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Incidence geometry from an algebraic graph theory point of view

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Preface

This thesis presents the result of my work as a PhD student at the Department of Mathematics at Ghent University, which started in October 2007 and was supported by the Research Foundation Flanders-Belgium (FWO-Vlaanderen).

When obtaining my master's degree, I had already very often encountered beautiful theorems and puzzling problems in incidence geometry. Chapter 1 introduces its basic notions, in particular projective and polar spaces. It was in this area that I was keen to do research afterwards. Very roughly speaking, the problems in finite geometry I am mostly interested in, usually come down to this:

"If something has to be like this, how large can it be, and what happens if it is that large?"

There are several ways to deal with such problems, and many are solved by use of clever combinatorial arguments. However, my supervisors Frank De Clerck and John Bamberg made me get into contact with several other mathematicians, such as Tim Penttila, Peter Cameron, Chris Godsil and Bill Martin, all of whom focus on algebraic graph theory and geometry. It became clear that such a point of view could be fruitful for me as well.

Chapter 2 essentially summarizes what I learned when digging into literature, learning new terminology,... A substantial part of this chapter deals with the theory of Delsarte, who made fundamental contributions to algebraic combinatorics. One of the most important concepts is that of an association scheme, which is a set together with relations on them, behaving in a very regular way. Graphs are distance-regular when they define an association scheme in a certain way, and such graphs appear very often in geometry. This chapter is then used as a reference throughout this thesis.

I enjoyed experimenting with these techniques. Their advantage is that they can sometimes very quickly lead to a result, but on the other hand, these results are very often already known. This explains why you will see many alternative proofs in this thesis. Most of the time, I was not really looking for a technique to solve a problem, but rather for some unsolved problem where the technique could be useful.

Chapter 3 deals with the Grassmann association schemes, or more geometrically: with the projective geometries. Delsarte linked algebraic and geometric properties of subsets in Grassmann schemes, as well as in many other schemes. Several examples of interesting subsets are given, and we can easily derive completely combinatorial properties of them.

Chapter 4 discusses the association schemes from classical finite polar spaces. Here, using an explicit description of the eigenspaces, eigenvalues of some specific graphs will be explicitly computed. One of the main applications is obtaining bounds for the size of substructures known as partial m-systems. In most cases the bound is already known, except in one specific case, where the partial m-systems are partial spreads in the polar space $H(2d-1, q^2)$ with d odd. In this case, the bound is new and even tight.

In late 2009, Valentina Pepe and Leo Storme started working on the Erdős-Ko-Rado problem in polar spaces, which can be described as the converse problem to that of studying partial spreads. Here, one looks at sets of pairwise non-trivially intersecting generators in a polar space, and one tries to describe those of the maximum size. It soon became clear that only a combination of geometric and algebraic arguments could solve this problem. We decided to join our efforts, and after months of trying, speculating, guessing... we were able to completely solve the problem, except for one specific polar space, which was ironically the easiest case when studying partial spreads. The results of this joint work are presented in Chapter 5.

In the mean time, I kept thinking about the bound for partial spreads, which I really wanted to "feel" to be true. In the end, it became clear that the related dual polar graphs have properties, not only appearing for generalized polygons as well, but in fact for all near polygons. I decided to learn more about these structures, and to see what could be done there. These results are discussed in the first half of Chapter 6 on near polygons. It is in this context that more properties of partial spreads of maximum size in $H(2d-1, q^2)$ with d odd will be presented, as well as results for spreads in the polar spaces $Q(2d, q)$ and

$W(2d-1,q)$ with d odd as well, where we in fact find the well-known bound "twice", which we then exploit. In the same way, this yields some results on similar extremal substructures in generalized hexagons and octagons.

For $d = 3$, the bound for partial spreads in $H(2d-1,q^2)$ was already obtained by De Beule and Metsch, and it puzzled me why their proof is so different from my own. It relied on a beautiful property of this specific polar space due to J.A. Thas, which I wanted to see in a graph-theoretical context. It turned out that the associated dual polar graph has these properties because it is extremal with respect to an inequality for near polygons. Further exploiting the case of equality has led to a possible construction of a distance-regular graph with specific parameters, namely with classical parameters, and of new type for $d \geq 3$. These new results form the main part of the second half of Chapter 6. For diameter $d = 2$, that inequality comes down to the well known Higman inequality for generalized quadrangles, and the construction to a result by Thas on hemisystems.

Although I certainly recognized the power of new techniques, I always tried to see if things could be proved in more than one way. In the end, it turned out to be possible to prove the bound for partial spreads in a completely combinatorial way. This alternative proof is given in Appendix A, and is intentionally written in a language which can be understood after just reading Chapter 1. The proof was inspired by concepts from algebraic graph theory though, and on the other hand served as a motivation to study extremal near polygons. This way the last part of the thesis should close the circle and express the appreciation I gained of both approaches.

Frédéric Vanhove
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Working in the Research Group "Incidence Geometry" gave me the advantage to have colleagues specializing in about any related area nearby. I thank Kris Coolsaet, Jan De Beule, Bart De Bruyn, Tom De Medts, Michel Lavrauw, Valentina Pepe, Leo Storme, Koen Struyve, Joseph Thas, Koen Thas, Geertrui Van de Voorde and Hendrik Van Maldeghem for their advice. I obtained my master's degree together with my colleagues Michiel De Smet, Nele Vandersickel and my office mate, Thomas Maes, and I am happy to have gone through the experience together with them.

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When visiting Worcester Polytechnic Institute in Massachusetts, Bill Martin was an excellent host to me. Not only did I learn so much from the mathematical discussions, I also enjoyed the warm hospitality of his family.

Special thanks go to Tim Penttila, who already gave me completely new perspectives on mathematics when meeting him for the very first time, which were a driving force in all of my research from that moment on.

Finally, I want to thank my family. Above all, I must mention my father Luc Vanhove, who always encourages, helps and inspires me as a scientist and as a friend.

Summary of main new results

The reader is referred the specified parts of the thesis and the Index for definitions. For the sake of brevity, some results are presented in a weaker form in this summary.

1. Corollary 3.3.14 on page 53: For any non-degenerate alternating form on $V(2n, q)$, the number of totally isotropic subspaces in any $t - (2n, t + 1, \lambda; q)$-design with $0 \leq t \leq n - 1$ and t even, can be computed explicitly.

2. Corollary 3.3.16 on page 54: If S is a $2n - (4n + 2, 2n + 1, \lambda; q)$-design in $V(4n + 2, q)$, and Q is a non-degenerate elliptic quadratic form, then the number of $(2n + 1)$-spaces in S on which Q has a restriction of a certain type can be computed explicitly.

3. Theorem 4.4.16 on page 89: A partial spread S in $H(2d - 1, q^2)$, d odd, has size at most $q^d + 1$ (algebraic proof). (This bound is tight.)

4. Section 5.10 on page 125: Classification of Erdős-Ko-Rado sets of generators of maximum size in classical finite polar spaces of rank $d \geq 3$, except for $H(2d - 1, q^2)$ for odd $d \geq 5$.

5. Theorem 6.4.19 on page 148: In a generalized hexagon of order (s, t), a maximal partial distance-3-ovoid has size at most $\min((\sqrt{st})^3 + 1, s^3 + 1)$, with equality if and only if S is completely regular in the point graph.

6. Theorem 6.4.20 on page 150: If S and S' are distance-2-ovoids in a generalized hexagon of order (s, s^3), $s > 1$, then $|S \cap S'|$ is 0 or $h(s^2 + s + 1)$ for some integer $h \geq s^3 - s + 1$.

7. Theorem 6.4.21 on page 151: If S is the point set of a suboctagon of order (s', t') in a generalized octagon of order (s, t), then $s = s'$ or $s \geq s't'$.

8. Theorem 6.4.27 on page 155: Partial spreads of size q^3+1 in $H(5,q^2)$ are completely regular codes in the dual polar graph.

9. Theorem 6.4.30 on page 158: Spreads in $Q(10,q)$ or $W(9,q)$ are completely regular codes in the dual polar graph.

10. Corollary 6.4.31 on page 159: If S is a spread in $Q(2d,q)$ or $W(2d-1,q)$ with d odd, and T is an m-ovoid of the dual polar space, then $|S \cap T| = m(q^d+1)/(q+1)$.

11. Theorem 6.6.1 on page 164: The intersection number c_i of the point graph of a regular near $2d$-gon of order (s,t), $d \geq 2$ and $s > 1$, satisfies:

$$c_i \leq (s^{2i}-1)/(s^2-1).$$

12. Theorem 6.7.8 on page 174: If S is a $(q+1)/2$-ovoid in the dual polar space on $H(2d-1,q^2)$ with q odd and $d \geq 2$, then the induced subgraph Γ' on S of the point graph Γ is distance-regular with classical parameters:

$$(d,b,\alpha,\beta) = \left(d, -q, -\left(\frac{q+1}{2}\right), -\left(\frac{(-q)^d+1}{2}\right)\right).$$

13. Corollary 6.7.12 on page 176: If S is a partial spread of size q^d+1 in $H(2d-1,q^2)$ for odd $d \geq 3$, and T is a $(q+1)/2$-ovoid of the dual polar space, then $|S \cap T| = (q^d+1)/2$.

14. Theorem 6.8.7 on page 183: Consider a partial quadrangle $\mathrm{PQ}(s,t,\mu)$ with $\mu = st/(s+1)$. If for a set of points S, χ_S is orthogonal to the eigenspace for s of the point graph, then every two parallel lines intersect S in the same number of points.

15. Theorem A.2.1 on page 188: Suppose S is a partial spread in $H(2d-1,q^2)$, d odd and $d \geq 3$. Then $|S|$ is at most q^d+1. If $|S| > 1$ and $\pi \in S$, then every generator intersecting π in a $(d-1)$-space intersects the same number of other elements of S in just a point, if and only if $|S| = q^d+1$. In that case, that number must be q^{d-1} (geometric proof).

Contents

Chapter 1

Incidence geometries

1.1 Incidence geometries

An *incidence geometry of rank n* is an ordered set $(S, \mathrm{I}, \Delta, \sigma)$, with S a non-empty set of *varieties*, I a binary symmetric *incidence relation*, Δ a finite set of size n and σ a surjective *type map* from S to Δ, such that no ordered pair of elements of S of the same type is in I. A *flag* is a set of pairwise incident varieties, and the type of a flag is its image under σ.

A *point-line geometry* is an incidence geometry of rank 2, where the varieties of the two types are referred to as *points* and *lines*. We will also denote such a geomety with set of points P and set of lines L by (P, L, I). If a point and line are incident, we say the line *passes through* the point or *contains* the point, or the point is *on* the line. Two distinct points incident with a common line are said to be *collinear*, and two distinct lines incident with a common point are said to *intersect* or *meet*, and are *skew* otherwise. If there is exactly one line incident with two distinct points p_1 and p_2, we will often denote this line by p_1p_2. A point-line geometry is called a *partial linear space* if every two distinct points are incident with at most one line and every line is incident with at least two points. A partial linear space is a *linear space* if every two distinct points are on exactly one line. The *dual* of (P, L, I) is (L, P, I).

An *isomorphism* from $(S, \mathrm{I}, \Delta, \sigma)$ to $(S', \mathrm{I}', \Delta, \sigma')$ is a bijection $\phi : S \to S'$ with $x\mathrm{I}y \iff \phi(x)\mathrm{I}'\phi(y)$ and $\sigma(x) = \sigma(y) \iff \sigma'(\phi(x)) = \sigma'(\phi(y))$, $\forall x, y \in S$. An *automorphism* of $(S, \mathrm{I}, \Delta, \sigma)$ is an isomorphism from this geometry to itself.

A *duality* from a point-line geometry (P, L, I) to (P', L', I') is an isomorphism from the first geometry to the dual of the second. A *correlation* of (P, L, I) is a duality from this geometry to itself. A *polarity* is an involutive correlation. A point-line geometry admitting a correlation or polarity is said to be *self-dual* or *self-polar*, respectively.

1.2 Projective geometries

The projective geometry $\mathrm{PG}(n, \mathbb{K})$ is the incidence geometry $(S, \mathrm{I}, \Delta, \sigma)$ of rank n, derived from a left vector space $V(n+1, \mathbb{K})$ of dimension $n+1$ over a division ring $\mathbb{K}$. The set S consists of the subspaces of $V(n+1, \mathbb{K})$ different from the trivial and full subspace, I is symmetrized strict inclusion, Δ is the set $\{1, \ldots, n\}$, and σ maps each subspace onto its vectorial[1] dimension over $\mathbb{K}$. For every prime power q, there is a finite field of order q, which is unique up to isomorphism, and which we will denote by $\mathrm{GF}(q)$. The projective geometry $\mathrm{PG}(n, \mathrm{GF}(q))$ will also be denoted by $\mathrm{PG}(n, q)$.

An axiomatic approach to projective geometries is also possible. A *projective space* is defined as a point-line geometry satisfying the following axioms.

(i) For every two distinct points, there is exactly one line incident with both.

(ii) If p_1, p_2, p_3 and p_4 are four distinct points, such that the lines p_1p_2 and p_3p_4 intersect, then the lines p_1p_3 and p_2p_4 intersect as well.

(iii) Every line contains at least three points.

A *subspace of a projective space* is a subset S of points, such that every line containing two elements of S, only contains points of S. The projective dimension of a projective space is the largest number n for which there is a strictly increasing chain $\emptyset \subset S_0 \subset \cdots \subset S_n = P$ of subspaces of the projective space, where P denotes the full set of points.

Veblen and Young [166] proved that if the projective dimension is an integer $n \geq 3$, the projective space is isomorphic to the point-line geometry derived

[1] Unless stated otherwise, dimensions are assumed to be vectorial and not projective in this thesis.

from $\mathrm{PG}(n, \mathbb{K})$ by restricting to the 1- and 2-dimensional subspaces. In particular, Wedderburn's Little Theorem[2], which states that every finite division ring is a field (see for instance [2]), implies that the only finite projective spaces of projective dimension at least three are those derived from a $\mathrm{PG}(n, q)$.

A projective space of projective dimension 2, also referred to as a *projective plane*, can alternatively be defined as a point-line geometry satisfying the following axioms:

(i) every two distinct points are on a unique common line,

(ii) every two distinct lines contain a unique common point,

(iii) there are four distinct points, no three of which on a common line,

and if it is finite, then one can easily prove that for some $n \geq 2$, there are exactly $n + 1$ points on each line and $n + 1$ lines through each point (see for instance [34]). The number n is the *order* of the projective plane.

A projective plane is said to be *Desarguesian* if it is isomorphic to a $\mathrm{PG}(2, \mathbb{K})$ for some division ring $\mathbb{K}$. Many constructions for finite non-Desarguesian planes are known, but the classification is far from done (see for instance [97]).

1.3 Polar spaces

Polar spaces are important types of incidence geometries from which many interesting association schemes and graphs (see Chapter 2 for the definitions) can be derived, such as the dual polar graphs. They will play a fundamental role in this thesis.

1.3.1 Axiomatic definition

Veldkamp [167] was the first to axiomatically describe polar spaces, and the theory was later refined by Tits [158]. A distinction must be made between polar spaces of rank two and those of higher rank.

[2] See [117] for a detailed discussion on the history of this result.

A *polar space of rank* n with $n \geq 3$ is an incidence geometry $(S, \mathrm{I}, \{1, \ldots, n\}, \sigma)$, with all varieties subsets of $\sigma^{-1}(1)$, the set of varieties of type 1 (referred to as the *points*), and incidence defined as symmetrized strict inclusion, satisfying the following axioms.

(i) The incidence structure obtained by considering all varieties strictly contained in one given variety of size at least two, is isomorphic to a projective space of projective dimension m with $1 \leq m \leq n-1$, in which case the variety is said to be of projective dimension m.

(ii) The intersection of two varieties is either a variety or empty.

(iii) If π is a variety of projective dimension $n-1$ and p is a point not in π, then there is a unique variety π' such that $p \in \pi'$ and $\pi \cap \pi'$ has projective dimension $n-2$. Moreover, it contains all points in π that are in a common variety with p.

(iv) There exist two disjoint varieties of projective dimension $n-1$.

The varieties of projective dimension $n-1$ and $n-2$ in a polar space of rank n will be referred to as the *generators* or *maximals*, and the *next-to-maximals*, respectively. We will also refer to the varieties of projective dimension 2 as the *planes* of the polar spaces.

A polar space of rank two or *generalized quadrangle* is a partial linear space satisfying the following axioms.

(i) For any point p not on a line ℓ, there is a unique point on ℓ collinear with p.

(ii) Every point is incident with at least two lines.

Here, we will also refer to the lines as *generators* or *maximals*, and to the points as *next-to-maximals*.

Generalized quadrangles were introduced by Tits [156]. Note that the above definition of generalized quadrangles is self-dual. A finite generalized quadrangle with $s+1$ points on each line and $t+1$ lines through each point is said to be of order (s,t) and is denoted as a $\mathrm{GQ}(s,t)$. A generalized quadrangle of order s is a $\mathrm{GQ}(s,s)$.

A fundamental difference between projective spaces and polar spaces is that two points do not have to be on a common line in the latter.

1.3.2 Classical polar spaces

Many polar spaces can be constructed by considering certain subspaces of a vector space. We will now describe the possible constructions of these *classical polar spaces*. It will turn out that in some sense, almost all finite polar spaces are constructed in this way. Proofs and much more information on polar spaces can be found in for instance [140, Chapters 7 and 9] or [34].

Consider a vector space $V = V(m, \mathbb{K})$ over some field $\mathbb{K}$[3].

- A *bilinear form* on V is a map $f : V \times V \to \mathbb{K}$ such that $f(av_1 + bv_2, v) = af(v_1, v) + bf(v_2, v)$ and $f(v, av_1 + bv_2) = af(v, v_1) + bf(v, v_2)$ for every $a, b \in \mathbb{K}$ and $v, v_1, v_2 \in V$.

- A *sesquilinear form* on V is a map $f : V \times V \to \mathbb{K}$ such that $f(av_1 + bv_2, v) = af(v_1, v) + bf(v_2, v)$ and $f(v, av_1 + bv_2) = a^\theta f(v, v_1) + b^\theta f(v, v_2)$ for every $a, b \in \mathbb{K}$ and $v, v_1, v_2 \in V$, for some field automorphism θ of $\mathbb{K}$.

- A *quadratic form* on V is a map $Q : V \to \mathbb{K}$ such that $Q(av) = a^2 Q(v)$ for every $a \in \mathbb{K}$ and $v \in V$, and with $f : V \times V \to \mathbb{K} : (v_1, v_2) \to Q(v_1 + v_2) - Q(v_1) - Q(v_2)$ a bilinear form on V.

A bilinear form f is *symmetric* if $f(v_1, v_2) = f(v_2, v_1), \forall v_1, v_2 \in V$, and *alternating* or *symplectic* if $f(v, v) = 0, \forall v \in V$. A sesquilinear form is *Hermitian* if the field automorphism θ is an involution and $f(v_1, v_2) = f(v_2, v_1)^\theta$, $\forall v_1, v_2 \in V$. Note that if the characteristic of $\mathbb{K}$ is different from 2, symmetric bilinear forms are in one-to-one correspondence with quadratic forms.

With respect to a bilinear form or sesquilinear form f, a vector v_0 is *singular* if $f(v_0, v) = 0, \forall v \in V$. With respect to a quadratic form Q, a vector v_0 is said to be *singular* if both $Q(v_0) = 0$ and $Q(v_0 + v) = Q(v), \forall v \in V$. The singular vectors form a subspace in both cases, known as the *singular subspace*, and we say the form is non-degenerate if the singular subspace is trivial.

A bijective linear map from one vector space to another, both equipped with a form, is an *isometry* if it transforms the first form into the second. For a proof of the following theorem, see for instance [140].

[3] More generally, one can use division rings, but for our purposes in this thesis, we can restrict ourselves to a field $\mathbb{K}$.

Theorem 1.3.1. [Witt's Theorem] *Suppose f is a non-degenerate quadratic, symmetric, alternating or Hermitian form on V. Any isometry between two subspaces U_1 and U_2 of V extends to an isometry of V.*

A non-zero vector v is *isotropic* if $f(v, v) = 0$ (for a bilinear or sesquilinear form f) or $Q(v) = 0$ (for a quadratic form Q). A form with no isotropic vectors is *anisotropic*. A subspace is said to be *isotropic* with respect to a form if it contains an isotropic vector, and a subspace W is *totally isotropic* if the restriction of the form to $W \times W$ is trivial (for a bilinear or sesquilinear form) or its restriction to W is trivial (for a quadratic form). For quadratic and Hermitian forms, the isotropic subspaces are precisely those subspaces, all non-zero vectors of which are isotropic. The *Witt index* is the maximal dimension of the totally isotropic subspaces. It follows from Theorem 1.3.1 that every totally isotropic subspace is in a totally isotropic subspace of this maximal dimension.

For any non-degenerate quadratic form Q on $V(m, \mathbb{K})$, a basis can be found such that

$$Q(x_1, \ldots, x_m) = x_1x_2 + \cdots + x_{2g-1}x_{2g} + Q'(x_{2g+1}, \ldots, x_m),$$

where Q' is some anisotropic quadratic form. The Witt index is then given by g. If $\mathbb{K} = \mathrm{GF}(q)$, then anisotropic forms can only exist in vector spaces of dimension at most two, and thus $2g$ must be $m-2, m-1$ or m. The quadratic form is said to be of *elliptic*, *parabolic* or *hyperbolic* type in these cases, respectively. More generally, a quadratic form on $V(m, q)$ is said to be of one of these three types if its (non-degenerate) restriction to some subspace, complementary to its singular subspace, is of that type. Up to a non-zero scalar, two non-degenerate quadratic forms on $V(m, q)$ of the same type can be transformed into each other by some linear transformation.

Non-degenerate alternating forms on $V(m, \mathbb{K})$ only exist for even m. For any non-degenerate alternating form on $V(2n, \mathbb{K})$, a basis $\{e_1, \ldots, e_n, e'_1, \ldots, e'_n\}$ can be found such that $f(e_i, e_j) = f(e'_i, e'_j) = 0$ and $f(e_i, e'_j) = \delta_{ij}, \forall i, j$. The Witt index is then given by n.

A finite field has an involutive automorphism if and only if the order is a square q^2 for some prime power q, and in that case the involution is the unique mapping $x \to x^q$ and fixes the unique subfield of order q. For any non-degenerate Hermitian form f on $V(m, q^2)$, a basis $\{e_1, \ldots, e_m\}$ can be found such that

$f(e_i, e_j) = \delta_{ij}$. The Witt index is given by $m/2$ for even m and $(m-1)/2$ for odd m.

For any non-degenerate form f, let $\tilde{f}$ denote $\tilde{f} : V \times V \to \mathbb{K} : (v_1, v_2) \to f(v_1 + v_2) - f(v_1) - f(v_2)$ (if f is quadratic) or simply f if it is symmetric, alternating or Hermitian. We define for any subspace U in V:

$$U^\perp := \{v \in V | \tilde{f}(u, v) = 0, \ \forall u \in U\}.$$

Note that $U \subseteq U^\perp$ for any totally isotropic subspace U.

Consider a non-degenerate form with Witt index $n \geq 2$. Consider the incidence structure $(S, \mathrm{I}, \{1, \ldots, n\}, \sigma)$, where S is the set of all non-trivial totally isotropic subspaces, I is symmetrized strict inclusion, and σ maps each element of S onto its dimension. This is the *classical polar space* of rank n induced by the form. Note that all classical finite polar spaces can be embedded in $\mathrm{PG}(m, q)$ for some m. We now introduce notation for these finite polar spaces of rank n, based on these embeddings in projective spaces over $\mathrm{GF}(q)$.

- The polar spaces induced by non-degenerate quadratic forms are the *quadrics*, and they are *elliptic*, *parabolic* or *hyperbolic* if the form is of that type. When constructed in $V(m, q)$ (and hence embedded in $\mathrm{PG}(m-1, q)$), they are denoted by $Q^-(m-1, q)$,$Q(m-1, q)$ and $Q^+(m-1, q)$, respectively.
- The polar space induced by a non-degenerate symplectic form on $V(2n, q)$ (and hence embedded in $\mathrm{PG}(2n-1, q)$), is known as the *symplectic space* $W(2n-1, q)$.
- The polar space induced by a non-degenerate Hermitian form on $V(m, q^2)$ (and hence embedded in $\mathrm{PG}(m-1, q^2)$), is the *Hermitian variety* $H(m-1, q^2)$.

For even q, the embedding of the parabolic quadric $Q(2n, q)$ is such that there is a unique point p in $\mathrm{PG}(2n, q)$, not on the quadric and all lines through which in $\mathrm{PG}(2n, q)$ have exactly one point in common with the quadric. This point p is known as the *nucleus of the parabolic quadric* (see for instance [94, Lemma 22.3.1 (Corollary 2)]). In this case, the projection of the totally isotropic subspaces of $Q(2n, q)$ from the nucleus onto any hyperplane in $\mathrm{PG}(2n, q)$ not through p yields an isomorphism between the polar spaces $Q(2n, q)$ and $W(2n-1, q)$.

The classical generalized quadrangles are the classical polar spaces of rank two. For each such $\mathrm{GQ}(s,t)$, we give its parameters in Table 1.1.

	(s,t)
$Q^-(5,q)$	(q,q^2)
$Q(4,q)$	(q,q)
$Q^+(3,q)$	$(q,1)$
$W(3,q)$	(q,q)
$H(3,q^2)$	(q^2,q)
$H(4,q^2)$	(q^2,q^3)

Table 1.1: The classical generalized quadrangles and their parameters

For proofs of the following fundamental results on isomorphisms between classical generalized quadrangles, we refer to [122, 3.2.1 and 3.2.3].

Theorem 1.3.2. *The generalized quadrangles $Q(4,q)$ and $W(3,q)$ are dual to each other, and are isomorphic (and hence self-dual) if and only if q is even.*

Theorem 1.3.3. *The generalized quadrangles $H(3,q^2)$ and $Q^-(5,q)$ are dual to each other.*

1.3.3 Characterizations

Polar spaces of rank at least three were characterized by Tits [158]. In particular, all finite[4] polar spaces of rank at least three are classical.

The situation is completely different for polar spaces of rank two. Many non-classical finite generalized quadrangles are known, some of which of the same order as certain classical generalized quadrangles, and some of order $(q-1,q+1)$ with q a prime power. Up to duality, no other orders (s,t) with $s,t>1$ are known, but a complete classification of all generalized quadrangles seems hopeless. We refer to [122] for much more details.

[4] A similar result holds in the infinite case, but there are non-embeddable infinite polar spaces of rank three.

1.4 SPBIBDs

The following definition is due to Bridges and Shrikhande [19].

Definition 1.4.1. *A special partially balanced incomplete block design (SPBIBD) with parameters* $(v, b, r, k, \lambda_1, \lambda_2)$ *of type* (α_1, α_2)*, with* $v, b, r, k \geq 2$*,* $\lambda_1, \lambda_2, \alpha_1, \alpha_2 \geq 0$*,* $\lambda_1 \neq \lambda_2$ *and* $r < b$*, is a finite point-line geometry* (P, B, I) *satisfying the following axioms.*

(i) *There are* v *points and* b *lines.*

(ii) *Every point is on exactly* r *lines and every line contains exactly* k *points.*

(iii) *Two distinct points are either on exactly* λ_1 *common lines (when they are* λ_1*-associated) or on exactly* λ_2 *common lines (when they are* λ_2*-associated).*

(iv) *A point* p *is* λ_1*-associated to exactly* α_1 *points on a line* ℓ *if* p *is on* ℓ*, and to* α_2 *points on* ℓ *if* p *is not on* ℓ*.*

We will be mostly interested in the case $\lambda_2 = 0$ and $\alpha_1 = k - 1$, where two points are either on exactly λ_1 common lines or none at all.

Definition 1.4.2. *A partial geometry* $\mathrm{pg}(s, t, \alpha)$ *is a partial linear space with* $s, t, \alpha \geq 1$*, satisfying the following axioms.*

(i) *Every line contains exactly* $s + 1$ *points, and every point is on exactly* $t + 1$ *lines.*

(ii) *If a point is not on a line* ℓ*, then it is collinear with exactly* α *points on* ℓ*.*

Note that the partial geometries $\mathrm{pg}(s, t, \alpha)$ with v points and b lines are precisely the SPBIBDs $(v, b, t+1, s+1, 1, 0)$ of type (s, α). The dual of a $\mathrm{pg}(s, t, \alpha)$ is a $\mathrm{pg}(t, s, \alpha)$. The generalized quadrangles of order (s, t) are precisely the partial geometries $\mathrm{pg}(s, t, 1)$.

We will later see how both projective and polar spaces give rise to SPBIBDs.

Chapter 2

Association schemes

Finite incidence geometries often yield interesting combinatorial structures, such as distance-regular graphs and association schemes. The goal of this thesis is to study geometric structures by use of techniques from algebraic combinatorics. In this chapter, we will give an overview of some of the most important concepts and techniques in this area of mathematics.

2.1 Graphs

A *graph* Γ is an ordered pair $(V(\Gamma), E(\Gamma))$ where $V(\Gamma)$ is a non-empty set of elements called *vertices*, and $E(\Gamma)$ is a set of subsets of $V(\Gamma)$ of size two, called *edges*. Hence our graphs are assumed to be undirected, without loops and without multiple edges. We say two vertices x and y are *adjacent* or *neighbours* if $\{x, y\}$ is an edge, and we denote this by $x \sim y$.

A path of length i from vertex x to vertex y is a sequence $x = x_0, \ldots, x_i = y$ of vertices of Γ with every two successive vertices adjacent. The subset of $V(\Gamma) \times V(\Gamma)$, consisting of those ordered pairs of vertices (x, y) such that either x and y are equal or there is a path from x to y, is an equivalence relation on $V(\Gamma)$, and its classes are the *connected components* of the graph. A graph with only one connected component is *connected*. The *distance* between two vertices x and y in the same connected component is the minimal length of all paths from x to y, and is denoted by $d(x, y)$. The *diameter* of a connected graph is the maximal distance between any two vertices in the graph. For any vertex x

and any non-empty subset S, the *distance from x to S* is $\min\{d(x,y)|y \in S\}$, denoted by $d(x,S)$. We will write $\Gamma_i(x)$ for the set of vertices in Γ at distance i from a given vertex, and we will refer to it as the *i-th subconstituent* of Γ with respect to x. The *sphere* of radius e around a vertex x is the set of vertices in the graph at distance at most e from x. A sequence of vertices $x_0, \ldots, x_i$ is a *circuit* of length i ($i \geq 3$) if $x_1, \ldots, x_i$ are distinct, $x_0 = x_i$ and every two successive vertices are adjacent. The girth of Γ is the length of its shortest circuit.

The subgraph *induced* by a subset of vertices X is the graph with vertex set X whose edges are the edges of Γ contained in X.

The *degree* of a vertex is its number of neighbours, and we say a graph is *regular with valency k* or simply *k-regular* if every vertex has degree k.

A graph is *complete* if every two distinct vertices are adjacent. We say a graph Γ is *bipartite* if there is a partition $\{V_1, V_2\}$ of $V(\Gamma)$ such that no two vertices in the same class of the partition are adjacent.

A *clique* in a graph is a set of pairwise adjacent vertices, and a *coclique* or *independent set* is a set of pairwise non-adjacent vertices. A *triangle* in a graph is a clique of size three.

The *complement* of a graph Γ is the graph $\bar{\Gamma}$ with the same set of vertices, but with two distinct vertices adjacent if and only if they are not adjacent in Γ.

A *graph isomorphism* from one graph $\Gamma = (V(\Gamma), E(\Gamma))$ to another $\Gamma' = (V(\Gamma'), E(\Gamma'))$ is a bijection $\theta : V(\Gamma) \to V(\Gamma')$ such that $x^\theta \sim y^\theta \iff x \sim y, \forall x, y \in V(\Gamma)$. An *automorphism* of a graph Γ is an isomorphism from Γ to itself.

The *point graph* of a point-line geometry (P, L, I) is the graph whose vertex set is P and in which two points are adjacent if they are incident with a common line. The *incidence graph* of a point-line geometry is the bipartite graph with vertex set $P \cup L$ and edge set $\{\{p, \ell\}|p\mathrm{I}\ell\}$. For a point p and line ℓ, the *distance $d(p,\ell)$ with respect to the point graph* is short for $d(p, \{p'|p'\mathrm{I}\ell\})$.

For any subset S of a finite non-empty set Ω, we will denote by χ_S its *characteristic vector* in $\mathbb{R}^\Omega$, with $(\chi_S)_\omega = 1$ if $\omega \in S$ and $(\chi_S)_\omega = 0$ if $\omega \notin S$. Note that the orthogonal projection of χ_S onto $\langle \chi_\Omega \rangle$ is given by $(|S|/|\Omega|)\chi_\Omega$.

The *adjacency matrix* of a finite graph on Ω is the symmetric $(0,1)$-matrix,

the rows and columns of which are indexed by its vertices, and with

$$(A_i)_{x,y} = \begin{cases} 1 \text{ if } x \sim y \\ 0 \text{ if } x \not\sim y \end{cases}.$$

Another ordering of the vertices yields an adjacency matrix that only differs by conjugation with a permutation matrix. Hence we can define the *eigenvalues of a graph* as the eigenvalues λ of its adjacency matrix A (i.e. $Av = \lambda v$ for some $v \neq 0$). Its *spectrum* is its set of eigenvalues together with their multiplicities. Since the adjacency matrix is real symmetric, all eigenvalues are real and the vector space $\mathbb{R}^\Omega$ orthogonally decomposes into eigenspaces[1]. The following results follow from the Perron-Frobenius theorems (see for instance [23, Chapter 3]).

Theorem 2.1.1. *Let Γ be a graph with largest eigenvalue θ_0.*

(i) *θ_0 is at most the maximum degree $k_{\max}$, and if Γ is connected, then equality holds if and only if the graph is regular with valency θ_0.*

(ii) *If Γ is k-regular, then the multiplicity of k is the number of connected components of Γ.*

(iii) *The smallest eigenvalue is bigger than or equal to $-\theta_0$, and if Γ is connected, then equality holds if and only if Γ is bipartite, and in that case the spectrum is symmetric around* 0.

Note that a k-regular graph always has the all-one vector χ_Ω as an eigenvector for k.

A simple but important observation is that, given the adjacency matrix A and a subset of vertices S, the vector $A\chi_S$ has the following interpretation:

$$(A\chi_S)_\omega = |\{\omega' | \omega' \in S, \omega \sim \omega'\}|.$$

The elements of a partition $\{C_1, \ldots, C_m\}$ of the set of vertices $V(\Gamma)$ are known as its *cells*. A partition is said to be *equitable* if every $x_i \in C_i$ is adjacent to exactly c_{ij} vertices in C_j, where c_{ij} only depends on i and j and not on x_i. The *quotient matrix* of the equitable partition is the $(m \times m)$-matrix C

[1] Kernels, eigenspaces and eigenvectors will be assumed to be right kernels, right eigenspaces and right eigenvectors, respectively.

with $(C)_{ij} = c_{ij}$. The *characteristic matrix* of the equitable partition is the $(|V(\Gamma)| \times m)$-matrix, the i-th column of which is the characteristic vector of C_i. The following fundamental properties are proved in [82, Chapter 5].

Theorem 2.1.2. *Consider a graph Γ with adjacency matrix A. A partition $\{C_1, \ldots, C_m\}$ is equitable if and only if the characteristic matrix Π satisfies $A\Pi = \Pi C$ for some $(m \times m)$-matrix C. In that case, C must be the quotient matrix, and the following holds.*

(i) *The characteristic polynomial of C divides that of A.*

(ii) *If $v \neq 0$ satisfies $Cv = \lambda v$, then Πv is an eigenvector of A for the eigenvalue λ.*

(iii) *The column span of Π is spanned by m eigenvectors of A, and contains the all-one vector if Γ is regular.*

In particular, the previous theorem yields that if the number of cells is less than the number of distinct eigenvalues of Γ, the characteristic vectors of the cells are orthogonal to some of the eigenspaces of its adjacency matrix.

We say a subset S of vertices in a regular graph is *intriguing* with parameters (h_1, h_2) if the number of neighbours in S of a vertex x is h_1 if $x \in S$, and h_2 if $x \notin S$. If $\emptyset \neq S \neq V(\Gamma)$, then this is the case if and only if $\{S, V(\Gamma)\backslash S\}$ is an equitable partition into two parts.

Lemma 2.1.3. *A subset S of vertices in a k-regular graph Γ on Ω is intriguing if and only if χ_S is a linear combination of χ_Ω and an eigenvector v for some eigenvalue λ of the adjacency matrix A. In that case, S is intriguing with parameters (h_1, h_2):*

$$h_1 = \frac{|S|}{|\Omega|}(k - \lambda) + \lambda, h_2 = \frac{|S|}{|\Omega|}(k - \lambda).$$

Proof. We may suppose $\emptyset \neq S \neq \Omega$. If S is intriguing, then $\{S, \Omega\backslash S\}$ is an equitable partition in two parts and hence it follows from Theorem 2.1.2 that χ_S can be written as a linear combination of two eigenvectors of A, one of which χ_Ω. Conversely, if S is a linear combination of χ_S and an eigenvector of v for the eigenvalue λ of Γ, then

$$\chi_S = \frac{|S|}{|\Omega|}\chi_\Omega + v.$$

Note that $A\chi_\Omega = k\chi_\Omega$ as A is k-regular. Hence:

$$\begin{aligned} A\chi_S &= A\left(\frac{|S|}{|\Omega|}\chi_\Omega + v\right) \\ &= \frac{|S|}{|\Omega|}k\chi_\Omega + \lambda v \\ &= \frac{|S|}{|\Omega|}k\chi_\Omega + \lambda\left(\chi_S - \frac{|S|}{|\Omega|}\chi_\Omega\right) \\ &= \left(\frac{|S|}{|\Omega|}(k-\lambda) + \lambda\right)\chi_S + \frac{|S|}{|\Omega|}(k-\lambda)(\chi_\Omega - \chi_S). \end{aligned}$$

□

2.2 Association schemes

2.2.1 Definitions

Bose and Shimamoto [16] introduced the notion of association schemes. A d-class *association scheme* on a finite non-empty set Ω is an ordered pair $(\Omega, \mathcal{R})$ with $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ a set of symmetric non-empty relations on Ω, such that the following axioms hold.

(i) R_0 is the identity relation.

(ii) $\mathcal{R}$ is a partition of Ω^2.

(iii) There are constants p_{ij}^k, known as the *intersection numbers*, such that for $(x, y) \in R_k$, the number of elements z in Ω for which $(x, z) \in R_i$ and $(z, y) \in R_j$ equals p_{ij}^k.

We give two simple and fundamental examples of association schemes (see for instance [23]).

(i) Let X be a set of size $v \geq 1$, and let Ω denote the set of all subsets of X of size k. We define the relation R_i as $\{(\omega_1, \omega_2) | |\omega_1 \cap \omega_2| = k - i\}$ with $0 \leq i \leq d = \min(k, v-k)$. Now $(\Omega, \{R_0, \ldots, R_d\})$ is a d-class association scheme, known as a *Johnson scheme*. This scheme is studied in design theory.

(ii) Let X be a set of size $q \geq 2$, and let Ω denote the Cartesian product X^d for some d. For every $i \in \{0, \ldots, d\}$, we define R_i such that $(\omega_1, \omega_2) \in R_i$ if and only if w_1 and w_2 differ in exactly i coordinates. Now $(\Omega, \{R_0, \ldots, R_d\})$ is a d-class association scheme, known as a *Hamming scheme*. This scheme is studied in coding theory.

2.2.2 The Bose-Mesner algebra of an association scheme

Proofs of the results in this subsection can be found in for instance [23, Chapter 2]. Consider an association scheme $(\Omega, \{R_0, \ldots, R_d\})$. For each R_i, we define its *adjacency matrix* A_i as the symmetric $(0,1)$-matrix with rows and columns indexed by the elements of Ω, such that $(A_i)_{\omega_1,\omega_2} = 1$ if $(\omega_1, \omega_2) \in R_i$ and $(A_i)_{\omega_1,\omega_2} = 0$ if not. The axioms of an association scheme now imply:

(i) A_0 is the identity matrix,

(ii) $A_0 + \cdots + A_d$ is the all-one matrix,

(iii) $A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$.

The last axiom implies that the vector space $\langle A_0, \ldots, A_d \rangle$ is closed under matrix multiplication, and forms a $(d+1)$-dimensional commutative algebra of symmetric matrices over $\mathbb{R}$, known as the *Bose-Mesner algebra* of the association scheme. We will usually use the notation A_i for the adjacency matrix for R_i.

An idempotent E in the Bose-Mesner algebra (i.e. $E^2 = E$) is *minimal* if it cannot be written as the sum of two non-zero idempotents.

Theorem 2.2.1. *The Bose-Mesner algebra of a d-class association scheme $(\Omega, \{R_0, \ldots, R_d\})$ has a unique basis $\{E_0, \ldots, E_d\}$ of minimal idempotents.*

Any two minimal idempotents E_i and E_j satisfy: $E_i E_j = \delta_{ij} E_i$. Every minimal idempotent defines an orthogonal projection, and the subspaces $\mathrm{Im}(E_j)$ form an orthogonal decomposition of $\mathbb{R}^\Omega$. These subspaces are the *strata* of the association scheme. Projection onto the all-one vector χ_Ω is always one of the minimal idempotents, and we will always denote it by E_0. The *matrix of eigenvalues* of the association scheme is the $(d+1) \times (d+1)$-matrix $P = (P_{ji})_{i,j=0\ldots d}$ with $A_i = \sum_{j=0}^{d} P_{ji} E_j, \forall i \in \{0, \ldots, d\}$. The reason for this term is

that if $v \in \mathrm{Im}(E_j)$, then $A_i v = A_i E_j v = P_{ji} E_j v = P_{ji} v$. Hence every non-zero vector in any stratum is an eigenvector for all A_i. The zeroeth column of P is an all-one column, and the zeroeth row consists of the valencies k_i of each relation R_i. The sum of entries in the zeroeth row is $|\Omega|$, and is zero for all other rows (see for instance Table 4.2 on page 70 for an example).

Note that matrix multiplication with respect to the basis $\{A_0, \ldots, A_d\}$ is rather complicated while it is very simple with respect to the basis $\{E_0, \ldots, E_d\}$. We will now introduce a second multiplication for which these two bases will switch roles. We will refer to entrywise multiplication as *Schur multiplication* and denote it by $\circ$. Since $A_i \circ A_j = \delta_{ij} A_i$, the Bose-Mesner algebra is certainly closed under this multiplication and $\{A_0, \ldots, A_d\}$ is a basis of minimal idempotents (i.e. $A_i \circ A_i = A_i$) with respect to it. Now we can also write $E_i \circ E_j = \frac{1}{|\Omega|} \sum_{k=0}^{d} q_{ij}^k E_k$ for certain real numbers q_{ij}^k known as the *Krein parameters* of the association scheme. Scott [127] proved that these parameters q_{ij}^k must be non-negative, yielding important conditions on the parameters of association schemes known as the *Krein conditions*.

We will also write $u \circ v$ for the entrywise product of two vectors $u, v \in \mathbb{R}^\Omega$.

The *dual matrix of eigenvalues* of the association scheme is the $(d+1) \times (d+1)$-matrix $Q = (Q_{ij})_{i,j=0\ldots d}$ with $E_j = \frac{1}{|\Omega|} \sum_{i=0}^{d} Q_{ij} A_i, \forall j \in \{0, \ldots, d\}$. The zeroeth column of Q is an all-one column, and the zeroeth row consists of the ranks m_j of each idempotent E_j. The sum of entries in the zeroeth row is $|\Omega|$, and is zero for all other rows.

Table 2.1 exhibits a nice duality between ordinary and Schur multiplication, and between the parameters p_{ij}^k and q_{ij}^k. The matrix J is the all-one matrix with $|\Omega|$ rows and columns.

If Δ_k and Δ_m denote diagonal $(d+1) \times (d+1)$-matrices with $(\Delta_k)_{ii} = k_i$ and $(\Delta_m)_{jj} = m_j$, then P and Q satisfy the following relations:

$$PQ = |\Omega| I, \quad \Delta_k Q = P^T \Delta_m.$$

These relations yield the following handy expression for minimal idempotents and their ranks.

Lemma 2.2.2. *Suppose E_j is a minimal idempotent with rank m_j of an association scheme $(\Omega, \{R_0, \ldots, R_d\})$, such that every relation R_i has eigenvalue*

λ_i for E_j and has valency k_i. Then:

$$E_j = \frac{m_j}{|\Omega|}\left(\frac{\lambda_0}{k_0}A_0 + \cdots + \frac{\lambda_d}{k_d}A_d\right),$$

and

$$m_j = \frac{|\Omega|}{\sum_{i=0}^d \lambda_i^2/k_i} = \frac{\sum_{i=0}^d k_i}{\sum_{i=0}^d \lambda_i^2/k_i}.$$

A_iA_j	$= \sum_{k=0}^d p_{ij}^k A_k$	$E_i \circ E_j$	$= \frac{1}{\lvert\Omega\rvert}\sum_{k=0}^d q_{ij}^k E_k$
$A_i \circ A_j$	$= \delta_{ij}A_i$	E_iE_j	$= \delta_{ij}E_i$
A_iE_j	$= P_{ji}E_j$	$E_j \circ A_i$	$= \frac{1}{\lvert\Omega\rvert}Q_{ij}A_i$
A_iE_0	$= k_iE_0$	$E_j \circ A_0$	$= \frac{m_j}{\lvert\Omega\rvert}A_0$
$A_0 + \cdots + A_d$	$= J$	$E_0 + \cdots + E_d$	$= I$
A_0	$= I$	E_0	$= J/\lvert\Omega\rvert$
$p_{ij}^k k_k$	$= p_{kj}^i k_i$	$q_{ij}^k m_k$	$= q_{kj}^i m_i$

Table 2.1: Duality between parameters of association schemes

2.2.3 P- and Q-polynomial association schemes

Roughly speaking, association schemes are P- and Q-polynomial if they have a meaningful ordering of their relations and minimal idempotents, respectively.

Let $(\Omega, \{R_0, \ldots, R_d\})$ be an association scheme. We say it is *metric* with respect to the ordering $R_0, \ldots, R_d$ if $p_{ij}^k \neq 0$ implies $k \leq i+j$ and $p_{ij}^{i+j} \neq 0$ if $i+j \leq d$. Dually, we say it is *cometric* with respect to the ordering $E_0, \ldots, E_d$ of its minimal idempotents if $q_{ij}^k \neq 0$ implies $k \leq i+j$ and $q_{ij}^{i+j} \neq 0$ if $i+j \leq d$. We refer to [23, Section 2.7] for proofs of the following two theorems.

Theorem 2.2.3. *For any association scheme $(\Omega, \{R_0, \ldots, R_d\})$ the following are equivalent:*

(i) *the scheme is metric with ordering $R_0, \ldots, R_d$,*

(ii) $p_{1i}^{i+1} \neq 0, p_{1i}^k = 0$ *for* $k > i+1$ *and* $0 \leq i \leq d-1$,

(iii) *for every* $i \in \{0, \ldots, d\}$*, there is a real polynomial* p_i *of degree* i *such that* $P_{ji} = p_i(P_{j1})$ *for every* $j \in \{0, \ldots, d\}$*,*

(iv) $(\omega_1, \omega_2) \in R_i$ *if and only if* $d(\omega_1, \omega_2) = i$ *with respect to* R_1*.*

Theorem 2.2.4. *For any association scheme* $(\Omega, \{R_0, \ldots, R_d\})$ *with minimal idempotents* $E_0, \ldots, E_d$*, the following are equivalent:*

(i) *the scheme is cometric with ordering* $E_0, \ldots, E_d$*,*

(ii) $q_{1i}^{i+1} \neq 0, q_{1i}^{k} = 0$ *for* $k > i+1$ *and* $0 \leq i \leq d-1$*,*

(iii) *for every* $j \in \{0, \ldots, d\}$*, there is a real polynomial* q_j *of degree* j *such that* $Q_{ij} = q_j(Q_{i1})$ *for every* $i \in \{0, \ldots, d\}$*.*

The two previous theorems justify the alternative terms *P-polynomial* and *Q-polynomial* association schemes for metric and cometric schemes, respectively, which were introduced by Delsarte [65].

It is worth noting that schemes can be (co)metric with respect to more than one ordering (see for instance [13, Section III.6]).

2.2.4 Subsets and vectors in association schemes

We will now give an introduction to the concepts and results by Delsarte regarding subsets in association schemes.

Definition 2.2.5. *Consider an association scheme* $(\Omega, \{R_0, \ldots, R_d\})$ *with* $v \in \mathbb{R}^{\Omega}$*. The inner distribution of* v*, if* $v \neq 0$*, is the* $(d+1)$*-vector* $\mathbf{a}$ *with:*

$$\mathbf{a}_i := \frac{v^T A_i v}{v^T v}, \ \forall i \in \{0, \ldots, d\}.$$

The outer distribution of v *is the* $|\Omega| \times (d+1)$*-matrix* $B = (B_{x,i})$ *with:*

$$B_{x,i} := \sum_{(x,x') \in R_i} v_{x'} = (\chi_{\{x\}})^T A_i v, \ \forall i \in \{0, \ldots, d\}, \forall x \in \Omega.$$

Note that the i-th column of B is given by $A_i v$, and $\mathbf{a}$ by $(v^T B)/(v^T v)$.

We warn the reader that different normalizations are used in other works. We will also write B_x for the row of the outer distribution B, corresponding to $x \in \Omega$.

The vector $\mathbf{a}Q$ is called the *MacWilliams transform* of $\mathbf{a}$.

Lemma 2.2.6. [23, Lemma 2.5.1] *Let $(\Omega, \{R_0, \ldots, R_d\})$ be an association scheme. Suppose every R_i has valency k_i and eigenvalue λ_i for some idempotent E_j with rank m_j. Consider a vector $v \neq 0$ in $\mathbb{R}^\Omega$ with inner distribution $\boldsymbol{a}$. Then:*

$$(\mathbf{a}Q)_j = \frac{|\Omega|}{v^T v} v^T E_j v = m_j \left(\frac{\lambda_0}{k_0} \mathbf{a}_0 + \cdots + \frac{\lambda_d}{k_d} \mathbf{a}_d \right).$$

The following result is due to Delsarte [65, 67]. It yields non-negativeness of the MacWilliams transform and discusses the meaning of its entries equal to zero.

Theorem 2.2.7. *Consider an association scheme $(\Omega, \{R_0, \ldots, R_d\})$. If every R_i has valency k_i and eigenvalue λ_i for some idempotent E_j, then for any non-zero vector $v \in \mathbb{R}^\Omega$ the inner distribution $\mathbf{a}$ of v satisfies:*

$$\frac{\lambda_0}{k_0} \mathbf{a}_0 + \cdots + \frac{\lambda_d}{k_d} \mathbf{a}_d \geq 0,$$

with equality if and only if $E_j v = 0$. In that case, the outer distribution B of v satisfies:

$$\frac{\lambda_0}{k_0} B_{x,0} + \cdots + \frac{\lambda_d}{k_d} B_{x,d} = 0,$$

for every $x \in \Omega$.

Proof. Since $v^T E_j v \geq 0$ with equality if and only if $E_j v = 0$, the inequality follows immediately from Lemma 2.2.6. Working out $(\chi_{\{x\}})^T E_j v = 0$ by use of Lemma 2.2.2 and applying $B_{x,i} = (\chi_{\{x\}})^T A_i v$ yields the last part. □

We define the inner distribution $\mathbf{a}$ of a non-empty subset S of Ω as simply the inner distribution of its characteristic vector, and hence:

$$\mathbf{a}_i = \frac{1}{|S|} |(S \times S) \cap R_i|, \forall i \in \{0, \ldots, d\}.$$

The *degree* of S is defined as the number of non-zero indices i with $\mathbf{a}_i \neq 0$.

In an association scheme with minimal idempotents $E_0, \ldots, E_d$, the *dual degree set* of a vector v in $\mathbb{R}^\Omega$ is the set of indices $j \in \{1, \ldots, d\}$ such that $E_j v \neq 0$. The *dual degree* of S is the cardinality of its dual degree set.

The outer distribution B of a subset S of Ω is simply the outer distribution of its characteristic vector, and hence:

$$B_{x,i} = |\{x' \in S | (x, x') \in R_i\}|.$$

Note that for a non-empty subset S, the entries of the inner distribution and the entries of each row in the outer distribution, must add up to $|S|$.

Lemma 2.2.8. [23, Section 2.5] *Let $(\Omega, \{R_0, \ldots, R_d\})$ be a d-class association scheme with minimal idempotents $E_0, \ldots, E_d$ and a subset $S \subseteq \Omega$ with inner distribution $\boldsymbol{a}$. For every $x \in \Omega$ and $i \in \{0, \ldots, d\}$, the following holds:*

$$B_{x,i} = \sum_{j=0}^{d} (E_j \chi_S)_x P_{ji}.$$

The row span of B is precisely the space spanned by the rows of P with an index j such that $(\boldsymbol{a}Q)_j \neq 0$.

If the association scheme is metric, then the *width* (with respect to the metric ordering of the relations) of a non-empty subset S with inner distribution $\mathbf{a}$ is defined as $\max\{i | \mathbf{a}_i \neq 0\}$ (i.e. the maximum distance between elements of S with respect to the first non-trivial relation). If the scheme is cometric, then the *dual width* (with respect to the cometric ordering of the idempotents) of a non-empty subset is the maximum index in its dual degree set.

Corollary 2.2.9. *Suppose S is a non-empty clique of R_i $(i > 0)$ in an association scheme $(\Omega, \{R_0, \ldots, R_d\})$, with k and λ the valency and eigenvalue for some non-trivial idempotent E, respectively. If $\lambda < 0$ then:*

$$|S| \leq 1 - k/\lambda,$$

and $E\chi_S = 0$ if and only if $\lambda < 0$ and $|S| = 1 - k/\lambda$. In that case, the outer distribution B of S satisfies:

$$\frac{\lambda_0}{k_0} B_{x,0} + \cdots + \frac{\lambda_d}{k_d} B_{x,d} = 0, \forall x \in \Omega.$$

Proof. The inner distribution $\mathbf{a}$ of S satisfies $\mathbf{a}_0 = 1$ and $\mathbf{a}_i = |S| - 1$, while all other entries are zero. The result now follows immediately from Theorem 2.2.7. $\square$

Two vectors v_1 and v_2 in $\mathbb{R}^\Omega$ are said to be *design-orthogonal* if their dual degree sets are disjoint. Dual degree sets and design-orthogonality of subsets is again defined by demanding these properties for their characteristic vectors.

Lemma 2.2.10. *Consider an association scheme* $(\Omega, \{R_0, \ldots, R_d\})$.

(i) *If* v_1 *and* v_2 *are design-orthogonal vectors, then:*

$$v_1^T v_2 = \frac{((v_1)^T \chi_\Omega) \cdot ((v_2)^T \chi_\Omega)}{|\Omega|}.$$

(ii) *If* S_1 *and* S_2 *are design-orthogonal subsets, then:*

$$|S_1 \cap S_2| = \frac{|S_1| \cdot |S_2|}{|\Omega|}.$$

Proof.

(i) Let $E_0, \ldots, E_d$ denote the minimal idempotents. Each vector $v \in \mathbb{R}^\Omega$ can be decomposed orthogonally as $v = E_0 v + \cdots + E_d v$ with $E_0 v = ((v^T \chi_\Omega)/|\Omega|)\chi_\Omega$. Working out $(v_1)^T v_2 = (E_0 v_1)^T (E_0 v_2)$ now yields the first part.

(ii) This follows from (i) by substituting χ_{S_1} and χ_{S_2} for v_1 and v_2, respectively, and by using the identities $(\chi_{S_1})^T \chi_{S_2} = |S_1 \cap S_2|$ and $(\chi_{S_i})^T \chi_\Omega = |S_i|$ with $i = 1, 2$. $\square$

The previous lemma motivates the following definitions.

Definition 2.2.11. *Let* $(\Omega, \{R_0, \ldots, R_d\})$ *be an association scheme, and let* T *be a subset of* $\{1, \ldots, d\}$. *We say* $S \subseteq \Omega$ *is a* T*-design if its dual degree set is disjoint from* T*, and a* T*-antidesign if its dual degree set is a subset*[2] *of* T.

[2]Some authors use a slightly different definition of T-antidesigns, and demand that the dual degree set is exactly T.

The concepts of T-designs and design-orthogonality were introduced by Delsarte [65, 67], and T-antidesigns were introduced by Roos [126]. In cometric schemes, we will also refer to T-designs and T-antidesigns with $T = \{1, \ldots, t\}$ as simply t-designs and t-antidesigns, respectively, if the cometric ordering of the minimal idempotents is clear.

There is also a very general converse of the last lemma. We first need a technical result.

Lemma 2.2.12. *Let $(\Omega, \{R_0, \ldots, R_d\})$ be an association scheme with idempotents $E_0, \ldots, E_d$. Let $v_1, \ldots, v_m$ be (possibly equal) vectors in $\mathbb{R}^\Omega$ and let M denote the $(|\Omega| \times m)$-matrix with these vectors as columns. If MM^T is in the Bose-Mesner algebra, then the column span of M is the sum of those* $\mathrm{Im}(E_j)$ *with $E_j v_k \neq 0$ for at least one $k \in \{1, \ldots, m\}$.*

Proof. If MM^T is in the Bose-Mesner algebra, we can write it as $\sum_{j=0}^d \mu_j E_j$. Now $\mathrm{Im}(M) = \mathrm{Im}(MM^T)$ is the span of those $\mathrm{Im}(E_j)$ with $\mu_j \neq 0$. We can now write:

$$\mu_j = 0 \Longleftrightarrow E_j MM^T = 0 \Longleftrightarrow E_j M = 0,$$

and the latter is equivalent to $E_j v_k = 0$ for every $k \in \{1, \ldots, m\}$. □

We now state a result by De Bruyn and Suzuki [57] in a slightly more general form.

Lemma 2.2.13. *Let $S_1, \ldots, S_m$, denote (possibly equal) subsets of the same size X in an association scheme $(\Omega, \{R_0, \ldots, R_d\})$. Suppose there are constants a_i such that if $(\omega_1, \omega_2) \in R_i$, there are exactly a_i indices λ with $\omega_1, \omega_2 \in S_\lambda$. A subset S intersects every S_λ in the same number of elements if and only if S is design-orthogonal to every S_λ.*

Proof. Let M denote the matrix with the characteristic vector χ_{S_i} as i-th column. Note that the assumption implies $MM^T = a_0 A_0 + \cdots + a_d A_d$.

For a subset S, there will be such a constant size of intersection c if and only if

$$\left(\chi_S - \frac{c}{X}\chi_\Omega\right)^T \chi_{S_\lambda} = 0, \forall \lambda \in \{1, \ldots, m\}.$$

This is equivalent to orthogonality of $\chi_S - \frac{c}{X}\chi_\Omega$ to the span of all χ_{S_λ}. Hence it follows from Lemma 2.2.12 that it must be orthogonal to all strata $\mathrm{Im}(E_j)$ with $j = 0$ or in the dual degree set of any of the S_λ. As the projection of χ_S onto $\langle \chi_\Omega \rangle$ must be given by $\frac{|S|}{|\Omega|}\chi_\Omega$, we also see that in that case the constant c must be $|S|X/|\Omega|$. □

We can say something about the dual degree sets of intersections as well.

Theorem 2.2.14. [106, Theorem 1] *Consider an association scheme* $(\Omega, \{R_0, \ldots, R_d\})$ *with minimal idempotents* $E_0, \ldots, E_d$. *If* $u \in \mathrm{Im}(E_i), v \in \mathrm{Im}(E_j)$ *and* $q_{ij}^k = 0$, *then* $E_k(u \circ v) = 0$.

Martin [106] observed that the above property has combinatorial applications.

Corollary 2.2.15. *Consider an association scheme* $(\Omega, \{R_0, \ldots, R_d\})$ *with a fixed cometric ordering. If* S *is a* t*-antidesign and* S' *is a* t'*-antidesign, then* $S \cap S'$ *is a* $(t + t')$*-antidesign.*

Proof. First note that the characteristic vector of $S \cap S'$ is given by $\chi_S \circ \chi_{S'}$. The cometric property of the ordering implies that $q_{ij}^k = 0$ if $k > i + j$. The result now follows from Theorem 2.2.14. □

2.3 Distance-regular graphs

We have already defined P-polynomial association schemes in Subsection 2.2.3 as those schemes with a certain type of ordering of the relations. The first non-trivial relation then determines the entire scheme, and the corresponding graph is known as a distance-regular graph.

2.3.1 Definitions

Definition 2.3.1. *A finite connected graph* Γ *with diameter* d *is distance-regular if there are numbers* b_i *and* c_i, *known as the intersection numbers, such that for any two vertices* x *and* y *at distance* i *in* Γ:

$$\begin{aligned} |\Gamma_{i-1}(x) \cap \Gamma_1(y)| &= c_i, \text{ if } i \in \{1, \ldots, d\}, \\ |\Gamma_{i+1}(x) \cap \Gamma_1(y)| &= b_i, \text{ if } i \in \{0, \ldots, d-1\}. \end{aligned}$$

Note that such a graph is regular with valency $k = b_0$ and that $c_1 = 1$. We will always use the notation b_i and c_i as in the above definition. We will also write a_i for $|\Gamma_i(x) \cap \Gamma_1(y)|$, and it is clear that $a_i + b_i + c_i = k$ for every $i \in \{1, \ldots, d-1\}$.

The following theorem explains the link between distance-regular graphs and metric association schemes (see for instance [23, Section 4.1]).

Theorem 2.3.2. *Let Γ be a connected graph of diameter d with set of vertices Ω, and let (x, y) be in R_i if and only if $d(x, y) = i$ in Γ. Then Γ is distance-regular if and only if $(\Omega, \{R_0, \ldots, R_d\})$ is an association scheme, and in that case, the scheme is metric with respect to the ordering $R_0, \ldots, R_d$.*

For a distance-regular graph Γ, we will refer to $(\Omega, \{R_0, \ldots, R_d\})$ as in Theorem 2.3.2 as *the association scheme defined by* Γ. We will often assume that the relations of this scheme are ordered in this natural way.

The relations R_1 from the Johnson and Hamming schemes as defined in Subsection 2.2.1 yield distance-regular graphs. They are known as the *Johnson graphs*, denoted by $J(v, k)$, and the *Hamming graphs*, denoted by $H(n, q)$, and they define the Johnson and Hamming schemes, respectively (with the ordering of the relations as given in Subsection 2.2.1).

In a distance-regular graph Γ of diameter d, every vertex x yields an equitable partition $\{\Gamma_0(x), \Gamma_1(x), \ldots, \Gamma_d(x)\}$. This allows us to compute the eigenvalues of Γ in a relatively easy way.

Theorem 2.3.3. [23, Section 4.1B] *A distance-regular graph Γ with diameter d has exactly $d+1$ distinct eigenvalues, and they are precisely the eigenvalues of the tridiagonal quotient matrix L of the equitable partition $\{\Gamma_0(x), \Gamma_1(x), \ldots, \Gamma_d(x)\}$ for any vertex x:*

$$L = \begin{pmatrix} 0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & O & \\ & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & O & & \cdot & \cdot & b_{d-1} \\ & & & & c_d & a_d \end{pmatrix}.$$

The eigenvectors of Γ are eigenvectors for all distance-i relations.

We say a distance-regular graph is *Q-polynomial* if it defines a P-polynomial scheme which is also Q-polynomial.

A graph with diameter d is *distance-transitive* if for every two ordered pairs of vertices (ω_1, ω_2) and (ω_1', ω_2') with $d(\omega_1, \omega_2) = d(\omega_1', \omega_2')$, there is a graph automorphism sending the first to the second. All finite distance-transitive graphs are distance-regular, but the converse is not true (the graph on 16 vertices constructed by Shrikhande [130] is the smallest counterexample in terms of the number of vertices).

Definition 2.3.4. *A graph Γ on v vertices is strongly regular if there are integers k, λ, μ with $v, k \geq 1$ such that:*

(i) *every vertex has exactly k neighbours,*

(ii) *every two adjacent vertices have exactly λ common neighbours,*

(iii) *every two non-adjacent vertices have exactly μ common neighbours.*

We then say Γ is an $\mathrm{srg}(v, k, \lambda, \mu)$.

The complement of an $\mathrm{srg}(v, k, \lambda, \mu)$ is an

$$\mathrm{srg}(v, v-k-1, v-2k+\mu-2, v-2k+\lambda).$$

The parameters of an $\mathrm{srg}(v, k, \lambda, \mu)$ satisfy: $k(k-\lambda-1) = (v-k-1)\mu$.

If $\mu = 0$, then an $\mathrm{srg}(v, k, \lambda, \mu)$ is a disjoint union of cliques, and if $\mu = k$, then it is the complement of such a graph. If $0 < \mu < k < v-1$, then both the strongly regular graph Γ and its complement are connected. The connected non-complete strongly regular graphs (i.e. with $k < v-1$ and $\mu > 0$) are precisely the distance-regular graphs of diameter two.

The graphs $J(n, 2)$ with $n \geq 4$, known as *triangular graphs* and denoted by $T(n)$, are strongly regular graphs:

$$\mathrm{srg}(n(n-1)/2, 2(n-2), n-2, 4).$$

Another well known example is the *Petersen graph*, which is an $\mathrm{srg}(10, 3, 0, 1)$ and can be constructed as the complement of $T(5)$.

We refer to [23, 27] for proofs and more information on strongly regular graphs.

Theorem 2.3.5. *If Γ is an $\mathrm{srg}(v, k, \lambda, \mu)$ with $0 < \mu < k < v-1$, then it has precisely 3 distinct eigenvalues: k and the roots x_1, x_2 of this equation in x:*

$$x^2 + (\mu - \lambda)x + (\mu - k) = 0,$$

and the multiplicities are respectively:

$$1, f = \frac{k(x_2+1)(k-x_2)}{(k+x_1x_2)(x_2-x_1)}, g = \frac{k(x_1+1)(k-x_1)}{(k+x_1x_2)(x_1-x_2)}.$$

The absolute bounds must hold:

$$v \leq f(f+3)/2 \text{ and } v \leq g(g+3)/2.$$

Many incidence geometries give rise to strongly regular graphs. For a more general version of the following result, we refer to [19].

Lemma 2.3.6. [75, Lemma 3.2] *The point graph of an SPBIBD* (P, B, I) *with parameters* $(v, b, r, k, \lambda_1, 0)$ *of type* $(k-1, \alpha)$ *with* $1 \leq \alpha < k$ *and* $1 \leq \lambda_1 < r$ *is an* $\mathrm{srg}(v, r(k-1)/\lambda_1, \lambda, \mu)$ *with*

$$\begin{aligned} \lambda &= (k-2) + \frac{(r-\lambda_1)(\alpha-1)}{\lambda_1}, \\ \mu &= \frac{r\alpha}{\lambda_1}. \end{aligned}$$

The vector space $\mathbb{R}^P$ *decomposes into three eigenspaces:*

$$\mathbb{R}^P = \langle \chi_P \rangle \perp (\mathrm{Im}(C^T) \cap \langle \chi_P \rangle^\perp) \perp \ker(C),$$

with eigenvalues $r(k-1)/\lambda_1$, $k-\alpha-1$ *and* $-r/\lambda_1$, *respectively, where* C *denotes the incidence matrix with columns and rows indexed by the points and lines, respectively.*

In particular, it follows that the point graph of a partial geometry $\mathrm{pg}(s, t, \alpha)$ is an $\mathrm{srg}((s+1)(1+st/\alpha), s(t+1), s-1+t(\alpha-1), (t+1)\alpha)$ with eigenvalues $s(t+1), s-\alpha$ and $-t-1$.

2.3.2 Codes in distance-regular graphs

A *code* in a distance-regular graph Γ with set of vertices Ω and of diameter d is a non-empty subset of vertices. The *minimum distance* $\delta(C)$ of a code C is

$$\min\{d(x, y) | x, y \in C, x \neq y\} \text{ if } |C| > 1,$$

and is $2d+1$ if $|C| = 1$.

