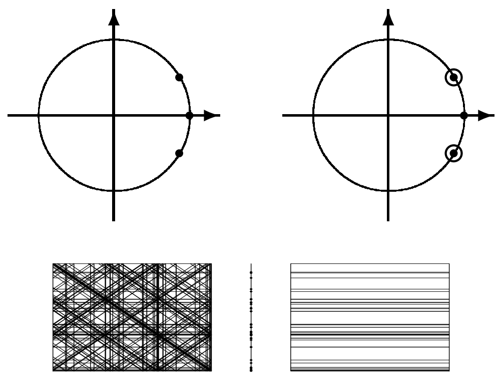
Bifurcation of periodic orbits and persistence of quasi periodic orbits in families of reversible systems.

Maria-Cristina Ciocci



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Voorwoord

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Gent, 4 oktober 2003 Maria-Cristina Ciocci

To Benny and my parents.

Contents

Opening note

This section is a general introduction to this thesis which consists of two main parts: Part I, *bifurcation of q-periodic points from a fixed point in families of reversible diffeomorphisms*, Part II, *persistence of quasi periodicity at a 1:1 resonance in families of reversible systems.*

The simplest example of a (time) reversible system is that of an equation of the form

$$\ddot{x} + f(x) = 0, \tag{1}$$

with $x \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ a given smooth function. Such systems frequently appear in mechanics and have the property that if $x(t)$ is a solution, then so is $x(-t)$ (*time reversibility*). Drawing the solution orbits of (1) in the $(x, \dot{x})$-plane gives the phase-portrait which for the particular case of

$$f(x) = \omega_0^2 x - x^2 \tag{2}$$

is sketched in Fig. 1.

This phase-portrait is invariant under the reflection $R_0 : (x, \dot{x}) \mapsto (x, -\dot{x})$; it also shows a typical phenomenon for reversible systems, namely the appearance of a one-parameter family of periodic orbits. In this particular example this family originates at the equilibrium $x = 0$, and terminates at an orbit which is homoclinic to the other equilibrium $x = \omega_0^2$. One can parametrise these periodic orbits by their intersection point $(x_0, 0)$ with the x-axis ($0 < x_0 < \omega_0^2$). Detailed calculations show that the corresponding period $T(x_0)$ is strictly increasing from $T(0) = 2\pi/\omega_0$ to $+\infty$.

If we add an appropriate forcing term to equation (1), for example

$$\ddot{x} + f(x) = A\cos(t), \tag{3}$$

the system remains reversible, i.e., if $x(t)$ is a solution then so is $x(-t)$. For $A \neq 0$ periodic solutions of (3) must necessarily have a period which is an integer multiple of 2π. Now, suppose that for all sufficiently small $A \neq 0$

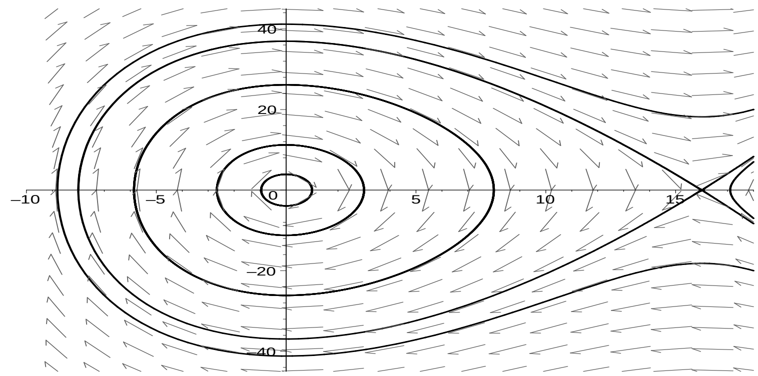

Figure 1: Phase-portrait of system (1) where $f(x)$ is given by (2) and $\omega_0 = 4$.

equation (3) has a $2\pi k$-periodic solution $\tilde{x}_A(t)$ that depends continuously on A. Taking the limit for $A \to 0$ then yields a $2\pi k$-periodic solution $\tilde{x}_0(t)$ of (1). In the particular case that f is of the form (2) this means that the orbit of $\tilde{x}_0(t)$ intersects the x-axis at the point $\tilde{x}_0 \in (0, \omega_0^2)$ such that $2\pi k = qT(\tilde{x}_0)$ for some integer $q \geq 1$. In such case we say that the branch $\{x_A(t)\}$ of periodic solutions of (3) bifurcates at $\tilde{x}_0(t)$ from the branch of periodic solutions of (1).

In order to calculate such bifurcating branch of periodic solutions we proceed as follows. Fix some $\tilde{x}_0 \in (0, \omega_0^2)$ such that

$$T_0 := T(\tilde{x}_0) = \frac{2\pi k}{q} \tag{4}$$

for some integers $k, q \geq 1$ such that $\gcd(\mathrm{k}, \mathrm{q}) = 1$ (one can show that this last condition is not a restriction). Denote by $\varphi(t; x_0, A)$ the unique solution of (3) that satisfies the initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = 0, \tag{5}$$

by reversibility we have that

$$\varphi(-t; x_0, A) = \varphi(t; x_0, A). \tag{6}$$

We then have to determine (x_0, A) near $\tilde{x}_0, 0$ such that

$$\varphi(t + 2\pi k; x_0, A) = \varphi(t; x_0, A), \quad \forall t \in \mathbb{R}. \tag{7}$$

By the uniqueness of solutions of the initial value problem for (3) and reversibility, the periodicity condition (7) can be rewritten as

$$\frac{\partial\varphi}{\partial t}(\pi k; x_0, A) = 0. \tag{8}$$

The fact that for $A = 0$ the solution of (1) satisfying (5) is $T(x_0)$-periodic implies that for all x_0

$$\frac{\partial\varphi}{\partial t}(\frac{q}{2}T(x_0); x_0, 0) = 0. \tag{9}$$

In combination with (4) this implies that

$$\frac{\partial\varphi}{\partial t}(k\pi; \tilde{x}_0, 0) = 0. \tag{10}$$

Differentiating (9) with respect to x_0 at $x_0 = \tilde{x}_0$ and using

$$\frac{\partial^2\varphi}{\partial t^2}(t; x_0, 0) = -f(\varphi(t; x_0, 0))$$

gives that

$$\frac{\partial^2\varphi}{\partial t\partial x_0}(k\pi; \tilde{x}_0, 0) = f(\varphi(k\pi; \tilde{x}_0, 0))\tfrac{q}{2}T'(\tilde{x}_0). \tag{11}$$

Since we have (implicitly) assumed that $\varphi(t, \tilde{x}_0, 0)$ is a non-trivial periodic solution (i.e., not an equilibrium), it follows that $f(\varphi(k\pi; \tilde{x}_0, 0)) \neq 0$; hence, if we assume that $T'(\tilde{x}_0) \neq 0$ (which is true in the example (2)) then $\dfrac{\partial^2\varphi}{\partial t\partial x_0}(k\pi; \tilde{x}_0, 0) \neq 0$ and we can use the Implicit Function Theorem to solve the equation (9) for x_0 as a function of A: $x_0 = x_0^*(A)$. This gives a one-parameter branch of $2\pi k$-periodic orbits

$$\tilde{x}_A(t) := \varphi(t; x_0^*(A), A), \tag{12}$$

that bifurcates from the branch of periodic orbits of (1) at the periodic orbit through $\tilde{x}_0$; the parameter is the amplitude A of the forcing term in (3).

Actually one can redo the foregoing analysis replacing x_0 everywhere by $x_1 := \varphi(\frac{1}{2}T(x_0); x_0, 0)$, i.e., the second intersection point of the x-axis with the periodic orbit of (1) through x_0. In the case $q \geq 3$ this leads to a second branch of periodic orbits bifurcating at $\tilde{x}_0$. Along one of these bifurcating branches the solutions are (weakly) stable, unstable along the other one.

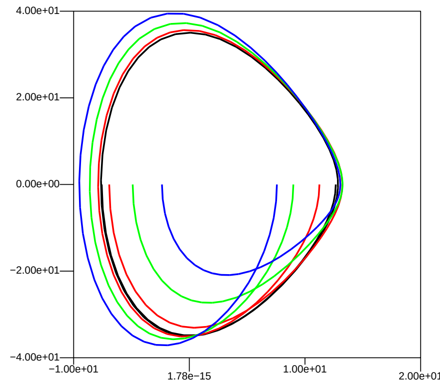

Figure 2: Branch of subharmonics: $q = 3$, $k = 1$, $l = 0$, $\omega_0^2 = 16$, $A = 13$. The period along this branch is $T = 2\pi/3$ and the Floquet multipliers are: 1 (double), $0.7367345 - 0.676182\,i$ and $0.7367345 + 0.676182\,i$ (simple, on the unit circle).

The following figures show some of these bifurcating periodic orbits in the example (2); these orbits were numerically calculated using AUTO [37].

Figure 4 summarizes Figure 3 and 2.

In an attempt to put the foregoing in a more general framework, we observe that all (periodic) solutions we have taken into account had to satisfy the condition $\dot{x}(0) = 0$, which by reversibility is equivalent to $x(t) = x(-t)$. We say that these solutions are *symmetric*. Also observe that the forcing $A\cos(t)$, see (3), is itself a symmetric periodic solution of the reversible equation

$$\ddot{y} + y = 0. \tag{13}$$

Hence we can reformulate our problem as follows: find the branches of symmetric periodic solutions of the autonomous system

$$\left\{ \begin{array}{lcl} \ddot{x} & = & -f(x) + y \\ \ddot{y} & = & -y, \end{array} \right. \tag{14}$$

and describe how different branches of such solutions connect to each other (*bifurcations*). System (14) reduces to (1) when $y(t) = 0$, and in that case we have a *primary branch* of periodic orbits as sketched in Figure 1. The

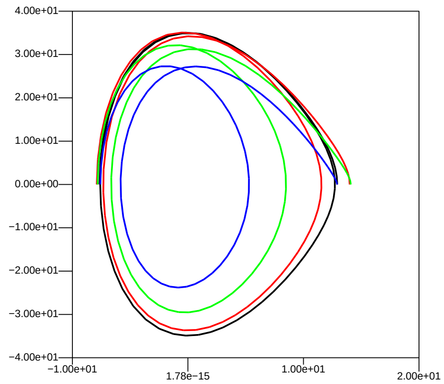

Figure 3: Branch of subharmonics: $q = 3$, $k = 1$, $l = 0$, $\omega_0^2 = 16$, $A = 13$. The period along the branch is $T = 2\pi/3$ and the Floquet multipliers are: 1 (double), 0.1235112 and 8.09643 (simple, off the unit circle).

other solution branches (with $A \neq 0$) bifurcate from the primary branch at certain specific solutions. In our example the condition for such a bifurcation point is that its minimal period is of the form (4).

Rewriting system (14) as a 4-dimensional first order system yields

$$\begin{cases} \dot{x}_1 &= & x_2 \\ \dot{x}_2 &= & -f(x_1) + y_1 \\ \dot{y}_1 &= & y_2 \\ \dot{y}_2 &= & -y_1, \end{cases} \tag{15}$$

or equivalently

$$\dot{z} = F(z), \tag{16}$$

where $z = (x_1, x_2, y_1, y_2) \in \mathbb{R}^4$ and $F : \mathbb{R}^4 \to \mathbb{R}^4$ is defined by the right hand side of (15). The reversibility of (15) (and (16)) is expressed by the fact that if $z(t) = (x_1(t), x_2(t), y_1(t), y_2(t))$ is a solution then so is $\tilde{z}(t) := (x_1(-t), -x_2(-t),\ y_1(-t), -y_2(-t))$. In other words, $\tilde{z}(t) = Rz(-t)$ where $R \in \mathcal{L}(\mathbb{R}^4)$ is a linear involution (i.e. $R^2 = \mathrm{I}$) given by

$$R(x_1, x_2, y_1, y_2) := (x_1, -x_2, y_1, -y_2). \tag{17}$$

Note also that

$$F(Rz) = -RF(z), \quad \forall z \in \mathbb{R}^4. \tag{18}$$

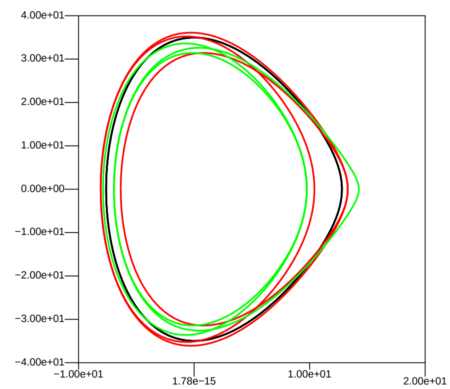

Figure 4: Subharmonic solutions for q=3: red is stable, green is unstable. The black orbit is the periodic one.

The symmetric solutions of (16) are those $z(t)$ such that $R(z(-t)) = z(t)$. This condition is equivalent to

$$z(0) \in \mathrm{Fix(R)} := \{\mathrm{z} \in \mathbb{R}^4 |\, \mathrm{Rz} = \mathrm{z}\} \tag{19}$$

by uniqueness of solution of the initial value problem (16). Note that $\mathrm{Fix(R)} = \{(\mathrm{x}_1, 0, \mathrm{y}_1, 0) \in \mathbb{R}^4 |\, \mathrm{x}_1, \mathrm{y}_1 \in \mathbb{R}\}$.

Let us now concentrate on a particular solution along the primary branch where bifurcation takes place. That is, consider a symmetric periodic solution with initial value $\tilde{z}_0 = (\tilde{x}_0, 0, 0, 0)$, where $\tilde{x}_0 \in (0, \omega_0^2)$ is such that (4) holds. The characteristic multipliers of this solution are 1, which has double multiplicity, and the complex conjugate pair $\{\mu, \overline{\mu}\}$, with $\mu = \exp(iT(\tilde{x}_0))$. From (4) it follows that $\mu^q = 1$, which illustrates the following result: a necessary condition for bifurcation at a particular periodic orbit is that this orbit has some characteristic multipliers which are roots of unity.

In our earlier proof of existence of branches bifurcating from the primary branch we used the fact that an orbit which intersects Fix(R) at two different points is, by reversibility, necessarily periodic (compare with (8)). In this thesis we will mainly use a different tool: a *Poincaré map* associated to a periodic orbit. To illustrate this consider again the initial point $\tilde{z}_0 = (\tilde{x}_0, 0, 0, 0)$ together with the 3-dimensional hyperplane $\Sigma_0 := \{(x_1, x_2, y_1, y_2) \in \mathbb{R}^4 |\, x_2 = 0\}$. Now, $\tilde{z}_0 \in \Sigma_0$, and $F(\tilde{z}_0)$ is transversal to

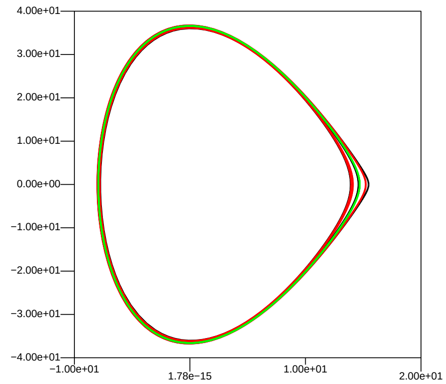

Figure 5: Subharmonics for $q = 5$: red is stable, black is unstable. The green orbit is the periodic one. We set $k = 2$, $l = 0$, $\omega_0^2 = 16$, $A = 1.5$. The Floquet multipliers of the unstable solutions are: 1 (double), 0.467929 and 2.137076 (simple and off the unit circle). Those of the stable solution are: 1 (double), $0.7316809 - 0.681647\,i$ and $0.7316809 + 0.681647\,i$ (simple and on the unit circle).

