On a Mathematical Method for Discovering Relations Between Physical Quantities: a Photonics Case Study

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Abstract: Quantity calculus defines the rules that apply to SI physical quantities used in physics and engineering. This research aims at the construction of a rigorous mathematical framework explaining the selection rule resulting in *realizable constitutive equations*. Here, we show that each SI physical quantity, that is represented by a lattice point in a seven dimensional integer lattice, has a *unique* $7D$-hypersphere. The lattice points incident on the $7D$-hypersphere are forming rectangles containing the origin $\boldsymbol{o}$, the lattice point $\boldsymbol{z}$ representing the selected physical quantity and the lattice point representations$\boldsymbol{x}, \boldsymbol{y}$ of a pair of *distinguishable* physical quantities $[x], [y]$ where $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$. The resulting rectangles are the geometric representations of the *realizable constitutive equations* for the selected physical quantity $[z]$. We apply the ``nD-hypersphere´´ method on the physical quantities $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}, \boldsymbol{B}$ and find the integral forms of Maxwell's equations. We find an integer sequence of non-degenerated *unique* rectangles formed by 4 lattice points $\boldsymbol{o}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ in $\mathbb{Z}^7$ as function of the infinity norm $\|\boldsymbol{z}\|_\infty = s$

Keywords: constitutive equations, SI physical quantities, integer lattice, $7D$-hypersphere, infinity norm.

1. Introduction

The SI [1] is used worldwide defining the semantics and syntax in the domains of science and technology. An algebraic structure for *quantity calculus* was proposed by R. Fleischmann [2], who also introduced the concept of "Verknüpfungsgleichung´´ that we translate as *constitutive equation*. This research addresses the question *What are realizable constitutive equations?*

2. Axioms of the SI physical quantities

We posit from the 8th edition of the SI [1] a set of axioms derived from *promoting* some of the SI conventions to mathematical axioms.

Axiom 1. *The base quantities are length, mass, time, electric current, thermodynamic temperature, amount of substance and luminous intensity.*

Axiom 2. *The base quantities are independent.*

Axiom 3. *The physical quantities are organized according to a system of dimensions.*

Axiom 4. *For each base quantity of the SI, there exists one and only one dimension.*

Axiom 5. *The product of two quantities is the product of their numerical values and units.*

Axiom 6. *The quotient of two quantities is the quotient of their numerical values and units.*

The uniqueness of the SI symbols forms an alphabet that is the base of any physical expression.

Definition 1. The dimension of a physical quantity q is expressed as a dimensional product [1] :

$$\dim q = \mathrm{L}^\alpha \mathrm{M}^\beta \mathrm{T}^\gamma \mathrm{I}^\delta \mathbf{\Theta}^\epsilon \mathrm{N}^\zeta \mathrm{J}^\eta ;$$

where the exponents $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta \in \mathbb{Z}$ are called *dimensional exponents*.

The dimensional exponents of the *common* SI physical quantities take small *integer* values. When all the dimensional exponents are zero, we call the physical quantity *dimensionless* or a physical quantity of *dimension one*. These dimensionless quantities occur in the celebrated *Buckingham theorem* [3] also known as the Π-theorem.