The *covering radius* of the code is

$$t(C) = \max\{d(x, C) | x \in \Omega\}.$$

Hence the covering radius is the minimal radius e such that every vertex in Γ is in at least one sphere of radius e around some vertex in C. It can easily be seen that $\delta(C) \leq 2t(C) + 1$, and equality occurs if and only if these spheres partition the vertex set of Γ. In that case, we say C is a *perfect* e*-code*.

We say a code C is *s-regular* if every entry $B_{x,i}$ of its outer distribution B with $d(x, C) = l \leq s$ only depends on i and l. If C is $t(C)$-regular, we say it is *completely regular*. We say a code C in Γ is *completely transitive* if there is an automorphism group of Γ with the sets $C_i = \{x|d(x, C) = i\}$ as orbits. All completely transitive codes are completely regular.

Neumaier gave the following equivalent definition of completely regular codes.

Theorem 2.3.7. [110] *Let C be a code in a distance-regular graph Γ with covering radius $t(C)$, and with $C_i = \{x|d(x, C) = i\}$ for every $i \in \{0, \ldots, t(C)\}$. Then C is a completely regular code if and only if $\{C_0, \ldots, C_{t(C)}\}$ is an equitable partition.*

The simplest example of a completely regular code in any distance-regular graph is a singleton.

In particular, note that the completely regular codes with covering radius 1 are precisely the proper non-empty intriguing sets.

The following lemma will allow us to determine the quotient matrix, given all possible rows of the outer distribution. For any completely regular code C with covering radius $t(C)$ and outer distribution B, we define the *reduced outer distribution* as the $(t(C) + 1) \times (d + 1)$-matrix B', with the i-th row of B' equal to any row of B for a vertex at distance i from C.

Lemma 2.3.8. *Consider a distance-regular graph Γ of diameter d on Ω, and write L for its tridiagonal matrix from Theorem 2.3.3. Let C be a completely regular code in Γ with reduced outer distribution B'. If $L_{t(C)}$ and $B'_{t(C)}$ are the submatrices of L and B' consisting of columns 0 up to $t(C)$, respectively, then the quotient matrix L_C of the equitable partition $\{C_0, \ldots, C_{t(C)}\}$ with $C_i = \{x|d(x, C) = i\}$ is given by:*

$$L_C = B' L_{t(C)} (B'_{t(C)})^{-1}.$$

Proof. Since L is the quotient matrix of the equitable partition with respect to the completely regular code $\{x\}$ for any vertex x, Theorem 2.1.2 yields:

$$A_1(A_0 v| \ldots |A_d v) = (A_0 v| \ldots |A_d v)L,$$

for any $v \in \mathbb{R}^\Omega$, and hence in particular:

$$A_1(A_0 v| \ldots |A_{t(C)} v) = (A_0 v| \ldots |A_d v)L_{t(C)}.$$

The outer distribution B of C is given by:

$$B = (A_0\chi_C|\dots|A_d\chi_C) = (\chi_{C_0}|\dots|\chi_{C_{t(C)}})B'.$$

and in particular:

$$(A_0\chi_C|\dots|A_{t(C)}\chi_C) = (\chi_{C_0}|\dots|\chi_{C_{t(C)}})B'_{t(C)}.$$

Note that $B'_{t(C)}$ is a square upper diagonal matrix with non-zero elements on the diagonal, and hence invertible. We can now write:

$$\begin{aligned} A_1(\chi_{C_0}|\dots|\chi_{C_{t(C)}}) &= A_1(A_0\chi_C|\dots|A_{t(C)}\chi_C)(B'_{t(C)})^{-1} \\ &= (A_0\chi_C|\dots|A_d\chi_C)L_{t(C)}(B'_{t(C)})^{-1} \\ &= (\chi_{C_0}|\dots|\chi_{C_{t(C)}})B'L_{t(C)}(B'_{t(C)})^{-1}, \end{aligned}$$

and now it follows from Theorem 2.1.2 that $B'L_{t(C)}(B'_{t(C)})^{-1}$ is the desired quotient matrix. □

The following theorem links the dual degree, covering radius and minimum distance of a code to certain regularity properties.

Theorem 2.3.9. [65, pp. 60-68] *Let C be a code in a distance-regular graph with dual degree $r(C)$.*

(i) *The rank of the outer distribution B of C is equal to $r(C)+1$.*

(ii) *The bound $t(C) \leq r(C)$ holds, with equality if C is completely regular.*

(iii) *If $s = \delta(C) - r(C) \geq 0$, then C is s-regular.*

(iv) *If C is s-regular and $s \geq r(C)-1$, and in particular if $\delta(C) \in \{2r(C)-1, 2r(C), 2r(C)+1\}$, then C is completely regular.*

(v) *C is a perfect code if and only if $\delta(C) = 2r(C)+1$. In that case C is completely regular, and $\sum_{i=0}^{r(C)} P_{ji} = 0$ holds precisely for those non-zero indices j in the dual degree set of C.*

Remark 2.3.10. Theorem 2.3.9(v) is known as Lloyd's Theorem (proved in [103] for a specific case in coding theory), and imposes strong conditions on the parameters of a distance-regular graph for a perfect code to exist. In particular, it says that perfect 1-codes cannot exist if -1 is not an eigenvalue of the distance-regular graph.

One of the main themes in this thesis is the link between the dual degree set of a subset and its geometric properties, if the graph is related to some geometric structure. We now give a useful result on intriguing sets in certain strongly regular graphs. We already described the eigenspaces of such graphs in Lemma 2.3.6.

Theorem 2.3.11. [75, Corollary 2.3 and Theorem 3.4] *Consider an SPBIBD* (P, B, I) *with parameters* $(v, b, r, k, \lambda_1, 0)$ *of type* $(k-1, \alpha)$ *with* $1 \leq \alpha < k$ *and* $1 \leq \lambda_1 < r$. *Let* C *denote the incidence matrix as in Lemma 2.3.6. Suppose* S *is a set of points.*

(i) χ_S *can be written as a linear combination of* χ_P *and an eigenvector of* $k - \alpha - 1$ *if and only if* $\chi_S \in \mathrm{Im}(C^T)$,

(ii) χ_S *can be written as a linear combination of* χ_P *and an eigenvector of* $-r/\lambda_1$ *if and only if* $\chi_S \in \langle \chi_\Omega \rangle \perp \ker(C)$, *and if and only if every line contains the same number of points in* S.

Every intriguing set must be of one of these two types, and every two intriguing sets S_1 *and* S_2 *of different types intersect in exactly* $|S_1| \cdot |S_2|/|P|$ *points.*

Note that the previous theorem in particular applies to partial geometries $\mathrm{pg}(s, t, \alpha)$, including the generalized quadrangles of order (s, t).

2.3.3 Distance-regular graphs with classical parameters

Brouwer, Cohen and Neumaier [23] observed that many well-known distance-regular graphs have parameters that can be expressed in terms of only four parameters. They introduced the terms classical graphs and classical parameters. We will give a construction of a certain type of such graphs in Section 6.7.

Definition 2.3.12. *A distance-regular graph* Γ *with diameter* d *has classical parameters* (d, b, α, β) *if:*

$$\begin{aligned} b_i &= \left(\begin{bmatrix} d \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b \right) \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_b \right), \\ c_i &= \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b \right), \end{aligned}$$

with $\left[{i \atop 1}\right]_b = b^{i-1} + \cdots + 1$ *if* $i \geq 1$, *and* $\left[{0 \atop 1}\right]_b = 0$.

Remark 2.3.13. One can prove that if Γ has classical parameters (d, b, α, β) with $d \geq 3$, then b is an integer $\neq 0, -1$ (see [23, Proposition 6.2.1]).

The reader should be warned that it is possible for a distance-regular graph to have more than one set of classical parameters (see for instance Theorem 4.1.8).

We give two examples of well-known graphs with classical parameters (see [23, Table 6.1] for many more examples).

(i) The Johnson graph $J(n, k)$ with

$$(d, b, \alpha, \beta) = (\min(k, n-k), 1, 1, \max(k, n-k)).$$

(ii) The Hamming graph $H(n, q)$ with

$$(d, b, \alpha, \beta) = (n, 1, 0, q-1).$$

Theorem 2.3.14. [23, Corollary 8.4.2] *Every distance-regular graph with classical parameters* (d, b, α, β) *is* Q*-polynomial, and the eigenvalues in the corresponding ordering are given by:*

$$\begin{bmatrix} d-j \\ 1 \end{bmatrix}_b \left(\beta - \alpha \begin{bmatrix} j \\ 1 \end{bmatrix}_b\right) - \begin{bmatrix} j \\ 1 \end{bmatrix}_b, j \in \{0, \ldots, d\}.$$

2.4 Spherical designs and association schemes

We will refer to the set of vectors in $\mathbb{R}^m$ with Euclidean norm 1 as the unit sphere $\mathbb{S}^{m-1}$.

Spherical designs were introduced by Delsarte, Goethals and Seidel [69], and are closely linked to association schemes.

Definition 2.4.1. *A finite non-empty subset* X *of* $\mathbb{S}^{m-1}$ *with* $A(X) = \{\langle x_1, x_2 \rangle | x_1 \neq x_2 \in X\}$ *is an* $|A(X)|$*-distance set with angle set* $A(X)$.

Definition 2.4.2. *A finite non-empty subset X of $\mathbb{S}^{m-1}$ is a spherical t-design if*

$$\frac{\int_{\mathbb{S}^{m-1}} f(u)du}{\int_{\mathbb{S}^{m-1}} 1du} = \frac{1}{|X|}\sum_{x\in X} f(x)$$

holds for all polynomials $f \in \mathbb{R}[u_1, \ldots, u_m]$ of degree at most t.

See [12] for several equivalent definitions and much more information on spherical t-designs.

Theorem 2.4.3. [69, Theorem 6.6] *If $X \subset \mathbb{S}^{m-1}$, $m \geq 2$, is an s-distance set and a spherical t-design, then $t \leq 2s$ and*

$$|X| \leq \binom{m+s-1}{m-1} + \binom{m+s-2}{m-1},$$

and X is a spherical $2s$-design if and only if the last bound is attained.

We now explain the link between association schemes and spherical designs. We first show how to construct an s-distance set with $s \leq d$ from a d-class association scheme.

Lemma 2.4.4. *Let $(\Omega, \{R_0, \ldots, R_d\})$ be an association scheme, with valencies k_i for R_i and corresponding eigenvalue λ_i for some fixed minimal idempotent $E \neq E_0$ of rank m. The image X of the map*

$$\omega \to \sqrt{\frac{|\Omega|}{m}} E\chi_{\{\omega\}}$$

is a set of unit vectors, with inner product λ_i/k_i between the images of ω_1 and ω_2 if $(\omega_1, \omega_2) \in R_i$.

Proof. Suppose $(\omega_1, \omega_2) \in R_i$. Note that $(\chi_{\{\omega_1\}})^T A_j \chi_{\{\omega_2\}} = \delta_{ij}$. Lemma 2.2.2 yields:

$$E = \frac{m}{|\Omega|}\left(\frac{\lambda_0}{k_0}A_0 + \cdots + \frac{\lambda_d}{k_d}A_d\right).$$

Hence $(E\chi_{\{\omega_1\}})^T(E\chi_{\{\omega_2\}}) = (\chi_{\{\omega_1\}})^T E\chi_{\{\omega_2\}} = \frac{m}{|\Omega|}\frac{\lambda_i}{k_i}$.

We conclude that all column vectors of E have squared length $\frac{m}{|\Omega|}$, and their inner products are $\frac{m}{|\Omega|}(\lambda_i/k_i)$ if $(\omega_1, \omega_2) \in R_i$. □

Remark 2.4.5. Because of Lemma 2.4.4, the ratios λ_i/k_i are often referred to as the *cosines* with respect to a minimal idempotent.

Note that if some of the ratios λ_i/k_i are equal, two different relations might correspond with the same angle between unit vectors. However, all ratios must be different if E is the first non-trivial idempotent in a cometric ordering.

We say a minimal idempotent is *non-degenerate* if exactly one of the cosines is equal to 1. It is precisely in this case that all its column vectors are different.

We already mentioned the Krein conditions $q_{ij}^k \geq 0$ in Subsection 2.2.2. The following result treats a particular case of equality.

Theorem 2.4.6. [38, Proposition 4.1] *Consider a d-class association scheme $(\Omega, \{R_0, \ldots, R_d\})$ with minimal idempotents $E_0, \ldots, E_d$, and let X be the set of normalized column vectors of an idempotent $E_j \neq E_0$ of rank m_j, seen as embedded in $\mathbb{S}^{m_j-1}$. Then X is a spherical 2-design, and it is a spherical 3-design if and only if $q_{jj}^j = 0$.*

2.5 Permutation groups and modules

Bannai and Ito described "Algebraic Combinatorics" as "group theory without groups" in the Preface of [13]. Indeed, in most cases the assumption of a group action with certain properties on the set of objects is unnecessary and would make results less general. However, many nice association schemes allow such group actions, and this often makes it easier to understand the relations, the eigenspaces, and the interplay with other sets of objects. In this section, we will give a brief introduction to the theory of permutation groups and modules.

2.5.1 Semisimple algebras and modules

We will first introduce some quite general results from algebra. We refer to [98] and [125] for proofs and more information.

Definition 2.5.1. *An algebra over a field F is a ring A with unit element, such that A is also an F-vector space with*

$$(\lambda x)y = x(\lambda y) = \lambda(xy),$$

for every $\lambda \in F$ and $x, y \in A$.

Definition 2.5.2. *An A-module, with A an F-algebra with unit element* 1*, is a finite-dimensional vector space V over F with a scalar multiplication $A \times V \to V : (a, v) \to av$, such that for every $c \in F$, for every $a, a' \in A$ and for every $v, v' \in V$:*

(i) $a(v + v') = av + av'$,

(ii) $(a + a')v = av + a'v$,

(iii) $a'(av) = (a'a)v$,

(iv) $a(cv) = c(av) = (ca)v$,

(v) $1v = v$.

A *submodule* of an A-module V is an A-invariant subspace W of V. A module is said to be *irreducible* or *simple* if it only has itself and its trivial subspace as submodules.

Definition 2.5.3. *An A-module V is semisimple if it can be written as the direct sum of irreducible submodules.*

Definition 2.5.4. *An algebra A is semisimple if it is semisimple as a left A-module over itself.*

Theorem 2.5.5. *An algebra A is semisimple if and only if every A-module is semisimple.*

If V and W are two A-modules, then the *A-homomorphisms* from V to W are the linear maps $\phi : V \to W$ such that $\phi(av) = a\phi(v)$ for all $a \in A, v \in V$. The set of all such homomorphisms has the structure of an F-vector space and is denoted by $\mathrm{Hom}_A(V, W)$. We say two A-modules V and W are *isomorphic* or *equivalent* if there is a bijection in $\mathrm{Hom}_A(V, W)$. The *A-endomorphisms* of an A-module V are simply the A-homomorphisms from V to itself. This set $\mathrm{Hom}_A(V, V)$, denoted by $\mathrm{End}_A(V)$, has the structure of an F-algebra, and is also known as the *centralizer ring* of the A-module V.

Lemma 2.5.6. [Schur's Lemma] *If V is an irreducible A-module, then the endomorphism ring* $\mathrm{End}_A(V)$ *is a division ring.*

In case A is an F-algebra with F an algebraically closed field, we can say more about $\mathrm{End}_A(V)$.

Lemma 2.5.7. *If V is an irreducible A-module with A an F-algebra and F an algebraically closed field, then $\mathrm{End}_A(V)$ is the ring of scalar multiplications by the elements of F on V.*

For any A-module V and any irreducible A-module M, the *M-homogeneous part* of V is the sum of all irreducible submodules of V that are isomorphic to M. The homogeneous parts of V are also known as the *isotypic components*. For any isomorphism class α of irreducible A-modules, we write V^α for the sum of irreducible submodules in V in that class.

Theorem 2.5.8. *Suppose V is a semisimple A-module.*

(i) *V can be written as the direct sum of its isotypic components.*

(ii) *Every M-homogeneous part can in turn be written as the direct sum of irreducibles isomorphic to M, and in every such decomposition, the number of irreducibles is the same.*

Note that the decomposition into isotypic components is unique, while the further decomposition into irreducible submodules need not be.

Theorem 2.5.9. *Let V and V' denote two semisimple A-modules.*

(i) *Any $f \in \mathrm{Hom}_A(V, V')$ will map V^α into $(V')^\alpha$ and induce an element of $\mathrm{Hom}_A(V^\alpha, (V')^\alpha)$.*

(ii) *If V decomposes into isotypic components as $V^{\alpha_1} \oplus \cdots \oplus V^{\alpha_n}$ then for every $(f_1, \ldots, f_n)$ with $f_i \in \mathrm{Hom}_A(V^{\alpha_i}, (V')^{\alpha_i})$ there is a unique $f \in \mathrm{Hom}_A(V, V')$ such that $f_{|V^{\alpha_i}} = f_i, \forall i \in \{1, \ldots, n\}$.*

Theorem 2.5.10. *Let V be a semisimple A-module, decomposing into isotypic components as: $V^{\alpha_1} \oplus \cdots \oplus V^{\alpha_n}$. Suppose that for every irreducible in V^{α_i} the endomorphism ring is isomorphic to the division ring D_i.*

(i) *$\mathrm{End}_A(V)$ is also semisimple, and V decomposes as an $\mathrm{End}_A(V)$-module into the same isotypic components.*

(ii) $\mathrm{End}_A(V) \cong \mathrm{End}_A(V^{\alpha_1}) \times \cdots \times \mathrm{End}_A(V^{\alpha_n})$ *and* $\mathrm{End}_A(V^{\alpha_i}) \cong M_{n_i}(D_i)$ *where* n_i *denotes the multiplicity of irreducibles in decompositions of* V^{α_i} *into irreducibles.*

Lemma 2.5.7 in particular applies to the field of complex numbers $\mathbb{C}$. We are also interested in working over the real numbers, and hence the following lemma is useful. For any A-module V with A an $\mathbb{R}$-algebra, we mean by *complexification* of V the module $\{v_1 + v_2 i | v_1, v_2 \in V\}$ over $\{a_1 + a_2 i | a_1, a_2 \in A\}$, such that $(a_1 + a_2 i) \cdot (a_1' + a_2' i) = (a_1 a_1' - a_2 a_2') + (a_1 a_2' + a_2 a_1')i$ and $(a_1 + a_2 i) \cdot (v_1 + v_2 i) = (a_1 v_1 - a_2 v_2) + (a_1 v_2 + a_2 v_1)i$.

Lemma 2.5.11. *Let A be an $\mathbb{R}$-algebra, and consider an irreducible A-module V. There are three possibilities.*

(i) $\mathrm{End}_A(V) \cong \mathbb{R}$, *and* $\mathrm{End}_A(V)$ *simply consists of scalar multiplication with the elements of* $\mathbb{R}$, *and* V *remains irreducible after complexification.*

(ii) $\mathrm{End}_A(V) \cong \mathbb{C}$, *and* V *splits into two non-isomorphic irreducible modules after complexification.*

(iii) $\mathrm{End}_A(V) \cong \mathbb{H}$ *(i.e. the quaternion algebra), and* V *splits into two isomorphic irreducible modules after complexification.*

We say that V *has type* $1, 0$ *or* -1 *in these cases, respectively.*

2.5.2 Group representations

A *representation* of a finite group G is a group morphism ρ from G to $\mathrm{GL}(V)$, where V is a finite-dimensional vector space over a field F and $\mathrm{GL}(V)$ denotes the group of invertible linear transformations of V.

The *group ring* of a finite group G over a field F is a the ring FG of all finite formal linear combinations $\sum_i c_i g_i$ with $g_i \in G$ and $c_i \in F$, where multiplication is defined by linear extension of multiplication in G.

Every representation $\rho : G \to \mathrm{GL}(V)$ endows V with the structure of an FG-module, with

$$\left(\sum_i c_i g_i\right) \cdot v := \sum_i c_i(\rho(g_i)(v)).$$

Conversely, every FG-module V yields a representation $\rho : G \to \mathrm{GL}(V)$ defined as $\rho(g)(v) := g \cdot v$. Hence representations of a group are in correspondence with the modules over its group ring.

Theorem 2.5.12. [Maschke's Theorem] *The group ring FG of a finite group G is a semisimple algebra if and only if the characteristic of F does not divide $|G|$.*

The *regular module* of a finite group G over a field F is the group algebra FG as a left module over itself.

2.5.3 Permutation groups

We will now discuss the link between group actions and association schemes. We refer to [36] for proofs.

Let G be a finite permutation group on a finite set Ω. We will denote the image of $\omega \in \Omega$ under $g \in G$ by ω^g, and $(\omega^g)^h = \omega^{(hg)}$.

The *orbits* are the classes of the equivalence relation $\{(\omega_1, \omega_2) | \exists g \in G : \omega_1^g = \omega_2\}$. We say G acts *transitively* on Ω if there is only one orbit, namely Ω itself.

The *orbitals* are the orbits of $\Omega \times \Omega$ under G. An orbital R is *self-paired* if $(\omega_1, \omega_2) \in R$ also implies $(\omega_2, \omega_1) \in R$. If all orbitals under a transitive group action are self-paired, we say G acts *generously transitively* on Ω. Note that this is equivalent to the condition that every $(\omega_1, \omega_2) \in \Omega \times \Omega$ is in the same orbit under G as (ω_2, ω_1).

If G acts on Ω with distinct orbitals $R_0, \ldots, R_d$ and $R_0 \subseteq \{(\omega, \omega) | \omega \in \Omega\}$, then $(\Omega, \{R_0, \ldots, R_d\})$ is an association scheme if and only if G acts generously transitively on Ω. The association schemes obtained in this way are said to be *Schurian*. We also say that this is the scheme *afforded* by (the group action of) the group G.

The *permutation module* over FG of a permutation group G is the module F^Ω, with $g \cdot \chi_{\{\omega\}} := \chi_{\{\omega^g\}}, \forall \omega \in \Omega$. Note that the corresponding representation of G can be represented by permutation matrices for every group element.

Now suppose G acts on Ω_1 and Ω_2 and consider the two permutation modules. With any orbit R_i of G on $\Omega_1 \times \Omega_2$ we can associate the matrix C_i with columns

and rows indexed by the elements of Ω_1 and Ω_2, respectively, such that:

$$(C_i)_{\omega_2,\omega_1} = \begin{cases} 1 \text{ if } (\omega_1, \omega_2) \in R_i \\ 0 \text{ if } (\omega_1, \omega_2) \notin R_i \end{cases}.$$

These C_i form a basis for $\mathrm{Hom}_{FG}(F^{\Omega_1}, F^{\Omega_2})$.

In particular, we can consider the endomorphism ring of a permutation module F^{Ω} over F. With any orbital $R_i \subseteq (\Omega \times \Omega)$ we can associate the adjacency matrix A_i over F with rows and columns indexed by Ω, with:

$$(A_i)_{\omega_2,\omega_1} = \begin{cases} 1 \text{ if } (\omega_1, \omega_2) \in R_i \\ 0 \text{ if } (\omega_1, \omega_2) \notin R_i \end{cases},$$

and now these A_i form a basis for the centralizer algebra $\mathrm{End}_{FG}(F^{\Omega})$.

We say a permutation module is *multiplicity-free* as a module over $\mathbb{C}G$ if it decomposes as a direct sum of non-isomorphic irreducible submodules.

Theorem 2.5.13. *Let G be a permutation group acting on Ω.*

(i) *The number of self-paired orbitals is equal to the number of irreducible submodules of type 1 minus twice the number of those of type -1, counting both with multiplicity, in the decomposition of $\mathbb{R}^{\Omega}$ as an $\mathbb{R}G$-module.*

(ii) *The endomorphism ring of the permutation module (over $\mathbb{R}G$ as well as $\mathbb{C}G$) is commutative if and only if $\mathbb{C}^{\Omega}$ is multiplicity-free.*

(iii) *The action is generously transitive if and only if the permutation module $\mathbb{R}^{\Omega}$ over $\mathbb{R}G$ decomposes into irreducible submodules of type 1, all with multiplicity one.*

Theorem 2.5.14. *Suppose G acts generously transitively on Ω with orbitals $R_0, \ldots, R_d$, with $R_0 = \{(\omega, \omega) | \omega \in \Omega\}$.*

(i) *The permutation module $\mathbb{R}^{\Omega}$ decomposes into $d+1$ irreducible orthogonal submodules of type 1, and these are the strata of the association scheme $(\Omega, \{R_0, \ldots, R_d\})$.*

(ii) *The centralizer ring $\mathrm{End}_{\mathbb{R}G}(\mathbb{R}^{\Omega})$ is the Bose-Mesner algebra of the association scheme, each element of which acts invariantly as scalar multiplication on every irreducible.*

Delsarte [66] developed a general theory of regular semilattices. In this way, both the relations and idempotents of certain metric and cometric association schemes can be described by use of other sets of objects. We refer to Stanton [136] for a discussion of many well-known schemes, including the Johnson and Hamming schemes. However, we will also use a somewhat different approach by linking two sets of objects by considering a group acting on both. This approach was suggested by Ito [99].

Theorem 2.5.15. *Suppose G acts generously transitively on X and on X', and consider the decompositions into isotypic components of the permutation modules $\mathbb{R}^X$ and $\mathbb{R}^{X'}$:*

$$\begin{aligned} \mathbb{R}^X &= (V_1 \perp \ldots \perp V_n) \perp (W_1 \perp \ldots \perp W_s) \\ \mathbb{R}^{X'} &= (V_1' \perp \ldots \perp V_n') \perp (W_1' \perp \ldots \perp W_t'), \end{aligned}$$

where V_i and V_i' are equivalent $\mathbb{R}G$-modules.
For every i, there is an isomorphism (unique up to scalar) $p_i : V_i \to V_i'$, extending to a unique homomorphism $\mathrm{Hom}_{\mathbb{R}G}(\mathbb{R}^X, \mathbb{R}^{X'})$ that vanishes on all other isotypic components. These $p_1, \ldots, p_n$ form a basis for $\mathrm{Hom}_{\mathbb{R}G}(\mathbb{R}^X, \mathbb{R}^{X'})$.

Proof. Theorem 2.5.13 and Lemma 2.5.11 yield that every irreducible has an endomorphism ring consisting of scalar multiplication with elements of $\mathbb{R}$ and appears at most once in each decomposition (up to isomorphism). The structure of the homomorphism space now follows from Theorem 2.5.9. □

For any $f \in \mathrm{Hom}_{\mathbb{R}G}(\mathbb{R}^X, \mathbb{R}^{X'})$, we write $\mathrm{Supp}(f)$ for the irreducible submodules in $\mathbb{R}^X$ not in the kernel of f (and hence trivially intersecting it).

If G acts generously transitively on both X and X', and $R \subseteq X \times X'$ is an orbit, then we say $S \subseteq X$ is an R-design if $|\{x \in S | (x, x') \in R\}|$ is constant for all $x' \in X'$. Following Ito, we say S is a *combinatorial design* if it is an R-design for every orbit R on $X \times X'$. We now state characterizations by Ito.

Theorem 2.5.16. *Suppose G acts generously transitively on both X and X'. Suppose R is an orbit in $X \times X'$, and let f_R be the corresponding element of $\mathrm{Hom}_{\mathbb{R}G}(\mathbb{R}^X, \mathbb{R}^{X'})$.*

- *A subset $S \subseteq X$ is an R-design if and only if χ_S is orthogonal to every irreducible submodule in $\mathrm{Supp}(f_R)$ different from $\langle \chi_X \rangle$.*
  In that case, the constant value for $|(S \times \{x'\}) \cap R|$ must be $\frac{|R||S|}{|X||X'|}$.

- *A subset $S \subseteq X$ is a combinatorial design if and only if it is orthogonal to every irreducible in $\mathbb{R}^X$ with an isomorphic copy in $\mathbb{R}^{X'}$, different from $\langle \chi_X \rangle$.*

Proof. For any $x' \in X'$, we can write:

$$\begin{aligned} \langle f_R(\chi_S), \chi_{\{x'\}} \rangle &= |(S \times \{x'\}) \cap R|, \\ \langle f_R(\chi_X), \chi_{\{x'\}} \rangle &= |(X \times \{x'\}) \cap R| = \frac{|R|}{|X'|}. \end{aligned}$$

Hence $|(S \times \{x'\}) \cap R|$ assumes a constant value c for every $x' \in X'$ if and only if $f_R(\chi_S - \frac{c|X'|}{|R|}\chi_X)$ is orthogonal to every $\chi_{\{x'\}}$, or thus zero. As f_R only vanishes on the trivial vector in every irreducible in $\mathrm{Supp}(f_R)$, this is possible if and only if $\chi_S - \frac{c|X'|}{|R|}\chi_X$ is orthogonal to every irreducible in $\mathrm{Supp}(f)$. Since the projection of χ_S onto the submodule $\langle \chi_X \rangle$ is given by $|S|/|X|\chi_X$, this also implies $c = \frac{|R||S|}{|X||X'|}$.

Finally, S is a combinatorial design if and only if $f_R(\chi_S)$ is a scalar multiple of $\chi_{X'}$ for every orbit R, and hence so is $f(\chi_S)$ for any $f \in \mathrm{Hom}_{\mathbb{R}G}(\mathbb{R}^X, \mathbb{R}^{X'})$. Theorem 2.5.15 now yields the desired result. □

We now mention a useful theorem by Delsarte as a consequence of a more general theorem.

Theorem 2.5.17. [67, Theorem 6.8]*If a group G acts generously transitively on Ω, then two subsets S and T are such that $|S \cap T^g|$ is independent of $g \in G$ if and only if S and T are design-orthogonal in the afforded association scheme on Ω.*

Proof. Since G acts transitively on ordered pairs (ω_1, ω_2) in the same relation of the association scheme, this follows from Theorem 2.2.13 by taking $S_g = T^g, \forall g \in G$. □

Chapter 3

Grassmann schemes

The aim of this chapter is to apply techniques from algebraic combinatorics to the finite projective geometries $\mathrm{PG}(n, q)$. Graph-theoretically, this means that we will discuss the Grassmann schemes. We will first describe the relations in these schemes and their eigenspaces. Next, we will discuss some interesting substructures and obtain rather short proofs for certain properties.

Delsarte [66] applied his theory of semiregular lattices to Grassmann schemes to characterize certain types of subsets of subspaces, known as designs, in a way very similar to his results for the Johnson scheme from (classical) design theory. Another example of a well-known type of subsets of subspaces that can be considered in this algebraic framework are the line classes introduced by Cameron and Liebler [40].

3.1 Grassmann schemes

In any vector space $V(n, q)$, we will write Ω_a for the set of a-dimensional subspaces (or simply a-spaces). We refer to the 1- and 2-dimensional subspaces as points and lines, respectively. We also define the *Gaussian coefficient* as follows:

$$\begin{bmatrix} n \\ a \end{bmatrix}_q = \prod_{i=1}^{a} \frac{q^{n+1-i} - 1}{q^i - 1} \text{ if } 0 \leq a \leq n,$$

and $\left[\begin{smallmatrix} n \\ a \end{smallmatrix}\right]_q = 0$ if $a < 0$ or $n < a$.

Note that $\begin{bmatrix} n \\ a \end{bmatrix}_q = \begin{bmatrix} n \\ n-a \end{bmatrix}_q$.

The following results are well known (see for instance [23, Lemma 9.3.2]).

Lemma 3.1.1. *Consider $V(n,q)$.*

(i) *The number of a-spaces in Ω_a is given by $\begin{bmatrix} n \\ a \end{bmatrix}_q$.*

(ii) *For any $\pi_a \in \Omega_a$, the number of b-spaces intersecting π_a trivially is given by $q^{ab} \begin{bmatrix} n-a \\ b \end{bmatrix}_q$.*

(iii) *For any $\pi_a \in \Omega_a$, the number of b-spaces intersecting π_a in an i-space is given by $q^{(a-i)(b-i)} \begin{bmatrix} n-a \\ b-i \end{bmatrix}_q \begin{bmatrix} a \\ i \end{bmatrix}_q$.*

We will write $\mathrm{GL}(n,q)$ for the group of invertible linear transformations of $V(n,q)$. More generally, a map $f : V \times V$ is *semi-linear* if $f(v_1+v_2) = f(v_1) + f(v_2), \forall v_1, v_2 \in V(n,q)$ and if there is an automorphism θ such that $f(\lambda v) = \lambda^\theta v, \forall \lambda \in \mathrm{GF}(q)$ and $\forall v \in V(n,q)$. The group of all such bijective semi-linear maps is $\Gamma\mathrm{L}(n,q)$. Finally, we also denote $\mathrm{GL}(n,q)/\mathrm{Sc}(n,q)$ by $\mathrm{PGL}(n,q)$ and $\Gamma\mathrm{L}(n,q)/\mathrm{Sc}(n,q)$ by $\mathrm{P}\Gamma\mathrm{L}(n,q)$, where $\mathrm{Sc}(n,q)$ denotes the subgroup of non-zero scalar linear transformations of $V(n,q)$. We will now consider the action of $\mathrm{GL}(n,q)$ on the sets of subspaces.

Lemma 3.1.2. *Two ordered pairs (π_a, π_b) and (π'_a, π'_b) in $\Omega_a \times \Omega_b$ are in the same orbit of $\mathrm{GL}(n,q)$ if and only if $\dim(\pi_a \cap \pi_b) = \dim(\pi'_a \cap \pi'_b)$.*

Proof. Since dimensions of subspaces are preserved under $\mathrm{GL}(n,q)$, the former certainly implies the latter. On the other hand, if the dimensions are equal, one easily finds an element $g \in \mathrm{GL}(n,q)$ mapping (π_a, π_b) to (π'_a, π'_b) by constructing appropriate ordered bases and using transitivity of $\mathrm{GL}(n,q)$ on ordered bases. □

We define the relation R^i_{ab} with $0 \leq i \leq \min(a, b, n-a, n-b)$ as:

$$R^i_{ab} = \{(\pi_a, \pi_b) \in (\Omega_a \times \Omega_b) | \dim(\pi_a \cap \pi_b) = \min(a,b) - i\}.$$

Note that for two subspaces π_a and π_b, $(\pi_a, \pi_b) \in R^0_{ab}$ if and only if one is included in the other.

Lemma 3.1.2 in particular yields that $\mathrm{GL}(n,q)$ acts generously transitively on every Ω_a. The afforded d-class association scheme $(\Omega_a, \{R^0_{aa}, \ldots, R^d_{aa}\})$ with

$d = \min(a, n-a)$ is called a *Grassmann scheme* or *q-Johnson scheme*. The graph defined by the first non-trivial relation R^1_{aa} is the *Grassmann graph*, and is denoted by $J_q(n,a)$. This notation and the second name is motivated by the similarities with the Johnson graph $J(n,a)$.

Theorem 3.1.3. [23, Lemma 9.4.1] *The Grassmann graph $J_q(n,k)$ has diameter $d = \min(k, n-k)$ and is distance-transitive. Two vertices are at distance i if and only they intersect in a subspace of codimension i.*
The intersection numbers are given by:

$$\begin{aligned} b_i &= q^{2i+1} \begin{bmatrix} k-i \\ 1 \end{bmatrix}_q \begin{bmatrix} n-k-i \\ 1 \end{bmatrix}_q, \forall i \in \{0, \dots, d-1\}, \\ c_i &= \begin{bmatrix} i \\ 1 \end{bmatrix}_q^2, \forall i \in \{1, \dots, d\}. \end{aligned}$$

The graph $J_q(n,k)$ has classical parameters $(d, q, q, \left[\begin{smallmatrix} n-d \\ 1 \end{smallmatrix}\right]_q q)$.

The Grassmann graph $J_q(n,2)$, $n \geq 4$, can be seen as the point graph of an SPBIBD satisfying the conditions of Lemma 2.3.6. The "points" of this SPBIBD are the 2-spaces, its "lines" are the 1-spaces, and incidence is just symmetrized containment. This is in fact a $\mathrm{pg}(\left[\begin{smallmatrix} n-1 \\ 1 \end{smallmatrix}\right]_q - 1, q, q+1)$.

3.2 Irreducible submodules and eigenvalues for Grassmann schemes

For each $R^i_{ab} \subseteq \Omega_a \times \Omega_b$, we will write C^i_{ab} for the corresponding matrix, with columns indexed by the elements of Ω_a, and rows indexed by those of Ω_b:

$$(C^i_{ab})_{\pi_b, \pi_a} = \begin{cases} 1 \text{ if } \dim(\pi_a \cap \pi_b) = \min(a,b) - i \\ 0 \text{ if } \dim(\pi_a \cap \pi_b) \neq \min(a,b) - i \end{cases}.$$

Note that C^i_{ab} and C^i_{ba} are mutually transposed matrices.

It follows from Subsection 2.5.3 and Lemma 3.1.2 that the C^i_{ab} with $0 \leq i \leq \min(a, b, n-a, n-b)$ form a basis for $\mathrm{Hom}_{\mathbb{R}G}(\mathbb{R}^{\Omega_a}, \mathbb{R}^{\Omega_b})$, with $G = \mathrm{GL}(n,q)$.

We now state a technical result by Kantor.

Theorem 3.2.1. [101] *If* $\min(a, n-a) \leq \min(b, n-b)$*, then the incidence matrix* C^0_{ab} *for* $V(n,q)$ *is injective.*

Theorem 3.2.2. *Consider* $V(n,q)$ *and* $G = \mathrm{GL}(n,q)$*. If* $a \in \{0, \ldots, n\}$*, then the permutation module* $\mathbb{R}^{\Omega_a}$ *over* $\mathbb{R}G$ *on a-spaces in* $V(n,q)$ *decomposes into irreducibles:*

$$\mathbb{R}^{\Omega_a} = \bigoplus_{i=0}^{\min(a,n-a)} V_i^a,$$

with V_i^a *and* V_i^b *equivalent* $\mathbb{R}G$*-modules if* $0 \leq a, b \leq n$*.*

Proof. Theorem 2.5.14 immediately implies that each permutation module $\mathbb{R}^{\Omega_a}$ decomposes into non-isomorphic irreducibles. We will now proceed by induction. For $a = 0$, the result is clear. Now we may assume the result holds up to $a-1$, with $1 \leq a \leq n/2$. We know from Lemma 3.1.2 that

$$\dim(\mathrm{Hom}_{\mathbb{R}G}(\mathbb{R}^{\Omega_{a-1}}, \mathbb{R}^{\Omega_{a-1}})) = \dim(\mathrm{Hom}_{\mathbb{R}G}(\mathbb{R}^{\Omega_{a-1}}, \mathbb{R}^{\Omega_a})) = a$$

and $\dim(\mathrm{Hom}_{\mathbb{R}G}(\mathbb{R}^{\Omega_a}, \mathbb{R}^{\Omega_a})) = a+1$. If $\mathbb{R}^{\Omega_{a-1}}$ decomposes into irreducibles as $\bigoplus_{i=0}^{\min(a-1)} V_i^{a-1}$, then Theorem 2.5.15 yields that there must be a unique irreducible submodule in $\mathbb{R}^{\Omega_a}$, not isomorphic to any of the a irreducibles in $\mathbb{R}^{\Omega_{a-1}}$, while all a irreducibles in $\mathbb{R}^{\Omega_{a-1}}$ have a unique isomorphic copy in $\mathbb{R}^{\Omega_a}$.

Now suppose $n/2 \leq a \leq n$. Lemma 3.1.2 tells us that

$$\begin{aligned} \dim(\mathrm{Hom}_{\mathbb{R}G}(\mathbb{R}^{\Omega_a}, \mathbb{R}^{\Omega_a})) &= \dim(\mathrm{Hom}_{\mathbb{R}G}(\mathbb{R}^{\Omega_{n-a}}, \mathbb{R}^{\Omega_a})) \\ &= \dim(\mathrm{Hom}_{\mathbb{R}G}(\mathbb{R}^{\Omega_{n-a}}, \mathbb{R}^{\Omega_{n-a}})) \\ &= (n-a)+1. \end{aligned}$$

Theorem 2.5.15 here yields that every irreducible in one of the permutation modules has a unique isomorphic copy in the second. □

Corollary 3.2.3. *Consider* $V(n,q)$*. If* $i \in \{0, \ldots, \min(a, b, n-a, n-b)\}$*, then:*

$$\begin{aligned} \mathrm{Im}(C^i_{ab}) &\subseteq \bigoplus_{j=0}^{\min(a,b,n-a,n-b)} V_j^b, \\ \left(\bigoplus_{j=0}^{\min(a,b,n-a,n-b)} V_j^a\right)^{\perp} &\subseteq \ker(C^i_{ab}). \end{aligned}$$

Equality holds for $i = 0$*.*

Proof. First note that $\mathrm{Im}(C^i_{ab}) = \ker((C^i_{ab})^T)^\perp = \ker(C^i_{ba})^\perp$. The inclusions now follow from Theorem 2.5.15 and Theorem 3.2.2.

Finally, if $V^a_i \subseteq \ker(C^0_{ab})$ with $0 \leq i \leq \min(a, b, n-a, n-b)$, then also $V^b_i \subseteq \ker(C^0_{ab}(C^0_{ab})^T) = \ker(C^0_{ba})$, because of Theorem 2.5.15. One of these inclusions contradicts Theorem 3.2.1. □

As an illustration, we visualize the decomposition for $V(6, q)$ in Figure 3.1. Each column corresponds with the permutation module on subspaces of $V(6, q)$ and its entries are the irreducibles in it, and we write isomorphic irreducibles on the same row.

$\mathbb{R}^{\Omega_0}$	$\mathbb{R}^{\Omega_1}$	$\mathbb{R}^{\Omega_2}$	$\mathbb{R}^{\Omega_3}$	$\mathbb{R}^{\Omega_4}$	$\mathbb{R}^{\Omega_5}$	$\mathbb{R}^{\Omega_6}$
V^0_0	V^1_0	V^2_0	V^3_0	V^4_0	V^5_0	V^6_0
	V^1_1	V^2_1	V^3_1	V^4_1	V^5_1	
		V^2_2	V^3_2	V^4_2		
			V^3_3			

Figure 3.1: Theorem 3.2.2: The decomposition into irreducibles for $V(6, q)$

Delsarte [66] described the eigenspaces of the Grassmann schemes using the theory of regular semilattices and computed the eigenvalues.

Theorem 3.2.4. *Consider the Grassmann scheme defined by $J_q(n, k)$ with $d = \min(k, n-k)$. The eigenvalue P_{ji} of the distance-i relation for V^k_j is given by:*

$$\sum_{s=0}^{i} (-1)^{i+s} \begin{bmatrix} d-s \\ i-s \end{bmatrix}_q \begin{bmatrix} d-j \\ s \end{bmatrix}_q \begin{bmatrix} n-d+s-j \\ s \end{bmatrix}_q q^{sj+(i-s)(i-s-1)/2},$$

and in particular, the eigenvalue of the Grassmann graph itself for V^k_j is:

$$q^{j+1} \begin{bmatrix} k-j \\ 1 \end{bmatrix}_q \begin{bmatrix} n-k-j \\ 1 \end{bmatrix}_q - \begin{bmatrix} j \\ 1 \end{bmatrix}_q.$$

Remark 3.2.5. Eisfeld [77] computed an alternative expression for P_{ji}:

$$\sum_{s=\max(0,j-i)}^{\min(j,k-i)} (-1)^{j+s} \begin{bmatrix} j \\ s \end{bmatrix}_q \begin{bmatrix} n-k+s-j \\ n-k-i \end{bmatrix}_q \begin{bmatrix} k-s \\ i \end{bmatrix}_q q^{i(i+s-j)+(j-s)(j-s-1)/2}.$$

Remark 3.2.6. We know from Theorem 3.1.3 that the Grassmann graph $J_q(n,k)$ has classical parameters $(d,q,q,\left[{n-d \atop 1}\right]_q q)$ with $d = \min(k, n-k)$. The corresponding cometric ordering of the eigenspaces (see Theorem 2.3.14) corresponds with the ordering $V_0^k, \ldots, V_d^k$ from Theorem 3.2.2.

3.3 Codes in Grassmann graphs

We will now consider subsets of subspaces with interesting algebraic properties, including t-designs and t-antidesigns with respect to the cometric ordering from Remark 3.2.6.

We first remark that Chihara [44] proved that no non-trivial perfect codes (i.e. different from a singleton or the full set of vertices) can be found in $J_q(n,k)$ (see also [107]). However, many completely regular codes are known, including the Cameron-Liebler line classes (see Subsection 3.3.2).

3.3.1 Designs

Definition 3.3.1. *A $t-(n,k,\lambda;q)$-design is a set S of k-spaces in $V(n,q)$ with $0 \leq t \leq k \leq n$, such that every t-space is in exactly λ elements of S.*

Hence, a design in $V(n,q)$ is a specific type of code in the Grassmann graph $J_q(n,k)$. The characterizations and applications in this subsection were obtained by Delsarte [66].

Theorem 3.3.2. *A set S of k-spaces in $V(n,q)$ is a $t-(n,k,\lambda;q)$-design for some λ if and only if $\chi_S \in (V_j^k)^\perp$ for every j with $1 \leq j \leq \min(t, n-t, k, n-k)$. In this case,*

$$|S| = \lambda \frac{\left[{n \atop k}\right]_q}{\left[{n-t \atop k-t}\right]_q}.$$

In particular, if $\min(k, n-k) \leq \min(t, n-t)$, then the only $t-(n,k,\lambda;q)$-designs are the empty set and the full set of k-spaces Ω_k.

Proof. Corollary 3.2.3 yields that $\mathrm{Supp}(C_{kt}^0)$ consists of the V_j^k with $0 \leq j \leq \min(k, t, n-k, n-t)$. Every t-space in $V(n,q)$ is contained in exactly $\left[{n-t \atop k-t}\right]_q$ k-spaces because of Lemma 3.1.1 (iii), and thus $|R_{tk}^0|/|\Omega_t| = \left[{n-t \atop k-t}\right]_q$. The desired result now follows from Theorem 2.5.16. □

We can also obtain a very similar restriction to the one from Theorem 3.3.2.

Corollary 3.3.3. *If S is a set of k-spaces in $V(n,q)$ such that every t-space contains exactly λ elements of S, with $0 \leq k \leq t \leq n-k$, then S is either empty or the full set of k-spaces.*

Proof. Dualization (i.e. for instance by mapping every $U \in S$ onto $U^{\perp}$ with respect to a fixed non-degenerate symmetric form on $V(n,q)$) yields an $(n-t)-(n,n-k,\lambda;q)$-design, which must be trivial because of Theorem 3.3.2. □

We can now derive some more properties of designs in $J_q(n,k)$, all of which are q-analogs of properties of classical designs (see for instance [171]).

Theorem 3.3.4. *Let S be a $t-(n,k,\lambda;q)$-design in $V(n,q)$, with $0 \leq t \leq k \leq n$. If $0 \leq t' \leq t$, then S is also a $t'-(n,k,\lambda';q)$-design, with:*

$$\lambda' = \lambda \frac{\begin{bmatrix} n-t' \\ k-t' \end{bmatrix}_q}{\begin{bmatrix} n-t \\ k-t \end{bmatrix}_q}.$$

Proof. The condition for t-designs from Theorem 3.3.2 implies the condition for t'-designs. Since $|R^0_{tk}| = |\Omega_t| \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$ and $|R^0_{t'k}| = |\Omega_{t'}| \begin{bmatrix} n-t' \\ k-t' \end{bmatrix}_q$ by Lemma 3.1.1(iii), the desired numbers follow from Theorem 2.5.16. □

Theorem 3.3.5. *A set of k-spaces S in $V(n,q)$ is a $t-(n,k,\lambda;q)$-design with $t \leq k \leq n-t$ for some λ if and only if every $(n-t)$-space contains exactly λ' elements of S for some constant λ'. In that case: $\lambda' = \lambda \frac{\begin{bmatrix} n-t \\ k \end{bmatrix}_q}{\begin{bmatrix} n-t \\ k-t \end{bmatrix}_q}$.*

Proof. Corollary 3.2.3 yields that $\mathrm{Supp}(C^0_{kt}) = \mathrm{Supp}(C^0_{k,n-t})$. The desired result now follows from Theorem 2.5.16 and by considering $|R^0_{tk}| = |\Omega_t| \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$ and $|R^0_{n-t,k}| = |\Omega_{n-t}| \begin{bmatrix} n-t \\ k \end{bmatrix}_q$ with $|\Omega_t| = |\Omega_{n-t}|$. □

Lemma 3.3.6. *Consider $(\pi_a, \pi_b) \in \Omega_a \times \Omega_b$ in $V(n,q)$ with $\pi_a \subseteq \pi_b$. If T is the set of vertices π in $J_q(n,k)$ with $\pi_a \subseteq \pi \subseteq \pi_b$, $0 \leq a \leq k \leq b \leq n$, then χ_T is orthogonal to every $V^k_j \subseteq \mathbb{R}^{\Omega_k}$ with $j > a + (n-b)$.*

Proof. If we let T_a and T_b denote the set of k-spaces incident with π_a and π_b, respectively, then we can write:

$$\chi_{T_a} = C^0_{ak}\chi_{\{\pi_a\}} \text{ and } \chi_{T_b} = C^0_{bk}\chi_{\{\pi_b\}}.$$

Corollary 3.2.3 yields that $\chi_{T_a} \in (V_0^k \perp \ldots \perp V^k_{\min(a,n-a,k,n-k)})$ and that $\chi_{T_b} \in (V_0^k \perp \ldots \perp V^k_{\min(b,n-b,k,n-k)})$. Note that $\min(a, n-a) + \min(b, n-b) \leq a + (n-b)$. Since $T = T_a \cap T_b$, the desired result follows from Corollary 2.2.15. □

Theorem 3.3.7. *Let S be a $t-(n,k,\lambda;q)$-design in $J_q(n,k)$. If π_a and π_b are incident a- and b-spaces, respectively, with $0 \leq a \leq k \leq b \leq n$ and $a+(n-b) \leq t \leq k$, then the number of elements of S through π_a and in π_b is given by:*

$$\lambda \frac{\begin{bmatrix} b-a \\ k-a \end{bmatrix}_q}{\begin{bmatrix} n-t \\ k-t \end{bmatrix}_q}.$$

Proof. Let T denote the set of all k-spaces incident with both π_a and π_b. Lemma 3.3.6 yields that $\chi_T \in (V_j^k)^\perp$ if $j > \min(a+(n-b), k, n-k)$. Theorem 3.3.2 yields that $\chi_S \in (V_1^k \perp \ldots \perp V^k_{\min(t,n-k)})^\perp$, and gives us $|S|$. Lemma 2.2.10 now yields:

$$|S \cap T| = \frac{|S||T|}{|\Omega_k|} = \frac{(\lambda \begin{bmatrix} n \\ k \end{bmatrix}_q / \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q) \begin{bmatrix} b-a \\ k-a \end{bmatrix}_q}{\begin{bmatrix} n \\ k \end{bmatrix}_q} = \lambda \frac{\begin{bmatrix} b-a \\ k-a \end{bmatrix}_q}{\begin{bmatrix} n-t \\ k-t \end{bmatrix}_q}.$$

□

Theorem 3.3.8. *If S is a $t-(n,k,\lambda;q)$-design in $J_q(n,k)$, then for every t-space π_t, the number of elements of S intersecting π_t in a $(t-i)$-space with $0 \leq i \leq t$ is given by:*

$$\lambda' = \lambda \frac{q^{i(k-t+i)} \begin{bmatrix} n-t \\ k-t+i \end{bmatrix}_q \begin{bmatrix} t \\ i \end{bmatrix}_q}{\begin{bmatrix} n-t \\ k-t \end{bmatrix}_q}.$$

Proof. Theorem 3.3.2 yields that S is also a combinatorial design with respect to Ω_k. Lemma 3.1.1(iii) yields that:

$$\begin{aligned} |R^0_{tk}| &= |\Omega_t| \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q, \\ |R^i_{tk}| &= |\Omega_t| q^{i(k-t+i)} \begin{bmatrix} n-t \\ k-t+i \end{bmatrix}_q \begin{bmatrix} t \\ i \end{bmatrix}_q, \end{aligned}$$

and hence the ratio between λ and the desired constant λ' can be computed using the formula from Theorem 2.5.16. □

Many constructions for 1-designs are known. In particular, a $1-(n,k,1;q)$-design is known as a *spread of k-spaces*: every point is in exactly one of its elements. Hence they consist of $(q^n-1)/(q^k-1)$ trivially intersecting k-spaces. They exist if and only if k divides n (see for instance [128]).

Lemma 3.3.9. *If S is a spread of k-spaces in $V(n,q)$, $0<k<n$, then χ_S is orthogonal to V_j^k if and only if $j=1$.*

Proof. Since S is a $1-(n,k,1;q)$-design, we know $\chi_S \in (V_1^k)^\perp$ because of Theorem 3.3.2, and that $n \geq 2k$. Note that S is a clique of the k-distance relation in the Grassmann scheme. If λ_k and k_k denote the eigenvalue for V_j and the valency of the k-distance relation, respectively, then it follows from Corollary 2.2.9 that $\chi_S \in (V_j^k)^\perp$ if and only if $1+\lambda_k/k_k(|S|-1)=0$. We know from Remark 3.2.5 that the eigenvalue of the k-distance relation for the subspace V_j^k is given by:

$$(-1)^j \begin{bmatrix} n-k-j \\ n-2k \end{bmatrix}_q q^{k(k-j)+j(j-1)/2}.$$

Dividing the absolute value of this eigenvalue for j by the one for $j+1$ yields: $q^{k-j}(q^{n-k-j}-1)/(q^{k-j}-1)>1$ for every $j \in \{0,\dots,k-1\}$, and hence all these eigenvalues are distinct. This implies that χ_S is only orthogonal to $(V_1^k)^\perp$. □

The following result already appeared in [76, Theorem 7], except for $n=2k \geq 8$.

Corollary 3.3.10. *Let S be any spread of k-spaces in $V(n,q)$. A set of k-spaces T is such that $|S \cap T^g|$ is independent of $g \in \mathrm{GL}(n,q)$ if and only if $\chi_T \in (V_0^k \perp V_1^k)$.*

Proof. Note that the group $\mathrm{GL}(n,q)$ affords the Grassmann scheme because of Lemma 3.1.2. The result now follows immediately from Theorem 2.5.17 and Lemma 3.3.9. □

Known results

Non-trivial t-designs in $J_q(n,k)$ (i.e. different from the empty or full set of k-spaces) with $t \geq 2$ are quite hard to find. Thomas [154] constructed $2-(n,3,7;2)$-designs for all $n \geq 7$ with n coprime to 6. Suzuki [137, 138]

generalized this by constructing $2-(n,3,q^2+q+1;q)$-designs for all prime powers q, with the same restrictions on n. Moreover, a $3-(8,4,11;2)$-design was found in [18].

No $t-(n,k,1;q)$-designs with $2 \leq t < k < n$ are known to exist. The existence of a $2-(7,3,1;q)$-design in $J_q(7,3)$, even for $q=2$, is still an open problem (see [79]).

3.3.2 Subsets in $J_q(n,2)$

For $n \geq 4$, the Grassmann graph $J_q(n,2)$ is a strongly regular graph that can be seen as the point graph of a partial geometry $\mathrm{pg}(\left[{n-1 \atop 1}\right]_q - 1, q, q+1)$, as explained at the end of Section 3.1. As observed by Eisfeld (see Theorem 2.3.6), the structure of the eigenspaces already follows from this setting. It follows from Theorem 2.3.11 that the intriguing sets S of lines in $J_q(n,2)$ are precisely those with a characteristic vector orthogonal to (at least) one of the non-trivial eigenspaces, and then at least one of the following must hold

(i) χ_S is in the image of the incidence matrix $C_{1,2}$: $V_0^2 \perp V_1^2$,

(ii) χ_S minus a scalar multiple of the all-one vector, is in the kernel of the incidence matrix $C_{2,1}$: V_2^2, and this holds if and only if every 1-space is in the same number of elements in S, i.e. S is a $1-(n,2,\lambda;q)$-design.

The intriguing sets of the first type in $J_q(4,2)$ are known as *Cameron-Liebler line-classes*. Cameron and Liebler [40] introduced this concept under the name *special line classes* when studying the actions of permutation groups on subspaces of $V(n,q)$. Note that Corollary 3.3.10 implies that they are also precisely the sets of 2-spaces in $V(4,q)$, intersecting every spread of 2-spaces in the same number of elements.

In order to explain their motivation, we need the following lemma, which is essentially part of Block's Lemma [14] (see also [70, p.21]).

Lemma 3.3.11. *Let G be a group acting on two finite sets X and X' with orbits $O_1,\ldots,O_m$ in X and orbits $O'_1,\ldots,O'_{m'}$ in X'. Suppose $R \subseteq X \times X'$ is a G-invariant relation with corresponding $(|X| \times |X'|)$-matrix A.*

(i) *The images $A\chi_{O_i}$ are linear combinations of the vectors $\chi_{O'_j}$.*

(ii) *If A has a trivial kernel, then $m \leq m'$, and if $m = m'$, then all characteristic vectors $\chi_{O'_j}$ are linear combinations of the vectors $A\chi_{O_i}$.*

Proof. If $x' \in \Omega'$ then $(A\chi_{O_i})_{x'} = |\{x | x \in O_i, (x, x') \in R\}|$, and hence this number only depends on the orbit O'_j that x' is in. This implies that $A\chi_{O_i}$ is a linear combination of the m' linearly independent vectors $\chi_{O'_j}$.

Now suppose A has a trivial kernel. As the subsets $O_1, \ldots, O_m$ partition the set X, the corresponding m characteristic vectors are linearly independent, and hence in this case the images $A\chi_{O_i}$ of these vectors must also be linearly independent. This implies that $m \leqslant m'$. If $m = m'$, then the space spanned by $\{A\chi_{O_1}, \ldots, A\chi_{O_m}\}$ is the same as the space spanned by $\{\chi_{O'_1}, \ldots, \chi_{O'_{m'}}\}$.□

An immediate consequence is the following.

Theorem 3.3.12. [40, Propositions 3.1 and 3.2] *Any subgroup of* $\mathrm{P\Gamma L}(4, q)$ *has at least as many orbits on lines as on points, and in case of equality all orbits on lines are Cameron-Liebler line classes in* $\mathrm{PG}(3, q)$.

Corollary 3.2.3 yields that the incidence matrices $C_{1,2}$ and $C_{n-1,2}$ have the same image. One now easily verifies that the following are examples of Cameron-Liebler line classes in $V(4, q)$: the empty set of lines, the full set of lines, all lines through a fixed point p, all lines in a fixed 3-space π, all lines through a point p or in a 3-space π with p not in π, and the complements of such sets. Cameron and Liebler conjectured that these are the only possibilities. However, Drudge [74] gave a counterexample for $q = 3$, which was then generalized for all odd q by Bruen and Drudge [29]. A counterexample for $q = 4$ was also found by Govaerts and Penttila [85].

Finally, we remark that Bamberg and Penttila [11] proved that if a subgroup of $\mathrm{P\Gamma L}(n, q)$, $n \geq 4$, is irreducible (i.e. not stabilizing any k-space with $0 < k < n$) and has equally many orbits on points as on lines, then it acts transitively on both.

We will use similar ideas to construct interesting subsets in the following subsection.

3.3.3 Embeddings of other geometries

We will first consider the embedding of the symplectic space $W(2n-1,q)$ in $\mathrm{PG}(2n-1,q)$ (see Subsection 1.3.2). Consider a non-degenerate alternating form f on a vector space $V(2n,q)$. The set S^k of totally isotropic k-spaces in $V(2n,q)$ with respect to f has size: $\begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{i=1}^{k}(q^{n-i+1}+1)$ if $1 \leq k \leq n$ (see for instance [23, Lemma 9.4.1] or Theorem 4.1.1). We will consider the corresponding group of isometries:

$$\mathrm{Sp}(2n,q) = \{g \in \mathrm{GL}(2n,q) | f(u^g, v^g) = f(u,v), \forall u, v \in V(2n,q)\}.$$

If $0 \leq k \leq n$, then the possible ranks of the restriction of f to k-spaces are the even numbers $2i$ with $0 \leq 2i \leq k$.

Theorem 3.3.13. *Let f be a non-degenerate alternating form on a vector space $V(2n,q)$. The set S of totally isotropic k-spaces with $1 \leq k \leq n$ is a completely transitive code in $J_q(2n,k)$ and its dual degree consists of the even indices in $\{1,\ldots,k\}$.*

Proof. All totally isotropic k-spaces are contained in the same number of totally isotropic n-spaces, while the other k-spaces are contained in no element of S^n. Hence we can write χ_{S^k} as $C^0_{n,k}\chi_{S^n}$ up to a non-zero scalar, for every $k \in \{0,\ldots,n\}$. Corollary 3.2.3 now yields that $\chi_{S^k} \in (V_i^k)^\perp$ if and only if $\chi_{S^n} \in (V_i^n)^\perp$, for every $i \in \{0,\ldots,k\}$.

Let S_i^k be the set of k-spaces with $0 \leq k \leq n$, such that the restriction of f to it has rank $2i$. A k-space π is at distance i from S^k in the Grassmann graph $J_q(2n,k)$ if and only if the Witt index of the restriction of f to π is $k-i$, and hence if and only if this restriction has rank $2i$. Hence, it follows from Theorem 1.3.1 that $\mathrm{Sp}(2n,q)$ has the sets S_i^k with $0 \leq 2i \leq k$ as orbits, and so S^k is a completely transitive code with covering radius $\lfloor k/2 \rfloor$.

Now consider any odd i with $1 \leq i \leq n$. We know from the above that $\mathrm{Sp}(2n,q)$ has $(i+1)/2$ orbits on $(i-1)$-spaces and on i-spaces. Lemma 3.3.11 now yields that $\chi_{S_0^i},\ldots,\chi_{S^i_{(i-1)/2}}$ are linear combinations of the images $C^0_{i-1,i}(\chi_{S_0^{i-1}}),\ldots,$ $C^0_{i-1,i}(\chi_{S^{i-1}_{(i-1)/2}})$. Hence $\chi_{S^i} \in \mathrm{Im}(C^0_{i-1,i}) = (V_i^i)^\perp$ by Corollary 3.2.3. Hence we can conclude that $\chi_{S^k} \in (V_i^k)^\perp$ for any odd i if $1 \leq k \leq n$. On the other hand, all even indices must be in the dual degree set, as the dual degree is the covering radius $\lfloor k/2 \rfloor$ by Theorem 2.3.9(iii). $\square$

Corollary 3.3.14. *For any non-degenerate alternating form on $V(2n,q)$, the number of totally isotropic subspaces in any $t-(2n,t+1,\lambda;q)$-design with $0 \le t \le n-1$ and t even, is given by:*

$$\frac{\lambda}{\left[{2n-t \atop 1}\right]_q} \prod_{i=0}^{t} \left(\frac{q^{2(n-i)}-1}{q^{i+1}-1} \right).$$

Proof. Let T denote the design, which has size $\lambda \left[{2n \atop t+1}\right]_q / \left[{2n-t \atop 1}\right]_q$ by Theorem 3.3.2. On the other hand, the set of totally isotropic $(t+1)$-spaces S has size: $\left[{n \atop t+1}\right]_q \prod_{i=1}^{t+1}(q^{n-i+1}+1)$. It follows from Theorems 3.3.2 and 3.3.13 that S and T are design-orthogonal subsets in $J_q(2n,t+1)$, and hence we can use Lemma 2.2.10 to compute $|S \cap T|$:

$$\begin{aligned} \frac{|S||T|}{|\Omega_{t+1}|} &= \frac{\left(\left[{n \atop t+1}\right]_q \prod_{i=1}^{t+1}(q^{n-i+1}+1) \right) \left(\lambda \left[{2n \atop t+1}\right]_q / \left[{2n-t \atop 1}\right]_q \right)}{\left[{2n \atop t+1}\right]_q} \\ &= \frac{\lambda}{\left[{2n-t \atop 1}\right]_q} \prod_{i=0}^{t} \left(\frac{q^{2(n-i)}-1}{q^{i+1}-1} \right). \end{aligned}$$

□

Applying the last corollary to one of the $2-(6,3,3;2)$-designs from [18], we see that exactly 27 of its 279 planes are totally isotropic with respect to any non-degenerate alternating form.

Now consider $V(4n+2,q), n \ge 1$, equipped with a non-degenerate quadratic form Q of elliptic type (hence with Witt index $2n$). We will consider the action of the group $\mathrm{GO}^-(4n+2,q)$, defined as:

$$\{g \in \mathrm{GL}(4n+2,q) | \exists \lambda \in \mathbb{F}_q : (Q(v^g) = \lambda Q(v), \forall v \in V(4n+2,q))\}.$$

Two m-dimensional subspaces are in the same orbit of $\mathrm{GO}^-(4n+2,q)$ if and only if the restriction of Q onto them are of the same type (see for instance [94, Theorem 22.6.6]).

Theorem 3.3.15. *All orbits of $\mathrm{GO}^-(4n+2,q)$ on Ω_{2n+1} in $V(4n+2,q)$ have characteristic vectors orthogonal to V_{2n+1}^{2n+1}.*

Proof. Computation of the number of possible types of restriction of Q to both $(2n)$- and $(2n+1)$-spaces can be done immediately by simply applying [94,

Theorem 22.8.3 (Corollary 1)]. One finds that the group $\mathrm{GO}^-(4n+2,q)$ has $3n+1$ orbits on $(2n)$-spaces, as well as on $(2n+1)$-spaces. Lemma 3.3.11 now implies that the characteristic vectors of the orbits of $\mathrm{GO}^-(4n+2,q)$ on $(2n+1)$-spaces are all in the image of $C^0_{2n,2n+1}$, and hence orthogonal to V^{2n+1}_{2n+1}, because of Corollary 3.2.3. □

Corollary 3.3.16. *Let S be a $2n-(4n+2,2n+1,\lambda;q)$-design in $V(4n+2,q)$, and let Q be a non-degenerate quadratic form with Witt index $2n$. Let $O_\alpha \subseteq \Omega_{2n+1}$ denote the set of $(2n+1)$-spaces on which Q has a restriction of type α. Then:*

$$|S \cap O_\alpha| = \frac{\lambda}{\begin{bmatrix} 2n+2 \\ 1 \end{bmatrix}_q} |O_\alpha|.$$

Proof. We know from Theorems 3.3.2 and 3.3.15 that S is design-orthogonal to any O_α, and that its size is given by $\lambda|\Omega_{2n+1}|/\begin{bmatrix} 2n+2 \\ 1 \end{bmatrix}_q$. Lemma 2.2.10 now yields the desired result. □

The numbers $|O_i|$ can be computed using for instance [94, Theorem 22.8.2]. We can now apply Corollary 3.3.16 to any $2-(6,3,3;2)$-design S, such as those constructed in [18]. Table 3.1 gives the four possible types of restrictions of the quadratic form to 3-spaces (or planes), together with the full number of such planes in $J_2(6,3)$, as well as their number in S.

dim. singular subspace	type	$J_2(6,3)$	S
0	parabolic	720	144
1	hyperbolic	270	54
1	elliptic	270	54
2	parabolic	135	27

Table 3.1: Corollary 3.3.16: Elliptic quadrics and $2-(6,3,3;2)$-designs

Chapter 4

Classical finite polar spaces

We will now consider the association schemes on the totally isotropic subspaces of a fixed dimension in a classical finite polar space. These schemes and the decompositions of the Bose-Mesner algebra were studied by Stanton [135].

Eisfeld [77] described a general method for an inductive computation of all eigenvalues in these association schemes. We will use similar techniques to compute explicit expressions for some specific eigenvalues instead.

We will then consider subsets in these association schemes with low dual degree. This will allow us to obtain bounds and information on the interaction between different substructures. In many cases, these results are already known but our techniques often shed a new light on the situation. Previous work on substructures in polar spaces by use of algebraic techniques was done by Stanton [135, 136], Eisfeld [75, 77], Drudge [73] and Bamberg, Kelly, Law and Penttila [9, 10].

One of the main new results in this chapter is a tight upper bound for partial spreads in the polar space $H(2d-1, q^2)$ with d odd, which was published in *Electronic Journal of Combinatorics* [165].

