline{R}$ as:

$$\overline{R} = R \cup \{(a, b) \in X \times X \,|a \geq b\}.$$

Consider a function $F : \mathcal{R} \to \mathcal{R}$ and assume that F is an algebraic closure operator that satisfies $C7$, e.g. the function T or C.

[6] See Mas-Colell et al. [1995, p.42] for a discussion of monotonicity and strict monotonicity for preference relations.

In this section, we will characterize the set of relations which have a complete and (strict) monotonic extension $R^* = F(R^*)$.

We begin by defining the function $\overline{F}$.

i. Define the function $\overline{F}$.

Consider a function $F : \mathcal{R} \to \mathcal{R}$. Then we can define the function $\overline{F}$ by:

$$\overline{F}\left(R\right) = F\left(\overline{R}\right).$$

Let us indicate how $\overline{F}$ relates to the property of monotonicity.

ii. $R = \overline{F}(R)$ if and only if R is monotonic and $R = F(R)$

Proof. ($\rightarrow$). (i) Monotonicity. If $a \geq b$, then immediately $(a, b) \in \overline{R}$. Furthermore, by $C4$, $\overline{R} \subseteq \overline{F}(R) = R$. Conclude that $(a, b) \in R$.

(ii) $R = F(R)$. By $C4$, $R \subseteq F(R)$. By $C3$ and $R \subseteq \overline{R}$, we derive that $F(R) \subseteq \overline{F}(R) = R$.

($\leftarrow$). Let $R = F(R)$ and assume that R is monotonic. Monotonicity implies $R = \overline{R}$. Then $\overline{R} = R = F(R) = F(\overline{R}) = \overline{F}(R)$. □

iii. The function $\overline{F}$ is an algebraic closure operator that satisfies condition $C7$.

iii.1 For all $R \in \mathcal{R} : \overline{F}(R) = \bigcap\{Q \supseteq R | Q = \overline{F}(Q)\}$.

Proof. Clearly $\overline{F}(X \times X) = X \times X$. Therefore, the set $\{Q \supseteq R | Q = \overline{F}(Q)\}$ is not empty.

As F is a closure operator, we know that $F(\overline{R}) = \{Q \supseteq \bar{R} | Q = F(Q)\}$. Therefore, it suffices to show that $\{Q \supseteq R | Q = \overline{F}(Q)\} = \{Q \supseteq \overline{R} | Q = F(Q)\}$.

($\subseteq$) Let $R' \in \{Q \supseteq R | Q = \overline{F}(Q)\}$. From section 2.3.3.ii, R' is monotonic and $R' = F(R')$. From monotonicity: $\overline{R} \subseteq R'$. Conclude that $R' \in \{Q \supseteq \overline{R} | Q = F(Q)\}$.

($\supseteq$) Let $R' \in \{Q \supseteq \bar{R} | Q = F(Q)\}$. From $\overline{R} \subseteq R'$, we derive that R' is monotonic. Together with $R' = F(R')$, we know from section 2.3.3.ii that $R' = \overline{F}(R')$. Conclude that $R' \in \{Q \supseteq R | R' = \overline{F}(R')\}$. □

iii.2 The function $\overline{F}$ satisfies condition $C6$.

Proof. Let $(a, b) \in \overline{F}(R) = F(\overline{R})$. As F satisfies condition $C6$, we know that there exists a finite subset R' of $\overline{R}$ such that $(a, b) \in F(R')$. As $R' \subseteq \overline{R'}$, we derive from $C3$ that $(a, b) \in F(\overline{R'}) = \overline{F}(R')$. Conclude that $\overline{F}$ satisfies $C6$. □

iii.3 The function $\overline{F}$ satisfies condition $C7$.

Proof. Let $R = \overline{F}(R)$ and assume that $N(R) \neq \emptyset$. As $R \subseteq \overline{R}$ and $\overline{R} \subseteq F(\overline{R}) = R$ we have that $R = \overline{R}$. Applying $C7$ to the function F verifies the existence of a set $T \subseteq N(R)$ for which $R \cup T \preceq F(R \cup T)$. Clearly, $R \cup T = \overline{R \cup T}$. Therefore $F(R \cup T) = F(\overline{R \cup T}) = \overline{F}(R \cup T)$. Conclude that $R \cup T \preceq \overline{F}(R \cup T)$. □

iv. Conclusion.

We know that $\overline{F}$ is an algebraic closure operator that satisfies condition $C4$. Using theorem 2.2, we establish that a relation R has a complete and monotonic extension $R' = F(R')$ if and only if $\overline{F}(R) \cap P(R)^{-1} = \emptyset$.

We can derive a similar result regarding the property of strict monotonicity:

If F is an algebraic closure operator satisfying $C7$, then a relation R has a strict monotonic and complete extension $R^* = F(R^*)$ if and only if $\overline{F}(R) \cap P(R)^{-1} = \emptyset$ and for all $b > a$:

$$(a, b) \notin \overline{F}(R).$$

Proof. ($\leftarrow$) First, notice that $\overline{F}(R) \cap P(R)^{-1} = \emptyset$ is a necessary condition to have a monotonic and complete extension $R^* = F(R^*)$. Hence, it is also necessary to have a strict monotonic and complete extension $R^* = F(R^*)$. Second, if on the contrary $b > a$ and $(a, b) \in \overline{F}(R)$, we have by $C3$ and $R \subseteq R^*$, that $(a, b) \in F(R^*) = R^*$, a contradiction.

($\rightarrow$) Assume that $\overline{F}(R) \cap P(R)^{-1} = \emptyset$ and for all $b > a$, $(a, b) \notin \overline{F}(R)$. From the first result in this section, we conclude that $\overline{F}(R)$ has a complete extension $R^* = \overline{F}(R^*)$. This relation also extends R. Let us verify that R^* is strict monotonic. Consider two elements a and $b \in X$ for which $a > b$. We have that $(a, b) \in \overline{F}(R)$ and $(b, a) \notin \overline{F}(R)$, hence, $(a, b) \in P(\overline{F}(R))$. Deduce that $(a, b) \in P(R^*)$ and that R^* is strict monotonic. □

2.3.4 Homothetic extensions

The final part in this section concentrates on the property of homotheticity[7] . We assume that X is a subset of $\mathbb{R}_+^m$, where $\mathbb{R}_+$ is the set of positive reals. Further, we assume that X is a cone, i.e., if $a \in X$, then for all $\alpha \in \mathbb{R}_{++} : \alpha a \in X$, where $\mathbb{R}_{++}$ is the set of strict positive reals.

Definition 2.8 (Homotheticity)**.** *A relation R is homothetic if for all elements a and $b \in X$ and all $\alpha \in \mathbb{R}_{++}$*

$$\text{if} \quad (a, b) \in R \quad \text{then} \quad (\alpha a, \alpha b) \in R.$$

It turns out that homotheticity is much easier to analyze jointly with monotonicity. Thus, we look for the necessary and sufficient conditions for a relation to have a complete, transitive, homothetic and monotonic extension.

We start by defining the function H.

[7]See Mas-Colell et al. [1995, p.45] for the main reasons why preferences are assumed to be homothetic.

i. Define the function H.

The function H is given by, $(a, b) \in H\,(R)$ if there is a sequence $s \in S$ with $s(1) = a$, $s(n_s) = b$ and for all $i = 1, \ldots, n_s - 1$:

- $s(i) \geq s(i + 1)$, or
- there is an $\alpha_i \in \mathbb{R}_{++}$ for which $(\alpha_i s(i), \alpha_i s(i + 1)) \in R$.

In the step (ii), we relate the function H to the properties of homotheticity and monotonicity.

ii. For all $R \in \mathcal{R}$, $R = H(R)$ if and only if R is transitive, homothetic and monotonic.

Proof. Sufficiency is straightforward, so we only verify necessity.

Assume that R is transitive, homothetic and monotonic. Obviously $R \subseteq H(R)$, hence, we only need to verify that $H(R) \subseteq R$. Let $(a, b) \in H(R)$. Then, there is a sequence s with $s(1) = a, s(n_s) = b$ and for all $i = 1, \ldots, n_s - 1$, $s(i) \geq s(i + 1)$ or there is an $\alpha_i \in \mathbb{R}_{++}$ for which $(\alpha_i s(i), \alpha_i s(i + 1)) \in R$. From monotonicity and homotheticity, we deduce that $(s(i), s(i + 1)) \in R$ for all $i = 1, \ldots, n_s - 1$. Transitivity combined with the result from section 2.3.1.ii implies that $(a, b) \in R$. □

iii. The function H is an algebraic closure operator which satisfies condition $C7$.

We start by showing that H is a closure operator.

iii.1 For all $R \in \mathcal{R} : H(R) = \bigcap\{Q \supseteq R | Q = H(Q)\}$.

Proof. From condition $C4$: $H(X \times X) = X \times X$. This implies that $\{Q \supseteq R | Q = H(Q)\}$ is non-empty for all $R \in \mathcal{R}$.

($\subseteq$) Let $(a, b) \in H(R)$ and let $R' \in \{Q \supseteq R | Q = H(Q)\}$. Then, there exists a sequence $s \in S$ with $s(1) = a$, $s(n_s) = b$ and for all $i = 1, \ldots, n_s - 1$, $s(i) \geq s(i + 1)$ or there is an $\alpha_i \in \mathbb{R}_{++}$ such that $(\alpha_i s(i), \alpha_i s(i + 1)) \in R \subseteq R'$. Then $(a, b) \in H(R') = R'$. Conclude that $(a, b) \in \bigcap\{Q \supseteq R | Q = H(Q)\}$.

($\supseteq$) Let us begin by demonstrating that $H(R)$ is transitive, homothetic and monotonic.

(i) Transitivity. Let $(a, b) \in H(R)$ and $(b, c) \in H(R)$. Then, there are sequences s and $s' \in S$ with $s(1) = a$, $s(n_s) = s'(1) = b$, $s'(n_{s'}) = c$, for all $i = 1, \ldots, n_s - 1$: $s(i) \geq s(i + 1)$ or there is an $\alpha_i > 0$ for which $(\alpha_i s(i), \alpha_i s(i + 1)) \in R$ and for all $j = 1, \ldots, n_{s'} - 1$: $s'(j) \geq s'(j + 1)$ or there is an $\alpha_j > 0$ for which $(\alpha_j s'(j), \alpha_j s'(j + 1)) \in R$. If we apply the definition of H to the sequence $s'' = s(1), \ldots, s(n_s), s'(2), \ldots, s'(n_{s'})$, we find that $(a, b) \in H(R)$. Hence, $H(R)$ is transitive.

(ii) Homotheticity. Let $(a, b) \in H(R)$ and $\beta > 0$. Then, there is a sequence $s \in S$ with $s(1) = a$, $s(n_s) = b$ and for each $i = 1, \ldots, n_s - 1$: $s(i) \geq s(i + 1)$ or there is an $\alpha_i \in \mathbb{R}_{++}$ for which $(\alpha_i s(i), \alpha_i s(i + 1)) \in R$. Construct the sequence $s' = \beta s(1), \beta s(2), \ldots, \beta s(n_s)$. If $s(i)$ satisfies $s(i) \geq s(i + 1)$, then $s'(i) \geq s'(i + 1)$, and if $s(i)$ satisfies $(\alpha_i s(i), \alpha_i s(i + 1)) \in R$ we can construct $\alpha_i' = \frac{\alpha_i}{\beta} > 0$, to find that $(\alpha_i' s'(i), \alpha_i' s'(i+1)) \in R$. Therefore, $(\beta a, \beta b) \in H(R)$. Conclude that $H(R)$ is homothetic.

(iii) Monotonicity. Let $a \geq b$. Using the sequence $s = a, b$, we immediately derive that $(a, b) \in H(R)$.

From 2.3.4.ii, we infer that $H(H(R)) = H(R)$. Conclude that $\bigcap\{Q \supseteq R | Q = H(Q)\} \subseteq H(R)$. □

Now we verify that H is algebraic.

iii.2 The function H satisfies condition $C6$.

Proof. Let $(a, b) \in H(R)$. Then, there is a sequence $s \in S$ with $s(1) = a$, $s(n_s) = b$ and for all $i = 1, \ldots, n_s - 1$, $s(i) \geq s(i+1)$ or there is an $\alpha_i \in \mathbb{R}_{++}$ for which $(\alpha_i s(i), \alpha_i s(i+1)) \in R$. The relation $R \cap (D \times D)$, with $D = \{s(1), s(2), \ldots, s(n_s)\}$ is finite and satisfies $(a, b) \in H(R \cap (D \times D))$. Conclude that H satisfies $C6$. □

Finally, we establish that H satisfies condition $C7$.

iii.3 The function H satisfies condition $C7$.

Proof. Let $R = H(R)$ and $(a, b) \in N(R)$. Construct the relation $R' = R \cup \{(a, b)\}$. We prove that $R' \in \mathcal{R}^*$. By $C4$ and lemma 2.1, this simplifies to $H(R') \cap P(R')^{-1} = \emptyset$. Assume, on the contrary, that $(c, d) \in P(R')$ and $(d, c) \in H(R')$. Then, there is a sequence $s \in S$ with $s(1) = d$, $s(n_s) = c$ and for all $i = 1, \ldots, n_s - 1$: $s(i) \geq s(i+1)$ or there is an $\alpha_i \in \mathbb{R}_{++}$ for which $(\alpha_i s(i), \alpha_i s(i+1)) \in R'$.

If for all $i = 1, \ldots, n - 1$ for which $(\alpha_i s(i), \alpha_i s(i+1)) \in R'$ also
$(\alpha_i s(i), \alpha_i s(i+1)) \in R$, then $(d, c) \in H(R) = R$, a contradiction. Hence, there must be at least one $i = 1, \ldots, n_s - 1$ with $(\alpha_i s(i), \alpha_i s(i+1)) = (a, b)$.

From finiteness of $\{s(1), \ldots, s(n_s)\}$, it follows that there is a number $q \in \mathbb{N}$ and a finite set $I = \{\beta_1, \ldots, \beta_q\}$ of elements in $\mathbb{R}_{++}$ such that for all $i = 1, \ldots, q - 1$:$\left(\frac{1}{\beta_i} b, \frac{1}{\beta_{i+1}} a\right) \in H(R) = R$, and $\left(\frac{1}{\beta_q} b, \frac{1}{\beta_1} a\right) \in H(R) = R$. Consider the smallest value from the set I, say β_j. If $j > 1$, by homotheticity of R, we get $\left(b, \frac{\beta_{j-1}}{\beta_j} a\right) \in R$ and by monotonicity, $\left(\frac{\beta_{j-1}}{\beta_j} a, a\right) \in R$. By transitivity of R, we derive that $(b, a) \in R$, a contradiction. If $j = 1$, we have that $\left(b, \frac{\beta_q}{\beta_1} a\right) \in R$ and $\left(\frac{\beta_q}{\beta_1} a, a\right) \in R$. Again by transitivity: $(b, a) \in R$, a contradiction. Conclude that H satisfies $C7$. □

iv. Conclusion.

The function H is an algebraic closure operator and satisfies $C7$. We can apply theorem 2.2 and conclude that a relation R has a homothetic, monotonic, complete and transitive extension if and only if $H(R) \cap P(R)^{-1} = \emptyset$.

2.4 F-rationalizability

Let X be a set of alternatives and let Σ be a collection of nonempty subsets of X. A choice function K is a correspondence

$$K : \Sigma \twoheadrightarrow X : S \rightarrow K(S) \subseteq S,$$

such that for all $S \in \Sigma$, $K(S)$ is non-empty.

Definition 2.9 (F-rationalizability). *A choice function K is F-rationalizable if there exists a complete relation $R^* = F(R^*)$, such that for all $S \in \Sigma$:*

$$K(S) = \{a \in S \,|(a, b) \in R^* \text{ for all } b \in S\},$$

i.e. the elements chosen from S are top-ranked according to R^*.

We assume that the function F contains the notion of transitivity, i.e. for all $R \in \mathcal{R}$, if $(a, b) \in T(R)$, then $(a, b) \in F(R)$. For a choice function K, we define the revealed preference relation R_v by $(a, b) \in R_v$ if there is a set $S \in \Sigma$ such that $a \in K(S)$ and $b \in S$. If also $b \notin K(S)$, we say that a is strictly revealed preferred to b and write $(a, b) \in P_v$.

We can now present the characterization result for F-rationalizability.

Theorem 2.3. *If F satisfies $C1$, $C2$ and $C3$ then a choice function K is F-rationalizable if and only if $R_v \cap P_v^{-1} = \emptyset$.*

Proof. First of all, notice that by $R \subseteq T(R)$ and $T(R) \subseteq F(R)$, we have that F satisfies $C4$: for all $R \in \mathcal{R} : R \subseteq F(R)$.

($\rightarrow$) If K is F-rationalizable, there exists a complete relation R^* such that $R^* = F(R^*)$, and $a \in K(S)$ implies that $(a, b) \in R^*$ for all $b \in S$. As $T(R^*) \subseteq F(R^*)$, we have that R^* is also transitive. Assume, on the contrary, that $(a, b) \in F(R_v) \cap P_v^{-1}$. It is easy to verify that $R_v \subseteq R^*$, hence, by $C3$, we find that $F(R_v) \subseteq F(R^*) = R^*$. Hence, $(a, b) \in R^*$. From $(b, a) \in P_v$, there is a $S \in \Sigma$ such that $b \in K(S)$ and $a \in S - K(S)$. Let us deduce that $(b, a) \in P(R^*)$.

From $R_v \subseteq R^*$, we derive that $(b, a) \in R^*$. If, on the contrary, also $(a, b) \in R^*$, then by transitivity of R^*, $(a, c) \in R^*$ for all $c \in S$. This implies, from rationalizability of K, that $a \in K(S)$, a contradiction.

($\leftarrow$) Let $F(R_v) \cap P_v^{-1} = \emptyset$. It is easy to verify that this implies that $P_v = P(R_v)$. Hence, by lemma 2.1 and $C4$: $R_v \preceq F(R_v)$. By theorem 2.1, R_v has a complete extension, $R^* = F(R^*)$. Let us establish that R^* rationalizes F. If $a \in K(S)$, by definition $(a, b) \in R_v$ for all $b \in S$ and hence $(a, b) \in R^*$ for all $b \in S$. On the other hand, if $a \notin K(S)$, by non-emptiness of K, there must be a $b \in S$ such that $b \in K(S)$. By definition, $(b, a) \in P_v = P(R_v)$. As R^* is an extension of R_v, we must have that $(b, a) \in P(R_v)$, hence it is not the case that $(a, b) \in R^*$ for all $b \in S$.□

This result is immediately applicable to the functions T, C, H, $\overline{T}$ and $\overline{C}$ defined in section 2.3. In particular, if we substitute F with T, we reproduce Richter's congruence result [Richter, 1966, thm 1]:

A choice function is rationalizable by a complete and transitive relation if and only if $(a, b) \in T(R_v)$ *implies* $(b, a) \notin P_v$.

Let us now return to the example given at the end of section 2.1. Consider the sequences $s^1 = x_1, x_2, x_3, x_4$ and $s^2 = x_1, x_5, x_6, x_4$. Observe that $x_2 = 0.5\,x_3 + 0.25\,x_4 + 0.25\,x_6$ and that $x_5 = 0.5\,x_6 + 0.25\,x_4 + 0.25\,x_3$. Then we have for sequence s^1 that that $(x_1, x_2) \in R_v$, $x_2 \in V(\{x_3, x_4, x_6\})$, and $(x_3, x_4) \in R_v$ and for sequence s^2 that $(x_1, x_5) \in R_v$, $x_5 \in V(\{x_6, , x_4, x_3\})$ and $(x_6, x_4) \in R_v$. From this, we can conclude that $(x_1, x_4) \in C(R_v)$. This contradicts with $(x_4, x_1) \in P_v$, hence, the observations are not rationalizable by a complete, transitive and convex relation.

CHAPTER 3

Absolute and relative time consistent revealed preferences

3.1 Introduction

Consider a preference relation over a set of alternatives, X. In order to pick a best element out of every two element choice set (subset of X), it is necessary that preferences are complete. If we want to choose a best element out of any larger choice set, we must also impose transitivity. Additional requirements on preferences commonly require further structure on the set of alternatives. This section investigates the significance of restrictions from intertemporal settings.

If we seek to impose properties that are linked to the time-instances at which the various alternatives are consumed, we can represent the set of alternatives by $X \times T$, where X is a set of consumption bundles and T is a set of time-instances. A preference relation is a transitive and complete binary relation on this extended set.

Besides transitivity and completeness, this research deals with the implication of three additional intertemporal properties.

Let us start with the property of impatience. An alternative a is labelled as a 'good' if for all time instances t and v with $t \leq v$, we have that the consumption of a at time t is at least as good as the consumption of a at time v. An alternative a is labelled as a 'bad' if for all time instances t and v with $t \geq v$, the consumption of a at time t is at least as good as the consumption of a at time v. We say that a preference relation R is impatient every alternative is either a 'good' or a 'bad' (or both). This excludes cases where we have time instances $t < v < w$ and an alternative a such that the consumption of a at time v is preferred to both the consumption of a at times t and the consumption of a at time w. Since impatience is a basic requirement in an intertemporal context, we maintain it throughout, and combine it successively with each of the two following properties.

For the next property, consider two bundles (a, t) and (b, v) and assume that (a, t) is at least as good as (b, v). After a certain amount of time $s \leq t, v$, the individual is asked to reconsider the two bundles, which have now become $(a, t - s)$ and $(b, v - s)$. It would seem natural to require that individuals do not change their judgement (preferences) as time goes by. Therefore, we should have that $(a, t - s)$ is at least as good as $(b, v - s)$. This property is called *absolute time-consistency* and it is one of the key assumptions in the characterization of the exponential discounted utility model (see for example Fishburn and Rubinstein [1982]).

The final property is a variant of the second and is called *relative time-consistency*. It states, contrary to absolute time-consistency, that preferences over alternatives depend on the 'relative' time differences between the two consumption periods instead of the absolute difference. Begin by fixing an element $\delta \in \mathbb{R}_{++}$. We say that preferences are relative time consistent if, whenever the bundle (a, t) is at least as good as the bundle (b, v), then for any strict positive real number $k \geq \max\left\{\frac{\delta}{t+\delta}, \frac{\delta}{v+\delta}\right\}$, the bundle $(a, k\cdot(t+\delta)-\delta)$ is at least as good as the bundle $(b, k\cdot(v+\delta)-\delta)$. This property is a key property for the characterization of the hyperbolic discounting model as in Loewenstein and Prelec [1992] (see also al-Nowaihi and Dhami [2006] for a correction of their paper)[1].