Σ_0. The orbit starting at $\tilde{z}_0$ returns to $\tilde{z}_0 \in \Sigma_0$ after time $T(\tilde{x}_0)$. Application of the Implicit Function Theorem implies that the same holds for the orbits starting at points $z \in \Sigma_0$ close to $\tilde{z}_0$, i.e., they return after time $\tau(z)$ close to $T(\tilde{x}_0)$ to a point $P(z) \in \Sigma_0$ that is again close to $\tilde{z}_0$. This defines a mapping $P : \Sigma_0 \to \Sigma_0$ (Poincaré map) for which $\tilde{z}_0$ is a fixed point:

$$P(\tilde{z}_0) = \tilde{z}_0. \tag{20}$$

One can show that (except for the multiplier 1 counted once) the eigenvalues of $DP(\tilde{z}_0)$ are precisely the characteristic multipliers of the periodic orbit through $\tilde{z}_0$. Also, since Σ_0 is R-invariant the map P inherits the reversibility of (16), that means

$$R \circ P \circ R = P^{-1}. \tag{21}$$

We say that P is *R-reversible*. The bifurcating solutions $\tilde{x}_A(t)$, calculated above, start at $\tilde{z}_A := (x_0^*(A), 0, A, 0) \in \Sigma_0$ and return to the same point after time $2\pi k = qT(\tilde{x}_0)$, in between they intersect Σ_0 in $(q-1)$ points close to $\tilde{z}_0$. Therefore, $\tilde{z}_A$ ($A \neq 0$) is a fixed point of the q-th iterate of P:

$$P^q(\tilde{z}_A) = \tilde{z}_A. \tag{22}$$

That is, $\tilde{z}_A$ is a q-periodic point of P. Hence, we have the bifurcation of a branch of q-periodic points $\{\ \tilde{z}_A\}$ from the primary branch $\{(x_0, 0, 0, 0)|\, 0 < x_0 < \omega_0^2\}$ of fixed points. The bifurcation takes place at $\tilde{z}_0$ where $DP(\tilde{z}_0)$ has a pair of complex conjugates eigenvalues which are qth roots of unity. This brings us to the main topic of the first part of this thesis:

To develop a general framework for the study of the bifurcation of q-periodic points from a fixed point of reversible diffeomorphisms.

In our study we allow the diffeomorphisms to depend on parameters, and next to possible bifurcation scenarios, we shall study the stability properties of the bifurcating periodic points.

To introduce the second part of the thesis, let us return to the condition (4), which gives a rational relation between the period (or equivalently the frequency) of the periodic motion of (1) at which bifurcation takes place and the period of the forcing term which causes the bifurcation. We say that there is a *resonance* between the two frequencies involved. Then one can of course ask what happens when there is no resonance, i.e., the two frequencies are rationally independent. Are there any dynamical characteristics of the unperturbed system (1) that survive under a perturbation such as (3)?

It appears that our set up until now is too narrow to give a positive answer to this question. Therefore we broaden the scope a little bit by replacing the forcing term in (3) by something more general, for example

$$\ddot{x} + f(x) = Ag(\omega t), \tag{23}$$

where both A and ω are parameters, and $g : \mathbb{R} \to \mathbb{R}$ is 2π-periodic and even. We can rewrite (23) as a first order system:

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -f(x_1) + Ag(y) \\ \dot{y} & = & \omega, \end{array} \right. \tag{24}$$

where we consider y as an element of $S^1 \cong \mathbb{R}/(2\pi\mathbb{Z})$, i.e., the state space is $\mathbb{R}^2 \times S^1$. System (24) is reversible with respect to $\tilde{R} : \mathbb{R}^2 \times S^1 \to \mathbb{R}^2 \times S^1$ given by $\tilde{R}(x_1, x_2, y) = (x_1, -x_2, -y)$. For $A \neq 0$ and fixed $\omega > 0$, (24) has a one-parameter family of invariant 2-tori of the form $\Gamma_{x_0} \times S^1$, where $\Gamma_{x_0} \subset \mathbb{R}^2$ is the periodic orbit of (1) passing through $(x_0, 0) \in \mathbb{R}^2$ $(0 < x_0 < \omega_0^2)$. The flow on such tori is periodic if there exist integers $(k, q) \neq (0, 0)$ such that $qT(x_0) = k2\pi/\omega$, and *quasi periodic* otherwise. A natural question is which of these invariant 2-tori persist for small $A \neq 0$. This is the kind of question treated in Part II of the thesis. It turns out that persistence can be guaranteed for those unperturbed tori whose flow is 'sufficiently' quasi periodic, i.e., the two frequencies involved satisfy an appropriate *Diophantine condition*. In the (x_0, ω)-space the subset of such tori forms a Cantor set of large measure.

Part II forms in several aspects an anti-pole of Part I:

- vector fields versus maps;
- quasi periodicity versus periodicity;
- continuation versus bifurcation;
- KAM techniques of rapid convergence versus application of the Implicit Function Theorem;
- diophantine conditions versus resonances;
- Cantor sets versus smooth branches.

However, the unifying theme is **reversibility**.

Part I

Bifurcation of periodic orbits for families of reversible maps

1

Setting of the Problem and Results

In the theory of dynamical systems, a broad interest is with the study of periodic solutions of ordinary differential equations. In particular the focus is on periodic solutions in the presence of resonance. It is known for a long time, compare with [3], that in such cases many bifurcations may occur, for example, of subharmonic periodic solutions. During the last century a lot of researchers have contributed to the development of theories of generic bifurcations in various contexts, such as for Hamiltonian systems, symmetric or reversible systems, and also for systems where no a priori structure has to be preserved, compare with [69, 70, 3, 41, 7, 72, 84, 21, 38, 43, 60, 62, 65]. There are two more or less standard approaches to bifurcation problems, namely, the Lyapunov-Schmidt (LS) reduction, and a reduction based on normal form theory. The LS reduction method concentrates (exclusively) on periodic solutions, ignoring other dynamic behaviour. It is usually applied to reduce existence problems for periodic solutions of a given system to solving algebraic equations on a lower dimensional space [81]. A major property of this approach is that it leads to equations which have an explicit circle-symmetry (S^1-symmetry) generated by the semisimple part of the linearization of the system. In contrast, the normal form approach keeps full track of the dynamics and essentially consists in making transformations which put the given equations in a 'simpler' form [43]. See also [41].

In [82] the authors present a method which in a sense combines both LS and normal form reduction to study periodic solutions near equilibria in Hamiltonian systems, with a period near a given T_0. The basic idea is to first bring the system into normal form up to appropriately high order and then to apply a *Generalized Lyapunov-Schmidt Reduction* (GLS). The term generalized refers to the fact that the problem is reduced to solving a set of algebraic equations on the (Jordan) *generalized* null-space corresponding to the purely imaginary eigenvalues of the linearization at the equilibrium which are in resonance with T_0, and not on the null-space as in the usual Lyapunov-Schmidt reduction, compare e.g. with [81]. The advantage is that these algebraic equations can be interpreted as the problem of finding relative equilibria of an S^1-equivariant Hamiltonian system (the *reduced* system) on the generalized null-space. So, the existence problem for periodic solutions of the original system reduces to the same problem for a reduced system which in general is lower dimensional, Hamiltonian and S^1-equivariant. Moreover, the reduced equations are up to higher order terms given by the restriction of the normal form of the original ones. This provides an additional circle-symmetry. Other (reasonable) structures of the original system such as reversibility, equivariance with respect to a symmetry group, etc. are inherited by the reduced equation. Somewhat similar approaches were described earlier in [38, 84, 65, 61]. The method was then sharpened to handle Hopf bifurcation problems at k-fold resonance for conservative, equivariant, or time reversible systems [52, 53, 54]. In (almost) all cases next to the theory for vector fields a similar approach exists for diffeomorphisms. Often, instead of directly dealing with (subharmonic) bifurcations of the periodic solution, one studies fixed points (periodic points) of a corresponding Poincaré return map. If the differential equations preserve an additional structure, this is reflected in structure properties of the return map. Note that subharmonic periodic solutions correspond to periodic points of the map. In [74] a result similar to [82] was proved for the bifurcation of periodic points from a fixed point of a family of diffeomorphisms. In that case the additional circle-symmetry of the reduced equation is replaced by $\mathbb{Z}_q$-equivariance, where q is the period. The term 'general' refers to the fact that no extra structure has to be preserved. A natural question is whether the results of [74] can be reformulated in a manner that applies in structure-preserving settings. In [32] the case of a family of symplectic diffeomorphisms was analysed. In this thesis we focus on the case of reversible diffeomorphisms. Our aim is fourfold.

1- To develop a structure-preserving Generalized Lyapunov-Schmidt Reduction (GLS) for bifurcations of periodic points from a fixed point in families of reversible diffeomorphisms. In particular we study bifurcation of periodic orbits of a given period q. The reduction leads to a similar problem on a lower dimensional space with an additional $\mathbb{D}_q$-symmetry.

2- To review a structure-preserving normal form theory for families of reversible diffeomorphisms as developed earlier in [69, 70, 18, 8, 84, 9, 72] and relating these normal forms to the reduced problem.

3- To exploit the $\mathbb{D}_q$-symmetry and the normal forms to prove existence of subharmonic bifurcations at resonances and also in a simple case of multiple resonance. Here the fixed point is called resonant when the derivative at the fixed point has roots of unity as eigenvalues. We have multiple resonance when a pair of complex conjugate resonant eigenvalues has higher multiplicity or when there is more than one such pair of resonant eigenvalues.

4- To discuss briefly how the reduction and normal form results can be used to determine stability properties for the bifurcating periodic points.

1.1 Preliminaries

A natural setting for subharmonic bifurcations as introduced above is that of germs of fixed points of diffeomorphisms depending on parameters. Therefore, consider a C^∞-smooth local map $\Phi : (\mathbb{R}^n, 0) \times (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$, $(x, \lambda) \mapsto \Phi_\lambda(x) = \Phi(x, \lambda)$ satisfying the following hypotheses: for all $\lambda \in \mathbb{R}^m$ in a neighbourhood of zero,

$$\boxed{\Phi_\lambda(0) = 0 \text{ and } D_x\Phi_\lambda(0) \in \mathcal{L}(\mathbb{R}^n) \text{ is invertible.}} \tag{H1}$$

Given an integer $q \geq 1$, our interests is with all small q-periodic points of Φ_λ, for λ near 0. Therefore, we want to determine all solutions (x, λ) near $(0, 0)$ of the equation

$$\boxed{x = \Phi_\lambda^q(x),} \tag{P}$$

where $\Phi_\lambda^q := \Phi_\lambda \circ \cdots \circ \Phi_\lambda$ (q times). We further restrict to the case where all (local) diffeomorphisms Φ_λ satisfy the *reversibility condition*

$$\boxed{R \circ \Phi_\lambda \circ R = \Phi_\lambda^{-1}.} \tag{R}$$

with respect to a given linear involution $R \in \mathcal{L}(\mathbb{R}^n)$. The main observation at this point is that the problem (P) has an *implicit* $\mathbb{D}_q$*-symmetry.* To explain this, let $\mathcal{S}_\lambda^q$ be the solution set of (P) defined by

$$\mathcal{S}_\lambda^q := \{x \in \mathbb{R}^n | \, \Phi_\lambda^q(x) = x\}. \tag{1.1}$$

Then $x \in \mathcal{S}_\lambda^q$ implies that $\Phi_\lambda(x) \in \mathcal{S}_\lambda^q$. Now, since Φ_λ^q acts as the identity on $\mathcal{S}_\lambda^q$, it follows that Φ_λ generates a $\mathbb{Z}_q$-action on $\mathcal{S}_\lambda^q$ [27], independent of

the reversibility. On the other hand $x \in \mathcal{S}_\lambda^q$ also implies that $Rx \in \mathcal{S}_\lambda^q$, and R generates a $\mathbb{Z}_2$-action on $\mathbb{R}^n$ since $R^2 = \mathrm{I}$. Then, Φ_λ and R together generate a $\mathbb{D}_q$-action on $\mathcal{S}_\lambda^q$. We call this $\mathbb{D}_q$-symmetry implicit because it appears on the yet to determine solution set $\mathcal{S}_\lambda^q$.
Our approach to solve (P) consists of a GLS reduction which lowers the dimension of the problem and leads to algebraic bifurcation equations that are *explicitly* $\mathbb{D}_q$-symmetric. This will be made clear in the next sections.

1.2 Reversible Generalized Lyapunov-Schmidt (GLS) Reduction

In this section we formulate the main GLS reduction result and explain in what sense the above $\mathbb{D}_q$-symmetry is made explicit. The key tool of our approach is to replace the equation $\Phi_\lambda^q(x) = x$ for q-periodic points of Φ_λ, by an equivalent equation for q-periodic *orbits* of Φ_λ on an appropriate *orbit space*, and then perform the LS reduction to the latter problem [74, 32]. Starting point is the observation that the orbit of a point $x \in \mathbb{R}^n$ under Φ_λ can be seen as a point $y := (\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots)$ in the sequence space $(\mathbb{R}^n)^{\mathbb{Z}}$ where $y_j := \Phi_\lambda^j(x)$, $j \in \mathbb{Z}$. Define the *orbit space* Y_q as the nq-dimensional subspace of q-periodic sequences, i.e.,

$$Y_q := \{y = (y_j)_{j\in\mathbb{Z}} \in (\mathbb{R}^n)^{\mathbb{Z}} \,|\, y_{j+q} = y_j, \, \forall i \in \mathbb{Z}\} \subset (\mathbb{R}^n)^{\mathbb{Z}}. \tag{1.2}$$

Obviously, Φ_λ can be lifted to Y_q by taking $\widehat{\Phi}_\lambda : Y_q \to Y_q$ as

$$\widehat{\Phi}_\lambda(y) := (\Phi_\lambda(y_i))_{i\in\mathbb{Z}}\,, \quad y \in Y_q. \tag{1.3}$$

If we also introduce the (left) *shift operator*

$$\sigma : Y_q \to Y_q, \quad (\sigma \cdot y)_i := y_{i+1}, \quad i \in \mathbb{Z}, \, y \in Y_q, \tag{1.4}$$

then solving (P) is equivalent to solving

$$\widehat{\Phi}_\lambda(y) = \sigma \cdot y, \tag{1.5}$$

for $(y, \lambda) \in Y_q \times \mathbb{R}^n$ in a neighbourhood of $(0, 0)$. The advantage of lifting equation (P) to the orbit space Y_q, is that equation (1.3) is equivariant under the $\mathbb{Z}_q$-action generated by σ on Y_q, i.e. $\widehat{\Phi}_\lambda \circ \sigma = \sigma \circ \widehat{\Phi}_\lambda$. To get the full $\mathbb{D}_q$-equivariance as we announced, we go a step further and define the lift γ of R (*reversor operator*) to the orbit space Y_q as follows. Let $\gamma \in \mathcal{L}(Y_q)$ be given by

$$(\gamma \cdot y)_i := Ry_{-i}, \quad \forall i \in \mathbb{Z}, \, \forall y \in Y_q, \tag{1.6}$$

then

$$\gamma \circ \widehat{\Phi}_\lambda \circ \gamma = \widehat{\Phi}_\lambda^{-1}, \tag{1.7}$$

Since $\gamma^2 = \mathrm{I}$, (1.7) means that $\widehat{\Phi}_\lambda$ is γ-reversible.
A straightforward application of the (classical) LS reduction [72, 41] to equation $\widehat{\Phi}_\lambda(y) = \sigma \cdot y$ (1.3) results in a bifurcation equation of the form $E(v, \lambda) = 0$, where $E(\cdot, \lambda)$ is a $\mathbb{Z}_q$-equivariant map from $\ker(D\widehat{\Phi}_0(0) - \sigma)$ into a complement of $\mathrm{Im}(D\widehat{\Phi}_0(0) - \sigma)$ satisfying $E(0, \lambda) = 0$ and $D_v E(0, 0) = 0$. In the case where $A_0 = D_x\Phi_0(0)$ and hence also $\widehat{A}_0 = D\widehat{\Phi}_0(0)$ are non-semisimple, the details of the reduction strongly depend on the nilpotent part of A_0. Since we do not want to impose any restriction on A_0 except that it has to be invertible, we perform a LS reduction with respect to the semisimple part $\widehat{S}_0$ of $\widehat{A}_0$, cf. [74, 32]. 'Semisimple' here means complex diagonalisable, [56]. The starting point of our GLS reduction is the decomposition

$$Y_q = \ker(\widehat{S}_0 - \sigma) \oplus \mathrm{Im}(\widehat{S}_0 - \sigma). \tag{1.8}$$

Introducing the *reduced phase space* $U \subset \mathbb{R}^n$ given by

$$U := \ker(S_0^q - \mathrm{I}), \tag{1.9}$$

where S_0 is the semisimple part of $D_x\Phi_0(0)$, one directly proves that

(i) U and $\ker(\widehat{S}_0 - \sigma)$ are isomorphic,

(ii) S_0 generates a natural $\mathbb{Z}_q$-action on U,

(iii) U is invariant under R.