4.1 The association schemes from classical finite polar spaces

We already introduced the different types of classical finite polar spaces in Subsection 1.3.2. Here, the points and lines are the totally isotropic 1- and 2-spaces, respectively. We will say a classical finite polar space of rank d has *parameters* (q, q^e) if each line is incident with $q+1$ points, and each $(d-1)$-space is incident with q^e+1 generators or d-spaces. Theorem 1.3.1 implies that each classical finite polar space must have such parameters. Table 4.1 gives these parameters for all possible types of classical finite polar spaces with rank d. The notation in the first column is based on the embedding in a projective space, the notation in the second column is the one related to Chevalley groups (see for instance [43]). In the context of polar spaces, we will always denote the set of totally isotropic n-spaces by Ω_n for any $n \in \{0, \ldots, d\}$.

		(s,t)	e
$Q^+(2d-1,q)$	$D_d(q)$	$(q,1)$	0
$H(2d-1,q^2)$	${}^2A_{2d-1}(q)$	(q^2,q)	1/2
$Q(2d,q)$	$B_d(q)$	(q,q)	1
$W(2d-1,q)$	$C_d(q)$	(q,q)	1
$H(2d,q^2)$	${}^2A_{2d}(q)$	(q^2,q^3)	3/2
$Q^-(2d+1,q)$	${}^2D_{d+1}(q)$	(q,q^2)	2

Table 4.1: The classical finite polar spaces with parameters $(s,t) = (s, s^e)$

Theorem 4.1.1. [23, Lemma 9.4.1] *In a classical finite polar space of rank d with parameters (q, q^e), the number of n-spaces is given by:*

$$\begin{bmatrix} d \\ n \end{bmatrix}_q \prod_{i=1}^{n} (q^{d+e-i} + 1).$$

Remark 4.1.2. The previous theorem allows computation of the number of b-spaces through a fixed a-space with $a < b \leq d$. This is the number of $(b-a)$-spaces in the residual polar space, which is isomorphic to a polar space of the same type and with the same two parameters but of rank $d-a$.

We will write G for the full automorphism group of the classical finite polar space in this chapter.

A fundamental difference with the projective geometries $\mathrm{PG}(n,q)$ is that the orbit under G of an ordered pair in $\Omega_a \times \Omega_b$ depends on $\dim(\pi_a \cap \pi_b)$, as well as on the Witt index of $\langle \pi_a, \pi_b \rangle$. In that span, the unique subspace of that dimension through π_a is given by $\langle \pi_a, \pi_b \cap \pi_a^{\perp} \rangle$. Obviously every automorphism of the polar space needs to preserve these two dimensions. The following theorem yields that these two dimensions indeed determine the orbit on ordered pairs.

Theorem 4.1.3. [135, Proposition 4.9] *In a classical finite polar space, the orbits of the full automorphism group on $\Omega_a \times \Omega_b$ are given by:*

$$R_{a,b}^{s,k} := \{(\pi_a, \pi_b) | \dim(\pi_a \cap \pi_b) = s, \dim(\langle \pi_a, \pi_b \cap \pi_a^{\perp} \rangle) = k\},$$

with $0 \leq s \leq \min(a,b)$ and $\max(a,b) \leq k \leq \min(d, a+b-s)$.

Theorem 4.1.4. [23, Lemma 9.4.2] *For any a-space π_a in a classical finite polar space of rank d with parameters (q, q^e), the number of b-spaces π_b intersecting π_a in an s-space and with Witt index k for $\langle \pi_a, \pi_b \rangle$, i.e. with $(\pi_a, \pi_b) \in R_{a,b}^{s,k}$, is given by:*

$$q^{(a-s)(k-a)+(a+b-s-k)(a-b-s-3k+4d+2e-1)/2} \times$$
$$\begin{bmatrix} a \\ s \end{bmatrix}_q \begin{bmatrix} a-s \\ k-b \end{bmatrix}_q \begin{bmatrix} d-a \\ k-a \end{bmatrix}_q \prod_{i=0}^{k-a-1} (q^{d-a-i+e-1} + 1).$$

Figures 4.1 and 4.2 visualize the possible orbits of G on $\Omega_a \times \Omega_b$, by giving all possible parameters (s,k) for the relations $R_{a,b}^{s,k}$.

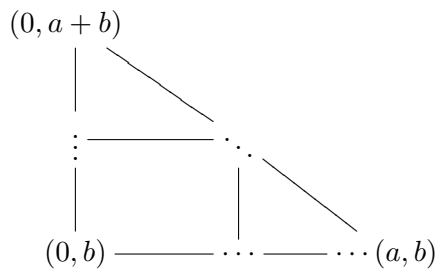

Figure 4.1: Theorem 4.1.3: (s,k) with $R_{a,b}^{s,k} \subseteq \Omega_a \times \Omega_b$ if $a \leq b$ and $a+b \leq d$ (triangular case)

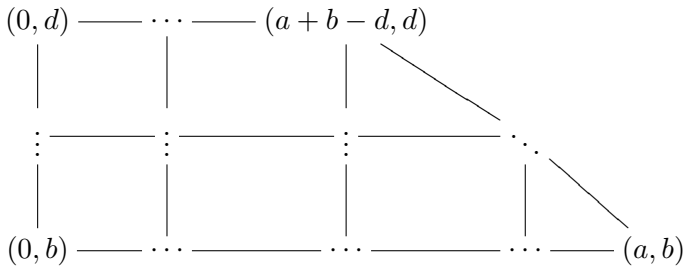

Figure 4.2: Theorem 4.1.3: (s,k) with $R^{s,k}_{a,b} \subseteq \Omega_a \times \Omega_b$ if $a \leq b$ and $a+b \geq d$ (trapezoidal case)

Theorem 4.1.3 in particular yields that the full automorphism group G of the polar space acts generously transitively on Ω_a and hence affords an association scheme. The number of relations in this scheme is given by $(a+1)(a+2)/2$ if $2a \leq d$ (the triangular case) and $(d-a+1)(3a-d+2)/2$ if $2a \geq d$ (the trapezoidal case).

If $(\pi_a, \pi_b) \in R^{\min(a,b),\max(a,b)}_{a,b}$ then π_a and π_b are incident (if $a \neq b$) or equal (if $a = b$). We simply write $R_{a,b}$ for this relation.

When (π_a, π_b) is in the orbit $R^{0,\max(a,b)}_{a,b}$, represented in the lower left corner in Figures 4.1 and 4.2, we say that π_a and π_b are *far away*. Note that this is the case if and only if π_a and π_b do no contain any common point, and no point on the subspace of the smallest dimension is collinear with all points in the other.

The relation $R^{a-1,\min(a+1,d)}_{a,a}$ yields the *graph of Lie type* on the a-spaces. In this graph on Ω_a, two elements are adjacent if they intersect a subspace of codimension one and have a totally isotropic span unless they are generators. For $a = 1$ and $a = d$, these graphs are also known under specific names.

Definition 4.1.5. *Consider a classical finite polar space $\mathcal{P}$ of rank d.*

(i) *The polar graph on $\mathcal{P}$ is the graph of Lie type on isotropic 1-spaces (with two vertices adjacent if they span a totally isotropic 2-space).*

(ii) *The dual polar graph on $\mathcal{P}$ is the graph of Lie type on totally isotropic d-spaces (with two vertices adjacent if they intersect in a $(d-1)$-space).*

The following theorem illustrates the importance of these two graphs.

Theorem 4.1.6. [23, pp. 334-336] *In a classical finite polar space of rank d, the graph of Lie type on a-spaces, $a \in \{1, \ldots, d\}$, is distance-regular if and only if $a = 1$ or $a = d$. It is also distance-transitive in these cases.*

The polar graph of a classical finite polar space of rank $d \geq 2$ with parameters (q, q^e) can be seen as the point graph of an SPBIBD satisfying the conditions of Lemma 2.3.6. The "points" of this SPBIBD are the isotropic 1-spaces of the polar space, while its "lines" are the maximal totally isotropic subspaces, and incidence is just symmetrized strict inclusion. This is an SPBIBD with parameters $(v, b, r, k, \lambda_1, 0)$ of type $(k-1, \alpha)$, with $k = (q^d - 1)/(q-1)$, $r = \prod_{i=1}^{d-1}(q^{i+e-1} + 1)$, $\lambda_1 = \prod_{i=1}^{d-2}(q^{i+e-1} + 1)$ and $\alpha = (q^{d-1} - 1)/(q-1)$.

Theorem 4.1.7. [23, Theorem 9.4.3] *The dual polar graph Γ on a classical finite polar space of rank d with parameters (q, q^e) is distance-regular with classical parameters $(d, q, 0, q^e)$. The intersection numbers are given by:*

$$b_i = q^{i+e} \begin{bmatrix} d-i \\ 1 \end{bmatrix}_q, \forall i \in \{0, \ldots, d-1\}; c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_q, \forall i \in \{1, \ldots, d\}.$$

Two vertices are at distance i if they intersect in a subspace of codimension i, and the valency of the distance-i relation is given by $\left[{d \atop i}\right]_q q^{i(i-1)/2} q^{ie}$.

Note that the above implies that the dual polar graph on a classical finite polar space of rank d defines the d-class association scheme on generators.

The dual polar graph on $H(2d-1, q^2)$ is very special, and this will be a recurring theme in this thesis. We give one of its exceptional properties.

Theorem 4.1.8. [23, Section 6.2] *The dual polar graph on $H(2d-1, q^2)$ has classical parameters $(d, b, \alpha, \beta) = (d, q^2, 0, q)$, as well as:*

$$(d, b, \alpha, \beta) = \left(d, -q, -\frac{q(q+1)}{q-1}, -\frac{q((-q)^d + 1)}{q-1}\right).$$

When two subspaces in Ω_a are far away, we also say they are *opposite*. Note that for generators, the oppositeness relation $R^{0,d}_{d,d}$ corresponds with the maximum distance relation with respect to the dual polar graph of diameter d.

The following notation is based on the one used in [77]. The numbers are well-defined because of Theorem 4.1.3.

Definition 4.1.9. *Consider a classical finite polar space of rank d.*

(i) *For any a-space π_a, we let $\alpha_{d,a,c,l}$ denote the number of c-spaces π_c with $(\pi_a, \pi_c) \in R^{0,l}_{a,c}$.*

(ii) *If $(\pi_a, \pi_b) \in R^{s,k}_{a,b}$, we let $\alpha_{d,(a,b),(s,k),c,(t,l)}$ denote the number of c-spaces π_c through π_b with $(\pi_a, \pi_c) \in R^{t,l}_{a,c}$.*

(iii) *If $(\pi_a, \pi_b) \in R^{s,k}_{a,b}$, we let $\gamma_{(a,b),(s,k),c,(t,l)}$ denote the number of c-spaces π_c in π_b with $(\pi_a, \pi_c) \in R^{t,l}_{a,c}$.*

For proofs of the following technical results, we refer to Theorem 4.1.4, [77, Theorem 3.6(d)] and [135, Proposition 6.5], respectively.

Lemma 4.1.10. *Consider a classical finite polar space of rank d with parameters (q, q^e).*

(i) $\alpha_{d,a,c,l} =$

$$q^{a(l-a)+(a+c-l)(a-c-3l+4d+2e-1)/2} \begin{bmatrix} a \\ l-c \end{bmatrix}_q \begin{bmatrix} d-a \\ l-a \end{bmatrix}_q \prod_{i=0}^{l-a-1} (q^{d-a-i+e-1}+1),$$

(ii) $\alpha_{d,(a,b),(s,k),c,(t,l)} =$

$$\begin{bmatrix} k-b \\ t-s \end{bmatrix}_q \alpha_{d-b+s-t,k-b+s-t,c-b+s-t,l-b+s-t},$$

(iii) $\gamma_{(a,b),(s,k),c,(t,l)} =$

$$\begin{bmatrix} s \\ t \end{bmatrix}_q \begin{bmatrix} k-a \\ l-a \end{bmatrix}_q \begin{bmatrix} a+b-s-k \\ a+c-t-l \end{bmatrix}_q q^{(s-t)(l-a)+(s+k-l-t)(a+c-l-t)}.$$

4.2 Irreducible submodules for polar spaces

For each relation $R^{s,k}_{a,b}$, we will denote the corresponding $(0,1)$-matrix, with columns and rows indexed by Ω_a and Ω_b, respectively, by $C^{s,k}_{a,b}$. Theorem 4.1.3

yields that the relations $R^{s,k}_{a,b}$ are the orbits of G on $\Omega_a \times \Omega_b$. We know from Subsection 2.5.3 that these $C^{s,k}_{a,b}$ form a basis for $\mathrm{Hom}_{\mathbb{R}G}(\mathbb{R}^{\Omega_a}, \mathbb{R}^{\Omega_b})$. Note that if $0 \leq a \leq b \leq d$, then $\dim(\mathrm{Hom}_{\mathbb{R}G}(\mathbb{R}^{\Omega_a}, \mathbb{R}^{\Omega_b}))$ is given by $(a+1)(a+2)/2$ if $a+b \leq d$ (the triangular case), and by $(2a+b-d+2)(d-b+1)/2$ if $a+b \geq d$ (the trapezoidal case). For the incidence relation, we will also write $C_{a,b}$ for the corresponding matrix $C^{\min(a,b),\max(a,b)}_{a,b}$. If there is no non-empty relation $R^{s,k}_{a,b}$ in $(\Omega_a \times \Omega_b)$, then we will agree that $C^{s,k}_{a,b} = 0$.

Stanton described the decomposition of $\mathbb{R}^{\Omega_n}$ into irreducible G-modules.

Theorem 4.2.1. [135, Theorem 6.23] *Consider a classical finite polar space of rank d. Under the action of G, every module $\mathbb{R}^{\Omega_n}$, $0 \leq n \leq d$, has a unique decomposition into irreducibles, which is multiplicity-free and orthogonal, and given by:*

$$\mathbb{R}^{\Omega_n} = \bigoplus_{\substack{0 \leq r \leq n \\ 0 \leq i \leq \min(r, d-n)}} V^n_{r,i},$$

where submodules $V^a_{r,i} \subseteq \mathbb{R}^{\Omega_a}$ and $V^b_{r,i} \subseteq \mathbb{R}^{\Omega_b}$ are isomorphic.

We now consider the vector space $V = \mathbb{R}^{\Omega_0} \perp \ldots \perp R^{\Omega_d}$ for any classical finite polar space of rank d. Each $C^{s,k}_{a,b}$ induces an endomorphism on V, vanishing on every component different from $\mathbb{R}^{\Omega_a}$ and mapping into $\mathbb{R}^{\Omega_b}$, that we will denote by $\tilde{C}^{s,k}_{a,b}$. The algebra of endomorphisms on V generated by all incidence maps $\tilde{C}_{a,b}$ is the *incidence algebra*, which was discussed by Terwilliger [142] in the much more general context of uniform posets.

Lemma 4.2.2. *The incidence algebra of a classical finite polar space of rank d has the $\tilde{C}_{t,b}\tilde{C}_{s,t}\tilde{C}_{a,s}$, $0 \leq s \leq a \leq b \leq t \leq d \leq s+t$, together with their transposes, as a basis, and is spanned by the elements $\tilde{C}^{s,k}_{a,b}$ with $C^{s,k}_{a,b} \in \mathrm{Hom}_{\mathbb{R}G}(\mathbb{R}^{\Omega_a}, \mathbb{R}^{\Omega_b})$.*

Proof. It follows from [142, Corollary 2.6 and Theorem 3.3] that the $\tilde{C}_{t,b}\tilde{C}_{s,t}\tilde{C}_{a,s}$ with $0 \leq s \leq a \leq b \leq t \leq d \leq s+t$, together with their transposes, form a basis for the incidence algebra.

Now suppose $a \leq b$. The endomorphisms in the incidence algebra, induced by a map from $\mathbb{R}^{\Omega_a}$ to $\mathbb{R}^{\Omega_b}$, form a space of dimension

$$\sum_{s=0}^{a} \sum_{t=\max(b,d-s)}^{d} 1 = \sum_{s'=0}^{a} \sum_{t=\max(b,d+s'-a)}^{d} 1 = \sum_{s'=0}^{a} \sum_{k=b}^{\min(a+b-s',d)} 1$$

(use the substitutions $s = a - s'$ and $t = b + d - k$, respectively), which is precisely $\dim(\mathrm{Hom}_{\mathbb{R}G}(\mathbb{R}^{\Omega_a}, \mathbb{R}^{\Omega_b}))$, and hence that space is precisely the space of maps induced by elements of $\mathrm{Hom}_{\mathbb{R}G}(\mathbb{R}^{\Omega_a}, \mathbb{R}^{\Omega_b})$, of which it must be a subspace.

If $a \geq b$, then again by considering transposes, one sees that every $\tilde{C}^{s,k}_{a,b}$ with $C^{s,k}_{a,b} \in \mathrm{Hom}_{\mathbb{R}G}(\mathbb{R}^{\Omega_a}, \mathbb{R}^{\Omega_b})$ is in the incidence algebra. □

Lemma 4.2.3. *Consider a classical finite polar space. The restriction of $C_{a,b}$ to $V^a_{r,i}$ has a trivial kernel if there is an isomorphic copy $V^b_{r,i}$.*

Proof. Suppose first that $a \leq b$. Since $V^a_{r,i}$ is irreducible, the restriction of $C_{a,b}$ is either trivial or has a trivial kernel. Suppose we are in the last case. We know from Lemma 4.2.2 that we can write every element in $\mathrm{Hom}_{\mathbb{R}G}(\mathbb{R}^{\Omega_a}, \mathbb{R}^{\Omega_b})$ as a linear combination of maps of the form $C_{t,b}C_{s,t}C_{a,s}$ with $0 \leq s \leq a \leq b \leq t \leq d \leq t + s$. For each such composition, the component $C_{s,t}$ can be written, up to a positive scalar, as $C_{b,t}C_{a,b}C_{s,a}$. Hence this would imply that every element of $\mathrm{Hom}_{\mathbb{R}G}(\mathbb{R}^{\Omega_a}, \mathbb{R}^{\Omega_b})$ vanishes on $V^a_{r,i}$, which is impossible by Theorem 2.5.15.

If $a > b$, then we can use the above argument since $(C_{a,b})^T = C_{b,a}$. □

We now state the main theorem of this subsection. We agree that $C_{d+1,n} = 0$ and $C_{-1,n} = 0$ for polar spaces of rank d.

Theorem 4.2.4. *Consider a classical finite polar space of rank d.*

(i) *Under the action of G, every module $\mathbb{R}^{\Omega_n}$ has a unique decomposition into irreducibles, which is multiplicity-free and orthogonal, and given by:*

$$\mathbb{R}^{\Omega_n} = \bigoplus_{\substack{0 \leq r \leq n \\ 0 \leq i \leq \min(r, d-n)}} V^n_{r,i},$$

with

$$V^n_{r,i} = \mathrm{Im}(C_{r,n}) \cap \mathrm{Im}(C_{r-1,n})^\perp \cap \mathrm{Im}(C_{d-i,n}) \cap \mathrm{Im}(C_{d-i+1,n})^\perp.$$

The submodules $V^a_{r,i} \subseteq \mathbb{R}^{\Omega_a}$ and $V^b_{r,i} \subseteq \mathbb{R}^{\Omega_b}$ are isomorphic.

(ii) *The restriction of the incidence map $C_{a,b} : \mathbb{R}^{\Omega_a} \to \mathbb{R}^{\Omega_b}$ to a submodule $V^a_{r,i} \subseteq \mathbb{R}^{\Omega_a}$ is trivial when there is no non-trivial $V^b_{r,i}$ in $\mathbb{R}^{\Omega_b}$, and is a bijection between the two isomorphic submodules in the other case.*

(iii) *The restriction of every map* $C^{s,k}_{a,b}$ *to* $V^a_{r,i} \subseteq \mathbb{R}^{\Omega_a}$ *is a scalar multiple of the restriction of* $C_{a,b}$.

Proof. This follows from Theorem 2.5.15, Theorem 4.2.1 and Lemma 4.2.3. □

The structure of the decomposition, described in Theorem 4.2.4 is of crucial importance, and therefore we draw the schemes for classical finite polar spaces of small rank in Figures 4.3, 4.4 and 4.5. Each column corresponds with the permutation module on the set of totally isotropic subspaces of a certain dimension, and isomorphic irreducible G-modules are written in the same row.

$\mathbb{R}^{\Omega_0}$	$\mathbb{R}^{\Omega_1}$	$\mathbb{R}^{\Omega_2}$
$V^0_{0,0}$	$V^1_{0,0}$	$V^2_{0,0}$
	$V^1_{1,0}$ $V^1_{1,1}$	$V^2_{1,0}$
		$V^2_{2,0}$

Figure 4.3: The decomposition for polar spaces of rank two

$\mathbb{R}^{\Omega_0}$	$\mathbb{R}^{\Omega_1}$	$\mathbb{R}^{\Omega_2}$	$\mathbb{R}^{\Omega_3}$
$V^0_{0,0}$	$V^1_{0,0}$	$V^2_{0,0}$	$V^3_{0,0}$
	$V^1_{1,0}$ $V^1_{1,1}$	$V^2_{1,0}$ $V^2_{1,1}$	$V^3_{1,0}$
		$V^2_{2,0}$ $V^2_{2,1}$	$V^3_{2,0}$
			$V^3_{3,0}$

Figure 4.4: The decomposition for polar spaces of rank three

$\mathbb{R}^{\Omega_0}$	$\mathbb{R}^{\Omega_1}$	$\mathbb{R}^{\Omega_2}$	$\mathbb{R}^{\Omega_3}$	$\mathbb{R}^{\Omega_4}$
$V^0_{0,0}$	$V^1_{0,0}$	$V^2_{0,0}$	$V^3_{0,0}$	$V^4_{0,0}$
	$V^1_{1,0}$ $V^1_{1,1}$	$V^2_{1,0}$ $V^2_{1,1}$	$V^3_{1,0}$ $V^3_{1,1}$	$V^4_{1,0}$
		$V^2_{2,0}$ $V^2_{2,1}$ $V^2_{2,2}$	$V^3_{2,0}$ $V^3_{2,1}$	$V^4_{2,0}$
			$V^3_{3,0}$ $V^3_{3,1}$	$V^4_{3,0}$
				$V^4_{4,0}$

Figure 4.5: The decomposition for polar spaces of rank four

4.3 Specific eigenvalues for polar spaces

We will now use the decomposition into irreducibles to compute several specific eigenvalues.

Theorem 4.2.4 allows us to define the following.

Definition 4.3.1. *In a classical finite polar space, we define $\theta_{(r,i),a,b,(s,k)}$ as the unique scalar such that*

$$C^{s,k}_{a,b}v = \theta_{(r,i),a,b,(s,k)}C_{a,b}v, \ \forall v \in V^a_{r,i}.$$

Note that since $(C^{s,k}_{a,b})^T = C^{s,k}_{b,a}$, we have $\theta_{(r,i),a,b,(s,k)} = \theta_{(r,i),b,a,(s,k)}$. If there is no irreducible submodule $V^a_{r,i}$ in $\mathbb{R}^{\Omega_a}$ or no $V^b_{r,i}$ in $\mathbb{R}^{\Omega_b}$, we write $\theta_{(r,i),a,b,(s,k)} = 0$.

In order to simplify our notation, we will also use the following notation based on that in [77].

Definition 4.3.2. *Consider a classical finite polar space of rank d.*

(i) $\psi_{d,r,i,s,k} = \theta_{(r,i),r,d-i,(s,k)}$,

(ii) $\chi_{d,i,r,n,t,l} = \theta_{(r,i),r,n,(t,l)}$.

The scalars $\psi_{d,r,i,s,k}$ are thus related to homomorphisms between the irreducible submodules $V^r_{r,i} \subseteq \mathbb{R}^{\Omega_r}$ and $V^{d-i}_{r,i} \subseteq \mathbb{R}^{\Omega_{d-i}}$, which are respectively the left- and right-most elements of an isomorphism class in Figures 4.3, 4.4 and 4.5.

Note that $\psi_{d,r,i,r,d-i} = 1$ if $0 \leq i \leq r \leq d-i$.

Lemma 4.3.3. *In a classical finite polar space of rank d with parameters (q, q^e), the following relations hold between homomorphisms if $a \leq b$:*

(i)

$$\begin{aligned} C_{b+1,b}C^{s,k}_{a,b+1} &= \begin{bmatrix} k-b \\ 1 \end{bmatrix}_q C^{s-1,k}_{a,b} + q^{2d-k-b-1+e} \begin{bmatrix} k-b \\ 1 \end{bmatrix}_q C^{s,k}_{a,b} \\ &\quad + q^{k-b-1} \begin{bmatrix} d-k+1 \\ 1 \end{bmatrix}_q (q^{d-k+e}+1) C^{s,k-1}_{a,b}, \end{aligned}$$

(ii)

$$\begin{aligned} C^{s,k}_{a-1,b}C_{a,a-1} &= q^{a-s-1} \begin{bmatrix} s+1 \\ 1 \end{bmatrix}_q C^{s+1,k}_{a,b} + \begin{bmatrix} a+b-s-k \\ 1 \end{bmatrix}_q C^{s,k}_{a,b} \\ &\quad + q^{a+b-s-k-1} \begin{bmatrix} k-b+1 \\ 1 \end{bmatrix}_q C^{s,k+1}_{a,b}. \end{aligned}$$

Proof.

(i) We can write this composition of homomorphisms as a linear combination of $C^{s',k'}_{a,b}$. If $(\pi_a, \pi_b) \in (\Omega_a \times \Omega_b)$ is in $R^{s',k'}_{a,b}$, then the corresponding coefficient is given by the number of $(b+1)$-spaces π'_{b+1} through π_b and with $(\pi_a, \pi'_{b+1}) \in R^{s,k}_{a,b+1}$. This is only possible if (s', k') is $(s-1, k)$, (s, k) or $(s, k-1)$. The desired coefficients are $\alpha_{d,(a,b),(s-1,k),b+1,(s,k)}$, $\alpha_{d,(a,b),(s,k),b+1,(s,k)}$ and $\alpha_{d,(a,b),(s,k-1),b+1,(s,k)}$, respectively, and follow from Lemma 4.1.10.

(ii) Similarly, we can write the left-hand side as a linear combination of $C^{s',k'}_{a,b}$, and if $(\pi_a, \pi_b) \in (\Omega_a \times \Omega_b)$ is in $R^{s',k'}_{a,b}$, then the corresponding coefficient is given by the number of $(a-1)$-spaces π'_{a-1} in π_a and with $(\pi'_{a-1}, \pi_b) \in R^{s,k}_{a-1,b}$. This is only possible if (s', k') is $(s+1, k)$, (s, k) or $(s, k+1)$, and the desired coefficients $\gamma_{(b,a),(s+1,k),a-1,(s,k)}$, $\gamma_{(b,a),(s,k),a-1,(s,k)}$ and $\gamma_{(b,a),(s,k+1),a-1,(s,k)}$ follow from Lemma 4.1.10(iii). □

Lemma 4.3.4. *In a classical finite polar space of rank d with parameters (q, q^e), the following relations hold:*

(i)

$$\begin{bmatrix} k-d+i \\ 1 \end{bmatrix}_q \psi_{d,r,i,s-1,k} + q^{d-k+i+e-1} \begin{bmatrix} k-d+i \\ 1 \end{bmatrix}_q \psi_{d,r,i,s,k}$$

$$+q^{k-d+i-1} \begin{bmatrix} d-k+1 \\ 1 \end{bmatrix}_q (q^{d-k+e}+1)\psi_{d,r,i,s,k-1} = 0,$$

(ii)

$$q^{r-s-1} \begin{bmatrix} s+1 \\ 1 \end{bmatrix}_q \psi_{d,r,i,s+1,k} + \begin{bmatrix} d-i+r-s-k \\ 1 \end{bmatrix}_q \psi_{d,r,i,s,k}$$

$$+q^{d-i+r-s-k-1} \begin{bmatrix} k+1-d+i \\ 1 \end{bmatrix}_q \psi_{d,r,i,s,k+1} = 0,$$

Proof.

(i) Consider any non-zero vector $v \in V^r_{r,i}$. It follows from Theorem 4.2.4 that there is no isomorphic irreducible submodule $V^{(d-i)+1}_{r,i}$ in $\mathbb{R}^{\Omega_{(d-i)+1}}$, and hence that $C^{s,k}_{r,(d-i)+1}v = 0$. The desired result follows from the expansion of $C_{(d-i)+1,d-i}C^{s,k}_{r,(d-i)+1}$ using Lemma 4.3.3(i).

(ii) Consider any non-zero vector $v \in V^r_{r,i}$. We know from Theorem 4.2.4 that $C_{r,r-1}v = 0$. The desired result now follows from the expansion of $C^{s,k}_{r-1,d-i}C_{r,r-1}v$ using Lemma 4.3.3(ii). □

If $0 \leq i \leq r \leq d-i$, then the diagram of orbits of G on $\Omega_r \times \Omega_{d-i}$ is trapezoidal (as in Figure 4.2). We now explicitly compute those values $\psi_{d,r,i,s,k}$ corresponding to the upper right, upper and left edge of the diagram (see Figure 4.6).

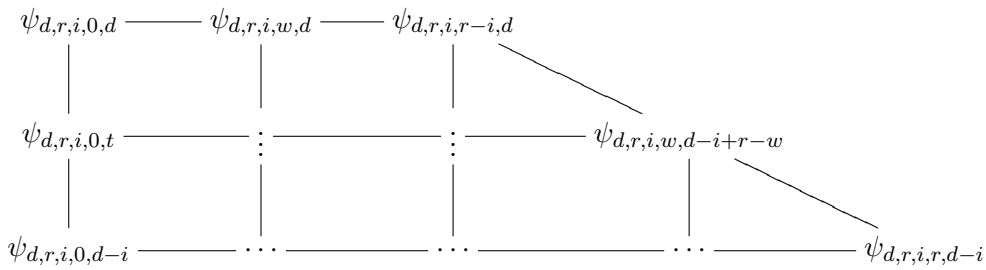

Figure 4.6: The values of ψ for fixed r and i

Lemma 4.3.5. *Consider a classical finite polar space of rank d with parameters (q, q^e). Suppose $0 \leq i \leq r \leq d - i$.*

(i) *If $(r - i) \leq w \leq r$:*

$$\psi_{d,r,i,w,d-i+r-w} = (-1)^{w+r} q^{(r-w-1)(r-w)/2} \begin{bmatrix} i \\ r-w \end{bmatrix}_q \prod_{1 \leq m \leq r-w} (q^{i-m+e} + 1),$$

(ii) *if $0 \leq w \leq (r - i)$:*

$$\psi_{d,r,i,w,d} = (-1)^{w+r} \begin{bmatrix} r-i \\ w \end{bmatrix}_q q^{(r-w)(r-w-1)/2} \prod_{1 \leq m \leq i} (q^{i-m+e} + 1),$$

(iii) *if $d - i \leq t \leq d$:*

$$\begin{aligned} \psi_{d,r,i,0,t} &= (-1)^{d+r+t} q^{r(r-1)/2+(d-t)^2+(d-t)(e-1)} \begin{bmatrix} i \\ t-d+i \end{bmatrix}_q \times \\ &\quad \prod_{1 \leq m \leq t-d+i} (q^{i-m+e} + 1). \end{aligned}$$

Proof.

(i) The result is clear for $w = r$ as $\psi_{d,r,i,r,d-i} = 1$. For any w with $r - i \leq w \leq r - 1$, Lemma 4.3.4(i) with $(s, k) = (w + 1, d - i + r - w)$ yields that:

$$\begin{bmatrix} r-w \\ 1 \end{bmatrix}_q \psi_{d,r,i,w,d-i+r-w} +$$

$$q^{r-w-1}\begin{bmatrix} i-r+w+1 \\ 1 \end{bmatrix}_q (q^{i-r+w+e}+1)\psi_{d,r,i,w+1,d-i+r-(w+1)} = 0,$$

and since $\left[\begin{smallmatrix} i \\ r-w \end{smallmatrix}\right]_q \left[\begin{smallmatrix} r-w \\ 1 \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} i-r+w+1 \\ 1 \end{smallmatrix}\right]_q \left[\begin{smallmatrix} i \\ r-w-1 \end{smallmatrix}\right]_q$, induction yields for every $w \in \{r-i, \dots, r\}$:

$$\psi_{d,r,i,w,d-i+r-w} = (-1)^{w+r} q^{(r-w-1)(r-w)/2} \begin{bmatrix} i \\ r-w \end{bmatrix}_q \prod_{1 \le m \le r-w} (q^{i-m+e}+1).$$

(ii) For $w = r-i$ the result follows from the above as

$$\psi_{d,r,i,r-i,d} = (-1)^i q^{i(i-1)/2} \prod_{1 \le m \le i} (q^{i-m+e}+1).$$

For any w with $0 \le w \le (r-i)-1$, Lemma 4.3.4(ii) yields with $(s,k) = (w,d)$ that:

$$\begin{bmatrix} w+1 \\ 1 \end{bmatrix}_q q^{r-w-1}\psi_{d,r,i,w+1,d} + \begin{bmatrix} r-i-w \\ 1 \end{bmatrix}_q \psi_{d,r,i,w,d} = 0.$$

By using $\left[\begin{smallmatrix} r-i \\ w+1 \end{smallmatrix}\right]_q \left[\begin{smallmatrix} w+1 \\ 1 \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} r-i \\ w \end{smallmatrix}\right]_q \left[\begin{smallmatrix} r-i-w \\ 1 \end{smallmatrix}\right]_q$, we obtain by induction that for every $w \in \{0, \dots, r-i\}$:

$$\psi_{d,r,i,w,d} = (-1)^{w+r} \begin{bmatrix} r-i \\ w \end{bmatrix}_q q^{(r-w)(r-w-1)/2} \prod_{1 \le m \le i} (q^{i-m+e}+1).$$

(iii) For $t = d$ the result again already follows from the above as

$$\psi_{d,r,i,0,d} = (-1)^r q^{r(r-1)/2} \prod_{1 \le m \le i} (q^{i-m+e}+1).$$

If $d-i \le t \le d-1$, then Lemma 4.3.4(i) yields with $(s,k) = (0, t+1)$ that:

$$q^{d+i-t-2+e}\begin{bmatrix} t+1-d+i \\ 1 \end{bmatrix}_q \psi_{d,r,i,0,t+1}$$

$$+q^{t+i-d}\begin{bmatrix} d-t \\ 1 \end{bmatrix}_q (q^{d-t-1+e}+1)\psi_{d,r,i,0,t} = 0$$

or:

$$\psi_{d,r,i,0,t} = -q^{2(d-t-1)+e} \frac{\left[{t+1-d+i \atop 1}\right]_q}{\left[{d-t \atop 1}\right]_q (q^{d-t-1+e}+1)} \psi_{d,r,i,0,t+1}.$$

Using the identity $\left[{i \atop t+1-d+i}\right]_q \left[{t+1-d+i \atop 1}\right]_q = \left[{i \atop t-d+i}\right]_q \left[{d-t \atop 1}\right]_q$, we obtain by induction that for every $t \in \{d-i, \ldots, d\}$, $\psi_{d,r,i,0,t}$ is given by:

$$(-1)^{d+r+t} q^{r(r-1)/2+(d-t)^2+(d-t)(e-1)} \begin{bmatrix} i \\ t-d+i \end{bmatrix}_q \prod_{1 \leq m \leq t-d+i} (q^{i-m+e}+1).$$

□

4.3.1 Eigenvalues for generators

We will write R_i for the relation $R_{d,d}^{d-i,d}$ between the d-spaces or generators, and A_i for the corresponding symmetric adjacency matrix. The relation R_1 corresponds with the distance-regular dual polar graph of diameter d (see Theorem 4.1.6), and $(\pi_d, \pi'_d) \in R_i$ if and only if $\pi_d \cap \pi'_d$ has dimension $d-i$.

Theorem 4.3.6. *Consider a classical finite polar space of rank d with parameters (q, q^e). The eigenvalue of the relation R_i between generators for the subspace $V_{j,0}^d$, $0 \leq j \leq d$, is given by:*

$$\sum_{0,j-i \leq u \leq d-i,j} (-1)^{j+u} \begin{bmatrix} d-j \\ d-i-u \end{bmatrix}_q \begin{bmatrix} j \\ u \end{bmatrix}_q q^{(u+i-j)(u+i-j+2e-1)/2+(j-u)(j-u-1)/2}.$$

Proof. Consider any non-zero vector $v \in V_{j,0}^d \subseteq \mathbb{R}^{\Omega_d}$. Theorem 4.2.4 implies that there is a (unique) vector $w \in V_{j,0}^j$, such that $v = C_{j,d}w$. As we are interested in $A_i v$, we have to consider the composition $A_i C_{j,d}$.

$$\begin{tikzcd} \mathbb{R}^{\Omega_j} \arrow[r, bend left=60, "C_{j,d}^{u,d}"] \arrow[r, "C_{j,d}"'] & \mathbb{R}^{\Omega_d} \arrow[loop right, "A_i"] \end{tikzcd}$$

If a j-space π_j is in a generator π_d, and π'_d is a generator intersecting π_d in a $(d-i)$-space, then π_j and π'_d must meet in a u-space with $\max(0, j-i) \leq u \leq \min(d-i, j)$. Hence we can write:

$$A_i C_{j,d} = \sum_{\max(0,j-i)\leq u\leq \min(d-i,j)} \alpha_{d,(d,j),(u,d),d,(d-i,d)} C^{u,d}_{j,d}.$$

It follows immediately from Definition 4.3.2 that $C^{u,d}_{j,d} w = \psi_{d,j,0,u,d} C_{j,d} w$. We can now determine $A_i v$:

$$\begin{aligned} A_i v &= A_i C_{j,d} w \\ &= \sum_{\max(0,j-i)\leq u\leq \min(d-i,j)} \alpha_{d,(d,j),(u,d),d,(d-i,d)} C^{u,d}_{j,d} w \\ &= \sum_{\max(0,j-i)\leq u\leq \min(d-i,j)} \alpha_{d,(d,j),(u,d),d,(d-i,d)} \psi_{d,j,0,u,d} C_{j,d} w \\ &= \left(\sum_{\max(0,j-i)\leq u\leq \min(d-i,j)} \alpha_{d,(d,j),(u,d),d,(d-i,d)} \psi_{d,j,0,u,d} \right) v. \end{aligned}$$

Lemma 4.1.10 gives us that $\alpha_{d,(d,j),(u,d),d,(d-i,d)} = \begin{bmatrix} d-j \\ d-i-u \end{bmatrix}_q q^{(u+i-j)(u+i-j+2e-1)/2}$. We also know from Lemma 4.3.5(ii) that $\psi_{d,j,0,u,d} = (-1)^{j+u} \begin{bmatrix} j \\ u \end{bmatrix}_q q^{(j-u)(j-u-1)/2}$. This proves the theorem. □

Remark 4.3.7. Stanton [135, Theorem 5.4] expressed these eigenvalues in terms of q-Krawtchouk polynomials.

As an example, we use Theorem 4.3.6 to compute the matrix of eigenvalues P for classical finite polar spaces of rank three with parameters (q, q^e). We use the ordering R_0, R_1, R_2, R_3 for the columns and $V^3_{0,0}, V^3_{1,0}, V^3_{2,0}, V^3_{3,0}$ for the rows in Table 4.2.

$$P = \begin{pmatrix} 1 & q^e(q^2+q+1) & q^{1+2e}(q^2+q+1) & q^{3+3e} \\ 1 & q^{1+e}+q^e-1 & q^{1+2e}-q^{1+e}-q^e & -q^{1+2e} \\ 1 & q^e-q-1 & q-q^{1+e}-q^e & q^{1+e} \\ 1 & -q^2-q-1 & q(q^2+q+1) & -q^3 \end{pmatrix}$$

Table 4.2: Theorem 4.3.6: Eigenvalues for generators in polar spaces of rank 3

4.3.2 Eigenvalues of the graph of Lie type

In this subsection, we will apply Theorem 4.2.4 to compute the eigenvalues of a graph of Lie type on n-spaces, defined by the relation $R_{n,n}^{n-1,\min(n+1,d)}$ (see Subsection 4.1). We first point out that the eigenvalues of many similar graphs, including polar and dual polar graphs, can be found in [26].

Lemma 4.3.8. *In a classical finite polar space of rank d with parameters (q, q^e), we have for every $r \geq 1$:*

$$\psi_{d,r,i,r-1,d-i} = -\begin{bmatrix} r \\ 1 \end{bmatrix}_q + \begin{bmatrix} i \\ 1 \end{bmatrix}_q (q^{i-1+e} + 1),$$

if $0 \leq i \leq r \leq d - i$.

Proof. From Lemma 4.3.4(ii) with $(s, k) = (r - 1, d - i)$, we obtain:

$$\begin{bmatrix} r \\ 1 \end{bmatrix}_q \psi_{d,r,i,r,d-i} + \psi_{d,r,i,r-1,d-i} + \psi_{d,r,i,r-1,d-i+1} = 0.$$

From Lemma 4.3.5(i) we know that $\psi_{d,r,i,r,d-i} = 1$ and $\psi_{d,r,i,r-1,d-i+1} = -\left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right]_q (q^{i-1+e} + 1)$, which yields the desired result. □

Lemma 4.3.9. *In a classical finite polar space of rank d with parameters (q, q^e), we have the following identities if $r \geq 1$ and $0 \leq i \leq r \leq d - i$.*

(i) $\chi_{d,i,r,r,r-1,r} = -q^{d-i-r}\Big(\left[\begin{smallmatrix} r \\ 1 \end{smallmatrix}\right]_q - \left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right]_q (q^{i-1+e} + 1)\Big)$,

(ii) $\chi_{d,i,r,n,r-1,n} = -q^{d-i-n}\Big(\left[\begin{smallmatrix} r \\ 1 \end{smallmatrix}\right]_q - \left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right]_q (q^{i-1+e} + 1)\Big)$ *if $r \leq n \leq d - i$,*

(iii) $\chi_{d,i,r,n,r-1,n+1} = \left[\begin{smallmatrix} r \\ 1 \end{smallmatrix}\right]_q (q^{d-i-n} - 1) - q^{d-i-n}(q^{i-1+e} + 1) \left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right]_q$ *if $r \leq n \leq d - i$.*

Proof.

(i) We consider the composition $C_{r,r}^{r-1,r} C_{d-i,r}$.

$$\begin{tikzcd}
\mathbb{R}^{\Omega_r} \arrow[loop left, "C_{r,r}^{r-1,r}"] & \mathbb{R}^{\Omega_{d-i}} \arrow[l, "C_{d-i,r}"'] \arrow[l, bend right, "C_{d-i,r}^{r-1,d-i}"']
\end{tikzcd}$$

For any $\pi_r \in \Omega_r$ and $\pi_{d-i} \in \Omega_{d-i}$, it is only possible that $(\pi_r, \pi_r') \in R_{r,r}^{r-1,r}$ and $\pi_r' \subseteq \pi_{d-i}$ if $(\pi_{d-i}, \pi_r) \in R_{d-i,r}^{r-1,d-i}$. Hence we can write

$$C_{r,r}^{r-1,r} C_{d-i,r} = \gamma_{(r,d-i),(r-1,d-i),r,(r-1,r)} C_{d-i,r}^{r-1,d-i} = q^{d-i-r} C_{d-i,r}^{r-1,d-i}$$

where we applied Lemma 4.1.10(iii) for the last step. Now choosing any non-zero $v \in V_{r,i}^r$, we know from Theorem 4.2.4 that $v = C_{d-i,r} w$ with $w \in V_{r,i}^{d-i}$. This yields:

$$\begin{aligned}
\chi_{d,i,r,r,r-1,r} v &= C_{r,r}^{r-1,r} v \\
&= (C_{r,r}^{r-1,r} C_{d-i,r}) w \\
&= (q^{d-i-r} C_{d-i,r}^{r-1,d-i}) w \\
&= q^{d-i-r} \psi_{d,r,i,r-1,d-i} C_{d-i,r} w \\
&= q^{d-i-r} \psi_{d,r,i,r-1,d-i} v,
\end{aligned}$$

and hence the result follows from Lemma 4.3.8.

(ii) Next, we consider the composition $C_{r,n} C_{r,r}^{r-1,r}$.

$$\begin{tikzcd}
\mathbb{R}^{\Omega_r} \arrow[loop left, "C_{r,r}^{r-1,r}"] \arrow[r, "C_{r,n}"'] \arrow[r, bend left, "C_{r,n}^{r-1,n}"] & \mathbb{R}^{\Omega_n}
\end{tikzcd}$$

For any $\pi_r \in \Omega_r$ and $\pi_n \in \Omega_n$, it is only possible that $(\pi_r, \pi_r') \in R_{r,r}^{r-1,r}$ and $\pi_r' \subseteq \pi_n$ if $(\pi_r, \pi_n) \in R_{r,n}^{r-1,n}$. Hence

$$C_{r,n}C_{r,r}^{r-1,r} = \gamma_{(r,n),(r-1,n),r,(r-1,r)}C_{r,n}^{r-1,n} = q^{n-r}C_{r,n}^{r-1,n},$$

where we used Lemma 4.1.10(iii) for the last step. Choosing any non-zero $v \in V_{r,i}^r$, we can write:

$$\begin{aligned} C_{r,n}(\chi_{d,i,r,r,r-1,r}v) &= C_{r,n}C_{r,r}^{r-1}v \\ &= q^{n-r}C_{r,n}^{r-1,n}v \\ &= q^{n-r}\chi_{d,i,r,n,r-1,n}C_{r,n}v. \end{aligned}$$

We can now use (i) to obtain $\chi_{d,i,r,n,r-1,n}$.

(iii) Applying Lemma 4.3.3(ii) with $(a, b) = (r, n)$ and $(s, k) = (r - 1, n)$, we can write:

$$C_{r-1,n}C_{r,r-1}v = \begin{bmatrix} r \\ 1 \end{bmatrix}_q C_{r,n} + C_{r,n}^{r-1,n} + C_{r,n}^{r-1,n+1}.$$

$$\begin{tikzcd} \mathbb{R}^{\Omega_{r-1}} & \mathbb{R}^{\Omega_r} \arrow[l, "C_{r,r-1}"'] \arrow[r, bend right, "C_{r,n}"'] \arrow[r, bend left, "C_{r,n}^{r-1,n}"] & \mathbb{R}^{\Omega_n} \arrow[ll, bend right, "C_{r-1,n}"'] \arrow[ll, bend left, "C_{r,n}^{r-1,n+1}"] \end{tikzcd}$$

If v is any non-zero vector in $V_{r,i}^r$, then $C_{r,r-1}v = 0$ by Theorem 4.2.4, and hence:

$$0 = \left(\begin{bmatrix} r \\ 1 \end{bmatrix}_q + \chi_{d,i,r,n,r-1,n} + \chi_{d,i,r,n,r-1,n+1} \right)(C_{r,n}v).$$

Since $C_{r,n}v \neq 0$, we can now compute $\chi_{d,i,r,n,r-1,n+1}$ using (i) and (ii). □

We can now finally compute the eigenvalue of a graph of Lie type in a classical finite polar space. Note that we have to treat the dual polar graph separately.

Theorem 4.3.10. *Consider a classical finite polar space of rank d with parameters (q, q^e), and the graph of Lie type Γ on n-spaces.*

(i) *If* $1 \leq n \leq d-1$*, then the eigenvalue of* Γ *for* $V^n_{r,i}$ *is given by:*

$$q \begin{bmatrix} n-r \\ 1 \end{bmatrix}_q \begin{bmatrix} d-n \\ 1 \end{bmatrix}_q (q^{d-n-1+e}+1)+$$

$$\begin{bmatrix} n+1-r \\ 1 \end{bmatrix}_q \left(\begin{bmatrix} r \\ 1 \end{bmatrix}_q (q^{d-i-n}-1) - q^{d-i-n}(q^{i-1+e}+1) \begin{bmatrix} i \\ 1 \end{bmatrix}_q \right).$$

(ii) *If* $n = d$*, then the eigenvalue of* Γ *for* $V^d_{r,0}$ *is given by:*

$$q^e \begin{bmatrix} d-r \\ 1 \end{bmatrix}_q - \begin{bmatrix} r \\ 1 \end{bmatrix}_q .$$

Proof.

(i) Suppose first that $1 \leq n \leq d-1$. Consider the composition $C^{n-1,n+1}_{n,n} C_{r,n}$.

$$\begin{tikzcd} \mathbb{R}^{\Omega_r} \arrow[r, bend left=60, "C^{r-1,n+1}_{r,n}"] \arrow[r, "C_{r,n}"'] & \mathbb{R}^{\Omega_n} \arrow[loop right, "C^{n-1,n+1}_{n,n}"] \end{tikzcd}$$

If $(\pi_r, \pi_n) \in (\Omega_r \times \Omega_n)$, then there can only be an n-space π'_n with $\pi_r \subseteq \pi'_n$ and $(\pi_n, \pi'_n) \in R^{n-1,n+1}_{n,n}$ if either $\pi_r \subseteq \pi_n$ or $(\pi_r, \pi_n) \in R^{r-1,n+1}_{r,n}$. Hence:

$$C^{n-1,n+1}_{n,n} C_{r,n}$$

$$= \alpha_{d,(n,r),(r,n),n,(n-1,n+1)} C_{r,n} + \alpha_{d,(n,r),(r-1,n+1),n,(n-1,n+1)} C^{r-1,n+1}_{r,n}$$

$$= q \begin{bmatrix} n-r \\ 1 \end{bmatrix}_q \begin{bmatrix} d-n \\ 1 \end{bmatrix}_q (q^{d-n-1+e}+1) C_{r,n} + \begin{bmatrix} n+1-r \\ 1 \end{bmatrix}_q C^{r-1,n+1}_{r,n},$$

where we used Lemma 4.1.10 for the last step. Now consider any non-zero $v \in V^n_{r,i}$. In order to compute the corresponding eigenvalue, we must

find $C_{n,n}^{n-1,n+1}v$. We know that $v = C_{r,n}w$ with $w \in V_{r,i}^r$, and hence:

$$\begin{aligned} C_{n,n}^{n-1,n+1}v &= C_{n,n}^{n-1,n+1}(C_{r,n}w) \\ &= q\begin{bmatrix} n-r \\ 1 \end{bmatrix}_q \begin{bmatrix} d-n \\ 1 \end{bmatrix}_q (q^{d-n-1+e}+1)C_{r,n}w + \\ &\quad \begin{bmatrix} n+1-r \\ 1 \end{bmatrix}_q C_{r,n}^{r-1,n+1}w \\ &= \Bigg(q\begin{bmatrix} n-r \\ 1 \end{bmatrix}_q \begin{bmatrix} d-n \\ 1 \end{bmatrix}_q (q^{d-n-1+e}+1) + \\ &\quad \begin{bmatrix} n+1-r \\ 1 \end{bmatrix}_q \chi_{d,i,r,n,r-1,n+1}\Bigg)(C_{r,n}w). \end{aligned}$$

The desired result now follows from Lemma 4.3.9(iii).

(ii) For $n = d$, the result already follows from Theorem 4.3.6 with $i = 1$. □

Remark 4.3.11. The eigenvalues of the graph of Lie type can also be extracted from the value $x_i(r,p)$ which was computed in [142, Theorems 2.5 and 3.3(6)].

Remark 4.3.12. We know from Theorem 4.1.7 that the dual polar graph on a classical finite polar space of rank d with parameters (q, q^e) has classical parameters $(d, q, 0, q^e)$. Applying Theorem 2.3.14 and comparing the eigenvalues, one easily sees that the ordering in the following decomposition for maximals:

$$\mathbb{R}^{\Omega_d} = V_{0,0}^d \perp V_{1,0}^d \perp \ldots \perp V_{d,0}^d,$$

is precisely the cometric ordering associated with this set of classical parameters.

4.3.3 Eigenvalues of oppositeness

The oppositeness relation between the subspaces of dimension n in a classical finite polar space, which we denote by $R_{n,n}^{0,n}$, is another particular relation, the eigenvalues of which can still be given explicitly in a more or less compact form. In order to compute these eigenvalues, we will also consider the far away relation between subspaces of different dimension.

Lemma 4.3.13. *Consider a classical finite polar space of rank d with parameters (q, q^e). Suppose $0 \leq i \leq r \leq n \leq d - i$.*

(i) $\chi_{d,i,r,r,0,r} = (-1)^{r+i} q^{i(i-1)+ie+r(d-i-r/2-1/2)}$,

(ii) $\chi_{d,i,r,n,0,n} = (-1)^{r+i} q^{i(i-1)+ie+r(d-n-i+r/2-1/2)}$.

Proof.

(i) We first consider the composition $C^{0,r}_{r,r} C_{d-i,r}$.

$$\begin{tikzcd}
\mathbb{R}^{\Omega_r} \arrow[loop left, "C^{0,r}_{r,r}"] & & \mathbb{R}^{\Omega_{d-i}} \arrow[ll, "C_{d-i,r}"'] \arrow[ll, bend right, "C^{0,d-i}_{d-i,r}"']
\end{tikzcd}$$

If $\pi_{d-i} \in \Omega_{d-i}$ contains an r-space π'_r that is far away from the r-space π_r, then π_r and π_{d-i} must also be far away, and hence:

$$C^{0,r}_{r,r} C_{d-i,r} = \gamma_{(r,d-i),(0,d-i),r,(0,r)} C^{0,d-i}_{d-i,r} = q^{r(d-i-r)} C^{0,d-i}_{d-i,r},$$

where we used Lemma 4.1.10(iii) for the last step. Now we consider any non-zero $v \in V^r_{r,i}$. We can write $v = C_{d-i,r} w$ with $w \in V^{d-i}_{r,i}$, and hence:

$$\begin{aligned}
\chi_{d,i,r,r,0,r} v &= C^{0,r}_{r,r} v \\
&= C^{0,r}_{r,r} (C_{d-i,r} w) \\
&= q^{r(d-i-r)} C^{0,d-i}_{d-i,r} w \\
&= q^{r(d-i-r)} \psi_{d,r,i,0,d-i} (C_{d-i,r} w) \\
&= q^{r(d-i-r)} \psi_{d,r,i,0,d-i} v.
\end{aligned}$$

It follows from Lemma 4.3.5(iii) that $\psi_{d,r,i,0,d-i} = (-1)^{r+i} q^{r(r-1)/2+i^2+ie-i}$, which allows us to compute $\chi_{d,i,r,r,0,r}$.

(ii) Next, we consider the composition $C_{r,n} C^{0,r}_{r,r}$.

$$\begin{tikzcd}
\mathbb{R}^{\Omega_r} \arrow[loop left, "C^{0,r}_{r,r}"] \arrow[r, "C_{r,n}"'] \arrow[rr, bend left, "C^{0,n}_{r,n}"] & \mathbb{R}^{\Omega_n}
\end{tikzcd}$$

Since any n-space π_n can only contain an r-space π'_r opposite to the r-space π_r, if π_r and π_n are far away, we can write:

$$C_{r,n}C^{0,r}_{r,r} = \gamma_{(r,n),(0,n),r,(0,r)}C^{0,n}_{r,n} = q^{r(n-r)}C^{0,n}_{r,n},$$

again using Lemma 4.1.10(iii) for the last step. For any non-zero $v \in V^r_{r,i}$, we can write:

$$\begin{aligned}
C_{r,n}(\chi_{d,i,r,r,0,r}v) &= C_{r,n}(C^{0,r}_{r,r}v) \\
&= q^{r(n-r)}(C^{0,n}_{r,n}v) \\
&= (q^{r(n-r)}\chi_{d,i,r,n,0,n})(C_{r,n}v).
\end{aligned}$$

We can now obtain $\chi_{d,i,r,n,0,n}$ using (i). □

We can now write the homomorphism associated with the far away relation as a scalar multiple of the incidence relation, when restricted to an irreducible submodule. The eigenvalues of oppositeness will then immediately follow as well.

Theorem 4.3.14. *Consider a classical finite polar space of rank d with parameters (q, q^e). If $0 \leq i \leq r \leq a \leq b \leq d - i$, then:*

$$\theta_{(r,i),a,b,(0,b)} = (-1)^{r+i}q^{i(i-1)+ie+r(d-b-i+r/2-1/2)+(a-r)(2d-b+e-r/2-a/2-1/2)}.$$

Proof. We start by considering the composition $C^{0,b}_{a,b}C_{r,a}$.

$$\begin{tikzcd}
\mathbb{R}^{\Omega_r} \arrow[r, "C_{r,a}"'] \arrow[rr, bend left=50, "C_{r,b}^{0,b}"] \arrow[rr, bend right=50, "C_{r,b}"'] & \mathbb{R}^{\Omega_a} \arrow[r, "C_{a,b}"'] \arrow[r, bend left=50, "C_{a,b}^{0,b}"] & \mathbb{R}^{\Omega_b}
\end{tikzcd}$$

For any r-space π_r and any b-space π_b, it is only possible that there is an a-space through π_r and far away from π_b, if π_r and π_b are far away. Hence we can write:

$$C_{a,b}^{0,b} C_{r,a} = \alpha_{d,(b,r),(0,b),a,(0,b)} C_{r,b}^{0,b}.$$

Any non-zero $v \in V_{r,i}^a$ can be written as $v = C_{r,a} w$ with $w \in V_{r,i}^r$, and then:

$$\begin{aligned}
\theta_{(r,i),a,b,(0,b)} C_{a,b} v &= C_{a,b}^{0,b} v \\
&= (C_{a,b}^{0,b} C_{r,a}) w \\
&= \alpha_{d,(b,r),(0,b),a,(0,b)} C_{r,b}^{0,b} w \\
&= \alpha_{d,(b,r),(0,b),a,(0,b)} \chi_{d,i,r,b,0,b}) (C_{r,b} w).
\end{aligned}$$

On the other hand, we can also write $C_{a,b} C_{r,a} = \left[{b-r \atop a-r}\right]_q C_{r,b}$, and hence:

$$\theta_{(r,i),a,b,(0,b)} (C_{a,b} v) = \left(\theta_{(r,i),a,b,(0,b)} \begin{bmatrix} b-r \\ a-r \end{bmatrix}_q \right) (C_{r,b} w).$$

This yields:

$$\theta_{(r,i),a,b,(0,b)} = \frac{\alpha_{d,(b,r),(0,b),a,(0,b)} \chi_{d,i,r,b,0,b}}{\left[{b-r \atop a-r}\right]_q}.$$

Lemma 4.1.10 yields that $\alpha_{d,(b,r),(0,b),a,(0,b)} = q^{(a-r)(2d-b+e-r/2-a/2-1/2)} \left[{b-r \atop a-r}\right]_q$, and now we can use Lemma 4.3.13(ii) to compute $\theta_{(r,i),a,b,(0,b)}$. □

For generators (i.e. $n = d$), the following theorem can also be found in [134].

Theorem 4.3.15. *The eigenvalue of the oppositeness relation $R_{n,n}^{0,n}$ between n-spaces in a classical finite polar space of rank d with parameters (q, q^e) for the eigenspace $V_{r,i}^n$, $0 \leq i \leq r \leq n \leq d-i$, is given by:*

$$(-1)^{r+i} q^{n(4d-3n-1)/2 - r(d-r) - i(r+1-i) + (n+i-r)e}.$$

Proof. This follows from Theorem 4.3.14 with a and b equal to n. □

Remark 4.3.16. In spite of our lengthy calculations, all eigenvalues of oppositeness turn out to be powers of q up to sign. This phenomenon appears in fact in the much more general context of oppositeness between flags in finite buildings, as was recently proved by Brouwer [22].

Corollary 4.3.17. *Consider a classical finite polar space of rank $d \geq 2$ with parameters (q, q^e). The eigenvalue of the oppositeness relation between n-spaces, $n \in \{1, \ldots, d\}$, for the subspace $V^n_{1,0}$ is:*

$$-q^{n(4d-3n-1)/2-(d-1)+(n-1)e},$$

and this is the minimal eigenvalue, appearing for exactly one subspace $V^n_{r,i}$, except in the following cases:

(i) *for $Q^+(2d-1, q)$ with d odd, $n = d$; the minimal eigenvalue appears for $V^d_{d,0}$: $-q^{d(d-1)/2}$,*

(ii) *for $Q^+(2d-1, q)$ with d even, $n = d$; the minimal eigenvalue appears for both $V^d_{1,0}$ and $V^d_{d-1,0}$: $-q^{(d-1)(d-2)/2}$,*

(iii) *for $Q^+(2d-1, q)$ with d even, $n = d-1$; the minimal eigenvalue appears for both $V^{d-1}_{1,0}$ and $V^{d-1}_{d-1,0}$: $-q^{d(d-1)/2}$,*

(iv) *for $H(2d-1, q^2)$, with d odd, $n = d$; the minimal eigenvalue appears for $V^d_{d,0}$:$-q^{d(d-1)}$,*

(v) *for $W(2d-1, q)$ and $Q(2d, q)$ with d odd, $n = d$; the minimal eigenvalue appears for both $V^d_{1,0}$ and $V^d_{d,0}$: $-q^{d(d-1)/2}$.*

Proof. If $\lambda_{r,i}$ denotes the eigenvalue of oppositeness for $V^n_{r,i}$, then Theorem 4.3.15 yields:

$$\lambda_{r,i}/\lambda_{1,0} = (-1)^{r+i+1} q^{(d-r-1)(1-r)-i(r+1-i)+e(1-r+i)}.$$

Considering the restrictions $0 \leq i \leq r \leq n \leq d-i$ and $e \in \{0, 1/2, 1, 3/2, 2\}$ one can now determine when the minimal value is attained. □

Some of the exceptions for generators in Corollary 4.3.17 will play an important role in Chapter 5 (see Theorem 5.3.1).

4.4 Interesting subsets in polar spaces

4.4.1 Sets of points

We already mentioned in Section 4.1 how the polar graph on the points of a classical finite polar space can be seen as the point graph of an SPBIBD, with the generators playing the role of the "lines" of the SPBIBD. It follows from Theorem 4.2.4 that there are only three eigenspaces in the decomposition:

$$\mathbb{R}^{\Omega_1} = V^1_{0,0} \perp V^1_{1,0} \perp V^1_{1,1}.$$

Hence for points, as soon as a proper non-empty subset is orthogonal to an eigenspace, it is intriguing because of Lemma 2.1.3. There are two possibilities for an intriguing set of points (see Lemma 2.3.11):

(i) $\chi_S \in V^1_{0,0} \perp V^1_{1,0}$: this is the case if and only if χ_S can be written as a linear combination of the characteristic vectors of point sets of generators (we say S is a *tight set*),

(ii) $\chi_S \in V^1_{0,0} \perp V^1_{1,1}$: this is the case if and only if every generator intersects S in a fixed number of elements m (we say S is an *m-ovoid*).

Payne [121] defined tight sets for generalized quadrangles, and Drudge [73] generalized the concept for polar spaces of higher rank. Thas [150] introduced m-ovoids for generalized quadrangles, and this was generalized for polar spaces of higher rank by Shult and Thas [132].

More properties and equivalent definitions of these types of sets of points can be found in [75]. A detailed discussion of these concepts in polar spaces in a unifying context can be found in [9] and [10]. See also [102] for an extensive list of constructions.

We will introduce another generalization of tight sets and m-ovoids in Subsection 6.4.

4.4.2 Designs with respect to subspaces of fixed dimension

The following theorem links algebraic and geometric properties of subsets of subspaces in a polar space, and can be found (in a somewhat implicit form) in

[77]. For maximal totally isotropic subspaces, it also follows from Delsarte's theory of regular semilattices [66] (see also Stanton [135, 136] for a more explicit treatment).

Theorem 4.4.1. *In a classical finite polar space of rank d with parameters (q, q^e), a set $S \subseteq \Omega_a$ is such that every b-space is incident with (or equal to) a fixed number of elements of S, if and only if χ_S is orthogonal to every $V^a_{r,i}$, $(r,i) \neq (0,0)$, with an isomorphic copy $V^b_{r,i}$ in $\mathbb{R}^{\Omega_b}$.*
In that case, for $a \geq b$ this number is given by:

$$\frac{|S|}{|\Omega_a|} \begin{bmatrix} d-b \\ a-b \end{bmatrix}_q \prod_{i=1}^{a-b} (q^{d-b+e-i} + 1),$$

and for $a \leq b$ it is given by:

$$\frac{|S|}{|\Omega_a|} \begin{bmatrix} b \\ a \end{bmatrix}_q,$$

and S is in fact a combinatorial design with respect to Ω_b.

Proof. In order to apply Theorem 2.5.16 we need to consider $\mathrm{Supp}(C_{a,b})$. The characterization now follows from Theorem 4.2.4. The ratio $|R_{a,b}|/|\Omega_b|$ can be computed using Theorems 4.1.1 and 4.1.4, and hence the desired constant follows from the formula $\frac{|R_{a,b}||S|}{|\Omega_a||\Omega_b|}$ from Theorem 2.5.16. □

Corollary 4.4.2. *Consider a set of a-spaces S in a classical finite polar space of rank d, such that every b-space contains a fixed number of elements of S, with $a \leq b$ and $a + b \leq d$. Now S must either be empty or the full set Ω_a.*

Proof. In this case, it follows from Theorem 4.2.4 that every irreducible submodule $V^a_{r,i}$ has an isomorphic copy in $V^b_{r,i}$, and hence Theorem 4.4.1 implies that $\chi_S \in V^a_{0,0} = \langle \chi_{\Omega_a} \rangle$. □

We will revisit combinatorial designs of maximals in classical finite polar spaces of rank d with respect to the $(d-1)$-spaces in Chapter 6, in the context of *m-ovoids in dual polar spaces.*

4.4.3 Embeddings

In classical finite polar spaces, several interesting geometric structures can be embedded, including other polar spaces. This yields special subsets in the bigger polar space.

Suppose the polar space consists of the totally isotropic subspaces with respect to a form on a vector space. By choosing a non-singular hyperplane (i.e. a hyperplane of the vector space onto which the restriction of the form is also non-degenerate), one can obtain the following embeddings:

(1) the hyperbolic quadric $Q^+(2d-1,q)$ in the parabolic quadric $Q(2d,q)$,

(2) the Hermitian variety $H(2d-1,q^2)$ in the Hermitian variety $H(2d,q^2)$,

(3) the parabolic quadric $Q(2d,q)$ in the elliptic quadric $Q^-(2d+1,q)$,

(4) the parabolic quadric $Q(2d-2,q)$ in the hyperbolic quadric $Q^+(2d-1,q)$,

(5) the Hermitian variety $H(2d-2,q^2)$ in the Hermitian variety $H(2d-1,q^2)$,

(6) the elliptic quadric $Q^-(2d-1,q)$ in the parabolic quadric $Q(2d,q)$.

In the cases (1)-(2)-(3), the maximal totally isotropic subspaces in both polar spaces are of the same dimension, while in the cases (4)-(5)-(6), they are of dimension one less in the smaller polar space. Theorem 1.3.1 yields that G acts transitively on the embeddings of one type.

Theorem 4.4.3. *Consider a classical finite polar space of rank $d \geq 2$ and one of the embeddings given above. Denote the set of n-spaces with $1 \leq n \leq d$ of the smaller polar space by Ω'_n. Now:*

(i) $\chi_{\Omega'_n} \in V^n_{0,0} \perp V^n_{1,0}$ *in the cases (1)-(2)-(3),*

(ii) $\chi_{\Omega'_n} \in V^n_{0,0} \perp V^n_{1,1}$ *if* $1 \leq n \leq d-1$ *and* $\chi_{\Omega'_d} = 0$ *in the cases (4)-(5)-(6).*

Proof. Suppose the bigger polar space has parameters (q,q^e). Each element of Ω_n either contains $\begin{bmatrix} n \\ 1 \end{bmatrix}_q$ elements of Ω'_1, if it is in $\chi_{\Omega'_n}$, or $\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q$ elements of Ω'_1, if it is not. Every element of Ω_n also contains $\begin{bmatrix} n \\ 1 \end{bmatrix}_q$ elements of Ω_1. Hence:

$$\begin{aligned} C_{1,n}\chi_{\Omega'_1} &= \begin{bmatrix} n \\ 1 \end{bmatrix}_q \chi_{\Omega'_n} + \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q (\chi_{\Omega_n} - \chi_{\Omega'_n}), \\ C_{1,n}\chi_{\Omega_1} &= \begin{bmatrix} n \\ 1 \end{bmatrix}_q \chi_{\Omega_n}. \end{aligned}$$

Hence $\chi_{\Omega'_n} \in \mathrm{Im}(C_{1,n})$, which is $V^n_{0,0} \perp V^n_{1,0} \perp V^n_{1,1}$ by Theorem 4.2.4.