Most research on the plausibility of the absolute or relative time-consistency (and impatience) assumption, start from a particular functional form for the (instantaneous) utility function, and try to fit the model to the observed data (e.g. Eisenhauer and Ventura [2006] and Angeletos et al. [2001], for a good overview see Frederick et al. [2002]). In this section, we use the revealed preference approach. This approach has a clear advantage: the axioms do not depend on a particular functional form of the preference relation. In fact, the preference relation does not even need to have a functional representation[2]. Choice theory departs from observations on choice sets and the choices from these sets. If an alternative is selected, it is top ranked according to the revealed preference relation. The transitive closure of this revealed preference relation is called the indirect revealed preference relation. A choice function is rationalizable by the relation R if the observed choices from a choice set agree with the alternatives that are top ranked by R. Richter [1966] established that a choice function is rationalizable by a complete and transitive relation if and only if it satisfies the congruence axiom:

For all alternatives a and b, if a is indirectly revealed preferred to b, then b is not strictly revealed preferred to a.

This section presents two revealed preference axioms which characterize the set of choice functions that are rationalizable by a complete, transitive, absolute or relative time-consistent and impatient relation. These axioms, which we call the absolute and relative time-consistent axiom

[1] The hyperbolic discounting model, as in al-Nowaihi and Dhami [2006], departs from the assumption that there exist functions $v : X \to \mathbb{R}$ and $\sigma : T \to \mathbb{R}$ such that the value (utility) of a at time t can be written as $v(a)\sigma(t)$. Their assumption (A3a) states that for $a, b \in X$ and $t \in T$, if $v(a) = v(b)\sigma(t)$ and $s > 0$, then $v(a)\sigma(s) = v(b)\sigma(s + t + \alpha st)$. It is easy to see that this is equivalent to the condition that $v(a)\sigma(t) = v(b)\sigma(v)$ if and only if for all $k \geq \max\left\{\frac{\delta}{\delta+t}, \frac{\delta}{v+\delta}\right\}$, $v(a)\sigma(k(t - 1/\alpha) - 1/\alpha) = v(b)\sigma(k(v - 1/\alpha) - 1/\alpha)$. Substituting $\delta = 1/\alpha$, and using their condition A0 ($\sigma(.)$ is strictly increasing and if $0 < x < y$, then there is a $t \in T$ such that $x = y\sigma(t)$), we derive the desired transformation.

[2] See also remark 3 in section 3.2.

of revealed preference (ATARP and RTARP), state that the absolute (relative) time-consistent, transitive and time-monotonic closure of the revealed preference relation does not conflict with the strict revealed preference relation. In particular:

For all alternatives (a, t) *and* (b, v)*, if* $((a, t), (b, v))$ *belongs to the absolute (relative) time-consistent, transitive and time-monotonic closure of the revealed preference relation, then* (b, v) *is not strictly revealed preferred to* (a, t)*.*

Let us provide an example. Assume that $X = \{a, b, c\}$ and assume that $(b, 3)$ is revealed preferred to $(a, 2)$, $(a, 9)$ is revealed preferred to $(b, 8)$, $(a, 24)$ is strictly revealed preferred to $(c, 20)$ and $(c, 3)$ is revealed preferred to $(a, 4)$. In section 3.2, we will reconsider this example and show that these observations satisfy ATARP. Hence, there exist a complete, transitive, absolute time-consistent and impatient relation that rationalizes the choice function. On the other hand, we can show that the observations violate RTARP for every $\delta < \frac{8}{3}$. Therefore, if this condition is satisfied, there does not exist a complete, transitive, relative consistent and impatient relation that rationalizes the observed choices.

Section 3.2 presents notation and introduces the results. Section 3.3 presents the proofs.

3.2 Notation and results

Consider a set A. A *binary relation*, R, in A is a subset of $A \times A$. Define R^{-1} as $(x, y) \in R^{-1}$ if and only if $(y, x) \in R$. The *asymmetric part*, $P(R)$, of the relation R is given by $R - R^{-1}$, the *symmetric part*, $I(R)$, of R is given by $R \cap R^{-1}$ and the *non-comparable part*, $N(R)$, of R is given by $(A \times A) - R \cup R^{-1}$. A relation R is *transitive* if for all x, y, z in A, $(x, y) \in R$ and $(y, z) \in R$ implies that $(x, z) \in R$. The relation R is *complete* if for all x and y in A, $(x, y) \in R$ or $(y, x) \in R$. A complete and transitive relation is called an *ordering*. An *extension* R' of R is a relation which satisfies $R \subseteq R'$ and $P(R) \subseteq P(R')$.

Let X be the set of alternatives and let $T = \mathbb{R}_+$ denote the universal set of time instances. The present is set at time equal to 0. An element (a, t) of $X \times T$ denotes the consumption of alternative a at time t. Instead of the set A, we will work on the set $X \times T$. A binary relation R on $X \times T$ is a subset of $(X \times T) \times (X \times T)$.

Consider the set $L(R)$,

$$L(R) = \{a \in X | \text{there exists an } ((a, t), (a, v)) \in R \text{ and } t < v\},$$

and the set $U(R)$,

$$U(R) = \{a \in X | \text{there exists an } ((a, t), (a, v)) \in R \text{ and } t > v\}.$$

Let us give the definitions of time-monotonicity, impatience, absolute time-consistency and relative time-consistency.

Definition 3.1 (time-monotonicity)**.** *A relation R in $X \times T$ is time-monotonic if for all $t, v \in T$ with $t \leq v$ and all $a \in X$,*

$$\text{if } a \in L(R), \text{ then } ((a,t),(a,v)) \in R, \text{ and if}$$

$$a \in U(R), \text{ then } ((a,v),(a,t)) \in R.$$

Definition 3.2 (impatience)**.** *A relation R in $X \times T$ is impatient if it is time-monotonic and for all $a \in X$, either $a \in U(R)$ or $a \in L(R)$ (or both).*

It is easy to see that every complete and time-monotonic relation is also impatient. Therefore, we will use the term of time-monotonicity only in cases where the relation is not required to be complete.

Definition 3.3 (Absolute time-consistency)**.** *The relation R is absolute time-consistent if for all $a, b \in X$; $t, v \in T$ and all $s \in \mathbb{R}$, $s \leq t, v$:*

$$\text{if } \; ((a,t),(b,v)) \in R \; \text{ then } \; ((a,t-s),(b,v-s)) \in R.$$

Let us fix a parameter $\delta \in \mathbb{R}_{++}$.

Definition 3.4 (Relative time-consistency)**.** *The relation R is relative time-consistent if for all $a, b \in X$; $t, v \in T$ and $k \geq \max(\frac{\delta}{t+\delta}, \frac{\delta}{v+\delta})$:*

$$\text{if } \; ((a,t),(b,v)) \in R \; \text{ then } \; ((a, k \cdot (t+\delta) - \delta),(b, k \cdot (v+\delta) - \delta)) \in R.$$

Let us define the *absolute time-consistent and transitive closure*[3] of R as the smallest transitive and absolute time-consistent relation that contains R. Before we give the formal definition, let us begin with an example. Let us introduce $B(R)$ as the absolute time-consistent and transitive closure of R. Consider elements a, b and c in X and assume that

$$((a,3),(b,2)) \in R \qquad \text{and} \qquad ((b,6),(c,4)) \in R.$$

We know that $B(R)$ contains R and satisfies absolute-time consistency, hence, we can add an equal amount, 4, to 3 and 2 to deduce:

$$((a,7),(b,6)) \in B(R) \qquad \text{and} \qquad ((b,6),(c,4)) \in B(R).$$

As $B(R)$ is transitivity, we establish that $((a,7),(c,4)) \in B(R)$. Following the calculations through, we see that 7 was obtained as $3 + (6-2)$, hence we can write $7-4$ as $(3-2)+(6-4)$. Notice that by absolute time-consistency of $B(R)$, only the absolute difference between 7 and

[3]Although it is possible to define the transitive closure of the absolute time-consistent closure and the absolute time-consistent closure of the transitive closure, these two closures do not satisfy both conditions: the first is not always absolute time-consistent and the second is not always transitive. Therefore, it is necessary to join the two conditions in the same closure operator

4 really matters. Therefore, we derive that for all t and v in T that satisfy $t - v = 7 - 4 = (3 - 2) + (6 - 4)$:

$$((a, t), (c, v)) \in B(R).$$

This example does not rely on the specific values of $3, 2, 6$ and 4. Hence, we can substitute $3 = t_1, 2 = v_1, 6 = t_2$ and $4 = v_2$, and conclude that:

$$((a, t_1), (b, v_1)) \in R \text{ and } ((b, t_2), (c, v_2)) \in R \text{ implies } ((a, t), (c, v)) \in B(R),$$

where t and v satisfy $t - v = (t_1 - v_1) + (t_2 - v_2)$.

Above example only relates to two element subsets of R. The generalization to all finite subsets leads to the following definition[4].

Definition 3.5 (Absolute time-consistent and transitive closure)**.** *The absolute time-consistent and transitive closure $B(R)$ of R is defined as:*
$((a, t), (b, v)) \in B(R)$ if there exist a number $n \in \mathbb{N}$, a sequence $a = x_1, \ldots, x_n = b$ of elements in X and sequences $t_1, \ldots, t_{n-1}$ and $v_1, \ldots, v_{n-1}$ of elements in T such that for all $i = 1, \ldots, n - 1$:

$$((x_i, t_i), (x_{i+1}, v_i)) \in R$$

and

$$t - v = \sum_{i=1}^{n-1} (t_i - v_i).$$

Let us abide to a similar deduction for the property of relative time-consistency. Let $\delta = 2$ and consider the relation R with:

$$((a, 3), (b, 2)) \in R \qquad \text{and} \qquad ((b, 6), (c, 4)) \in R.$$

Let $S(R)$ be the relative time-consistent and transitive closure of R. Then we know that:

$$((a, 2 \cdot (3 + 2) - 2), (b, 2 \cdot (2 + 2) - 2)) \in S(R),$$

Hence,

$$((a, 8), (b, 6)) \in S(R) \qquad \text{and} \qquad ((b, 6), (c, 4)) \in R.$$

By transitivity of $S(R)$, we derive that $((a, 8), (c, 4)) \in S(R)$. We see that the multiplicator 2 was obtained as the fraction $\frac{6+2}{2+2}$. Substituting $t_1 = 3$, $v_1 = 2$, $t_2 = 6$ and $v_2 = 4$, we see that 8 was obtained as

$$\frac{t_2 + 2}{v_1 + 2}(t_1 + \delta) - \delta.$$

From this it leads that:

$$\frac{8 + \delta}{4 + \delta} = \frac{t_1 + \delta}{v_1 + \delta} \frac{t_2 + \delta}{v_2 + \delta}.$$

[4]The proof that the relation $B(R)$ in definition 4 coincides with the smallest transitive and absolute time-consistent relation containing R is almost identical to the proof of Lemma 3.1 in section 3.3 and is therefore omitted.

It is also easy to see[5] that the fraction $\frac{t+\delta}{s+\delta}$ remains invariant if and only if we allow for time-shifts $\frac{f_k(t)+\delta}{f_k(t)+\delta}$ with $f_k(t) = k \cdot (t+\delta) - \delta$. Hence, we can conclude that:

$$((a,t_1),(b,t_2)) \in R \text{ and } ((b,t_2),(c,v_2)) \in R \text{ implies } ((a,t),(c,v)) \in S(R),$$

where t and v satisfy

$$\frac{t+\delta}{v+\delta} = \frac{(t_1+\delta)(t_2+\delta)}{(v_1+\delta)(v_2+\delta)}.$$

If we generalize this to all finite subsets of R, we are let to the following definition.

Definition 3.6 (Relative time-consistent and transitive closure)**.** *The relative time-consistent and transitive closure $S(R)$ of R is defined by: $((a,t),(b,v)) \in S(R)$ if there exist a number $n \in \mathbb{N}$, a sequence $a = x_1, \ldots, x_n = b$ of elements in X and sequences $t_1, \ldots, t_{n-1}$ and $v_1, \ldots, v_{n-1}$ of elements in T such that for all $i = 1, \ldots, n-1$:*

$$((x_i,t_i),(x_{i+1},v_i)) \in R$$

and

$$\frac{t+\delta}{v+\delta} = \prod_{i=1}^{n-1} \frac{t_i+\delta}{v_i+\delta}.$$

For a relation R, we define the relation $\widetilde{R}$ by,

$$\begin{aligned}\widetilde{R} = \quad & \{((a,t),(a,v))|a \in L(B(R)) \text{ and } t \leq v\} \\ & \cup \{((a,t),(a,v)|a \in U(B(R)) \text{ and } t \geq v\}.\end{aligned}$$

Then, we can define the relation $\overline{R}$ by

$$\overline{R} = R \cup \widetilde{R}.$$

We call $B(\overline{R})$ the absolute time-consistent, transitive and time-monotonic closure of R. Lemma 3.1 in section 3.3 shows that $B(\overline{R})$ is indeed the smallest transitive, absolute time-consistent and time-monotonic relation containing R.

Similarly, we define the relation $\widehat{R}$ by

$$\begin{aligned}\widehat{R} = \quad & \{((a,t),(a,v))|a \in L(S(R)) \text{ and } t \leq v\} \\ & \cup \{((a,t),(a,v)|a \in U(S(R)) \text{ and } t \geq v\}.\end{aligned}$$

Then $\overline{\overline{R}} = R \cup \widehat{R}$ and $S(\overline{\overline{R}})$ is the relative time-consistent, transitive and time-monotonic closure of R.

[5]Sufficiency is straightforward. To see necessity, let $\frac{f_k(t)+\delta}{f_k(v)+\delta} = \frac{t+\delta}{v+\delta}$. Setting $v = 0$ and setting $f_k(0) = \delta(k-1)$, we derive the desired result, namely, $f_k(t) = k(t+\delta) - \delta$.

Consider the set $\Lambda = 2^{X \times T} - \{\emptyset\}$ of all non-empty choice sets. A *choice function*, C, is a function from a set $\Sigma \subseteq \Lambda$ to Λ such that for all $A \in \Sigma$: $C(A) \subseteq A$. The set $C(A)$ is to be interpreted as the choices made from the set A.

A choice function C is *rationalizable by an absolute time-consistent and impatient ordering* if and only if there exists an absolute time-consistent and impatient ordering R in $X \times T$ such that for all $A \in \Sigma$:

$$C(A) = \{(a,t) \in A | \forall (b,v) \in A : ((a,t),(b,v)) \in R\}.$$

In words: the choices made from A are the ones that are top ranked according to R.

Analogously, a choice function C is *rationalizable by a relative time-consistent and impatient ordering* if and only if there exist a relative time-consistent and impatient ordering R on $X \times T$ such that for all $A \in \Sigma$:

$$C(A) = \{(a,t) \in A | \forall (b,v) \in A : ((a,t),(b,v)) \in R\}.$$

Given the choice function, C, we define the *revealed preference relation* R_v by $((a,t),(b,v)) \in R_v$ if and only if there is an $A \in \Sigma$ such that $(a,t) \in C(A)$ and $(b,v) \in A$. The *strict revealed preference relation* P_v is defined by $((a,t),(b,v)) \in P_v$ if and only if there is an $A \in \Sigma$ such that $(a,t) \in C(A)$ and $(b,v) \in A - C(A)$.

Consider a choice function C and assume that C is rationalizable by an absolute time-consistent and impatient ordering R. The choice function, C, is only defined over the set Σ, so in general it is impossible to reconstruct the ordering R from C. On the other hand, we do observe the relations R_v and P_v. From the rationalizability of C, we find that $R_v \subseteq R$ and $P_v \subseteq P(R)$. The ordering R is transitive, absolute time-consistent and impatient, hence it must include the absolute time-consistent, transitive and time-monotonic closure of R_v, i.e. $B(\overline{R_v}) \subseteq R$. Therefore, it must be that $B(\overline{R_v}) \cap P_v^{-1}$ is empty. If not, we would have that $R \cap P(R)^{-1}$ is non-empty which contradicts the definition of the asymmetric part. We call this property the absolute time-consistent axiom of revealed Preference.

Definition 3.7 (ATARP). *A choice function C satisfies the absolute time-consistent axiom of revealed preference (ATARP) if*

$$B(\overline{R_v}) \cap P_v^{-1} = \emptyset.$$

We define the relative time-consistent axiom of revealed preference similarly:

Definition 3.8 (RTARP). *A choice function C satisfies the relative time-consistent axiom of revealed preference (RTARP) if*

$$S(\overline{\overline{R_v}}) \cap P_v^{-1} = \emptyset.$$

As demonstrated above, the ATARP (RTARP) is a necessary condition for a choice function to be rationalizable by an absolute (relative) time-consistent and impatient ordering. Fortunately, it turns out that it is also sufficient. We state this in the following theorems. The proof is given in the next section.

Theorem 3.1. *A choice function C is rationalizable by an absolute time-consistent and impatient ordering if and only if it satisfies the ATARP.*

Theorem 3.2. *A choice function C is rationalizable by a relative time-consistent and impatient ordering if and only if it satisfies the RTARP.*

Let us return to the example in the introduction. We have that $((b,3),(a,2)) \in R_v$, $((a,9),(b,8)) \in R_v$, $((a,24),(c,20)) \in P_v$ and $((c,3),(a,4)) \in R_v$. Let us begin by showing that these observations satisfy ATARP.

Using the sequences b,a,b $3,9$ and $2,8$, in the definition of B, we see that:

$$a \in U(B(R_v)).$$

Using the sequences a,b,a $9,3$ and $8,2$, we see that $b \in U(B(R_v))$ and using the sequences c,a,c, $4,24$ and $3,20$, we see that $c \in U(B(R_v))$. It can also be verified that none of the elements a,b and c are in $L(B(R_v))$. Let us now show, by contradiction, that ATARP is satisfied. We take the case where $((c,20),(a,24)) \in B(\overline{R_v}) \cap P_v^{-1}$. The other cases are very similar and are left to the reader. Then there exist a sequence $c = x_1, \ldots, x_n = a$ and sequences $t_1, \ldots, t_{n-1}$ and $v_1, \ldots, v_{n-1}$ such that for all $i = 1, \ldots, n-1$:

$$((x_i,t_i),(x_{i+1},v_i)) \in R_v \cup \widetilde{R_v}$$

and

$$20 - 24 = \sum_{i=1}^{n-1}(t_i - v_i) < 0.$$

As a,b and c are all in $U(R_v) - L(R_v)$, we derive that the only factors $(t_i - v_i)$ which can be negative on the rhs are the ones where $((x_i,t_i),(x_{i+1},v_i)) = ((c,3),(a,4))$. Let L collect all such i. If $|L| = 1$, then the sum $\sum_{i=1}^{n-1}(t_i - v_i)$ is larger or equal to -1. However, $20 - 24 = -4$, which is strictly smaller. Therefore, we must conclude that $|L| > 1$. However, for every $j \in L$ such that there is an $i \in L$ with $j < i$ it must be the case that there exists a $k \leq n$ with ($j < k < i$ for all $i > j, i \in L$) and $((x_k,t_k),(x_{k+1},v_k)) = ((a,24),(c,20))$. Therefore, we can conclude that $\sum_{i=1}^{n-1}(t_i - v_i) \geq -1$, a contradiction.

Now, we demonstrate that RTARP is violated for every $\delta < \frac{8}{3}$. Consider the sequences a,c,a, $24,3$ and $20,4$. We see that for all $\delta < \frac{8}{3}$:

$$\frac{(24+\delta)(3+\delta)}{(20+\delta)(4+\delta)} < 1.$$

As such $a \in L(S(R_v))$. On the other hand, using the sequences a,b,a, $9,3$ and $8,2$, we see that $a \in U(S(R_v))$ also. Let us verify that RTARP is violated by showing that $((c,20),(a,24)) \in S(\overline{\overline{R_v}})$. Consider the sequences c,a,a, $3,t$ and $4,v$ such that t and v solves:

$$\frac{20+\delta}{24+\delta} = \frac{(3+\delta)(t+\delta)}{(4+\delta)(v+\delta)}.$$

It is always possible to find values of t and v in T that satisfy this condition. Conclude that RTARP is violated by noticing that $((a,t),(a,v)) \in \widehat{R_v}$.

We close this section with a few remarks.

Remark 1. We assumed that $T = \mathbb{R}_+$, but it is easy to alter T to to a smaller set $T' \subset \mathbb{R}_+$, e.g. $T' = \mathbb{N}$, while the theorems remain valid. To see this, notice that we can restrict the set Σ to select only sets Q that have alternatives with time instances in T'. Theorems 3.1 and 3.2 give rise to a rationalization R^*, which is defined over the entire set $X \times T$. The restriction of R^* to the set $X \times T'$ provides us with a rationalization on the smaller domain.

Remark 2. Sometimes, time-consistency is defined in term of sequences, $(x_i, t), (x_{i+1}, t+1)..., (x_{i+n}, t+n)$, of consumption bundles instead of single consumption bundles. In this sense, we can say that a relation R is n absolute time-consistent if for all $t, v \in \mathbb{N}$, $s \in \mathbb{Z}$ and $s \leq t, v$:

$$(((x_i, t), ..., (x_{i+n}, t+n)), ((y_i, v), ..., (y_{i+n}, v+n))) \in R$$

if and only if:

$$(((x_i, t-s), ..., (x_{i+n}, t+n-s)), ((y_i, v-s)..., (y_{i+n}, v+n-s))) \in R.$$

Let us take $X = \widetilde{X}^n$ with $\widetilde{X}$ the set of alternatives and let $(x_1, ..., x_n, t) = (x, t) \in X \times T$ denote the bundle $((x_1, t), ..., (x_n, t+n))$. We verify immediately that the concept of absolute time-consistency and of n absolute time-consistency coincide. If we take into account remark 1, Theorem 3.1 carries directly over to this alternative definition of time-consistency. An analogous result holds also for the property of relative time-consistency. Furthermore, setting $X = \widetilde{X}^\infty$, allows for infinite sequences.

Remark 3. The existence of an absolute (relative) time-consistent ordering on $X \times T$ does not imply that there exist a functional (real valued) representation of such ordering. For example: let $X = X_G$ and consider the lexicographic ordering R where:

$$((a,t),(b,v)) \in R \text{ if and only if } t < v \text{ or } [t = v \text{ and } (a,b) \in Q],$$

with Q an ordering on the set X. The ordering R is absolute time-consistent, relative time-consistent and impatient. Moreover, it is well known that such an ordering has no real valued representation (see Debreu [1954]).

3.3 Proof of theorem 3.1.

We provide the proof for the case of rationalizability by an absolute time-consistent and impatient ordering (theorem 3.1). The proof of theorem 3.2 is completely analogous and is left to the reader.

Consider the the absolute time consistent, transitive and time-monotonic closure $B(\overline{R})$ of R. We have the following result:

Lemma 3.1. *For a relation R, the closure $B(\overline{R})$ is the smallest, transitive, absolute time-consistent and time-monotonic relation containing R.*

Proof. Consider a relation R and its absolute time-consistent, transitive and time-monotonic closure $B(\overline{R})$. We begin by verifying that $B(\overline{R})$ is time-monotonic, absolute time-consistent and transitive.