Then, the following reduction result holds.

Theorem 1 (Reversible GLS Reduction). *Let $\Phi : (\mathbb{R}^n, 0) \times (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ be a local family of R-reversible diffeomorphisms, satisfying* (H1). *Let $S_0 \in GL_{-R}(n, \mathbb{R})$ be the semisimple part of $A_0 := D_x\Phi_0(0)$, let $q \geq 1$, and define the reduced phase space U as in* (1.9). *Then there exist a family of (reduced) diffeomorphisms $\Phi_{r,\lambda} : U \to U$ and a map $x^* : U \times \mathbb{R}^m \to \mathbb{R}^n$ such that for each sufficiently small $\lambda \in \mathbb{R}^m$ the following properties hold:*

(i) $\Phi_{r,\lambda}(0) = 0$, $D\Phi_{r,\lambda=0} = A_0|_U$, $x^*(0, \lambda) = 0$ *and, for all* $\tilde{u} \in U$, $D_u x^*(0,0) \cdot \tilde{u} = \tilde{u}$;

(ii) *$x^*(Ru,\lambda) = Rx^*(u,\lambda)$;*

(iii) *$\Phi_{r,\lambda}$ is $\mathbb{Z}_q$-equivariant: $\Phi_{r,\lambda}(S_0u) = S_0\Phi_{r,\lambda}(u)$;*

(iv) *$\Phi_{r,\lambda}$ is R-reversible: $R \circ \Phi_{r,\lambda} \circ R = \Phi_{r,\lambda}^{-1}$;*

(v) *for sufficiently small $(x,\lambda) \in \mathbb{R}^n \times \mathbb{R}^m$ the point x is q-periodic under Φ_λ if and only if $x = x^*(u,\lambda)$, with $u \in U$ q-periodic under $\Phi_{r,\lambda}$;*

(vi) *for sufficiently small $(u,\lambda) \in U \times \mathbb{R}^m$ the point u is q-periodic under $\Phi_{r,\lambda}$ if and only if*

$$\Phi_{r,\lambda}(u) = S_0u. \tag{1.10}$$

Moreover, let $\mathcal{B} : U \times \mathbb{R}^n \to U$ be defined by

$$\mathcal{B}(u,\lambda) := S_0^{-1}\Phi_{r,\lambda}(u) - S_0\Phi_{r,\lambda}^{-1}, \tag{1.11}$$

then

(vii) *a point $(u,\lambda) \in U \times \mathbb{R}^m$ is a solution of equation (1.10) if and only if it is a solution of*

$$\mathcal{B}(u,\lambda) = 0; \tag{1.12}$$

(viii) *the map $\mathcal{B}(\cdot,\lambda)$ is $\mathbb{D}_q$-equivariant:*

$$\mathcal{B}(S_0u,\lambda) = S_0\mathcal{B}(u,\lambda) \quad \text{and} \quad \mathcal{B}(Ru,\lambda) = -R\mathcal{B}(u,\lambda). \tag{1.13}$$

We call equation (1.10) the *determining equation*. While we refer to the map $\mathcal{B}(\cdot,\lambda)$ as to the *branching function*, and to equation (1.12) as to the *branching equation*.

Remarks

1- The GLS reduction as described above can also be worked out such that (reasonable) additional structures of Φ_λ are preserved. In [32] and [25] the symplectic case has been analysed.

2- It has not escaped our attention that an analogue result of Theorem 1 also hold when generalizing the definition of reversibility as follows. Let V be a finite dimensional state space and let $\Gamma \subset \mathcal{L}(V)$ be a compact group with non-trivial character $\chi : \Gamma \to \mathbb{Z}_2$. A diffeomorphism $\Phi \in C^\infty(V)$ is Γ- reversible if

$$\Phi(\gamma x) = \gamma \Phi(x)^{\chi(\gamma)}, \quad x \in V, \gamma \in \Gamma. \tag{1.14}$$

Note that the case we analysed in this thesis corresponds to the simplest situation of $\Gamma = \{I_V, R\}$, with $R \in \mathcal{L}(V)$ linear involution: $R^2 = \mathrm{I}_V$, and $\chi(R) = -1$.

1.3 Structure-preserving Parametrized Normal Forms

To apply Theorem 1 to concrete examples one needs a method to calculate or approximate the reduced diffeomorphism $\Phi_{r,\lambda}$. We approximate $\Phi_{r,\lambda}$ by a normal form of Φ_λ. Before exploring this further, let us first explain what 'normal form' means in the present context. Our main concern is to simplify the Taylor series of the diffeomorphism Φ_λ at the fixed point. 'Simplicity' here means symmetry: the normalized part is invariant under certain linear transformations and, in particular, is reversible. To this purpose we adapt the result of Takens [70] for this reversible case with parameters by generalizing Vanderbauwhede [74]. Note that the problem of structure preserving normal forms for vector fields near equilibria was addressed earlier by Broer [8, 9] in terms of graded and filtered Lie algebras and then extended to the case of germs of diffeomorphisms in [18]. We present a variation on these ideas. Recall that a linear operator $A \in gl(n, \mathbb{R})$ admits the unique Jordan-Chevalley decomposition (SN decomposition): $A = S + N$, where $S \in gl(n, \mathbb{R})$ is semisimple, $N \in gl(n, \mathbb{R})$ is nilpotent, and $SN = NS$, see [56]; S is called the semisimple part of A and N the nilpotent part. In the case of invertible operators, however, it can be more suitable to use the *semisimple-unipotent* decomposition, (SU decomposition). Namely, if $A \in GL(n, \mathbb{R})$ one writes

$$A = Se^{\mathcal{N}},$$

with S semisimple, $\mathcal{N}$ nilpotent and $S\mathcal{N} = \mathcal{N}S$. This decomposition is unique with S the same as in the SN decomposition and $e^{\mathcal{N}} = I + S^{-1}N$. For more details see Proposition 2.6 below and also [74]. Now, fix a scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ as given in the following lemma.

Lemma 1.1. *Let $S_0 \in GL_{-R}(n,\mathbb{R})$ be semisimple. Then there exists a scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$ such that when we denote the transpose of a linear operator $A \in GL_{-R}(n,\mathbb{R})$ with respect to this scalar product by A^T the following holds:*

(i) *the involution $R \in gl(n,\mathbb{R})$ is orthogonal, i.e. $R^T R = I$;*

(ii) *A linear operator $A \in \mathcal{L}(\mathbb{R}^n)$ commutes with S_0 if and only if it commutes with S_0^T: $\ker\left(\mathrm{ad}(S_0^T)\right) = \ker\left(\mathrm{ad}(S_0)\right)$.*

The proof of Lemma 1.1 is postponed to chapter 2. Observe that (i) implies that together with A also A^T belongs to $GL_{-R}(n,\mathbb{R})$.

Theorem 2 (Parametrized Reversible Normal Form). *Assume that Φ_λ satisfies (H1), and (R) and set $A_0 := D\Phi_0(0)$. Let $A_0 = S_0 e^{\mathcal{N}_0}$ be the SU-decomposition of A_0, and let $\langle\cdot,\cdot\rangle$ be a scalar product as in Lemma 1.1. Then, for each $k \geq 1$ there exists a parameter-dependent near-identity R-equivariant transformation $\Psi_{k,\lambda} : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\Psi_{k,\lambda}^{-1} \circ \Phi_\lambda \circ \Psi_{k,\lambda} = S_0 e^{\mathcal{N}_0 + Z_\lambda} + R_{k+1}, \tag{1.15}$$

with $R_{k+1}(x,\lambda) = O(||x||^{k+1})$ uniformly for λ in a neighbourhood of 0. Moreover, the smooth family of vector fields $Z_\lambda(x)$ is such that

$$Z_\lambda(0) = 0, \quad DZ_{\lambda=0}(0) = 0, \tag{1.16}$$

$$S_0 \circ Z_\lambda = Z_\lambda \circ S_0, \tag{1.17}$$

$$DZ_\lambda(x)\mathcal{N}_0^T x = \mathcal{N}_0^T Z_\lambda(x), \tag{1.18}$$

$$R \circ (\mathcal{N}_0 + Z_\lambda) = -\left(\mathcal{N}_0 + Z_\lambda\right) \circ R. \tag{1.19}$$

The exponential $e^{\mathcal{N}_0+Z_\lambda}$ denotes the time-one map of the vector field $\mathcal{N}_0+Z_\lambda$. We call

$$\Phi_\lambda^{NF} := S_0 e^{\mathcal{N}_0+Z_\lambda}$$

the *normal form* of Φ_λ *up to order* k or, briefly, the *truncated* normal form. Note that some authors call $\mathcal{N}_0 + Z_\lambda$ the Takens normal form. If we wish to underline that the property (1.18) holds, then we refer to (1.15) also as to the *reversible nilpotent normal form.* The Parametrized Reversible Normal Form (PRNF) Theorem 2 implies that the group of linear transformations generated by S_0 is a formal symmetry group of Φ_λ while Z_λ commutes with $e^{t\mathcal{N}_0^T}$ and is R-reversible. A property is called 'formal' if it holds up to a certain order (k in the theorem above). The proof of the PRNF Theorem 2, given below, is inductive and based on a combined use of the (standard) adjoint action of the group of diffeomorphisms satisfying (H1) and the Implicit Function Theorem.

Remarks

1- In [32] (see also [25]) the analogous of the PRNF Theorem 2 is proved in the case that Φ_λ is a symplectic diffeomorphism.

2- The result of the PRNF Theorem 2 also holds when generalizing the definition of reversibility as in (1.14).

Returning to the reduced map $\Phi_{r,\lambda}$, it turns out that the restriction of the normal form (1.15) of Φ_λ to the reduced phase space $U = \ker(S_0^q - \mathrm{I})$ gives a good approximation of $\Phi_{r,\lambda}$. Indeed, the following holds.

Corollary 1.2. *Suppose that Φ_λ satisfying (H1) and (R) is in normal form up to order k. Then,*

$$\Phi_{r,\lambda}(u) = \Phi_\lambda^{NF}(u) + O(||u||^{k+1}), \tag{1.20}$$

as $u \to 0$ uniformly for λ in a neighbourhood of $0 \in \mathbb{R}^m$. Moreover, one has

$$x^*(u, \lambda) = u + O(||u||^{k+1}), \tag{1.21}$$

as $u \to 0$, uniformly for λ in some neighbourhood of $0 \in \mathbb{R}^m$.

As a consequence, the solutions of the determining equation (1.10) can be approximated by the equilibria $u \in U$ of the normal form vector field $\mathcal{N}_0 + Z_\lambda(\cdot)$. Namely, (1.10) can be approximated by $e^{\mathcal{N}_0 + Z_\lambda}(u) = u$ and the (small) solutions $u \in U$ of this equations are given by equilibria of the normal form vector field $\mathcal{N}_0 + Z_\lambda(\cdot)$, i.e., the solutions of $\mathcal{N}_0 u + Z_\lambda(u) = 0$. See section 3.2.2 for the proof of Corollary 1.2.

1.4 Stability

In this section we describe how to obtain information on the stability of bifurcating periodic orbits. When $x \in \mathbb{R}^n$ generates a q-periodic orbit of Φ_λ then the (linear) stability of this orbit is determined by the eigenvalues of $D\Phi_\lambda^q(x)$: the orbit is stable if all eigenvalues are inside the unit circle, and unstable if there are any eigenvalues outside the unit circle. When the periodic orbit is symmetric (i.e. invariant under R) then together with $\mu \in \mathbb{C}$ also μ^{-1} will be an eigenvalue of $D\Phi_\lambda^q(x)$. In such case the orbit is unstable if an eigenvalue is off the unit circle, and there is a weak form of stability if all eigenvalues are on the unit circle. For bifurcating periodic orbits $x = x^*(u, \lambda)$ as given in Theorem 1 in order to establish stability one has to determine the eigenvalues of

$$\mathcal{D}(u, \lambda) := D\Phi_\lambda^q\left(x^*\left(u, \lambda\right)\right),$$

for all small $(u, \lambda) \in U \times \mathbb{R}^m$ satisfying the determining equation $\Phi_{r,\lambda}(u) = S_0 u$. For $(u, \lambda) = (0, 0)$ we find $\mathcal{D}(0, 0) = A_0^q$, which implies that 1 is an eigenvalue of $\mathcal{D}(0, 0)$ with algebraic multiplicity equal to dim U, and with geometric multiplicity equal to the sum of the geometric multiplicities of the resonant eigenvalues of A_0. We assume that

$$\boxed{\begin{array}{c}\text{all non-resonant eigenvalues of } A_0 \text{ are simple and}\\ \text{on the unit circle.}\end{array}} \tag{S}$$

Recall that the non-resonant eigenvalues μ are such that $\mu^q \neq 1$. For small (u, λ) the eigenvalues of $\mathcal{D}(u, \lambda)$ are close to those of $\mathcal{D}(0, 0)$; in particular if (S) holds then the eigenvalues of $\mathcal{D}(u, \lambda)$ not close to 1 will be simple. One then shows that the stability of the symmetric periodic orbits is determined by the eigenvalues of $\mathcal{D}(u, \lambda)$ close to 1. We call these eigenvalues *critical* . To calculate them we use the following result.

Proposition 1.3. *Assume that Φ_λ satisfies* (H1), (R), (S) *and is in normal form up to order k. Then there exists a smooth mapping $\widetilde{\mathcal{D}} : U \times \mathbb{R}^m \to \mathcal{L}(U)$, with*

$$\widetilde{\mathcal{D}}(u, \lambda) = D\Phi_\lambda^{NF}(u)|_U + O(||u||^k), \tag{1.22}$$

such that for all sufficiently small solutions (u, λ) of the determining equation (1.10) *the critical eigenvalues of $\mathcal{D}(u, \lambda)$ are given by the q^{th} powers of the eigenvalues of $\widetilde{\mathcal{D}}(u, \lambda)$.*

Now, up to terms of order k the solutions of (1.10) are given by the fixed points of $\Phi_\lambda^{NF}|_U$, or equivalently, by the zeros of $(\mathcal{N}_0 + Z_\lambda)|_U$. Therefore, according to Proposition 1.3, the stability of the corresponding periodic orbit of Φ_λ up to terms of order $(k-1)$ is determined by the eigenvalues of

$$\exp\left(\mathcal{N}_0 + DZ_\lambda(u)\right)|_U.$$

Hence, the stability properties of the bifurcating periodic solutions of Φ_λ are expected to be the same as those of the corresponding equilibria of the normal form vector field $\mathcal{N}_0 + Z_\lambda$ restricted to the reduced phase space U. In applications the challenge usually is to take the order k sufficiently large such that higher order terms do not change the qualitative picture obtained from the normal form.

1.5 Bifurcation of Periodic Points

The classical Lyapunov Center Theorem for reversible vector fields [36] (with $\dim(\mathrm{Fix}\, R) = \frac{1}{2}\dim V$, where V is the phase space) states that under appropriate non-resonance conditions to each pair of simple purely imaginary eigenvalues of the linearization at a symmetric equilibrium there corresponds a one-parameter family of symmetric periodic orbits. This family originates at the equilibrium and generates a two-dimensional invariant manifold filled with periodic orbits surrounding the equilibrium. This picture is persistent under small reversible perturbations. That is, due to reversibility, the simple eigenvalues remain simple and on the imaginary axis, and each of them still generates a one-parameter family of periodic orbits. The situation changes in presence of resonances: under a change of parameters certain purely imaginary eigenvalues can coalesce and split off the imaginary axis. The problem is to study what happens to the families of periodic orbits associated to the two pairs of purely imaginary eigenvalues when these leave the imaginary axis. This question is put in the form of a Hopf bifurcation problem as follows: study the bifurcation of periodic solutions from an equilibrium at a parameter value for which some eigenvalues of the linearization leave the imaginary axis. The related bifurcation scenario has been analysed in [54]. We aim at studying the corresponding situation for maps. That is, under suitable assumptions we want to solve problem (P) for various values of q. Since we are dealing with maps, the imaginary axis is replaced by the unit circle and the role of the pair of simple purely imaginary eigenvalues is taken over by a pair of simple eigenvalues on the unit circle which are roots of unity. We are mainly interested in two cases: bifurcations at a *simple root of unity* (SRU case) and bifurcations at a *resonant root of unit* (RRU case). In the former case the main assumption is that the linearization at $(x, \lambda) = (0, 0)$ of Φ_λ has got a pair of simple eigenvalues that are roots of unity and these are the only eigenvalues on the unit circle which are roots of unity. In the latter case the eigenvalues are no longer simple, indeed, the linearization at $(0,0)$ has a pair of non-semisimple eigenvalues on the unit circle that are roots of unity, with algebraic multiplicity 2 and geometric multiplicity 1.