(i) In the cases (1)-(2)-(3), an element of Ω_n can only be in any element of Ω'_d if it is in Ω'_n, and in that case the number of such elements of Ω'_d is a constant c. Hence:

$$C_{d,n}\chi_{\Omega'_d} = c\chi_{\Omega'_n},$$

yielding that $\chi_{\Omega'_n} \in \mathrm{Im}(C_{d,n}) = V^n_{0,0} \perp V^n_{1,0} \perp \ldots \perp V^n_{n,0}$ by Theorem 4.2.4. Hence $\chi_{\Omega'_n} \in V^n_{0,0} \perp V^n_{1,0}$ in these cases.

(ii) In the cases (4)-(5)-(6), there are no totally isotropic d-spaces in the smaller polar space, and hence $\chi_{\Omega'_d} = 0$. Each maximal $\pi_d \in \Omega_d$ intersects the non-singular hyperplane in a maximal π_{d-1} of the smaller polar space, which contains exactly $\begin{bmatrix} d-1 \\ n \end{bmatrix}_q$ elements of Ω'_n. Theorem 4.4.1 now implies that for any $n \in \{1, \ldots, d\}$, we must have $\chi_{\Omega'_n} \in (V^n_{1,0})^\perp$. Together with the above, this yields that $\chi_{\Omega'_n} \in V^n_{0,0} \perp V^n_{1,1}$. □

We visualize Theorem 4.4.3 for polar spaces of rank four in Figures 4.7 and 4.8 by crossing out those subspaces the characteristic vector of Ω'_n is orthogonal to.

For points, Theorem 4.4.3 gives us an intriguing set of points, which is tight in the cases (1)-(2)-(3) and an m-ovoid in the cases (4)-(5)-(6). This result was given in [9, Lemma 7].

$\mathbb{R}^{\Omega_0}$	$\mathbb{R}^{\Omega_1}$	$\mathbb{R}^{\Omega_2}$	$\mathbb{R}^{\Omega_3}$	$\mathbb{R}^{\Omega_4}$
$V^0_{0,0}$	$V^1_{0,0}$	$V^2_{0,0}$	$V^3_{0,0}$	$V^4_{0,0}$
	$V^1_{1,0}$ $\cancel{V^1_{1,1}}$	$V^2_{1,0}$ $\cancel{V^2_{1,1}}$	$V^3_{1,0}$ $\cancel{V^3_{1,1}}$	$V^4_{1,0}$
		$\cancel{V^2_{2,0}}$ $\cancel{V^2_{2,1}}$ $\cancel{V^2_{2,2}}$	$\cancel{V^3_{2,0}}$ $\cancel{V^3_{2,1}}$	$\cancel{V^4_{2,0}}$
			$\cancel{V^3_{3,0}}$ $\cancel{V^3_{3,1}}$	$\cancel{V^4_{3,0}}$
				$\cancel{V^4_{4,0}}$

Figure 4.7: Theorem 4.4.3 in the cases (1)-(2)-(3)

$\mathbb{R}^{\Omega_0}$	$\mathbb{R}^{\Omega_1}$	$\mathbb{R}^{\Omega_2}$	$\mathbb{R}^{\Omega_3}$	$\mathbb{R}^{\Omega_4}$
$V^0_{0,0}$	$V^1_{0,0}$	$V^2_{0,0}$	$V^3_{0,0}$	$\cancel{V^4_{0,0}}$
	$\cancel{V^1_{1,0}}$ $V^1_{1,1}$	$\cancel{V^2_{1,0}}$ $V^2_{1,1}$	$\cancel{V^3_{1,0}}$ $V^3_{1,1}$	$\cancel{V^4_{1,0}}$
		$\cancel{V^2_{2,0}}$ $\cancel{V^2_{2,1}}$ $\cancel{V^2_{2,2}}$	$\cancel{V^3_{2,0}}$ $\cancel{V^3_{2,1}}$	$\cancel{V^4_{2,0}}$
			$\cancel{V^3_{3,0}}$ $\cancel{V^3_{3,1}}$	$\cancel{V^4_{3,0}}$
				$\cancel{V^4_{4,0}}$

Figure 4.8: Theorem 4.4.3 in the cases (4)-(5)-(6)

We will now consider the embedding of the split Cayley hexagon $\mathrm{H}(q)$ in the parabolic quadric $Q(6,q)$. This is a point-line geometry, the points of which are simply the points of $Q(6,q)$, the lines of which are certain lines of $Q(6,q)$, known as the *hexagon lines*, and incidence is inherited from the polar space. This incidence structure satisfies the axioms of a generalized hexagon of order (q,q), consisting of equally many points and lines. We will introduce and discuss generalized polygons in detail in Chapter 6. The remaining lines of $Q(6,q)$ are called the *ideal lines*. We refer to [161] for the following basic properties.

(i) The hexagon lines through a point p of $Q(6,q)$ are precisely the $q+1$ lines in a fixed plane through p, that we denote by p^α. Such planes are the *hexagon planes*, and the other planes of $Q(6,q)$ are the *ideal planes*. No two distinct points yield the same hexagon plane.

(ii) A hexagon line is on $q+1$ hexagon planes and on no ideal planes. An ideal line is on exactly one hexagon plane, and on q ideal planes.

(iii) In a hexagon plane p^α, the hexagon lines are the $q+1$ lines through p, and the q^2 remaining lines are ideal lines. An ideal plane only contains ideal lines.

We will now show that the sets of hexagon lines and hexagon planes both have dual degree 1.

Theorem 4.4.4. *Consider the embedding of* $\mathrm{H}(q)$ *in* $Q(6,q)$ *given above. Let* HL*,* IL*,* HP *and* IP *be the sets of hexagon lines, ideal lines, hexagon planes and ideal planes, respectively. Then* $\chi_{HL}, \chi_{IL} \in V^2_{0,0} \perp V^2_{2,0}$ *and* $\chi_{HP}, \chi_{IP} \in V^3_{0,0} \perp V^3_{2,0}$.

Proof. As every point is on $q+1$ hexagon lines and on $q^2(q+1)$ ideal lines, Theorem 4.4.1 already implies that χ_{HL} and χ_{IL} are in $V^2_{0,0} \perp V^2_{2,0} \perp V^2_{2,1}$. It also follows from property (ii) of $\mathrm{H}(q)$ in $Q(6,q)$ that

$$\begin{aligned} C_{3,2}(\chi_{HP}) &= \chi_{IL} + (q+1)\chi_{HL} \\ C_{3,2}(\chi_{IP}) &= q\chi_{IL}. \end{aligned}$$

Hence χ_{HL} and χ_{IL} are both in $\mathrm{Im}(C_{3,2}) = V^2_{0,0} \perp V^2_{1,0} \perp V^2_{2,0}$ as well (see Theorem 4.2.4). Hence we can conclude that χ_{HL}, χ_{IL} are in $V^2_{0,0} \perp V^2_{2,0}$.

Finally, it follows from property (iii) that

$$\begin{aligned} C_{2,3}(\chi_{HL}) &= (q+1)\chi_{HP} \\ C_{2,3}(\chi_{IL}) &= q^2\chi_{HP} + (q^2+q+1)\chi_{IP}. \end{aligned}$$

Hence χ_{HP} and χ_{IP} are linear combinations of $C_{2,3}(\chi_{HL})$ and $C_{2,3}(\chi_{IL})$, and so they are both in the image of $V^2_{0,0} \perp V^2_{2,0}$ under $C_{2,3}$, i.e. in $V^3_{0,0} \perp V^3_{2,0}$ (see Theorem 4.2.4). □

In Figure 4.9, we illustrate Theorem 4.4.4 by crossing out the eigenspaces of the association schemes on lines and planes in $Q(6,q)$ that are orthogonal to the characteristic vectors of those sets.

$\mathbb{R}^{\Omega_0}$	$\mathbb{R}^{\Omega_1}$	$\mathbb{R}^{\Omega_2}$	$\mathbb{R}^{\Omega_3}$
$V^0_{0,0}$	$V^1_{0,0}$	$V^2_{0,0}$	$V^3_{0,0}$
	$V^1_{1,0}$	$\cancel{V^2_{1,0}}$	$\cancel{V^3_{1,0}}$
	$V^1_{1,1}$	$\cancel{V^2_{1,1}}$	
		$V^2_{2,0}$	$V^3_{2,0}$
		$\cancel{V^2_{2,1}}$	
			$\cancel{V^3_{3,0}}$

Figure 4.9: Theorem 4.4.4: the dual degree sets of the sets of hexagon lines and planes

We now demonstrate how some well-known properties of the embedding of $\mathrm{H}(q)$ can be seen using design-orthogonality.

Corollary 4.4.5. *Consider the embedding of* $\mathrm{H}(q)$ *in* $Q(6,q)$ *and a non-singular hyperplane* H *with respect to the quadratic form.*

(i) *If* H *intersects* $Q(6,q)$ *in a hyperbolic quadric, then it contains* $(q+1)(q^2+q+1)$ *hexagon lines and* $2(q^2+q+1)$ *hexagon planes.*

(ii) *If* H *intersects* $Q(6,q)$ *in an elliptic quadric, then it contains exactly* q^3+1 *hexagon lines, and they are pairwise opposite.*

Proof. The dual degree sets of the sets of lines and planes in H follows from Theorem 4.4.3. Their sizes follow from Theorem 4.1.1: $(q+1)(q^2+q+1)(q^2+1)$ lines and $2(q+1)(q^2+1)$ planes in the hyperbolic case, and $(q^2+1)(q^3+1)$ lines and no planes in the elliptic case, while $Q(6,q)$ itself has $(q^2+1)(q^2+q+1)(q^3+1)$ lines and $(q+1)(q^2+1)(q^3+1)$ planes. The sets of hexagon lines and planes are both of size $(q^6-1)/(q-1)$, and their dual degree sets follow from Theorem 4.4.4. The intersection sizes now follow from Lemma 2.2.10. Finally, since no two lines of $\mathrm{H}(q)$ can intersect in a line without spanning a totally isotropic plane (property (i)), and since $Q^-(5,q)$ contains no planes, the relations $R^{1,2}_{2,2}, R^{1,3}_{2,2}$ and $R^{0,3}_{2,2}$ cannot appear between lines of $\mathrm{H}(q)$ in H if the intersection is an elliptic quadric. Hence in the second case, we obtain q^3+1 pairwise opposite lines. □

Remark 4.4.6. Thas [147] used the property in Corollary 4.4.5(ii) to prove the existence of perfect 1-codes in the dual of $\mathrm{H}(q)$, which is also a generalized hexagon (see Subsection 6.4.2).

4.4.4 Partial m-systems

We will now focus on the cliques of one specific relation in the association scheme on Ω_n, namely oppositeness. We will give alternative proofs of several known results, and obtain a new and tight bound in one specific case.

Definition 4.4.7. *A partial ovoid in a finite polar space is a set of points, such that any two elements are non-collinear.*

Equivalently, one can define partial ovoids as sets of points, no two of which are on a common generator of the polar space.

Definition 4.4.8. *An ovoid in a finite polar space is a set of points, such that each generator is incident with exactly one of its elements.*

In a classical finite polar space of rank d with parameters (q, q^e), there are $(q^e + 1) \cdots (q^{d-1+e} + 1)$ generators, and $(q^e + 1) \cdots (q^{d-2+e} + 1)$ generators through each point, by Theorem 4.1.1. Hence the size of a partial ovoid is at most $q^{d-1+e} + 1$, with equality if and only if it is an ovoid. The number $q^{d-1+e} + 1$ is known as the *ovoid number* of the polar space.

We now introduce similar concepts for generators in a polar space.

Definition 4.4.9. *A partial spread in a finite polar space is a set of maximals, no two of which incident with a common point.*

Definition 4.4.10. *A spread in a finite polar space is a set of maximals, partitioning the set of points of the polar space.*

In a classical finite polar space of rank d with parameters (q, q^e), each generator is incident with $(q^d-1)/(q-1)$ points, and the full number of points is given by $(q^{d-1+e}+1)(q^d-1)/(q-1)$, again by Theorem 4.1.1. This yields that the size of a partial spread is also at most the ovoid number $q^{d-1+e} + 1$, with equality if and only if it is a spread.

In order to explain the phenomenon of the equal bounds for points and generators, Shult and Thas [132] examined the unifying concept of partial m-systems.

Definition 4.4.11. *A partial $(m-1)$-system in a polar space is a set of m-spaces, such that any two elements are opposite.*

Remark 4.4.12. Note that here we use the notation of Shult and Thas, who use projective dimensions.

One can also define partial $(m-1)$-systems as sets of m-spaces S, such that every generator through an element of S has a trivial intersection with any other element of S.

In a classical finite polar space of rank d, the partial ovoids and partial spreads are precisely the partial 0- and $(d-1)$-systems, respectively. Shult and Thas proved the following unifying bound.

Theorem 4.4.13. [132, Theorem 4] The size of a partial $(m-1)$-system in a classical finite polar space of rank d with parameters (q, q^e) is at most the ovoid number $q^{d-1+e}+1$.

Shult and Thas [132] defined the *m-systems* as those partial m-systems with size equal to the ovoid number $q^{d-1+e}+1$. In particular, the ovoids and spreads are precisely the 0- and $(d-1)$-systems, respectively. Many interesting properties of m-systems were also obtained. We will now give alternative proofs of some of these results.

Theorem 4.4.14. *Let S be a non-empty partial $(m-1)$-system in a classical finite polar space of rank d with parameters (q, q^e), $1 \leq m \leq d$, and let $\tilde{S}$ and $\bar{S}$ denote the sets of points and generators incident with or equal to a (necessarily unique) element of S, respectively. Then $|S| \leq q^{d-1+e}+1$, and the following are equivalent:*

(i) $|S| = q^{d-1+e}+1$, *i.e. S is an $(m-1)$-system,*

(ii) $\chi_S \in (V_{1,0}^m)^\perp$,

(iii) *$\tilde{S}$ is a k-ovoid for some k,*

(iv) *every point is on exactly r elements of $\bar{S}$ for some r.*

If any (and thus all) of the above statements hold, then $k = \left[{m \atop 1}\right]_q$ and $r = \prod_{i=1}^{d-m}(q^{i-1+e}+1)$.

Proof. We know from Theorem 4.3.15 and Corollary 4.3.17 that the valency k and the eigenvalue λ of oppositeness for $V_{1,0}^m$ are given by:

$$\begin{aligned} k &= q^{m(4d-3m-1)/2+me} \\ \lambda &= -q^{m(4d-3m-1)/2-(d-1)+(m-1)e}. \end{aligned}$$

Corollary 2.2.9 now yields that $|S| \leq 1 - k/\lambda = 1 + q^{d-1+e}$, with equality if and only if $\chi_S \in (V_{1,0}^m)^\perp$. Hence (i) and (ii) are equivalent.

We can write the characteristic vectors of $\tilde{S}$ and $\bar{S}$ as $C_{m,1}\chi_S$ and $C_{m,d}\chi_S$, respectively. We know from Theorem 4.4.1 that $\tilde{S}$ is a k-ovoid if and only if the characteristic vector of $\tilde{S}$ is orthogonal to $V_{1,0}^1$. Similarly, every point will be on equally many elements of $\bar{S}$ if and only if the characteristic vector of

$\bar{S}$ is orthogonal to $V^d_{1,0}$. Theorem 4.2.4 yields that both are equivalent to χ_S itself being orthogonal to $V^m_{1,0}$. This yields equivalence between (ii), (iii) and (iv). Every generator through some element π_m of S intersects $\tilde{S}$ in exactly the $\left[{m \atop 1}\right]_q$ points of π_m, and every point on π_m is on exactly $\prod_{i=1}^{d-m}(q^{i-1+e}+1)$ generators of $\bar{S}$, namely those that are through π_m itself. This quickly yields the constants in case of equality. □

It was also shown in [132] that in some particular cases, the m-systems also behave in a very nice way with respect to embedded polar spaces. We will now use design-orthogonality to give an alternative proof.

Theorem 4.4.15. *Let S be a non-empty partial $(m-1)$-system in a classical finite polar space of rank d with parameters (q, q^{e+1}), and consider one of the embeddings (1)-(2)-(3) by use of a non-singular hyperplane intersection from Subsection 4.4.3. The number of elements S in such a hyperplane is a constant if and only if S is an $(m-1)$-system. In that case, this constant is $q^{d-m+e}+1$.*

Proof. Let Ω'_m denote the set of m-spaces in the hyperplane. Note that the smaller polar space has parameters (q, q^e). We know from Theorem 4.4.3 that $\chi_{\Omega'_m} \in V^m_{0,0} \perp V^m_{1,0}$. Theorem 2.5.17 yields that a constant number of elements of S is contained in every element of the orbit of Ω'_m under G, if and only if χ_S is orthogonal to $V^m_{1,0}$. It follows from Theorem 4.4.14 that this is the case if and only if S is an $(m-1)$-system, and Theorem 2.5.17 then implies that $|S \cap \Omega'_m| = |S||\Omega'_m|/|\Omega_m|$. In this case $|S| = q^{(d-1)+(e+1)}+1$ and it follows from Theorem 4.1.1 that $|\Omega'_m|/|\Omega_m| = (q^{d+e-m}+1)/(q^{d+e}+1)$, which yields the desired size of intersection. □

The bound for partial $(m-1)$-systems from the subspace $V^m_{1,0}$ is exactly the same as the ovoid number. However, using Corollary 2.2.9 we can obtain an upper bound for the size from any subspace $V^m_{r,i}$, provided that the corresponding eigenvalue of the oppositeness relation is negative. Surprisingly, in one very specific case this leads to a sharper bound.

Theorem 4.4.16. *A non-empty partial spread S in $H(2d-1, q^2)$, d odd, has size at most q^d+1. Equality holds if and only if its characteristic vector χ_S is orthogonal to $V^d_{d,0}$.*

Proof. We know from Theorem 4.3.15 and Corollary 4.3.17 that here the valency and the minimal eigenvalue of the oppositeness relation between generators are given by $k = q^{d^2}$ and $\lambda = (-1)^d q^{d(d-1)}$. Corollary 2.2.9 now yields that $|S| \leq 1 - k/\lambda = 1 + q^d$, with equality if and only if $\chi_S \in (V^d_{d,0})^\perp$. □

Thas [152, Theorems 20 and 21] already proved that in $H(2d-1,q^2)$ spreads, or hence partial spreads of size $q^{2d-1}+1$, cannot exist, and gave a better upper bound for partial spreads if d is even. On the other hand, partial spreads of size q^d+1 in $H(2d-1,q^2)$ were constructed for all $d \geq 2$ in [1], by use of a symplectic polarity of the projective space $\mathrm{PG}(2d-1,q^2)$, commuting with the associated Hermitian polarity. In the Baer subgeometry of points on which these two polarities coincide, a spread of the induced symplectic polar space $W(2d-1,q)$ can always be found, and these q^d+1 generators extend to pairwise disjoint generators of $H(2d-1,q^2)$. Maximality of partial spreads of $H(2d-1,q^2)$ constructed in this way was also shown for $d=2,3$ in [1] and for all odd d in [105]. De Beule and Metsch [52] already obtained the bound of q^3+1 for $H(5,q^2)$.

We will further examine these partial spreads of maximum size in Chapter 6, and we will also give different proofs of the bound in other contexts in Subsection 6.4.3 and Appendix A. We also refer the reader to [104, Appendix A] for a good survey on existence results for m-systems in polar spaces in general.

Chapter 5

Erdős-Ko-Rado theorems for dual polar graphs

In this chapter, we will focus on one specific problem in polar spaces, which can be seen as the converse of studying partial spreads. It will be interpreted as one of the many Erdős-Ko-Rado problems appearing in the literature.

Erdős, Ko and Rado [78] considered sets of subsets of equal size in a finite set, such that every two elements have an intersection of at least a given size. Many variants of their theorem have been proved later, including one for subspaces in a vector space by Hsieh [96].

We will be concerned with those sets of maximal totally isotropic subspaces in classical finite polar spaces, such that every two elements are incident with at least one common point. It is our aim to determine the maximum size and to classify all sets attaining that bound. When translating our problem into the language of graph theory, these subsets will correspond with the cocliques of one relation of the association scheme defined by the dual polar graph. Stanton [135] already used algebraic techniques to obtain upper bounds for these cocliques, which turn out to be tight in most but not all cases. Brouwer, Godsil, Koolen and Martin [24] developed a general theory to obtain information on similar extremal sets in many association schemes. Tanaka [139] worked further on this and already classified extremal subsets satisfying certain conditions in many association schemes, providing new proofs of known results, as well as obtaining new results for graphs such as the dual polar graphs. We will use similar techniques, as well as geometric arguments, to obtain a

classification in all polar spaces, except $H(2d-1, q^2)$ with d odd and $d \geq 5$.

An overview can be found at the end of this chapter in Section 5.10.

The results in this chapter are joint work with Valentina Pepe and Leo Storme, and will be published in *Journal of Combinatorial Theory Series A* [124].

5.1 Erdős-Ko-Rado theorems

We start by giving the original *Erdős-Ko-Rado theorem*. We will refer to subsets of size k as simply *k-sets*, and we will say a set S of subsets is *t-intersecting* if any two elements of S have an intersection of size at least t.

Theorem 5.1.1. [78] *Suppose S is a t-intersecting set of k-sets in a set of size n, with $1 \leq t \leq k \leq n$.*

(i) *If $n \geq t + (k-t)\binom{k}{t}^3$, then $|S| \leq \binom{n-t}{k-t}$.*

(ii) *If $n \geq 2k$ and $t = 1$, then $|S| \leq \binom{n-1}{k-1}$.*

Note that the set of k-sets in an n-set through a fixed t-set is a t-intersecting set of size $\binom{n-t}{k-t}$.

Wilson obtained the following sharper result.

Theorem 5.1.2. [170] *Suppose S is a t-intersecting set of k-sets in a set of size n, with $1 \leq t \leq k$ and $n \geq (t+1)(k-t+1)$. Then $|S| \leq \binom{n-t}{k-t}$, and if $n > (t+1)(k-t+1)$, then equality holds if and only if S consists of all k-sets through a fixed t-set.*

We will say a set of k-spaces in a vector space is *t-intersecting* if the dimension of the intersection of every two elements is at least t. Hsieh proved the following analogue of the original Erdős-Ko-Rado theorem for vector spaces.

Theorem 5.1.3. [96] *If S is a t-intersecting set of k-spaces in $V(n,q)$ with $1 \leq t \leq k$ and $n \geq 2k+1$, and also with $(n,q) \neq (2k+1, 2)$ in case $t \geq 2$, then $|S| \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$. Equality holds if and only if S consists of all k-spaces through a fixed t-space.*

This result was improved in [81]. For $t = 1$, elegant proofs for both the tight upper bound and the classification in $V(n, q)$ can be found in [84] for $n \geq 2k+1$ and in [111] for $n \geq 2k$. We refer to Tanaka [139] for a proof of the following very general result.

Theorem 5.1.4. *If S is a set of t-intersecting k-spaces in $V(n, q)$ with $1 \leq t \leq k$ and $n \geq 2k$, then $|S| \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$, and equality holds if and only if S either consists of all k-spaces through a fixed t-space, or $n = 2k$ and S consists of all k-spaces in a fixed $(2k - t)$-space.*

It is our goal to study sets of generators in a classical finite polar space, no two elements of which have a trivial intersection. We will refer to such sets as *EKR sets of generators*. We will say that such a set is a *maximal EKR set of generators* if it is not a proper subset of another EKR set of generators. A simple example of an EKR set of generators is the *point-pencil* construction, consisting of all generators through a fixed point. This is in fact always a maximal EKR set of generators, as for every generator π not through a fixed point p, a generator through p disjoint from π can be found. We will see that in many polar spaces, these are the unique EKR sets of generators of maximum size.

Note that for polar spaces of rank two, i.e. generalized quadrangles, it is easy to see that the maximal EKR sets of generators are precisely the sets of lines through a fixed point. Therefore, we will only consider classical finite polar spaces of rank at least three.

5.2 Algebraic techniques

Theorems 5.1.1 and 5.1.2 can be seen as results on codes in the Johnson graphs $J(n, k)$, and similarly, Theorems 5.1.3 and 5.1.4 deal with codes in the Grassmann graphs $J_q(n, k)$. We have seen in Sections 2.3 and 3.1 that the distance between vertices in these graphs corresponds with the size or dimension of their intersection.

Recall the definitions of width and dual width from Subsection 2.2.4.

We know from Theorem 4.1.7 that two vertices are at distance i in the dual polar graph, associated with a classical finite polar space of rank d, if and only if their intersection has dimension $d-i$. Therefore, the EKR sets of generators

are the codes in the dual polar graph with width at most $d-1$. We can also describe them as the cocliques in the graph corresponding to the (maximum) distance-d relation in the d-class association scheme, which is the oppositeness relation between generators.

The bound in the following fundamental theorem on cocliques in general is known as *Hoffman's bound*.

Theorem 5.2.1. [95] *Let Γ be a k-regular graph, $k \geq 1$, on a set of vertices Ω, with minimal eigenvalue λ. If S is a coclique in Γ, then:*

$$|S| \leq \frac{|\Omega|}{1-k/\lambda},$$

and in case of equality:

(i) *the characteristic vector χ_S can be written as $\frac{|S|}{|\Omega|}\chi_\Omega + v$ with v an eigenvector of the adjacency matrix A for λ,*

(ii) *every vertex in $\Omega \backslash S$ is adjacent to exactly $-\lambda$ elements of S.*

Proof. Since λ is the minimal eigenvalue, we know that $A-\lambda \mathrm{I}$ has non-negative eigenvalues and thus is positive semidefinite. Note that $\lambda < 0$ since $\mathrm{Tr}(A) = 0$ and $A \neq 0$. Hence:

$$\left(\chi_S - \frac{|S|}{|\Omega|}\chi_\Omega\right)^T \left(A - \lambda \mathrm{I}\right)\left(\chi_S - \frac{|S|}{|\Omega|}\chi_\Omega\right) \geq 0,$$

Since S is a coclique, we can write $(\chi_S)^T A \chi_S = 0$. Using this and $A\chi_\Omega = k\chi_\Omega$, we can rewrite the above inequality to obtain the desired upper bound on $|S|$. If equality holds, then $\chi_S - \frac{|S|}{|\Omega|}\chi_\Omega$ is an eigenvector of A with eigenvalue λ, and thus intriguing with parameters (h_1, h_2) with $h_1 - h_2 = \lambda$, because of Lemma 2.1.3. As $h_1 = 0$, we find that every vertex not in S is adjacent to exactly $-\lambda$ elements of S. □

We now state two general results by Brouwer, Godsil, Koolen and Martin on subsets in metric/cometric schemes with a certain width/dual width.

Theorem 5.2.2. [24, Theorem 1] *If S is a non-empty subset in a metric d-class association scheme with dual degree s^* and width w, then $s^* + w \geq d$. If equality holds, then S is a completely regular code.*

Theorem 5.2.3. [24, Theorem 2] *If S is a non-empty subset in a cometric d-class association scheme with dual width w^* and degree s, then $s + w^* \geq d$. If equality holds, then S, together with the non-empty restrictions to S of the relations of the scheme, is a cometric s-class association scheme.*

If the scheme is both metric and cometric, then the width w of S is at least the degree s, and hence the previous theorem in particular implies that $w+w^* \geq d$, and if equality holds, a subscheme is induced by S.

Tanaka observed that the following result is in fact implicit in [24].

Theorem 5.2.4. [139, Proposition 2] *If S is a non-empty subset in a metric and cometric d-class association scheme with width w and dual width w^*, then $w + w^* \geq d$. If equality holds, then S, together with the non-empty restrictions to S of the scheme is a cometric w-class association scheme, the intersection numbers of which only depend on w.*

Theorem 4.2.4 implies that in a classical finite polar space of rank d, the set S of generators through a fixed i-space with $0 \leq i \leq d$ has dual degree set $\{1, \ldots, i\}$ and hence dual width i with respect to the cometric ordering from Remark 4.3.12. The width is given by $d - i$. Here, the induced scheme that follows from Theorem 5.2.4 is isomorphic to the one defined by the dual polar graph with the same parameters and of the same type but with diameter $d-i$.

Finally, we conclude this section by stating a general theorem, that we will use in Section 5.8, where the most difficult case in this chapter will be considered.

Definition 5.2.5. *Let G act generously transitively on both X and X', and let $R_1, \ldots, R_n$ be its orbits on $X \times X'$. The generalized outer distribution of $S \subseteq X$ with respect to X' is the matrix $B = (B_{x',i})_{x' \in X'; i=1,\ldots,n}$, with:*

$$B_{x',i} = |\{x \in X | (x, x') \in R_i\}|, \ \forall x' \in X', \forall i \in \{1, \ldots, n\}.$$

Note that for every $x' \in X'$, the sum of the entries $B_{x',i}$ in the corresponding row must be equal to $|S|$.

Under some assumptions, we will now prove a generalization of Lemma 2.2.8, which implies that the rank of the outer distribution of a non-empty subset in an association scheme is equal to the dual degree plus 1.

Theorem 5.2.6. *Suppose G acts generously transitively on both X and X', and consider the decompositions into isotypic components:*

$$\begin{aligned}\mathbb{R}^X &= (V_1 \perp \ldots \perp V_n) \perp (A_1 \perp \ldots \perp A_s),\\ \mathbb{R}^{X'} &= (V_1' \perp \ldots \perp V_n') \perp (B_1 \perp \ldots \perp B_t),\end{aligned}$$

where V_i and V_i' are isomorphic $\mathbb{R}G$-modules.
There is a set of row vectors $(\lambda_j^1, \ldots, \lambda_j^n)$ with $j \in \{1, \ldots, n\}$ such that for every $S \subseteq X$, the row span of the generalized outer distribution B with respect to X' is precisely spanned by those row vectors $(\lambda_j^1, \ldots, \lambda_j^n)$ with $j \in J$, where J is the set of indices j such that $\chi_S \notin V_j^\perp$.

Proof. We know from Theorem 2.5.15 that we can find a basis of homomorphisms p_j with $j \in \{1, \ldots, n\}$ of the space $\mathrm{Hom}_{\mathbb{R}G}(\mathbb{R}^X, \mathbb{R}^{X'})$, such that every p_j maps V_j into V_j' bijectively, while vanishing on all other irreducibles. For every orbit R_i, write C_i for the corresponding $(0,1)$-matrix with columns and rows indexed by X and X', respectively. We now define the scalars λ_j^i as follows:

$$C_i = \sum_{j=1}^{n} \lambda_j^i p_j, \forall i \in \{1, \ldots, n\}.$$

We denote by E_j the minimal idempotent of the association scheme on X, afforded by G, which projects onto V_j. We can now write for every $x' \in X'$:

$$\begin{aligned}B_{x',i} &= (C_i \chi_S)_{x'}\\ &= (C_i E_1 \chi_S)_{x'} + \cdots + (C_i E_n \chi_S)_{x'}\\ &= \sum_{j \in J} (C_i E_j \chi_S)_{x'}\\ &= \sum_{j \in J} \lambda_j^i (p_j(E_j \chi_S))_{x'}.\end{aligned}$$

Hence every row of the generalized outer distribution B of S is a linear combination of the row vectors $(\lambda_j^1, \ldots, \lambda_j^n)$ with $j \in J$. On the other hand, the column space is precisely the space spanned by all $C_i \chi_S$, and hence certainly contains all $p_j \chi_S = p_j(E_j \chi_S)$ with $j \in J$. As these $|J|$ column vectors are linearly independent, this concludes the proof. □

Calderbank and Delsarte [31] considered this concept for the specific case of the Johnson schemes, and called the generalized outer distribution with respect to

t-sets the t-distribution matrix. A very similar theory was developed for the Hamming schemes by Delsarte [68]. In these papers, the scalars from Theorem 5.2.6 are explicitly computed in terms of Hahn and Krawtchouk polynomials, respectively, but for our study of the dual polar graph, such computations will not be required.

5.3 Bounds for EKR sets of generators

We will now use Theorem 5.2.1 to obtain upper bounds on the size of EKR sets of generators in classical finite polar spaces. This computation was already done by Stanton [134]. It turns out that in almost all cases, the upper bound is exactly the number of generators through a fixed point, and hence also tight.

Theorem 5.3.1. *Let S be an EKR set of generators in a classical finite polar space $\mathcal{P}$ of rank d with parameters (q, q^e), and consider the decomposition $\mathbb{R}^{\Omega_d} = V^d_{0,0} \perp \ldots \perp V^d_{d,0}$.*

- *If $\mathcal{P} = Q^+(2d-1, q)$, d odd, then $|S|$ is at most half of the total number of generators in $\mathcal{P}$, and if this bound is attained, then $\chi_S \in V^d_{0,0} \perp V^d_{d,0}$.*
- *If $\mathcal{P} = Q^+(2d-1, q)$, d even, then $|S|$ is at most the number of generators through a fixed point, and if this bound is attained, then $\chi_S \in V^d_{0,0} \perp V^d_{1,0} \perp V^d_{d-1,0}$.*
- *If $\mathcal{P} = H(2d-1, q^2)$, d odd, then $|S|$ is at most the total number of generators in $\mathcal{P}$ divided by $q^d + 1$, and if this bound is attained, then $\chi_S \in V^d_{0,0} \perp V^d_{d,0}$.*
- *If $\mathcal{P} = Q(2d, q)$ or $\mathcal{P} = W(2d-1, q)$, with d odd in both cases, then $|S|$ is at most the number of generators through a fixed point, and if this bound is attained, then $\chi_S \in V^d_{0,0} \perp V^d_{1,0} \perp V^d_{d,0}$.*

For all other polar spaces, the size of S is at most the number of generators through a fixed point, and if this bound is attained, then $\chi_S \in V^d_{0,0} \perp V^d_{1,0}$.

Proof. We know the eigenvalue of the oppositeness relation between generators for $V^d_{j,0}$ from Theorem 4.3.15:

$$(-1)^j q^{d(d-1)/2+(d-j)(e-j)}.$$

For $j = 0$, one obtains the valency k of oppositeness. The minimal eigenvalue λ of oppositeness was considered in Theorem 4.3.17. Theorem 5.2.1 now yields that $|S|$ is at most $|\Omega_d|/(1 - k/\lambda)$, and that in case of equality, $\chi_S - \frac{|S|}{|\Omega_d|}\chi_{\Omega_d}$ is in the eigenspace for λ. □

5.4 General observations on maximal EKR sets of generators

We will first obtain some results by use of purely geometric arguments, which already hold when only assuming maximality of the EKR set of generators.

For the remainder of this chapter, we will also refer to the $(d-1)$-spaces in a classical finite polar space of rank d as the *dual lines*.

Lemma 5.4.1. *Let π_a, π_b and π_c be pairwise non-trivially intersecting generators in a classical finite polar space. The intersections $\pi_a \cap \pi_b$ and $\pi_a \cap \pi_c$ cannot be complementary subspaces of π_a.*

Proof. Suppose $\pi_a \cap \pi_b$ and $\pi_a \cap \pi_c$ are complementary subspaces of π_a. As π_b and π_c are assumed to intersect non-trivially, they must have a point p in common, not in π_a. This point would be collinear with all points in $\pi_a \cap \pi_b$ and with all points in $\pi_a \cap \pi_c$, and hence with all points in $\langle \pi_a \cap \pi_b, \pi_a \cap \pi_c \rangle = \pi_a$, which would contradict the assumption that π_a is a maximal totally isotropic subspace. □

Lemma 5.4.2. *Let S be a maximal EKR set of generators. If a dual line is incident with at least two elements of S, then all generators through it are in S.*

Proof. Let μ be a dual line, incident with two distinct elements π_a and π_b of S. Suppose a third generator π' through μ is not in S. As S is assumed to be maximal, there must be a generator $\pi_c \in S$ disjoint from π' and hence also from μ. As S is an EKR set, π_c must intersect both π_a and π_b non-trivially. Hence π_c intersects π_a in a point p not on μ. Hence the intersections $\pi_a \cap \pi_b = \mu$ and $\pi_a \cap \pi_c = p$ are complementary in π_a, contradicting Lemma 5.4.1. □

The previous lemma motivates us to introduce the following terminology. We say that a dual line in a classical finite polar space is *secant*, *tangent* or *external*

with respect to a maximal EKR set of generators S if all, one or none of the generators through it are in S, respectively.

Let S be a maximal EKR set of generators in a classical finite polar space with $\pi \in S$. Consider all secant dual lines with respect to S in π. We will refer to their intersection as the *nucleus* of π (with respect to S). If there are no secant dual lines in $\pi \in S$, the nucleus is simply π itself. The nuclei of the elements of S will play a crucial role in our classification of the EKR sets of generators of maximum size. In the following lemma, we prove fundamental properties of the nuclei.

Lemma 5.4.3. *Let S be a maximal EKR set of generators in a classical finite polar space of rank d and with parameters (q, q^e). Suppose π_s is the s-dimensional nucleus of $\pi \in S$.*

(i) *The secant dual lines in π are those through π_s, and the tangent dual lines in π are those not through π_s.*

(ii) *The number of elements of S that intersect π in any dual line is given by $q^e \begin{bmatrix} d-s \\ 1 \end{bmatrix}_q$.*

(iii) *If a generator $\pi' \in S$ intersects π in just a point, then this point must be in π_s.*

Proof. If $\pi' \in S$ intersects π in a point p, then p must belong to every secant dual line μ in π by Lemma 5.4.1, hence $p \in \pi_s$.

Let μ be a dual line through π_s. By Lemma 5.4.2, μ is either secant or tangent. Suppose that μ is tangent. This means that there is a π_1 through μ such that $\pi_1 \notin S$. Since S is maximal, there must be a $\pi_2 \in S$ disjoint from π_1. Now π_2 must intersect π, so this intersection is a point, not in μ and hence not in π_s either. This contradicts the above, and hence μ is not tangent.

The number of dual lines in π through π_s is given by $\begin{bmatrix} d-s \\ 1 \end{bmatrix}_q$, and through each such dual line there are q^e other elements of S, and hence there are exactly $q^e \begin{bmatrix} d-s \\ 1 \end{bmatrix}_q$ elements of S intersecting π in a dual line. □

As we will often count with respect to generators, the following corollary will be particularly useful in this chapter.

Corollary 5.4.4. *Let $\mathcal{P}$ be a classical finite polar space of rank d with parameters (q, q^e). The number of generators intersecting a fixed totally isotropic m-space π_m in a subspace of codimension i in π_m is given by:*

$$q^{i(2d-2m+2e+i-1)/2} \begin{bmatrix} m \\ i \end{bmatrix}_q \prod_{j=0}^{d-m-1} (q^{d-m-j-1+e} + 1).$$

For any generator, there are $q^{i(i-1)/2+ie}$ generators intersecting it in a fixed subspace of codimension i.

Proof. The first part follows from Theorem 4.1.4 with $a = m, b = d, s = m - i$ and $k = d$. The last number is the valency of oppositeness between generators in the residual polar space of the chosen subspace of dimension $d - i$, which has rank i. □

Note that in a classical finite polar space of rank d with parameters (q, q^e), the number of generators through a fixed point is $|\Omega_d|/(q^{d-1+e} + 1)$, where $q^{d-1+e} + 1$ is the ovoid number of the polar space.

5.5 Classification of maximum EKR sets of generators in most polar spaces

We know from Theorem 5.3.1 that in most classical finite polar spaces, the maximum size of an EKR set of generators is attained by the set of all generators through a fixed point, and that in those cases the dual degree set is $\{1\}$ with respect to the cometric ordering from Remark 4.3.12. Tanaka [139] classified all non-empty subsets of generators in the classical finite polar spaces of rank d with width w and dual degree w^* satisfying $w + w^* = d$, by proving that they are precisely the sets of generators through a fixed w^*-space. For the sake of completeness, we will use his technique for the specific case $w = 1$. We will then slightly change the argument in Section 5.6.

Lemma 5.5.1. *Consider a classical finite polar space of rank d, and a subset of generators S with width $d - 1$ and dual width 1. Suppose π_1 and π_2 are in S and $d(\pi_1, \pi_2) = i$ in the dual polar graph Γ with $1 \leq i \leq d$. Any neighbour π of π_1 with $d(\pi, \pi_2) = i - 1$ is also in S.*

Proof. Suppose the polar space has parameters (q, q^e). We know from Theorem 4.1.7 that there are exactly $c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_q$ neighbours of π_1 in Γ at distance $i-1$ from π_2. Theorem 5.2.4 yields that the distance-j relations with $j \in \{0, \ldots, d-1\}$ have non-empty restrictions to S and yield a $(d-1)$-class association scheme, with parameters only depending on w. Hence the induced scheme on S is isomorphic to the one defined by the dual polar graph of the same type, with the same parameters (q, q^e) and with diameter $d-1$. Two vertices in S are at the same distance with respect to the corresponding graph Γ' in the induced scheme, as in Γ itself. Hence, with respect to Γ, the number of neighbours of π_1 in S at distance $i-1$ from π_2 is also given by $c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_q$, which yields the desired result. □

Theorem 5.5.2. *Let $\mathcal{P}$ be a polar space of rank $d \geq 3$, either $H(2d, q^2)$, $H(2d-1, q^2)$ with d even, $Q(2d, q)$ with d even, $W(2d-1, q)$ with d even or $Q^-(2d+1, q)$. If S is an EKR set of generators of $\mathcal{P}$ with $|S|$ equal to the number of generators through a fixed point, then S consists of all generators through a fixed point.*

Proof. We know from Theorem 5.3.1 that in these polar spaces, sets of generators with width $w \leq d-1$ of this size have dual width $w^* = 1$. It follows from Theorem 5.2.3 that w is exactly $d-1$. Let π_0 and π_1 be any two vertices in S at distance $d-1$, hence intersecting in just a point p. Now suppose there is a $\pi_2 \in S$ not through p, with $d(\pi_1, \pi_2) = i$ in the dual polar graph. In that case, there is certainly a point $p_2 \in \pi_2$ not in $p^\perp$. Now consider the generator $\pi = \langle p_2, p_2^\perp \cap \pi_1 \rangle$. This generator intersects π_1 in a hyperplane not through p, and $d(\pi, \pi_2) = i-1$. It follows from Lemma 5.5.1 that π is also in S. However, this implies that π_0 and π cannot intersect trivially, while they intersect π_1 in the complementary subspaces p and $p_2^\perp \cap \pi_1$, respectively. This contradicts Lemma 5.4.1. We can hence conclude that all generators in S must go through p. The size $|S|$ now implies that S is exactly the set of generators through p.□

The forthcoming sections in this chapter will be devoted to the remaining polar spaces.

5.6 Hyperbolic quadrics

In the hyperbolic quadric $Q^+(2d-1, q)$, there are two systems of generators of the same size. We will refer to them as the *Latin and Greek* generators,

and use the symbols $\Omega_{d,1}$ and $\Omega_{d,2}$ for these sets. They have the property that two generators are in the same system if and only if the dimension of their intersection has the same parity as d. Moreover, a totally isotropic subspace of dimension $d-1$ is contained in exactly two generators: one in $\Omega_{d,1}$ and one in $\Omega_{d,2}$, since the polar space has parameters $(q,1)$.

The automorphism group of $Q^+(2d-1,q)$ acts transitively on the generators, but the dual polar graph is bipartite with diameter d, with the sets of Latins and Greeks as the two bipartite classes. Hence every automorphism either stabilizes both systems, or switches them. We refer to [140, Chapter 11] for proofs and more information.

For this particular dual polar graph, the eigenvalue for the subspace $V^d_{j,0}$ from Theorem 4.1.7, with $0 \leq j \leq d$, is given by $\left[{d-j \atop 1}\right]_q - \left[{j \atop 1}\right]_q$. Note that the eigenvalues for $V^d_{j,0}$ and $V^d_{d-j,0}$ are opposite values. The following relation holds between eigenspaces:

$$V^d_{j,0} = \{\iota_1(v_1) - \iota_2(v_2) | v_1 \in \mathbb{R}^{\Omega_{d,1}}, v_2 \in \mathbb{R}^{\Omega_{d,2}}, \iota_1(v_1) + \iota_2(v_2) \in V^d_{d-j,0}\},$$

where $\iota_i : \mathbb{R}^{\Omega_{d,i}} \to \mathbb{R}^{\Omega_d}$ is the injection with $(\iota_i(v))_\omega = v_\omega$ if $\omega \in \Omega_{d,i}$, and $(\iota_i(v))_\omega = 0$ if not.

In particular, $V^d_{0,0}$ and $V^d_{d,0}$ are one-dimensional eigenspaces of this dual polar graph, with $V^d_{0,0} = \langle \chi_{\Omega_{d,1}} + \chi_{\Omega_{d,2}} \rangle$ and $V^d_{d,0} = \langle \chi_{\Omega_{d,1}} - \chi_{\Omega_{d,2}} \rangle$.

We will first consider the case where the diameter d is odd. Here, two generators of the same system cannot intersect trivially, so the set of all Latins and the set of all Greeks are both EKR sets, and their sizes meet the eigenvalue bound from Theorem 5.3.1. The following algebraic argument quickly establishes that this is the only possibility.

Theorem 5.6.1. *If S is an EKR set of generators in $Q^+(2d-1,q)$, d odd, of size $|\Omega_d|/2$, then S is one of the two systems of the hyperbolic quadric.*

Proof. Theorem 5.3.1 yields that $\chi_S \in V^d_{0,0} \perp V^d_{d,0}$. The eigenspace $V^d_{0,0}$ is spanned by $\chi_{\Omega_{d,1}} + \chi_{\Omega_{d,2}}$, while $V^d_{d,0}$ is spanned by $\chi_{\Omega_{d,1}} - \chi_{\Omega_{d,2}}$. Hence χ_S can only be $\chi_{\Omega_{d,1}}$ or $\chi_{\Omega_{d,2}}$. □

Next, we consider the case where the diameter d is even. Here, two generators of two different systems cannot intersect trivially, so if S_1 is an EKR set contained in $\Omega_{d,1}$ and S_2 is an EKR set contained in $\Omega_{d,2}$, then the union $S_1 \cup S_2$

is still an EKR set for the polar space. The upper bound from Theorem 5.3.1 for an EKR set of generators S in $Q^+(2d-1,q)$ is $2(q+1)\cdots(q^{d-2}+1)$ if d is even. This bound can be attained by taking all generators through a single point, but one could for instance also take all Latins through one point p_1, and all Greeks through another point p_2 (either collinear with p_1 or not) to obtain an EKR set of generators of maximum size. We now introduce the *half dual polar graph* Γ', the vertices of which are the generators of one system, with two of them adjacent when intersecting in a subspace of codimension 2. We refer to [23, Section 9.4.C] for a discussion of this graph.

Theorem 5.6.2. [23, Corollary 8.4.2 and Theorem 9.4.8] *Let Γ' be the half dual polar graph on one system of generators $\Omega_{d,1}$ in the hyperbolic quadric $Q^+(2d-1,q)$.*

(i) *Γ' is distance-regular with diameter $d' := \lfloor \frac{d}{2} \rfloor$, and two vertices are at distance i if and only if they intersect in a subspace of codimension $2i$.*

(ii) *The valency of the distance-i relation for Γ' is given by $\left[{d \atop 2i}\right]_q q^{i(2i-1)}$, and the intersection numbers are given by:*

$$b_i = q^{4i+1}\begin{bmatrix} d-2i \\ 2 \end{bmatrix}_q, \forall i \in \{0,\dots,d'-1\}; c_i = \begin{bmatrix} 2i \\ 2 \end{bmatrix}_q, \forall i \in \{1,\dots,d'\}.$$

(iii) *If d is even, then Γ' has classical parameters:*

$$\left(d', q^2, q(q+1), q\begin{bmatrix} d-1 \\ 1 \end{bmatrix}_q\right),$$

and if d is odd, then Γ' has classical parameters:

$$\left(d', q^2, q(q+1), q\begin{bmatrix} d \\ 1 \end{bmatrix}_q\right).$$

(iv) *The vector space $\mathbb{R}^{\Omega_{d,1}}$ decomposes as $W_0 \perp W_1 \perp \ldots \perp W_{d'}$, where W_j is an eigenspace of Γ' for the eigenvalue:*

$$q^{2j+1}\begin{bmatrix} d-2j \\ 2 \end{bmatrix}_q - \frac{q^{2j}-1}{q^2-1},$$

and all $d'+1$ eigenvalues are distinct. The ordering of the spaces W_j is Q-polynomial.

Remark 5.6.3. If $v \in \mathbb{R}^{\Omega_{d,1}}$ is an eigenvector for the eigenvalue λ of the half dual polar graph Γ', then $\iota_1(v)$ is an eigenvector for λ of the distance-2 relation of Γ. One can now use Theorems 4.3.6 and 5.6.2 to verify that $\iota_1(W_j)$ is in the span of $V^d_{j,0}$ and $V^d_{d-j,0}$, with $0 \leq 2j \leq d$.

As an illustration of Theorem 5.6.2 and Remark 5.6.3, we give the matrix of eigenvalues for the P-polynomial association schemes defined by both the dual polar graph and the half dual polar graph for the hyperbolic quadric $Q^+(7,q)$ (i.e. $d = 4$) in Tables 5.1 and 5.2. The ordering of the rows corresponds with the cometric orderings from Remarks 4.3.12 and 5.6.3.

$$\begin{pmatrix} 1 & q^3+q^2+q+1 & (q^2+1)\,q\,(q^2+q+1) & q^3\,(q^3+q^2+q+1) & q^6 \\ 1 & (q+1)\,q & q^3-1 & -(q+1)\,q & -q^3 \\ 1 & 0 & -q^2-1 & 0 & q^2 \\ 1 & -(q+1)\,q & q^3-1 & (q+1)\,q & -q^3 \\ 1 & -q^3-q^2-q-1 & (q^2+1)\,q\,(q^2+q+1) & -q^3\,(q^3+q^2+q+1) & q^6 \end{pmatrix}$$

Table 5.1: Eigenvalues for all generators in $Q^+(7,q)$

$$\begin{pmatrix} 1 & (q^2+1)\,q\,(q^2+q+1) & q^6 \\ 1 & q^3-1 & -q^3 \\ 1 & -q^2-1 & q^2 \end{pmatrix}$$

Table 5.2: Eigenvalues for one system of generators in $Q^+(7,q)$

We will now consider the eigenvalues of the disjointness relation on the vertices of the half dual polar graph in order to obtain bounds.

Lemma 5.6.4. *An EKR set of generators S in one system of $Q^+(2d-1,q)$ with d even has size at most $(q+1)\cdots(q^{d-2}+1)$. In case of equality, $\chi_S \in W_0 \perp W_1$, using the same notation as in Theorem 5.6.2.*

Proof. If Γ' is the half dual polar graph on $Q^+(2d-1,q)$, then the eigenvalue of the distance-i relation for Γ' for the subspace W_j is the same as the eigenvalue of the distance-$2i$ relation for the original dual polar graph Γ for both the subspace $V^d_{j,0}$ and $V^d_{d-j,0}$. Hence, the ratio $1-k/\lambda$ from Theorem 5.2.1 remains the same, and we find that an EKR set of generators of the same system has size at most $(q+1)\cdots(q^{d-2}+1)$, and this bound can only be attained if the characteristic vector χ_S is in $W_0 \perp W_1$. □

It follows from Lemma 5.6.4 that an EKR set of maximum size in $Q^+(2d-1,q)$, d even, must contain exactly $(q+1)\cdots(q^{d-2}+1)$ elements of each system. The two systems of generators are equivalent with respect to the automorphism group, so it suffices to classify the EKR sets of size $(q+1)\cdots(q^{d-2}+1)$ of one system in $Q^+(2d-1,q)$.

It is our aim to show that an EKR set of generators of one system in the hyperbolic quadric $Q^+(2d-1,q)$, for even $d \geq 6$, consists of all generators of that system through one point, and that for $d = 4$, there is only one extra construction.

Lemma 5.6.5. *Let S be an EKR set of $(q+1)\cdots(q^{d-2}+1)$ generators of one system in $Q^+(2d-1,q)$, d even. The distance-i relations of the half dual polar graph Γ' with $0 \leq i \leq d/2-1$ induce an association scheme on S with the same intersection numbers as the scheme defined by the half dual polar graph on $Q^+(2(d-1)-1,q)$.*

Proof. We know from Lemma 5.6.4 that the assumptions imply that $\chi_S \in W_0 \perp W_1$, using the same notation as in Theorem 5.6.2. Hence, with respect to Γ' and its Q-polynomial ordering from Theorem 5.6.2(iv), the width w of S is at most $d-1$ and the dual width w^* is 1. Theorem 5.2.3 yields that the distance-i relations define a $(d/2-1)$-class association scheme on S.

Now let S_0 be the set of all generators of the same system through a fixed point. This set satisfies the same assumptions. We know that the association schemes induced by S and by S_0 have the same intersection numbers, and the latter is isomorphic to that on generators of one system in the hyperbolic quadric $Q^+(2(d-1)-1,q)$. □

We can now prove a result very similar to Lemma 5.5.1.

Lemma 5.6.6. *Let S be an EKR set of $(q+1)\cdots(q^{d-2}+1)$ generators of one system in $Q^+(2d-1,q)$ with d even, and suppose that π_1 and π_2 are two*

elements of S at distance i in the associated half dual polar graph Γ'. If π is a neighbour of π_1 in Γ' and at distance $i-1$ from π_2, then π must be in S as well.

Proof. If two generators π_1 and π_2 in S are at distance i in Γ', then the number of generators in S at distance $i-1$ from π_1 and at distance 1 from π_2 is given by $c_i = \begin{bmatrix} 2i \\ 2 \end{bmatrix}_q$, because of Lemma 5.6.5 and Theorem 5.6.2.

Moreover, $\begin{bmatrix} 2i \\ 2 \end{bmatrix}_q$ is also the number of generators in the full graph Γ', at distance $i-1$ from π_1 and at distance 1 from π_2. Hence every such generator in Γ' must belong to S. □

The proof of the following lemma was inspired by the proof of [139, Theorem 1] (see the proof of Theorem 5.5.2).

Lemma 5.6.7. *Let S be an EKR set of $(q+1)\cdots(q^{d-2}+1)$ generators of one system of $Q^+(2d-1,q)$, d even and $d \geq 4$. If π_0 and π_1 are elements of S intersecting in just a line ℓ, then no element of S can intersect ℓ trivially.*

Proof. Suppose π_0 and π_1 are elements of S intersecting in just the line ℓ. Suppose $\pi_2 \in S$ intersects π_1 in a subspace μ of codimension $2i$ in π_1, skew to ℓ. Let m be any line in π_2, skew to $\ell^\perp \cap \pi_2$. Consider the generator $\pi = \langle m, m^\perp \cap \pi_1 \rangle$. This generator intersects π_1 in a subspace of codimension 2, skew to ℓ, and is at distance $i-1$ with respect to the half dual polar graph from π_2. Hence π is in S as well, because of Lemma 5.6.6. Now π and π_0 must also intersect non-trivially, but then the triple $\{\pi_0, \pi_1, \pi\}$ would contradict Lemma 5.4.1 as $\pi_0 \cap \pi_1$ and $\pi \cap \pi_1$ are complementary subspaces in π_1. Hence π_2 cannot intersect ℓ trivially. □

We now come to our main result concerning hyperbolic quadrics. The proof of the following theorem uses the Erdős-Ko-Rado theorem for 1-intersecting sets of k-spaces in $V(n,q)$. We know from Theorem 5.1.4 that the case $n = 2k$ is special. This will have its consequences in the proof and will force us to assume that the rank d of the polar space is at least 6.

Theorem 5.6.8. *If S is an EKR set of Latins of size $(q+1)\cdots(q^{d-2}+1)$ in $Q^+(2d-1,q)$, d even and $d \geq 6$, then S is the set of Latins through a fixed point.*

Proof. Let π be in S. We know from Lemma 5.6.5 that the number of elements of S intersecting π in exactly a line is the same as the number of generators

in $Q^+(2(d-1)-1,q)$ that intersect a fixed generator in exactly a point. This is the valency of the distance-$(d/2-1)$ relation in the half dual polar graph on $Q^+(2(d-1)-1,q)$, and thus given by $\begin{bmatrix} d-1 \\ 1 \end{bmatrix}_q q^{(d-2)(d-3)/2}$ because of Theorem 5.6.2. On the other hand, Corollary 5.4.4 yields that there are exactly $q^{(d-2)(d-3)/2}$ generators of $Q^+(2d-1,q)$, intersecting π in just a fixed line. Hence the set A of lines that are intersections of π with some element of S has size at least $\begin{bmatrix} d-1 \\ 1 \end{bmatrix}_q$, and we know from Lemma 5.6.7 that no two of them can be disjoint. As π is d-dimensional with $d \geq 6$, we can now apply Theorem 5.1.4 to see that A is precisely the set of $\begin{bmatrix} d-1 \\ 1 \end{bmatrix}_q$ lines through some fixed point p in π.

Now suppose π' is an element of S not through p. This means that $\mu = \pi \cap \pi'$ is a subspace of codimension at least 2 in π and not through p. Let ℓ be a line in π, through p and skew to μ. Now $\ell \in A$ is the intersection of two elements of S, while $\pi' \in S$ is disjoint from ℓ, contradicting Lemma 5.6.7. □

Remark 5.6.9. One can actually avoid using Theorem 5.1.4 in the proof of Theorem 5.6.8, as it is in fact very easy to see that every maximal 1-intersecting set of 2-spaces in $V(n,q)$ with $n \geq 3$ consists of either all $\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q$ 2-spaces through a fixed point, or all $\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q$ 2-spaces in a fixed 3-space.

The hyperbolic quadric $Q^+(7,q)$ of rank 4 must be treated separately. Let $\mathcal{P}_0$ be the set of $(q+1)(q^2+1)(q^3+1)$ points in $Q^+(7,q)$, $\mathcal{P}_1$ the set of $(q+1)(q^2+1)(q^3+1)$ Latins, $\mathcal{P}_2$ the set of $(q+1)(q^2+1)(q^3+1)$ Greeks, and $\mathcal{L}$ the set of lines of $Q^+(7,q)$. We can define an incidence relation between two elements belonging to any couple of sets: a Latin and a Greek are incident if they intersect in a plane, and in all the other cases it is just symmetrized strict inclusion. It is well known that there exists a *triality*, i.e. an incidence preserving map of order three that maps $\mathcal{P}_0$ to $\mathcal{P}_1$, $\mathcal{P}_1$ to $\mathcal{P}_2$, $\mathcal{P}_2$ to $\mathcal{P}_0$, and $\mathcal{L}$ to $\mathcal{L}$ (see for instance [161, Section 2.4]).

Theorem 5.6.10. *Let S be a set of $(q+1)(q^2+1)$ Latins in $Q^+(7,q)$, pairwise intersecting non-trivially. Then either S consists of all Latins through one point, or S is the set of all Latins intersecting a fixed Greek in a plane.*

Proof. Let S be a set of pairwise intersecting Latins, and let τ be any triality. Then $S^{\tau^{-1}}$ is a set of mutually collinear points. It is well known that in every polar space the largest set of pairwise collinear points is the set of points in a generator (see for instance [73, Lemma 9.2]). Hence, there is a generator π containing all $(q+1)(q^2+1)$ points of $S^{\tau^{-1}}$.

If π is a Latin, then S itself consists of all Latins incident with the Greek π^τ, or hence of all Latins intersecting π^τ in a plane. If π is a Greek, then S itself consists of all Latins through the point π^τ. □

5.7 $Q(2d, q)$ with d odd

We will now treat the problem in the parabolic quadrics of odd rank. The bound from Theorem 5.3.1 is still attained by the point-pencil construction, but the properties of the characteristic vector are a bit weaker. We will make use of the embedding in the hyperbolic quadric. We have seen in Section 5.6 that $Q^+(7, q)$ is a special case, and therefore $Q(6, q)$, which has rank three, will also be exceptional.

Theorem 5.7.1. *Let S be an EKR set of generators in $Q(2d, q)$, d odd and $d \geq 3$, with $|S| = (q+1)\cdots(q^{d-1}+1)$. One of the following must hold:*

(i) *S is the set of all generators through a fixed point,*

(ii) *S is the set of all generators of one system of an embedded $Q^+(2d-1, q)$,*

(iii) *$d = 3$ and S consists of one fixed generator and all generators intersecting it in a line (i.e. S is a sphere of radius 1 in the dual polar graph).*

Proof. Consider the embedding of $Q(2d, q)$ in $Q^+(2d+1, q)$ as a non-singular hyperplane section, i.e. $Q(2d, q) = Q^+(2d+1, q) \cap H$ with H a hyperplane of the projective geometry $\mathrm{PG}(2d+1, q)$. Every generator of $Q(2d, q)$ is contained in a unique generator of a fixed system of $Q^+(2d+1, q)$, so let $\bar{S}$ be the set of Latin generators in $Q^+(2d+1, q)$ through an element of S. The elements of $\bar{S}$ cannot be disjoint either and $|\bar{S}| = |S| = (q+1)\cdots(q^{d-1}+1)$. Theorems 5.6.8 and 5.6.10 then yield that $\bar{S}$ is either the set of all Latins through a point p in $Q^+(2d+1, q)$, or $d = 3$ and $\bar{S}$ is the set of all Latins intersecting a fixed Greek γ in a plane. Suppose that we are in the first case. If p is in H, then S is simply the set of all generators through p in $Q(2d, q)$. If p is not in H, then $p^\perp \cap H$ intersects the parabolic quadric $Q(2d, q)$ in a non-singular hyperbolic quadric $Q^+(2d-1, q)$. Then S is one system of generators of that hyperbolic quadric. In the second case, we see that S consists of the plane $\gamma \cap H$ and the $(q^2+q+1)q$ planes of $Q(6, q)$ intersecting that plane in a line. □

5.8 $W(2d-1,q)$ with d odd

If q is even, parabolic and symplectic polar spaces with the same rank and the same parameters are isomorphic (see for instance [140, Chapter 11]), and hence we will also easily obtain the classification in those spaces. The case q odd will be considerably harder.

We first consider $Q(2d,q)$ and $W(2d-1,q)$ with d odd and q even. Recall the definition and the properties of the nucleus p of $Q(2d,q)$ from Subsection 1.3.2, which yields an isomorphism from $Q(2d,q)$ to $W(2d-1,q)$ by use of a hyperplane H_0 in $\mathrm{PG}(2d,q)$ not through p. If H is a non-singular hyperplane intersecting the parabolic quadric $Q(2d,q)$ in a hyperbolic quadric $Q^+(2d-1,q)$, then the generators of $Q(2d,q)$ in H will correspond with those of an embedded $Q^+(2d-1,q)$ in $W(2d-1,q)$ when projecting from the nucleus onto H_0.

Theorem 5.8.1. *Let S be an EKR set of generators in $W(2d-1,q)$, d odd, $d \geq 3$ and q even, with $|S| = (q+1)\cdots(q^{d-1}+1)$. One of the following must hold:*

(i) *S is the set of all generators through a fixed point,*

(ii) *S is the set of all generators of one system of an embedded $Q^+(2d-1,q)$,*

(iii) *$d=3$ and S consists of one fixed generator and all generators intersecting it in a line (i.e. S is a sphere of radius 1 in the dual polar graph).*

Proof. Suppose $Q(2d,q)$ has nucleus p in $\mathrm{PG}(2d,q)$, allowing an embedding of $W(2d-1,q)$ in the latter. An EKR set of generators S of maximum size yields an EKR set in $Q(2d,q)$, and now the result follows from the preceding paragraph and Theorem 5.7.1. □

The rest of this section is devoted to the case q odd.

In $W(2d-1,q)$, d odd, Theorem 5.3.1 does not yield that the characteristic vector χ_S of an EKR set of generators S of maximum size is in the span of the subspaces $V^d_{0,0}$ and $V^d_{1,0}$. This significantly weakens our control over this set. We also don't have an isomorphism $Q(2d,q)$ and $W(2d-1,q)$ if q is odd, but the intersection numbers and the eigenvalues of the association schemes on generators are still the same (see for instance [23, Section 9.4]).

With respect to the disjointness relation, we will still be able to use the following strong property.

Lemma 5.8.2. *Let S be an EKR set of generators in $Q(2d,q)$ or $W(2d-1,q)$ of size $(q+1)\cdots(q^{d-1}+1)$. Every generator $\pi \notin S$ is disjoint from exactly $q^{d(d-1)/2}$ elements of S.*

Proof. Theorem 4.3.15 and Corollary 4.3.17 yield that the valency and the minimal eigenvalue of oppositeness are given by $q^{d(d+1)/2}$ and $-q^{d(d-1)/2}$, respectively. The result now follows from Theorem 5.2.1. □

Recall our definition of the outer distribution B of a subset in an association scheme from Subsection 2.2.4, and our notation B_x for the row corresponding to x. We will now discuss the outer distribution B of an EKR set S of maximum size in $W(2d-1,q)$ with d odd. We first consider the two known constructions of EKR sets of generators of maximum size S in $Q(2d,q)$ with d odd, together with some element $\pi \in S$:

(i) Point-pencil construction: $B_\pi = v_1$ with $(v_1)_i := \left[{d-1 \atop i}\right]_q q^{i(i+1)/2}$ (see Theorem 4.1.7).
For instance, in $Q(10,q)$:

$$v_1 = \left(1, \begin{bmatrix}4\\1\end{bmatrix}_q q, \begin{bmatrix}4\\2\end{bmatrix}_q q^3, \begin{bmatrix}4\\3\end{bmatrix}_q q^6, \begin{bmatrix}4\\4\end{bmatrix}_q q^{10}, 0\right).$$

(ii) All Latins of an embedded $Q^+(2d-1,q)$: $B_\pi = v_2$ with $(v_2)_i := \left[{d \atop i}\right]_q q^{i(i-1)/2}$ if i is even, and $(v_2)_i := 0$ if i is odd (see Theorem 4.1.7).
For instance, the Latins of an embedded $Q^+(9,q)$ in $Q(10,q)$ yield:

$$v_2 = \left(1, 0, \begin{bmatrix}5\\2\end{bmatrix}_q q, 0, \begin{bmatrix}5\\4\end{bmatrix}_q q^6, 0\right).$$

Lemma 5.8.3. *Consider an EKR set of generators S with outer distribution B in $Q(2d,q)$ or $W(2d-1,q)$, with d odd and $|S| = (q+1)\cdots(q^{d-1}+1)$. For every $\pi \in S$, there is a parameter $\tau \in \mathbb{R}$ such that $B_\pi = (1-\tau)v_1 + \tau v_2$.*

Proof. We know from Theorem 5.3.1 that if $|S|$ attains the upper bound $(q+1)\cdots(q^{d-1}+1)$, then $\chi_S \in V^d_{0,0} \perp V^d_{1,0} \perp V^d_{d,0}$. Let P be the matrix of eigenvalues of the association scheme on generators, and let E_j denote the

idempotent projecting onto eigenspace $V^d_{j,0}$. Lemma 2.2.8 implies that every row of the outer distribution B of S is a linear combination of the rows of P, with

$$B_{\pi,i} = \sum_{j=0}^{d} (E_j \chi_S)_\pi P_{ji},$$

for every $\pi \in S$. As E_0 is just the orthogonal projection onto the all-one vector, the coefficient of row 0 of P is simply $|S|/|\Omega_d|$. On the other hand, the entry $B_{\pi,0}$ must be 1 for every $\pi \in S$, and $P_{j0} = 1$ for every $j \in \{0,\dots,d\}$. We can conclude that for every $\pi \in S$ there is an $\alpha \in \mathbb{R}$ such that:

$$\begin{aligned} B_{\pi,i} &= \frac{|S|}{|\Omega_d|} P_{0i} + \alpha P_{1i} + \left(1 - \frac{|S|}{|\Omega_d|} - \alpha\right) P_{di} \\ &= \frac{|S|}{|\Omega_d|} P_{0i} + \left(1 - \frac{|S|}{|\Omega_d|}\right) P_{di} + \alpha(P_{1i} - P_{di}). \end{aligned}$$

Instead of explicitly calculating these eigenvalues, we use two relatively simple vectors that must be of this form. As the parameters and the eigenvalues for generators in $W(2d-1,q)$ and $Q(2d,q)$ are the same, the row vectors v_1 and v_2 that were given are both of this form. Hence both $v_2 - v_1$ and $B_\pi - v_1$ with $\pi \in S$ are scalar multiples of the difference between row 1 and row d of P. As $v_1 \neq v_2$, this implies that $B_\pi - v_1$ can be written as $\tau(v_2 - v_1)$ for some $\tau \in \mathbb{R}$. □

Theorem 5.8.4. *Suppose S is an EKR set of generators in $W(2d-1,q)$ of size $(q+1)\cdots(q^{d-1}+1)$, d odd and $d \geq 3$. Let π be any element of S with s-dimensional nucleus π_s.*

(i) *The number of elements of S intersecting π in a subspace of codimension i, is given by*

$$\frac{q^{d-s}-1}{q^{d-1}-1} \begin{bmatrix} d-1 \\ i \end{bmatrix}_q q^{i(i+1)/2} \qquad \text{for odd } i,$$

and

$$\begin{bmatrix} d-1 \\ i \end{bmatrix}_q q^{i(i+1)/2} + \frac{q^{d-s}(q^{s-1}-1)}{q^{d-1}-1} \begin{bmatrix} d-1 \\ i-1 \end{bmatrix}_q q^{i(i-1)/2} \qquad \text{for even } i.$$

(ii) *For every point of π_s, there are exactly $q^{d(d-1)/2-s+1}$ elements of S intersecting π in just that point.*

Proof. Let B denote the outer distribution of S, and consider any $\pi \in S$. We already know from Lemma 5.4.3 that if an element of S intersects π in exactly one point p, then $p \in \pi_s$, and that $B_{\pi,1}$, the number of elements of S intersecting π in a dual line, is exactly $q \left[{d-s \atop 1}\right]_q$.