(i) Time-monotonicity. Assume that $a \in L(B(\overline{R}))$. (The case where $a \in U(B(\overline{R}))$ is very similar and is left to the reader.) Then, there exist a sequence $a = x_1, \ldots x_n = a$ and sequences $t_1, \ldots, t_{n-1}$ and $v_1, \ldots, v_{n-1}$ such that for all $i = 1, \ldots, n-1$

$$((x_i, t_i), (x_{i+1}, v_i)) \in R \cup \widetilde{R}$$

and

$$t - v = \sum_{i}^{n-1} (t_i - v_i) < 0$$

There are three cases to consider:

1) There is no $i = 1, ..., n-1$ such that $((x_i, t_i)(x_{i+1}, v_i)) \in \widetilde{R}$. In this case, we have that $((a, t), (a, v)) \in B(R)$ and by definition of $\widetilde{R}$, $((a, t'), (b, v')) \in \widetilde{R} \subseteq B(\overline{R})$ for all $t' - v' \leq 0$.

2) There is an i such that $((x_i, t_i), (x_{i+1}, v_i)) \in \widetilde{R}$. In this case, we remove all such elements x_i, t_i and v_i from the sequences and we reenumerate these sequences to obtain that $a = y_1, \ldots, y_m = a$ is a sequence in X and $u_1, \ldots, u_{m-1}$ and $w_1, \ldots, w_{m-1}$ are sequences in T such that $((a, t'), (b, v')) \in B(R)$ for all:

$$t' - v' = \sum_{i=1}^{m-1} (u_i - w_i).$$

If $t' - v' < 0$, we have that $((a, t''), (a, v'')) \in \widetilde{R} \subseteq B(\overline{R})$ for all $t'' - v'' \leq 0$, and we are done.

3) If $t' - v' \geq 0$, we must have that there is an $i = 1, ..., n-1$ such that $((x_i, t_i), (x_i, v_i)) \in \widetilde{R}$ and $t_i - v_i < 0$. Let us reintroduce the elements x_i, αt_i and αv_i ($\alpha \in \mathbb{R}_{++}$) in the sequences to get that $((a, t'), (a, v')) \in B(\overline{R})$ for all:

$$t' - v' = \sum_{i}^{m-1} (u_i - v_i) + \alpha(t_i - v_i),$$

Let us show that $((a, t''), (b, v'')) \in B(\overline{R})$ for all $t'' - v'' \in]-\infty, 0]$. Let $\alpha' \in \mathbb{R}$ solve:

$$t'' - v'' = \sum_{i}^{m-1} (u_i - v_i) + \alpha'(t_i - v_i) = (t' - v') + \alpha'(t_i - v_i).$$

As $t_i - v_i < 0$ and $t' - v' \geq 0$, we have that $\alpha' \in \mathbb{R}_{++}$. Therefore, $((a, t''), (b, v'')) \in B(\overline{R})$ for all $t'' - v'' < t' - v'$. This includes the cases where $t'' - v'' \in]-\infty, 0]$.

(ii) Absolute-time consistent. Let $((a,t),(b,v)) \in B(\overline{R})$. Then, there is a sequence $a = x_1,\dots,x_n = b$ in X and sequences $t_1,\dots,t_{n-1}$ and $v_1,\dots,v_{n-1}$ in T such that for all $i = 1,\dots,n-1$:

$$((x_i,t_i),(x_{i+1},v_i)) \in \overline{R}$$

and

$$t - v = \sum_{i=1}^{n-1}(t_i - v_i).$$

Immediately, we see that for any $s \in \mathbb{R}$, $s \leq t, v$:

$$(t-s)-(v-s) = \sum_{i=1}^{n-1}(t_i - v_i).$$

Therefore, $((a,t-s),(b,v-s)) \in B(\overline{R})$, hence $B(\overline{R})$ is absolute time-consistent.

(iii) Transitive. Let $((a,t),(b,v)) \in B(\overline{R})$ and $((b,v),(c,w)) \in B(\overline{R})$ so that there are sequences $a = x_1,\dots,x_n = b$ and $b = y_1,\dots,y_m = c$ in X and sequences $t_1,\dots,t_{n-1}$; $v_1,\dots,v_{n-1}$; $s_1,\dots,s_{m-1}$ and $w_1,\dots,w_{m-1}$ in T such that for all $i = 1,\dots,n-1$ and $j = 1,\dots,m-1$:

$$((x_i,t_i),(x_{i+1},v_i)) \in \overline{R},$$

$$((y_j,s_j),(y_{j+1},w_j)) \in \overline{R},$$

$$t - v = \sum_{i=1}^{n-1}(t_i - v_i)$$

and

$$v - w = \sum_{j=1}^{m-1}(s_j - w_j).$$

Construct the compound sequence $a = x_1,\dots,x_n,y_2,\dots,y_m = c$ in X and the compound sequences $t_1,\dots,t_{n-1},s_1,\dots,s_{m-1}$ and $v_1,\dots,v_{n-1},w_1,\dots,w_{m-1}$ in T. As:

$$t - v + v - w = t - w = \sum_{i=1}^{n-1}(t_i - v_i) + \sum_{j=1}^{m-1}(s_j - w_j),$$

...we can conclude that $((a,t),(c,w)) \in B(\overline{R})$, hence $B(\overline{R})$ is transitive.

We are left to show that $B(\overline{R})$ is the **smallest** transitive, absolute time-consistent and time-monotonic relation containing R. Consider a transitive, absolute time-consistent and time-monotonic relation R^*, that contains R. Let us show that $B(\overline{R}) \subseteq R^*$.

First, we show that for all $Q \subseteq R^*$, $B(Q) \subseteq R^*$. Assume, on the contrary that $((a,t),(b,v)) \in B(Q) - R^*$. Then, then is a sequence $a = x_1,\dots,x_n = b$ in X and sequences $t_1,\dots,t_{n-1}$ and $v_1,\dots,v_{n-1}$ in T such that for all $i = 1,\dots,n-1$:

$$((x_i,t_i),(x_{i+1},v_i)) \in Q$$

and

$$t - v = \sum_{i=1}^{n-1}(t_i - v_i).$$

We proceed by induction on n. For $n = 2$, we have that:

$$((a, t_1), (b, v_1)) \in Q$$

and

$$t - v = t_1 - v_1.$$

Setting $s = t_1 - t$ and applying absolute time-consistency of R^*, we obtain that $((a, t), (b, v)) \in R^*$, a contradiction. Suppose the result holds up to ℓ and there is a sequence $a = x_1, \ldots, x_\ell, x_{\ell+1} = b$ and sequences $t_1, \ldots, t_\ell$ and $v_1, \ldots, v_\ell$ such that for all $i = 1, \ldots, \ell$:

$$((x_i, t_i), (x_{i+1}, v_i)) \in Q$$

and

$$t - v = \sum_{i=1}^{\ell}(t_i - v_i),$$

we can take two elements t' and v' from T such that:

$$t' - v' = \sum_{i=1}^{\ell-1}(t_i - v_i).$$

Hence $t' - v' + t_\ell - v_\ell = t - v$. From the induction hypothesis, we have that $((a, t'), (x_\ell, v')) \in R^*$. Observe also that $((x_\ell, t_\ell), (b, v_\ell)) \in R^*$. If $v' \geq t_\ell$ put $s = t_\ell - v'$. From absolute time-consistency, we derive that: $((a, t'), (x_\ell, v')) \in R^*$ and $((x_\ell, v'), (b, v_\ell - t_\ell + v')) \in R^*$. From transitivity, we establish that $((a, t'), (b, t' - t + v)) \in R^*$. Put $s = t' - t$ and apply absolute time-consistency of R^* to verify that $((a, t), (b, v)) \in R^*$. The case where $v' \leq t_\ell$ is solved similarly. Conclude that $((a, t), (b, v)) \in R^*$, a contradiction. Hence, $B(Q) \subseteq R^*$.

From time-monotonicity of R^* and $B(R) \subseteq R^*$, it follows that $\widetilde{R} \subseteq R^*$. Hence, $R \cup \widetilde{R} \subseteq R^*$. Applying above rule once more, we derive that $B(R \cup \widetilde{R}) = B(\overline{R}) \subseteq R^*$. □

Consider a relation R on $X \times T$. Recall from section 3.2 that a relation R^* is an *extension* of R if $R \subseteq R^*$ and $P(R) \subseteq P(R^*)$.

Lemma 3.2. *A relation R has a time-consistent and impatient ordering extension if and only if $B(\overline{R}) \cap P(R)^{-1} = \emptyset$.*

Proof. The proof is similar to the proof of Szpilrajn's lemma [Szpilrajn, 1930], which states that every quasi-order (transitive and reflexive binary relation) has an ordering extension[6].

[6] In fact, lemma 3.2 is a special case of theorem 2.1.

To verify necessity assume that R has an absolute time-consistent and impatient ordering extension, R^*, and, on the contrary, $((a,t),(b,v)) \in B(\overline{R}) \cap P(R)^{-1}$. Lemma 3.1, states that $B(\overline{R})$ is the smallest absolute time-consistent, time-monotonic and transitive relation containing R. As R^* is an absolute time-consistent, impatient and transitive relation, we must have that $B(\overline{R}) \subseteq R^*$. This shows that $((a,t),(b,v)) \in R^*$. Further, as R^* is an extension of R, we derive that $((b,v),(a,t)) \in P(R^*)$, a contradiction. Hence, $B(\overline{R}) \cap P(R)^{-1} = \emptyset$.

For sufficiency, consider the set Ω which selects all extensions, R', of R for which $B(\overline{R'}) \cap P(R')^{-1} = \emptyset$. This set is non-empty as $R \in \Omega$. Let Ω' be a chain in Ω, i.e. for all $R', R'' \in \Omega$ either $R' \subseteq R''$ or $R'' \subseteq R'$. Consider the relation $Q = \bigcup_{R' \in \Omega'} R'$. Let us show that $Q \in \Omega$. It is easy to see that Q is an extension of R. To see that $B(\overline{Q}) \cap P(Q)^{-1} = \emptyset$, assume on the contrary that there exist elements $a, b \in X$ and $t, v \in T$ such that $((a,t),(b,v)) \in B(\overline{Q}) \cap P(Q)^{-1}$. Then, there exist a sequence $a = x_1, \ldots, x_n = b$ of elements in X and sequences $t_1, \ldots, t_{n-1}$ and $v_1, \ldots, v_{n-1}$ such that for all $i = 1, \ldots, n-1$:

$$((x_i, t_i), (x_{i+1}, v_i)) \in \overline{R'}$$

and

$$t - v = \sum_{i=1}^{n-1} (t_i - v_i).$$

From the construction of Q, we see that there must be relations $R_1, ..., R_{n-1}$ in Ω' such that for all $i \leq n-1$, $((x_i, t_i), (x_{i+1}, v_i) \in \overline{R_i}$. All these relations are ranked by set inclusion, thus, there must be a largest one, lets say $\overline{R_j}$. Further, from the definition of Q, there must be a relation $R_0 \in \Omega'$ such that $((b,v),(a,t)) \in R_0$ and for all $R' \in \Omega'$ it is not the case that $((a,t),(b,v)) \in R'$. The relations R_0 and R_j are ranked by set inclusion so either $R_j \subseteq R_0$ or $R_0 \subseteq R_j$. In the first case, we have that $((a,t),(b,v)) \in B(\overline{R_0}) \cap P(R_0)^{-1}$, contradicting the fact that $R_0 \in \Omega$. In the second case, we have that $((a,t),(b,v)) \in B(\overline{R_j}) \cap P(R_j)^{-1}$, contradicting the fact that $R_j \in \Omega$. Therefore, we can conclude that $R' \in \Omega$. By application of Zorn's lemma, the set Ω has a maximal element. Let R^* be such an element.

First of all, notice that by maximality of R^*: $R^* = B(\overline{R^*})$. Therefore, by lemma 3.1, R^* is absolute time-consistent, time-monotonic and transitive. Let us show that R^* is complete. Assume, on the contrary, that $N(R^*) \neq \emptyset$,
There are two cases.

i). There exist an element $a \in X - (U(R^*) \cup L(R^*))$.

ii). $X = U(R^*) \cup L(R^*)$.

Let begin by case (i). Consider two time instances t and v with $t > v$ and let $Q = \{((a,t),(b,v))\}$, for some $a \in X - (U(R^ast) \cup L(R^*))$. Let us show that $B(R^* \cup \widetilde{Q})$ is a time-monotonic, transitive and absolute time consistent extension of R^*. By a simple adaptation of the proof of lemma 3.1, it is easy to show that, $B(R^* \cup \widetilde{Q})$ is transitive and absolute time consistent. Let us show that it is time-monotonic.

First, take the case where there is an element $b \in L(B(R \cup \widetilde{Q}))$. Then there is a sequence $b = x_1, \ldots, x_n = b$ in X and sequences $t_1, \ldots, t_{n-1}$ and $v_1, \ldots, v_{n-1}$ in T such that for all $i = 1, \ldots, n-1$,

$$((x_i, t_i), (x_{i+1}, v_i)) \in R^* \cup \widetilde{Q},$$

and

$$t - v = \sum_{i}^{n-1} (t_i - v_i) < 0.$$

If there is no i such that $((x_i, t_i), (x_{i+1}, v_i)) \in \widetilde{Q}$, then $((b, t), (b, v)) \in B(R^*) = R^*$ and we are done. If there is an i such that $((x_i, t_i), (x_{i+1}, v_i)) \in \widetilde{Q}$ then we remove all such x_i, t_i and v_i from the sequences and renumber them such that we get a sequence $b = y_1, \ldots, y_m = b$ in X and sequences $u_1, \ldots, u_{m-1}$ and $w_1, \ldots, w_{m-1}$ in T where $((b, t'), (b, v')) \in B(R^*) = R^*$ for all

$$t' - v' = \sum_{i}^{m-1} (u_i - w_i).$$

If $t' - v' < 0$, we are done (by time-monotonicity of R^*)), hence assume that $t' - v' \geq 0$. Consider an $((x_i, t_i), (x_{i+1}, v_i)) \in \widetilde{Q}$ and let us reintroduce $x_i, \alpha t_i$ and αv_i ($\alpha \in \mathbb{R}_{++}$) in the sequences. Then we derive that for all $\alpha \in \mathbb{R}_{++}$ and:

$$t'' - v'' = \sum_{i}^{m-1} (u_i - w_i) + \alpha(t_i - v_i),$$

$((b, t''), (b, v'')) \in B(R \cup \widetilde{Q})$. This is especially true for all $t'' - v'' \in]-\infty, 0]$.

Consider now the case where there is a $b \in U(B(R^* \cup \widetilde{Q}))$. Then there is a sequence $b = x_1, \ldots, x_n = b$ in X and sequences $t_1, \ldots, t_{n-1}$ and $v_1, \ldots, v_{n-1}$ in T such that for all $i = 1, \ldots, n-1$:

$$((x_i, t_i), (x_{i+1}, v_i)) \in R^* \cup \widetilde{Q},$$

and

$$t - v = \sum_{i}^{n-1} (t_i - v_i) > 0.$$

If there is no i such that $((x_i, t_i), (x_i, t_{i+1}) \in \widetilde{Q}$, we have that $((b, t), (b, v)) \in B(R^*) = R^*$, hence, by time-monotonicity of R^*, $((b, t'), (b, v')) \in R^*$ for every $t' - v' \geq 0$. Now, assume that there is an i for which $((x_i, t_i), (x_{i+1}, v_i)) \in \widetilde{Q}$. Let L collect all such instances i. Then we have that:

$$t - v = \sum_{i \notin L} (t_i - v_i) + \sum_{i \in L} (t_i - v_i) \leq \sum_{i \notin L} (t_i - v_i).$$

By removing the elements x_i, t_i and v_i from the sequences for each $i \in L$, we derive that $((b, t'), (b, v')) \in B(R^*) = R^*$ for all:

$$t' - v' = \sum_{i \notin L} (t_i - v_i) > 0.$$

The conclusion follows from the fact that R^* is time-monotonic.

Let us now show that $B(R^* \cup \widetilde{Q})$ is an extension of R^*. If, on the contrary, $((b,t),(c,v)) \in B(R^* \cup \widetilde{Q})$ and $((c,v),(b,t)) \in P(R^*)$, we know that there exist a sequence $b = x_1, \ldots, x_n = c$ in X and sequences $t_1, \ldots, t_{n-1}$ and $v_1, \ldots, v_{n-1}$ in T such that for all $i = 1, \ldots, n-1$:

$$((x_i, t_i), (x_{i+1}, v_i)) \in R^* \cup \widetilde{Q},$$

and

$$t - v = \sum_{i=1}^{n-1} (t_i - v_i).$$

As $((b,t),(c,v)) \notin R^*$, there must be at least one instance of i for which $((x_i, t_i), (x_{i+1}, v_i)) \in \widetilde{Q}$. Let $L \subseteq \{1, \ldots, n-1\}$ be the set of all these instances. We have that,

$$t - v = \sum_{i \notin L} (t_i - v_i) + \sum_{i \in L} (t_i - v_i).$$

Introduce $t_n = v$ and $v_n = t$ and include the element $((b,v),(a,t)) = ((x_n, t_n), (x_1, v_n))$ into the sequence in order to make a loop joining $x_n = b$ back to $x_1 = a$. We can divide this loop into $|L|$ subsequences each starting with a and ending with a. Denote the set of i's falling in the l-th sequence by L_l. This gives us that $((a, t_\ell), (a, v_\ell) \in B(R^*)$ for all:

$$t_\ell - v_\ell = \sum_{i \in L_\ell} (t_i - v_i).$$

We also have that:

$$0 = \sum_{i \notin L} (t_i - v_i) + \sum_{i \in L} (t_i - v_i) = \sum_{\ell \in L} (t_\ell - v_\ell) + \sum_{i \in L} (t_i - v_i).$$

The second term is less than zero, which implies that there must be an $\ell \in L$ for which $t_\ell - v_\ell \neq 0$, contradicting the assumption that $a \in X - (U(R^*) \cup L(R^*))$. We can conclude that $B(R \cup \widetilde{Q})$ is a time-monotonic, transitive and absolute time consistent extension of R^*. This contradicts with the maximality of R^*. Therefore, case ii), i.e. $X = U(R^*) \cup L(R^*)$, must hold.

Now, let $((a,t),(b,v)) \in N(R)$ and consider $Q = B(R \cup \{((a,t),(b,v))\})$. Let us show that Q is a transitive, absolute time-consistent and time-monotonic extension of R^*. A simple adaptation of the proof in lemma 3.1 shows that Q is transitive and absolute time-consistent.

To see that Q is time-monotonic, consider first the case where there is an element $c \in L(Q)$. (the case where $c \in U(Q)$ is analogue and is left to the reader). If $c \in L(R^*)$, we are done, hence assume that $c \in U(R^*) - L(R^*)$.

Then there is a sequence $c = x_1, \ldots, x_n = c$ and sequences $t_1, \ldots, t_{n-1}$ and $v_1, \ldots, v_{n-1}$ such that for all $i = 1, \ldots, n-1$,

$$((x_i, t_i), (x_{i+1}, v_i)) \in Q,$$

and,

$$t' - v' = \sum_{i}^{n-1} (t_i - v_i) < 0.$$

From $c \in U(R^*) - L(R^*)$ we derive that $((c, v'), (c, t')) \in P(R^*)$. If there is no i such that $((x_i, t_i), (x_{i+1}, v_i)) = ((a, t), (b, v))$ then $((c, t'), (c, v')) \in B(R^*) = R^*$, a contradiction. Therefore, conclude that there is an i such that $((x_i, t_i), (x_{i+1}, v_i)) = ((a, t), (b, v))$. Let L collect all such instances of i. Then we have that:

$$t' - v' = \sum_{i \notin L} (t_i - v_i) + \sum_{i \in L} (t_i - v_i).$$

Introduce $t_n = v'$ and $v_n = t'$ and include the element $((c, v'), (c, t')) = ((x_n, t_n), (x_1, v_n))$ into the sequence in order to make a loop joining $x_n = b$ back to $x_1 = b$. We can divide this loop into $|L|$ subsequences each starting with b and ending with a. Denote the set of i's falling in the l-th sequence by L_l. This gives us that $((b, v_\ell), (a, t_\ell)) \in B(R^*)$ for all:

$$v_\ell - t_\ell = \sum_{i \in L_\ell} t_i - v_i.$$

Consider first the case where $a \in L(R^*)$. Observe that:

$$0 = \sum_{\ell \in L} (t_\ell - v_\ell) + |L|(t - v).$$

This implies that either $v - t$ is equal to $v_\ell - t_\ell$ for each $\ell \in L$ which cannot occur (because $((b, v), (a, t)) \notin B(R^*) = R^*$) or there are elements $\ell, \ell' \in L$ such that $v_\ell - t_\ell > v - t > v_{\ell'} - t_{\ell'}$. Together with $a \in L(R^*)$ we have that $((b, v_\ell), (a, t_\ell)) \in B(R^*)$ and $((a, v + t_\ell), (a, t + v_\ell)) \in R^*$. Hence $((b, v), (a, t)) \in B(R^*) = R^*$, a contradiction. If $a \in U(R^*)$, we derive that $((b, v_{\ell'}), (a, t_{\ell'}) \in R^*$ and $((a, v + t_{\ell'} + t), (a, t + v_{\ell'})) \in R^*$. Hence $((b, v), (a, t)) \in B(R^*)$, again a contradiction.

In order to show that Q is an extension of R^*, we can assume, on the contrary, that there exist a pair $((c, t'), (d, v')) \in Q$ and $((d, v'), (c, t')) \in P(R^*)$. The proof that this leads to a contradiction is very similar to the proof that Q is time-monotonic and is left to the reader.
In both cases i) and ii), we have that there exist an element of Ω that extends R^* and which contains elements that are not in R^*. This contradicts the maximality of R^*. Therefore, we can conclude that R^* is complete. □

Now, we confirm that ATARP is a necessary and sufficient condition for rationalizability by an absolute time-consistent an time-monotonic ordering.

Proof. Assume that R^* rationalizes the choice function C and assume, on the contrary, that $((a, t), (b, v)) \in B(\overline{R_v})$ and $((b, v), (a, t)) \in P_v$. By absolute time-consistency, transitivity and time-monotonicity of R^*, and from the definition of R_v and lemma 3.1, we establish that $\overline{R'_v} \subseteq$

R^*. Further, from the definition of P_v, we can deduce that $P_v \subseteq P(R^*)$. Conclude that that $((a,t),(b,v)) \in R^*$ and $((b,v),(a,t)) \in P(R^*)$, a contradiction.

To see the converse, let C satisfy ATARP, i.e. $B(\overline{R_v}) \cap P_v^{-1} = \emptyset$. First we show that $P_v = P(R_v)$. To verify $P_v \subseteq P(R_v)$, observe that from the definitions $P_v \subseteq R_v$. If, on the contrary, $((a,t),(b,v)) \in P_v$ and $((b,v),(a,t)) \in R_v$, we find a contradiction with $B(\overline{R_v}) \cap P_v^{-1} = \emptyset$, hence $P_v \subseteq P(R_v)$. To prove that $P(R_v) \subseteq P_v$, assume that $((a,t),(b,v)) \in P(R_v)$. Then, there is an $A \in \Sigma$ such that $(a,t) \in C(A)$ and $(b,v) \in A$. If, on the contrary $((a,t),(b,v)) \notin P_v$, we deduce that $(b,v) \in C(A)$. This implies that $((b,v),(a,t)) \in R_v$, a contradiction. Conclude that $P(R_v) = P_v$.