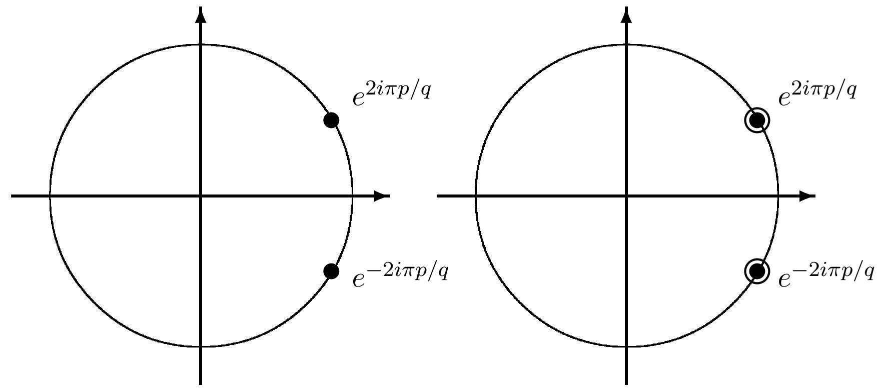

Figure 1.1: Eigenvalue configurations for the SRU case and RRU case. A dot denotes a simple eigenvalue; circle-dot denotes a double eigenvalue.

Remark Observe that when considering a m-parameter family of reversible maps Φ_λ satisfying (H1), by reversibility, if $\mu \in \mathbb{C}$ is an eigenvalue of the linearization $A_\lambda = D\Phi_\lambda(0)$ then so are μ^{-1}, $\overline{\mu}$ and $\overline{\mu}^{-1}$. Assuming that A_λ has a pair of simple eigenvalues on the unit circle (different from ± 1) it follows that the continuation of these eigenvalues stays on the unit circle for all nearby parameter values. Hence, one typically find an infinite number of parameter values for which these eigenvalues are roots of unity. Now, if A_λ has a pair of resonant eigenvalues on the unit circle (different from ± 1), these might split off the circle. Now, if $\lambda \in \mathbb{R}^2$ (i.e. $m = 2$), by varying one of the two parameters one can take care that the resonant eigenvalues are roots of unity, while the variation of the other parameter will imply that the eigenvalues split off the circle.

1.5.1 Bifurcation at a Simple Root of Unity (SRU)

In this section we specify the hypotheses of the SRU case and state the corresponding bifurcation result that solves problem (P) for this case. Consider a one-parameter family of maps Φ_λ (i.e., $\lambda \in \mathbb{R}$) satisfying (H1), (R). For fixed $q \geq 3$ and $0 < p < q$ with $\gcd(\mathrm{p}, \mathrm{q}) = 1$, assume that

$$\boxed{A_0 \text{ has a pair of simple eigenvalues } \exp(\pm 2\pi p/q),} \tag{H2}$$

and

$$\boxed{A_0 \text{ has no other eigenvalues } \mu \in \mathbb{C} \text{ such that } \mu^q = 1.} \tag{H2a}$$

Now, let $\beta_q(\lambda) \in \mathbb{R}$ with $\beta_q(0) = 0$, then the continuation of the eigenvalue $\exp(2\pi p/q)$ can be written as $\exp(i\beta_q(\lambda))\exp(2i\pi p/q)$. We assume the *transversality condition*

$$\boxed{\beta_q'(0) \neq 0.} \tag{T1}$$

Remark The usual hypothesis in the treatment of bifurcation of subharmonic solutions, see e.g. [81, 44], is that there is a pair of simple characteristic multipliers crossing the unit circle transversally at a root of unity. Here reversibility prevents such transversal crossing: the multipliers stay on the unit circle. The hypothesis (T1) then gives transversality along the unit circle.

Now, using the Reversible GLS Reduction Theorem 1 in combination with the $\mathbb{D}_q$-symmetry, one solves (P) by solving the bifurcation equation (1.12), which turns out to be equivalent to

$$\mathcal{B}(z, \lambda) = i\theta_1(z, \lambda)z + i\theta_2(z, \lambda)\overline{z}^{q-1}, \tag{1.23}$$

with $\theta_i : \mathbb{C} \times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ smooth real-valued $\mathbb{D}_q$-invariant functions, see section 6.1 for details. Assuming the *non-degeneracy condition*

$$\boxed{\theta_2(0, 0) \neq 0,} \tag{ND}$$

the main result can be stated as follows.

Theorem 3 (SRU). *Under the hypotheses* (H1), (R), (H2), (H2a),(T1) *and* (ND), *if $q \geq 3$, exactly two R-symmetric branches of q-periodic orbits of Φ_λ bifurcate at $\lambda = 0$ from the fixed point $x = 0$. The solutions within each branch are related to one another by the $\mathbb{D}_q$-action of R and S_0 on U. Also, under a further appropriate assumption, for $q \geq 5$, the solutions in one branch are stable, while those in the other are unstable.*

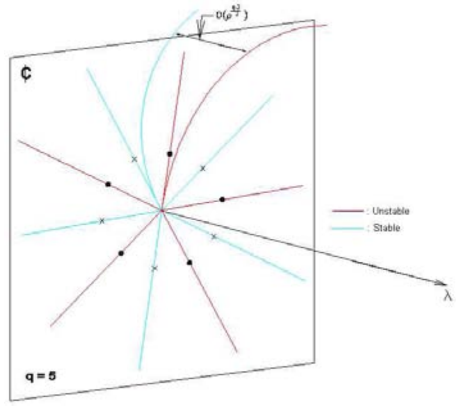

Figure 1.2: Illustration of Theorem 3 for q=5. The picture gives a section of the solution set for a fixed value of the parameter λ. The two whiskers represent two bifurcating solutions. The lines with dots and crosses symbolize the fact that the problem has a $\mathbb{Z}_q$-symmetry and therefore given 2 solutions one finds their orbits under Φ_λ (i.e. the other solutions) by rotations of $2\pi/q$.

The further hypothesis necessary for the stability result is specified later in Lemma 6.4.

Remarks

1- In [32] the symplectic versions of Theorem 1 and Theorem 2 are combined to prove a result similar to the SRU Theorem 3 in the symplectic setting. This bifurcation result generalizes the classical results of K. Meyer [57] on generic bifurcation of periodic points for symplectic diffeomorphisms.

2- For $q \geq 5$, the tangency properties of the two branches of Theorem 3 are the same of those of the two curves defining the boundary of the (classical) Arnold tongues, [3, 74]. See section 6.1 for further details.

1.5.2 Bifurcation at a Resonant Root of Unity (RRU)

In this section we state the bifurcation theorem that solves problem (P) in the RRU case. To this purpose, we consider a two-parameter family of maps

Φ_λ (i.e. $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$) satisfying (H1), (R) and

$$\boxed{A_0 = D\Phi_0(0) \text{ has eigenvalues } \exp(\pm 2i\pi p/q)} \tag{H3}$$

where $0 < p < q$ with $\gcd(\mathrm{p}, \mathrm{q}) = 1$. Under the further assumptions that

$$\boxed{e^{\pm 2i\pi p/q} \text{ have algebraic multiplicity 2 and geometric multiplicity 1}} \tag{H3a}$$

and that

$$\boxed{A_0 \text{ has no other eigenvalues that are } q\text{th roots of unity,}} \tag{H3b}$$

application of the GLS reduction as before shows that we are left with a 4-dimensional problem on U, i.e., $\dim U = 4$. Setting $\theta_0 = 2\pi p/q$ and identifying U with $\mathbb{C} \times \mathbb{C}$ the linearization $A_\lambda = D\Phi_\lambda(0)$ on U takes the form

$$A(\lambda) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \exp\left(i(\theta_0 + \vartheta(\lambda))\right) \begin{pmatrix} 1 & 1 \\ \sigma(\lambda) & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

with $\sigma : \mathbb{R}^2 \to \mathbb{R}$, $\vartheta : \mathbb{R}^2 \to \mathbb{R}$ smooth functions such that $\sigma(0) = 0$ and $\vartheta(0) = 0$. It follows that as $\sigma(\lambda)$ increases through zero the eigenvalues of $A(\lambda)$ (λ small) move towards each other on the unit circle, collide at $\sigma(\lambda) = 0$, and split off the unit circle for $\sigma(\lambda) > 0$. Under the *transversality condition*

$$\boxed{\frac{\partial(\vartheta, \sigma)}{\partial(\lambda_1, \lambda_2)}(0,0) \neq 0,} \tag{T2}$$

one has the following.

Theorem 4 (RRU). *Under the hypotheses* (H1), (R), (H3), (H3a), (H3b), (T2)*, if $q \geq 3$, for each (sufficiently small) λ_1 two branches of R-symmetric q-periodic orbits of Φ_λ bifurcate at some parameter value $\lambda_2 = \widehat{\lambda}_2(\lambda_1)$ near zero.*

1.6 Subharmonic Branching in Reversible Vector Fields

We consider the case when the map Φ is the Poincaré map of a $2k$-dimensional autonomous time-reversible vector field with a non-constant R-symmetric periodic solution γ_0 of period T_0. It turns out that in this case 1 is always an eigenvalue of $D\Phi(0)$. This gives the existence of a one-dimensional branch of R-symmetric fixed points originating at 0, i.e., a one-parameter family of periodic solutions with period close to the period in the original system. Taking a coordinate on this branch as parameter, for isolated values of the parameter one meets symmetric fixed points at which the derivative of Φ has eigenvalues that are q-roots of unity, for some $q \geq 3$. Then, the question again is whether this leads to the branching of q-periodic points, i.e. subharmonic branching of the original system. We are interested in two cases: *subharmonic branching at a simple root of unity* (SBSRU case) and *subharmonic branching at a resonant root of unity* (SBRRU case). In the former case the main assumption is that the linearization of Φ at 0 has, next to the eigenvalue 1, a pair of simple eigenvalues that are qth roots of unity . In the latter case the linearization of Φ at 0 has eigenvalue 1 and a pair of non-semisimple eigenvalues with algebraic multiplicity 2 and geometric multiplicity 1 that are qth roots of unity.

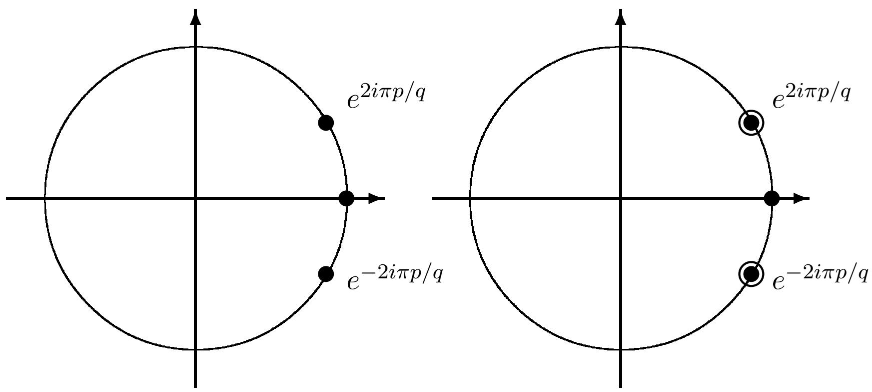

Figure 1.3: Eigenvalue configurations in the SBSRU case and SBRRU case. A dot denotes a simple eigenvalue; circle-dot denotes a double eigenvalue.

We briefly describe the set up, see also [33]. Let $R \in \mathcal{L}(\mathbb{R}^{2n})$ be a linear involution with $\dim \operatorname{Fix} R = n$. Let $X : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a R-reversible vector field, i.e., $X(Rx) = -RX(x)$, $x \in \mathbb{R}^n$ and denote by $\exp(tX)(x)$ the flow of the system $\dot{x} = X(x)$. The reversibility implies that $\exp(tX)(Rx) = R\exp(-tX)(x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^{2n}$. Let $p \in \mathrm{Fix}R$ and $T_0 > 0$ be such that $\exp(T_0X)(p) = p$ and $\exp(tX)(p) \neq p$ if $0 < t < T_0$. Then p generates a symmetric T_0-periodic orbit $\gamma_0 := \{\exp(tX)(p)|t \in \mathbb{R}\}$ which has a second intersection point with Fix$R$ given by $q := \exp(\frac{T_0}{2}X)(q) \in \mathrm{Fix}R$.

Now, $RX(p) = -X(Rp) = -X(p)$ and since R is semisimple then $\mathbb{R}^{2n} = \mathbb{R}X(p) \oplus S$ for some subspace S which is R-invariant and contains Fix$R$. It follows that $p \in S$ and that S can be used as a transversal section to γ_0 to construct the Poincaré map $P : S \to S$ so that $P(p) = p$. Moreover, from $R(S) = S$ and $\exp(tX)(Rx) = R\exp(-tX)(x)$ one obtains $RPR = P^{-1}$, i.e. P is R-reversible. Denoting $R|_S$ again by R and defining $\Phi : S \to S$ by

$$\Phi(x) := P(p + x) - p \quad \forall x \in S,$$

we have that $\dim S = 2n - 1$, $\dim \mathrm{Fix} R = n$, Φ is R-reversible and $\Phi(0) = 0$.

So, the map Φ satisfies (H1) with $n = 2n - 1$ and $m = 0$ and is reversible. The fixed point 0 corresponds to γ_0, other fixed points correspond to periodic orbits of the system close to γ_0 with minimal period close to T_0. Finally, q-periodic orbits of Φ correspond to subharmonic solutions of the system, i.e. periodic orbits near γ_0 with minimal period near qT_0. We are interested in those eigenvalues of $A_0 = D\Phi(0) = DP(p)$ that are qth roots of unity. It turns out that the possibilities for these eigenvalues are, [33]:

$q = 1$: 1 is typically simple eigenvalue and $\ker(A_0 - \mathrm{I}) = \mathbb{R}u_0$, where u_0 is the corresponding eigenvector in Fix R.

$q = 2$: generically, if -1 is an eigenvalue, then it has even multiplicity and is non-semisimple.

$q \geq 3$: eigenvalues of the form $\exp(\pm 2\pi p/q)$, with $q \geq 3$, $0 < p < q$ and $gcd(p, q) = 1$ can have any multiplicity but typically they are simple.

We analyse the bifurcation of q-periodic points from the fixed point for different choices of q, each time assuming generic hypotheses for the (resonant) eigenvalues of A_0. We start with the bifurcation of fixed points.

Indeed, assume (H1), (R) and consider problem (P) for $q = 1$. Assume that

$$\boxed{+1 \text{ is a simple eigenvalue of } A_0.} \tag{H4}$$

The space $U = \ker(S_0 - \mathrm{I})$ is identified with $U = \{\alpha u_0 | \, \alpha \in \mathbb{R}\}$, where $u_0 \in \mathbb{R}^n$ is an eigenvector corresponding to the eigenvalue 1 with $S_0 u_0 = u_0$, $\mathcal{N}_0 u_0 = 0$ and $Ru_0 = u_0$. It follows that the one-dimensional $\ker(S_0 - \mathrm{I})$ is such that $Ru = u$ for all $u \in \ker(S_0 - \mathrm{I})$. The reversible GLS reduction of section 1.2 now implies the following.

Theorem 5 (Primary Branch). *Assume* (H1), (R) *and* (H4). *Then there exists a smooth map* $x^* : \ker\,(S_0 - \mathrm{I}) \to \mathbb{R}^n$ *such that*

(i) $x^*(0) = 0$;

(ii) $Rx^*(u) = x^*(u)$, *for all* $u = \alpha u_0 \in U$;

(iii) $\Phi(x^*(u)) = x^*(u)$, *for all sufficiently small* $u = \alpha u_0 \in U$.

Moreover, Φ *has no other fixed points nearby the origin than those on the curve* $\{x^*(\alpha u_0) |\, \alpha \in \mathbb{R}\}$.

We call this branch of fixed points the *primary branch*.