3. Isomorphism between classes of physical quantities and the 7-dimensional integer lattice

Let the set of *all* physical quantities be denoted by Q. Physical quantities are described by tensors and we can without loss of generality consider a component of a tensor and denote it as q . We know that *concepts* in physics are labeled in many ways. The physics concept *energy* has the labels: potential energy, kinetic energy, work, Lagrange function, Hamilton function,…in the formulations of physics. To cope with this multitude of *labels*, we define an equivalence relation between the physical quantities $a, b \in$ Q with notation $a \sim b$ meaning "a is *dimensionally* equivalent to b". The set of all equivalence classes in Q, given the equivalence relation $\sim$, is the quotient set Q/$\sim$. The equivalence class for the *concept* energy has notation $[energy]_\sim$. We define the surjective function dim(q) from Q to Q/$\sim$ given by $\dim(q) = [q]_\sim = \mathrm{L}^\alpha \mathrm{M}^\beta \mathrm{T}^\gamma \mathrm{I}^\delta \mathbf{\Theta}^\epsilon \mathrm{N}^\zeta \mathrm{J}^\eta$. In the sequel of this article we omit the symbol for the equivalence relation $\sim$ and denote the equivalence class as $[q]$. The class of dimensionless physical quantities is denoted $[I]$. We consider a multiplicative binary operator $\{\cdot\}$ between the equivalence classes of $\mathsf{Q}/\!\sim$. The algebraic properties of the composition of the equivalence classes result in a *multiplicative commutative group* $\mathsf{Q}/\!\sim,\{\cdot\}$. We now consider the set of integer septuples $\mathbb{Z}^7 \doteq \{(f_1,\ldots,f_7) : f_i \in \mathbb{Z}\}$. We know that $\mathbb{Z}^7,\{+\}$ is an *additive commutative group*. We define a mapping dex ():

Definition 2 (Mapping dex ()). The mapping dex () is defined from $\mathsf{Q}/\!\sim$ into $\mathbb{Z}^7$ and formally as dex () : $\mathsf{Q}/\sim\,\to \mathbb{Z}^7$: dex $([q]) \doteq \boldsymbol{f} = (f_1,\ldots,f_7)$ where $f_i \in \mathbb{Z}$.

We relabel f_i such that $f_1 = \alpha$, $f_2 = \beta$, $f_3 = \gamma$, $\ldots f_7 = \eta$ being the dimensional exponents *taken in the correct order* of a physical quantity q . Observe that we map the unit element $[I]$ of $\mathsf{Q}/\sim,\{\cdot\}$ on the unit element $\boldsymbol{o} = (0,\ldots,0)$ of $\mathbb{Z}^7,\{+\}$ and thus we have dex $([I]) \doteq \boldsymbol{o} = (0,\ldots,0)$. Each element of $\mathbb{Z}^7$ is the image of *one and only one* class $[q]$ of dimensionally equivalent physical quantities. We define the inverse mapping dex^{-1} ():

Definition 3 (Mapping dex^{-1} ()). The inverse of the dex () mapping is a mapping of $\mathbb{Z}^7$ into $\mathsf{Q}/\!\sim$, and defined as dex^{-1} () : $\forall \boldsymbol{a} \in \mathbb{Z}^7$, $\exists [a] \in \mathsf{Q}/\sim : \text{dex}^{-1}(\boldsymbol{a}) \doteq [a]$.

A homomorphism f : $\mathsf{Q}/\sim\,\to \mathbb{Z}^7$ is an *isomorphism* if there exists a homomorphism g : $\mathbb{Z}^7 \to \mathsf{Q}/\sim$ such that $\mathrm{f}\circ\mathrm{g}$ and $\mathrm{g}\circ\mathrm{f}$ are the identity mappings of $\mathbb{Z}^7$ and $\mathsf{Q}/\sim$ respectively [4]. We identify f = dex () and g = dex^{-1} () and infer that a *group isomorphism* exists between $\mathsf{Q}/\!\sim$ and $\mathbb{Z}^7$ that we denote $\mathbb{Z}^7 \approx \mathsf{Q}/\sim$ [4]. The set $\mathbb{Z}^n$ is known as the n-dimensional integer lattice [5] that is a discrete subgroup of the real vector space $\mathbb{R}^n$. The properties of the integer lattice $\mathbb{Z}^n$ are found in several publications [5]. In the sequel of this article we choose $n = 7$. We select seven basis lattice points of $\mathbb{Z}^7$ and choose an orthonormal basis and write using the Conway notation [5]:

$$\begin{aligned}
\boldsymbol{e}_1 &\doteq \text{dex}\,([\textit{length}]) = (1,0^6)\,,\\
\boldsymbol{e}_2 &\doteq \text{dex}\,([\textit{mass}]) = (0,1,0^5)\,,\\
\boldsymbol{e}_3 &\doteq \text{dex}\,([\textit{time}]) = (0^2,1,0^4)\,,\\
\boldsymbol{e}_4 &\doteq \text{dex}\,([\textit{electric current}]) = (0^3,1,0^3)\,,\\
\boldsymbol{e}_5 &\doteq \text{dex}\,([\textit{thermodynamic temperature}]) = (0^4,1,0^2)\,,\\
\boldsymbol{e}_6 &\doteq \text{dex}\,([\textit{amount of substance}]) = (0^5,1,0)\,,\\
\boldsymbol{e}_7 &\doteq \text{dex}\,([\textit{luminous intensity}]) = (0^6,1)
\end{aligned}$$

with $\boldsymbol{e}_i \in \mathbb{Z}^7$. A set of lattice points is called a *lattice constellation* [6]. An arbitrary set of physical quantities is represented by a constellation of points in $\mathbb{Z}^7$ and not by a set of vectors. We are interested in the properties of these constellations of points and focus on the simplest nontrivial constellation consisting of 4 integer lattice points. Observe that the *parallelogram law* $\boldsymbol{x}+\boldsymbol{y} = \boldsymbol{z}$ where $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in \mathbb{Z}^7$ is valid. We can prove [7] that ternary equations $[z] = f(\Pi)[x][y]$ are geometrically represented by *parallelograms* in $\mathbb{Z}^7$. We can define [7] an inner product $\{\cdot\}$ and p-norm $\|\|_p$ in $\mathbb{Z}^7$ and write $\boldsymbol{f} = \sum_{i=1}^{7}(\boldsymbol{f}\cdot\boldsymbol{e}_i)\boldsymbol{e}_i$.

4. Decomposition of a lattice point in pairwise orthogonal lattice points

We define *distinguishable* physical quantities as orthogonal lattice points dex $([x])$ and dex $([y])$. The decomposition of a lattice point $\boldsymbol{z}$ in two pairwise orthogonal lattice points $\boldsymbol{x}$ and $\boldsymbol{y}$ assumes the existence of a system of Diophantine equations:

$$\text{parallelogram law: } \boldsymbol{x}+\boldsymbol{y}-\boldsymbol{z} = 0\,, \tag{1a}$$

$$\text{inner product: } \boldsymbol{x}\cdot\boldsymbol{y} = 0\,, \tag{1b}$$

where $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in \mathbb{Z}^7$. We eliminate $\boldsymbol{y}$ from the equation (1b) and find:

$$\boldsymbol{x}\cdot\boldsymbol{x}-\boldsymbol{x}\cdot\boldsymbol{z} = 0\,. \tag{2}$$

We apply the method of *completing the square* and write equation (2) as:

$$(\boldsymbol{x}-\frac{\boldsymbol{z}}{2})^2 = (\frac{\boldsymbol{z}}{2})^2\,, \tag{3}$$

that represents a $7D$-hypersphere in $\mathbb{R}^7$ with center at $\frac{\boldsymbol{z}}{2}$ and radius $\|\frac{\boldsymbol{z}}{2}\|_2$. The center of the $7D$-hypersphere is only a lattice point if its coordinates are *even*. Observe that there exists a *unique* $7D$-hypersphere (3) for each physical quantity $[z]$. This unique $7D$-hypersphere determines the *finite* set of pairwise *distinguishable* physical quantities $[x]$ and $[y]$ that satisfy the realizable constitutive equation $[z] = f(\Pi)[x][y]$. We call the above method the "nD-hypersphere method$''$ as it can be generalized to a n-dimensional integer lattice.