Now, it is possible to rewrite $B(\overline{R_v}) \cap P_v^{-1} = \emptyset$ as $B(\overline{R_v}) \cap P(R_v)^{-1} = \emptyset$. Lemma 3.2 implies that R_v has a time-consistent and impatient ordering extension R^*. If $(a,t) \in C(A)$, we find that $((a,t),(b,v)) \in R_v$ for all $(b,v) \in A$, hence $((a,t),(b,v)) \in R^*$ for all $(b,v) \in A$. If $(a,t) \notin C(A)$, we find, from the non-emptiness[7] of $C(A)$, that there is an $(b,v) \in A$ such that $((b,v),(a,t)) \in P_v = P(R_v)$. As R^* extends R_v, we establish that $((b,v),(a,t)) \in P(R^*)$. Conclude that

$$C(A) = \{(a,t) \in A | \forall (b,v) \in A, ((a,t),(b,v)) \in R^*\}.$$

Hence, R^* is an absolute time-consistent and impatient rationalization of C. □

[7]This follows from the assumption that the set Λ does not contain the empty set.

CHAPTER 4

Nash rationalization of collective choice over lotteries

Joint work with Luc Lauwers

4.1 Introduction

A recent track of research seeks to identify the testable implications of various theories of multi-agent decision making. Along these lines we set up a test to verify whether players have independent preference relations and select a Nash equilibrium. Let us start the exposition with an example.

Consider a two person game in normal form. Each player has two pure strategies: U(p) and D(own) for player 1, L(eft) and R(ight) for player 2. Each player is informed about the meaning of a mixture over pure strategies and about the payoff such a mixture generates. Denoting by x (resp. y) the weight attached to the pure strategy U (resp. L). The players may select the mixture

$$x\times U \ + \ (1-x)\times D \qquad \text{and} \qquad y\times L \ + \ (1-y)\times R,$$

with x and y in the closed interval $[0,1]$, and communicate the selected value of x (resp. y) to the experimental designer.[1] In this setup, we observe the values $x = 0.4$ and $y = 0.3$. Then, a second experiment is executed. For player 1, the set of pure strategies $\{U, D\}$, is modified to $\{U, D'\}$ with

$$D' = 0.4\times U \ + \ 0.6\times D.$$

For player 2, the set $\{L, R\}$ is modified to $\{L', R\}$ with

$$L' = 0.42\times L \ + \ 0.58\times R.$$

[1]Sopher and Narramore [2000] carry out an experiment to test consistency and (in)transitivity of individual choice over lotteries and mixtures of lotteries. In the spirit of their questionnaire, we propose the players to select from a menu of mixtures over pure strategies.

The lottery $x'{\times}U \;+\; (1-x'){\times}D'$ coincides with $(x'+(1-x')\,0.4){\times}U \;+\; (1-x')\,0.6{\times}D$ and the lottery $y'{\times}L \;+\; (1-y'){\times}R$ coincides with $0.42\,y'{\times}L \;+\; (1-0.42\,y'){\times}R$. Given these sets of pure strategies, player 1 selects $0.4\,U + 0.6\,D$ (i.e. $x' = 0$) and player 2 selects $0.42\,L + 0.58\,R$ (i.e. $y' = 1$). Similar experiments generate the following data:

pure strategies		$\longmapsto$ selected mixtures	
player 1	player 2	player 1	player 2
$\{U, D\}$	$\{L,\, R\}$	$.4\,U + .6\,D$	$.3\,L + .7\,R$
$\{U, D' = .4\,U + .6\,D\}$	$\{L' = .42\,L + .58\,R\,,\, R\}$	$.4\,U + .6\,D$	$.42\,L + .58\,R$
$\{U'' = .5\,U + .5\,D, D\}$	$\{L'' = R'' = .5\,L + .5\,R\}$	$.5\,U + .5\,D$	$.5\,L + .5\,R$

TABLE 1: OBSERVED DATA.

The following question arises. Given such data, is it possible to check whether or not these players are rational in the sense that they optimize with respect to an independent preference relation and select a Nash equilibrium? In section 4 we return to this example and we will argue that the above data are not Nash rationalizable. The remaining part of the introduction positions this research in the literature and introduces our main results.

Many theories on behavior start from assumptions on the individual preference relation over the feasible set of alternatives (e.g. transitivity, completeness). As soon as one accepts that binary relations are not observable while actual choices are observable; it is important to test whether the actual choices support or reject the assumptions. This issue has been discussed by, among others, Arrow [1959], and Sen [1971].

There are at least two ways to tackle this problem. One approach [Sen, 1971] studies how the selection reacts upon particular changes in the set of feasible alternatives. Obviously, if the individual consults a transitive and complete preference relation, then he should not reconsider his choice when the choice set shrinks while his selected alternative remains feasible. Analogously, when he selects the same alternative from two different choice sets, then he should select again this alternative from the union of these two choice sets. As such, the hypothesis of revealed preference becomes testable. A second approach is offered through the theory of revealed preferences. If an alternative is chosen from a set, then it is top ranked in this choice set according to the revealed preference relation. Thetransitive closure of this revealed preference relation is called the indirect revealed preference relation. Richter's [1966] congruence axiom provides necessary and sufficient conditions for a choice function to be rationalizable: if an alternative a is indirectly revealed preferred to b, then b should not be strictly revealed preferred to a.

Sprumont [2000] extends the problem of rationalizability to situations involving different and interacting individuals. He defines a joint choice function to be Nash rationalizable if there exists a profile of complete and transitive preference relations over the sets of actions, so that the observed outcomes coincide with the Nash equilibria based upon these preferences. In the spirit of Sen's approach, he characterizes Nash rationalization through the combination of an expansion and a contraction property. Ray and Zhou [2001] perform a similar study for subgame perfect Nash equilibria.

We extend one of the results of Sprumont [2000] and tackle the Nash rationalizability of collective choice when individuals have a menu of mixtures at their disposal (each mixture defines a probability distribution over the set of pure strategies). For example, Table 1 might result from an experiment. Following the tradition in game theory, we interpret the rational behavior of a player in terms of expected utility maximization. In particular, besides completeness and transitivity we impose an independence demand upon the preference relations of the (rational) players. This independence condition states that the relationships between two lotteries (over the set of pure strategies profiles) are not affected when they are mixed in the same way with a third lottery. Myerson [1997, p11] discusses the strength of the independence axiom in the expected utility maximization theorem. In addition, he indicates some of the difficulties that arise in decision theory when independence is dropped. As a matter of fact, Clark [2000] introduces a 'revealed Archimedian axiom' (a continuity axiom) to capture the difference between rationalization by an independent ordering and maximizing expected utility.

Furthermore, in contrast to Sprumont [2000], we follow the track of revealed preferences. If only one individual is involved, it is sufficient to check the transitive closure of the revealed preference relation. The present setting with more than one individual, however, is more demanding. We modify Richter's axiom and require that the 'transitive and independent' closure of the revealed preference relation does not conflict with the strict revealed preference relation. Besides that, we need an axiom that connects the individual behavior to the collective behavior. We assume that a strategy profile belongs to the collective choice if each player keeps his selected strategy when he is assured that he is the only player allowed to deviate. We refer to this condition as the collective choice being noncooperative. Later on, we will argue that this condition has some flavor of an expansion-contraction axiom. Our main result reads (see theorem 4.3 in Section 4.4 for the exact formulation):

Theorem. *A collective choice correspondence is Nash rationalizable if and only if it is noncooperative and satisfies the modified version of Richter's congruence axiom.*

Let us highlight two intermediate results towards this theorem. *First*, we need a condition that is strong enough to guarantee that a binary relation extends to a transitive, complete, and independent relation. Here, we learn from Suzumura [1976], who showed that consistency is sufficient and necessary for a relation to have an ordering extension. We shift Suzumura's result to a setup involving choices over lotteries, and we use the term 'lottery-consistency' as a reference. *Second*, we study the behavior of a single individual choosing from a set of lotteries. Here, we show that the extended version of Richter's congruence axiom—restricted to one player—is sufficient and necessary for the individual choice function to be rationalizable by an independent ordering.[2] Then, we broaden the setup from one individual to a finite number of interacting players. We apply the axiom of noncooperation behavior and conclude the above theorem.[3]

This theorem can be used in an experimental setting to test whether players have independent preference relations and select a Nash equilibrium. First, each player is told the structure of the game: the number of players, the sets of pure strategies, the concept of a mixture (and its in-

[2] Section 4.3 discusses similar results obtained by Clark [1993, thm 3] and by Taesung [1996, thm 3.1].

[3] Galambos [2005] considers a related set-up where the strategy space is restricted to consists only of pure strategies. He introduces an analogous condition (I-congruence) for the Nash rationalizability. See section 4.4 for a discussion.

terpretation as a probability distribution over the set of pure strategies), and the payoff function. Subsequently, each individual is informed about the menus available to all of the players, and is asked to choose from 'his' menu of mixtures. The size of such a menu is either finite (e.g. [MacDonald and Wall, 1989], [Conlisk, 1989], [Oliver, 2003]) or infinite ([Sopher and Narramore, 2000], [Shachat, 2002]). Once each player has independently chosen a mixture from his menu, the experimenter determines the resulting lottery over the pure strategy profiles and the corresponding outcome. This setup allows the experimental designer to control the sets of available mixtures. Moreover, the experimenter observes the mixtures selected by each individual separately. In order to check whether individuals play a Nash equilibrium, we propose to proceed in two stages. In a first experiment, the players are screened according to whether they maximize with respect to an independent preference relation. This can be done, for instance, through some Allais-paradox test (e.g. [Conlisk, 1989], [Oliver, 2003])[4] . This step filters out those individuals who violate the expected utility criterion. In the second step, one confronts the remaining players with a noncooperative game. As such, one can judge on the basis of observations whether in mixed strategies the Nash criterion is rejected or supported.

The next section introduces the notation and studies binary relations and their independent and transitive extensions. Section 4.3 introduces the concept of lottery-consistency as a test for the rational behavior of an individual choosing over lotteries, and discusses related results of Clark [1993] and Taesung [1996]. Section 4.4 extends the notation to collective choice and proves the main result. Here, we also return to the data in Table 1. Section 4.5 links our result to the analysis of Sprumont [2000].

4.2 Independent ordering extensions

This section establishes the notation, introduces the concept of independence of a binary relation, and provides conditions for a relation to have an independent ordering extension.

Let $H \subset \mathbb{R}^n$ be the hyperplane of n-vectors the coordinates of which add up to 1, and let $\Delta = \Delta^n \subset H$ be the $(n-1)$-dimensional simplex. An element $a = (a_1, a_2, \ldots, a_n)$ in Δ is an n-tuple of nonnegative real numbers adding up to 1, and is called a lottery. The i-th coordinate a_i of the lottery a gives the probability that state i occurs.

Throughout, the set D refers to either Δ or H. A binary relation R in the set D is a subset of the cartesian product $D \times D$. The symmetric component $R \cap R^{-1}$ is denoted by $I(R)$, the asymmetric part $R \setminus I$ by $P(R)$, and the non-comparable part $D \times D \setminus (R \cup R^{-1})$ by $N(R)$. For the binary relation R' we denote these induced relations by $I(R')$, $P(R')$, $N(R')$; for R^* we use $I(R^*)$, $P(R^*)$, $N(R^*)$; etc. A reflexive and transitive relation is said to be a quasi-ordering. A complete quasi-ordering is said to be an ordering. The binary relation R' in D extends the relation R if $R \subseteq R'$ and $P(R) \subseteq P(R')$.

Next, we introduce the notion of independence. This condition studies the behavior of a binary relation on compound vectors. For a and b in H and for α a nonnegative real number, the vector

[4] This step should be complemented with a power-test, e.g. [Bronars, 1987]

$[\alpha, a, b]$ denotes the linear combination $\alpha\, a + (1-\alpha)b$. For α in between 0 and 1, the compound vector $[\alpha, a, b]$ is a convex combination of a and b. And, for $\alpha > 1$ the compound vector is a point on the ray starting in b and going through a and does not belong to the closed interval $[a, b]$.

A relation R in D is said to be *independent* if for each pair in R the composition with a third vector in D preserves the initial relationships. Formally, R is independent if for each a, b, and c in D, we have

$$\text{if } (a,b) \in R,\ \alpha \geq 0,\ [\alpha, a, c] \in D, \text{ and } [\alpha, b, c] \in D;$$
$$\text{then } ([\alpha, a, c], [\alpha, b, c]) \in R. \tag{1}$$

This condition implies the reflexivity of R (put $\alpha = 0$). Observe that α is allowed to take values larger than 1. As a consequence of this, an independent relation satisfies the 'strict' version of condition (1):

$$\text{if } (a,b) \in P(R),\ \alpha > 0,\ [\alpha, a, c] \in D, \text{ and } [\alpha, b, c] \in D;$$
$$\text{then } ([\alpha, a, c], [\alpha, b, c]) \in P(R).$$

Indeed, let us assume that $(a, b) \in P(R)$ and $\alpha > 0$, while $([\alpha, a, c], [\alpha, b, c]) \notin P(R)$. Since R is independent and $(a, b) \in P(R) \subseteq R$, it follows that
$([\alpha, a, c], [\alpha, b, c]) \in R$. The assumption $([\alpha, a, c], [\alpha, b, c]) \notin P(R)$ implies that $([\alpha, b, c], [\alpha, a, c]) \in R$. Therefore,

$$(b,a) = \left(\left[\frac{1}{\alpha}, [\alpha, b, c], c\right], \left[\frac{1}{\alpha}, [\alpha, a, c], c\right] \right) \in R.$$

A contradiction is obtained: $(b, a) \in R$ and $(a, b) \in P(R)$. Note that α and $1/\alpha$ simultaneously occur (one of these values is larger than 1).

In case R happens to be a complete binary relation, a similar argument implies that R is independent if and only if R is reflexive and for each a, b, c in D, and each α, we have

$$\text{if } (a,b) \in R \text{ (resp. } P(R)) \text{ and } 0 < \alpha \leq 1,$$
$$\text{then } ([\alpha, a, c], [\alpha, b, c]) \in R \text{ (resp. } P(R)). \tag{2}$$

Obviously, condition (1) entails condition (2). Let us check that (2) implies (1). Suppose the antecedent clause of (1) holds, and let $\alpha > 1$. Then, the opposite conclusion—in the assumption that R is complete—reads: "$(\,[\alpha, b, c], [\alpha, a, c]\,) \in P(R)$". Again, we obtain a contradiction: $(b, a) \in P(R)$ while $(a, b) \in R$.
Condition (2) only considers convex combinations and is therefore, in the present setting, perhaps a more natural property.

There is an obvious relationship between the class of independent orderings on H and the class of independent orderings on Δ. If R is an independent ordering on H, then its restriction to Δ is an independent ordering on Δ. The next lemma looks at the reverse relationship.

Lemma 4.1. *An independent, transitive, and complete relation R in Δ, uniquely extends to an independent, transitive, and complete relation R' in H.*

Proof. Let a and b belong to H. Let c be an element in the interior of Δ. Choose $\alpha > 0$ sufficiently close to 0, such that $a' = [\alpha, a, c]$ and $b' = [\alpha, b, c]$ belong to Δ. Let the ordering R' on $\{a, b\}$ agree with the ordering R on $\{a', b'\}$. The ordering R' on $\{a, b\}$ does not depend upon the choice of c and α. We show this by contradiction. Let $a'' = [\beta, a, u] \in \Delta$ and $b'' = [\beta, y, u] \in \Delta$ and assume that $(a', b') \in R$ while $(b'', a'') \in P(R)$. By independence, we have

$$\Big(\underbrace{\Big[\frac{\beta}{\alpha+\beta}, a', b''\Big]}_{v_1}, \underbrace{\Big[\frac{\beta}{\alpha+\beta}, b', b''\Big]}_{v_2}\Big) \in R$$

$$\text{and} \quad \Big(\underbrace{\Big[\frac{\alpha}{\alpha+\beta}, b'', b'\Big]}_{v_3}, \underbrace{\Big[\frac{\alpha}{\alpha+\beta}, a'', b'\Big]}_{v_4}\Big) \in P(R).$$

Observe that v_2 and v_3 coincide. Transitivity of R implies that $(v_1, v_4) \in P(R)$. One can write v_1 and v_4 in terms of a, b, c, and u and verify that $v_1 = v_4$. Hence, we obtain $(v_1, v_1) \in P(R)$. This contradicts the definition of the asymmetric component $P(R)$ of the relation R. Therefore, R' is well defined. Transitivity and independence of R' follows from the definition of R' in combination with the transitivity and independence of R.

Finally, we show that the extension R' is unique. Let the independent, transitive, and complete relations R' and R'' extend R. Let a and b belong to H. Let c be an element in the interior of Δ. Choose $\alpha > 0$ sufficiently close to 0 such that $a' = [\alpha, a, c]$ and $b' = [\alpha, b, c]$ both belong to Δ. Since R' and R'' extend R and are independent, it follows that R' and R'' rank a and b in the same way. Hence, $R' = R''$. □

Now, we focus on conditions that are strong enough to guarantee that a binary relation has an extension that is complete, transitive, and independent.

Let us insert here a result of Suzumura [1976, thm 3] who solved a similar exercise. Suzumura started from a relation R and looked for a complete and transitive relation R^* such that $R \subset R^*$ and $P(R) \subset P(R^*)$. A natural way to proceed is to check whether the transitive closure $T(R)$ of R respects the asymmetric part, i.e. $P(R) \subset P(T(R))$. Apparently, this provides sufficient (and necessary) conditions: R has an ordering extension if and only if

$$\text{for each } a, b, \text{ we have } (a, b) \in T(R) \text{ implies } (b, a) \notin P(R).$$

Suzumura labelled this condition as consistency.

We proceed similarly. Let R be a relation in D. The independent order relation R^* in D is said to be an independent ordering extension of R if $R \subset R^*$ and $P(R) \subset P(R^*)$. The *transitive and independent closure* of R is the smallest (for inclusion) relation in D that includes R, satisfies transitivity and independence.

The next lemmas provide further insight in the transitive and independent closure of a relation in the hyperplane H. Let R be a relation in H. Define the binary relation R_{TI} in H by $(a, b) \in R_{TI}$

if there exists a natural number n, elements $x_1 = a, x_2, \ldots, x_{n+1} = b$ and $z_1, z_2, \ldots, z_n$ in H, and positive real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\text{for each } i = 1, 2, \ldots, n, \text{ we have } ([\alpha_i, x_i, z_i], [\alpha_i, x_{i+1}, z_i]) \in R.$$

Lemma 4.2. *Let R be a relation in H. The above defined relation R_{TI} is its transitive and independent closure.*

Proof. It is easy to see that R_{TI} is transitive. To check independence, let $(a, b) \in R_{TI}$, $\beta \geq 0$, and let $q \in H$. We have to show that $(a' = [\beta, a, q] \; ; b' = [\beta, b, q])$ belongs to R_{TI}. Since $(a, b) \in R_{TI}$, there exist elements x_i ($x_1 = a$ and $x_{n+1} = b$), z_i in H, and positive real numbers α_i such that

$$([\alpha_i, x_i, z_i], [\alpha_i, x_{i+1}, z_i]) \in R, \qquad \text{for each } i = 1, 2, \ldots, n.$$

Define $v_i = \frac{(1-\alpha_i)\beta z_i - (1-\beta)\alpha_i q}{\beta - \alpha_i}$ and $x'_i = [\beta, x_i, q]$ in H. In case $\beta = \alpha_i$, the element v_i has no role and can be chosen arbitrarily, e.g. put $v_i = 0$. It follows that

$$\left(\left[\frac{\alpha_i}{\beta}, x'_i, v_i \right], \left[\frac{\alpha_i}{\beta}, x'_{i+1}, v_i \right] \right) \in R, \qquad \text{for each } i = 1, 2, \ldots, n.$$

Hence, $(a', b') \in R_{TI}$.
Finally, we have to show that R_{TI} is the smallest (for inclusion) independent and transitive relation containing R. Let R' be an independent and transitive extension of R. For each (a, b) in R_{TI}, there exists elements $x_1 = a, x_2, \ldots, x_{n+1} = b$; $z_1, z_2, \ldots, z_n$ in H and positive real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$([\alpha_i, x_i, z_i], [\alpha_i, x_{i+1}, z_i]) \in R, \qquad \text{for each } i = 1, 2, \ldots, n.$$

The independence of R' and the fact that $R \subseteq R'$ imply that $(x_i, x_{i+1}) \in R'$ for each $i = 1, 2, \ldots, n$. The transitivity of R' implies that $(a, b) \in R'$. Therefore, $R_{TI} \subseteq R'$. □

Lemma 4.3. *Let R be a relation in H. Then, (a, b) belongs to the transitive and independent closure R_{TI} of R if and only if*

- *either, (a, b) belongs to the transitive closure of R;*
- *or, $a - b = \Sigma_{i=1}^{\ell} \beta_i (x_i - y_i)$, with (x_i, y_i) in R and $\beta_i > 0$ for each i, and $\beta_j \neq 1$ for at least one j.*

Proof. First, let $(a, b) \in R_{TI}$. The elements a and b are linked through a finite sequence $a = x_1, x_2, \ldots, x_{n+1} = b$; and there exist $z_1, z_2, \ldots, z_k$ in H and $\alpha_1, \alpha_2, \ldots, \alpha_k > 0$ such that

$$([\alpha_i, x_i, z_i], [\alpha_i, x_{i+1}, z_i]) \in R, \text{ for each } i = 1, 2, \ldots, n.$$

For each i we obtain $[\alpha_i, x_i, z_i] - [\alpha_i, x_{i+1}, z_i] = \alpha_i\,(x_i - x_{i+1})$. Multiply these equations by $1/\alpha_i > 0$, and add them up:

$$a - b \;=\; x_1 - x_{k+1} \;=\; \Sigma_{i=1}^{k}\,\frac{[\alpha_i, x_i, z_i] - [\alpha_i, x_{i+1}, z_i]}{\alpha_i}.$$

In case $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 1$, then (a, b) belongs to the transitive closure of R.

Next, assume $a - b = \Sigma_{i=1}^{\ell}\beta_i(x_i - y_i)$ with (x_i, y_i) in R, $\beta_i > 0$, and $\beta_j \neq 1$. We have to prove that $(a, b) \in R_{TI}$. We proceed by induction on ℓ.

For $\ell = 1$, it suffices to observe that the vector $z = (a - \beta_1 x_1)/(1 - \beta_1)$ in H allows us to write $a = [\beta_1, x_1, z]$ and $b = [\beta_1, y_1, z]$.