Remark The reversibility of the operator $D\Phi(x^*(u))$ implies that if $\mu \in \mathbb{C}$ is an eigenvalue then so is μ^{-1}. Assuming that $D\Phi(x^*(u))$ has a pair of simple eigenvalues on the unit circle (different from ± 1) at some $u = \hat{u}$, then these stay on the unit circle for all u near $\hat{u}$. Hence, one typically find that along the primary branch there are symmetric fixed points at which the linearization of Φ has eigenvalues which are qth roots of unity, for some $q \geq 3$. As it will be shown below, this leads to the branching of periodic points for Φ, which means subharmonic branching for the original system.

1.6.1 Subharmonic Branching at a Simple Root of Unity (SB-SRU)

Assume (H1), (R) and (H4). Our goal is to solve problem (P) for some fixed $q \geq 3$ in the further assumption that

$$\boxed{A_0 \text{ has the simple eigenvalues } \exp(\pm 2i\pi p/q),} \tag{H5}$$

where $0 < p < q$, $gcd(p, q) = 1$, and that

$$\boxed{A_0 \text{ has no other eigenvalues that are } q\text{th roots of unity.}} \tag{H5a}$$

From (H5)-(H5a) it follows that $U = \ker\,(S_0^q - \mathrm{I})$ is 3-dimensional. Also, $\ker\,(S_0 - \mathrm{I}) = \mathbb{R}u_0 \subset U$ as before. Denoting by U_q the real eigenspace corresponding to the eigenvalues $\exp(\pm 2i\pi p/q)$ (note that $\dim U_q = 2$), one has $U = \ker\,(S_0 - \mathrm{I}) \oplus U_q$. Now, identifying U with $\mathbb{R} \times \mathbb{C}$, the branching equation (1.12) takes the form

$$\mathcal{B}(\alpha, z) = (b_0(\alpha, z), b_1(\alpha, z)) = 0, \tag{1.24}$$

with $b_0 : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ and $b_1 : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ such that

$$\begin{aligned}
&b_0(0,0) = 0,\\
&b_1(0,0) = 0,\\
&b_0(\alpha, \exp(2i\pi \tfrac{p}{q})z) = b_0(\alpha, z),\\
&b_1(\alpha, \exp(2i\pi \tfrac{p}{q})z) = \exp(2i\pi \tfrac{p}{q}) b_1(\alpha, z),\\
&b_0(\alpha, \overline{z}) = -b_0(\alpha, z),\\
&b_1(\alpha, \overline{z}) = -\overline{b_1(\alpha, z)}.
\end{aligned}$$

The α-axis forms a line of equilibria, corresponding to the primary branch. Now, if A_α denotes the linearization of Φ along the primary branch, i.e., $A_\alpha := D\Phi(\alpha e_0)$, $\alpha \in \mathbb{R}$, using normal forms it follows that the eigenvalues of $A_\alpha|_{U_q}$ are of the form

$$\exp\left(\pm i(2\pi p/q + \beta_q(\alpha))\right), \quad \text{with } \beta_q(0) = 0.$$

We further introduce the *transversality condition*

$$\boxed{\beta_q'(0) \neq 0.} \tag{T3}$$

This means that as we move along the primary branch a pair of simple eigenvalues moves with non-zero speed along the unit circle, passing through the root of unity $\exp(\pm i 2\pi p/q)$ for $\alpha = 0$.

Theorem 6 (SBSRU). *Let $q \geq 3$. Assume* (H1), (R)*,* (H4)*,* (H5)*,* (H5a) *and* (T3)*. As we move along the primary branch, the reversible family Φ at $\alpha = 0$ undergoes the bifurcation of at least two different branches of q-periodic orbits. The solutions within each branch are related by the $\mathbb{D}_q$-action of R and S_0 on U.*

Also in this case one expects that , for $q \geq 5$, the solutions on the one branch are stable, and unstable on the other, cf. section 6.5.1.

Remarks

1- The two branches of Φ correspond to R-symmetric subharmonics of the original system bifurcating from the primary branch at the orbit γ_0; the limiting period at γ_0 along these branches is qT_0.

2- Note that the coordinate $\alpha \in \mathbb{R}$ on the primary branch plays the role of the parameter $\lambda \in \mathbb{R}$ in section 1.5.1.

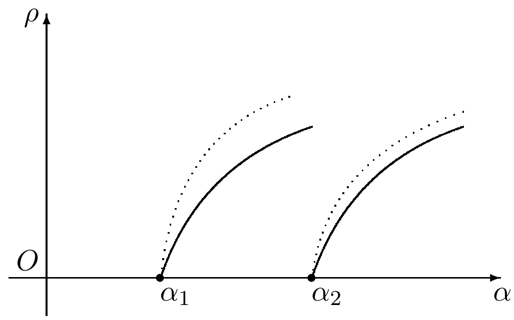

Figure 1.4: Bifurcation of subharmonics along the primary branch. The horizontal continuous line represents the primary branch. The points α_1, α_2 represent two points on the primary branch where a pair of semisimple eigenvalues of A_α pass through a root of unity (for different values of p and q). The whiskers at α_1 and α_2 represent the bifurcating branches: the dashed whisker is the unstable branch, the continuous one is stable.

1.6.2 Subharmonic Branching at a Resonant Root of Unity - (SBRRU)

In this section we state the bifurcation theorem that solves problem (P) in the SBRRU case. We consider a one-parameter family of maps Φ_λ satisfying (H1), (H4) and

$$\boxed{A_0 \text{ has eigenvalues } \exp(\pm 2i\pi p/q)} \tag{H6}$$

where $0 < p < q$ with $(p,q) = 1$. Under the further assumptions that

$$\boxed{e^{\pm 2i\pi p/q} \text{ have algebraic multiplicity 2 and geometric multiplicity 1}} \tag{H6a}$$

and that

$$\boxed{A_0 \text{ has no other eigenvalues that are } q\text{th roots of unity.}} \tag{H6b}$$

Application of the reduction as before shows that we are left with a 5-dimensional problem on U, $\dim U = 5$. Again $\ker(S_0 - \mathrm{I}) \subset U$ is one-dimensional, R-invariant and can be identified with $\mathbb{R}$. If V is a 4-dimensional

S_0-invariant complement of $\ker(S_0 - \mathrm{I})$ in U, then we identify U with $\mathbb{R} \times \mathbb{C} \times \mathbb{C}$ and the reduced branching equation consists of one real equation and two complex ones. Under an appropriate transversality condition, a combined use of the normal form results and the Implicit Function Theorem allows us to solve one of the two complex equation; we are then left with a problem similar to that of subharmonic branching described in section 1.6.1.

Theorem 7 (SBRRU). *Let $q \geq 3$ and assume* (H1), (R), (H4), (H6), (H6a-b). *Then, the reversible family Φ_λ undergoes the bifurcation of at least two different branches of q-periodic orbits from the fixed point $x = 0$.*

Remarks

1- In section 6.6.1 we actually prove that under two further non-degeneracy conditions the bifurcating families of q-periodic orbits are exactly two.

2- The two branches of Φ correspond to R-symmetric subharmonics of the original system bifurcating from the primary branch at the orbit γ_0.

2

Miscellanea

In this chapter we recall some basic definitions and technicalities we shall need throughout. The most of the results can be found in textbooks, therefore we shall be as brief as possible. We mainly quote from [1, 3, 4, 5, 12, 56, 63, 81].

2.1 Notations and Conventions

(i) Given two real vector spaces X and Y, $\mathrm{Lin}(X, Y)$ denotes the space of all linear operators $A : X \to Y$. In the case that $Y = X$ we write $\mathrm{Lin}(X)$ for $\mathrm{Lin}(X, X)$.

(ii) If X and Y are Banach spaces, $\mathcal{L}(X, Y)$ denotes the subspace of $\mathrm{Lin}(X, Y)$ consisting of all continuous linear operators $A : X \to Y$.

Remarks

1- Note that $\mathcal{L}(X) = \mathrm{Lin}(X)$ when X is finite dimensional.

2- In the particular case of $X = \mathbb{R}^n$, we also use the notation $gl(n, \mathbb{R})$ for $\mathcal{L}(\mathbb{R}^n)$ (the set of all $n \times n$ matrices). $GL(n, \mathbb{R})$ then is the subset of all invertible real $n \times n$ matrices, i.e.,

$$GL(n, \mathbb{R}) := \{A \in gl(n, \mathbb{R}) |\ A \text{ is invertible}\} \subset gl(n, \mathbb{R}). \tag{2.1}$$

The identity matrix on $\mathbb{R}^n$ is denoted by $I \in GL(n, \mathbb{R})$.

(iii) We denote a vector field X on a manifold M in local coordinates $(x_1, \ldots, x_m)$ by

$$X(x) = X_1(x)\frac{\partial}{\partial x_1} + \cdots + X_m(x)\frac{\partial}{\partial x_m}$$

where each component $X_i(x)$ is a function of x.

In the sequel we shall consider only smooth vector fields and by smooth we will always mean C^∞.

(iv) The set of all smooth vector fields on a manifold M is denoted by $\mathcal{X} = \mathcal{X}(M)$, while $\mathcal{X}_0 := \{X \in \mathcal{X} |\, X(0) = 0\}$ is the subspace of vector fields with a fixed point at the origin and $\mathcal{X}_0^k := \{X \in \mathcal{X}_0 |\, D^j X(0) = 0,\ 1 \leq j \leq k\}$.

(v) The flow $\exp(tX)(x)$ of a vector field $X \in \mathcal{X}$ is the solution of the initial value problem

$$\begin{cases} \frac{\partial}{\partial t}\exp(tX)(x) & = & X(\exp(tX)(x)) \\ \exp(tX)|_{t=0}(x) & = & x. \end{cases} \tag{2.2}$$

Note that if $X \in \mathcal{X}$ is linear, i.e. $X = A \in gl(n,\mathbb{R})$, then $e^{tX} = e^{tA}$ is the usual exponential of linear operators, and also that if $X \in \mathcal{X}_0$ then $\exp(tX)(0) = 0$ for all $t \in \mathbb{R}$.

(vi) The set of smooth diffeomorphisms on $\mathbb{R}^n$ is denoted by $\mathrm{Diff}(\mathbb{R}^n)$ while $\mathrm{Diff}_0(\mathbb{R}^n) := \{\varphi \in \mathrm{Diff}(\mathbb{R}^n) |\, \varphi(0) = 0\} \subset \mathrm{Diff}(\mathbb{R}^n)$ is the subset of diffeomorphisms with a fixed point at the origin.

Lemma 2.1 (cf. e.g. [63]). *Let $X(x)$ be a vector field on $\mathbb{R}^n$ with flow* $\exp(tX)(x)$. *Let $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. Then* $\Psi(\exp(tX)(\Psi^{-1}(y)))$ *is the flow of the equation*

$$\dot{y} = (\Psi_* X)(y), \quad y \in \mathbb{R}^n, \tag{2.3}$$

where $\Psi_* X := D\Psi(y) X(\Psi^{-1}(y)) \in \mathcal{X}(\mathbb{R}^n)$ is the push-forward *of X by Ψ.*

For later needs we introduce the notation $\Psi^* Y \in \mathcal{X}(\mathbb{R}^n)$ for the *pull-back* of Y by Ψ, where $\Psi^* := (\Psi^{-1})_*$.

Now, let $R \in GL(n,\mathbb{R})$ be a fixed linear involution, i.e. $R^2 = \mathrm{I}$. Then, one says that

(vii) a diffeomorphism $\phi \in \mathrm{Diff}(\mathbb{R}^n)$ is *R-reversible* if

$$R \circ \phi \circ R = \phi^{-1}. \tag{2.4}$$

(viii) a vector field $X \in \mathcal{X}(\mathbb{R}^n)$ is *R-reversible* if

$$R_* X = -X, \tag{2.5}$$

that is, $X(Rx) = -RX(x)$, for all $x \in \mathbb{R}^n$. The space of R-reversible vector fields is denoted by $\mathcal{X}_{-R} = \mathcal{X}_{-R}(\mathbb{R}^n)$. The subsets $\mathcal{X}_0^{-R}$ and $\mathcal{X}_0^{k,-R}$ are defined accordingly.

(ix) a vector field $X \in \mathcal{X}(\mathbb{R}^n)$ is *R-equivariant* if

$$R_* X = X, \tag{2.6}$$

that is, $X(Rx) = RX(x)$, for all $x \in \mathbb{R}^n$. The space of R-equivariant vector fields is denoted by $\mathcal{X}^{+R} = \mathcal{X}^{+R}(\mathbb{R}^n)$.

(x) Given an R-reversible autonomous system

$$\dot{x} = X(x) \tag{2.7}$$

with $X \in \mathcal{X}_0$, we say that a solution is (R-) symmetric if $x(t) = Rx(-t)$, for all $t \in \mathbb{R}$, which holds if and only if $x(0) = Rx(0)$.

Remark One can show that the following statements are equivalent for the orbit $\gamma = \{x(t) | \, t \in \mathbb{R}\}$ of a solution of (2.7):

(i) γ is the orbit of a symmetric solution;
(ii) γ is invariant under R, i.e., $R\gamma = \gamma$;
(iii) γ has a non-empty intersection with $\mathrm{Fix(R)} := \{\mathrm{x} \in \mathbb{R}^\mathrm{n} | \, \mathrm{Rx} = \mathrm{x}\}$.

When these conditions are satisfied, one says that γ is a *symmetric* orbit.

Lemma 2.2. *Given an autonomous R-reversible system*

$$\dot{x} = X(x) \tag{2.8}$$

on a vector space V, assume that $R \neq \pm \mathrm{I}$ and that γ is a non-trivial periodic R-symmetric orbit of (2.8). Then there exists a R-reversible Poincaré map associated to γ.

Proof. A periodic orbit γ of (2.8) is R-symmetric if $R\gamma = \gamma$. These periodic orbits have precisely two intersections x_1, x_2 with $\mathrm{Fix}(R) := \{x \in V | \, Rx = x\}$. To construct a Poincaré map associated with γ we consider two transversal sections Σ_1 and Σ_2 to γ at respectively x_1 and x_2 such that $R\Sigma_i = \Sigma_i$, $i = 1, 2$. The Poincaré map can then be written as

$$P = P_{2 \to 1} \circ P_{1 \to 2},$$

where $P_{2\to 1}$ is the 'halfway' Poincaré map from Σ_2 to Σ_1 and $P_{1\to 2}$ the halfway Poincaré map from Σ_1 to Σ_2. The reversibility implies that $P_{2\to 1} = R \circ P_{1\to 2}^{-1} \circ R$ and therefore

$$P := R \circ P_{1\to 2}^{-1} \circ R \circ P_{1\to 2} : \Sigma_1 \to \Sigma_1 \text{ is such that } RP = P^{-1}R, \quad (2.9)$$

i.e. P is R-reversible. Moreover, the following properties hold for the eigenvalues of P: if $\mu \in \mathbb{C}$ is an eigenvalue of P, then so is μ^{-1}; both eigenvalues have the same algebraic and geometric multiplicities and their respective eigenspaces and generalized eigenspaces are transformed into each other by the operator R. □

In the case of linear diffeomorphisms and vector fields we use the following notations, see also [12].

(ix) $GL_{+R}(n,\mathbb{R}) := \{A \in GL(n,\mathbb{R}) |\, RAR = A\}$.

(x) $GL_{-R}(n,\mathbb{R}) := \{A \in GL(n,\mathbb{R}) |\, RAR = A^{-1}\}$.

(xi) $gl_{\pm R}(n,\mathbb{R}) := \{A \in gl(n,\mathbb{R}) |\, RAR = \pm A\}$.