Lemma 5.8.3 also yields that the row vector B_π can be written as $(1-\tau)v_1+\tau v_2$ for some parameter $\tau \in \mathbb{R}$. In particular, $B_{\pi,1}$ gives us the following equation:

$$(1-\tau)\begin{bmatrix} d-1 \\ 1 \end{bmatrix}_q q + \tau.0 = \begin{bmatrix} d-s \\ 1 \end{bmatrix}_q q.$$

Hence: $\tau = q^{d-s}(q^{s-1}-1)/(q^{d-1}-1)$.

If i is odd, then the corresponding entry of v_2 is zero, and hence:

$$B_{\pi,i} = \frac{q^{d-s}-1}{q^{d-1}-1}\begin{bmatrix} d-1 \\ i \end{bmatrix}_q q^{i(i+1)/2}.$$

If i is even, then:

$$\begin{aligned} B_{\pi,i} &= (1-\tau)\begin{bmatrix} d-1 \\ i \end{bmatrix}_q q^{i(i+1)/2} + \tau \begin{bmatrix} d \\ i \end{bmatrix}_q q^{i(i-1)/2} \\ &= \begin{bmatrix} d-1 \\ i \end{bmatrix}_q q^{i(i+1)/2} + \tau\left(\begin{bmatrix} d \\ i \end{bmatrix}_q - \begin{bmatrix} d-1 \\ i \end{bmatrix}_q q^i\right) q^{i(i-1)/2}. \end{aligned}$$

Using the identity $\left[{d \atop i}\right]_q = \left[{d-1 \atop i}\right]_q q^i + \left[{d-1 \atop i-1}\right]_q$, the latter can also be written as:

$$B_{\pi,i} = \begin{bmatrix} d-1 \\ i \end{bmatrix}_q q^{i(i+1)/2} + \frac{q^{d-s}(q^{s-1}-1)}{q^{d-1}-1}\begin{bmatrix} d-1 \\ i-1 \end{bmatrix}_q q^{i(i-1)/2}.$$

In particular, we find that $B_{\pi,d-1}$, the number of elements of S intersecting π in just a point, is exactly $\left[{s \atop 1}\right]_q q^{d(d-1)/2-s+1}$.

For any point p in π_s, let $f(p)$ denote the number of elements of S intersecting π in just p. Consider any hyperplane π_{s-1} of π_s. We want to find $\sum_{p\in\pi_{s-1}} f(p)$. Consider any generator π' intersecting π in a dual line but intersecting π_s in just π_{s-1}. It follows from Lemma 5.4.3 that π' is not in S, and thus Lemma 5.8.2 implies that π' is disjoint from exactly $q^{d(d-1)/2}$ elements of S, all necessarily intersecting π in just a point in $\pi_s \backslash \pi_{s-1}$. Conversely, any generator of S intersecting π in just a point of $\pi_s \backslash \pi_{s-1}$ must be disjoint from π' because of Lemma 5.4.1.

Hence $\sum_{p\in\pi_{s-1}} f(p)$, the number of elements of S that intersect π in just a point of π_{s-1}, is given by $\left[{s \atop 1}\right]_q q^{d(d-1)/2-s+1} - q^{d(d-1)/2} = \left[{s-1 \atop 1}\right]_q q^{d(d-1)/2-s+1}$. Now let $\mathcal{H}$ denote the set of all hyperplanes in π_s, and consider any point p_0 in π_s. Note that each point $p \in \pi_s$ different from p_0 is in exactly $\left[{s-1 \atop 1}\right]_q - \left[{s-2 \atop 1}\right]_q = q^{s-2}$ hyperplanes of π_s not containing p_0. We obtain:

$$\begin{aligned}
\sum_{h\in\mathcal{H}, p_0\notin h}\left(\sum_{p\in h} f(p)\right) &= q^{s-2}\left(\sum_{p\in\pi_s\setminus\{p_0\}} f(p)\right)\\
&\Downarrow\\
q^{s-1}\left(\begin{bmatrix} s-1 \\ 1 \end{bmatrix}_q q^{d(d-1)/2-s+1}\right) &= q^{s-2}\left(\sum_{p\in\pi_s\setminus\{p_0\}} f(p)\right).
\end{aligned}$$

Hence

$$\begin{aligned}
f(p_0) &= \sum_{p\in\pi_s} f(p) - \sum_{p\in\pi_s\setminus\{p_0\}} f(p)\\
&= \begin{bmatrix} s \\ 1 \end{bmatrix}_q q^{d(d-1)/2-s+1} - \begin{bmatrix} s-1 \\ 1 \end{bmatrix}_q q^{d(d-1)/2-s+2}\\
&= q^{d(d-1)/2-s+1}. \qquad \square
\end{aligned}$$

Theorem 5.8.4 allow us to compute each row of the outer distribution of an EKR set of maximum size in $W(2d-1,q)$ with d odd, but only if we know the dimension s of the nucleus π_s. We will now consider the generalized outer distribution of S with respect to dual lines instead, as introduced in Subsection 5.2. It will turn out that these numbers are in fact easier to control.

Theorem 5.8.5. *Let S be an EKR set of generators of size $(q+1)\cdots(q^{d-1}+1)$ in $W(2d-1,q)$, d odd. For each secant dual line π_{d-1}, the number of elements of S intersecting π_{d-1} in a subspace of dimension $(d-1)-i$ is given by $\left[{d-2 \atop i}\right]_q (q+1)q^{i(i+3)/2}$.*

Proof. We know from Theorem 4.1.3 that the full automorphism group G has d orbits $R_{d-1,d}^{(d-1)-i,d}$ on $\Omega_{d-1}\times\Omega_d$, $0\le i\le d-1$, determined by the dimension $(d-1)-i$ of the intersection of the dual line and the generator. Let B denote the generalized outer distribution of S with respect to the set of dual lines Ω_{d-1}. We know from Theorem 5.3.1 that $\chi_S \in V_{0,0}^d \perp V_{1,0}^d \perp V_{d,0}^d$. It follows from Theorem 4.2.4 that the last subspace $V_{d,0}^d$ is not isomorphic to any submodule

in $\mathbb{R}^{\Omega_{d-1}}$. Theorem 5.2.6 now implies that there are two row vectors of scalars $(\lambda_0^0, \ldots, \lambda_0^{d-1})$ and $(\lambda_1^0, \ldots, \lambda_1^{d-1})$, independent of S, such that for any dual line π_{d-1} the row $B_{\pi_{d-1}}$ can be written as a linear combination of them. Instead of explicitly calculating these scalars, we consider two particular vectors spanned by these two vectors. Let S' be the set of all generators through a fixed point p', and denote by B' its generalized outer distribution with respect to the set of dual lines. This is certainly an EKR set of generators of the maximum size. Let u_1 denote the row of B' for some dual line through p', and let u_2 denote the row of B' for some dual line not through p' and not spanning a generator with p' either. In the first case, the dual line is secant, and in the second case it is external, so $(u_1)_0 = q+1$ and $(u_2)_0 = 0$. Hence the row vectors u_1 and u_2 are certainly different. So for any EKR set of generators S of the maximum size, and for any dual line π_{d-1} we can write: $B_{\pi_{d-1}} = \alpha u_1 + \beta u_2$ for some parameters $\alpha, \beta \in \mathbb{R}$. We know that the entries of $B_{\pi_{d-1}}$, u_1 and u_2 must all add up to $|S|$, and hence $\alpha + \beta = 1$. If the dual line π_{d-1} is assumed to be secant, then $B_{\pi_{d-1},0} = q+1$, and hence $\alpha = 1$ and $\beta = 0$. Hence we conclude that $B_{\pi_{d-1}} = u_1$.

Now we explicitly calculate $(u_1)_i$. Considering the residual geometry of p', which is isomorphic to $W(2(d-1)-1, q)$, and applying Corollary 5.4.4, we obtain $(u_1)_i = \begin{bmatrix} d-2 \\ i \end{bmatrix}_q (q+1) q^{i(i+3)/2}$. □

The following result is a first step towards our classification of EKR sets of maximum size of generators in $W(2d-1, q)$ with d and q odd. We will now prove that if S is an EKR set of maximum size and $\pi \in S$, then not all the neighbours of π are in S, except possibly in the smallest case $W(5, q)$. In other words, if π_s is the s-dimensional nucleus of $\pi \in S$, then $s \neq 0$, unless $d = 3$.

Lemma 5.8.6. *Let S be an EKR set of generators of size $(q+1)\cdots(q^{d-1}+1)$ in $W(2d-1, q)$, d odd and $d \geq 5$. There is no element $\pi \in S$ with a trivial nucleus.*

Proof. Suppose $\pi \in S$ has a trivial nucleus. Lemma 5.4.3 yields that no element of S intersects π in just a point, and Theorem 5.8.4 implies that exactly $q^{(d-1)(d-2)/2} \begin{bmatrix} d \\ 1 \end{bmatrix}_q$ elements of S intersect π in a line, and exactly $q^3 \begin{bmatrix} d-1 \\ 2 \end{bmatrix}_q - q^d$ intersect π in a subspace of codimension 2. We know from Corollary 5.4.4 that there are $q^{(d-1)(d-2)/2}$ generators in the polar space intersecting π in a fixed line, and hence the set A of lines in π appearing as such an intersection has size at least $\begin{bmatrix} d \\ 1 \end{bmatrix}_q$. Now consider any subspace ρ with codimension 2 in π. There are

exactly q^3 generators intersecting π in just ρ, again by Corollary 5.4.4. This implies that B, the set of all subspaces with codimension 2 in π arising from the intersection with an element of S, has cardinality at least $\left[{d-1 \atop 2}\right]_q - q^{d-3}$. Lemma 5.4.1 also yields that every element of A intersects every element of B non-trivially. The main idea will be that A and B are too large to have this property.

As there are only $\left[{d-1 \atop 1}\right]_q$ lines through a point in the projective geometry $\mathrm{PG}(d-1,q)$, no point in π can be on all lines in A. If a point $p \in \pi$ is not on a line $\ell \in A$, then it follows from Lemma 3.1.1 that there are precisely $\left[{d-1 \atop 2}\right]_q - q^{2d-6}$ subspaces with codimension 2 in π through p and intersecting ℓ non-trivially, which is less than $|B|$ as $d \geq 5$. Hence we can conclude that no point on π is on all elements of B. Since all elements of A must intersect every element of B non-trivially, there can be at most $\left[{d-2 \atop 1}\right]_q$ elements of A through each point of π.

Now let μ be any hyperplane of π. Let X denote the subset of elements of B contained in μ. Since μ is a secant dual line, we know from Theorem 5.8.5 that exactly $\left[{d-2 \atop 1}\right]_q (q+1)q^2$ elements of S intersect μ in a hyperplane of μ. These elements of S either intersect π in some element of X, or intersect π in some hyperplane, different from μ. Hence

$$\begin{bmatrix} d-2 \\ 1 \end{bmatrix}_q (q+1)q^2 \leq |X|q^3 + \left(\begin{bmatrix} d \\ 1 \end{bmatrix}_q - 1\right)q,$$

yielding $|X| \geq \left[{d-3 \atop 1}\right]_q$.

Next, consider two distinct lines ℓ_1 and ℓ_2 in A. Let μ be a hyperplane of π, intersecting these lines in distinct points p_1 and p_2, respectively (this always exists). We know from the above that μ contains at least $\left[{d-3 \atop 1}\right]_q$ elements of B. These elements must contain the points p_1 and p_2, and hence they are precisely the $\left[{d-3 \atop 1}\right]_q$ hyperplanes of μ through the line $\langle p_1, p_2\rangle$. Let ρ be a fixed hyperplane of μ not through the line $\langle p_1, p_2\rangle$. As $\rho \notin B$, there is certainly a generator π' with $\rho = \pi \cap \pi'$ and $\pi' \notin S$. Lemma 5.8.2 implies that there are exactly $q^{d(d-1)/2}$ elements of S that are disjoint from π'. These elements must intersect π in a line, disjoint from ρ. Hence we obtain at least $q^{d(d-1)/2}/q^{(d-1)(d-2)/2} = q^{d-1}$ of these lines of A in π, and they all intersect μ in just a point. As these lines must intersect all elements of B non-trivially, and hence certainly all hyperplanes of μ through $\langle p_1, p_2\rangle$, they must intersect μ in a point of that line, not on ρ. But we have seen that through each of

those q points on $\langle p_1, p_2\rangle$, there are at most $\left[{d-2 \atop 1}\right]_q$ elements of A. This yields $q^{d-1} \leq q\left[{d-2 \atop 1}\right]_q$, which is clearly a contradiction. $\square$

We will now focus on the other extremal situation, where no two elements of the EKR set in $W(2d-1,q)$, d odd, are neighbours in the dual polar graph. Even though $W(2d-1,q)$ and $Q(2d,q)$ are isomorphic if and only if q is even, the parameters of the corresponding association scheme are the same, regardless of the parity of q. However, we want to prove that the construction using an embedded $Q^+(2d-1,q)$ in $Q(2d,q)$ for odd d, which appeared in Theorem 5.7.1, has no analog in $W(2d-1,q)$ if q and d are odd. We will need the following fundamental results on the associated classical generalized quadrangles $W(3,q)$ and the dual $Q(4,q)$ (see for instance [122, 1.3.6, 3.2.1 and 3.3.1]). This will allow us to distinguish the symplectic spaces from the parabolic quadrics for odd q.

Theorem 5.8.7.

(i) *If three lines are pairwise skew in $W(3,q)$, then the number of lines of $W(3,q)$ meeting all three is 0 or 2 for odd q, and 1 or $q+1$ for even q.*

(ii) *If three lines are pairwise skew in $Q(4,q)$, then the number of lines of $Q(4,q)$ meeting all three is 1 or $q+1$.*

We will also need the following technical result.

Lemma 5.8.8. *For all integers n and q with $n \geq 0$ and $q \geq 3$, the following inequality holds:*

$$\prod_{i=1}^{n}(q^i+1) < 2q^{n(n+1)/2}.$$

Proof. The inequality clearly holds if $n=0$. We will prove that the left-hand side is at most $2q^{n(n+1)/2}(1-1/q^n)$ if $n \geq 1$ and $q \geq 3$. This is clearly true for every integer $q \geq 3$ if $n=1$, and if the claim holds for n, then for every $q \geq 3$:

$$\begin{aligned}\prod_{i=1}^{n+1}(q^i+1) &\leq 2q^{n(n+1)/2}\big(1-\frac{1}{q^n}\big)(q^{n+1}+1)\\ &= 2q^{(n+1)(n+2)/2}\big(1-\frac{1}{q^n}+\frac{1}{q^{n+1}}-\frac{1}{q^{2n+1}}\big)\\ &\leq 2q^{(n+1)(n+2)/2}\big(1-\frac{1}{q^{n+1}}\big),\end{aligned}$$

and thus it also holds for $n+1$. □

We can now prove that if d and q are odd, there is no EKR set of generators in $W(2d-1,q)$ of maximum size with minimum distance greater than or equal to two. This is in fact the only instance where the parity of q is of any importance in the proof.

Lemma 5.8.9. *Suppose S is an EKR set of generators in $W(2d-1,q)$ of size $|S| = (q+1)\cdots(q^{d-1}+1)$, d odd, $d \geq 3$ and q odd. Then there exist two elements of S intersecting in a subspace of codimension* 1.

Proof. Suppose that no two elements of S intersect in a subspace of codimension 1. Lemma 5.4.3 yields that in this case each element of S is its own nucleus. Theorem 5.8.4 now implies that for each $\pi \in S$, the corresponding row of the outer distribution B of S is v_2. In particular, the elements of S cannot intersect in a subspace with odd codimension. Let π be any element in S. We know that exactly $(v_2)_2 = \begin{bmatrix} d \\ 2 \end{bmatrix}_q q$ elements of S intersect π in exactly a subspace of codimension 2. As there are only $\begin{bmatrix} d \\ 2 \end{bmatrix}_q$ subspaces with codimension 2 in π, there must certainly be a subspace μ of codimension 2 in π, such that at least q elements ($q \geq 3$) of S intersect π in just μ. Let π_1 and π_2 be two such elements. Note that π_1 and π_2 cannot intersect in more than just μ, because their intersection cannot be a dual line as they are both in S. Hence the three generators π, π_1 and π_2 correspond with three pairwise skew lines ℓ, ℓ_1 and ℓ_2, respectively, in the residual geometry $W(3,q)$ of μ.

Now let S_0 denote the subset of generators in S, intersecting π in just a point, not in μ. Such a generator must intersect both π_1 and π_2 in a subspace of even codimension and skew to μ, thus in just a point not in μ. For every $\pi_0 \in S_0$, the generator $\langle \mu, \mu^\perp \cap \pi_0 \rangle$ through μ corresponds with a line meeting ℓ, ℓ_1 and ℓ_2 in $W(3,q)$. As q is odd, there are at most two such lines by Theorem 5.8.7(i). Hence, there are at most two possibilities for the generator $\langle \mu, \mu^\perp \cap \pi_0 \rangle$. As π_0 is skew to μ, it must intersect $\langle \mu, \mu^\perp \cap \pi_0 \rangle$ in a line. There are precisely $q^{2(d-2)}$ lines in a projective geometry $\mathrm{PG}(d-1,q)$, skew to a given subspace with codimension 2 (see Lemma 3.1.1). Finally, we consider the generators of S_0 that can contain that line. Since the elements of S pairwise intersect in a subspace with even codimension, all these generators must intersect in at least a plane, and hence in the residue of that line, which is isomorphic to $W(2(d-2)-1,q)$, we obtain a set of generators, pairwise intersecting in at least a point. This implies that we can apply the upper bound from Theorem

5.3.1 for EKR sets of generators in $W(2(d-2)-1,q)$, and see that there are at most $\prod_{i=1}^{d-3}(q^i+1)$ elements of S_0 through each such line (this can be seen directly if $d=3$). Hence, we obtain that $|S_0| \leq 2q^{2d-4}\prod_{i=1}^{d-3}(q^i+1)$.

Let us now explicitly calculate $|S_0|$. Theorem 5.8.4 yields that through each point of π, not in μ, there are precisely $q^{d(d-1)/2-d+1}$ elements of S that intersect π in just that point. Hence:

$$|S_0| = \left(\begin{bmatrix} d \\ 1 \end{bmatrix}_q - \begin{bmatrix} d-2 \\ 1 \end{bmatrix}_q\right) q^{(d-1)(d-2)/2} = (q^{d-1}+q^{d-2})q^{(d-1)(d-2)/2},$$

and thus we obtain the inequality:

$$(q^{d-1}+q^{d-2})q^{(d-1)(d-2)/2} \leq 2q^{2d-4}\prod_{i=1}^{d-3}(q^i+1),$$

which is equivalent to $\frac{q+1}{2}q^{(d-2)(d-3)/2} \leq \prod_{i=1}^{d-3}(q^i+1)$. As $2 \leq \frac{q+1}{2}$, this contradicts Lemma 5.8.8. □

We now prove a result on the nuclei of two neighbours of an EKR set of generators of maximum size in the dual polar graph on $W(2d-1,q)$ with d odd.

Lemma 5.8.10. *Let S be an EKR set of generators in $W(2d-1,q)$ of size $(q+1)\cdots(q^{d-1}+1)$, d odd and $d \geq 3$. If π_1 and π_2 are neighbours and both elements of S with a non-trivial nucleus, then they have the same nucleus.*

Proof. Let π_1 and π_2 have nuclei π_s and π_t with dimensions $s \geq 1$ and $t \geq 1$, respectively. It follows from the definition of nuclei that π_s and π_t are both in $\pi_1 \cap \pi_2$. If π_s is not contained in π_t, then $|\pi_s \backslash \pi_t| \geq q^{s-1}$. We know from Theorem 5.8.4 that for every $p \in \pi_s \backslash \pi_t$ there are $q^{d(d-1)/2-s+1}$ elements of S intersecting π_1 in just p, and by Lemma 5.4.3 these elements cannot intersect π_2 in just p. As $\pi_1 \cap \pi_2$ is a hyperplane in π_2, we see that these elements intersect π_2 in exactly a line. We also know from Theorem 5.8.4 that there are

exactly $\frac{q^{d-t}-1}{q-1}q^{(d-1)(d-2)/2}$ elements of S intersecting π_2 in a line, and hence:

$$q^{s-1}q^{d(d-1)/2-s+1} \leq \frac{q^{d-t}-1}{q-1}q^{(d-1)(d-2)/2},$$

which yields: $q^{d-1} \leq \frac{q^{d-t}-1}{q-1}$. This is impossible as $t \geq 1$. Hence $\pi_s \subseteq \pi_t$, and one shows in a completely analogous way that $\pi_t \subseteq \pi_s$. □

Lemma 5.8.11. *Let S be an EKR set of generators of size $(q+1)\cdots(q^{d-1}+1)$ in $W(2d-1,q)$ with d odd, q odd and $d \geq 3$. Suppose $\pi \in S$ has a point p as nucleus. Then S is the set of generators through p.*

Proof. By Theorem 5.8.4, there are $q^{d(d-1)/2}$ elements of S intersecting π in just p. Suppose that there exists a generator π' through p not in S, then by Lemma 5.8.2, there are $q^{d(d-1)/2}$ elements of S disjoint from π' that hence cannot pass through p. So there are at least $q^{d(d-1)/2}$ elements in S not through p, and at least $q^{d(d-1)/2}$ through p. Hence $|S| \geq 2q^{d(d-1)/2} > (q+1)\cdots(q^{d-1}+1)$ for $q \geq 3$ by Lemma 5.8.8, which is a contradiction. Hence every generator through p is in S, and now the result follows from the size $|S|$. □

Lemma 5.8.12. *Let S be an EKR set of generators of size $(q+1)\cdots(q^{d-1}+1)$ in $W(2d-1,q)$ with d odd, $d \geq 3$ and q odd. Suppose $\pi \in S$ has an s-dimensional nucleus π_s. Then $s \in \{0,1,2,d\}$. If $s = 2$, then for every dual line μ with $\pi_s \subseteq \mu \subset \pi$, an element of S intersects π in just a point if and only if it intersects μ in just a point.*

Proof. Suppose $s < d$. Then $\pi_s \neq \pi$, and now consider any dual line μ with $\pi_s \subseteq \mu \subset \pi$. Lemma 5.4.3 implies that μ is secant. We know from Theorem 5.8.4 that exactly $\begin{bmatrix} s \\ 1 \end{bmatrix}_q q^{d(d-1)/2-s+1}$ elements of S intersect π in just a point of π_s, and hence must intersect μ in exactly a point as well. Theorem 5.8.5 also yields that exactly $(q+1)q^{(d+1)(d-2)/2}$ elements of S intersect μ in just a point. Hence we obtain:

$$\frac{q^s-1}{q-1}q^{d(d-1)/2-s+1} \leq (q+1)q^{(d+1)(d-2)/2},$$

which is equivalent to $s \leq 2$. If $s = 2$, then the two sizes are equal, and hence the generators in S intersecting μ in just a point must be precisely those intersecting π in just a point. □

We can now finally complete the classification of EKR sets of generators of maximum size in $W(2d-1,q)$ with q and d odd.

Theorem 5.8.13. *Let S be an EKR set of generators in $W(2d-1,q)$ of size $(q+1)\cdots(q^{d-1}+1)$, with q odd, d odd and $d \geq 5$. Then S is the set of all generators through some point.*

Proof. It follows from Lemmas 5.8.6 and 5.8.12 that every element in S has a nucleus with a dimension s in $\{1,2,d\}$. If $s=1$ for some $\pi \in S$, then by Lemma 5.8.11, the set S consists of all the generators through a point. Hence, from now on we can assume that every $\pi \in S$ has a nucleus of dimension 2 or d, and we will prove that this leads to a contradiction.

First suppose some $\pi \in S$ has nucleus π_s of dimension $s=2$. Consider any dual line μ with $\pi_s \subseteq \mu \subset \pi$. Theorem 5.8.4 yields that there is certainly an element $\pi' \in S$ intersecting π in just a point of π_s. Consider the generator $\pi'' = \langle \mu, \mu^\perp \cap \pi' \rangle$, which intersects π' in a line. As π'' passes through the secant dual line μ, it is also in S. Since, by Lemma 5.8.6, π'' has a non-trivial nucleus, Lemma 5.8.10 yields that π'' also has π_s as nucleus. But this contradicts Lemma 5.8.12, as we now have the generator $\pi' \in S$ intersecting μ in just a point, while it intersects $\pi'' \in S$ in a line.

Hence the dimension s of the nucleus is d for every element of S, which contradicts Lemma 5.8.9 as q is odd. □

Just as for $Q(6,q)$, there is an extra construction for EKR sets of generators of the maximum size for $W(5,q)$, and hence this polar space must be treated separately.

Theorem 5.8.14. *Suppose S is an EKR set of $(q+1)(q^2+1)$ planes in $W(5,q)$, q odd. Then the elements of S are either all generators through a fixed point, or S consists of a plane π and all the planes intersecting π in a line (i.e. S is a sphere of radius 1 in the dual polar graph).*

Proof. By Lemma 5.8.9, there are at least two generators π and π_1 in S that intersect in a subspace of codimension 1. Hence the s-dimensional nucleus π_s of π is at most a line. Lemma 5.4.3 yields that if an element of S intersects π in a point p, then $p \in \pi_s$, and that the elements of S intersecting π in a line, are precisely those intersecting π in a line through π_s. Obviously, $s \in \{0,1,2\}$. If $s=0$, then all $q(q^2+q+1) = |S|-1$ planes intersecting π in a line, are in S, and hence S consists of these planes and π itself. If $s=1$, then π_s is a point

contained in all elements of S, and hence we are done again. Finally, suppose $s = 2$. Let a and b be distinct points on the line π_s. Theorem 5.8.4 yields that through both points, there are precisely q^2 elements of S, intersecting π in just that point. Suppose $\pi_a, \pi_b \in S$ with $\pi_a \cap \pi = \{a\}$ and $\pi_b \cap \pi = \{b\}$. As π_a and π_b cannot be disjoint, they must intersect by Lemma 5.4.1 in precisely one point c, necessarily outside of π. The points a, b and c span a plane π' in S by Lemma 5.4.3, as $\pi_s = \langle a, b\rangle \subseteq \pi'$. Since $\pi' \cap \pi = \langle a, b\rangle$, $\pi' \cap \pi_a = \langle a, c\rangle$ and $\pi' \cap \pi_b = \langle b, c\rangle$, π' has a trivial nucleus and then S consists of π' and all the planes intersecting π' in a line. □

5.9 $H(2d-1,q^2)$ with d odd

The size of the set of generators Ω_d in $H(2d-1,q^2)$ is given by $(q+1)(q^3+1)\cdots(q^{2d-1}+1)$. The number of generators through one point is $|\Omega_d|/(q^{2d-1}+1)$, but the eigenvalue bound from Theorem 5.3.1 is $|\Omega_d|/(q^d+1)$ for odd d, which is much larger.

In $H(5,q^2)$, where the diameter is three, there are $(q+1)(q^3+1)(q^5+1)$ generators, and $(q+1)(q^3+1)$ generators through one point. The upper bound arising from eigenvalue techniques in this case is $(q+1)(q^5+1)$. The following example shows that the point-pencil construction is in this case indeed not of maximum size. Let π be a plane in $H(5,q^2)$. Let S consist of π and all planes intersecting π in a line. In other words: S is the sphere of radius 1 around π in the dual polar graph. Now $|S| = q(q^4+q^2+1)+1$, and in particular: $(q+1)(q^3+1) < |S| < (q+1)(q^5+1)$.

It is possible that there is no simple answer for $H(2d-1,q^2)$ for odd d in general. However, we can already exclude the possibility of attaining the upper bound from Theorem 5.3.1.

Theorem 5.9.1. *Suppose S is an EKR set of generators in $H(2d-1,q^2)$ with d odd and $d \geq 3$. Then $|S| < |\Omega_d|/(q^d+1)$.*

Proof. We already know from Theorem 5.3.1 that $|S| \leq |\Omega_d|/(q^d+1)$, with equality if and only if $\chi_S \in V^d_{0,0} \perp V^d_{d,0}$. Suppose equality holds. In that case, every dual line or $(d-1)$-space would be incident with exactly $(q+1)/(q^d+1)$ elements of S, because of Theorem 4.4.1. As $d \geq 3$, this yields a contradiction as this number is not an integer. □

Nevertheless, we can determine the maximum size of an EKR set of planes in $H(5, q^2)$. We first state a general theorem[1] on generalized quadrangles (see for instance [122, 1.2.4]).

Theorem 5.9.2. *Let a, b and c be three mutually non-collinear points in a generalized quadrangle of order (s, s^2). The number of points collinear with a, b and c is exactly $s + 1$.*

Dualizing, this yields the following result for the classical generalized quadrangle $H(3, q^2)$ of order (q^2, q).

Corollary 5.9.3. *If ℓ_1, ℓ_2 and ℓ_3 are three mutually skew lines in $H(3, q^2)$, then there are precisely $q + 1$ lines of $H(3, q^2)$ meeting all of them.*

Theorem 5.9.4. *Let S be an EKR set of planes in $H(5, q^2)$. Then $|S| \leq q^5 + q^3 + q + 1$, and this bound can only be attained if S consists of a plane π and all planes intersecting π in a line.*

Proof. Assume that S is a maximal EKR set of generators.

Suppose that $\pi \in S$ intersects some element of S in a line. Lemma 5.4.3 yields that the nucleus π_s of π has dimension $s \leq 2$.

If $s = 0$, then Lemma 5.4.3 yields that S contains all $1 + q(q^4 + q^2 + 1)$ planes that are equal to π or intersecting π in a line, while there are no planes in S intersecting π in just a point.

If $s = 1$, then all elements of S must pass through the point π_s, and hence $|S| \leq (q + 1)(q^3 + 1)$, which is less than $q^5 + q^3 + q + 1$.

Now suppose $s = 2$. If no element of S intersects π in a point, then all other elements of S intersect π in the line π_s and hence $|S| \leq q + 1$. Similarly, if all elements of S either contain π_s or intersect π in the same point p, then again $|S| \leq (q + 1)(q^3 + 1)$. Finally, suppose that π' and π'' are elements of S, intersecting π in different points p' and p'' of the nucleus of π, respectively. Lemma 5.4.1 yields that π' and π'' intersect in just a point, say p. Consider the plane $\langle p', p'', p\rangle$, which is in S since it passes through the nucleus of π. Its nucleus is trivial since π, π' and π'' intersect it in three non-concurrent lines. Lemma 5.4.3 again yields that S consists of $\langle p', p'', p\rangle$ and all the planes intersecting it in a line.

[1] We will give a generalization of Theorem 5.9.2 in Corollary 6.6.7.

In the remainder of this proof, we can suppose that all elements of S intersect in just a point. We will also assume that $|S|$ is at least the desired bound q^5+q^3+q+1, and prove that this leads to a contradiction. Suppose $\pi \in S$ and let p be a point on π. In the residual geometry of p, isomorphic to $H(3,q^2)$, the elements of S through p correspond with different mutually skew lines $\ell_1,\dots,\ell_t$. Lemma 4.1.1 yields that there are $(q^2+1)(q^3+1)$ points in $H(3,q^2)$ and hence $t \leq q^3+1$. Hence there are at least q^5+q elements of S not through p. The elements of S not through p are projected onto lines in the residual geometry $H(3,q^2)$. In each plane of $H(5,q^2)$ through p, there are q^4 lines skew to p. Since two elements of S cannot intersect in a line, at most q^4 elements of S can be projected onto the same line of $H(3,q^2)$, so we have constructed in this way at least $q+1$ lines in $H(3,q^2)$, namely $m_j, j=1,\dots,q+1$. Since the elements of S pairwise intersect in just a point, an element of S through p cannot be projected onto any of the m_j, so $\ell_i \neq m_j$, $\forall i,j$. As the elements of S cannot be pairwise disjoint, the lines l_i and m_j must intersect in the residual geometry $H(3,q^2)$. If $m_j \cap m_k$ is a point, then a line ℓ_i in $H(3,q^2)$ intersecting both of them must pass through their intersection point. As the lines ℓ_i are pairwise skew, $t=1$ in this case and so through p there can be at most one element of S. If all these lines m_j are pairwise skew, then there are at most $q+1$ lines meeting all of them because of Corollary 5.9.3, so there are in this case at most $q+1$ elements of S through p. Hence through every point of π there are at most $q+1$ elements of S, but $|S| \geq q^5+q^3+q+1$ implies that through every point of π there are exactly $q+1$ elements of S and $|S| = q^5+q^3+q+1$. So now we can consider a point $p \in \pi \in S$ and two other elements $\pi_1, \pi_2 \in S$ through p, such that π, π_1 and π_2 correspond with three skew lines ℓ, ℓ_1 and ℓ_2, respectively, in the residual geometry of p. In this geometry, only $q+1$ points of ℓ, corresponding to the plane π, are on a line meeting ℓ, ℓ_1 and ℓ_2, and hence in $H(5,q^2)$ only the points on the corresponding $q+1$ lines through p in π can be on a plane intersecting the planes π, π_1 and π_2 non-trivially. This contradicts the fact that there are $q+1$ elements of S through each point in the plane π. □

Remark 5.9.5. Newman [111, Section 3.2] raised the question of whether or not maximal cocliques S can be found in a k-regular graph on Ω, such that $|S| = |\Omega|/(1-k/\lambda)$, where λ is a negative eigenvalue but not the minimal eigenvalue. In all classical finite polar spaces of rank d and with parameters (q,q^e), the point-pencil construction has size $|\Omega_d|/(q^{d-1+e}+1)$. We know from Theorem 4.3.15 that this is precisely $|\Omega_d|/(1-k/\lambda)$, with λ the eigenvalue of oppositeness between generators for the subspace $V^d_{1,0}$ and k its valency. This

is always a maximal EKR set of generators, but Corollary 4.3.17 yields that for $H(2d-1, q^2)$ with d odd, this eigenvalue is not the minimal eigenvalue. Here, the point-pencil construction is maximal, but it is not of maximum size for $d = 3$, and it is an open problem if it is of maximum size for higher odd d.

5.10 Summary

We now summarize the results in this chapter for classical finite polar spaces of rank at least three. We write p.-p. to denote the point-pencil construction of an EKR set of generators consisting of all generators through one point. For the hyperbolic space of even rank we will focus only on one system of generators, namely the *Latins*. Finally, 1-*sphere* will refer to the construction in a polar space of rank three, consisting of one plane and all those intersecting it in a line.

Polar space	Maximum size	Classification
$Q^-(2d+1,q)$	$(q^2+1)\cdots(q^d+1)$	p.-p., Th.5.5.2
$Q(2d,q), d$ even	$(q+1)\cdots(q^{d-1}+1)$	p.-p., Th.5.5.2
$Q(2d,q), d$ odd and $d \geq 5$	$(q+1)\cdots(q^{d-1}+1)$	p.-p., Latins $Q^+(2d-1,q)$, Th.5.7.1
$Q(6,q)$	$(q+1)(q^2+1)$	p.-p., Latins $Q^+(5,q)$, 1-sphere, Th.5.7.1
$Q^+(2d-1,q), d$ odd	$(q+1)\cdots(q^{d-1}+1)$	one system, Th.5.6.1
Latins $Q^+(2d-1,q), d$ even and $d \geq 6$	$(q+1)\cdots(q^{d-2}+1)$	p.-p., Th.5.6.8
Latins $Q^+(7,q)$	$(q+1)(q^2+1)$	p.-p., intersecting Greek in plane, Th.5.6.10
$W(2d-1,q), d$ odd and $d \geq 5, q$ odd	$(q+1)\cdots(q^{d-1}+1)$	p.-p., Th.5.8.13
$W(2d-1,q), d$ odd and $d \geq 5, q$ even	$(q+1)\cdots(q^{d-1}+1)$	p.-p., Latins $Q^+(2d-1,q)$, Th.5.8.1
$W(5,q), q$ odd	$(q+1)(q^2+1)$	p.-p., 1-sphere, Th.5.8.14
$W(5,q), q$ even	$(q+1)(q^2+1)$	p.-p., 1-sphere, Latins $Q^+(5,q)$,Th.5.8.1
$W(2d-1,q), d$ even	$(q+1)\cdots(q^{d-1}+1)$	p.-p., Th.5.5.2
$H(2d,q^2)$	$(q^3+1)(q^5+1)$ $\cdots(q^{2d-1}+1)$	p.-p., Th.5.5.2
$H(2d-1,q^2), d$ even	$(q+1)(q^3+1)$ $\cdots(q^{2d-3}+1)$	p.-p., Th.5.5.2
$H(2d-1,q^2), d$ odd and $d \geq 5$	$< \lvert\Omega_d\rvert/(q^d+1)$	?,Th.5.9.1
$H(5,q^2)$	$q(q^4+q^2+1)+1$	1-sphere, Th.5.9.4

Chapter 6

Near polygons

Near polygons were introduced by Shult and Yanushka [133]. Our study of regular near polygons is motivated by the search for a better understanding of the algebraic proof of the bound for partial spreads from Theorem 4.4.16. The techniques to study that problem in (dual) polar spaces turn out to be available for near polygons in general, including the generalized polygons, which is why we treat all these structures in the same chapter.

We will discuss subsets of points in generalized polygons in Subsection 6.4.2. These results are inspired by the work of Martin [106], De Wispelaere and Van Maldeghem [63], and Bamberg, Law and Penttila [10]. We will very often give alternative proofs or generalizations.

In Subsection 6.4.3, we use the same techniques to study sets of maximal totally isotropic subspaces in classical finite polar spaces, seen as sets of points in dual polar spaces. Our results include a more elegant proof for the bound on partial spreads in $H(2d-1, q^2)$ for odd d, and more information on those attaining the bound, as well as some results on spreads in $Q(2d, q)$ and $W(2d-1, q)$ for odd d. These results on dual polar spaces have been accepted for publication in *Journal of Combinatorial Designs* [162].

We then focus on the parameters of a regular near $2d$-gon itself. Higman [87] proved that the point graph of a regular near 4-gon of order (s, t) with $s > 1$ satisfies $t \leq s^2$. We generalize this for regular near $2d$-gons in Section 6.6., and we consider the case of equality. In Section 6.7, we use this to prove that a specific type of subset of maximal totally isotropic subspaces in $H(2d-1, q^2)$, q odd, induces a distance-regular graph with classical parameters

$(d, -q, -(q+1)/2, -((-q)^d + 1)/2)$. This generalizes a result by Thas [148] on strongly regular graphs induced by hemisystems in the dual polar space on $H(3, q^2)$. The results referred to in this paragraph have been accepted for publication in *Journal of Algebraic Combinatorics* [164].

We conclude by discussing near pentagons in Section 6.8.

6.1 Definitions and basic properties

Definition 6.1.1. *A near polygon is a partial linear space $\mathcal{P}$ satisfying the following axioms.*

(i) *The point graph is connected with diameter $d \geq 1$.*

(ii) *With respect to the point graph, for each point p and line ℓ with $d(p, \ell) < d$, there is a unique point p' on ℓ at minimal distance from p.*

We say $\mathcal{P}$ is a near $(2d+1)$-gon if there is a point p at distance d from some line ℓ with respect to the point graph, and a near $2d$-gon otherwise.

The definition of near $2d$-gons is due to Shult and Yanushka [133]. We will be mostly concerned with this type of near polygons. Near polygons with an infinite number of points or lines are also studied, but from now on, the sets of points and lines will always be assumed to be finite.

A near 2-gon consists of simply one line with all (and at least two) points on it. A near 3-gon is the same as a linear space with more than one line. The near 4-gons with every point on at least two lines are precisely the generalized quadrangles as defined in Section 1.3.

The simplest example of a near n-gon, $n \geq 3$, is the *ordinary n-gon*, the incidence graph of which is just a circuit on $2n$ vertices.

We will also refer to near 4-gons, 5-gons, 6-gons and 8-gons as *near quadrangles*, *near pentagons*, *near hexagons* and *near octagons*, respectively.

We will consider both the point graph and the (bipartite) incidence graph of near polygons. Unless stated otherwise, distance between points or between a point and a line will refer to the distance in the point graph.

We say a near polygon is *of order* (s,t) if every line contains exactly $s+1$ points, and every point is on exactly $t+1$ lines.

Definition 6.1.2. *A near polygon is regular if its point graph is distance-regular.*

Theorem 6.1.3. [23, Section 6.4] *A partial linear space $\mathcal{P}$ with distance-regular point graph Γ with valency k and diameter $d \geq 2$ is a regular near polygon if for some integer $s \geq 1$, the intersection numbers satisfy:*

$$b_i = k - sc_i, \ \forall i \in \{1, \ldots, d-1\}.$$

In that case $k \geq sc_d$ holds, and $\mathcal{P}$ is a near $2d$-gon in case of equality, and a near $(2d+1)$-gon otherwise.

In any near n-gon, $n \geq 4$, the common neighbours in the point graph of any two adjacent vertices are precisely the remaining points on the unique line through them. Hence the maximal cliques in the point graph are precisely the sets of points on one line, and so a near n-gon, $n \geq 4$, is completely determined[1] by its point graph. It follows from Theorem 6.1.3 that in a regular near n-gon with $n \geq 4$, the number of common neighbours of two adjacent vertices is $a_1 = k - b_1 - c_1 = s - 1$ for some integer s. As the neighbours of a point p in the point graph are all other points on the lines through p, this yields that in a regular near n-gon with $n \geq 4$, the number of lines through each point is also a constant, namely k/s. Hence every regular near n-gon, $n \geq 4$, has an order (s,t), with the valency of the point graph given by $k = s(t+1)$. For any $i \in \{1, \ldots, d\}$, we will also use the standard notation t_i for $c_i - 1$. Note that the point graph of a regular near $2d$-gon of order (s,t) also satisfies $c_d = t+1$.

The regular near $2d$-gons of order $(1,t)$ have bipartite point graphs. Conversely, every bipartite distance-regular graph of diameter d and valency $k \geq 2$ gives rise to a regular near $2d$-gon of order $(1, k-1)$, with the vertices as points, the edges as lines, and symmetrized containment as incidence relation (see also [53, Theorem 1.24]).

Definition 6.1.4. *A sub near n-gon of a near n-gon $\mathcal{P} = (P, L, \mathrm{I})$ is a near n-gon $\mathcal{P}' = (P', L', \mathrm{I}')$ with $P' \subseteq P, L' \subseteq L$ and $\mathrm{I}' = ((P' \times L') \cup (L' \times P')) \cap \mathrm{I}$.*

[1] Some authors introduce regular near polygons as a type of distance-regular graphs, and refer to the maximal cliques as the singular lines.

A sub near polygon $\mathcal{P}' = (P', L', \mathrm{I}')$ of $\mathcal{P} = (P, L, \mathrm{I})$ is *proper* when $P' \neq P$ or $L' \neq L$. We say a sub near polygon $\mathcal{P}'$ of $\mathcal{P}$ is *isometrically embedded* if the distance between any two points of $\mathcal{P}'$ is the same with respect to both point graphs.

6.2 Types of near polygons

6.2.1 Generalized polygons

The definition of generalized polygons is due to Tits [156] and predates that of near polygons.

Definition 6.2.1. *A generalized n-gon, $n \geq 2$, is a point-line geometry with an incidence graph of diameter n and girth $2n$.*

Remark 6.2.2. One can easily prove that the conditions imply that each point is incident with at least two lines, and every line with at least two points (see for instance [161, Lemmas 1.3.6 and 1.5.10]).

Note that the above definition of generalized polygons is self-dual. All generalized n-gons with $n \geq 3$ are partial linear spaces. The simplest example of a generalized n-gon is again an ordinary n-gon.

The generalized 3-gons with at least three points on each line and at least three lines through each point are the projective planes (see [161, p. 11]).

For $n = 4$, the above definition coincides with the one given for generalized quadrangles in Section 1.3 (see [161, Lemma 1.4.1]).

Note that the axioms of a generalized n-gon imply that for any two vertices x and y at distance $i < n$ in the incidence graph, there is a unique path of shortest length from x to y.

Lemma 6.2.3. [161, Lemmas 1.3.6 and 1.5.10] *For any $n \geq 2$, a point-line geometry is a generalized n-gon if and only if the following axioms hold.*

(i) *There exist no ordinary k-gon as a subgeometry with $2 \leq k < n$.*

(ii) *Any two varieties are contained in some ordinary n-gon.*

Theorem 6.2.4. *Every generalized n-gon is a near n-gon for $n \geq 3$.*

Proof. We know from the definition that a generalized n-gon is a partial linear space. Suppose $n = 2d$ if n is even, and $n = 2d + 1$ if n is odd. The incidence graph has diameter n, so two points are at distance at most $2d$ in this graph. As the girth of the incidence graph is $2n$, we can indeed find points at distance $2d$ in any circuit of length $2n$ in the incidence graph. Hence the diameter of the point graph is d. If $n = 2d$, then the maximum distance between a point and a line is $2d - 1$ in the incidence graph, and hence $d - 1$ in the point graph, but if $n = 2d + 1$, then there exist a point and a line at distance $2d + 1$ in the incidence graph, hence at distance d in the point graph.

Since the girth of the incidence graph is $2n$, there is a unique path of shortest length between any two vertices in the incidence graph at distance at most $n - 1$, which is $2d - 1$ or $2d$. Hence if a point p is at distance at most $d - 1$ from a line ℓ in the point graph, or hence if $d(p, \ell) \leq 2d - 1$ in the incidence graph, then there is a unique point on ℓ at minimal distance from p in the point graph. □

Theorem 6.2.5. [23, Section 6.5][161, Lemma 1.5.4] *For $d \geq 2$, the generalized $2d$-gons of order (s, t) are precisely the regular near $2d$-gons order (s, t) with point graph Γ with parameters $c_1 = \ldots = c_{d-1} = 1$. They satisfy:*

$$c_d = t + 1 \text{ and } b_0 = s(t + 1), b_1 = \cdots = b_{d-1} = st.$$

The number of points is given by:

$$(s + 1)((st)^{d-1} + \cdots + 1),$$

and the valency k_i of the distance-i relation of the point graph Γ by:

$$k_i = s^i t^{i-1}(t + 1) \text{ if } 1 \leq i \leq d - 1, k_d = s^d t^{d-1}.$$

The *double* of a generalized n-gon is the point-line geometry, the points of which are the points and lines of the generalized n-gon, and the lines of which are the pairs of incident points and lines of the generalized n-gon. Incidence is just symmetrized inclusion. For a generalized n-gon of order (s, s), the double is a generalized $2n$-gon of order $(1, s)$, and all generalized $2n$-gons of order $(1, s)$ with $n \geq 2$ can be obtained in this way ([159], see also [161, Theorem 1.6.2]).

A *sub-n-gon* of a generalized n-gon $\mathcal{P}$ is a sub near polygon $\mathcal{P}'$ that is also a generalized n-gon. Note that in this case, the embedding of $\mathcal{P}'$ is always isometric: the distance between any two points of $\mathcal{P}'$ is the same in both point graphs (see [161, Remark 1.3.4]).

The following famous theorem by Feit and G. Higman says that generalized $2n$-gons of order (s,t) can only exist for very few values of n if $s,t>1$, and imposes additional restrictions on s and t.

Theorem 6.2.6. [80] *Suppose $\mathcal{P}$ is a generalized n-gon of order (s,t) with $n \geq 3$. One of the following must hold.*

(i) $s=t=1$ *and $\mathcal{P}$ is an ordinary n-gon,*

(ii) $n=3$,

(iii) $n=4$, *with $st(st+1)/(s+t)$ an integer,*

(iv) $n=6$, *with $\sqrt{st}$ an integer if $s,t>1$,*

(v) $n=8$, *with $\sqrt{2st}$ an integer if $s,t>1$,*

(vi) $n=12$, *and $s=1$ or $t=1$.*

Theorem 6.2.7. *Let $\mathcal{P}$ be a generalized n-gon of order (s,t) with $s,t>1$. The following holds:*

(i) ([87]) *if $n=4$, then $s \leq t^2$ and $t \leq s^2$ (D. Higman's bound),*

(ii) ([86]) *if $n=6$, then $s \leq t^3$ and $t \leq s^3$ (Inequality of Haemers and Roos),*

(iii) ([88]) *if $n=8$, then $s \leq t^2$ and $t \leq s^2$ (D. Higman's bound).*

We already discussed the known possible orders for generalized quadrangles in Subsection 1.3.3.

All known generalized hexagons and octagons of order (s,t) with $s,t>1$ are classical, in the sense that they are related to Chevalley groups in a natural way (see [43] for more information on the latter).

Up to duality, the only known generalized hexagons of order (s,t) with $s,t>1$ are those constructed by Tits [156]: the *split Cayley hexagons* $\mathrm{H}(q)$ of order (q,q) (related to $G_2(q)$) and the *twisted triality hexagons* $\mathrm{T}(q^3,q)$ (related to

${}^3D_4(q)$), where q denotes a prime power. We also denote the dual of $\mathrm{T}(q^3, q)$ by $\mathrm{T}(q, q^3)$. See [161] for a coordinatization of $\mathrm{H}(q)$ and $\mathrm{T}(q^3, q)$. We have already discussed the embedding of $\mathrm{H}(q)$ in the parabolic quadric $Q(6, q)$ in Subsection 4.4.3. The split Cayley hexagon $\mathrm{H}(q)$ is self-dual if and only if $q = 3^h$, and admits a polarity if and only if $q = 3^{2e+1}$ with h and e integers (see for instance [161, Corollary 3.5.7 and Subsection 7.3.8]).

Finally, up to duality, only one class of generalized octagons of order (s, t) with $s, t > 1$ is known, namely the *Ree-Tits octagons* of order (q, q^2) (related to ${}^2F_4(q)$) with $q = 2^{2e+1}$ for some integer e, which were constructed by Tits [157].

6.2.2 Dual polar spaces

Definition 6.2.8. *The dual polar space on a polar space is the point-line geometry, with the maximals as points, the next-to-maximals as lines, and incidence the restriction of incidence in the polar space.*

Theorem 6.2.9. [33, Theorem 1] *The dual polar space on a polar space of rank $d \geq 2$ is a near $2d$-gon.*

For a classical finite polar space of rank $d \geq 2$ with parameters (q, q^e), the associated dual polar space is a regular near $2d$-gon of order $\left(q^e, \left[{d \atop 1}\right]_q - 1\right)$. The point graph is simply the dual polar graph, with two points of the dual polar space adjacent if and only if they intersect in a subspace of codimension one, and more generally, with two points of the dual polar space at distance i if they intersect in a subspace of codimension i (see Theorem 4.1.7). If π and ℓ are points and lines in the dual polar space, respectively, then the unique point in the dual polar space on ℓ at minimal distance from π is given by $\langle \ell, \ell^\perp \cap \pi \rangle$. Cameron [33] in fact characterized the dual polar spaces as a class of near $2d$-gons.

6.2.3 Sporadic regular near $2d$-gons

We will now describe some of the sporadic regular near $2d$-gons.

- By [172], there is (up to isomorphism) a unique Steiner system $S(5, 8, 24)$. There are 24 points in this system, each block contains 8 points and every 5 different points are in a unique block. Two distinct blocks can intersect in $0, 2$ or 4 points. When two blocks are disjoint, the complement of their union is also a block. The points of the related near hexagon are the 759 blocks, and its lines are the triples of pairwise disjoint blocks. Incidence is symmetrized containment. Shult and Yanuskha [133] proved that this is a regular near hexagon with parameters $(s, t_2, t) = (2, 2, 14)$, and Brouwer [21] proved its characterization by its parameters. The point graph is known as the *large Witt graph* and has classical parameters $(3, -2, -4, 10)$. Two vertices are at distance $1, 2, 3$ in the point graph whenever they intersect in $0, 4, 2$ elements, respectively.

- The *extended ternary Golay code* is a subspace C of $\mathrm{GF}(3)^{12}$ (see for instance [41]). Consider the following point-line geometry. The points are the 729 cosets of C. The lines are the triples of cosets $\{C_1, C_2, C_3\}$, such that every two elements C_i and C_j have representatives which differ in only one position. Incidence is just symmetrized containment. Shult and Yanushka [133] proved that this is a regular near hexagon with parameters $(s, t_2, t) = (2, 1, 11)$ and Brouwer [20] proved its characterization by its parameters. The point graph has classical parameters $(3, -2, -3, 8)$.

- The Hall-Janko group is a sporadic simple group of order 604800 (see for instance [46]). It has a unique conjugacy class of 315 involutions whose centralizers contain Sylow-2-subgroups. Consider the following point-line geometry. The points are these 315 involutions, the lines are the triples of pairwise commuting involutions, and incidence is symmetrized containment. This is the unique regular near octagon with parameters $(s, t_2, t_3, t) = (2, 0, 3, 4)$ (see [45]).

6.3 Eigenvalues of near $2d$-gons

We will now give a result on the Bose-Mesner algebra of the association scheme defined by the point graph of a regular near $2d$-gon of order (s, t). It turns out that, even without knowing all parameters of the near $2d$-gon, one specific idempotent with an elegant expression always exists, and we will use this to obtain a wide variety of results. The following theorem is due to Brouwer and Wilbrink [28]. We give a proof for the sake of completeness.

Theorem 6.3.1. *Let Γ be the point graph of a regular near $2d$-gon of order (s,t) with $d \geq 2$. If A_i denotes the adjacency matrix of the distance-i relation in the point graph, then $M = \sum_{i=0}^{d}(-1/s)^i A_i$ is a minimal idempotent up to a positive scalar and is positive semidefinite, with*

$$\operatorname{rank}(M) = \frac{\sum_{i=0}^{d} k_i}{\sum_{i=0}^{d} k_i/s^{2i}},$$

where k_i denotes the valency of the distance-i relation, which has eigenvalue $\lambda_i = k_i/(-s)^i$ for this idempotent.

The column span of M is the eigenspace for $-(t+1)$, which is the minimal eigenvalue of the point graph, and is also the kernel of the incidence matrix C (with columns and rows indexed by the points and lines, respectively).

Proof. Let b_i and c_i be the intersection numbers of Γ and set $b_{-1} = b_d = c_0 = c_{d+1} = 0$, and let k denote the valency of Γ. We also define A_{-1} and A_{d+1} as zero matrices. This allows us to algebraically express the properties of intersection numbers:

$$A_1 A_i = b_{i-1}A_{i-1} + (k - b_i - c_i)A_i + c_{i+1}A_{i+1}, \forall i \in \{0, \ldots, d\}.$$

We can now write:

$$\begin{aligned}
A_1 M &= A_1\left(\sum_{i=0}^{d} \frac{(-1)^i}{s^i} A_i\right) \\
&= \sum_{i=0}^{d} \frac{(-1)^i}{s^i}(A_1 A_i) \\
&= \sum_{i=0}^{d} \frac{(-1)^i}{s^i}(b_{i-1}A_{i-1} + (k - b_i - c_i)A_i + c_{i+1}A_{i+1}) \\
&= \sum_{i=0}^{d} \frac{(-1)^{i+1}}{s^{i+1}}(b_i A_i) + \sum_{i=0}^{d} \frac{(-1)^i}{s^i}((k - b_i - c_i)A_i) + \sum_{i=0}^{d} \frac{(-1)^{i-1}}{s^{i-1}}(c_i A_i) \\
&= \sum_{i=0}^{d} \frac{(-1)^i}{s^i}\left(-\frac{b_i}{s} + (k - b_i - c_i) + (-sc_i)\right)A_i \\
&= \sum_{i=0}^{d} \frac{(-1)^i}{s^i}\left(-\frac{k}{s}\right)A_i \\
&= -(t+1)M.
\end{aligned}$$

where we used the identities $b_i = k - sc_i$, $\forall i \in \{0, \ldots, d\}$, and $k = s(t+1)$ from Theorem 6.1.3 in the last two steps. Now $-(t+1)$ must be an eigenvalue of A_1 and the column span of M is in the corresponding eigenspace. Note that M is clearly a non-zero element of the Bose-Mesner algebra, which can be written as a linear combination of the minimal idempotents. As A_1 has a different eigenvalue for each of the $d+1$ minimal idempotents (see Theorem 2.3.3), M must be a scalar multiple of the corresponding minimal idempotent E and the eigenspace for $-(t+1)$ must be precisely the column span of M. As both $\mathrm{Tr}(M) = \mathrm{Tr}(A_0)$ and $\mathrm{Tr}(E)$ are positive, this scalar must be positive, and thus M is also positive semidefinite. The eigenvalues λ_i and $\mathrm{rank}(M)$ now follow from Lemma 2.2.2.

Finally, two points can only be on a common line if they are equal, when they are on $t+1$ common lines, or at distance 1 in the point graph, when they are on a unique common line. Using the incidence matrix C, we can write this algebraically as:

$$C^T C = (t+1)A_0 + A_1.$$

This yields that $A_1 + (t+1)\mathrm{I}$ is positive semidefinite and hence has positive eigenvalues, so every eigenvalue of A_1 is bigger than or equal to $-(t+1)$. It also follows that $Cv = 0 \iff C^T Cv = 0 \iff A_1 v = -(t+1)v$, which completes the proof. □

Remark 6.3.2. The above is not true in general for regular near $(2d+1)$-gons of order (s,t), as the point graph need not have the eigenvalue $-(t+1)$. For instance, the ordinary $(2d+1)$-gons of order $(1,1)$ with $d \geq 2$ have non-bipartite 2-regular point graphs, which cannot have the eigenvalue -2 because of Theorem 2.1.1.

In the remainder of this chapter, M will denote the element $\sum_{i=0}^{d}(-1/s)^i A_i$ of the Bose-Mesner algebra for regular near $2d$-gons of order (s,t). Note that $\mathrm{rank}(M) = 1$ if and only if $s = 1$.

A formula for the full matrix of eigenvalues corresponding to the dual polar space of order $(s,t) = \left(q^e, \begin{bmatrix} d \\ 1 \end{bmatrix}_q - 1\right)$ on a classical finite polar space of rank d with parameters (q, q^e) was given in Theorem 4.3.6. Note that the eigenspace for $-t-1 = -\begin{bmatrix} d \\ 1 \end{bmatrix}_q$ is precisely the subspace $V^d_{d,0}$ from Theorem 4.2.4, which is the last eigenspace with respect to the cometric ordering from Remark 4.3.12.

We now also give the full matrix of eigenvalues for generalized n-gons of order (s,t) with $n \in \{4, 6, 8\}$. Note that Theorem 6.2.6 implies that if $s, t > 1$, then

all eigenvalues are integers.

Theorem 6.3.3. [23, Section 6.5] *Let P denote the matrix of eigenvalues of the scheme defined by the point graph of a generalized n-gon of order (s,t), with the i-th column corresponding to the distance-i relation in the point graph, and the last row with the column span of M.*

(i) *If $n = 4$:*

$$P = \begin{pmatrix} 1 & s(t+1) & s^2t \\ 1 & s-1 & -s \\ 1 & -t-1 & t \end{pmatrix}$$

and $\operatorname{rank}(M) = s^2(st+1)/(s+t)$.

(ii) *If $n = 6$:*

$$P = \begin{pmatrix} 1 & s(t+1) & s^2t(t+1) & s^3t^2 \\ 1 & s-1+\sqrt{st} & -s+(s-1)\sqrt{st} & -s\sqrt{st} \\ 1 & s-1-\sqrt{st} & -s-(s-1)\sqrt{st} & s\sqrt{st} \\ 1 & -t-1 & t(t+1) & -t^2 \end{pmatrix}$$

and $\operatorname{rank}(M) = s^3(s^2t^2+st+1)/(s^2+st+t^2)$.

(iii) *If $n = 8$:*

$$P = \begin{pmatrix} 1 & s(t+1) & s^2t(t+1) & s^3t^2(t+1) & s^4t^3 \\ 1 & s-1+\sqrt{2st} & st-s+(s-1)\sqrt{2st} & s^2t-st-s\sqrt{2st} & -s^2t \\ 1 & s-1 & -st-s & -s^2t+st & s^2t \\ 1 & s-1-\sqrt{2st} & st-s-(s-1)\sqrt{2st} & s^2t-st+s\sqrt{2st} & -s^2t \\ 1 & -t-1 & t(t+1) & -t^2(t+1) & t^3 \end{pmatrix}$$

and $\operatorname{rank}(M) = s^4(st+1)(s^2t^2+1)/((s+t)(s^2+t^2))$.

Remark 6.3.4. In each case, when dividing the entry in the last row and i-th column by the entry P_{0i} in the same column, one obtains $(-1/s)^i$, which is to be expected from Lemma 2.2.2 and Theorem 6.3.1.

6.4 Point sets in regular near $2d$-gons

6.4.1 Point sets in regular near $2d$-gons in general

We start by defining a special type of subsets of points in regular near $2d$-gons, which will play a fundamental role in this chapter.

Definition 6.4.1. *In a regular near $2d$-gon of order (s,t), a point set S is tight if χ_S is orthogonal to the eigenspace for $-(t+1)$ of the point graph.*

The following theorem motivates our definition.

Theorem 6.4.2. *Consider a regular near $2d$-gon, $d \geq 2$, of order (s,t) with incidence matrix C (with columns and rows indexed by the points and lines, respectively). Let S be a non-empty point set with inner distribution $\boldsymbol{a}$ with respect to the scheme defined by the point graph.*

(i)

$$\sum_{i=0}^{d} \left(-\frac{1}{s}\right)^i \boldsymbol{a}_i \geq 0,$$

and the following are equivalent:

(a) *equality holds in the above,*

(b) *S is tight,*

(c) *χ_S is in the column span of C^T.*

(ii) *In that case, the outer distribution B of S satisfies:*

$$\sum_{i=0}^{d} \left(-\frac{1}{s}\right)^i B_{p,i} = 0$$

for every point p, and $|S|$ is divisible by $s+1$.

Proof. We know from Theorem 6.3.1 that $M = \sum_{i=0}^{d}(-1/s)^i A_i$ is a minimal idempotent up to a positive scalar. Applying Theorem 2.2.7, we see that the desired inequality must hold, with equality if and only if $M\chi_S = 0$, and that the outer distribution must satisfy the stated equation in that case. Up to a positive scalar, $M\chi_S$ is orthogonal projection of χ_S onto the eigenspace for $-(t+1)$, and thus zero if and only if S is tight, yielding equivalence between (i)(a) and (i)(b). This eigenspace is $\ker(C)$ because of Theorem 6.3.1, and since $\ker(C)$ and $\mathrm{Im}(C^T)$ are orthogonal complements, we also obtain equivalence between (i)(b) and (i)(c).

Finally, for every point p the following holds:

$$\sum_{i=0}^{d}(-s)^{d-i}B_{p,i} = 0,$$

and hence:

$$|S| = \sum_{i=0}^{d} B_{p,i} \equiv 0 \bmod s+1.$$

□

If S is a tight set of size $i(s+1)$, we will also say S is an *i-tight set*. For generalized quadrangles, the concept of tight sets was introduced by Payne [119]. We first show how Theorem 6.4.2 yields a result of his.

Theorem 6.4.3. *Let S be a non-empty point set in a generalized quadrangle of order (s,t), and let a be the average number of points in S collinear with a fixed point of S. Then:*

$$a \leq s - 1 + \frac{|S|}{s+1},$$

with equality if and only if every point in S is collinear with exactly $s-1+|S|/(s+1)$ elements of S, and if and only if S is tight. In that case, every point not in S is collinear with exactly $|S|/(s+1)$ points in S.

Proof. The inner distribution $\mathbf{a}$ of S is given by $(1, a, |S|-a-1)$. Theorem 6.4.2 now yields that

$$1 - \frac{a}{s} + \frac{|S|-a-1}{s^2} \geq 0$$

with equality if and only if S is tight. The above inequality can be rewritten as $a \leq s-1+|S|/(s+1)$. If equality holds, then Theorem 6.4.2 also yields that the outer distribution B of S satisfies:

$$B_{p,0} - \frac{1}{s}B_{p,1} + \frac{1}{s^2}B_{p,2} = 0$$

for every point p. If b denotes the number of points in S collinear with p, then the corresponding row $(B_{p,0}, B_{p,1}, B_{p,2})$ of the outer distribution is given by $(1, b, |S|-b-1)$ if $p \in S$ and by $(0, b, |S|-b)$ if $p \notin S$. Substitution in the above equation and solving for b yields the solution $s-1+|S|/(s+1)$ in the first case, and $|S|/(s+1)$ in the second case.

Finally, if every point in S is collinear with exactly $s-1+|S|/(s+1)$ elements of S, then the average number of points in S collinear with a fixed point of S is also $s-1+|S|/(s+1)$ and hence the bound is certainly attained. □

We will now generalize some basic properties of tight sets in generalized quadrangles for regular near $2d$-gons.

Lemma 6.4.4. *Consider a regular near $2d$-gon, $d \geq 2$, of order (s,t).*

(i) *If an i_1-tight S_1 and an i_2-tight set S_2 are disjoint, then their union is an (i_1+i_2)-tight set.*

(ii) *The complement of a tight set is also tight.*

(iii) *The points on a set of m pairwise disjoint lines form an m-tight set.*

(iv) *A 1-tight point set consists of the points on one fixed line.*

Proof.

(i) The characteristic vectors χ_{S_1} and χ_{S_2} are orthogonal to the eigenspace for $-(t+1)$, and hence so is $\chi_{S_1 \cup S_2} = \chi_{S_1} + \chi_{S_2}$.

(ii) The complement of S has characteristic vector $\chi_\Omega - \chi_S$, where Ω denotes the full set of points. Since χ_Ω is also orthogonal to the eigenspace of $-(t+1)$, the result now follows.

(iii) The inner distribution of the set of points S on one fixed line is given by $(1, s, 0, \ldots, 0)$, and hence S is 1-tight because of Theorem 6.4.2(i). It now follows from (i) that the set of points on m pairwise disjoint lines is m-tight.

(iv) Consider a 1-tight set S with outer distribution B. Let p be any point in S. Let $w \in \{0, \ldots, d\}$ be the maximum index i with $B_{p,i} \neq 0$. Note that $w \neq 0$. Theorem 6.4.2(ii) now yields:

$$\sum_{i=0}^{w} (-1)^i s^{w-i} B_{p,i} = 0.$$

This implies that $B_{p,w}$ must be divisible by s, but we also know that $0 < B_{p,w} \leq |S| - B_{p,0} = s$, and thus $B_{p,w} = s$. This implies that all entries $B_{p,i}$ with $i \notin \{0, w\}$ are zero, and thus that $w = 1$. As p was an arbitrary point in S, we now see that S is a clique in the point graph of size $s+1$, and hence the set of points on a fixed line. □

We can also generalize a very simple construction due to Payne [119, III.3].