Suppose the result holds up to ℓ. Consider a positive linear combination of length $\ell + 1$. Assume that $\beta_1 \neq 1$. Consider $(a' - b') = (1/\beta) \times \Sigma_{i=2}^{\ell+1}\,\beta_i(x_i - y_i)$, with $0 < \beta \neq 1$ such that at least one of the coefficients β_i/β differs from 1. From the induction basis, we obtain $(a', b') \in R_{TI}$. Hence, we can write

$$a - b \;=\; \beta_1(x_1 - y_1) + \beta(a' - b'), \qquad \text{with } 0 < \beta_1 \neq 1 \text{ and } 0 < \beta \neq 1.$$

Let z and $\tilde{b}$ in H solve the equations $x_1 = [1/\beta_1, a, z]$ and $y_1 = [1/\beta_1, \tilde{b}, z]$. Independence implies $(a, \tilde{b}) \in R_{TI}$. Next, let z' and y^* in H solve the equations $a' = [1/\beta, \tilde{b}, z']$ and $b' = [1/\beta, y^*, z']$. Then, $(\tilde{b}, y^*) \in R_{TI}$. The transitivity of R_{TI} implies $(a, y^*) \in R_{TI}$. Finally, the equations $x_1 - y_1 = (a - \tilde{b})/\beta_1$ and $a' - b' = (\tilde{b} - y^*)/\beta$ imply that $y^* = b$. □

Now, we are able to shift the result of Suzumura towards the present setting. We extend the definition of consistency and we state the main result of this section.

Definition 4.1 (Lottery consistency)**.** *The relation R in D is said to be lottery-consistent if for each a and b in D, we have that $(a, b) \in R_{TI}$ implies $(b, a) \notin P(R)$.*

Theorem 4.1. *Let R be a relation in Δ. Then, R has an independent ordering extension in Δ if and only if R is lottery-consistent.*

Proof. Let R^* be an independent ordering extension of R. Then, by lemma 4.2, $R \subseteq R_{TI} \subseteq R^*$ and by the definition of extension $P(R) \subseteq P(R^*)$. Hence, it cannot happen that $(a, b) \in R_{TI}$ and $(b, a) \in P(R)$; otherwise the combination $(a, b) \in R^*$ and $(b, a) \in P(R^*)$ would occur. Conclude that R is lottery-consistent.

The proof of the reverse implication is more involved. We indicate that the non-comparable part of an incomplete, independent, and transitive extension of R can be further reduced by adding one single couple. We use this result, in combination with a free ultrafilter on an appropriate set, to define a complete, independent, and transitive extension of R.

Hence, let R^* be an incomplete, independent, and transitive extension of R. Let $(a, b) \in N(R^*)$. Define the relation $Q = R^* \cup \{(a, b)\}$ and let Q_{TI} be the transitive and independent closure of Q. We show that Q_{TI} extends R. First, observe the inclusions $R \subseteq R^* \subseteq Q_{TI}$. The inclusion

$P(R) \subseteq P(R_{TI})$ is shown by contradiction. Therefore, assume the existence of a couple (c,d) in $P(R)$ such that $(d,c) \in Q_{TI}$. Apply lemma 4.3 upon Q_{TI} and obtain

$$d - c \;=\; \Sigma_{i=1}^{\ell}\,\beta_i(x_i - y_i), \quad \text{with } (x_i, y_i) \in Q, \text{ and } \beta_i > 0 \text{ for each } i.$$

As R^* extends R and $(c,d) \in P(R) \subseteq P(R^*)$, the pair (a,b) occurs at the right hand side; say $(x_1, y_1) = (a,b)$. Rewrite the previous equation:

$$b - a \;=\; \gamma(c-d) \,+\, \Sigma_{i=2}^{\ell}\,\gamma_i(x_i - y_i), \qquad \text{with } \gamma > 0, \gamma_i > 0 \text{ for each } i.$$

lemma 4.3 implies that R^* is able to compare a and b. This conflicts with $(a,b) \in N(R^*)$.

Next, let $[D]^{<\infty}$ be the collection of all finite subsets of D. For each A in $[D]^{<\infty}$, let $S(A)$ collect all the finite supersets of A. For example, $S(\varnothing) = [D]^{<\infty}$. Let $\mathcal{U}$ be an ultrafilter that extends the filter generated by the family $\{S(A) \,|\, A \in [D]^{<\infty}\}$.[5]

For each set $A \in [D]^{<\infty}$, let the relation Q be an independent and transitive extension of R that is able to compare all pairs in A. The relation Q either coincides with R_{TI} or can be obtained by adding a finite number of couples to R_{TI} (as explained above). Denote the restriction of Q to the set A by R_A, i.e. $R_A = Q \cap (A \times A)$. Finally, define the relation R^* in D as follows:

$$(a,b) \in R^* \qquad \text{if and only if} \qquad \{A \in [D]^{<\infty} \,|\, (a,b) \in R_A\} \in \mathcal{U}.$$

We check that R^* extends R, is independent and transitive, and complete.

(i) R^* *extends* R. Let R be able to compare a and b. For each set A in $S(\{a,b\})$ the relation R_A agrees with R on the pair $\{a,b\}$. As $S(\{a,b\})$ belongs to the ultrafilter $\mathcal{U}$, the relation R^* extends R.

(ii) R^* *is transitive and independent.* Let $a - b = \sum_{i=1}^{\ell} \alpha_i(x_i - y_i)$ with $(x_i, y_i) \in R^*$ and $\alpha_i > 0$ for each $i = 1, 2, \ldots, \ell$. By definition, the sets $U_i = \{A \,|\, (x_i, y_i) \in R_A\}$ belong to $\mathcal{U}$. The finite intersection property implies that $U = U_1 \cap U_2 \cap \ldots \cap U_\ell$ and $U \cap S(\{a,b\})$ both belong to $\mathcal{U}$. Since each relation R_A is transitive and independent, we have $(a,b) \in R_{A\cup\{a,b\}}$ for each A in U. Hence, $(a,b) \in R^*$.

(iii) R^* *is complete.* Consider the pair $\{a,b\}$. The collection $S(\{a,b\})$ splits up into three parts,

$$\begin{aligned} S(\{a,b\}) \;&=\; \{A \,|\, (a,b) \in P(R_A)\} \,\cup\, \{A \,|\, (b,a) \in P(R_A)\} \\ &\qquad\qquad \cup \{A \,|\, (a,b), (b,a) \in R_A\}. \end{aligned}$$

Since $\mathcal{U}$ is an ultrafilter and $S(\{a,b\}) \in \mathcal{U}$, exactly one of these three parts belongs to $\mathcal{U}$. Conclude that R^* is able to compare a and b. $\square$

[5] A filter $\mathcal{F}$ on a set Ω is a subset of 2^{Ω} that (i) does not contain the empty set ($\varnothing \notin \mathcal{F}$), (ii) satisfies the intersection property (if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$), and (iii) is closed for supersets (if $A \in \mathcal{F}$ and $A \subset B \subset \Omega$, then $B \in \mathcal{F}$). If the filter $\mathcal{F}$ contains, for each $A \subset \Omega$, either A or its complement $\Omega - A$; then $\mathcal{F}$ is said to be an ultrafilter. An ultrafilter is a maximal (for inclusion) filter. Zorn's lemma implies that each filter extends to an ultrafilter. An ultrafilter that does not contain finite sets, is said to be free.

4.3 Rationalizability of choice over lotteries

This section extends Richter's result towards the rationalizability of individual choice over lotteries. At the end of this section we shortly discuss similar studies by Clark [1993] and by Taesung [1996].

Consider the $(n-1)$-dimensional simplex Δ and let $\mathcal{S}$ be a collection of nonempty subsets of Δ. A choice correspondence C is a correspondence

$$C : \mathcal{S} \twoheadrightarrow \Delta : S \longmapsto C(S) \subseteq S.$$

The choice correspondence C is said to be rationalizable if there exists an independent ordering R^* in Δ such that for each S in $\mathcal{S}$ the set $C(S)$ collects the maximizers of the restriction of R^* to S, i.e.

$$\text{for each } S \text{ in } \mathcal{S}: \; C(S) \;=\; M(R^*|S) \;=\; \{a \in S \,|\, \text{for all } b \text{ in } S : (a,b) \in R^*\}\,.$$

Observe that for a (rationalizable) choice correspondence the choice set $C(S)$ might be empty; e.g. if $S \subset \Delta$ is an open (in the Euclidean topology) set and if the ordering R^* happens to be continuous, the set $M(R^*|S)$ of maximizers might be empty. As it is unclear what one should conclude on the basis of an empty choice set, we impose the choice correspondence to be decisive on $\mathcal{S}$, i.e. a set S for which $C(S) = \varnothing$ is excluded from $\mathcal{S}$.

For a choice correspondence $C : \mathcal{S} \twoheadrightarrow \Delta$, the revealed preference relations R_v and P_v in Δ are defined as follows. The pair (a,b) belongs to the revealed preference relation R_v if and only if there is a set S in $\mathcal{S}$ such that $a \in C(S)$ and $b \in S$. Furthermore, the pair (a,b) belongs to the strict revealed preference relation P_v if and only if there is a set S in $\mathcal{S}$ such that $a \in C(S)$ while $b \in S \setminus C(S)$.

We extend the congruence axiom of Richter [1966]. A choice correspondence $C : \mathcal{S} \twoheadrightarrow \Delta$ is said to satisfy *the congruence axiom* if for each x and y in Δ we have

$$(a,b) \in R_{v,TI} \quad \text{implies} \quad (b,a) \notin P_v,$$

where $R_{v,TI}$ is the transitive and independent closure of the revealed preference relation R_v.

We will show that this congruence axiom is strong enough to guarantee the choice correspondence to be rationalizable. The next lemma is a first step towards this result.

Lemma 4.4. *If the choice correspondence $C : \mathcal{S} \twoheadrightarrow \Delta$ satisfies the congruence axiom, then the asymmetric part $P(R_v)$ of the revealed preference relation R_v coincides with the strict revealed preference relation P_v.*

Proof. $(i) : P(R_v) \subseteq P_v$. If $(a,b) \in P(R_v)$, then $(a,b) \in R_v$ and $(b,a) \notin R_v$. Hence, there exists a set S such that $a \in C(S)$ and $b \in S$; and for each set T containing a and b, it holds that $b \notin C(T)$. Put $T = S$ and conclude that $a \in C(S)$ while $b \in S \setminus C(S)$, i.e. $(a,b) \in P_v$.

$(ii) : P_v \subset P(R_v)$. If $(a,b) \in P_v$, then $(a,b) \in R_v$. In case also $(b,a) \in R_v$, the congruence axiom is violated: $(b,a) \in R_v \subset R_{v,TI}$ and $(a,b) \in P_v$. Therefore, $(b,a) \notin R_v$ and $(a,b) \in P(R_v)$. $\square$

As a corollary we obtain that if a choice correspondence satisfies the congruence axiom, then the revealed preference relation is lottery-consistent. The main result of this section reads:

Theorem 4.2. *Let the choice correspondence $C : \mathcal{S} \twoheadrightarrow \Delta$ be decisive on $\mathcal{S}$. Then, C is rationalizable if and only if it satisfies the congruence axiom.*

Proof. Let the independent ordering R^* in Δ rationalize the choice correspondence C. Obviously, R^* extends the revealed preference relation: $R_v \subseteq R^*$ and $P_v \subset P(R^*)$. As R^* is transitive and independent, R^* includes the transitive and independent closure $R_{v,TI}$ of R_v (use lemma 4.2). Suppose now that $(b, a) \in P_v$. Then, $(b, a) \in P(R^*)$ and $(a, b) \notin R^*$. As a consequence, if $(b, a) \in P_v$, then $(a, b) \notin R_{v,TI}$.

Let C satisfy the congruence axiom. By lemma 4.4, the revealed preference relation is lottery-consistent. Apply theorem 4.1 and extend the revealed preference relation R_v to an independent ordering R^* in Δ. Now, we have to verify whether $C(S) = M(R^*|S)$ holds for each set S in $\mathcal{S}$. Let $a \in C(S)$. Hence, for each b in S we have $(a, b) \in R_v \subseteq R^*$, i.e. $a \in M(R^*|S)$. Next, let $a \in S \setminus C(S)$. By assumption, C is decisive on S: there exists a b in S such that $b \in C(S)$. It follows that $(b, a) \in P_v \subseteq P(R^*)$. Conclude that $a \notin M(R^*|S)$. □

The ultimate goal is to establish a test for the null hypothesis

$$H_0 : \text{the individual choice correspondence } C : \mathcal{S} \longrightarrow\!\!\!\!\!\rightarrow \Delta \text{ is rationalizable.}$$

Of course, one can extract the binary relation behind the choice correspondence (by checking all the pairs in Δ) and verify whether this relation is an independent ordering. In an empirical setting, however, this is impossible to manage. Theorem 4.2 allows us to test on the basis of a finite data set whether or not the null hypothesis should be rejected. As usual, not rejecting H_0 does not imply that H_0 is shown to hold. The next section returns to this issue.

We now point out some differences with the work of Taesung [1996], who also studied the preference relation on lotteries revealed through a choice correspondence. Where Taesung [1996, Appendix] uses a generalization of the theorem of the alternative, we follow the axiomatic approach and start from the theory of binary extensions. Furthermore, Taesung [1996, thm 3.1] restricts the attention to finite choice sets. We do not impose restrictions on the size of the choice set. However, recall from theorem 4.2 that we need the choice correspondence to be decisive on the choice sets. Finally, as the revision of this paper was being completed, we learned of a result of Clark [1993, thm 3] which is very similar to our theorem 4.2. Let us highlight the main differences. First, Clark formulates different independence axioms. In the presence of transitivity and completeness, however, the combination of these axioms turn out to coincide with our independence condition. Second, Clark applies the Hausdorff maximality principle to obtain a complete relation. In contrast, we obtain completeness by means of a free ultrafilter (Cf. theorem 4.1). Hence, both proofs rely on non-constructive methods. The existence of a free ultrafilter, however, is a weaker assumption than the Hausdorff principle (which is equivalent to the axiom of choice). Third, we believe that our lemma 4.3 provides additional insights to the concept of independency.

4.4 Nash rationalization of collective choice

Assume an experimental setting with individuals playing a game allowing mixtures over the set of pure strategies. The experimenter observes the mixtures selected by each individual separately. In case the profile of revealed preferences extends to a profile of independent orderings such that the selection corresponds to a Nash equilibrium, then we say that the observations support the hypothesis of Nash rationalizable behavior. If the data reject this hypothesis, then *either* some player does not consult a complete, transitive, and independent binary relation, *or* the Nash equilibrium is not the right equilibrium concept. This section develops such a test procedure.

We start by introducing some further notation. Let $J = \{1, 2, \ldots, k\}$ be the set of players, $k \in \mathbb{N}$. Individual j has n_j pure strategies, his strategy space Δ_j is the $(n_j - 1)$-dimensional simplex Δ^{n_j}. A strategy profile is a vector $a = (a_1, a_2, \ldots, a_k)$ with a_j in Δ_j the strategy of player j. The product set $\Delta_J = \Delta_1 \times \Delta_2 \times \cdots \times \Delta_k$ collects all the strategy profiles:

$$\Delta_J \; = \; \left\{ a = (a_1, a_2, \ldots, a_k) \,|\, a_j = (a_{j\,1}, a_{j\,2}, \ldots, a_{j\,n_j}) \in \Delta_j \right\}.$$

In order to distinguish the strategy a_j in Δ_j of player j from the strategies of his opponents, we denote the strategy profile a also by (a_j, a_{-j}) with $a_{-j} = (a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k)$ collecting the strategies of j's opponents.

A choice set S is a cartesian product $S_1 \times S_2 \times \cdots \times S_k \subset \Delta_J$ with S_j a nonempty subset of Δ_j for each j. A choice set represents an experiment in which players are confronted with restrictions within their strategy spaces. In the example in Section 4.1 (Table 1) the choice sets are convex hulls of finite sets of points. The results below do not hinge on this convexity assumption.

For a choice set S, a strategy profile a in S, and a player j in J, we denote the cartesian product $S_j \times \{a_{-j}\}$ by S_j^a. In the choice set S_j^a the strategy space of player j is reduced to S_j while the opponents only have one option (opponent i selects a_i from his strategy space $\{a_i\}$).

Let $\mathcal{S}$ be a collection of choice sets. A joint choice correspondence C is a correspondence

$$C : \mathcal{S} \longrightarrow\!\!\!\!\rightarrow \Delta_J : S \longmapsto C(S) \subseteq S.$$

We assume that the choice correspondence C is *individually decisive*, that is, we assume that $C(S_j^a)$ is nonempty for each choice set in $\mathcal{S}$ of the form S_j^a. In words, when the choice of all but one players is limited to only one option, then this one player should be able to select a strategy.[6]

In order to employ individual decisiveness, we impose that for each choice set S in $\mathcal{S}$ all one-person choice sets S_j^a derived from S also belong to $\mathcal{S}$; Sprumont [2000] and Galambos [2005] impose the same condition.

In contrast to the previous section, we do not equip the players with a preference relation on the set Δ_J of strategy profiles. Instead, we assume that the players have preferences over the probability distributions of pure strategy profiles (e.g. via the payoffs corresponding to the pure strategies). As each player j has n_j pure strategies, these pure strategies generate $m = n_1\, n_2\, \cdots\, n_k$

[6] Decisiveness is usually assumed in this context. Clark [1995] discusses *in*decisive choice functions.

pure strategy profiles.[7] The $(m-1)$-dimensional simplex Δ^m collects all the distributions over these profiles. Let d denote the map that converts a strategy profile in Δ_J into a probability distribution in Δ^m:

$$d : \Delta_J \longrightarrow \Delta^m : a \longmapsto d(a), \text{ with } d_{i_1,i_2,\ldots,i_k}(a) = a_{1\,i_1}\, a_{2\,i_2} \,\cdots\, a_{k\,i_k},$$

where i_j runs over the pure strategies 1 to n_j of player j. Within this notation, we can define Nash rationalizability of choice over lotteries.

Definition 4.2 (Nash rationalizability)**.** *The joint choice correspondence $C : \mathcal{S} \twoheadrightarrow \Delta_J$ is said to be Nash rationalizable if there exists a profile $(R_1^*, R_2^*, \ldots, R_k^*)$ of independent orderings in Δ^m such that for each S in $\mathcal{S}$, we have*

$$a \in C(S) \quad \text{if and only if} \quad d(a) \in M\big(R_j^* \,|\, d(S_j^a)\big) \text{ for each } j \text{ in } J.$$

In words, a joint choice correspondence is Nash rationalizable if each player consults an independent ordering to select his own strategy conditional upon his opponents' strategies.

For a Nash rationalizable choice correspondence it holds that, whenever S_j^a is in the domain $\mathcal{S}$, $a \in C(S_i^a)$ if and only if $d(a) \in M(R_i^* \,|\, d(S_i^a))$. Hence, if C is Nash rationalizable and $S_j^a \in \mathcal{S}$ for each j in J, then $a \in C(S)$ if and only if $a \in C(S_j^a)$ for each j in J. The noncooperative behavior of the players is clearly incorporated in the definition of Nash rationalizability: a joint strategy is chosen if no single player has an incentive to deviate.

We modify the definitions of the revealed preference relations from the previous section towards the present setting. Let $x, y \in \Delta^m$.

We start with the revealed preference relations $R_{v,1}, R_{v,2}, \ldots, R_{v,k}$. We have $(x, y) \in R_{v,j}$ if there exist an a in Δ_J and an S_j^a in $\mathcal{S}$ such that $b \in S_j^a$, $a \in C(S_j^a)$, and $(x, y) = (d(a), d(b))$.

Next, we consider the strict revealed preference relations $P_{v,1}, P_{v,2}, \ldots, P_{v,k}$. We have $(x, y) \in P_{v,j}$ if there exist an a in Δ_J, S_j^a in $\mathcal{S}$, and b in S_j^a such that $a \in C(S_j^a)$, $b \in S_j^a \setminus C(S_j^a)$, and $(x, y) = (d(a), d(b))$.

These modifications imply that a player is only able to reveal preferences conditional upon a status quo of his opponents' strategies. A player is able to select x above y only if he has x and y at 'his' disposal, i.e. only if he is able to switch between x and y without the cooperation of any other player.

Similar to the previous section, we search for conditions upon the revealed preferences to guarantee the Nash rationalizability of a choice correspondence

$$C : \mathcal{S} \twoheadrightarrow \Delta_J.$$

Definition 4.3 (Congruence axiom)**.** *The joint choice correspondence C is said to satisfy the congruence axiom if for all x and y in Δ^m and for each j in J, we have*

$$(x, y) \in R_{v,j\,TI} \quad \text{implies} \quad (y, x) \notin P_{v,j},$$

with $R_{v,j\,TI}$ the transitive and independent closure of the revealed preference relation $R_{v,j}$.

[7] The example in Section 4.1 exhibits four pure strategy profiles: (U, L), (U, R), (D, L), and (D, R).

The next lemma states that if a joint choice correspondence satisfies the congruence axiom, then the revealed preference relations are lottery-consistent. Its proof only involves minor modifications of the proof of lemma 4.4 and is omitted.

Lemma 4.5. *If C satisfies the congruence axiom, then for each player j the asymmetric part $P(R_{v,j})$ of the revealed preference relation $R_{v,j}$ coincides with the strict revealed preference relation $P_{v,j}$.*

At this point we are ready to provide conditions for the rationalizability of the individual choice correspondence $S_j^a \mapsto C(S_j^a)$. In order to obtain rationalizability of the joint choice correspondence $S \twoheadrightarrow \Delta_J$, we need some 'local-global' condition to link the collective choice from a set S with the individual choices from the sets S_j^a. Here, we return to the noncooperative nature of the Nash equilibrium.

Definition 4.4 (Noncooperative)**.** *The correspondence $C : \mathcal{S} \twoheadrightarrow \Delta_J$ is said to be* noncooperative *if for each S in $\mathcal{S}$ we have*

$$a \in C(S) \quad \text{if and only if} \quad a \in C(S_j^a) \text{ for each } j \text{ in } J.$$

In words, if a strategy profile a is selected from S then each player j selects this profile when the choice set S contracts or shrinks into his individual choice set S_j^a. And, if the group of players jointly select a from the choice sets S_j^a, then the group of players jointly select a from the union $S = S_1^a \cup S_2^a \cup \ldots \cup S_k^a$. As such, this axiom has some flavor of a contraction-expansion property.

Noncooperation might be observed even in those cases where the individuals do not select a Nash equilibrium or do not consult an ordering. The axioms of congruence and noncooperation are independent: the former axiom only relies on choice sets of the type S_j^a for some a and j, in contrast the latter axiom is a global-local condition and always depend on other types of choice sets. Hence, a correspondence may satisfy one axiom and violate the other axiom. Also, the axioms of congruence and noncooperation only depend on the choice sets and the choices made from these sets. As such, these axioms are testable. The combination of noncooperation and the congruence axiom implies the rationalizability of the joint choice correspondence.

Theorem 4.3. *Let the joint choice correspondence $C : \mathcal{S} \twoheadrightarrow \Delta_J$ be individually decisive and assume that $\mathcal{S}$ satisfies the domain condition. Then, C is Nash rationalizable if and only if C is noncooperative and satisfies the congruence axiom.*

Proof. Let C be Nash rationalizable through the profile $(R_1^*, R_2^*, \ldots, R_k^*)$ of independent orderings in Δ^m. To prove that C satisfies the congruence axiom, one can apply theorem 4.2 upon the individual choice correspondences $C : \mathcal{S}_j \twoheadrightarrow \Delta_j$, where $\mathcal{S}_j$ collects all the choice sets of the form S_j^a with S running through the collection $\mathcal{S}$. That C is noncooperative has been argued above (see definition 4.4).