Observe that $GL_{+R}(n,\mathbb{R})$ is the group of R-equivariant linear diffeomorphisms on $\mathbb{R}^n$, while the set $GL_{-R}(n,\mathbb{R})$ is not even a group. Both $gl_{+R}(n,\mathbb{R})$ and $gl_{-R}(n,\mathbb{R})$ are subspaces of $gl(n,\mathbb{R})$, and

$$gl(n,\mathbb{R}) = gl_{+R}(n,\mathbb{R}) \oplus gl_{-R}(n,\mathbb{R}). \quad (2.10)$$

The projections on the first and second component of (2.10) are denoted respectively by $\pi_R^+ : gl(n,\mathbb{R}) \to gl_{+R}(n,\mathbb{R})$ and $\pi_R^- : gl(n,\mathbb{R}) \to gl_{-R}(n,\mathbb{R})$. Explicitly,

$$\pi_R^{\pm}(A) = \tfrac{1}{2}(A \pm RAR), \quad \forall\, A \in gl(n,\mathbb{R}). \quad (2.11)$$

Lemma 2.3. *There exists a scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$ such that R is orthogonal with respect to this scalar product, i.e. $\langle Rx, Ry\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{R}^n$, or $R = R^T$, where R^T is the transpose with respect to $\langle\cdot,\cdot\rangle$. Using this transpose and defining the scalar product $\langle A, B\rangle := tr(A^TB)$ on $gl(n,\mathbb{R})$, the decomposition (2.10) is also orthogonal.*

Proof. Given any scalar product $(\cdot,\cdot)$ on $\mathbb{R}^n$ we can define $\langle\cdot,\cdot\rangle$ by

$$\langle x, y\rangle = \tfrac{1}{2}\left[(x,y) + (Rx,Ry)\right].$$

The second part of the statement follows from the fact that for all $A, B \in gl(n,\mathbb{R})$ one has $tr(AB) = tr(BA)$, $tr(A+B) = tr(A) + tr(B)$ and therefore $\langle \pi^+(A), \pi^-(B)\rangle = 0$. □

It follows from the definitions that $\lambda \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of $A \in GL_{-R}(n, \mathbb{R})$ if and only if λ^{-1} is. This implies that next to possibly $+1$ and -1 the eigenvalues of A come in either pairs $\{\lambda, \overline{\lambda}\}$ on the unit circle ($|\lambda| = 1$), or in quadruples $\{\lambda, \lambda^{-1}, \overline{\lambda}, \overline{\lambda}^{-1}\}$ off the unit circle. Similarly, the eigenvalues of $B \in gl_{-R}(n, \mathbb{R})$ come in real pairs $\{\lambda, -\lambda\}$ or in complex quadruples $\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda}$ (with the possible exception of zero).

2.2 Elements of Lie Theory

We mainly quote from [3, 5, 1, 63, 86].

Lie algebras The space $gl(n, \mathbb{R})$ of all $n \times n$ matrices is a finite dimensional Lie algebra with Lie bracket given by the commutator of matrices; i.e.

$$[M_1, M_2] := M_1 M_2 - M_2 M_1, \quad \forall M_1, M_2 \in gl(n, \mathbb{R}). \tag{2.12}$$

In the sequel, a Lie algebra will usually be a subspace of the general Lie algebra $gl(n, \mathbb{R})$ and the role of the Lie bracket will indeed be played by the commutator (2.12).

The only infinite-dimensional Lie algebra we will consider is that given by the space of all smooth vector fields on $\mathbb{R}^n$, $\mathcal{X}$. The Lie bracket is in this case defined in terms of the action of vector fields as derivation on functions. Specifically, if X and Y are vector fields on $\mathbb{R}^n$, then their Lie bracket $[X, Y]$ is the unique vector field satisfying

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \tag{2.13}$$

for all smooth function $f : \mathbb{R}^n \to \mathbb{R}$. In local coordinates, if $X = \sum_{i=1}^n \xi_i(x)\partial_{x_i}$ and $Y = \sum_{i=1}^n \eta_i(x)\partial_{x_i}$ then

$$[X, Y] = \sum_{i=1}^n \sum_{j=1}^n \left\{ \xi_j \frac{\partial \eta_i}{\partial x_j} - \eta_j \frac{\partial \xi_i}{\partial x_j} \right\} \frac{\partial}{\partial x_i}.$$

Let $\mathcal{A}$ be an arbitrary (real) Lie algebra. To each $x \in \mathcal{A}$ one can associate the *adjoint operator*

$$\mathrm{ad}(x) : \mathcal{A} \to \mathcal{A}, \quad y \mapsto \mathrm{ad}(x)(y) := [x, y]. \tag{2.14}$$

Observe that $\mathrm{ad}([x, y]) = [\mathrm{ad}(x), \mathrm{ad}(y)]$, for all $x, y \in \mathbb{R}^n$, by the Jacobi identity. Define the *center* of $\mathcal{A}$ by

$$\ker(\mathrm{ad}) = \{x \in \mathcal{A} |\ [x, y] = 0, \forall y \in \mathcal{A}\}. \tag{2.15}$$

Lie groups. We refer to [3, 5, 1, 63, 86] for all the details.

The set $GL(n, \mathbb{R})$ of all invertible real $n \times n$ matrices is a group with respect to the product of matrices. The unity e is the $n \times n$ identity matrix I, and $GL(n, \mathbb{R})$ can be identified with a subset of $\mathbb{R}^{n^2}$. The function det : $gl(n, \mathbb{R}) \to \mathbb{R}$, $A \mapsto \det(A)$ is a C^∞ mapping and it holds

$$GL(n, \mathbb{R}) = \mathbb{R}^{n^2} \setminus \det^{-1}(\{0\}). \tag{2.16}$$

As a consequence $GL(n, \mathbb{R})$ is a (disconnected) n^2-dimensional sub variety of $\mathbb{R}^{n^2}$. The product of matrices and the inverse are smooth mappings with respect to this variety structure. Hence, $GL(n, \mathbb{R})$ is a n^2-dimensional real Lie group.

Lie algebra of a Lie group A corner stone in the theory of Lie groups is that to each Lie group G there corresponds a Lie algebra $\mathcal{G}$. Direct calculations show that the Lie algebra of the general Lie group $GL(n, \mathbb{R})$ is the space $gl(n, \mathbb{R})$ of all $n \times n$ matrices with Lie bracket the matrix commutator, [23]. It is also readily checked that $gl_{-R}(n, \mathbb{R})$ is not a Lie algebra, while $gl_{+R}(n, \mathbb{R})$ is. The associated Lie group is the group of all R-equivariant transformations: $GL_{+R}(n, \mathbb{R}) = GL(n, \mathbb{R}) \cap gl_{+R}(n, \mathbb{R})$.

We now introduce the Adjoint representation of the Lie group $GL(n, \mathbb{R})$ on its Lie algebra $gl(n, \mathbb{R})$.

For each $A \in GL(n, \mathbb{R})$ the *Adjoint operator*

$$\mathrm{Ad}(A) : GL(n, \mathbb{R}) \to GL(n, \mathbb{R}), \quad B \mapsto ABA^{-1}, \tag{2.17}$$

is an isomorphism of $GL(n, \mathbb{R})$ on itself. Moreover, it induces a linear mapping on $gl(n, \mathbb{R})$ given by

$$\mathrm{Ad}(A) : gl(n, \mathbb{R}) \to gl(n, \mathbb{R}), \quad \mathrm{Ad}(A)X := AXA^{-1}, \quad \forall X \in gl(n, \mathbb{R}). \tag{2.18}$$

In fact, if we consider $\mathrm{Ad}(A)\Phi(s) = A\Phi(s)A^{-1}$ ($s \in \mathbb{R}$) where $\Phi(0) = \mathrm{I} \in GL(n, \mathbb{R})$ and $\dfrac{d}{ds}\Phi(0) = X \in gl(n, \mathbb{R})$, then

$$\left.\frac{d}{ds}\right|_{s=0} \mathrm{Ad}(A)\Phi(s) = \mathrm{Ad}(A)X.$$

Note also that $\{\mathrm{Ad}(A) \in \mathcal{L}(gl(n,\mathbb{R}))|\ A \in GL(n,\mathbb{R})\}$ forms a (Lie) group of linear transformations on $gl(n,\mathbb{R})$, with corresponding Lie algebra

$$\mathrm{ad}(gl(n,\mathbb{R})) := \{\mathrm{ad}(X)|\ X \in gl(n,\mathbb{R})\}. \tag{2.19}$$

Proposition 2.4. *The mappings $Ad : GL(n,\mathbb{R}) \to \mathcal{L}(gl(n,\mathbb{R}))$ and $ad : gl(n,\mathbb{R}) \to \mathcal{L}(gl(n,\mathbb{R}))$ verify*

$$Ad \circ \exp = \exp \circ ad, \tag{2.20}$$

i.e. $Ad(\exp(X)) = \exp(ad(X))$, for all $X \in gl(n,\mathbb{R})$.

Proof. For each $X \in gl(n,\mathbb{R})$ we have that

$$\exp(tX) = \lim_{j\to\infty} \sum_{k=0}^{j} \frac{1}{k!}(tX)^k, \quad \forall t \in \mathbb{R}.$$

Since for each $A \in GL(n,\mathbb{R})$ the mapping $\mathrm{Ad}(A) : GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$ is in particular continuous, then, $\forall t \in \mathbb{R}$ and $\forall Y \in gl(n,\mathbb{R})$, the mapping

$$\eta(t) := \mathrm{Ad}\,(\exp(tX))\,Y = \exp(tX)Y\exp(-tX), \tag{2.21}$$

is continuous and $\eta(0) = Y$, $\eta(1) = \mathrm{Ad}\,(\exp(X))\,Y$. Moreover,

$$\begin{aligned} \frac{d}{dt}\eta(t) &= X\exp(tX)Y\exp(-tX) - \exp(tX)Y\exp(-tX)X \\ &= \mathrm{ad}(X)\eta(t), \quad \forall t \in \mathbb{R}. \end{aligned}$$

Hence, $\eta(t) = \exp(t\mathrm{ad}(X))$, for all $t \in \mathbb{R}$. Setting $t = 1$ proves (2.20). □

It is immediate that

$$\mathrm{Ad}(A)\exp(X) = A\exp(X)A^{-1} = \exp(\mathrm{Ad}(A)X) = \exp(AXA^{-1}), \tag{2.22}$$

for all $A \in GL(n,\mathbb{R})$ and for all $X \in gl(n,\mathbb{R})$.

The following lemma is a direct consequence of the definitions.

Lemma 2.5. *Let $GL_{\pm R}(n,\mathbb{R})$ and $gl_{\pm R}(n,\mathbb{R})$ be defined as in section 2.1. Then,*

(i) $\mathrm{Ad}(\Psi) : gl_{-R}(n,\mathbb{R}) \to gl_{-R}(n,\mathbb{R})$, *for all* $\Psi \in GL_{+R}(n,\mathbb{R})$. *In particular,* $\mathrm{Ad}(\Psi) \in \mathcal{L}(gl_{-R}(n,\mathbb{R}))$;

(ii) *if* $X \in gl_{+R}(n,\mathbb{R})$ *then* $\mathrm{ad}(X)\,(gl_{\pm R}(n,\mathbb{R})) = gl_{\pm R}(n,\mathbb{R})$;

(iii) $\mathrm{ad}(X)\,(gl_{\pm R}(n,\mathbb{R})) = gl_{\mp R}(n,\mathbb{R})$, *for all* $X \in gl_{-R}(n,\mathbb{R})$;

(iv) *if* $\Psi \in gl(n,\mathbb{R})$ *is semisimple with eigenvalues* $\lambda_1,\dots,\lambda_n$ *then* $\mathrm{ad}(\Psi) \in \mathcal{L}(gl(n,\mathbb{R}))$ *is semisimple with eigenvalues* $\lambda_k - \lambda_j$, $j,k \in \{1,\dots,n\}$;

(v) *the group* $GL_{+R}(n,\mathbb{R})$ *is algebraic.*

Remark Expressing the similarity of matrices in terms of the Adjoint action one can define the *orbit* of an operator $X \in gl_{-R}(n,\mathbb{R})$ under the Adjoint action of $GL_{+R}(n,\mathbb{R})$ as the similarity class of X, that is

$$\mathrm{Orb}(X) = \{\,\mathrm{Ad}(A)X|\ A \in GL_{+R}(n,\mathbb{R})\}\,. \tag{2.23}$$

The tangent space at $X \in gl_{-R}(n,\mathbb{R})$ to the orbit $\mathrm{Orb}(X)$ is precisely

$$T_X\mathrm{Orb}(X) = \{[Y,X] : Y \in gl_{+R}(n,\mathbb{R})\} = ad(X)\,(gl_{+R}(n,\mathbb{R}))\,; \tag{2.24}$$

which is a subspace of $gl_{-R}(n,\mathbb{R})$. Notice also that one can define the codimension c_X of an element $X \in gl_{-R}(n,\mathbb{R})$ as the codimension of its orbit in $gl_{-R}(n,\mathbb{R})$ under the Adjoint action. Since the eigenvalues are invariants of this action, it follows that all operators $X \in gl_{-R}(n,\mathbb{R})$ have positive codimension: $c_X > 0$, ([39]).

Denote by $\mathcal{H}_k$ the space of mappings on $\mathbb{R}^n$ whose components are homogeneous polynomials of degree k in the components of $x \in \mathbb{R}^n$, (note that $\mathcal{H}_1 = gl(n,\mathbb{R})$). Consider $\Phi \in \mathrm{Diff}_0(\mathbb{R}^n)$, and let

$$\Phi = \Phi_1 + \cdots + \Phi_k + \cdots, \quad \Phi_k(x) := \frac{1}{k!}D^k\Phi(0)\cdot(x,...,x) \in \mathcal{H}_k. \tag{2.25}$$

be its Taylor expansion at $x = 0$[1]. If $A_0 \in GL(n,\mathbb{R})$ and $\Phi \in \mathrm{Diff}_0(\mathbb{R}^n)$ has the Taylor expansion (2.25), then the Taylor expansion of $\mathrm{Ad}(A_0)\,\Phi$ is given by

$$\mathrm{Ad}(A_0)\,\Phi = \mathrm{Ad}_1(A_0)\,\Phi_1 + \cdots + \mathrm{Ad}_k(A_0)\,\Phi_k + \cdots \tag{2.26}$$

[1]The equality (2.25) is only formal, meaning that the Taylor series of Φ need not converge.

where $\mathrm{Ad}_k\,(A_0)$ is a linear operator on $\mathcal{H}_k$ defined by

$$\mathrm{Ad}_k\,(A_0)\,\Phi_k := A_0\Phi_k A_0^{-1}, \quad \forall\Phi_k \in \mathcal{H}_k, \forall\, k \geq 1. \tag{2.27}$$

If $A_0 \in gl(n,\mathbb{R})$, we define $\mathrm{ad}_k(A_0) : \mathcal{H}_k \to \mathcal{H}_k$ by

$$(\mathrm{ad}_k(A_0)\Phi_k)\,(x) := A_0\Phi_k(x) - D\Phi_k(x)\,(A_0x)\,, \quad \forall x \in \mathbb{R}^n, \forall\Phi_k \in \mathcal{H}_k. \tag{2.28}$$

Then, one can verify that

$$\mathrm{Ad}_k\big(\exp(A_0)\big) = \exp\big(\mathrm{ad}_k\,(A_0)\big), \quad \forall\, k \geq 1. \tag{2.29}$$

for each $\Phi_k \in \mathcal{H}_k$ and for all $x \in \mathbb{R}^n$. In the sequel we will usually omit the subindex k in the notation of Ad_k and ad_k.

2.3 Elements of Linear Algebra

In this section we give a brief overview of the linear algebra results we will need throughout. Recall that an operator $A \in gl(n,\mathbb{R})$ is said *semisimple* if it is complex diagonalisable, while it is *nilpotent* if $A^k = O$ for some integer k, [56]. We shall frequently use the following theorem about the decomposition of real linear operators; we refer to [56] for the proof.

Theorem 8. *For any linear operator $A \in gl(n,\mathbb{R})$ there exist unique operators $S, N \in gl(n,\mathbb{R})$ such that*

$$A = S + N, \quad \textit{with } SN = NS, \tag{2.30}$$

S is semisimple and N is nilpotent.

The decomposition (2.30) is called the semisimple-nilpotent (SN) decomposition[2] of A. The operator S is the semisimple part of A and N is the nilpotent part.

We proceed by generalizing Theorem 8.