Lemma 6.4.5. *In a regular near $2d$-gon $\mathcal{P} = (P, L, \mathrm{I})$ of order (s, t) with $d \geq 2$, let ℓ be a line and let $\ell^* = \{p_0, \ldots, p_s\}$ be the set of points on ℓ. Let m be an integer with $1 \leq m \leq t$. For every $k \in \{0, \ldots, s\}$, let $\ell_{k,1}, \ldots, \ell_{k,m}$ be m distinct lines through p_k, different from ℓ. The set S of points that are on some $\ell_{k,j}$ but not on ℓ is an ms-tight set.*

Proof. Every point in S is on exactly one line $\ell_{k,j}$, and every point on ℓ is on exactly m such lines. If T is the set of all lines $\ell_{k,j}$ and C is the incidence matrix, with columns and rows indexed by the points and lines of $\mathcal{P}$, respectively, then we can write:

$$\begin{aligned} C^T \chi_T &= \chi_S + m\chi_{\ell^*} \\ C^T \chi_{\{\ell\}} &= \chi_{\ell^*}. \end{aligned}$$

Hence $\chi_S \in \mathrm{Im}(C^T)$, and as $|S| = (s+1)ms$, it follows from Theorem 6.4.2 that S is an ms-tight set. □

Lemma 6.4.6. *Consider a regular near $2d$-gon $\mathcal{P}$ of order (s, t), $d \geq 2$, with an isometrically embedded sub near $2d$-gon $\mathcal{P}'$ of order (s, t'). A set of points S in $\mathcal{P}'$ is tight if and only if S is tight in $\mathcal{P}$.*

Proof. As the embedding is isometric, the inner distribution of S with respect to both point graphs is the same, say $\mathbf{a}$. The criterion from Theorem 6.4.2 for S to be tight is the same for both near polygons: $\sum_{i=0}^{d} \mathbf{a}_i/(-s)^i = 0$. This completes the proof. □

We now introduce another type of point sets, which will turn out to be design-orthogonal to the tight sets.

Definition 6.4.7. *An m-ovoid in a regular near $2d$-gon is a set of points, such that each line is incident with exactly m of its elements.*

Thas [150] introduced m-ovoids for generalized quadrangles.

Theorem 6.4.8. *Consider a regular near $2d$-gon $\mathcal{P} = (P, L, \mathrm{I})$, $d \geq 2$, of order (s, t). A subset of points S is an m-ovoid for some m if and only if χ_S is a linear combination of χ_P and an eigenvector v of the point graph for $-(t+1)$. In that case:*

$$|S| = \frac{m}{s+1}|P|, \ \chi_S = \frac{m}{s+1}\chi_P + v,$$

and for any point p, the number of points in S at distance i from p is

$$k_i\left(\frac{m}{s+1}+\left(1-\frac{m}{s+1}\right)\left(-\frac{1}{s}\right)^i\right) \text{ if } p\in S,$$

$$k_i\frac{m}{s+1}\left(1-\left(-\frac{1}{s}\right)^i\right) \text{ if } p\notin S,$$

where k_i denotes the valency of the distance-i relation of the point graph.

Proof. If C denotes the incidence matrix, with columns and rows indexed by points and lines, respectively, then $C\chi_P=(s+1)\chi_L$, as every line contains exactly $s+1$ points. Similarly, S is an m-ovoid if and only if $C\chi_S=m\chi_L$. Hence S is an m-ovoid if and only if

$$C\left(\chi_S-\frac{m}{s+1}\chi_P\right)=0.$$

We know from Theorem 6.3.1 that the kernel of C is precisely the eigenspace of $-(t+1)$ with respect to the point graph, and thus S is an m-ovoid if and only if $\chi_S=m/(s+1)\chi_P+v$ with v in the eigenspace of $-(t+1)$. As the projection of S onto $\langle\chi_P\rangle$ is given by $|S|/|P|\chi_P$, this then implies that $|S|=m/(s+1)|P|$. In that case, Lemma 2.1.3 yields that S is in fact intriguing with respect to the distance-i relation for every $i\in\{0,\dots,d\}$, and with parameters (h_1,h_2):

$$h_1=\frac{|S|}{|P|}(k_i-\lambda_i)+\lambda_i, h_2=\frac{|S|}{|P|}(k_i-\lambda_i),$$

with $\lambda_i/k_i=(-1/s)^i$ because of Theorem 6.3.1. □

Note that Lemma 2.1.3 and the above imply that a point set is an m-ovoid for some m in a regular near $2d$-gon of order (s,t), if and only if it is intriguing with parameters (h_1,h_2) satisfying $h_1-h_2=-(t+1)$.

Corollary 6.4.9. *In a regular near $2d$-gon $\mathcal{P}=(P,L,\mathrm{I})$ of order (s,t), $d\geq 2$, an i-tight set S_1 and an m-ovoid S_2 intersect in exactly mi points.*

Proof. It follows from Theorems 6.4.2 and 6.4.8 that S_1 and S_2 are design-orthogonal. Hence Lemma 2.2.10 yields that $|S_1\cap S_2|=|S_1||S_2|/|P|$. As $|S_1|=i(s+1)$ and $|S_2|=m/(s+1)|P|$, this yields the desired result. □

If $d = 2$, then the tight sets and m-ovoids are precisely the sets with dual degree less than d. See [10] for many examples.

Theorem 6.4.10. *If S is a non-empty set of points in a regular near $2d$-gon of order (s,t), $d \geq 2$, pairwise at distance i with i odd, then $|S| \leq 1 + s^i$, and equality holds if and only if S is a tight set.*

Proof. The inner distribution $\mathbf{a}$ of S satisfies $\mathbf{a}_0 = 1$ and $\mathbf{a}_i = |S| - 1$, while all other entries of $\mathbf{a}$ are zero. Theorem 6.4.2 now yields that

$$1 - \frac{|S| - 1}{s^i} \geq 0,$$

with equality if and only if S is a tight set. □

Corollary 6.4.11. *If S is a set of $s^d + 1$ points, pairwise at distance d, in a regular near $2d$-gon of order (s,t) for odd $d \geq 3$, then S is a 1-regular code in the point graph, and:*

(i) *if a point p is at distance 1 from an element of S, then p is at distance $d-1$ from exactly s^{d-1} points in S, and at distance d from the $s^d - s^{d-1}$ remaining elements of S,*

(ii) *if a point p is at distance $d-1$ from S, then p is at distance $d-1$ from exactly $(s^d+1)/(s+1)$ elements of S, and at distance d from the $s(s^d+1)/(s+1)$ remaining elements of S.*

Proof. We know from Theorem 6.4.10 that in case of equality, the outer distribution B of S satisfies:

$$\sum_{i=0}^{d} \left(-\frac{1}{s}\right)^i B_{p,i} = 0$$

for every point p. The corresponding row of the outer distribution is clear if $p \in S$. If p is at distance 1 from S, then $B_{p,0} = 0$ and $B_{p,1} = 1$ and $B_{p,i} = 0$ if $2 \leq i \leq d-2$. If p is at distance $d-1$ from S, then $B_{p,i} = 0$ if $0 \leq i \leq d-2$. The sum of the entries in each row of B must be $|S|$, and thus we can use the above equation to compute the row B_p in each case. In particular, this establishes 1-regularity of S. □

We can say a bit more if $d = 3$. Recall from Subsection 2.3.2 the definition of the covering radius $t(C)$, complete regularity and the reduced outer distribution of a code C.

Corollary 6.4.12. *If S is a non-empty set of points in a regular near hexagon of order (s,t), pairwise at distance three, then $|S| \leq 1+s^3$, with equality if and only if S is tight. In that case, S is completely regular with reduced outer distribution B' with respect to the point graph:*

$$B' = \begin{pmatrix} 1 & 0 & 0 & s^3 \\ 0 & 1 & s^2 & s^3-s^2 \\ 0 & 0 & s^2-s+1 & s(s^2-s+1) \end{pmatrix} \qquad \text{if } t(S)=2,$$

or

$$B' = \begin{pmatrix} 1 & 0 & 0 & s^3 \\ 0 & 1 & s^2 & s^3-s^2 \end{pmatrix} \qquad \text{if } t(S)=1.$$

Proof. The bound and the equivalence between equality and tightness immediately follow from Theorem 6.4.10. The reduced outer distribution B' follows immediately from Corollary 6.4.11. □

Before we move on to discuss point sets in specific types of regular near $2d$-gons, we slightly generalize a property of tight point sets. The outer distribution of a point set S tells us how S intersects the cells of the partition with respect to distance from any fixed point. More generally, we will now consider partitions with similar properties. We first need another definition, very similar to that of *valuations* as introduced by De Bruyn and Vandecasteele [58], but with weaker assumptions.

Definition 6.4.13. *Let $\mathcal{P} = (P, L, \mathrm{I})$ be a near $2d$-gon. A function f from P to $\mathbb{N}$ is a weak valuation*[2] *if it satisfies the following conditions.*

(i) *There exists at least one point p with value $f(p) = 0$.*

(ii) *Every line ℓ contains a unique point x_ℓ with smallest value for f, and $f(x) = f(x_\ell) + 1$ for every point $x \neq x_\ell$ on ℓ.*

Lemma 6.4.14. *If f is a weak valuation of a near $2d$-gon, then for every two points x and y, $|f(x) - f(y)| \leq d(x,y)$ holds with respect to the point graph, and in particular $0 \leq f(x) \leq d$ for every point x.*

Proof. The value for f of two collinear points differs by at most 1. If $d(x,y) = i$ and $(x = x_0, \ldots, x_i = y)$ is a path in the point graph, then

$$|f(x) - f(y)| \leq \sum_{j=1}^{i} |f(x_{j-1}) - f(x_j)| \leq i.$$

[2]This should not be confused with the concept of valuations in generalized polygons from [160].

The last claim follows from the assumption that there is a point p with $f(p) = 0$. □

We now give two important examples of weak valuations, which were discussed in [58].

Theorem 6.4.15. *Consider a regular near $2d$-gon $\mathcal{P} = (P, L, \mathrm{I})$.*

(i) *For any point p, the function $f_p : P \to \mathbb{N} : x \to d(x, p)$ (with respect to the point graph) is a weak valuation of $\mathcal{P}$.*

(ii) *For any 1-ovoid S, the function $f_S : P \to \mathbb{N}$ with $f_S(x) := 0$ if $x \in S$ and $f_S(x) := 1$ if $x \notin S$ is a weak valuation of $\mathcal{P}$.*

Proof. In both cases, there is clearly a point with value 0. In the first case, condition (ii) from Definition 6.4.13 is satisfied because of the definition of near $2d$-gons. In the second case, it is satisfied since by the definition of 1-ovoids, every line will contain exactly one point of S. □

Lemma 6.4.16. *Let f be a weak valuation of a regular near $2d$-gon of order (s, t), $d \geq 2$, with $F_i = \{p | f(p) = i\}$, $\forall i \in \{0, \ldots, d\}$. Then*

$$v = \sum_{i=0}^{d} \left(-\frac{1}{s}\right)^i \chi_{F_i}$$

is an eigenvector for $-(t + 1)$ with respect to the point graph.

Proof. Take $F_{-1} = F_{d+1} = \emptyset$. Consider any point $p \in F_i$. Now p can only be collinear with points in F_{i-1}, F_i and F_{i+1}. Let $c_{f,i}$ denote the number of lines through p with a (necessarily unique) point in F_{i-1}. All s other points on such a line are in F_i. All s points, distinct from p and on the remaining $(t+1) - c_{f,i}$ lines through p must be in F_{i+1}. Hence:

$$(A_1\chi_{F_{i-1}})_p = c_{f,i}, (A_1\chi_{F_i})_p = (s - 1)c_{f,i}, (A_1\chi_{F_{i+1}})_p = s((t + 1) - c_{f,i}),$$

$$(A_1\chi_{F_j})_p = 0 \text{ if } |i - j| > 1.$$

We can now write:

$$\begin{aligned}
(A_1 v)_p &= \Big(A_1\Big(\sum_{j=0}^{d}\Big(-\frac{1}{s}\Big)^j \chi_{F_j}\Big)\Big)_p \\
&= \Big(-\frac{1}{s}\Big)^{i-1} c_{f,i} + \Big(-\frac{1}{s}\Big)^{i}(s-1)c_{f,i} + \Big(-\frac{1}{s}\Big)^{i+1} s((t+1)-c_{f,i}) \\
&= \Big(-\frac{1}{s}\Big)^{i}\Big((-s)c_{f,i} + (s-1)c_{f,i} + \Big(-\frac{1}{s}\Big)(s(t+1-c_{f,i}))\Big) \\
&= \Big(-\frac{1}{s}\Big)^{i}(-(t+1)) = -(t+1)v_p.
\end{aligned}$$

As p was an arbitrarily chosen point, we can conclude that $A_1 v = -(t+1)v$.□

Theorem 6.4.17. *Let f be a weak valuation of a regular near $2d$-gon, $d \geq 2$, of order (s,t), with $F_i = \{p | f(p) = i\}$, $\forall i \in \{0,\ldots,d\}$. If S is a tight set of points, then*

$$\sum_{i=0}^{d}\Big(-\frac{1}{s}\Big)^i |S \cap F_i| = 0.$$

In particular:

$$\sum_{i=0}^{d}\Big(-\frac{1}{s}\Big)^i |F_i| = 0.$$

Proof. We know from Lemma 6.4.16 that $v = \sum_{i=0}^{d}(-1/s)^i \chi_{F_i}$ is an eigenvector for $-(t+1)$ with respect to the point graph. On the other hand, χ_S is orthogonal to the eigenspace for $-(t+1)$ by Theorem 6.4.2. Lemma 2.2.10 now yields:

$$0 = (\chi_S)^T v = \sum_{i=0}^{d}\Big(-\frac{1}{s}\Big)^i (\chi_S)^T \chi_{F_i} = \sum_{i=0}^{d}\Big(-\frac{1}{s}\Big)^i |S \cap F_i|.$$

The last part of the theorem follows from the observation that the full set of points is a tight set. □

6.4.2 Point sets in generalized $2d$-gons

Martin [106] considered the eigenspaces of the bipartite incidence graph of symmetric designs, and in particular of the finite projective planes. The results

for subsets in finite projective planes can be seen in the context of generalized hexagons of order $(1, q)$. De Wispelaere and Van Maldeghem [63] applied algebraic techniques, including design-orthogonality, to obtain a wide variety of results on subsets of points in generalized $2d$-gons. Similar work for generalized quadrangles was done by Bamberg, Law and Penttila [10]. This will serve as our inspiration, and we will give several similar results in this subsection. We will start with generalized quadrangles, then move on to the hexagons and conclude with the octagons.

A *partial distance-j-ovoid* in a generalized $2d$-gon, $2 \leq j \leq d$, of order (s, t) is a set S of points such that the distance in the point graph between any two distinct elements is at least j. We say S is a *maximal partial distance-j-ovoid* if it is not a proper subset of another partial distance-j-ovoid. We call S a *distance-j-ovoid* if for every point or line there is at least one element of S at distance j or less in the incidence graph from it. In particular, a partial distance-2-ovoid is a set of pairwise non-collinear points, and it is a distance-2-ovoid if and only if every line contains a (necessarily unique) element of this set. Hence the distance-2-ovoids are precisely the 1-ovoids. Two points in a generalized $2d$-gon are called *opposite* if they are at distance d. The partial distance-d-ovoids in a generalized $2d$-gon are thus precisely the sets of pairwise opposite points. These sets are distance-d-ovoids if and only if their size is $st+1$ for $d = 2$, $(s + 1)(1 + st + s^2t^2)/(1 + s + st)$ for $d = 3$ (but see below), and $s^2t^2 + 1$ for $d = 4$ (see for instance [63]).

We start by recovering results by Payne and Thas (see for instance [122, 2.2.1] and [120]) as an example.

Theorem 6.4.18. *If S is the set of points of a subquadrangle $\mathcal{P}'$ of order (s', t') in a generalized quadrangle $\mathcal{P}$ of order (s, t), then $s = s'$ or $s \geq s't'$ and one of the two equalities holds if and only if S is tight in $\mathcal{P}$. Moreover, if $s = s'$, then a subset in the subquadrangle is tight in $\mathcal{P}'$ if and only if it is tight in $\mathcal{P}$.*

Proof. The average number of points in S, collinear with a fixed point in S, is given by $s'(t' + 1)$ (in both point graphs). Theorem 6.2.5 yields that $|S| = (s' + 1)(s't' + 1)$. The inequality and the equivalence between equality as stated and tightness of S now follow from Theorem 6.4.3. The last part follows from Lemma 6.4.6 as $\mathcal{P}'$ must be isometrically embedded. □

In a generalized hexagon of order (s, t), a set of mutually opposite points is a distance-3-ovoid if and only if every point is either in it or collinear with

a (necessarily unique) point in it. Hence they are the perfect 1-codes in the point graph. Offer [112] proved that these perfect 1-codes can only exist if $s = t$, when they are precisely the partial distance-3-ovoids of size $s^3 + 1$. It is precisely in this case that -1 is an eigenvalue of the point graph (see Theorem 6.3.3 (ii)) and hence this condition also follows from Lloyd's theorem (Theorem 2.3.9(v)).

The split Cayley hexagon $\mathrm{H}(q)$ has distance-3-ovoids if q is a power of 3 (see for instance [152]) but never if q is even (see [148]) or a prime $p > 3$ (see [114] and [3]). The dual of $\mathrm{H}(q)$ always has a distance-3-ovoid (see [147] as well as Remark 4.4.6).

Coolsaet and Van Maldeghem [47] proved that in a generalized hexagon of order (s, s^3), $s > 1$, a partial distance-3-ovoid has size at most $s^5 - s^3 + s - 1$. We will now prove a general result, very similar to [63, Theorems 4.3 and 4.13].

Theorem 6.4.19. *Suppose S is a maximal partial distance-3-ovoid in a generalized hexagon of order (s, t). Then $|S| \leq \min((\sqrt{st})^3 + 1, s^3 + 1)$, and*

- *if $s < t$, then S is completely regular if and only if S is tight, and if and only if $|S| = 1 + s^3$, with*

$$B' = \begin{pmatrix} 1 & 0 & 0 & s^3 \\ 0 & 1 & s^2 & s^3 - s^2 \\ 0 & 0 & s^2 - s + 1 & s(s^2 - s + 1) \end{pmatrix},$$

$$L_S = \begin{pmatrix} 0 & s(t+1) & 0 \\ 1 & s^2 + s - 1 & (t-s)s \\ 0 & s^2 - s + 1 & st - (s-1)^2 \end{pmatrix},$$

  and in this case every m-ovoid intersects S in exactly $m(s^2 - s + 1)$ points, and in particular, every distance-2-ovoid intersects S in exactly $s^2 - s + 1$ points.

- *if $s = t$, then S is completely regular if and only if S is tight, and if and only if $|S| = 1 + s^3$ (and thus a distance-3-ovoid), and then every m-ovoid intersects S in exactly $m(s^2 - s + 1)$ points.*

- *if $s > t$, then S is completely regular if and only if $|S| = 1 + (\sqrt{st})^3$, with*

$$B' = \begin{pmatrix} 1 & 0 & 0 & (\sqrt{st})^3 \\ 0 & 1 & t\sqrt{st} & \sqrt{st}(s-1)t \\ 0 & 0 & (t+1)(\sqrt{st}+1) & \sqrt{st}(st - t - 1) - t \end{pmatrix},$$

$$L_S = \begin{pmatrix} 0 & s(t+1) & 0 \\ 1 & s-1+t\sqrt{st} & t(s-\sqrt{st}) \\ 0 & (t+1)(\sqrt{st}+1) & (t+1)(s-1-\sqrt{st}) \end{pmatrix},$$

(B' and L_S denote the reduced outer distribution of S and the quotient matrix of the corresponding equitable partition of the point graph, respectively.)

Proof. The matrix of eigenvalues for the scheme on the points was given in Theorem 6.3.3(ii). Let V_0, V_1, V_2 and V_3 denote the eigenspaces corresponding to the eigenvalues $s(t+1), s-1+\sqrt{st}, s-1-\sqrt{st}$ and $-(t+1)$, respectively. Maximality of S as a partial distance-3-ovoid means that the covering radius $t(S)$ is 1 or 2. If C is completely regular, then Theorem 2.3.9 yields that its dual degree $r(S)$ is precisely $t(S)$, and hence $r(S) \in \{1,2\}$. Hence χ_S must be orthogonal to at least one of the eigenspaces. Corollary 2.2.9 now yields that $|S| \leq 1 - k_3/\lambda_3$ for any eigenspace, with k_3 the valency of the distance-3-relation between points, and with λ_3 its eigenvalue for the corresponding eigenspace . Hence $|S| \leq 1 + (\sqrt{st})^3$ with equality if and only if $\chi_S \in V_1^{\perp}$, and $|S| \leq 1 + s^3$ with equality if and only if $\chi_S \in V_3^{\perp}$.

If $s < t$, then $1 + s^3 < 1 + (\sqrt{st})^3$. If $|S| = 1 + s^3$, then complete regularity of S and its reduced outer distribution B' follow from Corollary 6.4.12. In that case, S is $(s^2 - s + 1)$-tight, and thus the result with respect to m-ovoids follows from Corollary 6.4.9.

If $s = t$, then the two bounds are equal. Hence S is completely regular if and only if $|S| = 1 + s^3$, and in that case the dual degree and the covering radius are both 1, i.e. S is a perfect 1-code and thus a distance-3-ovoid. The result with respect to m-ovoids again follows from Corollary 6.4.9.

If $s > t$ then $1 + s^3 > 1 + (\sqrt{st})^3$, and hence S is completely regular if and only if $S = 1 + (\sqrt{st})^3$. In that case, for every point $p \notin S$ the corresponding row B_p of the outer distribution B of S is of the form $(0, 1, x, (\sqrt{st})^3 - x)$ or $(0, 0, x, (\sqrt{st})^3 + 1 - x)$, depending on whether p is at distance 1 or 2 from S. Theorem 2.2.9 and Theorem 6.3.3(ii) then also yield:

$$\frac{1}{1}B_{p,0} + \frac{s-1+\sqrt{st}}{s(t+1)}B_{p,1} + \frac{-s+(s-1)\sqrt{st}}{s^2t(t+1)}B_{p,2} + \frac{-s\sqrt{st}}{s^3t^2}B_{p,3} = 0.$$

Solving for x now allows explicit computation of the reduced outer distribution B' of S.

Finally, if $s < t$ or $s > t$, we can use the formula $L_S = B' L_{t(S)} (B'_{t(S)})^{-1}$ from Lemma 2.3.8 to compute the quotient matrix. Because of Theorem 6.2.5, the required matrix L is given by:

$$L = \begin{pmatrix} 0 & s(t+1) & 0 & 0 \\ 1 & (s-1) & st & 0 \\ 0 & 1 & (s-1) & st \\ 0 & 0 & t+1 & (s-1)(t+1) \end{pmatrix}.$$

□

De Wispelaere and Van Maldeghem [63, Example 4.15] constructed a partial distance-3-ovoid of size q^3+1 in $\mathrm{T}(q, q^3)$. It follows from Theorem 6.4.19 that this is a completely regular code, and they in fact proved that it is completely transitive.

The split Cayley hexagon $\mathrm{H}(2)$ has a unique distance-2-ovoid while its dual has none, and the self-dual $\mathrm{H}(3)$ has a unique distance-2-ovoid (see [61]). Two distance-2-ovoids have been found in $\mathrm{H}(4)$ (see [62, 64]), and they are the only ones (see [123]). The dual twisted triality hexagons $\mathrm{T}(2,8)$ and $\mathrm{T}(3,27)$ have no distance-2-ovoids, and none are known in any $\mathrm{T}(q, q^3)$ (see [61]). Although we don't have any examples of the latter, we can still give a little bit of information on the intersection of two distance-2-ovoids in generalized hexagons of order (s, s^3). Our proof is inspired by the techniques applied by Martin [106].

Theorem 6.4.20. *If S and S' are distance-2-ovoids in a generalized hexagon of order (s, s^3), $s > 1$, then $|S \cap S'|$ is 0 or $h(s^2+s+1)$ for some integer $h \geq s^3 - s + 1$.*

Proof. The scheme defined by the point graph has a cometric ordering V_0, V_1, V_2 and V_3 of eigenspaces, corresponding to the eigenvalues $s(s^3+1), -s^3-1, s-1+s^2$ and $s-1-s^2$, respectively (see for instance [23, Section 6.5] or Remark 6.5.3). Theorem 6.4.8 also yields that χ_S and $\chi_{S'}$ are both in $V_0 \perp V_1$, and thus S and S' are both 1-antidesigns with respect to this cometric ordering. Hence $S \cap S'$ is a 2-antidesign because of Theorem 2.2.15, i.e. $\chi_{S \cap S'}$ is orthogonal to V_3.

Now suppose $S \cap S'$ contains a point p. Consider the row of the outer distribution B of $S \cap S'$, corresponding to p. As both S and S' are cocliques in the point graph, we know that $(B_{p,0}, B_{p,1}, B_{p,2}, B_{p,3}) = (1, 0, x_2, x_3)$, with $1 + x_2 + x_3 = |S \cap S'|$. Finally, as $\chi_{S \cap S'} \in V_3^{\perp}$, Theorem 2.2.7 yields that B

satisfies:

$$\frac{1}{1}B_{p,0} + \frac{s-1-s^2}{s(s^3+1)}B_{p,1} + \frac{-s-s^3+s^2}{s^5(s^3+1)}B_{p,2} + \frac{s^3}{s^9}B_{p,3} = 0,$$

where we used Theorem 6.3.3(ii) to obtain the desired eigenvalues. Solving for x_2 and x_3 yields:

$$\begin{aligned} x_2 &= \frac{|S \cap S'|(s+1)}{s^2+s+1} + (s^2-1)(s^3+1), \\ x_3 &= s^2\left(\frac{|S \cap S'|}{s^2+s+1} - (s^3-s+1)\right). \end{aligned}$$

As x_2 and x_3 must be non-negative integers, and as $s+1$ and s^2+s+1 are coprime, we now find that $|S \cap S'| = h(s^2+s+1)$ for some integer $h \geq s^3-s+1$.□

Yanushka [173] proved that if a generalized octagon of order (s,t) with $t > s$ has a proper suboctagon of order (s',t'), then $st \geq (s')^2t'$. More restrictions on the parameters of suboctagons were obtained by Thas [146]. They imply in particular that if $s = s'$ or $t = t'$, then s' or t' must be 1 (see [161, Theorem 1.8.8]). We now generalize a result by De Bruyn [55].

Theorem 6.4.21. *Suppose $\mathcal{P}$ is a generalized octagon of order (s,t).*

(i) *If S is the point set of a suboctagon of order (s',t'), then $s = s'$ or $s \geq s't'$, and S is tight if and only if $s = s'$ or $s = s't'$. Every m-ovoid of $\mathcal{P}$ intersects S in $m(st'+1)((st')^2+1)$ points in the first case, and in $m(s'+1)(s^2+1)$ points in the second case.*

(ii) *Suppose $(T_\lambda)_{\lambda\in\Lambda}$, Λ a finite non-empty set of indices, is a set of m-ovoids of $\mathcal{P}$ with $0 < m < s+1$, such that the number of $\lambda \in \Lambda$ with $p_1, p_2 \in T_\lambda$ only depends on the distance $d(p_1,p_2)$ in the point graph of $\mathcal{P}$. A proper suboctagon of order (s',t') of $\mathcal{P}$ will intersect every T_λ in the same number of points if and only if $s = s'$ or $s = s't'$.*

(iii) *Suppose $(S_\lambda)_{\lambda\in\Lambda}$, Λ a finite non-empty set of indices, is a set of proper suboctagons of the same order (s',t'), such that the number of $\lambda \in \Lambda$ with $p_1, p_2 \in S_\lambda$ only depends on the distance $d(p_1,p_2)$ in the point graph. An m-ovoid with $0 < m < s+1$ will intersect every S_λ in the same number of points if and only if $s = s'$ or $s = s't'$.*

Proof. The inner distribution $\mathbf{a}$ of the point set S of the (isometrically embedded) suboctagon of order (s', t') consists of simply the valencies of the distance-relations corresponding to its point graph, and hence Theorem 6.2.5 yields:

$$\mathbf{a} = (1, s'(t'+1), (s')^2 t'(t'+1), (s')^3 (t')^2 (t'+1), (s')^4 (t')^3).$$

Theorem 6.4.2 now yields:

$$1 - \frac{s'(t'+1)}{s} + \frac{(s')^2 t'(t'+1)}{s^2} - \frac{(s')^3 (t')^2 (t'+1)}{s^3} + \frac{(s')^4 (t')^3}{s^4} \geq 0$$

with equality if and only if S is tight. We can rewrite the inequality as:

$$\frac{(s-s')(s-s't')(s^2+(s't')^2)}{s^4} \geq 0.$$

If $s = s'$, then S is $(st'+1)(s^2(t')^2+1)$-tight, and if $s = s't'$, then S is $(s'+1)(s^2+1)$-tight. In both cases, the intersection property with respect to m-ovoids follows immediately from Corollary 6.4.9.

Finally, we know from Theorem 6.4.8 that the characteristic vector of an m-ovoid with $0 < m < s+1$ has a non-zero component in an eigenspace if and only if the corresponding eigenvalue is the valency $s(t+1)$ of the point graph or its minimal eigenvalue $-(t+1)$. Hence a suboctagon is design-orthogonal to it if and only if it is tight. Now (ii) and (iii) follow from Theorem 2.2.13.□

Distance-3-ovoids in generalized octagons of order (s, t) are precisely the perfect 1-codes in the point graph. Lloyd's Theorem (see Theorem 2.3.9(v)) implies that these can only exist if -1 is an eigenvalue of the point graph, which is the case if and only if $s = 2t$ (see Theorem 6.3.3 (iii)). This restriction was obtained by Offer and Van Maldeghem [113], who constructed a distance-3-ovoid in the unique generalized octagon of order $(2, 1)$. The only other known generalized octagon of order $(2t, t)$ is the dual of the Ree-Tits octagon of order $(2, 4)$, and here the question of existence of distance-3-ovoids is still an open problem. The only result on existence of distance-4-ovoids in generalized octagons of order (s, t) with $s, t > 1$ is the sharper bound of 27 for partial distance-4-ovoids in the Ree-Tits generalized octagon of order $(2, 4)$ by Coolsaet and Van Maldeghem [47].

6.4.3 Point sets in dual polar spaces

An extensive list of equitable partitions of point graphs of dual polar spaces can be found in [42]. We will now discuss several examples of point sets in dual polar spaces, many of which are extremal in some sense, and some of which are completely regular and hence also yield equitable partitions of the dual polar graph.

We first state a very handy q-analog of the Binomial Theorem (see for instance [35, Theorem 9.2.5] for a proof).

Theorem 6.4.22. [q-Binomial Theorem] *For any integer $q \geq 2$ and $z \in \mathbb{R}$:*

$$\sum_{i=0}^{d} q^{i(i-1)/2} z^i \begin{bmatrix} d \\ i \end{bmatrix}_q = \prod_{j=1}^{d} (1 + q^{j-1} z).$$

Dual polar spaces can often be isometrically embedded in another, such as the embedding of the dual polar space on $W(2d-1, q)$ in that on $H(2d-1, q^2)$ (see [54]).

Theorem 6.4.23. *Let $\mathcal{P}$ and $\mathcal{P}'$ be dual polar spaces on classical finite polar spaces of the same rank d, with the latter isometrically embedded in the first. If the smaller polar space has parameters (q, q^{e_1}) and the bigger polar space has parameters $(q^a, (q^a)^{e_2})$, then the points of $\mathcal{P}'$ form a tight set in $\mathcal{P}$ if and only if $ae_2 - e_1$ is an integer with $0 \leq ae_2 - e_1 \leq d-1$.*

Proof. The inner distribution $\mathbf{a}$ of the point set of $\mathcal{P}'$ as a subset in $\mathcal{P}$ consists of the valencies of the relations between maximals in the smaller polar space: $\mathbf{a}_i = q^{i(i-1)/2} q^{ie_1} \left[{d \atop i} \right]_q$, for any $i \in \{0, \ldots, d\}$, by Theorem 4.1.7. Theorem 6.4.2 now implies that the set of points of $\mathcal{P}'$ is tight in $\mathcal{P}$ if and only if:

$$\sum_{i=0}^{d} (-1/q^{ae_2})^i q^{i(i-1)/2} q^{ie_1} \begin{bmatrix} d \\ i \end{bmatrix}_q = 0.$$

Using Theorem 6.4.22 with $z = -q^{e_1 - ae_2}$, we can rewrite the left-hand side:

$$\prod_{j=1}^{d} (1 - q^{j-1} q^{e_1 - ae_2}).$$

Hence the set of points of $\mathcal{P}'$ is tight in $\mathcal{P}$ if and only if $\prod_{j=1}^{d} (1 - q^{j-1+e_1-ae_2}) = 0$, which is the case if and only if $ae_2 - e_1$ is an integer with $0 \leq ae_2 - e_1 \leq d-1$. $\square$

Remark 6.4.24. If the rank is 2, then the smaller and bigger dual polar spaces are generalized quadrangles of orders $(s', t') = (q^{e_1}, q)$ and $(s, t) = (q^{ae_2}, q^a)$, respectively. Theorem 6.4.23 now yields that the point set of the subquadrangle is tight if and only if $s/s' = 1$ or $s/s' = t'$, which also follows from Theorem 6.4.18.

Remark 6.4.25. It is in fact possible that the dual degree of the point set of the embedded dual polar space is much less than $d - 1$. A particularly nice example is the isometric embedding of the dual polar space on $W(2d-1, q)$ in that on $H(2d-1, q^2)$. Cardinali and De Bruyn [42, Class 6] showed that this point set is in fact a completely regular code with covering radius and dual degree $\lfloor d/2 \rfloor$.

Recall the definition of (partial) spreads in polar spaces from Subsection 4.4.4. We know from Theorem 4.4.16 that in $H(2d-1, q^2)$ with d odd, which has parameters (q^2, q), a partial spread S has size at most $q^d + 1$, and hence S certainly cannot be a spread. However, the next theorem shows that partial spreads attaining this bound still possess interesting properties. We have chosen to formulate the result in the most straightforward language, rather than in terms of near polygons and algebraic graph theory, although our proof relies on this context. We also write explicit (vectorial) dimensions in order to avoid confusion between points of the polar space and points of the dual polar space.

Theorem 6.4.26. *A non-empty partial spread S in $H(2d-1, q^2)$ has size at most $q^d + 1$ for odd $d \geq 3$, and equality holds if and only if S is tight. In that case, if for any totally isotropic d-space π we denote by $B_{\pi,i}$ the number of elements of S intersecting π in a $(d-i)$-space, then:*

$$\sum_{i=0}^{d} \left(-\frac{1}{q}\right)^i B_{\pi,i} = 0.$$

In particular, S is a 1-regular code in the dual polar graph in that case, and for any totally isotropic d-space π:

(i) *if π intersects an element of S in a $(d-1)$-space, then it intersects exactly q^{d-1} elements of S in a 1-space and all other elements of S trivially,*

(ii) *if π intersects no element of S in a subspace of dimension more than 1, then it intersects exactly $(q^d + 1)/(q + 1)$ elements of S in a 1-space.*

Proof. The dual polar space is a regular near $2d$-gon of order $(q, \begin{bmatrix} d \\ 1 \end{bmatrix}_{q^2} - 1)$ in this case, and its point graph is a dual polar graph with set of vertices Ω_d. We know from Theorem 4.1.7 that two totally isotropic d-spaces are at distance i in the dual polar graph if and only if they intersect in a $(d-i)$-space, and thus the matrix $B = (B)_{\pi \in \Omega_d, i=0,\ldots,d}$ is precisely the outer distribution of S. Since S is a clique of the distance-d relation, the bound for $|S|$ and equivalence between equality and tightness follow immediately from Theorem 6.4.10. The rest follows from Theorem 6.4.2(ii) and Corollary 6.4.11. □

We obtain stronger properties for small rank. The following result can be seen as an analog of Theorem 6.4.19 for dual polar graphs of diameter 3.

Theorem 6.4.27. *A maximal partial spread S in $H(5, q^2)$ has size at most $q^3 + 1$, and equality holds if and only if S is tight, and if and only if S is completely regular. In that case, for any totally isotropic 3-space π:*

(i) *if π intersects an element of S in a 2-space, then it intersects exactly q^2 elements of S in a 1-space and all other elements of S trivially,*

(ii) *if π intersects no element of S in a subspace of dimension more than 1, then it intersects exactly $q^2 - q + 1$ elements of S in a 1-space,*

and the corresponding equitable partition of the dual polar graph has quotient matrix:

$$L_S = \begin{pmatrix} 0 & q(q^4+q^2+1) & 0 \\ 1 & q^4+q^2+q-1 & q^2(q-1)(q^2+1) \\ 0 & (q^2-q+1)(q^2+1) & (q^2-q+1)(q^3+q-1) \end{pmatrix}.$$

Proof. Theorem 6.4.12 immediately yields the bound and the equivalence between equality and tightness, and complete regularity in case of equality. On the other hand, maximality of S as a partial spread means that the covering radius $t(S)$ is 1 or 2. If S is completely regular, then Theorem 2.3.9 yields that its dual degree $r(S)$ is precisely $t(S)$. Hence χ_S must be orthogonal to at least one of the eigenspaces of the point graph of the dual polar space. Theorem 2.2.9 now yields that for any eigenspace, χ_S is orthogonal to it if and only if $|S| = 1 - k_3/\lambda_3$, with k_3 the valency of the oppositeness relation between totally isotropic 3-spaces, and with λ_3 its eigenvalue for the corresponding eigenspace. We know from Theorem 4.3.15 and Corollary 4.3.17 that the minimal eigenvalue of oppositeness is $-q^6$ (only appearing for one eigenspace, namely $V^3_{3,0}$)

and the valency is q^9, and thus S can only be completely regular if $|S| = 1+q^3$, when the covering radius $t(S)$ and dual degree is exactly 2.

In that case, the reduced outer distribution B' also follows from Theorem 6.4.12, and we can use the formula $L_S = B'L_{t(S)}(B'_{t(S)})^{-1}$ from Lemma 2.3.8 to compute the quotient matrix L_S. The required matrix L follows from Theorem 4.1.7:

$$L = \begin{pmatrix} 0 & q(q^4+q^2+1) & 0 & 0 \\ 1 & q-1 & q^3(q^2+1) & 0 \\ 0 & q^2+1 & (q-1)(q^2+1) & q^5 \\ 0 & 0 & q^4+q^2+1 & (q^6-1)/(q+1) \end{pmatrix}. \qquad \square$$

Remark 6.4.28. De Beule and Metsch [52] obtained the bound of q^3+1 for a partial spread in $H(5,q^2)$, and their combinatorial argument involved the vertices at distance 2 from S (the so-called *free planes*). This way, they also obtained the number $(q^3+1)/(q+1)$ in case of equality.

We now move on to spreads of parabolic quadrics and symplectic spaces. In $Q(2d,q)$ and $W(2d-1,q)$, a partial spread is a spread when it has size q^d+1. The symplectic space $W(2d-1,q)$ has a spread for all $d \geq 2$. If q is even, then the parabolic quadric $Q(2d,q)$ is isomorphic to $W(2d-1,q)$ and hence has a spread for all $d \geq 2$ as well. If q is odd, then $Q(2d,q)$ has no spreads for all even $d \geq 2$. The existence of an ovoid of 1-spaces in $Q(6,q)$ implies the existence of such a spread, and therefore $Q(6,q)$ is known to have spreads for many odd values of q, including all powers of 3. However, no spreads of $Q(2d,q)$, q odd, d odd and $d \geq 5$, are known. We refer to [152] for proofs of these results.

In any classical finite polar space of rank d with parameters (q,q^e), a spread S is always a 1-regular code in the dual polar graph. Indeed, if a maximal π is adjacent to some element π_0 of S, it is at distance $d-1$ or d from any other element of the set, and hence it can intersect the other elements of S in at most a 1-space. As all $\begin{bmatrix} d \\ 1 \end{bmatrix}_q - \begin{bmatrix} d-1 \\ 1 \end{bmatrix}_q = q^{d-1}$ of the 1-spaces in π but not in π_0 must be contained in some element of S, the maximal π is at distance $d-1$ from exactly q^{d-1} elements of S, and at distance d from the remaining $|S| - q^{d-1} - 1 = q^{d-1+e} - q^{d-1}$ elements of S. However, the next result will improve this regularity in $Q(2d,q)$ and $W(2d-1,q)$ in case the rank d is odd. We know from Theorem 4.3.17 that for odd rank d, the minimal eigenvalue for

oppositeness between generators appears twice in the matrix of eigenvalues. For the Erdős-Ko-Rado problem, this weakened our control over the EKR sets (see Theorem 5.3.1), but here this phenomenon will actually work in our favour.

Theorem 6.4.29. *For any odd $d \geq 3$, a non-empty partial spread S of the parabolic quadric $Q(2d, q)$ or of the symplectic space $W(2d-1, q)$ is tight if and only if it is a spread. In that case, for any totally isotropic d-space π:*

$$\sum_{i=0}^{d} \left(-\frac{1}{q}\right)^{i} B_{\pi,i} = 0,$$

where $B_{\pi,i}$ denotes the number of elements of S intersecting π in a $(d-i)$-space:

In particular, S is 2-regular, and if π is a totally isotropic d-space, then:

(i) *if π intersects an element of S in a $(d-1)$-space, it intersects exactly q^{d-1} elements of S in a 1-space and intersects $q^d - q^{d-1}$ elements trivially,*

(ii) *if π intersects an element of S in a $(d-2)$-space (with $d \geq 5$), then π intersects exactly q^{d-3} elements of S in a 2-space, $q^{d-1} - q^{d-3}$ elements in a 1-space, and intersects the remaining $q^d - q^{d-1}$ elements of S trivially,*

(iii) *if π intersects every element of S in a subspace of dimension at most 2, then π intersects $(q^{d-1}-1)/(q^2-1)$ elements of S in a 2-space, q^{d-1} elements in a 1-space and intersects the remaining $q^d - q^2(q^{d-1}-1)/(q^2-1)$ elements of S trivially.*

In particular, for odd d, every totally isotropic d-space must intersect at least one element of the spread in $Q(2d, q)$ or $W(2d-1, q)$ in a subspace of dimension at least[3] 2.

Proof. The dual polar space is a regular near $2d$-gon of order $(q, \begin{bmatrix} d \\ 1 \end{bmatrix}_q - 1)$ in this case, and its point graph is a dual polar graph with set of vertices Ω_d. We know from Theorem 4.1.7 that two totally isotropic d-spaces are at distance i in the dual polar graph if and only if they intersect in a $(d-i)$-space, and thus the matrix $B = (B)_{\pi \in \Omega_d, i=0,\ldots,d}$ is precisely the outer distribution of S. Since S is a clique of the distance-d relation, the bound for $|S|$ and equivalence

[3] In the terminology of [15]: every maximal is non-scattered with respect to the spread (although only spreads in projective spaces were considered there).

between equality and tightness follow immediately from Theorem 6.4.10. The equation for B follows from Theorem 6.4.2(ii).

Suppose from now on that S is a spread. Consider any maximal π. We know that

$$B_{\pi,0} + \cdots + B_{\pi,d} = |S|.$$

We also know that the outer distribution B of S satisfies:

$$\sum_{i=0}^{d} (-1/q)^i B_{\pi,i} = 0.$$

Finally, as each of the $\left[{d \atop 1}\right]_q$ points in π must be on a unique element of S:

$$\sum_{i=0}^{d} B_{\pi,i} \begin{bmatrix} d-i \\ 1 \end{bmatrix}_q = \begin{bmatrix} d \\ 1 \end{bmatrix}_q .$$

If π is at distance 1 from S, then $B_{\pi,0} = 0, B_{\pi,1} = 1$ and $B_{\pi,i} = 0$ if $2 \leq i \leq d-2$. If π is at distance 2 from S and $d \geq 5$, then $B_{\pi,0} = 0, B_{\pi,1} = 0, B_{\pi,2} = 1$ and $B_{\pi,i} = 0$ if $2 \leq i \leq d-3$. Finally, if π is at distance at least $d-2$ from S, then $B_{\pi,i} = 0$ if $0 \leq i \leq d-3$. In all cases, the three equations given above allow explicit computation of the remaining entries $B_{\pi,d-2}, B_{\pi,d-1}$ and $B_{\pi,d}$ of the outer distribution. In particular, we see that for no maximal π it is possible that $B_{\pi,0} = \ldots = B_{\pi,d-2} = 0$, and hence the covering radius of S is at most $d-2$.

This establishes 2-regularity of S (if $d = 3$, then every maximal is in S or at distance $d-2 = 1$ from S). □

Just as for $H(2d-1, q^2)$ with d odd, we can say more for small d. For $Q(6,q)$ and $W(5,q)$, we will recover a result by Thas [145].

Theorem 6.4.30. *The spreads in $Q(6,q)$ or $W(5,q)$ are precisely the perfect 1-codes. A spread in $Q(10,q)$ or $W(9,q)$ is completely regular with covering radius 3, and the reduced outer distribution B' and quotient matrix L_S of the corresponding equitable partition of the dual polar graph are given by:*

$$B' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & q^5 \\ 0 & 1 & 0 & 0 & q^4 & q^5 - q^4 \\ 0 & 0 & 1 & q^2 & q^4 - q^2 & q^5 - q^4 \\ 0 & 0 & 0 & q^2+1 & q^4 & q^5 - q^4 - q^2 \end{pmatrix},$$

$$L_S = \begin{pmatrix} 0 & q(q^4+q^3+q^2+q+1) & 0 & 0 \\ 1 & q-1 & q^2(q+1)(q^2+1) & 0 \\ 0 & q+1 & q^4+q^3+2q^2-1 & q^2(q^3-1) \\ 0 & 0 & (q^2+1)(q^2+q+1) & q^5-q^2-1 \end{pmatrix}.$$

Proof. Let S be a spread. If the diameter d is 3 or 5, we can use Theorem 6.4.29 to determine every row of the outer distribution B of S, since then every maximal is at distance $0, 1, 2$ or $d-2$ from S. This yields B'.

If $d = 3$, then every maximal is either in S or at distance 1 from a unique element of S, and thus S is a perfect 1-code. Conversely, every perfect 1-code consists of $(q+1)(q^2+1)(q^3+1)/(1+q(q^2+q+1)) = q^3+1$ maximals that are pairwise at distance 3 (i.e. intersect trivially), and hence is a spread.

If $d = 5$, then we know from Theorem 6.4.29 that the covering radius $t(S)$ is at most $d-2 = 3$. Since S is completely regular, Theorem 2.3.9 yields that the covering radius $t(S)$ is the dual degree $r(S)$. Corollaries 2.2.9 and 4.3.17 yield that χ_S is orthogonal to exactly 2 eigenspaces, and hence $r(S) = t(S) = d-2$ (alternatively, one can see that S is not a perfect 2-code by counting). We can use the formula $L_S = B' L_{t(S)} (B'_{t(S)})^{-1}$ from Lemma 2.3.8 to compute the quotient matrix L_S. The required matrix $L_{t(S)}$ follows from Theorem 4.1.7:

$$L_{t(S)} = \begin{pmatrix} 0 & q(q^4+q^3+q^2+q+1) & 0 & 0 \\ 1 & q-1 & q^2(q+1)(q^2+1) & 0 \\ 0 & q+1 & q^2-1 & q^3(q^2+q+1) \\ 0 & 0 & q^2+q+1 & q^3-1 \\ 0 & 0 & 0 & (q+1)(q^2+1) \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

□

Corollary 6.4.31. *If S is a spread in $Q(2d,q)$ or $W(2d-1,q)$ with d odd, and T is an m-ovoid of the dual polar space (i.e. every totally isotropic $(d-1)$-space is in exactly m elements of T), then $|S \cap T| = m(q^d+1)/(q+1)$.*

Proof. We know Theorem 6.4.29 that S is a $(q^d+1)/(q+1)$-tight set. The result now follows immediately from Corollary 6.4.9. □

In classical finite polar spaces of rank d, no combinatorial design in the dual polar graph with respect to t-spaces with $t \geq 2$ (different from the empty or full

set) seems to be known, except for the halves in the bipartite dual polar graph on $Q^+(2d-1,q)$. As m-ovoids in the dual polar space on a classical finite polar space of rank d are the combinatorial designs with respect to $(d-1)$-spaces, this suggests that non-trivial m-ovoids might be very hard to find if $d \geq 3$. If S is an m-ovoid, the elements of S through a fixed isotropic 1-space induce an m-ovoid in the residual polar space, the associated dual polar graph of which has diameter $d-1$. Hence one should start by considering the polar spaces of rank three.

A 1-ovoid S in a dual polar space on a classical finite polar space of rank 3 with parameters (q,q^e) gives rise to a partial geometry $\mathrm{pg}(q^{e+1}, q(q+1), q+1)$, with the elements of S as points, the isotropic 1-spaces as lines, and incidence inherited from the polar space (see for instance [131, Subsection 2.4]). The point graph of the dual of this partial geometry is simply the (strongly regular) polar graph on the isotropic 1-spaces. This graph is the point graph of a partial geometry if and only if the dual polar space has a 1-ovoid (see for instance [116]). For these to exist, the polar space with the same parameters (q,q^e) but of rank two, certainly must have spreads of 2-spaces. Non-existence of 1-ovoids in the dual polar spaces on $W(5,q)$ was shown by Thomas [155] (see also [118],[48] and [56]). Non-existence for $Q^-(7,2)$ was conjectured in [60] and proved by Panigrahi [116]. For 1-ovoids in dual polar spaces of rank three, this leaves the question of existence open for $Q^-(7,q)$ and for $H(6,q^2)$, with $q \geq 3$ in both cases.

Non-existence of non-trivial perfect codes (i.e. different from a singleton or the full set) in dual polar graphs on classical finite polar spaces was shown by Chihara [44], except for perfect 1-codes for $Q(2d,q)$ and $W(2d-1,q)$ with $d = 2^m - 1$. Thas [145] proved that they exist for $W(5,q)$, but for $Q(6,q)$ it is not completely settled yet for odd q, and nothing seems to be known for $d = 2^m - 1$ with $m \geq 3$.

6.5 Krein conditions and spherical designs

In this subsection, we will briefly consider the spherical representation of dual polar spaces using the normalized columns of M. For the sake of completeness, we will mention known results on near polygons and generalized polygons as well.

The following inequality is due to Brouwer and Wilbrink [28].

Theorem 6.5.1. *Consider a regular near $2d$-gon of order (s,t), $d \geq 2$ and $s > 1$, with valency k_i for the distance-i relation with respect to the point graph. Then*

$$\sum_{i=0}^{d} \left(-\frac{1}{s}\right)^{3i} k_i \geq 0.$$

The normalized columns of $M = \sum_{i=0}^{d}(-1/s)^i A_i$ are a d-distance set with angle set $\{-1/s, \ldots, (-1/s)^d\}$, and they form a spherical 3-design if and only if equality holds in the above inequality.

Proof. We know from Theorem 6.3.1 that M is up to a positive scalar a minimal idempotent, with $\lambda_i/k_i = (-1/s)^i$. Hence $A_i M = \lambda_i M = k_i/(-s)^i M$. Using $A_i \circ A_j = \delta_{ij} A_i$, we can now write:

$$(M \circ M)M = \left(\sum_{i=0}^{d} \frac{A_i}{s^{2i}}\right) M = \left(\sum_{i=0}^{d} \frac{k_i}{(-s)^{3i}}\right) M.$$

The coefficient on the right-most side is, up to a positive scalar, one of the Krein parameters q^j_{jj} and thus is at least zero. Lemma 2.4.4 now yields that X is up to normalization a d-distance set with angle set $\{-1/s, \ldots, (-1/s)^d\}$, and Theorem 2.4.6 provides the desired equivalence. □

A matrix-free proof of the above inequality was given by Neumaier [109], when looking for an alternative proof of D. Higman's restriction for generalized octagons (see Theorem 6.2.7). The well-known inequalities of D. Higman for generalized polygons are an immediate consequence of this.

Corollary 6.5.2. *For a generalized $2d$-gon of order (s,t), $d \geq 2$ and $s > 1$, let X be the set of normalized column vectors of $M = \sum_{i=0}^{d}(-1/s)^i A_i$.*

(i) *If $d = 2$ or $d = 4$, then $t \leq s^2$ with equality if and only if X is a spherical 3-design. If $d = 3$, then X is never a spherical 3-design.*

(ii) *For $d = 2$, X is a spherical 4-design if and only if $(s,t) = (2,4)$.*

Proof.

(i) The valencies k_i follow from Theorem 6.2.5. Calculating $\sum_{i=0}^{d} k_i/(-s)^{3i}$ yields:

$$\begin{aligned} &\text{for } d=2: && (s^2-1)(s^2-t)/s^4, \\ &\text{for } d=3: && (s^2-1)(s^4-s^2t+t^2)/s^6, \\ &\text{for } d=4: && (s^2-1)(s^2-t)(s^4+t^2)/s^8. \end{aligned}$$

The bound and the criterion for spherical 3-designs now follow from Theorem 6.5.1.

(ii) If X is a spherical 4-design, then it is also a 3-design so (i) implies that $d = 2$ and $t = s^2$ must hold. In that case, the number of points is $v = (s+1)(s^3+1)$ because of Theorem 6.2.5, and the rank of M is given by $m = s(s^2 - s + 1) > 1$ because of Theorem 6.3.3(i). Theorem 2.4.3 then yields that X is a spherical 4-design if and only if

$$\frac{m(m+3)}{2} - v = \frac{(s-2)(s^2+1)(s^3+1)}{2} = 0,$$

or hence if and only if $s = 2$. □

Remark 6.5.3. The inequality $t \leq s^3$ for generalized hexagons of order (s,t) with $s > 1$ (see Theorem 6.2.7), also follows from a Krein condition, but not of the form $q^j_{jj} \geq 0$. The point graph of a generalized hexagon of order (s, s^3) with $s > 1$ has classical parameters $(3, -s, -s/(s-1), s^2+s)$ (see [23, Section 8.5]). The corresponding cometric ordering of the eigenvalues from Theorem 2.3.14 is given by: $s(s^3+1), -s^3-1, s-1+s^2, s-1-s^2$.

There is a unique generalized quadrangle of order $(2,4)$, namely $Q^-(5,2)$ (see for instance [122, 6.1.3]). Hence here we obtain a set of 27 vectors in $\mathbb{S}^5$ with angle set $\{-1/2, 1/4\}$. The complement of the point graph is known as the *Schläfli graph*.

We now move on to dual polar spaces.

Theorem 6.5.4. *Consider a classical finite polar space of rank $d \geq 2$ with parameters (q, q^e) and $e \in \{1/2, 1, 3/2, 2\}$, and let X be the set of normalized column vectors of $M = \sum_{i=0}^{d}(-1/q^e)^i A_i$ with respect to its dual polar graph. Then X is a d-distance set with angle set $\{-1/q^e, \ldots, (-1/q^e)^d\}$, and a spherical 3-design if and only if $d \geq 2e+1$.*

Proof. The dual polar space is a regular near $2d$-gon of order $(q^e, \left[{d \atop 1}\right]_q - 1)$. The valencies k_i follow from Theorem 4.1.7. We can now compute:

$$\begin{aligned}\sum_{i=0}^{d} \frac{k_i}{(-s)^{3i}} &= \sum_{i=0}^{d} q^{i(i-1)/2}(-q^{-2e})^i \begin{bmatrix} d \\ i \end{bmatrix}_q, \\ &= \prod_{j=1}^{d}(1 + q^{j-1}(-q^{-2e}))\end{aligned}$$

where we used Theorem 6.4.22 with $z = -q^{-2e}$ for the last step. The result now follows from Theorem 6.5.1. □

Note that the dual polar space on $H(2d-1, q^2)$ with $d = 2$ is isomorphic to the classical generalized quadrangle $Q^-(5, q)$ of order (q, q^2) (see Theorem 1.3.3). We know from Theorem 6.5.2 that the case $q = 2$ plays a special role here. We conclude this section by mentioning a similar, much more recent result by Munemasa for higher rank d.

Theorem 6.5.5. [108] *Consider the dual polar graph on $H(2d-1, 2^2)$, $d \geq 3$, and let X be the set of normalized column vectors of $M = \sum_{i=0}^{d}(-1/2)^i A_i$. Then X is a spherical 5-design (and not a spherical 6-design).*

6.6 Higman inequalities for regular near $2d$ -gons

Several restrictions on the parameters of regular near polygons have already been given in literature. We already mentioned the results for generalized polygons by D. Higman [87, 88] and by Haemers and Roos [86] (see Theorem 6.2.7). Inequalities for regular near $2d$-gons were given by Brouwer and Wilbrink [28] and by Neumaier [109], and for regular near hexagons by Mathon (unpublished, see [28]). Hiraki and Koolen [91, 92, 93] also obtained several bounds. In particular, it is proved in [91] that if Γ is a regular near $2d$-gon of order (s, t) with $s > 1$, then $t < s^{4d/r-1}$ for a certain integer $r \geq 1$. We will give another bound for t in terms of s in Corollary 6.6.7. We also note that Terwilliger and Weng [144] gave a bound, which is attained only by Hamming and dual polar graphs.

We now give one of the main results in this chapter.

Theorem 6.6.1. *Consider a regular near $2d$-gon of order (s,t), $d \geq 2$ and $s > 1$, with point graph Γ. Then:*

$$c_i \leq \frac{s^{2i}-1}{s^2-1}, \forall i \in \{1,\dots,d\}.$$

Consider two points a and b at distance i with $1 \leq i \leq d$. Suppose $v = \alpha\chi_{\{a\}} + \beta\chi_{\{b\}} + \gamma\chi_T$ with $(\alpha,\beta,\gamma) \in \mathbb{R}^3\backslash\{(0,0,0)\}$ and $T = \Gamma_1(a) \cap \Gamma_{i-1}(b)$. Now $Mv = 0$ if and only if both $c_i = (s^{2i}-1)/(s^2-1)$ holds and (α,β,γ) is a scalar multiple of

$$\left(s\frac{s^{2i-2}-1}{s^2-1}, (-1)^i s^{i-1}, 1\right).$$

Proof. Given a and b at distance i with $1 \leq i \leq d$, there are exactly c_i points on a common line with a and at distance $i-1$ from b. Hence T has size c_i, and no two points in T are on the same such line.

We will now consider $v^T A_j v$ for every $j \in \{0,\dots,d\}$ (i.e. we will consider the inner distribution of v). Note that for any two subsets of points S_1 and S_2, the value of $(\chi_{S_1})^T A_j \chi_{S_2} = (\chi_{S_2})^T A_j \chi_{S_1}$ is given by the number of ordered pairs $(\omega_1,\omega_2) \in (S_1 \times S_2)$ with $d(\omega_1,\omega_2) = j$. Our assumptions immediately yield:

$$(\chi_{\{a\}})^T A_0 \chi_{\{a\}} = (\chi_{\{b\}})^T A_0 \chi_{\{b\}} = 1,$$
$$(\chi_{\{a\}})^T A_j \chi_{\{a\}} = (\chi_{\{b\}})^T A_j \chi_{\{b\}} = 0 \text{ if } 1 \leq j \leq d,$$
$$(\chi_{\{a\}})^T A_i \chi_{\{b\}} = 1, (\chi_{\{a\}})^T A_j \chi_{\{b\}} = 0 \text{ if } j \neq i,$$
$$(\chi_{\{a\}})^T A_1 \chi_T = |T| = c_i, (\chi_{\{a\}})^T A_j \chi_T = 0 \text{ if } j \neq 1,$$
$$(\chi_{\{b\}})^T A_{i-1} \chi_T = |T| = c_i, (\chi_{\{b\}})^T A_j \chi_T = 0 \text{ if } j \neq i-1.$$

Finally, as every two distinct points in T are on distinct lines through a, they cannot be collinear, and hence they are at distance 2. This yields:

$$(\chi_T)^T A_0 \chi_T = |T| = c_i, (\chi_T)^T A_2 \chi_T = |T|(|T|-1) = c_i(c_i-1),$$
$$(\chi_T)^T A_j \chi_T = 0 \text{ if } j \notin \{0,2\}.$$

We will now work out the following:

$$\begin{aligned} s^i(v^T M v) &= \sum_{j=0}^{d} (-1)^j s^{i-j} (v^T A_j v) \\ &= \sum_{j=0}^{d} (-1)^j s^{i-j} (\alpha\chi_{\{a\}} + \beta\chi_{\{b\}} + \gamma\chi_T)^T A_j (\alpha\chi_{\{a\}} + \beta\chi_{\{b\}} + \gamma\chi_T). \end{aligned}$$

We obtain

$$s^i(\alpha^2+\beta^2+\gamma^2c_i)-s^{i-1}(2\alpha\gamma)c_i+s^{i-2}\gamma^2c_i(c_i-1)+(-1)^{i-1}s(2\beta\gamma)c_i+(-1)^i(2\alpha\beta),$$

and hence $s^i(v^TMv)$ can be rewritten as $(\alpha,\beta,\gamma)F(\alpha,\beta,\gamma)^T$ with:

$$F=\begin{pmatrix} s^i & (-1)^i & -s^{i-1}c_i \\ (-1)^i & s^i & (-1)^{i-1}sc_i \\ -s^{i-1}c_i & (-1)^{i-1}sc_i & c_is^{i-2}(s^2+c_i-1) \end{pmatrix}.$$

We compute the determinant of F:

$$\begin{aligned}
\mathrm{Det}(F) &= (-1)^ic_i\mathrm{Det}\begin{pmatrix} s^i & (-1)^i & -s^{i-1}c_i \\ 1 & (-1)^is^i & -sc_i \\ -s^{i-1} & (-1)^{i-1}s & s^{i-2}(s^2+c_i-1) \end{pmatrix} \\
&= (-1)^ic_i\mathrm{Det}\begin{pmatrix} 0 & -(-1)^i(s^{2i}-1) & c_is^{i-1}(s^2-1) \\ 1 & (-1)^is^i & -sc_i \\ 0 & (-1)^is(s^{2i-2}-1) & -(c_i-1)s^{i-2}(s^2-1) \end{pmatrix} \\
&= -c_is^{i-2}(s^2-1)\mathrm{Det}\begin{pmatrix} -(s^{2i}-1) & c_is \\ s(s^{2i-2}-1) & -(c_i-1) \end{pmatrix} \\
&= c_is^{i-2}(s^2-1)((s^{2i}-1)-c_i(s^2-1)).
\end{aligned}$$

We know from Lemma 6.3.1 that M is a minimal idempotent up to a positive scalar and thus positive semidefinite. Hence $v^TMv \geq 0$ for all $(\alpha,\beta,\gamma) \in \mathbb{R}^3\backslash\{(0,0,0)\}$. Thus F is positive semidefinite, and hence its determinant must be non-negative, and F is positive definite if and only if this determinant is positive. We find that $c_i \leq (s^{2i}-1)/(s^2-1)$ since $s>1$. We can also write:

$$Mv=0 \Longleftrightarrow v^TMv=0 \Longleftrightarrow (\alpha,\beta,\gamma)F(\alpha,\beta,\gamma)^T=0.$$

As F is positive semidefinite, the latter will hold if and only if both F is not positive definite and $F(\alpha,\beta,\gamma)^T=0$. This is possible if and only if both $c_i=(s^{2i}-1)/(s^2-1)$ holds and (α,β,γ) is a scalar multiple of the vector $(s(s^{2i-2}-1)/(s^2-1),(-1)^is^{i-1},1)$. □

We will now focus on the case of equality. First, we introduce another property of association schemes.

Definition 6.6.2. *Suppose $(\Omega,\{R_0,\ldots,R_d\})$ is an association scheme with cometric ordering $E_0,E_1,\ldots,E_d$.*

(i) *The scheme is dual bipartite with respect to* E_1 *if* $q^j_{1j} = 0$, $\forall j \in \{0, \ldots, d\}$.

(ii) *The scheme is almost dual bipartite with respect to* E_1 *if* $q^j_{1j} = 0$, $\forall j \in \{0, \ldots, d-1\}$, *and* $q^d_{1d} \neq 0$.

We now mention a result by Terwilliger on (almost) dual bipartite association schemes that are also P-polynomial.

Theorem 6.6.3. [141] *Suppose* Γ *is a distance-regular graph on* Ω *with diameter* $d \geq 2$ *and with a non-degenerate minimal idempotent* E_1. *The following are equivalent.*

(i) *The association scheme defined by* Γ *is dual bipartite or almost dual bipartite with respect to* E_1.

(ii) *For all* $\omega_1, \omega_2 \in \Omega$ *and* $i_1, i_2 \in \{0, \ldots, d\}$, *the vector* $E_1\chi_T$ *is a linear combination of* $E_1\chi_{\{\omega_1\}}$ *and* $E_1\chi_{\{\omega_2\}}$, *with* $T = \Gamma_{i_1}(\omega_1) \cap \Gamma_{i_2}(\omega_2)$.

(iii) *For all* $\omega_1, \omega_2 \in \Omega$, *the vector* $E_1\chi_T$ *is a linear combination of* $E_1\chi_{\{\omega_1\}}$ *and* $E_1\chi_{\{\omega_2\}}$, *with* $T = \Gamma_1(\omega_1) \cap \Gamma_1(\omega_2)$.

See [71, 72] for more on (almost) dual bipartite P-polynomial schemes.

Theorem 6.6.4. *Consider a regular near* $2d$*-gon* $\mathcal{P}$, $d \geq 2$, *of order* (s, t) *with* $s > 1$. *The following are equivalent.*

(i) $c_2 = s^2 + 1$.

(ii) *The point graph* Γ *defines an (almost) dual bipartite association scheme with respect to the eigenspace for* $-(t+1)$.

(iii) *For all* $\omega_1, \omega_2 \in \Omega$ *and* $i_1, i_2 \in \{0, \ldots, d\}$, *the vectors* $M\chi_{\{\omega_1\}}$, $M\chi_{\{\omega_2\}}$, $M\chi_T$ *are linearly dependent, with* $T = \Gamma_{i_1}(\omega_1) \cap \Gamma_{i_2}(\omega_2)$.

(iv) $c_i = (s^{2i} - 1)/(s^2 - 1), \forall i \in \{1, \ldots, d\}$.

(v) *If* $d = 2$, *then* $\mathcal{P}$ *is a generalized quadrangle of order* (s, s^2), *and if* $d \geq 3$, *then* s *is a prime power* q *and* $\mathcal{P}$ *is the dual polar space on* $H(2d-1, q^2)$.

Proof. Recall from Theorem 6.3.1 that $M = \sum_{j=0}^{d} (-1/s)^j A_j$ is, up to a positive scalar, the non-degenerate idempotent corresponding to the eigenvalue $-(t+1)$ of Γ.

(i)→(ii): We will verify that (iii) from Theorem 6.6.3 is satisfied, i.e. that for every two points ω_1 and ω_2, the vector $M\chi_T$ with $T = \Gamma_1(\omega_1) \cap \Gamma_1(\omega_2)$ is a linear combination of $M\chi_{\{\omega_1\}}$ and $M\chi_{\{\omega_2\}}$. If $d(\omega_1, \omega_2) \geq 3$, then $\Gamma_1(\omega_1) \cap \Gamma_1(\omega_2) = \emptyset$, so there is nothing to check. If $d(\omega_1, \omega_2) = 0$, then $M\chi_T = M(A_1\chi_{\{\omega_1\}}) = -(t+1)M\chi_{\{\omega_1\}}$. If $d(\omega_1, \omega_2) = 1$, then $\{\omega_1\} \cup \{\omega_2\} \cup T$ is precisely the set ℓ^* of $s+1$ points on the line $\omega_1\omega_2$, and hence $M\chi_{\{\omega_1\}} + M\chi_{\{\omega_2\}} + M\chi_T = M\chi_{\ell^*} = 0$ because of Lemma 6.4.2. Finally, if $d(\omega_1, \omega_2) = 2$, then the desired linear combination follows from Theorem 6.6.1 and our assumption (i).

(ii)→(iii): This follows immediately from Theorem 6.6.3.