Now, suppose that C is noncooperative and satisfies the congruence axiom. Then, each revealed preference relation $R_{v,j}$ is lottery-consistent and extends to an independent ordering R_j^* in Δ^m (use theorem 4.2). We have to check whether for each S in $\mathcal{S}$, for each a in S, it holds that

$$a \in C(S) \text{ if and only if } d(a) \in M(R_j^* | d(S_j^a)) \text{ for each } j \text{ in } J.$$

Let $a \in C(S)$. As C is noncooperative, it follows that $a \in C(S_j^a)$ for each j in J. Hence, for each b in S_j^a we have $(d(a), d(b)) \in \tilde{R}_{v,j} \subseteq R_j^*$. It follows that $d(a) \in M(R_j^* | S_j^a)$ for each j in J.

Finally, let $a \in S \setminus C(S)$ and assume that $d(a) \in d(S)$. As C is noncooperative, there exists at least one player j for which $a \notin C(S_j^a)$. Since C is individually decisive, there exists a b in S_j^a such that $b \in C(S_j^a)$. Therefore $(d(b), d(a)) \in P_{v,j} \subseteq P(R_j^*)$. It follows that for player j we have that $d(a) \notin M(R_j^* | d(S_i^a))$. $\square$

This theorem establishes a rule to judge whether or not the hypothesis

$$H_0 : \begin{array}{l} \text{the collective choice correspondence } C : \mathcal{S} \longrightarrow\!\!\!\!\rightarrow \Delta_J \\ \text{is Nash rationalizable} \end{array}$$

should be rejected. The test is exact in the sense that as soon as the observations conflict with the axiom of congruence, the null hypothesis is false with certainty. The probability to reject the hypothesis when it is actually true is zero. Let us apply the test upon the data (Table 1) presented in Section 4.1.

Denote $a = C(S)$, $a' = C(S')$, and $a'' = C(S'')$. Let us list the four pure strategy profiles: $(U, L), (U, R), (D, L)$, and (D, R). We have that $d(a) =$ $(0.12, 0.28, 0.18, 0.42)$.

Use the axiom of noncooperation to conclude that player 1 reveals to (weakly) prefer $(0.4, 0.6)$ above any other strategy available to him, such as $(0.3, 0.7)$. Let us write $b = (0.3, 0.7) \times (0.3, 0.7)$, and $d(b) = (0.09, 0.21, 0.21, 0.49)$. As such we learn that $(d(a), d(b)) \in R_{v,1TI}$.

Similarly, $d(a') = (0.168, 0.232, 0.252, 0.348)$. Since also the strategy $(0.42, 0.58)$ is at the disposal of player 1, it follows (again, use the axiom of noncooperation) that $(d(a'), d(b')) \in R_{v,1TI}$, with $d(b') = (0.2205, 0.3045, 0.1995, 0.2755) \in$ Δ^4.

Finally, $d(a'') = (0.25, 0.25, 0.25, 0.25)$. The available strategy $(0.2, 0.8)$ leads to the distribution $d(b'') = (0.1, 0.1, 0.4, 0.4)$. The data imply $(d(a''), d(b'')) \in P_{v,1}$.

One can check that $2(d(a) - d(b)) + 4(d(a') - d(b')) + (d(a'') - d(b'')) = 0$. Solve this equation for $d(b'') - d(a'')$ and conclude (use lemma 4.3) that $(d(b''), d(a''))$ belongs to the independent and transitive closure of $R_{v,1}$. This contradicts our extended version of Richter's congruence axiom. Therefore, the data reject the hypothesis H_0.

We close this section with a discussion of work by Galambos [2005], who obtains a single condition—labeled I-congruence—for the Nash rationalizability in pure strategies. Define the binary relation R_j^* in the product strategy space Δ_J by $(a, b) \in R_j^*$ if (i) the strategies a_i and b_i

coincide for each $i \neq j$ and (ii) there exists an S in $\mathcal{S}$ for which $a \in C(S)$. In the assumption that the axiom of noncooperation holds, the relations R_j^* and $R_{v,j}$ coincide. Let $R_{j\,TI}^*$ be the transitive and independent closure of R_j^*. We rephrase the I-congruence axiom as follows. For each choice set S and for each strategy profile a,

$$\text{if } (a,b) \in R_{j\,TI}^* \text{ for each } b \text{ in } S_j^a \text{ and each } j \text{ in } N, \text{ then } a \in C(S).$$

This I-congruence condition combines the axioms of congruence and noncooperation and therefore provides an alternative rationalizability condition. The above theorem 4.3 uses two axioms that represent two separate ideas. The axiom of congruence reflects the idea that each individual, independently of the behavior of his opponents, consults an independent preference relation. The axiom of noncooperation reflects the idea of the Nash equilibrium that each individual takes the behavior of the opponents as given.

4.5 Persistence axioms of Sprumont

In this section we show that the persistence conditions of Sprumont are equivalent to our conditions for Nash rationalizability (when restricted to the setting of pure strategies). As such we indicate that our theorem 4.3 extends theorem 4.2 of Sprumont [2000] to cases involving mixtures over pure strategies.

Let A_j be the set of pure strategies available to player j and let $A = A_1 \times A_2 \times \cdots \times A_k$ be the set of all joint pure strategies. When restricted to the pure strategies, the map d from the space Δ_J of strategy profiles to the space $d(\Delta^n)$ of distributions over the pure strategy profiles remains one-to-one. Observing a (degenerate) distribution in Δ^m boils down to observing the pure strategies selected by the players.

Sprumont [2000] considers the collection $\mathcal{S}$ of cartesian products $S_1 \times S_2 \times \cdots \times S_k$ with $\varnothing \neq S_j \subset A_j$ and studies joint choice correspondences $C : \mathcal{S} \twoheadrightarrow A$ that are decisive on $\mathcal{S}$.

Such a correspondence C is said to be *persistent under expansion* if for each S and T in $\mathcal{S}$ it holds that $C(S) \cap C(T) \subset C(S \vee T)$, with $S \vee T$ the smallest choice set in $\mathcal{S}$ that includes S and T.

Furthermore, C is said to be *persistent under contraction* if (i) for each S and T in $\mathcal{S}$ with $T \subset S$ it holds that $C(S) \cap T \subset C(T)$ and (ii) for each S and T in $\mathcal{S}$ with $T \subset S_j^a$ and $C(S_j^a) \cap T \neq \varnothing$, it holds that $C(T) \subset C(S_j^a)$.

The next proposition phrases the equivalence between the two approaches. Of course, this proposition mutually supports our results and those of Sprumont.

Proposition. *Let $C : \mathcal{S} \twoheadrightarrow A$ be a decisive joint choice correspondence. Then, C is noncooperative and satisfies the congruence axiom (taking only the transitive closure into account) if and only if C is persistent under expansion and persistent under contraction.*

Proof. First, assume C is noncooperative and satisfies the congruence axiom. Let us check whether C is persistent under expansion. Let S and T in $\mathcal{S}$. If $a \in C(S) \cap C(T)$, then (use noncooperation) $a \in C(S_j^a) \cap C(T_j^a)$ for each j in J. Hence, the players reveal $(a, b) \in R_{v,j}$ for each b in $S_j^a \cup T_j^a$. If for player i in J we have $a \notin C\left((S \vee T)_i^a\right)$, then this player reveals to strictly prefer some action b (the decisiveness of C implies the existence of such an action) over a, i.e. $(b, a) \in P_{v,i}$. This contradicts the congruence axiom. Hence, $a \in C\left((S \vee T)_j^a\right)$ for each j in J. Noncooperation implies $a \in C(S \vee T)$.

We now verify persistence under contraction. Condition (i). Let $T \subset S$ and $a \in C(S) \cap T$. Noncooperation implies that each player j selects a from the individual choice set S_j^a. The congruence axiom implies that each player j selects a from the smaller choice set T_j^a. Conclude that $a \in C(T)$.

Contraction condition (ii). Let $T \subset S_j^a$, $b \in C(S_j^a) \cap T$, and $a \in C(T)$. As a consequence, $(a, b) \in R_{v,j}$. Hence, if this player does not select a from S_j^a, there exists a d in S_j^a such that $(d, a) \in P_{v,j}$. As $b \in C(S_j^a)$ and $d \in S_j^a$, it follows that $(b, d) \in R_{v,j}$. These observations contradict the congruence axiom: (a, d) belongs to the transitive closure of $R_{v,j}$, while $(d, a) \in P_{v,j}$.

Next, suppose that C satisfies the persistence axioms. Let us check the congruence axiom. Hence, assume (a, b) belongs to the transitive closure of $R_{v,j}$ with j in J. Denote the sequence from a to b by $a = x_1, x_2, \ldots, x_{t+1} = b$, i.e. we have $(x_1, x_2), (x_2, x_3), \ldots, (x_t, x_{t+1}) \in R_{v\,j}$. As player j is only able to reveal preferences conditional upon a status quo of his opponents, it must be the case that $x_1, x_2, \ldots, x_{t+1} \in A_j \times \{a_{-j}\}$, remember that a_{-j} collects the strategies of j's opponents. Persistence under contraction (part i) allows us to focus on the sets $S_\ell = \{a_1, a_2, \ldots, a_\ell\}$ with $\ell = 2, 3, \ldots, t+1$. One can check that $C(S_\ell) \cap S_{\ell-1} \neq \varnothing$. From persistence under contraction (part ii) it follows that $C(S_{\ell-1}) \subset C(S_\ell)$. Therefore, $a \in C(S_{t+1})$, and $a \in C(\{a, b\})$. Conclude that $(b, a) \notin P_{v,j}$ and $(a, b) \notin R_{v,j}$.

Finally, we check for noncooperation. Let $a \in C(S_j^a)$ for each j in J. Persistence under expansion implies $a \in C(S_1^a \vee S_2^a \vee \ldots \vee S_k^a) = C(S)$. And, if $a \in C(S)$, then $a \in C(S_j^a)$ for each j (use persistence under contraction). □

CHAPTER 5

Conclusion

This chapter presents some generalizations (section 5.1), implementation issues (section 5.2) and concluding remarks (section 5.3).

5.1 Generalizations

In subsection 5.1.1, we show that all the results from chapters 2 and 3 can be derived from the assumption of the existence of a free ultrafilter instead of the stronger assumption of Zorn's lemma. In subsection 5.1.2, we discuss the generalization of Nash rationalizability (Cf. section 4.4) towards other properties. In section 5.1.3 we generalize the extension results that were derived for the properties of monotonicity and strict monotonicity (Cf. section 2.3.3).

5.1.1 Ultrafilters and Zorn's lemma

This section shows that we can strengthen the rationalizability results in chapters 1 and 2 by using the existence of ultrafilters instead of Zorn's lemma to derive theorem 2.2.

Theorems 3.1 and 3.2 in section 3.2 (ATARP and RTARP) can be derived from theorem 2.3 in section 2.4 (F-rationalizability). In particular, lemma 3.1 in section 3.3 shows that the absolute time consistent, transitive and impatient closure is a closure operator and the proof of lemma 3.2 in section 3.3 shows that this closure satisfies C7 (see section 2.2). Theorem 2.3 is derived from theorem 2.1 in section 2.2 and this theorem relies on Zorn's lemma.

It would appear that theorem 4.2 in section 4.3 (rationalizability by an independent ordering) is also a special case of theorem 2.3 in section 2.4. However, this is not true, for the following reason.

Theorem 4.2 is based on theorem 4.1 which is based on the existence of a free ultrafilter[1]. The existence of a free ultrafilter is guaranteed by Zorn's lemma. However, Zorn's lemma can not be derived from the existence of free ultrafilters alone.

It turns out that it is still possible to prove the rationalizability results in chapters 2 and 3 by only using the existence of free ultrafilters. In order to do this, we adjust theorem 2.2 from section 2.2 in the following way.

Theorem 5.1. *Let F be an algebraic closure operator and let $\mathcal{R}^* = \{R \in \mathcal{R} | R \preceq F(R)$. If F satisfies:*

C7' For all $R = F(R)$, if $N(R) \neq \emptyset$, then for all $(a, b) \in N(R)$, there exists a relation $T \subseteq N(R)$ that contains (a, b) and $R \cup T \in \mathcal{R}^*$.

Then, in order that a relation R has a complete extension $R^* = F(R^*)$ it is necessary and sufficient that $F(R) \cap P(R)^{-1} = \emptyset$.

The proof is very similar to the proof of theorem 4.1 in section 4.2 and is left to the reader. Condition $C7'$ is stronger than condition $C7$ but it is satisfied for every closure operator encountered in this monograph[2]. The difference between $C7$ and $C7'$ is that the latter requires that R can be extended to a larger relation (in $\mathcal{R}^*$) in any possible 'direction'. This allows us to find for any finite set of alternatives an extension of R in $\mathcal{R}^*$ which is complete over this finite set. $C7$ only states that R should be extendible to a larger relation in $\mathcal{R}^*$.

A straightforward consequence of Theorem 5.1 is the following corollary:

Corollary 5.1. *Let $F : \mathcal{R} \to \mathcal{R}$ be an algebraic closure operator that satisfies $C7'$ and assume that $T(R) \subseteq F(R)$ for all $R \in \mathcal{R}$.*
Then, the choice function C is rationalizable by a complete relation $R^* = F(R^*)$ if and only if:

$$F(R_v) \cap P_v^{-1} = \emptyset.$$

Corollary 5.1 implies that we can reproduce all the rationalizability results in chapter 2 and 3 via the existence of free ultrafilters instead of Zorn's lemma. This strengthens the derived rationalizability results. However, we should keep in mind that the results remain non-constructive.

5.1.2 Nash rationalizability

In this section, we generalize the Nash rationalizability characterization (theorem 4.3 in section 4.4) towards other properties. We derive Nash rationalizability results for profiles of preference

[1] A filter $\mathcal{F}$ on the set X is a collection of subsets of X such that i) $\emptyset \notin \mathcal{F}$, ii) for all $A, B \in \mathcal{F}, A \cap B \in \mathcal{F}$ and iii) if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$. An ultrafilter is a maximal filter and a free ultrafilter is an ultrafilter which does not contain a finite subset of X.

[2] The method for proving the validity of condition C7 in chapters 2 and 3 was to take any element $(a, b) \in N(R)$ and verify that $R \cup \{(a, b)\} \in \mathcal{R}^*$.

orderings which are convex, homothetic, monotonic or absolute (relative) time-consistent and impatient.

We develop a similar frameword as in section 4.4. Let $J = \{1, ..., k\}$ be the set of players and correspond to each player j a set of actions A_j. The set of action profiles is given by $A = A_1 \times A_2 \times ... \times A_k$. We denote a strategy profile $a \in A$ also as (a_j, a_{-j}) where a_{-j} collects the strategies of j's opponents. An outcome function $d : A \rightarrow X$ corresponds with each element $a \in A$ an element, $d(a)$ from the set of alternatives X. A choice set S is a cartesian product of $S_1 \times S_2 \times \ldots \times S_k \subseteq A$ with S_j a non-empty subset of A_j. For $a_j \in S_j$, we denote the choice set $\{a_1\} \times \ldots \times \{a_{j-1}\} \times S_j \times \{a_{j+1}\} \times ... \times \{a_k\}$ also by S_j^a.

Let $\mathcal{S}$ be a collection of choice sets. A joint choice function K is a correspondence

$$K : \mathcal{S} \twoheadrightarrow \Delta_J : S \longmapsto K(S) \subseteq S.$$

We assume that for all $S \in \mathcal{S}$ and all $a \in S, j \in J$, $S_j^a \in \mathcal{S}$. Furthermore, we assume that for all $S \in \mathcal{S}, j \in J$ and $a \in S$, $K(S_j^a) \neq \emptyset$. Let $F : \mathcal{R} \rightarrow \mathcal{R}$ be a function that satisfies $C1$, $C2$ and $C3$ (Cf. section 2.2) and assume that $T(R) \subseteq F(R)$ for all relations R in X (where $T(R)$ is the transitive closure of R).

Following section 4.4, we say that a choice function K is *Nash-rationalizable* if there exist complete relations $R_1^* = F(R_1^*), R_2^* = F(R_2^*), \ldots, R_k^* = F(R_k^*)$ in X such that for all $j \in J$ and all $S \in \mathcal{S}$:

$$a \in K(S) \quad \text{if and only if} \quad d(a) \in M(R_j^* | d(S_j^a)).$$

Where $a \in M(R_j^*|d(S))$ if and only if $a \in S$ and $(d(a), d(b)) \in R_j^*$ for all $b \in S$. We define the revealed preference relations $R_{v,j}$ and $P_{v,j}$ as in section 4.4: $(x, y) \in R_{v,j}$ if there exist an $a \in A$ and an $S_j^a \in \mathcal{S}$ such that $b \in S_j^a$, $a \in K(S_j^a)$, and $(x, y) = (d(a), d(b))$. We say that $(x, y) \in P_{v,j}$ if there exist an $a \in A$ and an $S_j^a \in \mathcal{S}$ such that $a \in K(S_j^a)$, $b \in S_j^a - K(S_j^a)$, and $(x, y) = (d(a), d(b))$.

We can now present following generalization of theorem 4.3 in section 4.4.

Theorem 5.2. *Let $F : \mathcal{R} \rightarrow \mathcal{R}$ be an algebraic closure operator and assume that $T(R) \subseteq F(R)$ for all $R \in \mathcal{R}$.*

Then, the choice function K is Nash rationalizable if and only if K is noncooperative ($a \in K(S) \leftrightarrow a \in K(S_j^a)$ for all $j \in J$) and for all $j \in J$: $F(R_{v,j}) \cap P_{v,j}^{-1} = \emptyset$.

The proof is a simple adaptation of the proof of theorem 4.3 in section 4.4 (with the use of theorem 2.3 in section 2.4) and is left to the reader. Theorem 5.2 can easily be applied to the closures C, H, $\bar{C}$, and B from chapter 2 and 3.

5.1.3 Generalizing (strict) monotonicity

In this section, we generalize the extension results for the properties of monotonicity and strict monotonicity. This leads to a characterizations that gives necessary and sufficient conditions for a set of relations $R_1, \dots R_n$ to have a common extension $R^* = F(R^*)$ which satisfies the additional property that $R_j \subseteq R^*$ for all relations R_j in some set $\{R_{n+1}, \dots R_m\}$. We give an example at the end of this section.

Recall from definitions 2.6 and 2.7 in section 2.3.3 that a relation R is monotonic if $(a, b) \in R$ for all $a, b \in X$ with $a \geq b$ and it is strict monotonic if in addition $a > b$ implies $(a, b) \in P(R)$. Now, let $(a, b) \in Q$ if and only if $a \geq b$. Then, R is monotonic if and only if $Q \subseteq R$ and R is strict monotonic if and only if $Q \preceq R$ (i.e. R is an extension of Q).

Let $F : \mathcal{R} \to \mathcal{R}$ be an algebraic closure operator that satisfies condition C7 (see section 2.2). From section 2.3.3 we know that a relation R has a complete and monotonic extension $R^* = F(R^*)$ if and only if:

$$F(R \cup Q) \cap P(R)^{-1} = \emptyset.$$

And a relation has a complete and strict monotonic extension $R^* = F(R^*)$ if and only if:

$$F(R \cup Q) \cap P(R)^{-1} = \emptyset \qquad \text{and} \qquad F(R \cup Q) \cap P(Q)^{-1} = \emptyset.$$

Let us abstain from the property of monotonicity and assume that Q represents an arbitrary relation in X. The above extension results generalize to:

Lemma 5.1. *Let $F : \mathcal{R} \to \mathcal{R}$ be an algebraic closure operator. A relation R has a complete extension $R^* = F(R^*)$ with $Q \subseteq R^*$ if and only if*

$$F(R \cup Q) \cap P(R)^{-1} = \emptyset.$$

A relation R has a complete extension $R^* = F(R^*)$ with $Q \preceq R^*$ if and only if

$$F(R \cup Q) \cap P(R)^{-1} = \emptyset \qquad \text{and} \qquad F(R \cup Q) \cap P(Q)^{-1} = \emptyset.$$

The validity of this lemma can be established along the lines of the argument developed in section 2.3.3 and is left to the reader. Let us insert a result from Suzumura [2004], which is a special case of lemma 5.1:

Proposition. *Let R be a binary relation on X, S a subset of X such that $x \neq y$ and $x, y \in S$ then $x, y \notin T(R)$, and Q an ordering on S. Then there exists an ordering extension R^* of R such that the restriction of R^* on S coincides with Q if and only if*

$$T(R) \cap P(R)^{-1} = \emptyset.$$

To see that this proposition is indeed a special case of lemma 5.1, observe first that $T(R) \cap P(R)^{-1} = \emptyset$ is a necessary condition. To see sufficiency, notice that R^* can be defined as an ordering extension of both R and Q. The existence of such extension is, by lemma 5.1, characterized by the conditions

i. $T(R \cup Q) \cap P(R)^{-1} = \emptyset$ and,

ii. $T(R \cup Q) \cap P(Q)^{-1} = \emptyset$.

Let us show that a violation of one of these conditions implies that $T(R) \cap P(R)^{-1} \neq \emptyset$.

i) Assume that $(a, b) \in T(R \cup Q)$ and $(b, a) \in P(R)$. Then there is a sequence[3] s such that $s(1) = a$, $s(n_s) = b$ and for all $i = 1, \ldots, n_{s-1}$, $(s(i), s(i+1)) \in R \cup Q$. Let ℓ be the largest integer such that $(s(\ell), s(\ell+1)) \in Q$ and let f be the smallest integer such that $(s(f), s(f+1)) \in Q$. (If there is no such ℓ and f, then $(a, b) \in T(R)$, which provides the required contradiction.) Then, we derive that $(s(\ell+1), s(f)) \in T(R)$ which contradicts $\{s(\ell+1), s(f)\} \subseteq S$.

ii) Assume that $(a, b) \in T(R \cup Q)$ and $(b, a) \in P(Q)$. Then there is a sequence s such that $s(1) = a, s(n_s) = b$ and for all $i = 1, \ldots, n_s$, $(s(i), s(i+1)) \in R \cup Q$. Let ℓ be the largest integer such that $(s(\ell), s(\ell+1)) \in Q$. (If there is no such ℓ, then $(a, b) \in T(R)$, contradicting $\{a, b\} \subseteq S$.) Conclude that $(s(\ell+1), b) \in T(R)$, contradicting $\{s(\ell+1), b\} \subseteq S$.