Proposition 2.6. *For any given $A_0 \in GL(n,\mathbb{R})$ there exists a unique semisimple $S_0 \in GL(n,\mathbb{R})$ and a unique nilpotent $\mathcal{N}_0 \in gl(n,\mathbb{R})$ such that $S_0\mathcal{N}_0 = \mathcal{N}_0 S_0$ and*

$$A_0 = S_0 e^{\mathcal{N}_0}. \tag{2.31}$$

[2] The decomposition (2.30) is also called the Jordan-Chevalley decomposition

Moreover,

$$\operatorname{Im}(A_0 - I) = \operatorname{Im}(S_0 - I) \oplus \Big(\ker(S_0 - I) \cap \operatorname{Im}(\mathcal{N}_0)\Big). \tag{2.32}$$

Proof. The equality (2.31) follows from the exponentiation of the result in Theorem 8, i.e. one sets $\mathcal{N}_0 := \log(\mathrm{I} + S_0^{-1}N_0)$. The uniqueness follows from the invertibility of the exponential map. To prove (2.32), observe that $A_0 - I = (S_0 - I) + S_0(e^{\mathcal{N}_0} - I)$ is the SN-decomposition of $A_0 - I$, hence

$$\operatorname{Im}(A_0 - I) = \operatorname{Im}(S_0 - I) \oplus \left[\ker(S_0 - I) \cap \operatorname{Im}\big(S_0(e^{\mathcal{N}_0} - I)\big)\right].$$

From the assumptions and the definition of $\mathcal{N}_0$ it then follows that

$$S_0(e^{\mathcal{N}_0} - I) = \mathcal{N}_0\left(I - \tfrac{1}{2!}\mathcal{N}_0 + \cdots + \tfrac{1}{(k-1)!}\mathcal{N}_0{}^{k-2}\right) S_0,$$

with both S_0 and $\left(I - \frac{1}{2!}\mathcal{N}_0 + \cdots + \frac{1}{(k-1)!}\mathcal{N}_0{}^{k-2}\right)$ invertible. Hence, $\operatorname{Im}(\mathcal{N}_0)$ $= \operatorname{Im}\big(S_0(e^{\mathcal{N}_0} - I)\big)$ and (2.32) follows. □

The right handside of (2.31) is called the *semisimple-unipotent* (SU) *decomposition* of A_0. Also, since $\big(e^{\mathcal{N}_0} - I\big) \in gl(n, \mathbb{R})$ is nilpotent, we say that $e^{\mathcal{N}_0} = I + \big(e^{\mathcal{N}_0} - I\big)$ is *unipotent.*

Corollary 2.7. *Assume the hypotheses of Proposition 2.6 and also that $A_0 \in GL_{-R}(n, \mathbb{R})$, then $S_0 \in GL_{-R}(n, \mathbb{R})$ and $\mathcal{N}_0 \in gl_{-R}(n, \mathbb{R})$.*

Proof. The result follows from the uniqueness of the decomposition. □

Lemma 2.8 ([74]). *Let $S_0 \in GL\,(n, \mathbb{R})$ be semisimple and $\mathcal{N}_0 \in gl\,(n, \mathbb{R})$ be nilpotent. Then,* $\operatorname{Ad}(S_0)$ *is semisimple and* $ad\,(\mathcal{N}_0)$ *is nilpotent.*

Remarks

1- Note that $B \in gl(n, \mathbb{R})$ commutes with $e^{\mathcal{N}_0}$ if and only if it commutes with $\mathcal{N}_0$.

2- If $A = Se^{\mathcal{N}}$ is the SU-decomposition of a given $A \in GL(n, \mathbb{R})$, then $B \in gl(n, \mathbb{R})$ commutes with A if and only if B commutes with S and $\mathcal{N}$. This statement is trivial if B is also invertible. Indeed, $BA = AB$ implies that $B^{-1}AB = A$ from which it follows that

$(B^{-1}SB)(B^{-1}e^{\mathcal{N}}B) = Se^{\mathcal{N}}$, where $(B^{-1}SB)$ is semisimple, $(B^{-1}e^{\mathcal{N}}B)$ is unipotent and they commute. The uniqueness of the SU decomposition of A implies that B commutes with S and $e^{\mathcal{N}}$. The converse is obvious.
Now, the argument above does not apply if B is not invertible. In this case, one has to use the fact that the semisimple part S of A can be written as a polynomial in A; hence, if B commutes with A then it also commutes with S and with $e^{\mathcal{N}} = S^{-1}A$.

3- The SU-decomposition of $A_0 = S_0 e^{\mathcal{N}_0}$ of $A_0 \in GL(n,\mathbb{R})$ induces the SU-decomposition

$$\mathrm{Ad}(A_0) = \mathrm{Ad}(S_0)e^{\mathrm{ad}\mathcal{N}_0} \tag{2.33}$$

of $\mathrm{Ad(A_0) \in GL(n^2,\mathbb{R})}$.

4- Lemma 2.8 also holds in $\mathcal{H}_k$, i.e., for all $k \geq 1$, $\mathrm{Ad}_k\,(S_0)$ is semisimple and $Ad_k\,(\mathcal{N}_0)$ is nilpotent. Moreover, if $A_0 = S_0 \exp(\mathcal{N}_0)$ is the SU-decomposition of $A_0 \in GL\,(n,\mathbb{R})$, then

$$\mathrm{Ad}_k\,(A_0) = \mathrm{Ad}_k\,(S_0) \exp\left(\mathrm{ad}_k\,(\mathcal{N}_0)\right) \tag{2.34}$$

is the SU-decomposition of $\mathrm{Ad}_k(A_0) \in GL\,(n,\mathbb{R})$, $\forall k \geq 1$, (compare with (2.33)). For a proof see e.g. [25], pg 58-59.

Lemma 2.9. *Let $A_0 = S_0 \exp(\mathcal{N}_0)$ be the SU-decomposition of $A_0 \in gl\,(n,\mathbb{R})$. Then there is a scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$ such that*

(i) $S_0\mathcal{N}_0^T = \mathcal{N}_0^T S_0$;

(ii) *a linear operator $A \in gl\,(n,\mathbb{R})$ commutes with S_0 if and only if the transpose A^T of A with respect to this scalar product commutes with S_0:*

$$AS_0 = S_0 A \Longleftrightarrow AS_0^T = S_0^T A \Longleftrightarrow A^T S_0 = S_0 A^T. \tag{2.35}$$

(iii) *the following direct sum decomposition holds:*

$$\mathbb{R}^n = \mathrm{Im}\,(A_0 - I) \oplus \Big(\ker\,(S_0 - I) \cap \ker\left(\mathcal{N}_0^T\right)\Big). \tag{2.36}$$

Proof. For a proof we refer to [74] (Lemma 10 and Corollary 11). □

Notice that both the operators $\mathcal{N}_0$ and $\mathcal{N}_0^T$ leave the subspace $\ker(S_0 - I)$ invariant. Therefore

$$\left(\mathcal{N}_0|_{\ker(S_0-I)}\right)^T = \mathcal{N}_0^T|_{\ker(S_0-I)}$$

and

$$\ker(S_0 - I) = \Big(\ker\,(S_0 - I) \cap \mathrm{Im}\,(\mathcal{N}_0)\Big) \oplus \Big(\ker(S_0 - I) \cap \ker\,\left(\mathcal{N}_0^T\right)\Big). \tag{2.37}$$

Lemma 2.9 can be refined to fit the reversible context as formulated in Lemma 1.1. The remainder of this section is devoted to the proof of Lemma 1.1

Proof of Lemma 1.1. Here we use $\langle x, Ay\rangle = \langle A^T x, y\rangle$ for all $x, y \in \mathbb{R}^n$. Recall that the reversibility implies that if λ is an eigenvalue of S_0 then so are $\lambda^{-1}, \overline{\lambda}, \overline{\lambda}^{-1}$. Let $\alpha_i, \beta_i \in \mathbb{R}$ be such that the set of eigenvalues of S_0 is given by

$$spec(S_0) = \big\{\alpha_j \pm i\beta_j,\ \frac{1}{\alpha^2+\beta^2}\,(\alpha_j \mp i\beta_j)\big\}.$$

Since S_0 is semisimple we can write $\mathbb{R}^n = \sum_{j=1}^{l} V_j$ where

$$V_j := \ker\Big(\big((S_0 - \alpha_j \mathrm{I})^2 + \beta_j^2 \mathrm{I}\big)\big(\left(S_0^{-1} - \alpha_j \mathrm{I}\right)^2 + \beta_j^2 \mathrm{I}\big)\Big). \tag{2.38}$$

Each V_j $(1 \le j \le l)$ is S_0- and R-invariant. Indeed, for example,

$$R\left[(S_0 - \alpha_j I)^2 + \beta_j^2 I\right] = \left[\left(S_0^{-1} - \alpha_j I\right)^2 + \beta_j^2 I\right] R.$$

Let $\langle *, *\rangle$ be a scalar product on $\mathbb{R}^n$ such that the V_j are mutually orthogonal; that is,

$$\langle v_1, v_2\rangle := \sum_{j=1}^{l} \langle \pi_j v_1, \pi_j v_2\rangle_j, \quad \forall v_1, v_2 \in V,$$

where $\pi_j \in \mathcal{L}(\mathbb{R}^n, V_j)$ is the projection of $\mathbb{R}^n$ onto V_j associated with the decomposition $\mathbb{R}^n = \sum_{j=1}^{l} V_j$, and $\langle *, *\rangle_j$ is some scalar product in V_j.
Observe that if some $A \in gl(n, \mathbb{R})$ is such that $A(V_j) \subset V_j$, $(1 \le j \le l)$, then $\pi_j A = A_j \pi_j$ for some $A_j \in \mathcal{L}(V_j)$ and $\pi_j A^T = A_j^T \pi_j$. Therefore it is

sufficient to prove the existence of a convenient scalar product within each V_j. We consider then the case

$$V = \ker\Big(\big((S_0 - \alpha\mathrm{I})^2 + \beta^2\mathrm{I}\big)\big(\big(S_0^{-1} - \alpha\mathrm{I}\big)^2 + \beta^2\mathrm{I}\big)\Big),$$

for some $\alpha, \beta \in \mathbb{R}$. Different cases are distinguished.

Case 1. Suppose $\alpha, \beta \neq 0$ and $\alpha^2 + \beta^2 \neq 1$.
Recall that from the reversibility of S_0 it follows that if v is a λ-eigenvector of S_0 then Rv is a λ-eigenvector of S_0^{-1}. Set

$$V_+ := \ker\left[(S_0 - \alpha I)^2 + \beta^2 I\right], \qquad V_- := \ker\left[(S_0^{-1} - \alpha I)^2 + \beta^2 I\right], \tag{2.39}$$

then $V = V_+ \oplus V_-$. Moreover, $R(V_\pm) = V_\mp$ and $V_\pm$ are S_0-invariant. Let $R_\pm := R|_{V_\pm}$,then $R_- R_+ = \mathrm{I}_{V_+}$ and $R_+ R_- = \mathrm{I}_{V_-}$. Define

$$J_+ := \frac{1}{\beta}(S_0 - \alpha\mathrm{I})|_{V_+}) \in \mathcal{L}(V_+),$$

and

$$J_- := \frac{1}{\beta}(S_0^{-1} - \alpha\mathrm{I})|_{V_-}) \in \mathcal{L}(V_-).$$

Obviously, $J_+^2 = -\mathrm{I}|_{V_+}$, $J_-^2 = -\mathrm{I}|_{V_-}$, and

$$S_0 v_+ = \alpha v_+ + \beta J_+ v_+, \quad \forall v_+ \in V_+, \tag{2.40}$$
$$S_0^{-1} v_- = \alpha v_- + \beta J_- v_-, \quad \forall v_- \in V_-. \tag{2.41}$$

Hence,

$$S_0 v_- = \frac{\alpha}{\alpha^2 - \beta^2} v_- - \frac{\beta}{\alpha^2 + \beta^2} J_- v_-.$$

Since $J_+^2 = -\mathrm{I}_{V_+}$, we see that J_+ generates a finite group of linear operations on V_+, and therefore we can find a scalar product $(\cdot,\cdot)$ on V_+ such that J_+ is orthogonal:

$$(J_+ v_+, J_+ v'_+) = (v_+, v'_+), \quad \forall v_+, v'_+ \in V_+.$$

Thus, $J_+^T = -J_+$. Also, $R_+ J_+ R_- = R\frac{1}{\beta}(S_0 - \alpha I)R_- = \frac{1}{\beta}(S_0^{-1} - \alpha I)|_{V_-} = J_-$ and similarly $R_- J_- R_+ = J_+$. Define a scalar product $\langle\cdot,\cdot\rangle$ on V by

$$\langle (v_+, v_-), (v'_+, v'_-)\rangle := (v_+, v'_+) + (R_- v_-, R_- v'_-).$$

Then,

$$\begin{aligned}
\langle S_0^T(v_+,v_-),(v'_+,v'_-)\rangle & \\
&= \langle (v_+,v_-), S_0(v'_+,v'_-)\rangle \\
&= \langle (v_+,v_-), \\
&\qquad (\alpha v'_+ + \beta J_+ v'_+, \frac{1}{\alpha^2+\beta^2}(\alpha v'_- - \beta J_- v'_-)\rangle \\
&= (v_+, \alpha v'_+ + \beta J_+ v'_+) \\
&\qquad + \frac{1}{\alpha^2+\beta^2}(R_- v_-, \alpha R_- v'_- - \beta R_- J_- v'_-) \\
&= (\alpha v_+ - \beta J_+ v_+, v'_+) \\
&\qquad + \frac{1}{\alpha^2+\beta^2}(\alpha R_- v_- + \beta R_- J_- v_-, R_- v'_-),
\end{aligned}$$

since $(R_- v_-, R_- J_- v'_-) = -(R_- J_- v_-, R_- v'_-)$. Hence,

$$S_0^T(v_+,v_-) = (\alpha v_+ - \beta J_+ v_+, \frac{1}{\alpha^2+\beta^2}(\alpha v_- + \beta J_- v_-))$$

and therefore $V_+ = \ker\big((S_0^T - \alpha \mathrm{I})^2 + \beta^2 \mathrm{I}\big)$. Also, $\langle R(v_+,v_-), R(v'_+,v'_-)\rangle$, i.e. $R^T = R$. Suppose now that $AS_0 = S_0 A$, then A leaves $V\pm$ invariant and $A_+J_+ = J_+A_+$, $A_-J_- = J_-A_-$, where we set $A(v_+,v_-) = (A_+v_+, A_-v_-)$ for all $(v_+,v_-)V_+ \times V_-$. It follows that

$$AS_0^T(v_+,v_-) = S_0^T A(v_+,v_-,)$$

that is $AS_0^T = S_0^T A$, which in turn implies that $AS_0 = S_0 A$. Therefore (ii) follows.

Case 2. Suppose $\alpha \neq 0$, $\beta \neq 0$, $\alpha^2+\beta^2 = 1$. Then we have that $S_0^T = S_0^{-1}$. So it is sufficient to take any scalar product on V for which R is orthogonal and J is antisymmetric.

Case 3. Suppose $\alpha \neq 0, 1$, $\beta = 0$. Then $V = \ker\Big((S_0 - \alpha I)\left(S_0^{-1} - \alpha I\right)\Big)$ and $S_0 = S_0^T$. Therefore it is sufficient to take any scalar product for which V_+ and V_- are orthogonal. □

The following consequence of Lemma 1.1 is essential in the proof of the PRNF Theorem 2.

Corollary 2.10. *Let* $A_0 = S_0 e^{\mathcal{N}_0}$ *be the SU-decomposition of* $A_0 \in GL_-(n,\mathbb{R})$ *and let* $\langle \cdot, \cdot \rangle$ *be a scalar product as in Lemma 1.1. Then also* A_0^T, S_0^T *belong to* $GL_{-R}(n,\mathbb{R})$, *and* ${\mathcal{N}_0}^T$ *belongs to* $gl_{-R}(n,\mathbb{R})$. *Moreover,*

$$\begin{aligned} &\ker\left(Ad(S_0) - I\right) \cap gl_{-R}(n,\mathbb{R}) \\ &\quad = \left[ad\left(\mathcal{N}_0\right)\left(gl_{+R}(n,\mathbb{R}) \cap \ker\left(Ad(S_0) - I\right)\right)\right] \\ &\qquad \oplus \left[\ker\left(Ad(S_0) - I\right) \cap gl_{-R}(n,\mathbb{R}) \cap \ker\left(ad\left(\mathcal{N}_0^T\right)\right)\right]. \end{aligned} \tag{2.42}$$

Proof. The first assertion follows from Lemma 1.1(i) by taking the transpose of the relation $A_0 R = R A_0^{-1}$. The splitting (2.42) follows from (2.37), (2.10) and the fact that $ad\,(\mathcal{N}_0) : gl_{\pm R}(n,\mathbb{R}) \to gl_{\mp R}(n,\mathbb{R})$. □

Note that if $A_0 = S_0 \exp\left(\mathcal{N}_0\right)$ is the SU-decomposition of $A_0 \in GL\,(n,\mathbb{R})$, then, for each $k \geq 1$,

$$\mathcal{H}_k = \mathrm{Im}\left(\mathrm{Ad}_k(A_0) - I\right) \oplus \left(\ker\left(\mathrm{Ad}_k(S_0) - I\right) \cap \ker\left(\mathrm{ad}_k\left(\mathcal{N}_0^T\right)\right)\right). \tag{2.43}$$

Also,

$$\mathcal{H}_k = \mathrm{Im}\left(\mathrm{Ad}_k(S_0) - I\right) \oplus \ker\left(\mathrm{Ad}_k\left(S_0\right) - I\right) \tag{2.44}$$

since $\mathrm{Ad}_k\left(S_0\right)$ is semisimple.