(iii)→(iv): If (iii) holds, then in particular it holds for $i_1 = 1$ and $i_2 = i - 1$ and for two points ω_1, ω_2 with $d(\omega_1, \omega_2) = i$, $1 \leq i \leq d$. Hence it follows from Theorem 6.6.1 that $c_i = (s^{2i} - 1)/(s^2 - 1)$.

(iv)→(i): Obvious.

(iv)↔(v): For $d = 2$, this follows immediately (recall from Theorem 6.1.3 that $c_d = t + 1$). Next assume $d \geq 3$. Assume (iv). Theorem 6.1.3 now also yields: $b_i = s(s^{2d} - s^{2i})/(s^2 - 1)$. Hence Γ has classical parameters $(d, q^2, 0, q)$. A regular near $2d$-gon of order (s, t) with classical parameters $(d, b, 0, \beta)$, $s > 1$ and $d \geq 3$ must be a dual polar space on a classical finite polar space with parameters (q, q^e) with $b = q$ and $\beta = q^e$, or a Hamming graph with $b = 1$ (see [23, Theorem 9.4.4]). This establishes (iv)→(v) for $d \geq 3$, while (v)→(iv) follows from Theorem 4.1.7.

Remark 6.6.5. Equivalence between (i) and (iv) in Theorem 6.6.4 can be seen in another way. In any regular near $2d$-gon of order (s, t), $s > 1$, the parameters satisfy $t_{i+1} \geq t_2(t_i + 1)$, $\forall i \in \{1, \ldots, d-1\}$ (see [28, Lemma 26 (Corollary)]). Hence if $s > 1$ and $t_2 = s^2$, then Theorem 6.6.1 yields by induction that $c_i = t_i + 1 = (s^{2i} - 1)/(s^2 - 1), \forall i \in \{1, \ldots, d\}$.

We now prove the existence of certain parameters, often referred to as *triple intersection numbers*, for those regular near $2d$-gons attaining one of the bounds from Theorem 6.6.1.

Theorem 6.6.6. *Consider a regular near $2d$-gon of order (s, t), $s > 1$ and $d \geq 2$, with point graph Γ. Suppose $c_i = (s^{2i} - 1)/(s^2 - 1)$ for some $i \in \{1, \ldots, d\}$, and consider three points a, b, c with $d(a, b) = i, d(a, c) = d, d(b, c) = k$. The*

set $\Gamma_1(a) \cap \Gamma_{i-1}(b) \cap \Gamma_{d-1}(c)$ *has size:*

$$\frac{s^{2i-1} + (-1)^{i+k+d} s^{d-k+i} - (-1)^{i+k+d} s^{d-k+i-1} - 1}{s^2 - 1}.$$

Proof. Let T be $\Gamma_1(a) \cap \Gamma_{i-1}(b)$. We know from Theorem 6.6.1 that $v = s(s^{2i-2} - 1)/(s^2 - 1)\chi_{\{a\}} + (-1)^i s^{i-1}\chi_{\{b\}} + \chi_T$ satisfies $Mv = 0$, where M denotes, up to a positive scalar, the minimal idempotent $\sum_{j=0}^{d}(-1/s)^j A_j$. If B is the outer distribution of v, then:

$$B_{c,j} = (\chi_{\{c\}})^T A_j \left(s\frac{s^{2i-2} - 1}{s^2 - 1}\chi_{\{a\}} + (-1)^i s^{i-1}\chi_{\{b\}} + \chi_T \right).$$

Working out $(\chi_{\{c\}})^T Mv = 0$ (or applying Theorem 2.2.7) we find that B must satisfy:

$$\sum_{j=0}^{d} \left(-\frac{1}{s}\right)^j B_{c,j} = 0.$$

As $d(a, c) = d$, all elements of T are at distance at least $d - 1$ from c. Hence if x denotes $|T \cap \Gamma_{d-1}(c)|$, then $|T \cap \Gamma_d(c)| = |T| - x = c_i - x$. The assumptions now imply:

$$(\chi_{\{c\}})^T A_d \chi_{\{a\}} = (\chi_{\{c\}})^T A_k \chi_{\{b\}} = 1,$$

$$(\chi_{\{c\}})^T A_j \chi_{\{a\}} = 0 \text{ if } j \neq d, (\chi_{\{c\}})^T A_j \chi_{\{b\}} = 0 \text{ if } j \neq k,$$

$$(\chi_{\{c\}})^T A_{d-1} \chi_T = x, (\chi_{\{c\}})^T A_d \chi_T = c_i - x, (\chi_{\{c\}})^T A_j \chi_T = 0 \text{ if } j \notin \{d-1, d\}.$$

Hence we obtain:

$$\left(\left(-\frac{1}{s}\right)^d s\frac{s^{2i-2} - 1}{s^2 - 1}\right) + \left(\left(-\frac{1}{s}\right)^k (-1)^i s^{i-1}\right) + \left(\left(-\frac{1}{s}\right)^{d-1} x + \left(-\frac{1}{s}\right)^d (c_i - x)\right) = 0.$$

Using $c_i = (s^{2i} - 1)/(s^2 - 1)$ and multiplying by $(-s)^d$, we obtain:

$$s\frac{s^{2i-2} - 1}{s^2 - 1} + (-1)^{d+k+i} s^{d-k+i-1} + \frac{s^{2i} - 1}{s^2 - 1} = (s + 1)x.$$

This yields $|(\Gamma_1(a) \cap \Gamma_{i-1}(b)) \cap \Gamma_{d-1}(c)| = |T \cap \Gamma_{d-1}(c)| = x =$

$$\frac{s^{2i-1} + (-1)^{i+k+d} s^{d-k+i} - (-1)^{i+k+d} s^{d-k+i-1} - 1}{s^2 - 1}.$$

□

The following corollary generalizes the Higman inequality between the parameters (s,t) of generalized quadrangles (see Theorem 6.2.7), and also gives a property in case of equality.

Corollary 6.6.7. *Consider a regular near $2d$-gon of order (s,t), $s>1$ and $d\geq 2$, with point graph Γ. Then:*

$$t+1\leq \frac{s^{2d}-1}{s^2-1},$$

If equality holds, then for any three points a,b and c, pairwise at distance d:

$$|\Gamma_1(a)\cap\Gamma_{d-1}(b)\cap\Gamma_{d-1}(c)|=\frac{(s^d-(-1)^d)(s^{d-1}+(-1)^d)}{s^2-1}.$$

Proof. As $t+1=c_d$, this follows immediately from Theorems 6.6.1 and 6.6.6 with $i=d$ and $k=d$. □

For $d=2$, the property in case of equality is due to Bose and Shrikhande [17]. Cameron [32] gave a combinatorial proof of the inequality, which also shows that in a generalized quadrangle of order (s,t) with $s>1$, the property holds if and only if $t=s^2$.

We will come back to these triple intersection numbers for the dual polar space on $H(2d-1,q^2)$ in Appendix A.

6.7 Subgraphs in extremal near $2d$-gons

In this section, we will consider substructures in those regular near $2d$-gons attaining bounds from Theorem 6.6.1. We will first demonstrate how to use Theorem 6.6.6 to prove distance-regularity of a last subconstituent, just by use of the parameters.

Theorem 6.7.1. *Suppose the point graph of Γ of a regular near $2d$-gon of order (s,t), $d\geq 2$ and $s>1$, has parameters $c_i=(s^{2i}-1)/(s^2-1)$, $\forall i\in\{1,\ldots,d\}$. For any point p, the last subconstituent $\Gamma'=\Gamma_d(p)$ is distance-regular with parameters b'_i and c'_i given by:*

$$\begin{aligned} b'_i &= \frac{s^{2d}-s^{2i}}{s+1}, \forall i\in\{0,\ldots,d-1\},\\ c'_i &= s^{i-1}\left(\frac{s^i-(-1)^i}{s+1}\right), \forall i\in\{1,\ldots,d\},\end{aligned}$$

and with classical parameters $(d, -s, -s-1, -(-s)^d - 1)$.
The distance between any two vertices in Γ' *is the same as in* Γ.

Proof. Note that $t+1 = c_d = (s^{2d}-1)/(s^2-1)$. Consider two vertices a and b in $\Gamma_d(p)$ at distance i with $0 \leq i \leq d$. There are c_i lines through a with a unique point at distance $i-1$ from b, and with all other s points at distance i from b (we write $c_0 = 0$). All points on the remaining $(t+1) - c_i$ lines through a, different from a, are at distance $i+1$ from b in Γ. We can compute $|\Gamma_1(a) \cap \Gamma_{i-1}(b) \cap \Gamma_{d-1}(p)|$ using Theorem 6.6.6, and hence if $1 \leq i \leq d$:

$$\begin{aligned} |\Gamma_1(a) \cap \Gamma_{i-1}(b) \cap \Gamma_d(p)| &= |\Gamma_1(a) \cap \Gamma_{i-1}(b)| - |\Gamma_1(a) \cap \Gamma_{i-1}(b) \cap \Gamma_{d-1}(p)| \\ &= c_i - \frac{s^{2i-1} + (-1)^i s^i - (-1)^i s^{i-1} - 1}{s^2-1} \\ &= \frac{s^{i-1}(s^i - (-1)^i)}{s+1}. \end{aligned}$$

In particular, the above number is non-zero, and so it follows by induction that vertices of Γ' are at the same distance in Γ and Γ'. Hence we have obtained the parameter c'_i of Γ'. On each of the $(t+1) - c_i = c_d - c_i$ lines through the point a at distance i from b, there is a unique point at distance $d-1$ from p, while all $s-1$ remaining points different from a are at distance d from p in Γ. Hence if $0 \leq i \leq d-1$:

$$\begin{aligned} |\Gamma_1(a) \cap \Gamma_{i+1}(b) \cap \Gamma_d(p)| &= (s-1)\left(\frac{s^{2d}-1}{s^2-1} - \frac{s^{2i}-1}{s^2-1}\right) \\ &= \frac{s^{2d} - s^{2i}}{s+1}, \end{aligned}$$

yielding the parameter b'_i of Γ'. This is non-zero if $0 \leq i \leq d-1$ and hence it follows by induction that the diameter of Γ' is indeed d. The last part follows immediately from Definition 2.3.12. □

We know from Theorem 6.6.4 that regular near $2d$-gons satisfying the conditions in Theorem 6.7.1 have specific vanishing Krein parameters. The link between vanishing Krein parameters and distance-regularity of subconstituents was studied in detail for strongly regular graphs in general by Cameron, Goethals and Seidel [39], who gave several examples. For diameter $d \geq 3$, we also know from Theorem 6.6.4 that we are in fact very much restricted. We first need to introduce another graph.

Definition 6.7.2. *The Hermitian forms graph* $\mathrm{Her}(d,q)$ *on* $V(d,q^2)$ *is the graph with as vertices the Hermitian forms on* $V(d,q^2)$*, and with two vertices* h_1 *and* h_2 *adjacent if the rank of* $h_1 - h_2$ *is* 1.

Theorem 6.7.3. [23, pp. 285-287]

(i) $\mathrm{Her}(d,q)$ *has classical parameters* $(d,-q,-q-1,-(-q)^d-1)$ *and is distance-transitive. For any two vertices* h_1 *and* h_2*,* $d(h_1,h_2)$ *is the rank of* $h_1 - h_2$*.*

(ii) *If* Γ *is the dual polar graph on* $H(2d-1,q^2)$*, then for any vertex* p *of* Γ*, the last subconstituent* $\Gamma_d(p)$ *is isomorphic to* $\mathrm{Her}(d,q)$*.*

Ivanov and Shpectorov [100] proved that any distance-regular graph Γ with classical parameters $(d,-s,-s-1,-(-s)^d-1)$, $s>1$, is constructed as in Theorem 6.7.1, under the assumption that all maximal cliques have size s. Terwilliger [143] proved that this assumption may be dropped if $d \geq 3$. It thus follows from Theorems 6.6.4 and 6.7.3 that if $d \geq 3$, then s must be a prime power q and $\Gamma \cong \mathrm{Her}(d,q)$.

Hence we cannot find new distance-regular graphs as a last subconstituent using Theorem 6.7.1. However, we will now prove that another induced subgraph is also distance-regular. We will first severely restrict the size of m-ovoids in a regular near $2d$-gon if at least one of the non-trivial bounds from Theorem 6.6.1 is attained.

Lemma 6.7.4. *Suppose* S *is an* m*-ovoid in a regular near* $2d$*-gon* $\mathcal{P}=(P,L,\mathrm{I})$ *of order* (s,t)*,* $d \geq 2$ *and* $s>1$*. Suppose* $c_i = (s^{2i}-1)/(s^2-1)$ *for some* $i \in \{1,\ldots,d\}$*, and* a *and* b *are two elements of* S *at distance* i *in* Γ*, then*

$$\begin{aligned} |S \cap \Gamma_1(a) \cap \Gamma_{i-1}(b)| &= m\frac{(s^i-(-1)^i)(s^{i-1}+(-1)^i)}{s^2-1} - \\ & \quad s\frac{(s^i-(-1)^i)(s^{i-2}+(-1)^i)}{s^2-1}. \end{aligned}$$

Proof. Let T denote the subset $\Gamma_1(a)\cap\Gamma_{i-1}(b)$ and take $\alpha = s(s^{2i-2}-1)/(s^2-1)$ and $\beta = (-1)^i s^{i-1}$. We know from Theorem 6.6.1 that $v = \alpha\chi_{\{a\}} + \beta\chi_{\{b\}} + \chi_T$ satisfies $Mv = 0$. We now consider $(\chi_S)^T v$:

$$(\chi_S)^T v = (\chi_S)^T(\alpha\chi_{\{a\}} + \beta\chi_{\{b\}} + \chi_T) = \alpha + \beta + |S \cap T|.$$

On the other hand, Theorem 6.4.8 implies that for some $w \in \mathbb{R}^P$, χ_S can be written as $(m/(s+1))\chi_P + Mw$. Hence:

$$\begin{aligned}
(\chi_S)^T v &= \left(\frac{m}{s+1}\chi_P + Mw\right)^T v \\
&= \frac{m}{s+1}(\chi_P)^T v + w^t(Mv) \\
&= \frac{m}{s+1}(\chi_P)^T(\alpha\chi_{\{a\}} + \beta\chi_{\{b\}} + \chi_T) \\
&= \frac{m}{s+1}(\alpha+\beta+|T|) = \frac{m}{s+1}(\alpha+\beta+c_i).
\end{aligned}$$

Hence we obtain:

$$|S \cap (\Gamma_1(a) \cap \Gamma_{i-1}(b))| = |S \cap T| = \frac{m}{s+1}(\alpha+\beta+c_i) - (\alpha+\beta),$$

which yields the desired result after substituting for α, β and c_i. □

Theorem 6.7.5. *If Γ is a regular near $2d$-gon of order (s,t), $d \geq 2$ and $s > 1$, with $c_i = (s^{2i}-1)/(s^2-1)$ for some $i \in \{2,\ldots,d\}$, then m-ovoids with $0 < m < s+1$ can only exist for $m = (s+1)/2$.*

Proof. Suppose S is an m-ovoid with $0 < m < s+1$. Consider any point b in S. We will count the number N of ordered pairs (p,a) of adjacent points in $(\Gamma_{i-1}(b) \cap S) \times (\Gamma_i(b) \cap S)$ in two ways. The size of $\Gamma_{i-1}(b) \cap S$ is given by Theorem 6.4.8. For each point p in $\Gamma_{i-1}(b) \cap S$, there are b_{i-1}/s lines through p such that the distance from b to this line is $d(p,b) = i-1$. The s other points on those lines are precisely the neighbours of p at distance i from b. Each such line contains exactly $m-1$ points in $S \backslash \{p\}$, all at distance i from b. Hence if k_{i-1} denotes $|\Gamma_{i-1}(b)|$:

$$N = k_{i-1}\left(\frac{m}{s+1} + \left(1 - \frac{m}{s+1}\right)\left(-\frac{1}{s}\right)^{i-1}\right)\frac{b_{i-1}}{s}(m-1).$$

We also know the size of $\Gamma_i(b) \cap S$ from Theorem 6.4.8, and for each point a in that subset, the number of its neighbours in S at distance $i-1$ from b is given by Lemma 6.7.4. Hence if k_i denotes $|\Gamma_i(b)|$:

$$\begin{aligned}
N &= k_i\left(\frac{m}{s+1} + \left(1 - \frac{m}{s+1}\right)\left(-\frac{1}{s}\right)^{i}\right) \times \\
&\quad \left(m\frac{(s^i-(-1)^i)(s^{i-1}+(-1)^i)}{s^2-1} - s\frac{(s^i-(-1)^i)(s^{i-2}+(-1)^i)}{s^2-1}\right).
\end{aligned}$$

When putting $m = x(s+1)$ and using the identity $k_{i-1}b_{i-1} = k_i c_i$ and the assumption $c_i = (s^{2i}-1)/(s^2-1)$, we see that x must be a root of the following polynomial in x:

$$\left(x + (1-x)\left(-\frac{1}{s}\right)^{i-1}\right)\frac{s^{2i}-1}{s(s^2-1)}(x(s+1)-1)-$$

$$\left(x + (1-x)\left(-\frac{1}{s}\right)^{i}\right)\left(x\frac{(s^i-(-1)^i)(s^{i-1}+(-1)^i)}{s-1} - s\frac{(s^i-(-1)^i)(s^{i-2}+(-1)^i)}{s^2-1}\right),$$

which can be rewritten as:

$$\frac{(-1)^i(s^i-(-1)^i)(s^{i-1}+(-1)^i)}{s^i(s-1)}(x-1)(2x-1).$$

Since $i \geq 2$ and $0 < m < s+1$, we see that $m/(s+1) = x = 1/2$. □

Definition 6.7.6. *A hemisystem in a generalized quadrangle of order (s, s^2), $s > 1$, is an $(s+1)/2$-ovoid.*

For the dual polar space on $H(3, q^2)$, which is a classical generalized quadrangle of order (q, q^2), Theorem 6.7.5 was first obtained for odd q by Segre [129] and for even q by Bruen and Hirschfeld [30]. Segre also proved that there is a unique hemisystem (up to equivalence) if $q = 3$. The restriction on m was obtained for all generalized quadrangles of order (s, s^2) in [150]. A major breakthrough was the construction of hemisystems in the dual polar space on $H(3, q^2)$ for every odd prime power q by Cossidente and Penttila [49]. A hemisystem in a non-classical generalized quadrangle of order $(5, 5^2)$ was constructed in [5]. Very recently, it was proved in [8] that hemisystems in fact exist in all flock generalized quadrangles (see [149] for more information on the latter), and new hemisystems were found in the dual polar spaces on $H(3, q^2)$ for small q in [7], suggesting the existence of other infinite families of hemisystems.

Cameron [37] proved that a hemisystem in any generalized quadrangle of order (s, s^2), $s > 1$, induces a strongly regular graph with parameters

$$\mathrm{srg}((s+1)(s^3+1)/2, (s-1)(s^2+1)/2, (s-3)/2, (s-1)^2/2).$$

For the dual polar space on $H(3, q^2)$, this result was already obtained by Thas [148], and for $q = 3$ the induced graph on the unique hemisystem is isomorphic to the triangle-free Gewirtz graph. The following lemma generalizes these facts to regular near $2d$-gons attaining the bounds from Theorem 6.6.1. Our proof only requires assumptions on the parameters.

Lemma 6.7.7. *Let Γ be the point graph of a regular near $2d$-gon of order (s,t), $d \geq 2$ and $s > 1$, with $c_i = (s^{2i}-1)/(s^2-1)$ for every $i \in \{1,\ldots,d\}$. Suppose S is an $(s+1)/2$-ovoid. The induced subgraph of Γ on S, denoted by Γ', is distance-regular with diameter d and intersection numbers:*

$$\begin{aligned} b'_i &= \frac{s^{2d}-s^{2i}}{2(s+1)}, \forall i \in \{0,\ldots,d-1\}, \\ c'_i &= \frac{(s^i-(-1)^i)(s^{i-1}-(-1)^i)}{2(s+1)}, \forall i \in \{1,\ldots,d\}. \end{aligned}$$

The distance between any two vertices in Γ' is the same as in Γ.

Proof. Consider any elements $a, b \in S$ at distance i in Γ with $i \in \{1,\ldots,d\}$. Lemma 6.7.4 yields, after substituting $(s+1)/2$ for m, that:

$$|S \cap (\Gamma_1(a) \cap \Gamma_{i-1}(b))| = \frac{(s^i-(-1)^i)(s^{i-1}-(-1)^i)}{2(s+1)},$$

which is in particular at least 1. Induction on i now yields that the distance between a and b in the induced subgraph is also i.

Now consider any two elements a and b of S at distance i in Γ with $0 \leq i \leq d-1$. There are precisely b_i/s lines through a at distance i from b. Only on these lines through a, points at distance $i+1$ from b and adjacent to a (in Γ) can be found, and each such line contains exactly $(s-1)/2$ points of $S \backslash \{a\}$. Hence:

$$|S \cap (\Gamma_1(a) \cap \Gamma_{i+1}(b))| = \frac{b_i}{s}\frac{s-1}{2} = \frac{k-sc_i}{s}\frac{s-1}{2} = (c_d - c_i)\frac{s-1}{2} = \frac{s^{2d}-s^{2i}}{2(s+1)},$$

where $k = sc_i + b_i$ followed from Theorem 6.1.3 (we let c_0 be zero). Note also that the last value is non-zero if $0 \leq i \leq d-1$, so the diameter of Γ' is precisely d. □

Because of Theorem 6.6.4 the result from Lemma 6.7.7 comes down to the following if the diameter is at least three, which is one of the main results in this chapter.

Theorem 6.7.8. *Suppose S is a $(q+1)/2$-ovoid in the dual polar space on $H(2d-1,q^2)$ with q odd and $d \geq 2$. The induced subgraph Γ' on S of the point graph Γ is distance-regular with classical parameters:*

$$(d,b,\alpha,\beta) = \left(d, -q, -\left(\frac{q+1}{2}\right), -\left(\frac{(-q)^d+1}{2}\right)\right).$$

The distance between any two vertices in Γ' is the same as in Γ.

Proof. Theorems 4.1.7 and 6.2.9 imply that the corresponding dual polar space is a regular near $2d$-gon of order $(q, \begin{bmatrix} d \\ 1 \end{bmatrix}_{q^2} - 1)$ with $c_i = (q^{2i} - 1)/(q^2 - 1)$, $\forall i \in \{1, \ldots, d\}$. The result now follows immediately from Lemma 6.7.7 and Definition 2.3.12. □

Theorems 6.7.3 and 6.7.8 both yield induced subgraphs of the dual polar graph on $H(2d-1, q^2)$, each with classical parameters (d, b, α, β) with $b < -1$. This particular dual polar graph itself also has such classical parameters (see Theorem 4.1.8). We now mention a related result by Weng.

Theorem 6.7.9. [169]*Let Γ denote a distance-regular graph with classical parameters (d, b, α, β) and $d \geq 4$. Suppose $b < -1$, and suppose the intersection numbers satisfy $c_2 > 1$ and $a_1 \neq 0$ (i.e. Γ has triangles). Then $b = -q$ with q a prime power, and precisely one of the following must hold.*

(i) *Γ is the dual polar graph on $H(2d-1, q^2)$ and*

$$(d, b, \alpha, \beta) = \left(d, -q, -\frac{q(q+1)}{q-1}, -\frac{q((-q)^d+1)}{q-1}\right).$$

(ii) *Γ is the Hermitian forms graph* $\mathrm{Her}(d, q)$ *and*

$$(d, b, \alpha, \beta) = (d, -q, -q-1, -(-q)^d - 1).$$

(iii) *q is odd and*

$$(d, b, \alpha, \beta) = \left(d, -q, -\left(\frac{q+1}{2}\right), -\left(\frac{(-q)^d+1}{2}\right)\right).$$

Remark 6.7.10. The assumption that $d \geq 4$ in the above theorem is indeed necessary, as the two sporadic regular near hexagons from Subsection 6.2.3 provide counterexamples. The large Witt graph has classical parameters $(3, -2, -4, 10)$ with $a_1 = 1$ and $c_2 = 3$. The point graph of the near hexagon related to the extended ternary Golay code has classical parameters $(3, -2, -3, 8)$ with $a_1 = 1$ and $c_2 = 2$.

Graphs with classical parameters $(d, -q, -(q+1)/2, -((-q)^d+1)/2)$ are known to exist for every odd prime power q if $d = 2$, as we have seen that we can

construct an $\mathrm{srg}((q+1)(q^3+1)/2, (q-1)(q^2+1)/2, (q-3)/2, (q-1)^2/2)$ from a hemisystem. No such graphs with $d \geq 3$ are known. It follows from Definition 2.3.12 that a graph of type (iii) from Theorem 6.7.9 has intersection number $a_1 = (q-3)/2$. Hence particularly interesting is the case $q = 3$, where the graph is triangle-free. Recall that for $d = 2$ and $q = 3$ one obtains the Gewirtz graph. We now mention the following conjecture by Pan, Lu and Weng, which says that "the Gewirtz graph does not grow".

Conjecture 6.7.11. [115, Conjecture 4.11] *There is no distance-regular graph for $d \geq 3$ with classical parameters*

$$(d, -3, -2, -((-3)^d + 1)/2).$$

The information in [168] and [169] strongly suggests that graphs with classical parameters $(d, b, \alpha, \beta) = (d, -q, -(q+1)/2, -((-q)^d+1)/2)$, if they exist, must be constructed as in Theorem 6.7.8.

We have already discussed the problem of finding m-ovoids in dual polar spaces at the end of Subsection 6.4.3. An intersection property was given in Corollary 6.4.31 for m-ovoids with respect to spreads in $W(2d-1, q)$ and $Q(2d, q)$ with d odd. We will now give a similar result for partial spreads of the maximum size $q^d + 1$ in $H(2d-1, q^2)$ with d odd. We postponed this result in order to mention the restriction on m from Theorem 6.7.5 first, and also because this might be useful when trying to find or prove non-existence of the desired $(q+1)/2$-ovoids in the dual polar space (we already mentioned a construction of such partial spreads in Subsection 4.4.4).

Corollary 6.7.12. *If S is a partial spread of size $q^d + 1$ in $H(2d-1, q^2)$ for odd $d \geq 3$, and T is a $(q+1)/2$-ovoid of the dual polar space (i.e. for every totally isotropic $(d-1)$-space, exactly half of the $q+1$ maximals through it are in T), then $|S \cap T| = (q^d + 1)/2$.*

Proof. We know from Theorem 6.4.26 that S is a $(q^d + 1)/(q + 1)$-tight set. The result now follows immediately from Theorem 6.4.9. □

6.8 Regular near pentagons

Until now, we have only discussed near $2d$-gons. Near $(2d + 1)$-gons are quite different, and were in fact not even included in the original definition of near

polygons from [133]. However, near pentagons were studied by Cameron [32] under the name *partial quadrangles* (see below for our precise definition).

6.8.1 Definitions and examples

Definition 6.8.1. *A partial quadrangle* $\mathrm{PQ}(s,t,\mu)$ *with* $s,t,\mu \geq 1$ *is a partial linear space satisfying the following axioms.*

(i) *Every line contains exactly* $s+1$ *points and every point is on exactly* $t+1$ *lines.*

(ii) *For every point* p *not on a line* ℓ*, there is at most one point on* ℓ *collinear with* p*.*

(iii) *If two points are not collinear, then exactly* μ *points are collinear with both.*

The partial quadrangles $\mathrm{PQ}(s,t,\mu)$ with $\mu = t+1$ are precisely the generalized quadrangles of order (s,t). A $\mathrm{PQ}(s,t,\mu)$ with $\mu < t+1$ is a regular near pentagon of order (s,t) with $c_2 = t_2 + 1 = \mu$. Conversely, every regular near pentagon is a partial quadrangle.

The point graph of a partial quadrangle $\mathrm{PQ}(s,t,\mu)$ is a strongly regular graph (see [32]):

$$\mathrm{srg}\Big(1 + s(t+1)\big(1 + \frac{st}{\mu}\big), s(t+1), s-1, \mu\Big).$$

Note that the maximal cliques in the point graph are precisely the sets of points on a line, and hence the partial quadrangle is determined by its point graph.

For a $\mathrm{PQ}(s,t,\mu)$ with $s = 1$, the point graph is a triangle-free strongly regular graph, the edges of which correspond to the lines of the partial quadrangle. Conversely, every triangle-free strongly regular graph $\mathrm{srg}(v,k,0,\mu)$ with $k \geq 2$ and $\mu \geq 1$ yields a $\mathrm{PQ}(1,k-1,\mu)$ when taking the vertices as points and the edges as lines, with incidence just symmetrized containment. An $\mathrm{srg}(v,k,0,\mu)$ with $\mu = k$ is just the complement of a union of two disjoint cliques of size k. Apart from such graphs, the only known triangle-free strongly regular graphs are those given in Table 6.1. These graphs are all characterized by their parameters (see also [27]).

Name	(v, k, λ, μ)
Pentagon	$(5, 2, 0, 1)$
Petersen graph	$(10, 3, 0, 1)$
Clebsch graph	$(16, 5, 0, 2)$
Hoffman-Singleton graph	$(50, 7, 0, 1)$
Gewirtz graph	$(56, 10, 0, 2)$
M_{22} graph	$(77, 16, 0, 4)$
Higman-Sims graph	$(100, 22, 0, 6)$

Table 6.1: The known triangle-free strongly regular graphs with $1 \leq \mu < k$

For more information on the Hoffman-Singleton graph, the M_{22} graph and the Higman-Sims graph, see for instance [23, Chapter 13]. We have already introduced the Petersen graph in Subsection 2.3.1 as the complement of the triangular graph $T(5)$, and the Gewirtz graph in Section 6.7. The Clebsch graph is isomorphic to the *folded* 5*-cube*. The vertices of the folded 5-cube are the sixteen 5-tuples in $\{0, 1\}^5$ with an even number of ones (i.e. with "even weight"), and with two vertices adjacent when differing in all but one position. Its complement is known as the *halved* 5*-cube*.

In this context, the smallest open case (in terms of the number of vertices) appears to be the existence of an $\mathrm{srg}(162, 21, 0, 3)$. See also [83] for a discussion of this problem.

We now derive the most important construction for partial quadrangles, due to Cameron [32]. Note that in order to avoid confusion with s, we use a parameter q, which need not be a prime power.

Theorem 6.8.2. *Consider a generalized quadrangle $\mathcal{P}$ of order (q, q^2), $q > 1$. For any point p, let $\mathcal{P}\backslash p^\perp$ denote the incidence structure, the points of which are those in $\mathcal{P}$ not collinear with p, the lines of which are those of $\mathcal{P}$, not through p, and with incidence inherited from $\mathcal{P}$. Then $\mathcal{P}\backslash p^\perp$ is a* $\mathrm{PQ}(q-1, q^2, q^2-q)$.

Proof. Every line in $\mathcal{P}$ not through p has exactly one point collinear with p, and hence is incident with exactly q points of $\mathcal{P}\backslash p^\perp$. If p_1 is not collinear with p in $\mathcal{P}$, then all $q^2 + 1$ lines through it are in $\mathcal{P}\backslash p^\perp$. It is clear that $\mathcal{P}\backslash p^\perp$ is also a partial linear space. If (p_1, ℓ_1) is a non-incident point-line pair in $\mathcal{P}\backslash p^\perp$, then there is a unique point p' in $\mathcal{P}$ on ℓ_1 collinear with p_1. Hence in $\mathcal{P}\backslash p^\perp$, p_1 is collinear with one or no point on ℓ_1, depending on whether p' is in $\mathcal{P}\backslash p^\perp$ or

not. Finally, if p_1 and p_2 are two non-collinear points in $\mathcal{P}\backslash p^\perp$, then there are q^2+1 points collinear with both, and it follows from Theorem 6.6.7 that in $\mathcal{P}$, precisely $q+1$ of them are collinear with p as well, and hence not in $\mathcal{P}\backslash p^\perp$. $\square$

This construction can be carried out for any prime power q, as there is always the dual polar space on $H(3,q^2)$ of order (q,q^2). In that specific case, it follows from Theorem 6.7.3 that the point graph of the partial quadrangle is isomorphic to $\mathrm{Her}(2,q)$, the vertices of which are the (2×2)-Hermitian matrices over $\mathrm{GF}(q^2)$, with two vertices adjacent if the rank of their difference is 1. We already mentioned the work by Ivanov and Shpectorov [100], which implies that, conversely, any $\mathrm{PQ}(q-1,q^2,q^2-q)$ with $q\geq 2$ (q not necessarily a prime power) can be constructed from some generalized quadrangle of order (q,q^2). Note that $\mathrm{Her}(2,2)$ is isomorphic to the Clebsch graph. Finally, we remark that Brouwer and Haemers [25] gave another short proof of Ivanov and Shpectorov's result for $d=2$, and obtain that there is only one $\mathrm{PQ}(2,9,6)$, and that $\mathrm{Her}(2,3)$ is the unique $\mathrm{srg}(81,20,1,6)$.

In its most general form, the following construction is due to Cameron [33]. We prove it as a consequence of Theorem 6.7.7.

Theorem 6.8.3. *Consider a generalized quadrangle $\mathcal{P}=(P,L,\mathrm{I})$ of order (q,q^2), $q>1$, and suppose S is a hemisystem of $\mathcal{P}$. The incidence structure $(S,L,((S\times L)\cup(L\times S))\cap\mathrm{I})$ is a $\mathrm{PQ}((q-1)/2,q^2,(q-1)^2/2)$.*

Proof. This is clearly a partial linear space with $(q+1)/2$ points on each line and q^2+1 lines through each point. If $p\in S$ is not on a line ℓ, then it is collinear in $\mathcal{P}$ with exactly one point on ℓ, which can be in S or not. Finally, the desired constant μ is the constant c_2' given by Theorem 6.7.7. $\square$

As mentioned earlier, if $\mathcal{P}$ is the dual polar space on $H(3,3^2)$, then the point graph of the constructed partial quadrangle $\mathrm{PQ}(1,9,2)$ is isomorphic to the Gewirtz graph.

We conclude this subsection by giving a list of the known partial quadrangles that are not generalized quadrangles:

- the triangle-free strongly regular graphs from Table 6.1,
- the partial quadrangles $\mathrm{PQ}(q-1,q^2,q^2-q)$ constructed as $\mathcal{P}\backslash p^\perp$ with $\mathcal{P}$ a generalized quadrangle of order (q,q^2), $q>1$ (see Theorem 6.8.2)

- the partial quadrangles $\mathrm{PQ}((q-1)/2, q^2, (q-1)^2/2)$ induced by a hemisystem of a generalized quadrangle of order (q, q^2), q odd and $q > 1$ (see Theorem 6.8.3)
- the three exceptional quadrangles related to the Coxeter 11-cap [50] ($\mathrm{PQ}(2, 10, 2)$), the Hill 56-cap [89] ($\mathrm{PQ}(2, 55, 20)$) or the Hill 78-cap [90] ($\mathrm{PQ}(3, 77, 14)$), all arising via linear representations (see for instance [153] for more information).

6.8.2 Parallelism in near pentagons

We say two lines of a partial quadrangle are *parallel*[4] if they are either equal, or skew with no lines intersecting both. Note that the latter is impossible in generalized quadrangles.

Lemma 6.8.4. *Consider a* $\mathrm{PQ}(s, t, \mu)$.

(i) *If two distinct lines are parallel, then* $\mu \leq st/(s+1)$.

(ii) *If* $\mu = st/(s+1)$, *and* ℓ *and* ℓ' *are distinct parallel lines with* p *a point on* ℓ, *then all points collinear with* p *and not on* ℓ *are collinear with a point on* ℓ'.

Proof. Consider two distinct parallel lines ℓ and ℓ' with p on ℓ. We count the number N of ordered pairs (a, p') with p' on ℓ' and a collinear with both p and p', in two ways. There are $s+1$ points on ℓ', and for each such point p', there are μ common neighbours to p and p' in the point graph. Hence $N = (s+1)\mu$. On the other hand, on each of the t lines through p, different from ℓ, there are s neighbours of p, each collinear with at most one point on ℓ'. Hence $N \leq st$, and equality holds if and only if every neighbour of p, not on ℓ, is collinear with a point on ℓ'. □

Note that the bound on μ from Lemma 6.8.4 is quite weak. The only known partial quadrangle that is not a generalized quadrangle and for which this bound is not satisfied, is the ordinary pentagon, which is a $\mathrm{PQ}(1, 1, 1)$ and (consequently) has no parallel lines. We will now consider the case of equality.

Theorem 6.8.5. *Consider a* $\mathrm{PQ}(s, t, \mu)$ *with* $\mu = st/(s+1)$.

[4] This should not be confused with the concept of parallelism in near $2d$-gons from [28].

(i) *Parallelism is an equivalence relation.*

(ii) *The parallel class $\mathcal{M}$ of any line ℓ has size at most $\mu + s + 1$. Equality holds if and only if through every point p, not collinear with any point on ℓ, there is a unique line in $\mathcal{M}$.*

(iii) *The matrix of eigenvalues P of the association scheme defined by the point graph Γ is:*

$$P = \begin{pmatrix} 1 & \mu(s+1)+s & (s+1)(\mu s+\mu+s) \\ 1 & s & -s-1 \\ 1 & -\mu-1 & \mu \end{pmatrix}.$$

(iv) *If ℓ_1 and ℓ_2 are two distinct parallel lines with sets of points S_1 and S_2, respectively, then $\chi_{S_1} - \chi_{S_2}$ is an eigenvector of s with respect to the point graph Γ. Every point p, not on any of these lines, is either collinear with one point of S_1 and one of S_2, or with none of S_1 and none of S_2.*

Proof.

(i) Suppose ℓ, ℓ' and ℓ'' are pairwise distinct lines, with $\{\ell, \ell'\}$ and $\{\ell', \ell''\}$ pairs of parallel lines. Suppose ℓ is not parallel to ℓ''. In that case, there must be a point p on ℓ collinear with some point p'' on ℓ''. As ℓ and ℓ' are parallel, it follows from Lemma 6.8.4(ii) that p'' must be collinear with a point on ℓ'. This contradicts the assumption that ℓ' and ℓ'' are parallel.

(ii) Let ℓ be a line in the parallel class $\mathcal{M}$. The number of points in the partial quadrangle is $1 + s(s+2)(t+1)$. The number of points on ℓ is $s+1$, and the number of those points not on ℓ but collinear with a unique point on ℓ is $(s+1)st$. Hence the number of points not collinear with any point on ℓ is $s(s+t+1) = (s+1)(\mu+s)$. These points are either on a unique or on no line parallel to ℓ. Hence the size of $\mathcal{M}$ is at most $(\mu+s)+1$, with equality if and only if all points of the last type are on a unique line in $\mathcal{M}$.

(iii) This follows immediately from Theorem 2.3.5.

(iv) Every point on one of the two lines is collinear with the s remaining points on that line, and at distance 2 from the $s+1$ points on the other

line. Hence the inner distribution $\mathbf{a}$ of the vector $\chi_{\ell_1} - \chi_{\ell_2}$ is $(1, s, -s-1)$. We now use Theorem 2.2.7 with respect to the last row of P:

$$\mathbf{a}_0 - \frac{\mu+1}{\mu s+\mu+s}\mathbf{a}_1 + \frac{\mu}{(s+1)(\mu s+\mu+s)}\mathbf{a}_2 = 0, \qquad \square$$

and thus $\chi_{\ell_1} - \chi_{\ell_2}$ is orthogonal to the last eigenspace. As $|S_1| = |S_2| = s+1$, this vector is also orthogonal to the all-one vector, and hence it is an eigenvector of s. The last part follows from Lemma 6.8.4(ii).

For the partial quadrangles $\mathcal{P}\backslash p^\perp$ constructed from generalized quadrangles $\mathcal{P}$ of order (q, q^2), the construction of an eigenvector as in Theorem 6.8.5(iv) was already observed in [4, Corollary 6.1].

Almost all known examples of partial quadrangles $\mathrm{PQ}(s, t, \mu)$ satisfying $\mu = st/(s+1)$ are those of the form $\mathcal{P}\backslash p^\perp$ as in Theorem 6.8.2, with $s = q-1, t = q^2, \mu = q^2-q$. The partial quadrangles $\mathrm{PQ}(s, q, \mu)$ with $\mu = sq/(s+1)$, the dual of which can be embedded in the projective space $\mathrm{PG}(3, q)$, are isomorphic to $\mathcal{P}\backslash p^\perp$ with $\mathcal{P}$ the dual polar space on $H(3, q)$, or correspond to the Petersen graph if $q = 2$ (see [59]).

We now consider those $\mathrm{PQ}(s, t, \mu)$ with $s = 1$, i.e. corresponding to the triangle-free strongly regular graphs.

Lemma 6.8.6. *If Γ is an $\mathrm{srg}(v, t+1, 0, \mu)$ with $\mu = t/2 \geq 1$, then Γ is either the Petersen graph (with $\mu = 1$) or the Clebsch graph (with $\mu = 2$).*

Proof. It follows from Theorem 2.3.5 that the multiplicity of the eigenvalue $-\mu-1$ of the point graph is given by $8 - 12/(\mu+2)$, and thus $\mu \in \{1, 2, 4, 10\}$. We know from the above that the Petersen graph and the Clebsch graph are the only possibilities for $\mu = 1$ and $\mu = 2$, respectively, while the existence of an $\mathrm{srg}(28, 9, 0, 4)$ or an $\mathrm{srg}(64, 21, 0, 10)$ is ruled out by the absolute bound (see Theorem 2.3.5) with $f = 6$ and $f = 7$, respectively. $\square$

Constructing the Petersen graph as the complement of the triangular graph $T(5)$ for the set $\{1, 2, 3, 4, 5\}$, one easily sees that parallel lines (i.e. edges) indeed exist: take for instance the edges $\{\{1, 2\}, \{3, 4\}\}$ and $\{\{1, 3\}, \{2, 4\}\}$. Using the construction of the Clebsch graph from the above, we also easily find parallel edges here: take for instance the edges $\{(0, 0, 0, 0, 0), (1, 1, 1, 1, 0)\}$ and $\{(1, 1, 0, 0, 0), (0, 0, 1, 1, 0)\}$. In both cases, one can verify that indeed every vertex, not on any of the two edges, is adjacent to either one vertex of each edge, or none of each edge, as implied by Theorem 6.8.5(iv).

6.8.3 Subsets in near pentagons

Intriguing sets in the point graphs of partial quadrangles were studied in depth by Bamberg, De Clerck and Durante [6]. We will now consider similar concepts.

Theorem 6.8.7. *Consider a* $\mathrm{PQ}(s,t,\mu)$ *with* $\mu = st/(s+1)$. *If for a set of points* S, χ_S *is orthogonal to the eigenspace for* s *of the point graph, then every two parallel lines intersect* S *in the same number of points.*

Proof. If two lines ℓ_1 and ℓ_2 with sets of points S_1 and S_2, respectively, are parallel, then we know from Theorem 6.8.5(iv) that $\chi_{S_1} - \chi_{S_2}$ is an eigenvector of s. Lemma 2.2.10 now yields:

$$(\chi_S)^T(\chi_{S_1} - \chi_{S_2}) = 0,$$

and hence $|S \cap S_1| = |S \cap S_2|$. □

If the parallel classes are "sufficiently large", we can also prove a converse.

Theorem 6.8.8. *Consider a* $\mathrm{PQ}(s,t,\mu)$ *with* $\mu = st/(s+1)$ *and with each parallel class of size* $\mu+s+1$. *Every two parallel lines intersect a point set* S *in the same number of points, if and only if* χ_S *is orthogonal to the eigenspace for* s *of the point graph. In that case, every point* p *is collinear with* $|S|/(s+1)-\mu-1$ *points in* S *if* $p \in S$, *and to* $|S|/(s+1)$ *if* $p \notin S$, *and* $|S|(t+1)$ *must be divisible by* $\mu+s+1$.

Proof. If χ_S is orthogonal to the eigenspace for s of the point graph, then it follows from Theorem 6.8.7 that every two parallel lines intersect S in the same number of points.

Now suppose for the remainder of this proof that every two parallel lines intersect S in the same number of points. Let p be any point, and let x denote the number of points in S collinear with it. Let N_1 denote the number of ordered triples (p_1, ℓ_1, ℓ_2) with $p_1 \in S$, with p and p_1 on ℓ_1, and with ℓ_1 distinct from and parallel to ℓ_2. There are $x+t+1$ or x possibilities for such (p_1, ℓ_1), if p is in S or not, respectively. As each parallel class has size $\mu+s+1$, then there are $\mu+s$ possibilities for ℓ_2. Now let N_2 denote the number of ordered triples (p_2, ℓ_1, ℓ_2) with $p_2 \in S$, p on ℓ_1, p_2 on ℓ_2, and with ℓ_1 parallel to and distinct from ℓ_2. Here p cannot be collinear with p_2, and hence we have $|S|-x-1$ or $|S|-x$ possibilities for p_2, if p is in S or not, respectively. Through such a point p_2 there are precisely $(t+1)-\mu$ lines such

that p is collinear with no point on them, and then it follows from Theorem 6.8.5(ii) that there is a unique line through p parallel to it. However, if for any pair of distinct, parallel lines (ℓ_1, ℓ_2) with ℓ_1 through p, we let $f_1(\ell_1, \ell_2)$ and $f_2(\ell_1, \ell_2)$ denote the number of points in S on ℓ_1 and on ℓ_2 respectively, then $f_1(\ell_1, \ell_2) = f_2(\ell_1, \ell_2)$ because of the assumption, and hence:

$$N_1 = \sum_{(\ell_1,\ell_2)} f_1(\ell_1, \ell_2) = \sum_{(\ell_1,\ell_2)} f_2(\ell_1, \ell_2) = N_2,$$

and thus:

$$\begin{aligned} (x+t+1)(\mu+s) &= (|S|-x-1)(t+1-\mu) \qquad \text{if } p \in S, \\ x(\mu+s) &= (|S|-x)(t+1-\mu) \qquad \text{if } p \notin S. \end{aligned}$$

This yields $x = |S|/(s+1) - \mu - 1$ if $p \in S$ and $x = |S|/(s+1)$ if $p \notin S$. Hence S is intriguing with parameters (h_1, h_2) with $h_1 - h_2 = -\mu - 1$, and so now the desired orthogonality follows from Lemma 2.1.3 and from the eigenvalues from Theorem 6.8.5(iii).

Now count ordered pairs (p, ℓ) with $p \in S$ and ℓ through p. Through each of the $|S|$ points in S, there are $t+1$ lines. On the other hand, the assumption implies that for every parallel class $\mathcal{M}$, the number of such ordered pairs (p, ℓ) with $\ell \in \mathcal{M}$ must be divisible by $\mu + s + 1$, the size of the parallel class. Hence $|S|(t+1)$ is a sum of multiples of $\mu + s + 1$. □

We now discuss the intriguing sets from Theorem 6.8.8 in those $\mathrm{PQ}(s, t, \mu)$ determined in Lemma 6.8.6. Much more information on intriguing sets in the known triangle-free strongly regular graphs in general can be found in [6].

As our first example, we consider the Petersen graph on 10 vertices (with $s = 1$ and $\mu = 1$) once more. For any edge $\{\{a, b\}, \{c, d\}\}$, one easily verifies that the remaining edges in its parallel class are $\{\{a, c\}, \{b, d\}\}$ and $\{\{a, d\}, \{b, c\}\}$, and hence the size of each class is $\mu + s + 1 = 3$. We now consider the non-empty intriguing sets S with a characteristic vector orthogonal to the eigenspace for $s = 1$. We may assume $2|S| \leq 10$, as the complement is also such a set. It follows from Theorem 6.8.8 that S must be intriguing with parameters $(|S|/2 - 2, |S|/2)$, and thus S must be a coclique in the point graph of size 4. Hence S consists of four pairs in $\{1, 2, 3, 4, 5\}$ with no two pairs disjoint, and one now easily verifies that S must consist of all 4 pairs containing a fixed singleton (use for instance the Erdős-Ko-Rado Theorem 5.1.2.)

Now consider the Clebsch graph on 16 vertices (with $s = 1$ and $\mu = 2$). One easily verifies that each parallel class consists of the $\mu + s + 1 = 4$ edges, consisting of vertices with a fixed entry in a fixed position. For instance, the following 4 edges form one parallel class:

$$\{(0,0,0,0,0),(1,1,1,1,0)\},\{(1,1,0,0,0),(0,0,1,1,0)\},$$

$$\{(1,0,1,0,0),(0,1,0,1,0)\},\{(1,0,0,1,0),(0,1,1,0,0)\}.$$

We again consider the non-empty intriguing sets S with a characteristic vector orthogonal to the eigenspace for $s = 1$, and now with $2|S| \leq 16$. It follows from Theorem 6.8.8 that S must have parameters $(|S|/2 - 3, |S|/2)$, and that $5|S|$ is divisible by $\mu + s + 1 = 4$. Hence $|S| = 8$, and each element p_1 of S differs from exactly one other element of S in all but one position. In that case Theorem 6.8.8 yields that all 4 edges in the parallel class defined by the equal entry in that specific position must contain 2 elements of S. One now easily sees that these intriguing sets are precisely the 10 sets of size 8 with a fixed entry in some fixed position.

For partial quadrangles constructed from generalized quadrangles of order (q, q^2), we obtain a simpler divisibility condition.

Corollary 6.8.9. *Let $\mathcal{P}$ be a generalized quadrangle of order (q, q^2), $q > 1$, and consider the partial quadrangle $\mathcal{P}\backslash p^\perp$ for some point p. If S is a set of points in $\mathcal{P}\backslash p^\perp$, then the following are equivalent:*

- *every two lines ℓ_1 and ℓ_2 of $\mathcal{P}$, not through p but intersecting in a point collinear with p, contain the same number of elements of S,*
- *χ_S is orthogonal to the eigenspace for $q - 1$ of the point graph of $\mathcal{P}\backslash p^\perp$.*

and in that case, $|S|$ must be divisible by q^2.

Proof. The partial quadrangle $\mathcal{P}\backslash p^\perp$ is a $\mathrm{PQ}(s, t, \mu)$, with $s = q - 1, t = q^2, \mu = q^2 - q$, and with two distinct lines parallel if and only if they intersect in $\mathcal{P}$ in a point collinear with p. Through each point of $\mathcal{P}$ collinear with p, there are q^2 lines of $\mathcal{P}\backslash p^\perp$, and hence the size of every parallel class is $q^2 = \mu + s + 1$. Theorem 6.8.8 now immediately yields the equivalence, and implies that $|S|(q^2 + 1)$ is divisible by q^2. □

For a partial quadrangle $\mathcal{P}\backslash p^\perp$ with $\mathcal{P}$ a generalized quadrangle of order (q, q^2), $q > 1$, the simplest example of a set of points with a characteristic vector

orthogonal to the eigenspace of $q-1$ is a *cone*: the set of q^3 points in $\mathcal{P}\backslash p^\perp$ collinear with a fixed point p_0 of $\mathcal{P}$, with p_0 collinear with p in $\mathcal{P}$ (see [6, Lemma 5.7]).

Corollary 6.8.10. *A non-empty set S of mutually non-collinear points in a* $\mathrm{PQ}(s,t,\mu)$ *with $\mu = st/(s+1)$ has size at most $(\mu+1)(s+1)$, and this bound is attained if and only if S is intriguing in the point graph.*

Proof. The eigenvalues of the point graph were given in Theorem 6.8.5(iii). The bound, and orthogonality of χ_S to the eigenspace for s in case of equality, now follow immediately from Lemma 2.2.9, and the intriguing property from Lemma 2.1.3 and Theorem 6.8.8. □

Remark 6.8.11. Thas [151] proved that a partial spread of $H(3,q^2)$ (i.e. a partial distance-2-ovoid of the corresponding dual polar space) has size at most q^3-q^2+q+1, and remarked that the proof works for partial distance-2-ovoids in any generalized quadrangle of order (q,q^2) with $q>1$. If S is a partial distance-2-ovoid in a generalized quadrangle $\mathcal{P}$ of order (q,q^2), $q>1$, with $p\in S$, then we obtain a coclique $S\backslash\{p\}$ in the point graph of the partial quadrangle $\mathcal{P}\backslash p^\perp$. Applying Corollaries 6.8.9 and 6.8.10, we see that $|S|-1$ is at most $(\mu+1)(s+1)=q^3-q^2+q$, and is divisible by q^2 in case of equality. Hence the bound q^3-q^2+q+1 for $|S|$ cannot be attained. Other combinatorial arguments, such as counting the points collinear with p and at least one other element of S, also yield a contradiction. However, more recently the sharper bound $(q^3+q+2)/2$ for the dual polar space on $H(3,q^2)$ was proved in [51]. That proof actually works for partial distance-2-ovoids in all generalized quadrangles of order (q,q^2), $q>1$ (De Beule, personal communication).

Appendix A

A geometric proof for partial spreads in $H(2d-1,q^2)$ for odd d

One of the main results in this thesis is the tight upper bound for partial spreads in $H(2d-1,q^2)$ for odd d from Theorem 4.4.16. Here we will give an alternative proof, together with a characterization. The proof is completely geometric, although heavily inspired by concepts from algebraic graph theory, especially 1-regularity of codes (see Theorem 6.4.26).

This alternative proof was accepted for publication in *Advances in Mathematics of Communications* [163].

A.1 Triples of disjoint generators in $H(2d-1,q^2)$

We will only use vectorial dimensions. We also refer to 1-spaces and 2-spaces as *points* and *lines*, respectively.

The following beautiful lemma is due to Thas.

Lemma A.1.1. [152, pp. 538-539]*Let π_1, π_2 and π be three mutually disjoint generators in $H(2d-1,q^2)$. The set of points on π_1, that are on a (necessarily unique) line of $H(2d-1,q^2)$ intersecting both π and π_2 in a point, form a non-singular Hermitian variety $H(d-1,q^2)$ in π_1.*

Corollary A.1.2. *Let π_1, π_2 and π be three mutually disjoint generators in $H(2d-1,q^2)$. The number of generators intersecting π in a $(d-1)$-space, and*

intersecting both π_1 and π_2 in a point is

$$|H(d-1,q^2)| = \frac{(q^d-(-1)^d)(q^{d-1}+(-1)^d)}{q^2-1}.$$

Proof. For any point $p \in \mathrm{PG}(2d-1,q^2)$, we write $p^\perp$ for the subspace of $V(2d,q^2)$ orthogonal to p with respect to the associated Hermitian form.

It is obvious that every generator intersecting π in a $(d-1)$-space, can intersect π_1 and π_2 in at most one point. On the other hand, through any point $p_1 \in \pi_1$, there is a unique generator $\langle p_1, p_1^\perp \cap \pi\rangle$ intersecting π in a $(d-1)$-space. Hence we have to determine the number of points $p_1 \in \pi_1$ such that the generator $\langle p_1, p_1^\perp \cap \pi\rangle$ also intersects π_2 in a point.

First suppose that a point $p_1 \in \pi_1$ is such that the generator $\langle p_1, p_1^\perp \cap \pi\rangle$ intersects π_2 in a point p_2. In that case, the line p_1p_2 is a line of $H(2d-1,q^2)$, intersecting π as well, as $p_1^\perp \cap \pi$ is a hyperplane of $\langle p_1, p_1^\perp \cap \pi\rangle$. Conversely, suppose a point $p_1 \in \pi_1$ is on a line of $H(2d-1,q^2)$, intersecting π in p and π_2 in p_2. In that case, both p_1 and p are in the generator $\langle p_1, p_1^\perp \cap \pi\rangle$, and hence the entire line p_1p, including the point p_2, is in the generator. The desired result now follows from Lemma A.1.1. □

A.2 The proof

Theorem A.2.1. *Suppose S is a partial spread in $H(2d-1,q^2)$, d odd and $d \geq 3$. Then $|S|$ is at most q^d+1. If $|S|>1$ and $\pi \in S$, then every generator intersecting π in a $(d-1)$-space intersects the same number of other elements of S in just a point, if and only if $|S| = q^d+1$. In that case, that number must be q^{d-1}.*

Proof. Let S be a partial spread of size at least 2 in $H(2d-1,q^2)$. Consider a fixed element $\pi \in S$. Let $\{N_i | i \in I\}$ be the set of generators intersecting π in a $(d-1)$-space. As the number of $(d-1)$-spaces in a generator equals $(q^{2d}-1)/(q^2-1)$, and the number of generators through any $(d-1)$-space in $H(2d-1,q^2)$ is given by $q+1$, the cardinality of I is $q(q^{2d}-1)/(q^2-1)$.

Note that any generator N_i and any generator in $S\backslash\{\pi\}$, are either disjoint or intersect in a point. For every $N_i, i \in I$, let t_i denote the number of generators in $S\backslash\{\pi\}$, intersecting N_i in a point. We now count in two ways the number

of pairs (N_i, π'), with π' an element of $S\backslash\{\pi\}$ intersecting N_i in a point. As through every point p' on an element π' of $S\backslash\{\pi\}$, there is a unique generator intersecting π in a $(d-1)$-space, we obtain:

$$\sum_{i\in I} t_i = (|S|-1)\frac{q^{2d}-1}{q^2-1}.$$

Now we count the number of ordered triples (N_i, π_1, π_2), with π_1 and π_2 two distinct elements of $S\backslash\{\pi\}$, both intersecting N_i in a point. We know from Corollary A.1.2 that for every two distinct elements of $S\backslash\{\pi\}$, there will be exactly $|H(d-1, q^2)|$ generators N_i, intersecting both of them in a point. Hence we obtain:

$$\sum_{i\in I} t_i(t_i-1) = (|S|-1)(|S|-2)\frac{(q^d+1)(q^{d-1}-1)}{q^2-1}.$$

Combining the above, we find:

$$\sum_{i\in I} t_i^2 = (|S|-1)\frac{q^d+1}{q^2-1}\Big((q^d-1)+(|S|-2)(q^{d-1}-1)\Big).$$

As $(\sum_{i\in I} t_i)^2 \leq (\sum_{i\in I} t_i^2)|I|$, with equality if and only if all t_i are equal, this implies:

$$(|S|-1)^2\left(\frac{q^{2d}-1}{q^2-1}\right)^2 \leq (|S|-1)\frac{q^d+1}{q^2-1}\Big((q^d-1)+(|S|-2)(q^{d-1}-1)\Big)q\frac{q^{2d}-1}{q^2-1},$$

with equality if and only if all t_i are equal. Since we assumed that $|S| > 1$, we can cancel factors on both sides to obtain:

$$(|S|-1)(q^d-1) \leq \Big((q^d-1)+(|S|-2)(q^{d-1}-1)\Big)q, \qquad \square$$

implying that $|S| \leq q^d+1$, with equality if and only if all t_i are equal. In that case, their constant value must equal $(\sum_{i\in I} t_i)/|I| = (|S|-1)/q = q^{d-1}$.

A.3 Remarks

Remark A.3.1. This technique fails for partial spreads in $H(2d-1, q^2)$ with d even, where it yields a negative lower bound on the size instead.

Remark A.3.2. Corollary A.1.2 in fact already follows from the parameters of the dual polar graph on $H(2d-1, q^2)$. The dual polar space on $H(2d-1, q^2)$ is a regular near $2d$-gon of order $(q, (q^{2d}-1)/(q^2-1)-1)$ (see Subsection 6.2.2) and hence the desired constant is the so-called triple intersection number from Corollary 6.6.7.

Appendix B

Open problems

The following problems have caught my attention during my research. They are either old problems related to the topics in this thesis, problems that might be solved by use of similar techniques, or possible improvements of this work.

We refer to spaces with vectorial dimension k as *k-spaces*. A classical finite polar space has rank d if the maximal totally isotropic subspaces or *maximals* are d-spaces.

Note that two notations are often used for the different types of polar spaces (see Sections 1.3 and 4.1):

$$Q^+(2d-1,q) = D_d(q), H(2d-1,q^2) = {}^2A_{2d-1}(q), Q(2d,q) = B_d(q),$$

$$W(2d-1,q) = C_d(q), H(2d,q^2) = {}^2A_{2d}(q), Q^-(2d+1,q) = {}^2D_{d+1}(q).$$

Problem 1. *Are there any $t-(n,k,1;q)$-designs with $2 \leq t < k < n$?*

A $t-(n,k,\lambda;q)$-design is a set of k-spaces in $V(n,q)$, such that each t-space is in exactly λ of its elements. See Subsection 3.3.1 for a discussion. For $t \geq 2$, non-trivial designs are quite hard to construct. The $t-(n,k,1;q)$-designs are also known as the Steiner structures $S_q[t,k,n]$. For $t=1$, they are known as the spreads, which exist if and only if k divides n. No examples of Steiner structures $S_q[t,k,n]$ with $2 \leq t < k < n$ are known. A recent discussion of this problem can be found in [79]. Even the case $n=7, k=3, t=2, q=2$

(hence the existence of a set of 381 planes in $\mathrm{PG}(6,2)$ such that each line is in exactly one of its elements) is still open. An important partial non-existence result on Steiner structures $S_q[t,k,n]$ was obtained by Thomas [155].

Problem 2. *What is the maximum size of a partial spread in the polar space $H(2d-1,q^2)$ for even d?*

A partial spread in a classical finite polar space is a set of pairwise trivially intersecting maximals. In other words, a partial spread is a code in the dual polar graph with minimum distance equal to the rank d. Thas [152] proved that in $H(2d-1,q^2)$, their size is less than $q^{2d-1}+1$, i.e. they cannot partition the set of isotropic 1-spaces. One of the main new results in this thesis is the upper bound of q^d+1 for odd d (see Theorems 4.4.16 and 6.4.26). This bound is tight as a construction for partial spreads of size q^d+1 in $H(2d-1,q^2)$ is known for every d. Recently, results for even d were obtained in [51]: an upper bound of $(q^3+q+2)/2$ for $d=2$, and of $q^{2d-1}-q^{(3d+1)/2}+q^{3d/2}$ for even $d \geq 4$.

Problem 3. *Can the polar space $Q(2d,q)$ have spreads for odd $d \geq 5$ if q is odd?*

A spread in a classical finite polar space is a set of maximals, such that each isotropic 1-space is in exactly one of its elements. In the polar space $Q(2d,q)$, these are the sets of size q^d+1, consisting of pairwise trivially intersecting maximals. Theorems 6.4.29 and 6.4.30 provide information on such spreads for odd d, and the problem of their existence is also discussed in the preceding paragraphs. Thas [152] proved that $Q(2d,q)$ has spreads for all $d \geq 2$ if q is even, and has no spreads for even $d \geq 2$ if q is odd. Nothing is known for odd $d \geq 5$ if q is odd. The polar space $W(2d-1,q)$ has spreads for every prime power q and has the same rank and parameters (q,q) as $Q(2d,q)$, but is only isomorphic to it for even q.

Problem 4. *What is the maximum size of a set of pairwise non-trivially intersecting maximals in the polar space $H(2d-1,q^2)$ for odd $d \geq 5$?*

Sets of pairwise non-trivially intersecting sets of maximals, or *Erdős-Ko-Rado (EKR) sets of maximals*, were studied in Chapter 5. In most cases, the set of maximals through a fixed isotropic 1-space is the unique construction of maximum size, known as the *point-pencil construction*. However, the upper bound from Theorem 5.3.1 for $H(2d-1, q^2)$, d odd, is much larger. The EKR sets of maximals of maximum size for $d = 3$ were classified in Theorem 5.9.4, but we do not know if the point-pencil construction is of maximum size for odd $d \geq 5$. The EKR sets of maximals of maximum size in all other classical finite polar spaces of rank $d \geq 3$ were characterized (see Section 5.10 for a summary).

Problem 5. *Can the polar space $Q(2d, q)$ or $W(2d-1, q)$, $d = 2^m - 1$ and $m \geq 3$, have a perfect 1-code of maximals?*

In this context, a perfect e-code is a set S of maximals, such that for each maximal in the polar space, there is a unique element of S intersecting it in a subspace of dimension at least $d - e$. In most cases, its existence is excluded by the parameters of the polar space itself, by use of Lloyd's Theorem (see Theorem 2.3.9(v)). Chihara [44] proved that no non-trivial perfect e-codes can be found (i.e. different from a singleton or the full set of maximals), except in the polar spaces $Q(2d, q)$ or $W(2d-1, q)$ with $d = 2^m - 1$ and $e = 1$ in both cases. For $d = 3$, this comes down to a set of $q^3 + 1$ pairwise trivially intersecting maximals, i.e. a spread. Thas [145] proved that spreads always exist for $W(5, q)$. The case $Q(6, q)$, q odd, is not completely settled yet. Nothing seems to be known for $d = 2^m - 1$ if $m \geq 3$. Note that a perfect 1-code of maximals in $Q(2d, q)$ or $W(2d-1, q)$ is the same as a set of $(q+1)\cdots(q^d+1)(q-1)/(q^{d+1}-1)$ maximals, no two distinct elements of which intersect in a subspace of dimension $d-1$ or $d-2$.

Problem 6. *In a classical finite polar space of rank at least three, are there any non-trivial combinatorial designs of maximals with respect to t-spaces if $t \geq 2$?*

Here, a combinatorial design of maximals with respect to t-spaces is a set S of maximals, such that every totally isotropic t-space is in exactly m elements of

S for some m. Note that if this holds for t, then it also holds for $t' \leq t$. See Subsection 4.4.2 and in particular Theorem 4.4.1. Trivial examples for $t = d-1$ are the empty set, the full set, and one of the two types of maximals in the bipartite dual polar graph on $Q^+(2d-1, q)$. In a classical finite polar space of rank d, such designs with $t = d-1$ are precisely the m-ovoids of the dual polar space. The case $d = 3, t = 2$ and $m = 1$ is especially interesting, as one obtains partial geometries in this case, as explained at the end of Subsection 6.4.3. Here, the unsettled cases are $Q^-(7, q)$ and $H(6, q^2)$ with $q \geq 3$ in both cases, where one would obtain a $\mathrm{pg}(q^3, q(q+1), q+1)$ and a $\mathrm{pg}(q^5, q^2(q^2+1), q^2+1)$, respectively. The problem of finding 1-ovoids in dual polar spaces for rank three is equivalent to that of finding a partial geometry, the point graph of which is isomorphic to the polar graph on the isotropic 1-spaces.

Problem 7. *Do any $(q+1)/2$-ovoids in the dual polar space on $H(2d-1, q^2)$, q odd, exist if $d \geq 3$?*

This problem is a specific case of Problem 6. A $(q+1)/2$-ovoid is a set of maximals in $H(2d-1, q^2)$ such that each totally isotropic $(d-1)$-space is in exactly $(q+1)/2$ of its elements. For $d = 2$, such a $(q+1)/2$-ovoid is known as a hemisystem, and Thas [152] proved that they induce strongly regular graphs. This was generalized in Theorem 6.7.8, where it is proved that $(q+1)/2$-ovoids in the dual polar space on $H(2d-1, q^2)$ induce a distance-regular graph with classical parameters $(d, b, \alpha, \beta) = (d, -q, -(q+1)/2, -((-q)^d+1)/2)$. No such graphs are known with diameter $d \geq 3$. Cossidente and Penttila [49] recently proved that hemisystems for $H(3, q^2)$ exist for all odd q. If such a $(q+1)/2$-ovoid exists for $H(2d-1, q^2)$ with $d \geq 3$, then a $(q+1)/2$-ovoid also exists in $H(2d'-1, q^2)$ with $2 \leq d' \leq d$. For $q = 3$, the induced graph is triangle-free, and it was conjectured in [115] that graphs with classical parameters $(d, -3, -2, -((-3)^d+1)/2)$ do not exist if $d \geq 3$.

Problem 8. *Are all distance-regular graphs with classical parameters*

$$(d, b, \alpha, \beta) = (d, -q, -(q+1)/2, -((-q)^d+1)/2), q \text{ odd},$$

subgraphs of the dual polar graph on $H(2d-1, q^2)$ (for sufficiently large diameter)?

This problem is related to Problem 7, and asks when Theorem 6.7.8 provides the only construction for such graphs. Weng [169] proved that, under certain assumptions, classical parameters (d, b, α, β) with $b < -1$ fall into three classes (see Theorem 6.7.9). A lot of structural information on such graphs with classical parameters $(d, b, \alpha, \beta) = (d, -q, -(q+1)/2, -((-q)^d + 1)/2)$ comes with the proof, including the appearance of projective geometries $\mathrm{PG}(n, q^2)$ as posets of subgraphs (see the proofs of [168, Theorem 4.2] and [169, Theorem 10.3]).