Lemma 5.1 above can be further generalized to include any finite number of relations. In particular:

Theorem 5.3. *Let $F : \mathcal{R} \to \mathcal{R}$ be an algebraic closure operator. Then, a set of relations $R_1, \ldots, R_n$ has a (common) complete extension $R^* = F(R^*)$ with $R_j \subseteq R^*$ $(j = n+1, \ldots, m)$ if and only if:*

$$F\left(\bigcup_{j \leq m} R_j\right) \cap P(R_i)^{-1} = \emptyset \text{ for all } i \leq n.$$

Theorem 5.3 may have some importance for establishing existence results. Let us give an example from social choice theory.

Consider a finite number of individuals $J = \{1, \ldots, k\}$ each with a complete and transitive relation R_i over $X \times J$. We say that $((a, l), (b, j)) \in R_i$ if individual i judges being in position l when state a prevails at least as good as being in position j when state b prevails. The preference relations R_i are called extended preference orderings (see, among others, Suzumura [1983, chapter 5]). Let $\overline{R_i} = R_i \cap (X \times \{i\})$. The Pareto relation S on X is given by $(a, b) \in S$ if $(a, b) \in \bigcap_{i \in J} \overline{R_i}$. We say that the social welfare ordering R^* on X satisfies the Pareto property if $S \preceq R^*$.

Let $H(x) \subseteq J \times J$ be the relation given by $(i, j) \in H(x)$ if $((x, j), (x, i)) \in P(R_i)$, i.e. individual i judges being in position j better then being in position i when x prevails. In other words, individual i envies j at x. We can define the equity relation E on X by $(x, y) \in E$ if $H(x) \subseteq H(y)$. We have that $(x, y) \in E$ if and only if (if i envies j at state x) then (i envies j at state y). We say that the social welfare ordering, R^* on X satisfies the equity property if $E \preceq R^*$.

We can use theorem 5.3 to characterizes the set of relations $\{R_i\}_{i \in J}$ such that there exists a Pareto and equitable social welfare ordering, i.e.

$$T(S \cup E) \cap P(S)^{-1} = \emptyset \qquad \text{and} \qquad T(S \cup E) \cap P(E)^{-1} = \emptyset.$$

[3] The definition of a sequence is given by definition 2.3 in section 2.3.1

5.2 Implementation issues

The ultimate goal for the rationalizability tests in this dissertation is to be used in empirical research to test whether the rational choice hypothesis holds (see also chapter 1). An empirical setting consists of (a finite number of) observations $(A, A'), A \in \Sigma \subseteq 2^X - \{\emptyset\}$ where A is a set of alternatives and A' is the set of alternatives that are chosen from A. In terms of a choice function K, we can write $K(A) = A'$. From this choice function we can construct the revealed preference relations R_v and P_v and verify whether the rationalizability test, $F(R_v) \cap P_v^{-1} = \emptyset$ (Cf. section 2.4), holds.

The verification of this test may be computationally very demanding. In most occasions (when the choice sets are infinite) we have that R_v, and hence also, $F(R_v)$ are infinite. In this chapter, we give some results, which considerably simplify the rationalizability tests. We also point out which issues remain to be solved in order to operationalize the tests.

5.2.1 Monotonicity

Let $X \subseteq \mathbb{R}^m$. From section 2.4, we know that a choice function can be rationalized by a transitive, monotonic and complete relation if and only if:

$$\overline{T}(R_v) \cap P_v^{-1} = \emptyset.$$

Let Q represent the transitive relation $\geq$, i.e. $(a, b) \in Q$ if and only if $a \geq b$. Then we can rewrite the rationalizability condition as:

$$T(R_v \cup Q) \cap P_v^{-1} = \emptyset.$$

Let $D = \bigcup_{A \in \Sigma} K(A)$. The aim of this section is to rewrite the above rationalizability conditions as a condition involving only subrelations of $D \times D$. For a relation R, define $\underline{R} = R \cap (D \times D)$ and for two relations R and T let us write $(a, b) \in R \circ T$ if there exits a $c \in X$ such that $(a, c) \in R$ and $(c, b) \in T$. Obviously $R \circ (T \circ S) = (R \circ T) \circ S$, hence, we can abuse notation without introducing ambiguities and write also $R \circ T \circ S$. Furthermore, it is easy to see that $(a, b) \in T(R)$ if and only if there exist an $n \in \mathbb{N}$ such that

$$(a, b) \in \underbrace{R \circ R \circ \ldots \circ R}_{n \text{ times}}.$$

We have the following result.

Lemma 5.2.

$$T(R \cup Q) = Q \cup [Q \circ T(\underline{R_v \circ Q}) \circ R_v \circ Q].$$

Proof. ($\subseteq$). Let $(a, b) \in T(R_v \cup Q)$. Then there exist an $n \in \mathbb{N}$ such that:

$$(a, b) \in \underbrace{(R_v \cup Q) \circ (R_v \cup Q) \circ \ldots \circ (R_v \cup Q)}_{n \text{ times}}.$$

We show that $(a,b) \in Q \cup [Q \circ T(\underline{R_v \circ Q}) \circ R_v \circ Q]$ by induction on n. If $n = 1$, we derive that $(a,b) \in R_v \cup Q$. If $(a,b) \in Q$, we are done, so assume that $(a,b) \in R_v$. By reflexivity of Q and $a \in D$, we derive that $(a,a) \in Q$, $(a,a) \in T(\underline{R_v \circ Q})$, $(a,b) \in R_v$ and $(b,b) \in Q$. Conclude that $(a,b) \in Q \cup [Q \circ T(\underline{R_v \circ Q}) \circ R_v \circ Q]$.

Assume that the property holds for $n = \ell$ and take the case where $n = \ell + 1$. From the induction hypothesis we derive that there is an element c such that $(a,c) \in Q \cup [Q \circ T(\underline{R_v \circ Q}) \circ R_v \circ Q]$ and $(c,b) \in Q \cup R_v$. If $(a,c) \in Q$ and $(c,b) \in Q$, we derive from transitivity of Q that $(a,b) \in Q$. If $(a,c) \in Q$ and $(c,b) \in R_v$, we derive from reflexivity of Q and $c \in D$ that $(a,c) \in Q$, $(c,c) \in T(\underline{R_v \circ Q})$, $(c,b) \in R_v$ and $(b,b) \in Q$. Conclude that $(a,b) \in [Q \circ T(\underline{R_v \circ Q}) \circ R_v \circ Q]$. If $(a,c) \in [Q \circ T(\underline{R_v \circ Q}) \circ R_v \circ Q]$ and $(c,b) \in Q$, we derive that $(a,b) \in [Q \circ T(\underline{R_v \circ Q}) \circ R_v \circ Q] \circ Q = [Q \circ T(\underline{R_v \circ Q}) \circ R_v \circ Q]$. If $(a,c) \in [Q \circ T(\underline{R_v \circ Q}) \circ R_v \circ Q]$ and $(c,b) \in R_v$. Then $(a,b) \in [Q \circ T(\underline{R_v \circ Q}) \circ R_v \circ Q] \circ R_v \subseteq [Q \circ T(\underline{R_v \circ Q}) \circ R_v \circ Q]$.

($\supseteq$). This follows from observing that $Q \subseteq T(R_v \cup Q)$, $T(\underline{R_v \circ Q}) \subseteq T(R_v \cup Q)$ and $R_v \circ Q \subseteq T(R_v \cup Q)$. $\square$

Theorem 5.4. *A choice function is rationalizable by a complete, monotonic and transitive relation if and only if*

$$T(\underline{R \circ Q}) \cap (\underline{P_v \circ Q})^{-1} = \emptyset,$$

and

$$Q \cap P_v^{-1} = \emptyset.$$

Proof. From lemma 5.2 we know that a choice function is rationalizable by a complete, transitive and monotonic relation if and only if

$$\big(Q \cup [Q \circ T(\underline{R_v \circ Q}) \circ R_v \circ Q]\big) \cap P_v^{-1} = \emptyset.$$

This holds if $Q \cap P_v^{-1} = \emptyset$ and $[Q \circ T(\underline{R_v \circ Q}) \circ R_v \circ Q] \cap P_v^{-1} = \emptyset$. Let us rewrite this second condition.

$$Q \circ T(\underline{R_v \circ Q}) \circ R_v \circ Q \cap P_v^{-1} = \emptyset$$
$$\updownarrow$$
$$Q \circ T(\underline{R_v \circ Q}) \circ \underline{R_v \circ Q} \cap P_v^{-1} = \emptyset$$
$$\updownarrow$$
$$Q \circ T(\underline{R_v \circ Q}) \cap P_v^{-1} = \emptyset$$
$$\updownarrow$$
$$T(\underline{R_v \circ Q}) \cap (P_v \circ Q)^{-1} = \emptyset$$
$$\updownarrow$$
$$T(\underline{R_v \circ Q}) \cap (\underline{P_v \circ Q})^{-1} = \emptyset$$

$\square$

The advantage of the condition in theorem 5.4, i.e. $T(\underline{R_v \circ Q}) \cap (\underline{P_v \circ Q}) = \emptyset$ is that the relations $\underline{R_v \circ Q}$ and $\underline{P_v \circ Q}$ contain, in general, less elements then the relations $R \cup Q$ and P_v. In order to test the rationalizability criterion, we can take the following procedure:

i. *Compute the relations* $\underline{R_v \circ Q}$ *and* $\underline{P_v \circ Q}$.

The easiest way to do this is by a reverse procedure. For each $b \in D$, consider

$$A(b) = \left\{ a \in D \,\middle|\, \begin{array}{c} \exists A \in \Sigma \text{ with } a \in K(a) \\ \text{and } \exists c \in A \text{ with } c \geq b \end{array} \right\}.$$

Then $(a, b) \in \underline{R_v \circ Q}$ if and only if $b \in D$, and $a \in A(b)$. Observe that for all $a \in D$, $(a, a) \in \underline{R_v \circ Q}$. Consider the set

$$B(b) = \left\{ a \in D \,\middle|\, \begin{array}{l} \exists A \in \Sigma \text{ with } a \in K(A) \\ \text{and } \exists c \in A - K(A) \text{ with } c \geq a \end{array} \right\}.$$

Then $(a, b) \in \underline{P_v \circ Q}$ if and only if $b \in D$ and $a \in B(b)$[4].

ii. *Compute the transitive closure of* $\underline{R_v \circ Q}$.

The transitive closure $T(\underline{R_v \circ Q})$ can be computed along the lines of Warschall's algorithm (see Varian [1982]). Consider an enumeration of the elements in $D = \{d_1, d_2, \ldots d_n\}$ and consider the $n \times n$ matrix r^1 where $r_{i,j} = 1$ if $(d_i, d_j) \in \underline{R_v \circ Q}$ and $r_{i,j} = 0$ if $(d_i, d_j) \notin \underline{R_v \circ Q}$. Consider following algorithm:

1. Initialize $t = 1$. Go to step 2.
2. Construct r^{t+1} such that for all i and j in $\{1, \ldots, n\}$:

$$r^{t+1}_{i,j} = \max_{k \in \{1,\ldots,n\}} \{\min\{r^t_{i,k}\,,\ r^t_{k,j}\}\}.$$

   Go to step 3.
3. If $r^{t+1} = r^t$, then we define $r = r^t$ and we stop. Else, we augment t by one ($t = t+1$) and return to step 2.

The matrix r satisfies that $r_{i,j} = 1$ if and only if $(d_i, d_j) \in T(\underline{R_v \circ Q})$.

iii. *Verify that* $T(\underline{R_v \circ Q}) \cap (\underline{P_v \circ Q})^{-1} = \emptyset$ *and that* $Q \cap \underset{v}{P}^{-1} = \emptyset$.

[4]Observe that if all choice sets $A \in \Sigma$ are comprehensive (i.e. if $x \in A$ and $x \geq y$ then $y \in A$), then $A(b) = \{a \in D | (a, b) \in R_v\}$ and $B(b) = \{a \in D | (a, b) \in P_v\}$, which would imply that $\underline{R_v \circ Q} = \underline{R_v}$, $\underline{P_v \circ Q} = \underline{P_v}$, and as a consequence $T(\underline{R_v \circ Q}) \cap \underline{P^{-1} \circ Q}_v = \emptyset$ if and only if $T(\underline{R_v}) \cap \underline{P_v}^{-1} = \emptyset$.

5.2.2 Strict monotonicity

A choice function is rationalizable by a strict monotonic, transitive and complete relation if and only if

$$\overline{T}(R_v) \cap P_v^{-1} = \emptyset \text{ and } a > b \text{ implies } (b, a) \notin \overline{T}(R_v).$$

Define the relations W as $(a, b) \in W$ if and only if $a > b$. A choice function is rationalizable by a strict monotonic, transitive and complete relation if and only if:

$$T(R \cup Q) \cap P_v^{-1} = \emptyset \quad \text{and} \quad T(R \cup Q) \cap W^{-1} = \emptyset.$$

Theorem 5.5. *A choice function is rationalizable by a strict monotonic, transitive and complete relation if and only if*

$$T(\underline{R_v \circ Q}) \cap (\underline{P_v \circ Q})^{-1} = \emptyset,$$
$$Q \cap P_v^{-1} = \emptyset,$$

and

$$T(\underline{R_v \circ Q}) \cap (\underline{R_v \circ W})^{-1} = \emptyset.$$

Proof. The first two conditions are similar to theorem 5.4, so we only verify the last.

$$T(R_v \circ Q) \cap W^{-1} = \emptyset$$
$$\updownarrow$$
$$Q \circ T(\underline{R_v \circ Q}) \circ R_v \circ Q \cap W^{-1} = \emptyset$$
$$\updownarrow$$
$$T(\underline{R_v \circ Q}) \circ R_v \cap (Q \circ W \circ Q)^{-1} = \emptyset$$
$$\updownarrow$$
$$T(\underline{R_v \circ Q}) \circ R_v \cap W^{-1} = \emptyset$$
$$\updownarrow$$
$$T(\underline{R_v \circ Q}) \cap (R_v \circ W)^{-1} = \emptyset$$
$$\updownarrow$$
$$T(\underline{R_v \circ Q}) \cap (\underline{R_v \circ W})^{-1} = \emptyset.$$

□

The relation $\underline{R_v \circ W}$ can be computed in the same manner as the relation $\underline{R_v \circ Q}$: for each $b \in D$, construct the set,

$$C(b) = \{a \in D | \exists A \in \Sigma \text{ with } a \in K(A) \text{ and } \exists c \in A \text{ with } c > b\}.$$

We have that $(a, b) \in \underline{R_v \circ W}$ if and only if $b \in D$, and $a \in C(b)$.

5.2.3 Homotheticity

Let X be a cone in $\mathbb{R}^m_+$. From section 2.4, we know that a choice function is rationalizable by a complete, transitive, homothetic and monotonic relation if and only if

$$H(R_v) \cap P_v^{-1} = \emptyset.$$

Let $X \subseteq \mathbb{R}^n_+$ and define the relation $R \bullet S$ as $(a, b) \in R \bullet Q$ if there exist a $c \in X$ and $\alpha, \beta \in \mathbb{R}_{++}$ such that $(\alpha a, \alpha c) \in R$ and $(\beta c, \beta b) \in S$. Clearly $(R \bullet S) \bullet T = R \bullet (S \bullet T)$ hence we can again abuse notation without introducing ambiguities and write this also as $R \bullet S \bullet T$. It is easy to establish that $(a, b) \in H(R)$ if and only if there exist an $n \in \mathbb{N}$ such that:

$$(a, b) \in \underbrace{(R \cup Q) \bullet (R \cup Q) \bullet \ldots \bullet (R \cup Q)}_{n \text{ times}}.$$

Define the function $\widetilde{T}$ by $(a, b) \in \widetilde{T}(R)$ if there exists an $n \in \mathbb{N}$ such that:

$$(a, b) \in \underbrace{R \bullet R \bullet \ldots \bullet R}_{n \text{ times}}.$$

Consider the set

$$D = \left\{ a \in X | \exists \alpha \in \mathbb{R}_{++} \text{ such that } a = \alpha\, a' \text{ and } a' \in \bigcup_{A \in \Sigma} K(A) \right\}.$$

For a relation R in X, define $\underline{R} = R \cap (D \times D)$. Observe that if $(a, b) \in R_v \bullet Q$, then $a \in D$.

Lemma 5.3.

$$H(R_v) = Q \cup [Q \bullet \widetilde{T}(\underline{R_v \bullet Q}) \circ R_v \bullet Q].$$

Proof. ($\subseteq$) Let $(a, b) \in H(R_v)$. Then there exists an $n \in \mathbb{N}$ such that:

$$(a, b) \in \underbrace{(R_v \cup Q) \bullet (R_v \cup Q) \bullet \ldots \bullet (R_v \cup Q)}_{n \text{ times}}.$$

We validate the proof by induction on n. If $n = 1$, we have that $(a, b) \in R_v \cup Q$. If $(a, b) \in Q$, there is nothing to proof, so assume that $(a, b) \in R_v$. From reflexivity of Q and $a \in D$, we derive that $(a, a) \in Q$, $(a, a) \in \widetilde{T}(\underline{R_v \bullet Q})$, $(a, b) \in R_v$ and $(b, b) \in Q$.

Assume that the result holds for $n = \ell$ and consider the case where $n = \ell + 1$. Then there is an element $c \in X$ and elements α and $\beta \in \mathbb{R}_{++}$ such that $(\alpha a, \alpha c) \in Q \cup [Q \bullet \widetilde{T}(\underline{R_v \bullet Q}) \bullet R_v \bullet Q]$ and $(\beta c, \beta a) \in R_v \cup Q$. If $(\alpha a, \alpha c) \in Q$ and $(\beta c, \beta b) \in Q$, we derive that $(a, b) \in Q$ and we are done. If $(\alpha a, \alpha c) \in Q$ and $(\beta c, \beta b) \in R_v$, we derive from reflexivity of Q and $c \in D$ that $(\alpha a, \alpha a) \in Q$, $(\beta c, \beta c) \in \widetilde{T}(\underline{R_v \circ Q})$, $(\beta c, \beta b) \in R_v$ and $(\beta b, \beta b) \in Q$. Conclude that $(a, b) \in [Q \bullet \widetilde{T}(\underline{R_v \bullet Q}) \bullet R_v \bullet Q]$. If $(\alpha a, \alpha c) \in [Q \bullet \widetilde{T}(\underline{R_v \bullet Q}) \bullet R_v \bullet Q]$ and $(\beta c, \beta b) \in Q$, we derive that $(a, b) \in [Q \bullet \widetilde{T}(\underline{R_v \bullet Q}) \bullet R_v \bullet Q] \bullet Q = [Q \bullet \widetilde{T}(\underline{R_v \bullet Q}) \bullet R_v \bullet Q]$. If $(\alpha a, \alpha c) \in$

$[Q \bullet \widetilde{T}(\underline{R_v \bullet Q}) \bullet R_v \bullet Q]$ and $(\beta c, \beta b) \in R_v$, then $(a,c) \in [Q \bullet \widetilde{T}(\underline{R_v \bullet Q}) \bullet R_v \bullet Q] \bullet R_v \subseteq [Q \bullet \widetilde{T}(\underline{R_v \bullet Q}) \bullet R_v \bullet Q]$.

($\supseteq$) The conclusion follows from the fact that Q, R_v and $\widetilde{T}(\underline{R_v \cup Q})$ are subsets of $\widetilde{T}(R_v \cup Q)$. $\square$

Theorem 5.6. *A choice function is rationalizable by a complete, homothetic and complete relation if and only if*

$$\widetilde{T}(\underline{R_v \bullet Q}) \cap (\underline{P_v \bullet Q})^{-1} = \emptyset.$$

and

$$Q \cap P_v^{-1} = \emptyset.$$

The proof of this theorem is very similar to the proof of theorem 5.4 and is left to the reader. In order to test the rationalizability criterion one can follow following procedure:

i. *Compute* $\underline{R_v \bullet Q}$ *and* $\underline{P_v \bullet Q}$.

   The relation $\underline{R_v \bullet Q}$ can be computed in the following way: for each $b \in \bigcup_{A \in \Sigma} K(A)$ let

   $$A(b,\gamma) = \{a \in X | \exists A \in \Sigma \text{ with } a \in K(A) \text{ and } \exists c \in A \text{ with } \gamma\, c \geq b\}.$$

   Then for all $b \in \bigcup_{A\in\Sigma} K(A)$, $\gamma \in \mathbb{R}_{++}$, $a \in A(b,\gamma)$ and $\alpha \in \mathbb{R}_{++}$, we have that[5] $(\alpha\, a, \frac{\alpha}{\gamma}\, b) \in \underline{R_v \bullet Q}$. Observe that if $a \in A(b,\gamma)$, then $a \in A(b,\delta)$ for all $\delta \geq \gamma$.

   Therefore, if $\bigcup_{A\in\Sigma} K(A)$ is finite, then we can construct a finite set V of elements $(a,b,\widetilde{\gamma}) \in \left(\bigcup_{A\in\Sigma} K(A) \times \bigcup_{A\in\Sigma} K(A) \times \mathbb{R}_{++}\right)$ such that $(a,b,\widetilde{\gamma}) \in V$ if and only if for all $\alpha \in \mathbb{R}_{++}$ and all $\gamma \in \mathbb{R}_{++}$ with $\gamma \geq \widetilde{\gamma}$ we have that $(\alpha\, a, \frac{\alpha}{\gamma}\, b) \in \underline{R_v \bullet Q}$.

   The relation $\underline{P_v \bullet Q}$ can be computed in a similar manner by using the set $B(b,\gamma)$ instead of $A(b,\gamma)$.

   $$B(b,\gamma) = \left\{ a \in X \;\middle|\; \begin{array}{l} \exists A \in \Sigma \text{ with } a \in K(A) \\ \text{and } \exists c \in A - K(A) \text{ with } \gamma\, c \geq b \end{array} \right\}.$$

ii. *Compute* $\widetilde{T}(\underline{R_v \bullet Q})$.

   This function may be computed in the following way. Consider an enumeration of the elements in $\bigcup_{A\in\Sigma} K(A)$, i.e. $\{d_1, \ldots, d_n\}$ and let r^1 be the $n \times n$ matrix where $r_{i,j} = \gamma$ if $(a,b,\gamma) \in V$ and $r^1_{i,j} = 0$ if there is not an element $\gamma \in \mathbb{R}_{++}$ such that $(a,b,\gamma) \in V$. Observe that for all $i \in \{1,\ldots,n\}$, $0 < r^1_{ii} \leq 1$. Consider the following algorithm:

[5] In cases where all sets $A \in \Sigma$ are comprehensive (i.e. if $x \in A$ then for all $y \leq x$, $y \in A$), this simplifies to $(\alpha\, a, \frac{\alpha}{\gamma}\, b) \in \underline{R_v \bullet Q}$ if and only if $(a, \frac{1}{\gamma}\, b) \in R_v$.

1. Initialize $t = 1$. Go to step 2.
2. Construct the matrix r^{t+1} such that for all $i, j \in \{1, \ldots, n\}$:

$$r^{t+1}_{i,j} = \min_{k \in \{1,\ldots,n\}} \{\gamma \cdot \delta | r^k_{i,k} = \gamma > 0 \text{ and } r^t_{k,j} = \delta > 0\},$$

   if it exists and set $r^{t+1}_{i,j} = 0$ otherwise. Go to step 3.
3. if $r^{t+1} = r^t$ then we define $r = r^t$ and we stop. Else, we return to step 2.