5. Applications in photonics

We apply the "nD-hypersphere method$''$ to the physical quantities $[H],[B],[E],[D]$ occurring in the celebrated Maxwell's equations and infer relations between the physical quantities. The SI coordinates of the physical quantities $[H],[B],[E],[D]$ are:

Magnetic field strength: dex $([H]) = (-1,0,0,1,0,0,0)$
Magnetic induction: dex $([B]) = (0,1,-2,-1,0,0,0)$
Electric field: dex $([E]) = (1,1,-3,-1,0,0,0)$
Electrical displacement: dex $([D]) = (-2,0,1,1,0,0,0)$

The results are summarized in an isoperimetric distribution (Table 1) giving the frequency of occurrence of rectangles having a perimeter with value p formed by the lattice points $\{\boldsymbol{o},\boldsymbol{x}_i,\boldsymbol{y}_i,\boldsymbol{z}\}$ where $\{\boldsymbol{x}_i,\boldsymbol{y}_i\}$ are the i-th pair

Table 1: Isoperimetric distributions for $[H], [B], [E], [D]$.

$[q]$	Perimeter p	Frequency f
$[H]$	2.828	1
$[H]$	4	1
$[B]$	4.489	1
$[B]$	6.472	2
$[B]$	6.828	9
$[B]$	6.928	8
$[E]$	6.928	1
$[E]$	8.633	3
$[E]$	9.152	6
$[E]$	9.464	19
$[E]$	9.656	39
$[E]$	9.763	42
$[E]$	9.797	18
$[D]$	4.489	1
$[D]$	6.472	2
$[D]$	6.828	9
$[D]$	6.928	8

of lattice points representing uncorrelated physical quantities. We denote electric current I, electric charge q, electric charge density ρ_f, volume V, area S, time t, length l, electric current density J. We select the smallest non-degenerated rectangle of $[D]$ having $p = 6.472$. We find the lattice points $\boldsymbol{x} = (-2, 0, 0, 1, 0, 0, 0)$ and $\boldsymbol{y} = (0, 0, 1, 0, 0, 0, 0)$. We suggest the realizable constitutive equation:

$$
\begin{aligned}
D &= f_1(\Pi)\left(\frac{I}{S}\right)(t) \\
DS &= f_1(\Pi) I t \\
D \oiint_{S(V)} \mathrm{d}S &= f_1(\Pi) I \int \mathrm{d}t = f_1(\Pi)\, q \\
\oiint_{S(V)} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} &= f_1(\Pi) \iiint_V \rho_f \,\mathrm{d}V \\
\iiint_V \nabla \cdot \boldsymbol{D} \,\mathrm{d}V &= f_1(\Pi) \iiint_V \rho_f \,\mathrm{d}V
\end{aligned}
$$

that is the integral form of $\nabla \cdot \boldsymbol{D} = \rho_f$ where $f_1(\Pi) = 1$. We select the smallest non-degenerated rectangle of $[H]$ having $p = 4$. Observe that only $[H]$ has an unique non-degenerated rectangle. We find the lattice points $\boldsymbol{x} = (-1, 0, 0, 0, 0, 0, 0)$ and $\boldsymbol{y} = (0, 0, 0, 1, 0, 0, 0)$. We suggest the realizable constitutive equation:

$$
\begin{aligned}
H &= f_2(\Pi)\left(\frac{1}{l}\right)(I) \\
Hl &= f_2(\Pi) I \\
H \oint_{L(S)} \mathrm{d}l &= f_2(\Pi) \oiint_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S} \\
\oint_{L(S)} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} &= f_2(\Pi) \oiint_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S} \\
\oiint_S (\nabla \times \boldsymbol{H}) \cdot \mathrm{d}\boldsymbol{S} &= f_2(\Pi) \oiint_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S}
\end{aligned}
$$

that is the integral form of $\nabla \times \boldsymbol{H} = \boldsymbol{J}$ where $f_2(\Pi) = 1$. We select a non-degenerated rectangle of $[E]$ having $p = 9.152$. We find the lattice points $\boldsymbol{x} = (-1, 0, -1, 0, 0, 0, 0)$ and $\boldsymbol{y} = (2, 1, -2, -1, 0, 0, 0)$. We suggest the realizable constitutive equation:

$$
\begin{aligned}
E &= f_3(\Pi)\left(\frac{1}{lt}\right)(BS) \\
El &= f_3(\Pi)\frac{1}{t} BS \\
E \oint_{L(S)} \mathrm{d}l &= f_3(\Pi)\frac{\mathrm{d}}{\mathrm{d}t} \oiint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} \\
\oint_{L(S)} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} &= f_3(\Pi)\frac{\partial}{\partial t} \oiint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} \\
\oiint_S (\nabla \times \boldsymbol{E}) \cdot \mathrm{d}\boldsymbol{S} &= f_3(\Pi)\frac{\partial}{\partial t} \oiint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}
\end{aligned}
$$

that is the integral form of $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$ where $f_3(\Pi) = -1$. We have not found a realizable constitutive equation as basis for $\nabla \cdot \boldsymbol{B} = 0$. Observe that only $[E]$ and $[H]$ form a pair of orthogonal lattice points because the inner product $\mathrm{dex}([E]) \cdot \mathrm{dex}([B]) = 0$. We find that the physical quantities $[D]$ and $[B]$ have the same isoperimetric distributions and thus we find a matrix M such that $\mathrm{dex}([D])^{\intercal} = \mathsf{M}\,\mathrm{dex}([B])^{\intercal}$ where:

$$
\mathsf{M} = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

represents a signed permutation matrix. The automorphism group of the 7-dimensional cubic lattice $\mathrm{Aut}(\mathbb{Z}^7)$

contains all permutations and sign changes of the 7 coordinates and has order $2^7\,7! = 645120$. Each signed permutation matrix is an orthogonal matrix [8]. It is known from linear vector quantization [9] that the ℓ_2-norm and the phase of a lattice point are used to partition a lattice. However, this norm and phase are not the correct classifiers for the physical quantities. If we use as classifier the ℓ_∞-norm we obtain equivalence classes for which the elements of the class have the *same* isoperimetric distribution. In the framework of information theory we state that the lattice points dex $([D])$ and dex $([B])$ are elements of the absolute leader class $[21^2 0^4]$ that has cardinality 840.

6. Distribution of unique rectangles in the $7D$ integer lattice

We determine the distribution of non-degenerated *unique* rectangles formed by 4 lattice points $\boldsymbol{o}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ in $\mathbb{Z}^7$ as function of the infinity norm $\|\boldsymbol{z}\|_\infty = s$. We define a sample space Ω consisting of $7D$-hyperspheres with infinity norm $\|\boldsymbol{z}\|_\infty = s$, with $s \in \mathbb{N}$ and search for the event of an unique perimeter p in each hypercube with $\|\boldsymbol{z}\|_\infty = s$. Table 2 gives the result of the search for unique rectangles. We find in the $7D$-hypercube where $\|\boldsymbol{z}\|_\infty \leq 7$ a total of 1321 unique rectangles that represent *unique realizable constitutive equations* of the ternary type $[z] = f(\Pi)[x][y]$ for the selected physical quantity $[z]$. This sequence of integers is not listed in the OEIS [10] and we suggest further research on it.

Table 2: Distribution of *unique* rectangles in $\mathbb{Z}^7$ as function of the infinity norm $\|\boldsymbol{z}\|_\infty = s$.

Infinity norm $\|\boldsymbol{z}\|_\infty = s$	Frequency f
1	1
2	7
3	26
4	79
5	182
6	333
7	693

7. Conclusion

We show that each SI physical quantity, that is represented by a lattice point in a seven dimensional integer lattice $\mathbb{Z}^7$, has a *unique* $7D$-hypersphere. The lattice points incident on the $7D$-hypersphere are rectangles formed by 4 lattice points $\boldsymbol{o}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ in $\mathbb{Z}^7$ where $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$. The resulting rectangles are the geometric representation of the *realizable constitutive equations* of the ternary type $[z] = f(\Pi)[x][y]$ for the selected physical quantity $[z]$. We apply the "nD-hypersphere´´ method on the physical quantities $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}, \boldsymbol{B}$ and find the integral forms of Maxwell's equations. We find in the $7D$-hypercube, where $\|\boldsymbol{z}\|_\infty \leq 7$, a total of 1321 unique rectangles that represent *unique* realizable constitutive equations.

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