Appendix C

Nederlandstalige samenvatting

In deze Nederlandstalige samenvatting zullen we kort de resultaten in deze thesis bespreken. Hiertoe zullen ook de belangrijkste definities herhaald worden. We zullen de structuur volgen van de Engelstalige tekst. Bij de resultaten zal ook steeds de nummering van de stelling in die tekst tussen haakjes vermeld worden, waar het bewijs kan gevonden worden. In sommige gevallen is de stelling uit de Engelstalige versie niet volledig overgenomen met het oog op bondigheid.

C.1 Incidentiemeetkundes

In dit hoofdstuk geven we een overzicht van de belangrijkste incidentiemeetkundes. Het is ons doel om combinatorische eigenschappen van deze structuren en hun deelstructuren te vinden.

Een *incidentiemeetkunde van rang* n is een geordende verzameling $(S, \mathrm{I}, \Delta, \sigma)$, waarbij S een niet-ledige verzameling van variëteiten is, I een binaire symmetrische *incidentierelatie*, Δ een eindige verzameling van grootte n en σ een surjectieve *type-afbeelding*, zodanig dat geen twee variëteiten van hetzelfde type incident zijn. Een *vlag* is een verzameling van paarsgewijs incidente variëteiten, en het type van een vlag is zijn beeld onder σ.

Een *punt-rechte* meetkunde is een incidentiemeetkunde van rang 2, waarbij de variëteiten van de twee types respectievelijk de *punten* en *rechten* zijn. Een dergelijke meetkunde, met punten- en rechtenverzamelingen respectievelijk P

en L, en met incidentierelatie I, zullen we ook met (P, L, I) noteren. Wanneer een punt en een rechte incident zijn, dan zeggen we dat de rechte *door het punt gaat* of *het punt bevat*, of dat het punt *op de rechte ligt*. Twee verschillende punten die incident zijn met eenzelfde rechte noemen we *collineair*, en twee rechten incident met een gemeenschappelijk punt zijn *snijdend*. Een punt-rechte meetkunde is een *partieel lineaire ruimte* wanneer elke twee verschillende punten incident zijn met ten hoogste één gemeenschappelijke rechte en elke rechte ook incident is met minstens twee punten. Een partieel lineaire ruimte is een *lineaire ruimte* wanneer elke twee verschillende punten op precies één rechte liggen.

C.1.1 Projectieve meetkundes

De projectieve meetkunde $\mathrm{PG}(n, \mathbb{K})$ is de incidentiemeetkunde $(S, \mathrm{I}, \Delta, \sigma)$ van rang n, afgeleid van een linkse vectorruimte $V(n+1, \mathbb{K})$ van dimensie $n+1$ over een delingsring $\mathbb{K}$. De verzameling S bestaat uit de deelruimten $V(n+1, \mathbb{K})$ die niet de triviale of volledige vectorruimte zijn, I is symmetrische strikte inclusie, Δ is $\{1, \ldots, n\}$, en σ beeldt iedere deelruimte af op haar vectoriële[1] dimensie over $\mathbb{K}$. De projectieve meetkunde $\mathrm{PG}(n, \mathrm{GF}(q))$ over een eindig veld $\mathrm{GF}(q)$ met als grootte een priemmacht q, zullen we ook noteren met $\mathrm{PG}(n, q)$.

C.1.2 Polaire ruimten

De eindige klassieke polaire ruimten worden opgebouwd aan de hand van niet-singuliere kwadratische vormen en niet-singuliere bilineaire vormen, met name de symmetrische, alternerende en Hermitische vormen. De *totaal isotrope deelruimten* zijn de deelruimten waarop de vorm verdwijnt. De *Witt index* is de maximale dimensie van totaal isotrope deelruimten, die we de *generatoren* noemen. De geassocieerde klassieke eindige polaire ruimte van rang n is dan de incidentiemeetkunde $(S, \mathrm{I}, \Delta, \sigma)$, met S de verzameling van totaal isotrope deelruimten, I de symmetrische strikte inclusie, Δ de verzameling $\{1, \ldots, n\}$, en σ de afbeelding die iedere deelruimte op haar dimensie afbeeldt.

Aan de hand van vormen met Witt index d op een vectorruimte V over $\mathrm{GF}(q)$ kunnen we de volgende klassieke eindige polaire ruimten van rang d opbouwen: de hyperbolische kwadriek $Q^+(2d-1, q)$ (met f kwadratisch op $V(2d, q)$), de

[1] Tenzij anders vermeld zijn de dimensies steeds vectorieel en niet projectief in deze thesis.

parabolische kwadriek $Q(2d, q)$ (met f kwadratisch op $V(2d+1, q)$), de elliptische kwadriek $Q^-(2d+1, q)$ (met f kwadratisch op $V(2d+2, q)$), de symplectische ruimte $W(2d-1, q)$ (met f alternerend op $V(2d, q)$), en de Hermitische variëteiten $H(2d-1, q^2)$ en $H(2d, q^2)$ (met f Hermitisch op respectievelijk $V(2d, q^2)$ en $V(2d+1, q^2)$).

De polaire ruimten van rang twee of *veralgemeende vierhoeken* worden in het algemeen gedefinieerd als partieel lineaire ruimten die voldoen aan de volgende axioma's.

(i) Voor elk punt p dat niet op een rechte ℓ ligt, is er een uniek punt op ℓ collineair met p.

(ii) Elk punt is incident met minstens twee rechten.

Een veralgemeende vierhoek is van orde (s, t) met $s, t \geq 1$ als iedere rechte incident is met juist $s+1$ punten, en elk punt met juist $t+1$ rechten. Een dergelijke veralgemeende vierhoek noteren we dan met $\mathrm{GQ}(s, t)$.

Tits [158] bewees dat alle eindige polaire ruimten van rang minstens drie klassiek zijn. De classificatie van de veralgemeende vierhoeken lijkt echter een hopeloos probleem.

C.1.3 SPBIBD

De *SPBIBDs* of *speciaal partieel gebalanceerde incomplete blok designs* zijn een specifiek soort eindige punt-rechte meetkundes, waaronder in het bijzonder de partiële meetkundes $\mathrm{pg}(s, t, \alpha)$.

C.2 Associatieschema's

De concepten en technieken uit de algebraïsche grafentheorie die wij zullen toepassen op meetkundige structuren, worden uitvoerig besproken in Hoofdstuk 2, dat nadien voortdurend als referentie wordt gehanteerd.

C.2.1 Grafen

Een *graaf* is een geordend paar $(V(\Gamma), E(\Gamma))$, waarbij $V(\Gamma)$ een verzameling van *toppen* is en $E(\Gamma)$ een verzameling van deelverzamelingen van $V(\Gamma)$ van grootte 2, met name de *bogen*. Twee toppen zijn *adjacent* als ze samen een boog vormen. Een *pad met lengte* i is een reeks $x = x_0, \ldots, x_i = y$ waarbij twee opeenvolgende toppen telkens adjacent zijn. De *afstand* tussen twee toppen is de kortste lengte van alle paden tussen beide. De *diameter* van de graaf is de maximale afstand tussen twee toppen.

Een deelverzameling van toppen is een *kliek* als elke twee elementen adjacent zijn, en een *cokliek* als elke twee elementen niet-adjacent zijn. Een deelverzameling van toppen S is *intrigerend* als iedere top x juist h_1 buren heeft in S als $x \in S$, en juist h_2 als $x \notin S$.

De *adjacentiematrix* van een eindige graaf Γ is de symmetrische $(0,1)$-matrix A, met rijen en kolommen geïndexeerd door de toppenverzameling. De *eigenwaarden* van Γ zijn dan de eigenwaarden van A.

De *puntgraaf* van een punt-rechte meetkunde $\mathcal{P}$ is de graaf met de punten als toppen, waarbij twee punten adjacent zijn als ze collineair zijn. De *incidentiegraaf* heeft als toppen zowel de punten als rechten, en hier bepaalt de incidentierelatie de adjacentie.

De karakteristieke vector van een deelverzameling S van een eindige deelverzameling Ω, genoteerd met χ_S, is de vector in $\mathbb{R}^\Omega$ met $(\chi_S)_\omega = 1$ als $\omega \in S$ en $(\chi_S)_\omega = 0$ als $\omega \notin S$.

C.2.2 Associatieschema's

Associatieschema's werden ingevoerd door Bose en Shimamoto [16]. Een d-klasse *associatieschema* op een eindige niet-ledige verzameling Ω is een geordend paar $(\Omega, \mathcal{R})$ met $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ een verzameling van symmetrische niet-ledige relaties op Ω, zodanig dat aan de volgende axioma's voldaan is.

(i) R_0 is de identieke relatie.

(ii) $\mathcal{R}$ is een partitie van Ω^2.

(iii) Er zijn constanten p_{ij}^k, bekend als de *intersectiegetallen*, zodanig dat voor

elke $(x, y) \in R_k$, het aantal elementen z in Ω met $(x, z) \in R_i$ en $(z, y) \in R_j$ gelijk is aan p_{ij}^k.

Met elke R_i kunnen we ook een symmetrische $(0, 1)$-matrix A_i associëren (de adjacentiematrix) met rijen en kolommen geïndexeerd door de elementen van Ω, zodanig dat $(A_i)_{\omega_1,\omega_2} = 1$ als $(\omega_1, \omega_2) \in R_i$ en $(A_i)_{\omega_1,\omega_2} = 0$ in het andere geval. Uit de axioma's van een associatieschema volgt dat deze adjacentiematrices een $(d + 1)$-dimensionale vectorruimte over $\mathbb{R}$ vormen die gesloten is onder zowel gewone matrixvermenigvuldiging als onder elementsgewijze vermenigvuldiging (*Schur-vermenigvuldiging*, genoteerd met $\circ$). Deze matrices zijn de elementen van de algebra die we de *Bose-Mesner algebra* noemen.

Men kan bewijzen dat deze algebra ook nog een (unieke) basis $\{E_0, \ldots, E_d\}$ van minimale idempotenten heeft (i.e. $E_j^2 = E_j$ en E_j is geen som van niet-triviale idempotenten). Elke E_j bepaalt een orthogonale projectie op een deelruimte van $\mathbb{R}^\Omega$, en elke A_i werkt invariant als scalaire vermenigvuldiging met $P_{ji} \in \mathbb{R}$ op deze deelruimte. We spreken af dat E_0 telkens orthogonale projectie op $\langle \chi_\Omega \rangle$ voorstelt. De matrix $P = (P_{ji})_{i,j=0,\ldots,d}$ wordt de *eigenwaardenmatrix* genoemd. De matrix $Q = |\Omega| P^{-1}$ wordt de *duale eigenwaardenmatrix* genoemd. Merk op de i-de kolom van P de matrix A_i uitdrukt ten opzichte van $\{E_0, \ldots, E_d\}$, en duaal dat de j-de kolom van Q de matrix $|\Omega| E_j$ uitdrukt ten opzichte van $\{A_0, \ldots, A_d\}$.

Een *P-polynomiale* of *metrische* ordening van een d-klasse associatieschema $(\Omega, \{R_0, \ldots, R_d\})$ is een ordening $R_0, \ldots, R_d$, zodanig dat $(\omega_1, \omega_2) \in R_i$ als en slechts als $d(\omega_1, \omega_2) = i$ met betrekking tot R_1. Dit is equivalent met de voorwaarde dat A_i als een veelterm van graad i in A_1 kan geschreven worden (met gewone matrixvermenigvuldiging), $\forall i \in \{0, \ldots, d\}$.

Duaal is een *Q-polynomiale* of *cometrische* ordening van een associatieschema een ordening van de idempotenten $E_0, \ldots, E_d$, zodanig dat E_j geschreven kan worden als een veelterm van graad j in E_1 (met Schur-vermenigvuldiging).

Het is onze bedoeling om deelverzamelingen van Ω te bestuderen. Delsarte [65] ontwikkelde hiertoe krachtige technieken. Wij zullen hier een selectie weergeven.

Definitie C.2.1. *Beschouw een associatieschema $(\Omega, \{R_0, \ldots, R_d\})$ met $S \subseteq \Omega$. De inwendige distributie van S (als $S \neq \emptyset$) is de $(d + 1)$-vector $\mathbf{a}$ met*

$$\mathbf{a}_i = \frac{1}{|S|} |(S \times S) \cap R_i|, \forall i \in \{0, \ldots, d\}.$$

De uitwendige distributie van S is de $|\Omega| \times (d+1)$-matrix $B = (B_{x,i})$ met

$$B_{x,i} = |\{x' \in S | (x, x') \in R_i\}|, \forall i \in \{0, \ldots, d\}, \forall x \in \Omega.$$

Stelling C.2.2. *Als in het associatieschema $(\Omega, \{R_0, \ldots, R_d\})$ elke relatie R_i als valentie k_i heeft, en als eigenwaarde λ_i voor een zekere idempotent E_j, dan zal van iedere niet-ledige $S \subseteq \Omega$ de inwendige distributie $\boldsymbol{a}$ voldoen aan*

$$\frac{\lambda_0}{k_0}\mathbf{a}_0 + \cdots + \frac{\lambda_d}{k_d}\mathbf{a}_d \geq 0,$$

met gelijkheid als en slechts als $E_j\chi_S = 0$. In dat geval voldoet de uitwendige distributie B van S aan

$$\frac{\lambda_0}{k_0}B_{x,0} + \cdots + \frac{\lambda_d}{k_d}B_{x,d} = 0, \forall x \in \Omega.$$

De *duale graad verzameling* van een deelverzameling S in een associatieschema met idempotenten $E_0, \ldots, E_d$ is de verzameling van alle niet-nul indices j waarvoor $E_j\chi_S \neq 0$.

De vorige stelling toont reeds het belang aan van idempotenten die verdwijnen wanneer toegepast op een karakteristieke vector. Wanneer voor twee deelverzamelingen $S_1, S_2 \subseteq \Omega$ in een associatieschema $(\Omega, \{R_0, \ldots, R_d\})$ de duale graad verzamelingen disjunct zijn, dan zijn S_1 en S_2 *design-orthogonaal.* Het volgende lemma toont het combinatorische belang hiervan aan.

Lemma C.2.1. *Twee design-orthogonale deelverzamelingen S_1 en S_2 in een associatieschema $(\Omega, \{R_0, \ldots, R_d\})$ hebben precies $|S_1||S_2|/|\Omega|$ elementen gemeen.*

C.2.3 Afstandsreguliere grafen

Definitie C.2.3. *Laat Γ een eindige graaf zijn met diameter d. We zeggen dat Γ afstandsregulier is als er getallen b_i en c_i zijn, genaamd de intersectiegetallen, zodanig dat voor elke x en y op afstand i in Γ:*

$$\begin{aligned} |\Gamma_{i-1}(x) \cap \Gamma_1(y)| &= c_i, \forall i \in \{1, \ldots, d\}, \\ |\Gamma_{i+1}(x) \cap \Gamma_1(y)| &= b_i, \forall i \in \{0, \ldots, d-1\}. \end{aligned}$$

Wij zullen de notatie b_i en c_i altijd gebruiken zoals hierboven. We zullen met a_i ook $|\Gamma_i(x) \cap \Gamma_1(y)|$ aanduiden. Merk op dat $a_i + b_i + c_i = k$ voor elke $i \in \{1, \ldots, d-1\}$.

Uit de volgende stelling blijkt dat de studie van de afstandsreguliere grafen in feite neerkomt op de studie van de P-polynomiale associatieschema's.

Stelling C.2.4. *Laat Γ een eindige graaf zijn met diameter d en toppenverzameling Ω, en laat (x, y) in R_i zijn als $d(x, y) = i$ in Γ. Dan is Γ afstandsregulier als en slechts als $(\Omega, \{R_0, \ldots, R_d\})$ een associatieschema is.*
In dat geval is dit schema metrisch met ordening $R_0, \ldots, R_d$.

Een *code* in een afstandsreguliere graaf is een niet-ledige deelverzameling van toppen. Een code S is *s-regulier* is als de uitwendige distributie B van S zo is dat als $d(x, S) \leq s$, $B_{x,i}$ enkel afhangt van $d(x, S)$ en i. De *bedekkingsstraal* $t(S)$ van een code S is de maximale afstand tot S. Indien S een $t(S)$-reguliere code is, dan zeggen we dat S *compleet regulier* is.

Een graaf met v toppen is sterk regulier, genoteerd als $\mathrm{srg}(v, k, \lambda, \mu)$, wanneer iedere top k buren heeft, en wanneer twee toppen precies λ of μ gemeenschappelijke buren hebben, als de twee toppen respectievelijk wel of niet adjacent zijn. Afstandsreguliere grafen met diameter twee zijn de niet-triviale sterk reguliere grafen.

We definiëren nu een bepaald type van afstandsreguliere grafen, waartoe veel van de bekende grafen behoren.

Definitie C.2.5. *Een afstandsreguliere graaf Γ met diameter d heeft klassieke parameters (d, b, α, β) als*

$$\begin{aligned} b_i &= \left(\begin{bmatrix} d \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right), \\ c_i &= \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b\right), \end{aligned}$$

met $\left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right]_b = i$ als $b = 1$ en $\left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right]_b = (b^i - 1)/(b - 1)$ als $b \neq 1$.

C.2.4 Sferische designs en associatieschema's

Sferische designs zijn ingevoerd door Delsarte, Goethals en Seidel [69]. Er is een sterke link met de theorie van de associatieschema's, maar ze zijn ook een

studieobject op zich geworden.

Definitie C.2.6. *Een eindige niet-ledige deelverzameling X van $\mathbb{S}^{m-1}$ is een sferisch t-design als*

$$\frac{\int_{\mathbb{S}^{m-1}} f(u)du}{\int_{\mathbb{S}^{m-1}} 1du} = \frac{1}{|X|}\sum_{x\in X} f(x)$$

geldt voor elke veelterm $f \in \mathbb{R}[u_1, \ldots, u_m]$ met graad hoogstens t.

De verzameling van genormalizeerde kolommen van een minimale idempotent ($\neq E_0$) van een associatieschema is steeds een sferisch 2-design. Het is een sferisch t-design voor hogere t als het associatieschema en de idempotent aan bepaalde voorwaarden voldoen.

C.2.5 Permutatiegroepen en modulen

Als een groep G op een eindige verzameling Ω werkt, dan kunnen we met elke $g \in G$ een endomorfisme $\rho(g)$ van $\mathbb{R}^\Omega$ laten overeenkomen. Zo bekomt men het *permutatiemoduul*. Een *deelmoduul* is een deelruimte van $\mathbb{R}^\Omega$ die invariant is onder elke $\rho(g)$, en het is een *irreduciebel* deelmoduul als het zelf geen eigenlijke niet-triviale deelmodulen heeft.

We zeggen dat een groep G *vrijelijk transitief* op Ω werkt als (ω_1, ω_2) en (ω_2, ω_1) steeds tot dezelfde baan van G behoren. In dit geval leidt de groepsactie tot een associatieschema, waarvan de idempotenten overeenkomen met de orthogonale projecties op de irreduciebele deelmodulen van $\mathbb{R}^\Omega$.

C.3 Grassmann schema's

In Hoofdstuk 3 behandelen we de associatieschema's op de verzamelingen van deelruimten met een vaste dimensie in $V(n, q)$. We wijzen er eerst en vooral op dat dit neerkomt op een studie van $\mathrm{PG}(n-1, q)$. Om de eenvoud van de formules enigszins te bewaren, verkiezen wij om die laatste notatie niet verder te hanteren.

C.3.1 Grassmann schema's

In dit hoofdstuk stellen we de verzameling van a-ruimten in $V(n,q)$ voor met Ω_a. De elementen van Ω_1 en Ω_2 worden respectievelijk de punten en rechten genoemd. De *Gauss coëfficiënt* $\begin{bmatrix} n \\ a \end{bmatrix}_q$ wordt gedefinieerd als volgt:

$$\begin{bmatrix} n \\ a \end{bmatrix}_q = \prod_{i=1}^{a} \frac{q^{n+1-i}-1}{q^i-1} \text{ als } 0 \leq a \leq n,$$

en $\begin{bmatrix} n \\ a \end{bmatrix}_q = 0$ als $a < 0$ of $a > n$.

De groep $\mathrm{GL}(n,q)$ werkt vrijelijk transitief op Ω_a in $V(n,q)$, waarbij de baan van $(\pi_a, \pi'_a) \in \Omega_a \times \Omega_a$ enkel afhangt van $\dim(\pi_a \cap \pi'_a)$. Het hiermee overeenkomende associatieschema $(\Omega, \{R_0, \ldots, R_d\})$, $d = \min(a, n-a)$, is het *Grassmann* of *q-Johnson* schema, met $R_i = \{(\pi_a, \pi'_a) \in \Omega_a^2 | \dim(\pi_a \cap \pi'_a) = a - i\}$. Dit schema is P-polynomiaal met ordening $R_0, \ldots, R_d$, en de afstandsreguliere graaf die overeenkomt met R_1 staat bekend als de *Grassmann graaf* $J_q(n,a)$.

C.3.2 Irreduciebele deelmodulen en eigenwaarden voor Grassmann schema's

In Sectie 3.2 bespreken we de eigenruimten van de Grassmann schema's gedefinieerd door $J_q(n,a)$. Delsarte [66] ontwikkelde een algemene theorie omtrent *reguliere semitralies*, die toelaat om de eigenruimten in associatieschema's met een specifieke onderliggende structuur te begrijpen. Wij volgen de meer groepentheoretische aanpak van Ito [99].

Stelling C.3.1. *Beschouw $V(n,q)$ en $G = \mathrm{GL}(n,q)$. Voor elke $a \in \{0, \ldots, n\}$ wordt het permutatiemoduul $\mathbb{R}^{\Omega_a}$ over $\mathbb{R}G$ op a-ruimten ontbonden in irreduciebele deelmodulen als volgt:*

$$\mathbb{R}^{\Omega_a} = \bigoplus_{i=0}^{\min(a,n-a)} V_i^a,$$

waarbij V_i^a en V_i^b isomorfe $\mathbb{R}G$-modulen zijn.

Bij wijze van illustratie stellen we de ontbinding voor $V(6,q)$ voor in Figuur C.3.2. Elke kolom komt overeen met het permutatiemoduul op deelruimten van

$V(6,q)$ en bevat de irreduciebele deelmodulen daarin, en we schrijven isomorfe deelmodulen op dezelfde rij.

$\mathbb{R}^{\Omega_0}$	$\mathbb{R}^{\Omega_1}$	$\mathbb{R}^{\Omega_2}$	$\mathbb{R}^{\Omega_3}$	$\mathbb{R}^{\Omega_4}$	$\mathbb{R}^{\Omega_5}$	$\mathbb{R}^{\Omega_6}$
V_0^0	V_0^1	V_0^2	V_0^3	V_0^4	V_0^5	V_0^6
	V_1^1	V_1^2	V_1^3	V_1^4	V_1^5	
		V_2^2	V_2^3	V_2^4		
			V_3^3			

Figuur C.1: De ontbinding in irreduciebelen voor $V(6,q)$

C.3.3 Codes in Grassmann grafen

In deze sectie bespreken we deelverzamelingen van deelruimten in $V(n,q)$ die interessant zijn, met name wat betreft orthogonaliteit ten opzichte van de eigenruimten van het Grassmann schema.

Definitie C.3.2. *Een $t-(n,k,\lambda;q)$-design is een verzameling S van k-ruimten in $V(n,q)$ met $0 \leq t \leq k \leq n$, zodanig dat elke t-ruimte in juist λ elementen van S ligt.*

Uit Delsarte's theorie van de semireguliere tralies [66] volgt dat $S \subseteq \Omega_k$ in $V(n,q)$ een t-design is als en slechts als

$$\chi_S \in (V_i^k)^\perp, 1 \leq i \leq \min(k, n-k, t, n-t).$$

Cameron en Liebler [40] bestudeerden groepen met evenveel banen op punten als rechten in $\mathrm{PG}(n,q)$. We geven enkele voorbeelden van groepen met evenveel banen op deelruimten van twee verschillende dimensies, en daaruit leiden we combinatorische eigenschappen af.

Stelling C.3.3. (Theorem 3.3.13) *Laat f een niet-ontaarde alternerende vorm op $V(2n,q)$ zijn. De verzameling S van totaal isotrope deelruimten in $J_q(2n,k)$, $k \leq n$, heeft als duale graad de even indices in $\{1,\ldots,k\}$.*

Beschouw nu een vectorruimte $V(4n+2,q), n \geq 1$, samen met een niet-ontaarde kwadratische vorm Q van elliptisch type (of dus met Witt index $2n$). We

beschouwen de actie van de groep $\mathrm{GO}^-(4n+2,q)$:

$$\{g \in \mathrm{GL}(4n+2,q) | \exists \lambda \in \mathbb{F}_q : (Q(v^g) = \lambda Q(v), \forall v \in V(4n+2,q))\}.$$

Twee $(2n+1)$-ruimten liggen in dezelfde baan van $\mathrm{GO}^-(4n+2,q)$ op Ω_{2n+1} als en slechts als de restrictie van de vorm Q tot deze ruimten van hetzelfde type is.

Stelling C.3.4. (Theorem 3.3.15) *De banen van* $\mathrm{GO}^-(4n+2,q)$ *op* Ω_{2n+1} *in* $V(4n+2,q)$ *hebben karakteristieke vectoren die orthogonaal zijn met* V_{2n+1}^{2n+1}.

Uit het voorgaande kan zonder zwaar rekenwerk respectievelijk het volgende afgeleid worden.

Gevolg C.3.5. (Corollary 3.3.14) *Van een niet-ontaarde alternerende vorm op* $V(2n,q)$ *is het aantal totaal isotrope deelruimten in een* $t-(2n,t+1,\lambda;q)$*-design met* $0 \leq t \leq n-1$ *en* t *even gegeven door:*

$$\frac{\lambda}{\left[{2n-t \atop 1}\right]_q} \prod_{i=0}^{t} \left(\frac{q^{2(n-i)}-1}{q^{i+1}-1} \right).$$

Gevolg C.3.6. (Corollary 3.3.16) *Zij* S *een* $2n-(4n+2,2n+1,\lambda;q)$*-design in* $V(4n+2,q)$*, en zij* Q *een niet-ontaarde kwadratische vorm met Witt index* $2n$*. Als* $O_\alpha \subseteq \Omega_{2n+1}$ *de verzameling is van* $(2n+1)$*-ruimten waarop* Q *een restrictie van type* α *heeft, dan geldt:*

$$|S \cap O_\alpha| = \frac{\lambda}{\left[{2n+2 \atop 1}\right]_q} |O_\alpha|.$$

C.4 Klassieke eindige polaire ruimten

C.4.1 De associatieschema's van klassieke eindige polaire ruimten

We zeggen dat een klassieke eindige polaire ruimte van rang d *parameters* (q,q^e) heeft als iedere rechte incident is met $q+1$ punten, en elke $(d-1)$-ruimte met q^e+1 generatoren. Elke klassieke eindige polaire ruimte heeft dergelijke parameters, en die worden weergegeven in Tabel C.1 voor rang d (hier worden twee gebruikelijke notaties vermeld). In de context van polaire ruimten, zullen we de verzameling van totaal isotrope a-deelruimten voorstellen met Ω_a.

		(s,t)	e
$Q^+(2d-1,q)$	$D_d(q)$	$(q,1)$	0
$H(2d-1,q^2)$	${}^2A_{2d-1}(q)$	(q^2,q)	1/2
$Q(2d,q)$	$B_d(q)$	(q,q)	1
$W(2d-1,q)$	$C_d(q)$	(q,q)	1
$H(2d,q^2)$	${}^2A_{2d}(q)$	(q^2,q^3)	3/2
$Q^-(2d+1,q)$	${}^2D_{d+1}(q)$	(q,q^2)	2

Tabel C.1: De klassieke eindige polaire ruimten met parameters $(s,t) = (s,s^e)$

In een klassieke eindige polaire ruimte zijn de banen van de volledige automorfismengroep op $\Omega_a \times \Omega_b$ gegeven door:

$$R^{s,k}_{a,b} := \{(\pi_a,\pi_b)|\dim(\pi_a \cap \pi_b) = s, \dim(\langle \pi_a, \pi_b \cap \pi_a^{\perp}\rangle) = k\},$$

met $0 \leq s \leq \min(a,b)$ en $\max(a,b) \leq k \leq \min(d, a+b-s)$. Met iedere $R^{s,k}_{a,b}$ komt een $(0,1)$-matrix $C^{s,k}_{a,b}$ overeen, met kolommen en rijen geïndexeerd door respectievelijk Ω_a en Ω_b, zodanig dat $(C^{s,k}_{a,b})_{\pi_a,\pi_b} = 1$ is als (π_a,π_b) tot deze relatie behoort, en anders nul. Zo definieert elke relatie een afbeelding $\mathbb{R}^{\Omega_a} \to \mathbb{R}^{\Omega_b}$.

Uit het voorgaande volgt in het bijzonder dat de volledige automorfismengroep vrijelijk transitief op Ω_a werkt en dus tot een associatieschema leidt. Het aantal relaties in dit schema is $(a+1)(a+2)/2$ als $2a \leq d$ en $(d-a+1)(3a-d+2)/2$ als $2a \geq d$.

De relatie $R^{a-1,\min(a+1,d)}_{a,a}$ bepaalt de *graaf van Lie type* op de a-ruimten. Voor $a = 1$ is dit de *polaire graaf* op de punten van de polaire ruimte, waarbij twee verschillende toppen adjacent zijn als ze op een gemeenschappelijke rechte liggen. Voor $a = d$ is dit de *duale polaire graaf* op de generatoren, waarbij twee verschillende toppen adjacent zijn als ze door een gemeenschappelijke $(d-1)$-ruimte gaan.

De duale polaire graaf speelt een belangrijke rol in deze thesis. Voor een klassieke eindige polaire ruimte met rang d en parameters (q,q^e) is dit een afstandsreguliere graaf met klassieke parameters $(d,q,0,q^e)$, waarin twee toppen op afstand i zijn als ze in een $(d-i)$-ruimte snijden. Merk op dat dit betekent dat deze graaf het associatieschema op de generatoren definieert.

Een terugkerend fenomeen in deze thesis is de speciale rol van de duale polaire

graaf op $H(2d-1, q^2)$. Deze heeft ook nog eens klassieke parameters

$$(d, b, \alpha, \beta) = \Big(d, -q, -q(q+1)/(q-1), -q((-q)^d+1)/(q-1)\Big).$$

Een andere bijzondere relatie is de *oppositierelatie* $R^{0,a}_{a,a}$. Twee a-ruimten π_a en π'_a zijn opposite als geen punt op de ene collineair is met alle punten op de andere. Merk op dat dit voor generatoren neerkomt op het triviaal snijden, wat dan overeenkomt met de maximale afstandsrelatie met betrekking tot de duale polaire graaf.

C.4.2 Irreduciebele deelmodulen voor polaire ruimten

Net zoals bij de Grassmann schema's kan men de ontbinding in irreduciebele deelruimten begrijpen door de meetkundige objecten van verschillende types gelijktijdig te beschouwen. De volgende stelling kan afgeleid worden uit het werk van Stanton [135], Terwilliger [142] en Eisfeld [77].

Stelling C.4.1. *Beschouw een klassieke eindige polaire ruimte van rang d.*

(i) *Onder de actie van de volledige automorfismengroep heeft elk moduul $\mathbb{R}^{\Omega_n}$ een unieke (orthogonale) ontbinding in niet-isomorfe irreduciebele deelmodulen:*

$$\mathbb{R}^{\Omega_n} = \bigoplus_{\substack{0\le r\le n\\ 0\le i\le \min(r,d-n)}} V^n_{r,i}.$$

De deelmodulen $V^a_{r,i} \subseteq \mathbb{R}^{\Omega_a}$ en $V^b_{r,i} \subseteq \mathbb{R}^{\Omega_b}$ zijn isomorf.

(ii) *De restrictie van de incidentie-afbeelding $C_{a,b} : \mathbb{R}^{\Omega_a} \to \mathbb{R}^{\Omega_b}$ tot een deelmoduul $V^a_{r,i} \subseteq \mathbb{R}^{\Omega_a}$ is triviaal als er geen $V^b_{r,i}$ is in $\mathbb{R}^{\Omega_b}$, en in het andere geval is het een bijectie tussen de 2 isomorfe deelmodulen.*

(iii) *De restrictie van iedere afbeelding $C^{s,k}_{a,b}$ tot $V^a_{r,i} \subseteq \mathbb{R}^{\Omega_a}$ is een scalair veelvoud van de restrictie van $C_{a,b}$.*

De structuur van de ontbinding, zoals beschreven in Stelling C.4.1, wordt als voorbeeld weergegeven in Figuur C.2 voor klassieke eindige polaire ruimten van rang 4. Elke kolom komt overeen met een permutatiemoduul op een verzameling van totaal isotrope deelruimten met een vaste dimensie, en isomorfe deelmodulen worden op dezelfde rij geplaatst.

$\mathbb{R}^{\Omega_0}$	$\mathbb{R}^{\Omega_1}$	$\mathbb{R}^{\Omega_2}$	$\mathbb{R}^{\Omega_3}$	$\mathbb{R}^{\Omega_4}$
$V^0_{0,0}$	$V^1_{0,0}$	$V^2_{0,0}$	$V^3_{0,0}$	$V^4_{0,0}$
	$V^1_{1,0}$ $V^1_{1,1}$	$V^2_{1,0}$ $V^2_{1,1}$	$V^3_{1,0}$ $V^3_{1,1}$	$V^4_{1,0}$
		$V^2_{2,0}$ $V^2_{2,1}$ $V^2_{2,2}$	$V^3_{2,0}$ $V^3_{2,1}$	$V^4_{2,0}$
			$V^3_{3,0}$ $V^3_{3,1}$	$V^4_{3,0}$
				$V^4_{4,0}$

Figuur C.2: De ontbinding in irreduciebelen voor klassieke eindige polaire ruimten van rang 4

C.4.3 Specifieke eigenwaarden voor polaire ruimten

Eisfeld [77] gebruikte de ontbinding in irreduciebelen zoals beschreven in Stelling C.4.1 om op inductieve wijze de eigenwaarden te berekenen. In Sectie 4.3 werken wij dit uit voor specifieke gevallen waarbij blijkt dat er uiteindelijk toch een al bij al eenvoudige, expliciete uitdrukking kan gegeven worden.

C.4.4 Interessante deelverzamelingen in polaire ruimten

In Sectie 4.4 gebruiken we de eigenruimten en eigenwaarden om interessante deelverzamelingen van deelruimten in de polaire ruimte met een vaste dimensie te bespreken.

Het associatieschema op de punten heeft als ontbinding $\mathbb{R}^{\Omega_1} = V^1_{0,0} \perp V^1_{1,0} \perp V^1_{1,1}$. Eisfeld [75] besprak de twee types van intrigerende verzamelingen van de sterk reguliere polaire graaf in een algemenere context. De puntenverzamelingen S met $\chi_S \in V^1_{0,0} \perp V^1_{1,0}$ werden door Drudge [73] *dichte verzamelingen* genoemd, en die met $\chi_S \in V^1_{0,0} \perp V^1_{1,1}$ de *m-ovoïdes* door Thas [132]. Deze twee types werden samen uitvoerig behandeld in [9].

Het volgende resultaat is ook (impliciet) vermeld in [77]. Voor generatoren volgt dit ook uit de theorie van de reguliere semitralies (zie ook [135, 136]).

Stelling C.4.2. (Theorem 4.4.1) *In een klassieke eindige polaire ruimte van rang d met parameters (q, q^e) zal een verzameling $S \subseteq \Omega_a$ zo zijn dat elke b-ruimte incident is met (of gelijk aan) een vast aantal elementen van S, als en slechts als χ_S orthogonaal is met elke $V^a_{r,i}$, $(r, i) \neq (0, 0)$, die een isomorfe kopie $V^b_{r,i}$ heeft in $\mathbb{R}^{\Omega_b}$.*

Wij wijzen er eerst op dat dit in veel gevallen impliceert dat de verzameling S triviaal is.

Gevolg C.4.3. (Corollary 4.4.2) *Beschouw een verzameling van a-ruimten S in een klassieke eindige polaire ruimte, zodanig dat elke b-ruimte incident is met (of gelijk aan) een vast aantal elementen in S, met $a \leq b$ en $a + b \leq d$. Dan moet S leeg zijn of de volledige verzameling Ω_a.*

Een *partiële ovoïde* (*partiele spread*) in een polaire ruimte is een verzameling van punten (generatoren) die niet incident zijn met een gemeenschappelijke generator (punt). In een klassieke eindige polaire ruimte met rang d en parameters (q, q^e) kunnen beide structuren hoogstens $q^{d-1+e} + 1$ elementen bevatten, en dit is respectievelijk zo als iedere generator juist één element van de partiële ovoïde bevat, of ieder punt op juist één element van de partiële spread ligt. In dit geval spreken we respectievelijk over *ovoïdes* en *spreads*.

Shult en Thas [132] veralgemeenden dit door *partiële $(m-1)$-systemen* te definiëren als verzamelingen van paarsgewijs opposite m-ruimten (wij kiezen ervoor om vectoriële dimensies te gebruiken). Wanneer men deze opvat als klieken van de oppositierelatie en Stelling C.2.2 toepast, dan vindt men een alternatief bewijs van hun resultaat.

Stelling C.4.4. (Theorem 4.4.14) *Laat S een niet-ledig partieel $(m-1)$-systeem zijn in een klassieke eindige polaire ruimte met rang d en parameters (q, q^e). Nu is $|S| \leq q^{d-1+e} + 1$, met gelijkheid als en slechts als $\chi_S \in (V^m_{1,0})^\perp$.*

De partiële $(m-1)$-systemen die deze grens $q^{d-1+e} + 1$ bereiken worden de $(m-1)$-*systemen* genoemd.

De deelruimte $V^m_{1,0}$ van $\mathbb{R}^{\Omega_m}$ speelt een belangrijke rol, en levert via Stelling C.2.2 vrijwel altijd de beste bovengrens voor klieken van de oppositierelatie. Na zorgvuldig rekenwerk blijkt er echter één niet-triviale verbetering mogelijk, namelijk voor partiële spreads van $H(2d-1, q^2)$, d oneven. Deze stelling is het belangrijkste resultaat van Hoofstuk 4.

Stelling C.4.5. (Theorem 4.4.16) *Een partiële spread S in $H(2d-1,q^2)$ met d oneven bevat hoogstens q^d+1 elementen. Gelijkheid geldt als en slechts als de karakteristieke vector χ_S orthogonaal is tot $V^d_{d,0}$.*

Thas [152] bewees reeds dat in $H(2d-1,q^2)$ spreads, of dus partiële spreads met grootte $q^{2d-1}+1$, niet kunnen voorkomen. Partiële spreads met grootte q^d+1 werden echter in $H(2d-1,q^2)$ voor alle $d \geq 2$ geconstrueerd in [1], en bijgevolg is onze grens scherp.

We komen terug op deze grens en vooral het geval van gelijkheid in Subsectie 6.4.3 en de Appendix A.

C.5 Erdős-Ko-Rado stellingen in klassieke eindige polaire ruimten

In Hoofdstuk 5 bekijken we verzamelingen van generatoren in klassieke eindige polaire ruimten die paarsgewijs niet-triviaal snijden, m.a.w het omgekeerde probleem van de studie van partiële spreads. Dit kan opgevat worden als één van de vele Erdős-Ko-Rado problemen die opduiken in de wiskunde (zie hieronder voor een korte bespreking). Wij zullen voor iedere klassieke eindige polaire ruimte de maximale grootte van dergelijke verzamelingen bepalen, als ook de classificatie in het geval van gelijkheid, behalve in $H(2d-1,q^2)$ voor oneven $d \geq 5$.

De resultaten in dit hoofdstuk zijn bekomen in samenwerking met Valentina Pepe en Leo Storme.

C.5.1 Erdős-Ko-Rado stellingen

We geven eerst het oorspronkelijke resultaat van Erdős-Ko-Rado. Met k-verzamelingen bedoelen we deelverzamelingen van grootte k.

Stelling C.5.1. [78] *Zij S een verzameling van k-verzamelingen in een verzameling van grootte n, die paarsgewijs minstens t elementen gemeen hebben, met $1 \leq t \leq k \leq n$.*

(i) *Als $n \geq t + (k-t)\binom{k}{t}^3$, dan is $|S| \leq \binom{n-t}{k-t}$.*

(ii) *Als* $n \geq 2k$ *en* $t = 1$*, dan geldt* $|S| \leq \binom{n-1}{k-1}$.

Een verzameling S van k-verzamelingen door een vaste t-verzameling is een voorbeeld waarbij de grens bereikt wordt.

We zeggen dat een verzameling van k-ruimten *t-snijdend* is als de dimensie van de doorsnede van elke twee elementen minstens t is. Hsieh bewees het volgende analogon van Stelling C.5.1 voor vectorruimten.

Stelling C.5.2. [96] *Als S een t-snijdende verzameling van k-ruimten is in $V(n,q)$ met $1 \leq t \leq k$ en $n \geq 2k+1$, en ook met $(n,q) \neq (2k+1,2)$ als $t \geq 2$, dan is $|S| \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$. Gelijkheid geldt als en slechts als S bestaat uit alle k-ruimten door een vaste t-ruimte.*

Later zijn deze resultaten door verscheidene auteurs verbeterd.

We zullen *EKR verzamelingen van generatoren* bestuderen: verzamelingen van generatoren in klassieke eindige polaire ruimten die paarsgewijs niet-triviaal snijden. We zeggen dat een dergelijke verzameling *maximaal* is als het geen echte deelverzameling is van een andere EKR verzameling. Een eenvoudig voorbeeld van dit laatste is de verzameling van alle generatoren door een vast punt van de polaire ruimte. We zullen zien dat dit in de meeste gevallen ook daadwerkelijk de enige EKR verzamelingen van maximale grootte zijn.

Voor $d = 2$ zijn de maximale EKR verzamelingen de verzamelingen van rechten door een punt, en dus hoeven we enkel rang $d \geq 3$ te beschouwen.

C.5.2 Algebraïsche technieken

In de duale polaire graaf op een klassieke eindige polaire ruimte met rang d, liggen twee toppen op afstand d als en slechts als ze triviaal snijden. Bijgevolg zijn de EKR verzamelingen precies de coklieken van de maximale afstandsrelatie van de duale polaire graaf. In deze sectie geven we de technieken aan uit de algebraïsche grafentheorie waarmee we dit probleem zullen aanpakken.

Het belangrijkste hulpmiddel is *Hoffman's stelling.*

Stelling C.5.3. [95] *Beschouw een k-reguliere graaf Γ, $k \geq 1$,met toppenverzameling Ω en kleinste eigenwaarde λ. Als S een cokliek is in Γ, dan geldt:*

$$|S| \leq \frac{|\Omega|}{1 - k/\lambda},$$

en wanneer gelijkheid geldt kan χ_S geschreven worden als $\frac{|S|}{|\Omega|}\chi_\Omega + v$ met v een eigenvector voor λ van de adjacentiematrix A.

Verder zullen we ook steunen op een algemene theorie omtrent deelverzamelingen in associatieschema's van Brouwer, Godsil, Koolen en Martin [24]. Tanaka [139] ontwikkelde dit verder, en bekeek de parameters van associatieschema's geïnduceerd door deelverzamelingen van een specifiek type. Dit kan dan toegepast worden om classificatie in verscheidene Erdős-Ko-Rado problemen te bekomen.

Tenslotte beschouwen we in een associatieschema $(\Omega, \{R_0, \ldots, R_d\})$ een veralgemening van het concept van de uitwendige distributie, waarbij we tellen ten opzichte van een vast element dat tot een verzameling Ω' behoort, en waarbij een groepactie de verzamelingen Ω en Ω' linkt. Dergelijke ideeën werden voor specifieke associatieschema's reeds expliciet uitgewerkt door Calderbank en Delsarte (zie [31] en [68]).

C.5.3 Grenzen voor EKR verzamelingen van generatoren

Stanton [134] gebruikte Stelling C.5.3 om bovengrenzen voor EKR verzamelingen te berekenen. Op die manier komt men tot het volgende resultaat.

Stelling C.5.4. (Theorem 5.3.1) *Zij S een EKR verzameling van generatoren in een klassieke eindige polaire ruimte $\mathcal{P}$ van rang d met parameters (q, q^e), en beschouw de ontbinding $\mathbb{R}^{\Omega_d} = V^d_{0,0} \perp \ldots \perp V^d_{d,0}$.*

- *Als $\mathcal{P} = Q^+(2d-1, q)$ met d oneven, dan is $|S|$ hoogstens $(q+1)\cdots(q^{d-1}+1)$, en bij gelijkheid geldt: $\chi_S \in V^d_{0,0} \perp V^d_{d,0}$.*
- *Als $\mathcal{P} = Q^+(2d-1, q)$ met d even, dan is $|S|$ hoogstens $2(q+1)\cdots(q^{d-2}+1)$, en bij gelijkheid geldt: $\chi_S \in V^d_{0,0} \perp V^d_{1,0} \perp V^d_{d-1,0}$.*
- *Als $\mathcal{P} = H(2d-1, q^2)$ met d oneven, dan is $|S|$ hoogstens $|\Omega_d|/(q^d+1)$, en bij gelijkheid geldt: $\chi_S \in V^d_{0,0} \perp V^d_{d,0}$.*
- *Als $\mathcal{P} = Q(2d, q)$ of $\mathcal{P} = W(2d-1, q)$, met d oneven in beide gevallen, dan is $|S|$ hoogstens $(q+1)\cdots(q^{d-1}+1)$, en bij gelijkheid geldt: $\chi_S \in V^d_{0,0} \perp V^d_{1,0} \perp V^d_{d,0}$.*

Voor alle andere polaire ruimten is $|S|$ hoogstens het aantal generatoren door een punt, en bij gelijkheid geldt: $\chi_S \in V^d_{0,0} \perp V^d_{1,0}$.

C.5.4 Algemene observaties omtrent maximale EKR verzamelingen van generatoren

In Sectie 5.4 worden een aantal belangrijke observaties gedaan omtrent maximale EKR verzamelingen S. Hier wordt het belangrijke concept van de *kern* van een element van S (ten opzichte van S) ingevoerd.

C.5.5 Classificatie van maximum EKR verzamelingen van generatoren in de meeste polaire ruimten

In de gevallen die niet als uitzondering in Stelling C.5.4 optreden, volgt de classificatie vrijwel onmiddellijk uit het werk van Tanaka [139].

Stelling C.5.5. (Theorem 5.5.2) *Zij $\mathcal{P}$ één van de volgende polaire ruimten van rang $d \geq 3$: $H(2d, q^2)$, $H(2d-1, q^2)$ met d even, $Q(2d, q)$ met d even, $W(2d-1, q)$ met d even of $Q^-(2d+1, q)$. Als S een EKR verzameling van generatoren in $\mathcal{P}$ is, dan is $|S|$ hoogstens het aantal generatoren door een punt, met gelijkheid als en slechts als S bestaat uit alle generatoren door een punt.*

C.5.6 Hyperbolische kwadrieken

In de hyperbolische kwadriek $Q^+(2d-1, q)$ zijn er twee systemen van generatoren: *Latijnse* en *Griekse*. Twee generatoren behoren tot hetzelfde systeem als en slechts als hun doorsnede even codimensie heeft.

Als d oneven is, dan kunnen twee generatoren van eenzelfde type niet triviaal snijden.

Stelling C.5.6. (Theorem 5.6.1) *Als S een EKR verzameling van generatoren is met grootte $|\Omega_d|/2$ in $Q^+(2d-1, q)$, d oneven, dan is S één van de twee systemen.*

Als d even is, dan kunnen twee generatoren van een verschillend type niet triviaal snijden. Men kan met Stelling C.5.3 aantonen dat een EKR verzameling

in één systeem hoogstens $(q+1)\cdots(q^{d-2}+1)$ elementen bevat. Aangezien de automorfismengroep van de polaire ruimte transitief werkt op generatoren, kunnen we ons dus concentreren op het probleem in één systeem.

Stelling C.5.7. (Theorem 5.6.8) *Als S een EKR verzameling van Latijnse generatoren in $Q^+(2d-1,q)$, d even en $d \geq 6$, dan is $|S| \leq (q+1)\cdots(q^{d-2}+1)$, met gelijkheid als en slechts als S de verzameling is van alle Latijnse generatoren door een vast punt.*

Stelling C.5.8. (Theorem 5.6.10) *Als S een EKR verzameling is van Latijnse generatoren in $Q^+(7,q)$ met $|S| = (q+1)(q^2+1)$, dan bestaat S ofwel uit alle Latijnse generatoren door een vast punt, ofwel uit alle Latijnse generatoren die een vaste Griekse generator in een vlak snijden.*

C.5.7 $Q(2d,q)$ voor oneven d

Aangezien $Q(2d,q)$ ingebed kan worden in $Q^+(2d+1,q)$ aan de hand van een niet-singulier hypervlak, kunnen we het Erdős-Ko-Rado probleem hier ook vrij eenvoudig oplossen.

Stelling C.5.9. (Theorem 5.7.1) *Als S een EKR verzameling van generatoren is in $Q(2d,q)$, met $d \geq 3$ oneven, dan is $|S| \leq (q+1)\cdots(q^{d-1}+1)$, met gelijkheid als en slechts als:*

(i) *S de verzameling van generatoren door een vast punt is,*

(ii) *S de verzameling van alle generatoren van één systeem van een ingebedde $Q^+(2d-1,q)$ is,*

(iii) *$d=3$ en S bestaat uit één vlak en alle generatoren die het in een rechte snijden.*

C.5.8 $W(2d-1,q)$ voor oneven d

De duale polaire grafen op $W(2d-1,q)$ en $Q(2d,q)$ hebben dezelfde parameters, maar ze zijn enkel isomorf voor even q. Voor even q is de classificatie dus reeds gebeurd in Sectie 5.7. De classificatie voor oneven q is veel minder eenvoudig. Door gebruik te maken van de uitwendige distributie, als ook de veralgemeende uitwendige distributie met betrekking tot de $(d-1)$-ruimten van de polaire ruimte, kan men het volgende bewijzen.

Stelling C.5.10. (Theorem 5.8.13) *Zij S een EKR verzameling van generatoren in $W(2d-1,q)$ met q oneven, d oneven en $d \geq 5$. Dan is $|S| \leq (q+1)\cdots(q^{d-1}+1)$, met gelijkheid als en slechts als S de verzameling van alle generatoren door een vast punt is.*

Stelling C.5.11. (Theorem 5.8.14) *Zij S een EKR verzameling in $W(5,q)$ met q oneven. Dan is $|S| \leq (q+1)(q^2+1)$, met gelijkheid als en slechts als S bestaat uit ofwel alle generatoren door een vast punt van de polaire ruimte, ofwel één vlak π en alle generatoren die π snijden in een rechte.*

C.5.9 $H(2d-1,q^2)$ voor oneven d

Opnieuw speelt de duale polaire graaf op $H(2d-1,q^2)$ een speciale rol, maar hier is dit in ons nadeel voor oneven d: Stelling C.5.3 geeft ons hier een grens voor de grootte van de EKR verzamelingen van generatoren (nl. $|\Omega_d|/(q^d+1)$) die veel groter is dan het aantal generatoren door één punt (nl. $|\Omega_d|/(q^{2d-1}+1)$). Met behulp van Stelling C.4.2 vindt men nog vrij eenvoudig het volgende resultaat.

Stelling C.5.12. (Theorem 5.9.1) *Als S een EKR verzameling is van generatoren in $H(2d-1,q^2)$ met d oneven en $d \geq 3$, dan geldt $|S| < |\Omega_d|/(q^d+1)$.*

Het is mogelijk dat voor oneven $d \geq 5$ de EKR verzamelingen van generatoren door een vast punt nog steeds de enige zijn van maximale grootte, maar algebraïsche technieken lijken hier minder toepasbaar. Met puur meetkundige argumenten kunnen we het probleem echter nog oplossen als $d = 3$.

Stelling C.5.13. (Theorem 5.9.4) *Zij S een EKR verzameling van vlakken in $H(5,q^2)$. Nu is $|S| \leq q^5+q^3+q+1$, met gelijkheid als en slechts als S bestaat uit één vlak π en alle vlakken die π in een rechte snijden.*

C.5.10 Overzicht

In Sectie 5.10 vatten we alle resultaten over EKR verzamelingen van generatoren in klassieke eindige polaire ruimten samen, met verwijzingen naar de bewijzen.

C.6 Schier veelhoeken

In Hoofdstuk 6 worden de veralgemeende veelhoeken en duale polaire ruimten samen in de algemenere context van schier veelhoeken bestudeerd.

C.6.1 Definities en basiseigenschappen

Definitie C.6.1. *Een schier veelhoek is een partieel lineaire ruimte $\mathcal{P}$ die aan de volgende axioma's voldoet.*

(i) *De puntgraaf is samenhangend met diameter $d \geq 1$.*

(ii) *Voor elk punt p en elke rechte ℓ met $d(p, \ell) < d$ in de puntgraaf is er een uniek punt p' op ℓ dat op minimale afstand van p ligt.*

We zeggen dat $\mathcal{P}$ een schier $(2d+1)$-hoek is als er een punt op afstand d in de puntgraaf van een zekere rechte ℓ ligt, en een schier $2d$-hoek in het andere geval.

Schier $2d$-hoeken zijn ingevoerd door Shult en Yanushka [133]. We zullen vooral die structuren bekijken.

De schier vierhoeken met minstens twee rechten door ieder punt zijn precies de veralgemeende vierhoeken.

We zeggen dat een schier n-hoek $\mathcal{P}$ *regulier* is als de puntgraaf afstandsregulier is. Voor $n \geq 4$ impliceert dit dat $\mathcal{P}$ een *orde* (s, t) heeft: iedere rechte bevat juist $s + 1$ punten, en ieder punt ligt op juist $t + 1$ rechten.

Tenzij anders vermeld, bedoelen we met de afstand tussen 2 punten of tussen een punt en een rechte, de afstand in de puntgraaf.

C.6.2 Types van schier veelhoeken

We zullen twee belangrijke types van schier veelhoeken bekijken.

De *veralgemeende veelhoeken* zijn door Tits [156] ingevoerd. De veralgemeende $2d$-hoeken van orde (s, t) zijn precies die reguliere schier $2d$-hoeken van orde (s, t) waarvan de puntgraaf voldoet aan $c_1 = \ldots = c_{d-1} = 1$, met andere

woorden: tussen elke twee punten op niet-maximale afstand is er een uniek kortste pad in de puntgraaf. Dit zijn zelf-duale structuren.

Een beroemde stelling van Feit en G. Higman [80] stelt dat indien $s, t > 1$, veralgemeende $2d$-hoeken met $d \geq 2$ van orde (s, t) enkel kunnen bestaan als $d \in \{2, 3, 4\}$. Voor $d = 2$ bestaan er de klassieke polaire ruimten van rang 2, maar er zijn ook andere voorbeelden. Voor $d = 3$ zijn, op dualiteit na, enkel de *split Cayley zeshoeken* $\mathrm{H}(q)$ van orde (q, q) en de *getwiste trialiteitszeshoeken* $T(q^3, q)$ van orde (q^3, q) gekend. Tenslotte zijn voor $d = 4$, opnieuw op dualiteit na, enkel de *Ree-Tits achthoeken* van orde (q, q^2) met $q = 2^{2e+1}$, e geheel, gekend.

C.6.3 Eigenwaarden van schier $2d$-hoeken

In Sectie 6.3 bespreken we eigenwaarden van de puntgraaf van een schier $2d$-hoek. De idempotent voor de kleinste eigenwaarde kan heel elegant uitgedrukt worden en is één van de belangrijkste instrumenten in deze thesis.

Stelling C.6.2. *Als Γ de puntgraaf is van een reguliere schier $2d$-hoek van orde (s, t) met $d \geq 2$, en A_i is de adjacentiematrix van de afstand-i relatie, dan is*

$$M := \sum_{i=0}^{d} \left(-\frac{1}{s}\right)^i A_i$$

op een positive scalair na, een minimale idempotent en dus positief semidefiniet. De overeenkomstige eigenwaarde is $-(t + 1)$.

C.6.4 Puntenverzamelingen in schier $2d$-hoeken

In Sectie 6.4 bespreken we puntenverzamelingen in reguliere schier $2d$-hoeken en hun interactie. We doen dit eerst algemeen, en dan voor veralgemeende $2d$-hoeken en duale polaire ruimten.

Stelling C.6.3. (Theorem 6.4.2) *Beschouw een reguliere schier $2d$-hoek, $d \geq 2$, van orde (s, t). Laat S een niet-ledige puntenverzameling met inwendige distributie $\boldsymbol{a}$ zijn. Nu geldt:*

$$\sum_{i=0}^{d} \left(-\frac{1}{s}\right)^i \boldsymbol{a}_i \geq 0,$$

met gelijkheid als en slechts als $M\chi_S = 0$, en in dat geval voldoet de uitwendige distributie B van S voor elk punt p aan:

$$\sum_{i=0}^{d} \left(-\frac{1}{s}\right)^i B_{p,i} = 0.$$

We zullen alle puntenverzamelingen S met $M\chi_S = 0$ *dichte puntenverzamelingen* noemen. Voor veralgemeende vierhoeken valt dit samen met het gelijknamige concept ingevoerd door Payne [121].

Een *m-ovoïde* in een reguliere schier $2d$-hoek is een puntenverzameling waarvan iedere rechte juist m punten bevat.

In veralgemeende 2d-hoeken $\mathcal{P}$ zijn *partiële afstands-j-ovoïdes* puntenverzamelingen S waarbij de afstand tussen elke twee elementen van S in de puntgraaf minstens j is. We zeggen dat S een *afstands-j-ovoïde* is als er van ieder punt en van iedere rechte in $\mathcal{P}$ minstens één element op afstand hoogstens j in de incidentiegraaf ligt.

Stelling C.6.4. (Theorem 6.4.19) *Zij S een maximale partiële afstands-3-ovoïde in een veralgemeende zeshoek van orde (s,t).*

- *Als $s \leq t$, dan is $|S| \leq 1+s^3$, met gelijkheid als en slechts als S compleet regulier is, en als en slechts als S dicht is. In dat geval snijdt S elke m-ovoïde in $m(s^2-s+1)$ punten.*
- *Als $s > t$, dan is $S \leq 1+(\sqrt{st})^3$, met gelijkheid als en slechts als S compleet regulier is.*

Stelling C.6.5. (Theorem 6.4.20) *Als S en S' twee afstands-2-ovoïdes zijn in een veralgemeende zeshoek van orde (s,s^3) met $s>1$, dan is $|S \cap S'|$ gelijk aan 0 of $h(s^2+s+1)$ met h geheel en $h \geq s^3-s+1$.*

Stelling C.6.6. (Theorem 6.4.21) *Beschouw een veralgemeende achthoek $\mathcal{P}$ van orde (s',t'). Als S de puntenverzameling is van een deelachthoek van orde (s',t'), dan geldt $s=s'$ of $s \geq s't'$, en S is dicht in $\mathcal{P}$ als en slechts als $s=s'$ of $s=s't'$. Elke m-ovoïde van $\mathcal{P}$ snijdt S in $m(st'+1)((st')^2+1)$ punten in het eerste geval, en in $m(s'+1)(s^2+1)$ punten in het tweede geval.*

In duale polaire ruimten bekomen we vooral informatie over (partiële) spreads.

Stelling C.6.7. (Theorem 6.4.26) *In $H(2d-1,q^2)$ bevat een niet-ledige partiële spread S voor oneven $d \geq 3$ hoogstens q^d+1 elementen, met gelijkheid als en slechts als S dicht is. In dat geval is S ook 1-regulier, en geldt voor iedere totaal isotrope d-ruimte π:*

$$\sum_{i=0}^{d} \left(-\frac{1}{q}\right)^i B_{\pi,i} = 0.$$

waarbij $B_{\pi,i}$ het aantal elementen in S voorstelt dat π snijdt in een $(d-i)$-ruimte.

Stelling C.6.8. (Theorem 6.4.27) *In $H(5,q^2)$ bevat een maximale partiële spread S hoogstens q^3+1 elementen, en gelijkheid geldt als en slechts als S dicht is, en als en slechts als S compleet regulier is.*

Stelling C.6.9. (Theorem 6.4.29) *Voor elke oneven $d \geq 3$ is een niet-ledige partiële spread S in de parabolische kwadriek $Q(2d,q)$ of de symplectische ruimte $W(2d-1,q)$ dicht als en slechts als S een spread is. In dat geval is S ook 2-regulier, en geldt voor iedere totaal isotrope d-ruimte π:*

$$\sum_{i=0}^{d} \left(-\frac{1}{q}\right)^i B_{\pi,i} = 0,$$

waarbij $B_{\pi,i}$ het aantal elementen van S voorstelt dat π in een $(d-i)$-ruimte snijdt. Elke totaal isotrope d-ruimte snijdt minstens één element in een deelruimte met dimensie minstens 2.

Stelling C.6.10. (Theorem 6.4.30) *Een spread in $Q(10,q)$ of $W(9,q)$ is compleet regulier met bedekkingsstraal 3.*

C.6.5 Krein condities en sferische designs

In Sectie 6.5 tonen we hoe we zonder lastig rekenwerk kunnen nagaan wanneer de kolommen van M voor duale polaire ruimten na normalizering een sferisch 3-design vormen.

C.6.6 Higman ongelijkheden voor reguliere schier $2d$-hoeken

D. Higman [87] bewees dat als een veralgemeende vierhoek van de orde (s,t) is met $s > 1$, dan moet $t \leq s^2$, of dus dat de puntgraaf dan voldoet aan $c_2 \leq s^2 + 1$. De volgende veralgemening voor schier $2d$-hoeken is één van de belangrijkste resultaten in dit hoofdstuk.

Stelling C.6.11. (Theorem 6.6.1) *Beschouw een reguliere schier $2d$-hoek van orde (s,t) met $s > 1$. De puntgraaf voldoet aan:*

$$c_i \leq \frac{s^{2i}-1}{s^2-1}, \ \forall i \in \{1,\ldots,d\}.$$

Voor $d \geq 3$ zijn de reguliere schier $2d$-hoeken met $c_i = (s^{2i}-1)/(s^2-1)$ voor elke $i \in \{1,\ldots,d\}$ precies de duale polaire ruimten op $H(2d-1,q^2)$.

Gevolg C.6.12. (Corollary 6.6.7) *Beschouw een reguliere schier $2d$-hoek van orde (s,t), $s > 1$ en $d \geq 2$, met puntgraaf Γ. Dan geldt:*

$$t + 1 \leq \frac{s^{2d}-1}{s^2-1},$$

Bij gelijkheid geldt voor elk drietal punten a, b en c, paarsgewijs op afstand d:

$$|\Gamma_1(a) \cap \Gamma_{d-1}(b) \cap \Gamma_{d-1}(c)| = \frac{(s^d-(-1)^d)(s^{d-1}+(-1)^d)}{s^2-1}.$$

C.6.7 Subgrafen in extremale schier $2d$-hoeken

In Sectie 6.7 bekijken we subgrafen in de reguliere schier $2d$-hoeken die extremaal zijn met betrekking tot Stelling C.6.11.

Het is reeds gekend dat als Γ de duale polaire graaf is op $H(2d-1,q^2)$, of de puntgraaf van een veralgemeende vierhoek van orde (q,q^2), $\Gamma_d(p)$ (met $d = 2$ in het laatste geval) dan voor ieder punt afstandsregulier is met klassieke parameters.

Hier is ons hoofdresultaat de volgende constructie. Het is een veralgemening van een resultaat van Thas [148] voor rang $d = 2$.

Stelling C.6.13. (Theorem 6.7.8) *Zij S een $(q+1)/2$-ovoïde in de duale polaire ruimte op $H(2d-1,q^2)$ voor oneven q. De geïnduceerde subgraaf Γ' op S is afstandsregulier met klassieke parameters:*

$$(d,b,\alpha,\beta) = \left(d, -q, -\left(\frac{q+1}{2}\right), -\left(\frac{(-q)^d+1}{2}\right)\right).$$

Een $(s+1)/2$-ovoïde in een veralgemeende vierhoek van orde (s,s^2), $s>1$, wordt een *hemisysteem* genoemd. Cameron [37] bewees algemener dat deze in elke veralgemeende vierhoek van orde (s,s^2) een sterk reguliere graaf induceren.

C.6.8 Reguliere schier vijfhoeken

Definitie C.6.14. *Een partiële vierhoek* $\mathrm{PQ}(s,t,\mu)$ *met $s,t,\mu \geq 1$ is een partieel lineaire ruimte die aan de volgende axioma's voldoet.*

(i) *Elke rechte bevat juist $s+1$ punten en elk punt ligt op juist $t+1$ rechten.*

(ii) *Als een punt p niet op de rechte ℓ ligt, dan is er hoogstens één punt op ℓ collineair met p.*

(iii) *Als twee punten niet collineair zijn, dan zijn er juist μ punten collineair met beide punten.*

Deze structuren werden ingevoerd door Cameron [32]. De partiële vierhoeken van orde (s,t) die geen veralgemeende vierhoeken zijn, zijn precies de reguliere schier vijfhoeken.

Wij zeggen dat twee rechten in een $\mathrm{PQ}(s,t,\mu)$ *parallel* zijn als ze ofwel gelijk zijn, ofwel disjunct terwijl er geen rechte ze allebei snijdt. We tonen aan dat wanneer twee verschillende rechten parallel zijn, de ongelijkheid $\mu \leq st/(t+1)$ moet gelden, en bij gelijkheid is parallelisme onder meer een equivalentierelatie, waarbij elke klasse hoogstens $\mu+s+1$ elementen bevat.

De intrigerende puntenverzamelingen in partiële vierhoeken werden bestudeerd door Bamberg, De Clerck en Durante [6], en vooral in de $\mathrm{PQ}(s,t,\mu)$ die ingebed zijn in een veralgemeende vierhoek van orde (q,q^2). Wij zullen de $\mathrm{PQ}(s,t,\mu)$ met $\mu = st/(s+1)$ beschouwen. Hier heeft de puntgraaf als eigenwaarden $\mu(s+1)+s, s$ en $-\mu-1$.

Stelling C.6.15. (Theorem 6.8.7) *Beschouw een* $\mathrm{PQ}(s,t,\mu)$ *waarvoor* $\mu = st/(s+1)$*. Als voor een puntenverzameling* S *geldt dat* χ_S *orthogonaal is tot de eigenruimte voor* s *van de puntgraaf, dan zullen elke twee parallelle rechten* S *snijden in hetzelfde aantal punten.*

Stelling C.6.16. (Theorem 6.8.8) *Beschouw een* $\mathrm{PQ}(s,t,\mu)$ *waarvoor* $\mu = st/(s+1)$ *en waarin elke parallelklasse* $\mu+s+1$ *rechten bevat. Een puntenverzameling* S *is zo dat elke twee parallelle rechten* S *snijden in eenzelfde aantal punten, als en slechts als* χ_S *orthogonaal is met de eigenruimte voor* s *van de puntgraaf.*
In dat geval is elk punt collineair met $|S|/(s+1)-\mu-1$ *punten in* S *als* $p \in S$*, en met* $|S|/(s+1)$ *punten in* S *als* $p \notin S$*, en moet* $|S|(t+1)$ *deelbaar zijn door* $\mu+s+1$*.*

Appendix A: Een meetkundig bewijs voor partiële spreads in $H(2d-1,q^2)$ voor oneven d

In Appendix A geven we een bewijs dat volledig geschreven is in de taal van de eindige meetkunde. Nochtans is het sterk geïnspireerd door concepten uit de algebraïsche grafentheorie zoals 1-regulariteit van codes en drietal intersectiegetallen.

Stelling C.6.17. (Theorem A.2.1) *Zij* S *een partiële spread in* $H(2d-1,q^2)$ *met* d *oneven en* $d \geq 3$*. Dan is* $|S|$ *hoogstens* q^d+1*. Als* $|S| > 1$ *en* $\pi \in S$*, dan zal iedere generator die* π *in een* $(d-1)$*-ruimte snijdt, hetzelfde aantal elementen van* S *in juist één punt snijden, als en slechts als* $|S| = q^d+1$*. In dat geval is dit aantal* q^{d-1}*.*

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