We have that $r_{i,j} = \widetilde{\gamma}$ if and only if for all $\alpha \in \mathbb{R}_{++}$ and all $\gamma > \widetilde{\gamma}$, $(\alpha\, d_i, \frac{\alpha}{\gamma}\, d_j) \in \widetilde{T}(\underline{R_v \bullet Q})$.

iii. *Verify that* $\widetilde{T}(\underline{R_v \bullet Q}) \cap \underline{P_v \bullet Q} = \emptyset$ *and that* $Q \cap P_v^{-1} = \emptyset$.

5.2.4 Convexity

From section 2.4 we know that a choice function is rationalizable by a convex, transitive and complete relation, if and only if:

$$C(R_v) \cap P_v^{-1} = \emptyset.$$

Consider the function C':

Definition 5.1. *The function* $C' : \mathcal{R} \to \mathcal{R}$ *is given by* $(a, b) \in C'(R)$ *if there exists a sequence* $s \in S$ *where* $s(1) = a$, $s(n_s) = b$ *and for all* $i = 1, ..., n_s - 1$, $(s(i), s(i+1)) \in R$ *or* $s(i) \in V(A)$ *for some* $A \in \mathcal{A}(s(i))$.

The function C' only uses one sequence in contrast to the function C that uses a finite number of sequences. However, it turns out that $C(R_v) \cap P_v^{-1} = \emptyset$ if and only if $C'(R_v) \cap P_v^{-1} = \emptyset$.

Lemma 5.4. *A choice function satisfies* $C(R_v) \cap P_v^{-1} = \emptyset$ *if and only if* $C'(R_v) \cap P_v^{-1} = \emptyset$.

Proof. ($\rightarrow$) Straightforward as $C'(R_v) \subseteq C(R_v)$.

($\leftarrow$) Let $C'(R_v) \cap P(R)^{-1} = \emptyset$ and assume that, on the contrary, $(a, b) \in C(R_v)$ and $(b, a) \in P_v$. Then there are sequences $s_1, \ldots, s_m$ where for all $j = 1, \ldots, m$, $s_j(1) = a$, $s_j(n_{s_j}) = b$ and for all $i = 1, \ldots, n_{s_j} - 1$, $(s_j(i), s_j(i + 1)) \in R_v$ or $s_j(i) \in V(A)$ where $A \in \mathcal{A}(s_j(i))$.

Construct the sequence $s' = s_1 \oplus s_2 \oplus s_3 \oplus ... \oplus s_m$ and observe that $(a, b) \in C'(R_v)$. Hence, $(a, b) \in C'(R_v)$ and $(b, a) \in P_v$, a contradiction. $\square$

The function C' has the advantage that it uses only a single sequence. The computation of the relation $C'(R_v)$ may still be very complicated, so it should be interesting to see if it is possible to further simplify the condition $C'(R_v) \cap P_v^{-1} = \emptyset$. We leave this for future research.

5.2.5 Time-consistency and impatience

Let us return to chapter 3 where our set of alternatives was given by $X \times T$. Theorem 3.1 in section 3.2 states that a choice function can be rationalized by a complete, absolute-time consistent, transitive and impatient relation if and only if:

$$B(\overline{R_v}) \cap P_v^{-1} = \emptyset.$$

Where $\overline{R_v} = R_v \cup \widetilde{R_v}$ and

$$\begin{aligned} \widetilde{R_v} &= \{((a,t),(a,v)) | a \in L(B(R_v)) \text{ and } t \leq v\} \\ &\cup \{((a,t),(a,v)) | a \in U(B(R_v)) \text{ and } t \geq v\}. \end{aligned}$$

Let us define $\widehat{R_v}$ as $\widetilde{R_v} \cup \{((a,t),(a,t)) | a \in X, t \in T\}$. For two relations R and S we write $((a,t),(b,v)) \in R * S$ if there exist an element $c \in X$ and elements t_1, t_2, v_1 and $v_2 \in T$ such that $((a,t_1),(c,v_1)) \in R$, $((c,t_2),(b,v_2)) \in S$ and $t - v = t_1 - v_1 + t_2 - v_2$. It is easy to establish that $(R * S) * T = R * (S * T)$, hence we can abuse notation without introducing ambiguities and write also $R * S * T$. Further, we have that $((a,t),(b,v)) \in B(R)$ if there exist an $n \in \mathbb{N}$ such that:

$$((a,t),(b,v)) \in \underbrace{R * R * \ldots * R}_{n \text{ times}}.$$

Let

$$D = \left\{ (a,t) | \exists v \in T \text{ such that } (a,v) \in \bigcup_{A \in \Sigma} K(A) \right\}.$$

We have the following result:

Lemma 5.5.

$$B(R_v \cup \widetilde{R_v}) = \widehat{R_v} \cup [\widehat{R_v} * B(\underline{R_v * \widehat{R_v}}) * R_v * \widehat{R_v}].$$

Proof. ($\subseteq$). Let $((a,t),(b,v)) \in B(R_v \cup \widehat{R_v})$. Then there exist a number $n \in \mathbb{N}$ such that:

$$((a,t),(b,v)) \in \underbrace{(R_v \cup \widehat{R_v}) * (R_v \cup \widehat{R_v}) * \ldots * (R_v \cup \widehat{R_v})}_{n \text{ times}}.$$

We work by induction on n. For $n = 1$, we have that $((a,t),(b,v)) \in R_v \cup \widehat{R_v}$. If $((a,t),(b,v)) \in \widehat{R_v}$, there is nothing to prove so assume that $((a,t),(b,v)) \in R_v$. From reflexivity of $\widehat{R_v}$ and $(a,t) \in D$, we derive that $((a,t),(a,t)) \in \widehat{R_v}$, $((a,t),(a,t)) \in B(\underline{R_v * \widehat{R_v}})$, $((a,t),(b,v)) \in R_v$ and $((b,v),(b,v)) \in \widehat{R_v}$. From $t - v = t - t + t - t + t - v + v - v$, we conclude that $((a,t),(b,v)) \in [\widehat{R_v} * B(\underline{R_v * \widehat{R_v}}) * R_v * \widehat{R_v}]$.

Assume that the result holds for $n = \ell$ and take the case where $n = \ell + 1$. Then there exists an element $c \in X$ and t_1, v_1, t_2, v_2, such that $((a,t_1),(c,v_1)) \in \widehat{R_v} \cup [\widehat{R_v} * B(\underline{R_v * \widehat{R_v}}) * R_v * \widehat{R_v}]$, $((c,t_2),(b,v_2)) \in \widehat{R_v} \cup R_v$ and $t - v = t_1 - v_1 + t_2 - v_2$.

If $((a,t_1),(c,v_1)) \in \widehat{R_v}$ and $((c,t_2),(b,v_2)) \in \widehat{R_v}$, we derive that $a = c$ and $c = b$. If $t_1 - v_1 = 0$ and $t_2 - v_2 = 0$ then $t - v = (t_1 - v_1) + (t_2 - v_2) = 0$ and $((a,t),(a,v)) \in \widehat{R_v}$ by definition of $\widehat{R_v}$. If $t_1 - v_1 > 0$ and $t_2 - v_2 \geq 0$ (or $t_1 - v_1 \geq 0$ and $t_2 - v_2 > 0$), we derive that $t - v = (t_1 - v_1) + (t_2 - v_2) > 0$ and $((a,t),(a,v)) \in \widehat{R_v}$ by definition of $\widehat{R_v}$. The same applies if both are negative. If $t_1 - v_1 < 0$ and $t_2 - v_2 > 0$ (or $(t_1 - v_1) > 0$ and $(t_2 - v_2) < 0$), we have that $((a,t),(a,v)) \in \widehat{R_v}$ for all t and v hence also for $t - v = (t_1 - v_1) + (t_2 - v_2)$.

If $((a,t_1),(c,v_1)) \in \widehat{R_v}$ and $((c,t_2),(b,v_2)) \in R_v$, we derive from reflexivity of O and $(c,t_2) \in D$ that $((a,t_1),(c,v_1)) \in \widehat{R_v}$, $((c,t_2),(c,t_2)) \in B(\underline{R_v * O})$, $((c,t_2),(b,v_2)) \in R_v$ and $((b,v_2),(b,v_2)) \in \widehat{R_v}$. Noticing that $t - v = t_1 - v_1 + t_2 - t_2 + t_2 - v_2 + v_2 - v_2$, we may conclude that $((a,t),(b,v)) \in [\widehat{R_v} * B(\underline{R_v * \widehat{R_v}}) * R_v * \widehat{R_v}]$. If $((a,t_1),(c,v_1)) \in [\widehat{R_v} * B(\underline{R_v * \widehat{R_v}}) * R_v * \widehat{R_v}]$ and $((c,t_2),(b,v_2)) \in \widehat{R_v}$, we derive that $((a,t),(b,v)) \in [\widehat{R_v} * B(\underline{R_v * \widehat{R_v}}) * R_v * \widehat{R_v}] * \widehat{R_v} = [\widehat{R_v} * B(\underline{R_v * \widehat{R_v}}) * R_v * \widehat{R_v}]$. If $((a,t_1),(c,v_1)) \in [\widehat{R_v} * B(\underline{R_v * \widehat{R_v}}) * R_v * \widehat{R_v}]$ and $((c,t_2),(b,v_2)) \in R_v$, then $((a,t),(b,v)) \in [\widehat{R_v} * B(\underline{R_v * \widehat{R_v}}) * R_v * \widehat{R_v}] * R_v \subseteq [\widehat{R_v} * B(\underline{R_v * \widehat{R_v}}) * R_v * \widehat{R_v}]$.

$(\supseteq)$ The result follows from noticing that $\widehat{R_v}$, R_v and $B(\underline{R_v * \widehat{R_v}})$ are subsets of $B(R_v \cup \widehat{R_v})$. □

Theorem 5.7. *A choice function is rationalizable by a complete, absolute time-consistent, transitive and impatient relation if and only if,*

$$B(\underline{R_v * \widehat{R_v}}) \cap (\underline{P_v * \widehat{R_v}})^{-1} = \emptyset.$$

and

$$\widehat{R_v} \cap P_v^{-1} = \emptyset.$$

The proof is analogous to the proof of theorems 5.4 and 5.5 and is left to the reader. The (first two) conditions for theorem 5.7 may be verified by following algorithm:

i. *Compute the relations* $B(\underline{R_v * \widehat{R_v}})$ *and* $\underline{P_v * \widehat{R_v}}$.

Observe that $\underline{R_v * \widehat{R_v}} = \underline{R_v} * \widehat{R_v}$. The relation $\underline{R_v}$ may be computed in the following way: compute for any $(b,v),(a,t) \in \bigcup_{A \in \Sigma} K(A)$, the set

$$A((a,t)(b,v)) = \left\{ q \in \mathbb{R} \,\middle|\, \begin{array}{l} \exists A \in \Sigma \text{ with } (a,s) \in K(A) \\ \text{and } (b, s-q) \in A \end{array} \right\}.$$

Notice that $A((a,t),(b,v))$ does not depend on t and v, hence, we can write $A((a,t),(b,v)) = A(a,b)$. Consider the set D',

$$D' = \left\{ a \in X | \exists v \in T, (a,v) \in \bigcup_{A \in \Sigma} K(A) \right\}.$$

Assume that the set D' is finite. Consider an enumeration of the elements in $D' = \{d_1, \ldots, d_n\}$ and consider the $n \times n$ matrix r^1 with elements in $2^{\mathbb{R}}$ (i.e. subsets of $\mathbb{R}$). If $A(d_i,d_j)$ exists, we set $r^1_{i,j} = A(d_i,d_j)$ and if $A(d_i,d_j)$ does not exist, we set $r^1_{i,j} = \emptyset$. Observe that $0 \in r^t_{i,i}$ for all $i \leq n$. Consider the following algorithm:

1. Initialize $t = 1$. Go to step 2.
2. Construct the matrices r^{t+1} and s such that for all $i, j \in \{1, \dots, n\}$:
   a If for all $d_k \in D'$ either $r^t_{i,k} = \emptyset$ or $r^t_{k,j} = \emptyset$, then $s_{i,j} = \emptyset$.
   b If (a) does not hold, then
   $$s_{i,j} = \{a + b \in \mathbb{R} | \exists d_k \in D' \text{ with } a \in r^t_{i,k} \text{ and } r^t_{k,j} = b\}.$$
   c For all $i \leq n$ if $s_{i,i} \cap \mathbb{R}_{++} \neq \emptyset$, we set $r^{t+1}_{i,i} = s_{i,i} \cup \mathbb{R}_+$. If $s_{i,i} \cap \mathbb{R}_{--} \neq \emptyset$, we set $r^{t+1}_{i,i} = s_{i,i} \cup \mathbb{R}_+$. If $s_{i,i} \cap \mathbb{R}_{++} \neq \emptyset$ and $s_{i,i} \cap \mathbb{R}_{--} \neq \emptyset$, we set $s_{i,i} = \mathbb{R}$.
   d For all $i, j \leq n$ with $i \neq j$, we set $r^{t+1}_{i,j} = s_{i,j}$.
3. If $r^{t+1} = r^t$, we stop and define $r = r^t$. Else, we augment t by one ($t = t + 1$) and we return to step 2.

The matrix r satisfies that $q \in r_{i,j}$ if and only if $((d_i, t), (d_j, v)) \in B(\underline{R_v * \widehat{R_v}})$ for all $t - v = q$.

The set $\underline{P_v \circ \widehat{R_v}}$ may be computed in the following way. Observe first that for all $i \leq n$, $((d_i, t), (d_i, v)) \in \underline{\widehat{R_v}}$ if and only if $t - v \in B(\underline{R_v * \widehat{R_v}})$. Consider also the set $B(a, b)$.

$$B(a,b) = \left\{ q \in \mathbb{R} \;\middle|\; \begin{array}{l} \exists A \in \Sigma \text{ for which } (a,t) \in K(A) \\ \text{and } (b, t-q) \in A - K(A) \end{array} \right\}.$$

Construct the $n \times n$ matrix s^1 with elements in $2^{\mathbb{R}}$ such that $s^1_{i,j} = B(d_i, d_j)$ and define for all $i, j \neq n$: $s_{i,j} = \{a + b \in \mathbb{R} | a \in s^1_{i,j}, b \in r_{j,j}\}$. We have that $q \in s_{i,j}$ if and only if $((d_i, t), (d_j, v)) \in \underline{P_v * \widehat{R_v}}$ for all $t - v = q$.

ii. *Verify whether* $B(\underline{R_v * \widehat{R_v}}) \cap (\underline{P_v * O})^{-1} = \emptyset$ *and* $\widehat{R_v} \cap P_v = \emptyset$.

The results in this section carry directly over to the property of relative time-consistency. In order to do this, we need to redefine $R * S$ by $(a, b) \in R * Q$ if there exist a $c \in X$ and t_1, t_2, v_1 and $v_2 \in T_0$ such that $((a, t_1), (c, v_1)) \in R$, $((c, t_2), (b, v_2)) \in S$ and $\frac{t+\delta}{v+\delta} = \frac{(t_1+\delta)(t_2+\delta)}{(v_1+\delta)(v_2+\delta)}$.

5.2.6 Independence

From section 4.3, we know that a choice function is rationalizable by a complete, transitive and independent relation if and only if

$$(a, b) \in R_{v,TI} \quad \text{implies} \quad (b, a) \notin P_v,$$

where $(a, b) \in R_{v,TI}$ if there exists a finite set $\{(x_i, y_i)_{i \leq n}\} \in R_v$ and elements $\alpha_1, \dots, \alpha_n$ in $\mathbb{R}_+$ such that

$$a - b = \sum_{i=1}^{n} \alpha_i (x_i - y_i).$$

Consider the set $D = \{x \in \mathbb{R}^n | \sum_i x_i = 0\}$ of n-dimensional vectors whose elements sum to zero. The rationalizability condition can be verified by the following algorithm:

i. For each set $A \in \Sigma$ and for each $a \in K(A)$ and $b \in A$, compute the vectors $a - b \in D$.

ii. For each set $A \in \Sigma$, compute the smallest convex cone[6] in D that contains the elements from step i), i.e. compute the set $\{x = \sum_i \alpha_i(a_i - b_i), \alpha_i \geq 0, a_i \in K(A), b_i \in A\}$.

iii. Compute the smallest convex cone in D that contains all convex cones from step ii).

iv. Compute for each set $A \in \Sigma$ and each $a \in K(A)$, and $b \in A - K(A)$, the element $x = a - b$ in D. Verify if these elements are not contained in the cone constructed in step iii).

5.3 Concluding remarks

In the previous chapters, we developed extension (and rationalizability) characterizations for orderings which satisfy the additional property of convexity, monotonicity, homotheticity, absolute time-independence and impatience or independence.

These characterizations have in common that they can be described by the condition:

$$F(R) \cap P(R)^{-1} = \emptyset,$$

for some algebraic closure $F : \mathcal{R} \to \mathcal{R}$.

The collection of properties which were not discused can (roughly) be divided in three groups: i) properties for which there exist characterizations that are easily deduced from characterizations provided in this text, ii) properties for which the characterizations (if they exist) do not fit the above framework and iii) properties which may have characterizations that fit into the above framework but require further research. We conclude this thesis by giving an example for each group of properties.

i). As an example of the first group of properties, we can refer to section 5.1.3 which discussed generalizations for the extension results for (strict) monotonic relations.

ii). A relation R is said to be continuous if the sets $L_R(a) = \{b \in X | (a, b) \in P(R)\}$ and $U_R(a) = \{b \in X | (b, a) \in P(R)\}$ are open sets for each $a \in X$. The relation R is upper semicontinuous if $L_R(a)$ is open for each $a \in X$ and it is lower semicontinuous if $L_R(a)$ is open for each $a \in X$.

The existence of continuous ordering extensions depends on the topology under consideration, e.g. if we take the discrete topology[7], then every relation is continuous and the property of continuity does not impose any additional requirement.

Jaffray [1975] and Bossert et al. [2002] start from a binary relation R in a set X and consider the topology $\mathcal{L}_R$ generated by the sets $\{L_R(a) | a \in X\}$. Obviously, this implies that R is upper semicontinuous. Their question reads:

[6]A cone is a subset $A \subseteq D$ such that if x in A and $\alpha \geq 0$, then $\alpha x \in A$. A cone, A is convex if for all $a, b \in A$ and $\alpha \in [0, 1]$, $(\alpha a + (1 - \alpha)b) \in A$.

[7]A topology on X is discrete if every subset of X is open

When does a relation R have an ordering extension which is upper semicontinuous with respect to the topology $\mathcal{L}_R$.

Jaffray showed that transitivity of R is a sufficient condition. Bossert et al. relaxed this condition towards consistency (i.e. $T(R) \cap P(R)^{-1} = \emptyset$) in combination with a condition called PI-comparability (i.e. if $(a,b) \in P(R)$ and $(b,c) \in I(R)$, then $(a,c) \notin N(R)$) and towards consistency in combination with $P\overline{I}$-continuity (i.e. for all $x \in X$, the set $\{y \in X | \exists z \in X \text{ such that } (x,z) \in P(R) \text{ and } (z,y) \in T(I(R))\}$ is open in X). Similar results can be established for the property of lower semi-continuity.

These results provide sufficient conditions but do not give a characterization. Two other disadvantages are that they focus only on semicontinuity and that they start from relations that are already semi continuous. In order to relax these assumptions we have to start with a predefined topology $\mathcal{L}$ on a set X, and an arbitrary relation R in X. The question then becomes:

When does the relation R have an ordering extension which is continuous with respect to $\mathcal{L}$.

Herden and Pallack [2002] offer a first step towards an answer but they do not get to a 'full characterization'. In particular, they characterize the existence of continuous ordering extensions by the existence of another mathematical structure[8] (R-separable systems).

Unfortunately, the property of continuity is not easily applicable to the framework that we developed. To grasp some intuition for this, consider for instance an algebraic closure operator, F, that relates to the property of continuity, i.e. we would like to have that R has a continuous ordering extension if and only if $F(R) \cap P(R)^{-1} = \emptyset$.

Consider a relation R in the set $\mathbb{R}^m$ that violates continuity, e.g. $(b,a) \in P(R)$ and there is a sequence $\{a_i\}_{i \in \mathbb{N}}$ that converges to a, such that for all $n \in \mathbb{N}$, $(a_n, b) \in R$. As F is a closure operator, we should have that $F(R) = \bigcap\{Q \supseteq R | Q = F(Q)\}$ (see lemma 2.5). Each $Q = F(Q)$ should not violate continuity. Indeed if it does, then it has no continuous ordering extension (which contradicts $F(Q) \cap P(Q)^{-1} = \emptyset$). Hence, we should have that $(b,a) \notin P(Q')$ for all $Q' \in \{Q \supseteq R | Q = F(Q)\}$. Using $(b,a) \in P(R)$, we can conclude that $(a,b) \in Q'$ for all $Q' \in \{Q \supseteq R | Q = F(Q)\}$, hence $(a,b) \in F(R)$.

If F is algebraic, there should exist a finite subset R' of R for which $(a,b) \in F(R')$. However, from finiteness of R', there does not exist a sequence $\{a_i\}_{i \in \mathbb{N}}$ converging to a that satisfies $(a_n, b) \in R'$ for all $n \in \mathbb{N}$. As such there seems to be no particular reason why $(a,b) \in F(R')$ for any finite $R' \subseteq R$.

iii). For the third set of properties, we give the example of separability. A relation R in $X \times X$ is separable if $((a,c),(b,c)) \in R$ implies that $((a,d),(b,d)) \in R$ for all $d \in X$.

Define $(a,b) \in Q_R$ if there exist an element $c \in X$ such that $((a,c),(b,c)) \in R$. Then clearly, for the existence of a separable ordering extension R^* it should be the case that both $T(R) \cap P(R)^{-1} = \emptyset$ and that $T(Q_R) \cap P(Q_R)^{-1} = \emptyset$. However, these conditions are not sufficient.

[8] In contrast, our characterization results demonstrates the equivalence between a statement (in prenex normal form) that involves an existential quantifier: <u>there exists</u> a complete extension $R^* = F(R^*)$ of R, and a statement (in prenex normal form) that involves only universal quantifiers: <u>for all</u> $(a,b) \in F(R)$, $(b,a) \notin P(R)$.

Indeed, let $X = \{a, b, c, d, e\}$ and assume that

$$R = \{((a, b), (c, d)), ((c, d), (e, b)), ((e, d), (a, d))\}.$$

Then $T(R) \cap P(R)^{-1} = \emptyset$ and $T(Q_R) \cap P(Q_R)^{-1} = \emptyset$. But R has no separable ordering extension. If, on the contrary, R^* were such an extension then $((a, b), (e, b)) \in T(R) \subseteq R^*$ and $((e, d), (a, d)) \in P(R) \subseteq P(R^*)$, contradicting separability.

The development of an algebraic closure operator that relates to the property of separability and satisfies C7 is, to our knowledge, until now not yet established. We leave this for future research.

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