3

Reversible GLS Reduction

This chapter contains a proof of the Reversible GLS reduction Theorem 1. Recall that the goal is to develop a general structure-preserving reduction method for the study of bifurcation of periodic points from a resonant fixed point of an m-parameter family of *reversible* diffeomorphisms.

In section 3.1 we briefly recall the main reduction result for the case of general diffeomorphisms from [74], i.e. without preservation of structure. In section 3.2.1 we adapt it so that reversibility is preserved.

3.1 Summary of the General Case

Consider a smooth local map $\Phi : (\mathbb{R}^n, 0) \times (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0),\ (x, \lambda) \mapsto \Phi_\lambda(x) = \Phi(x, \lambda)$ satisfying the hypotheses (H1). Given an integer $q \geq 1$, we show how to solve (P) by applying a generalized Lyapunov-Schmidt reduction which makes the implicit $\mathbb{Z}_q$-symmetry of the problem explicit. Recall from section 1.1 that problem (P) has an implicit $\mathbb{Z}_q$-symmetry generated by Φ_λ on the solution set $\mathcal{S}^q_\lambda$ yet to be determined, see (1.1). The main result can be phrased as follows.

Theorem 9 ([74]). *Let Φ_λ be a m-parameter family of diffeomorphisms satisfying* (H1) *and let $S_0 \in GL(n, \mathbb{R})$ be the semisimple part of $A_0 := D_x\Phi_0(0)$. Let $q \geq 1$ and define $U := \ker\,(S_0^q - I) \subset \mathbb{R}^n$. Then, there exist smooth maps $x^* : U \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ and $\Phi_r : U \times \mathbb{R}^m \longrightarrow U$ such that*

(i) $x^*(0, \lambda) = 0$, *for all $\lambda \in \mathbb{R}^m$ and $D_u x^*(0, 0)\cdot\ \overline{u} = \overline{u}$, for all $\overline{u} \in U$;*

(ii) $\Phi_{r,\lambda}(0) = 0$, *for all $\lambda \in \mathbb{R}^m$ and $D_u\Phi_r(0, 0) = A_0|_U$;*

(iii) Φ_r *is $\mathbb{Z}_q$-equivariant: $\Phi_{r,\lambda}(S_0 u) = S_0\Phi_{r,\lambda}(u)$, for all $(u, \lambda) \in U \times \mathbb{R}^m$;*

(iv) *for all sufficiently small $(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m$ the point x is q-periodic for $\Phi_\lambda(\cdot)$ if and only if $x = x^*_\lambda(u)$ for some sufficiently small $u \in U$ which itself is a q-periodic point of $\Phi_{r,\lambda}(\cdot)$*

(v) *for all sufficiently small* $(u,\lambda) \in U \times \mathbb{R}^m$ *the point* u *is* q*-periodic point for* $\Phi_r(.,\lambda)$ *if and only if* $\Phi_r(u,\lambda) = S_0 u$; *i.e. all small* q*-periodic orbits of* $\Phi_{r,\lambda}(\cdot)$ *are necessarily* $\mathbb{Z}_q$*-orbits.*

The approach, used in [74], to approximate the reduced diffeomorphism $\Phi_{r,\lambda}(\cdot)$ consists of bringing Φ_λ in an appropriate normal form to obtain the reduced diffeomorphism $\Phi_{r,\lambda}$. We briefly describe how this is done.
Applying an analogue of Theorem 2, we may assume that, up to a near-identity transformation, the map Φ_λ has the form

$$\Phi_\lambda(x) = \Phi_\lambda^{NF}(x) + R_{k+1}(x,\lambda), \tag{3.1}$$

with

$$\Phi_\lambda^{NF} := S_0 e^{\mathcal{N}_0 + Z_\lambda}, \quad \text{where } \Phi_\lambda^{NF}(S_0 x) = S_0 \Phi_\lambda^{NF}(x),$$

and

$$R_{k+1}(x,\lambda) = O\left(|x|^{k+1}\right), \tag{3.2}$$

as $x \to 0$ uniformly in λ.

Proposition 3.1. *Assume* (H1) *and* (3.1) *then*

$$x^*(u,\lambda) = u + O(|u|^{k+1}) \quad \textit{and} \quad \Phi_{r,\lambda}(u) = \Phi_\lambda^{NF}(u) + O(|u|^{k+1}) \tag{3.3}$$

as $u \to 0$*, uniformly for* λ *in some neighborhood of the origin of* $\mathbb{R}^m$*. Moreover,* $D_u\Phi_r(0,\lambda) = D_x\, \Phi_\lambda^{NF}(0)\big|_U = A_\lambda|_U$ *; so the eigenvalues of* $D_u\Phi_{r,\lambda}(0)$ *coincide with the eigenvalues of* A_λ *which are close to* q*th roots of unity.*

Remark The solutions of the determining equation for q-periodic points of $\Phi_{r,\lambda}$ can be approximated by the equilibria $u \in U$ of the normal form vector field $\mathcal{N}_0 + Z_\lambda(\cdot)$. Indeed, setting $\Psi_\lambda^{NF} := S_0^{-1}\Phi_\lambda^{NF} = e^{\mathcal{N}_0 + Z_\lambda}$ one has

$$\Phi_{r,\lambda}(u) = S_0 \Psi_\lambda^{NF}(u) + O(||u||^{k+1}).$$

Up to terms of order k the determining equation then takes the form

$$\Psi_\lambda^{NF}(u) = u,$$

which for (u,λ) small enough is equivalent to

$$\mathcal{N}_0(u) + Z_\lambda(u) = 0.$$

3.2 Reversible GLS Reduction

We prove the reversible GLS Reduction Theorem 1.

3.2.1 Proof of the Reversible GLS Reduction Theorem 1

Recall that it is our interest to find all small q-periodic points of Φ_λ, for λ near 0, under the assumption that (H1) and the reversibility condition (R) are satisfied. The main goal is to preserve the reversibility condition (R) during the reduction (compare with Theorem 1-(iv)). For completeness we first briefly recall certain elements from section 1.2.

We first replace the equation $\Phi_\lambda^q(x) = x$, see (P), by an equivalent equation for q-periodic *orbits* of Φ_λ on an appropriate *orbit space*, and then perform the LS reduction to the latter problem [74, 32]. The orbit space is defined by (1.2) and the *shift operator* $\sigma \in \mathcal{L}(Y_q)$, the *reversal operator* $\gamma \in \mathcal{L}(Y_q)$, and the *lift* $\widehat{\Phi}_\lambda$ of Φ_λ to Y_q are given by (1.4), (1.7) and (1.3) respectively.

The following lemma is then a direct consequence of the definitions.

Lemma 3.2. *Let* $\Phi_\lambda : \mathbb{R}^n \to \mathbb{R}^n$, $\lambda \in \mathbb{R}^m$ *satisfy* (H1) and (R). Define Y_q, $\widehat{\Phi}_\lambda$, σ and γ as above. Then,

(i) $\widehat{\Phi}_\lambda$ is $\mathbb{Z}_q$-equivariant: $\widehat{\Phi}_\lambda(\sigma y) = \sigma \widehat{\Phi}_\lambda(y)$, $y \in Y_q$;

(ii) $\widehat{\Phi}_\lambda$ is γ-reversible, i.e., $\gamma \circ \widehat{\Phi}_\lambda \circ \gamma = \widehat{\Phi}_\lambda^{-1}$;

(iii) let $(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m$ be a solution of (P), and let $y \in Y_q$ be given by $y_i := \Phi_\lambda^{(i)}(x)$, $i \in \mathbb{Z}$. Then (y, λ) satisfies

$$\widehat{\Phi}_\lambda(y) = \sigma \cdot y. \tag{3.4}$$

Conversely, if $(y, \lambda) \in Y_q \times \mathbb{R}^m$ solves (3.4) then $x := y_0 \in \mathbb{R}^n$ satisfies (P).

Observe that

$$\sigma^q = \gamma^2 = I \quad \text{and} \quad \sigma^{-1}\gamma = \gamma\sigma. \tag{3.5}$$

It follows that the operators γ and σ generate a finite group $\Gamma \subset \mathcal{L}(Y_q)$, which has $2q$ elements and is isomorphic to the dihedral group $\mathbb{D}_q$ (the symmetry group of a regular q-polygon).

Remark If $(y, \lambda) \in Y_q \times \mathbb{R}^m$ is a solution of (3.4), then it also solves the equation $\sigma^{-1} \cdot y = \widehat{\Phi}_\lambda^{-1}(y)$.

Linearizing (3.4) at $(y, \lambda) = (0, 0)$ gives the linearized problem

$$\widehat{A}_0 y = \sigma \cdot y, \qquad \text{where } \widehat{A}_0 y := (A_0 y_1, ..., A_0 y_q). \tag{3.6}$$

Note that $\widehat{A}_0$ belongs to $GL_{-\gamma}(Y_q) := \left\{ \widehat{A} \in \mathcal{L}(Y_q) \middle| \; \gamma \widehat{A} = \widehat{A}^{-1}\gamma \right\}$, as this follows directly from Lemma 3.2-(ii). Also, if $A_0 = S_0 + N_0$ is the SN-decomposition of A_0, then $\widehat{A}_0 = \widehat{S}_0 + \widehat{N}_0$ is the SN-decomposition of $\widehat{A}_0$ with in particular

$$\widehat{S}_0 \sigma = \sigma \widehat{S}_0, \quad \widehat{N}_0 \sigma = \sigma \widehat{N}_0, \quad \text{and} \quad \gamma \widehat{S}_0 = \widehat{S}_0^{-1} \gamma.$$

Some further straightforward algebra gives the following lemma.

Lemma 3.3. *Let the subspaces $U \subseteq \mathbb{R}^n$ and $\widehat{U} \subseteq Y_q$ be given by*

$$U := \ker\,(S_0^q - I) \subseteq \mathbb{R}^n, \quad \widehat{U} := \ker\left(\widehat{S}_0 - \sigma\right) \subseteq Y_q. \tag{3.7}$$

Define, for all $u \in U$, $\xi : U \to \widehat{U}$ by

$$\xi(u) := \left(S_0^i u\right)_{i \in \mathbb{Z}}, \tag{3.8}$$

Then

(i) *U is invariant under S_0 and A_0;*

(ii) *ξ is a linear isomorphism from U onto $\widehat{U}$;*

(iii) *$\xi(S_0 u) = \widehat{S}_0 \xi(u) = \sigma \cdot \xi(u)$, $u \in U$;*

(iv) *$\xi(A_0 u) = \widehat{A}_0 \xi(u)$, $u \in U$;*

(v) *$\gamma \cdot \xi(u) = \xi(Ru)$, $u \in U$;*

(vi) *$Y_q = \xi(U) \oplus \operatorname{Im}\left(\hat{S}_0 - \sigma\right)$, and this decomposition is invariant under $\widehat{A}_0$, $\widehat{S}_0$, and σ.*

(vii) $\left(\widehat{A}_0 - \sigma\right)$ *is invertible on* $\mathrm{Im}(\widehat{S}_0 - \sigma)$.

Proof. The statements (i),(iii) and (iv) directly follow from the definitions and the fact that A_0 and S_0 commute. Item (ii) is proved as follows. An element $y = (x_1, ..., x_q)$ of Y_q belongs to $\ker\left(\widehat{S}_0 - \sigma\right)$ if and only if $x_{j+1} = S_0 x_j$ $(1 \le j \le q-1)$ and $x_1 = S_0 x_q$, i.e. if and only if $x_j = S_0^{j-1} x_1$ $(1 \le j \le q)$ and $S_0^q x_1 = x_1$. To show (vi), recall that σ is semisimple, σ and $\widehat{S}_0$ commute, and therefore also $\widehat{S}_0 - \sigma$ is semisimple. It follows that

$$Y_q = \ker\left(\widehat{S}_0 - \sigma\right) \oplus \mathrm{Im}\left(\hat{S}_0 - \sigma\right),$$

which in combination with $\ker\left(\widehat{S}_0 - \sigma\right) = \xi\left(U\right)$ implies (vi).

The SN-decomposition $\left(\widehat{A}_0 - \sigma\right) = (\widehat{S}_0 - \sigma) + \widehat{N}_0$ implies that the restriction of $\left(\widehat{A}_0 - \sigma\right)$ to $\mathrm{Im}\left(\widehat{S}_0 - \sigma\right)$ is invertible. Finally, property (v) is proved as follows. For all $u \in U$, using $S_0^{-1} = S_0^{q-1}$ and $S_0^q = \mathrm{I}$ on U, one has that

$$\begin{aligned} \gamma \cdot \xi\left(u\right) &= \gamma\left(u, S_0, \ldots, S_0^{q-1} u\right) = \left(Ru, RS_0^{q-1} u, \ldots, RS_0 u\right) \\ &= \left(Ru, S_0 Ru, S_0^2 Ru, \ldots, S_o^{q-1} Ru\right) = \xi\left(Ru\right). \end{aligned}$$

This completes the proof. □

By performing a LS reduction of (3.4) using the decomposition (vi) of Lemma 3.3, problem (P) reduces to an equation on the space $\xi\left(U\right)$. Indeed, each $y \in Y_q$ can be written in a unique way as $y = \xi(u) + v$, with $u \in U$ and $v \in \mathrm{Im}(\widehat{S}_0 - \sigma)$. Then $\sigma \cdot y = \xi(S_0 u) + \sigma \cdot v$, and equation (3.4) splits into a system of two equations

$$\begin{cases} S_0 u &= \Psi_\lambda\left(u, v\right) \quad \text{(a)} \\ \sigma \cdot v &= \Sigma_\lambda\left(u, v\right) \quad \text{(b)} \end{cases} \tag{3.9}$$

where the maps $\Psi_\lambda : U \times \mathrm{Im}\left(\widehat{S}_0 - \sigma\right) \longrightarrow U$ and $\Sigma_\lambda : U \times \mathrm{Im}\left(\widehat{S}_0 - \sigma\right) \longrightarrow \mathrm{Im}\left(\widehat{S}_0 - \sigma\right)$ are uniquely determined by the relation

$$\widehat{\Phi}_\lambda\left(\xi(u) + v\right) = \xi\left(\Psi_\lambda(u, v)\right) + \Sigma_\lambda(u, v). \tag{3.10}$$

Obviously,

$$\begin{aligned}
&\Psi_\lambda(0,0) = 0, \quad \Sigma_\lambda(0,0) = 0\\
&D_u\Psi_0(0,0) = A_0|_U\,, \quad D_v\Psi_0(0,0) = 0\\
&D_u\Sigma_0(0,0) = 0, \quad D_v\Sigma_0(0,0) = \left.\widehat{A}_0\right|_{\mathrm{Im}\left(\hat{S}_0-\sigma\right)}.
\end{aligned}$$

So, the Implicit Function Theorem applies, hence there exists a unique mapping $v^* : U \times \mathbb{R}^m \to \mathrm{Im}\left(\widehat{S}_0 - \sigma\right)$, smooth near the origin, with $v^*(0,0) = 0$ and equation (3.9)(b) holds for all $(u,v,\lambda) \in U \times \mathrm{Im}\left(\widehat{S}_0 - \sigma\right) \times \mathbb{R}^m$ if and only if $v = v^*(u,\lambda)$. Observe that the $\mathbb{Z}_q$-equivariance of $\widehat{\Phi}_\lambda$ (see Lemma 3.2-(i)) decomposes as

$$\Psi_\lambda(S_0u, \sigma\cdot v) = S_0\Psi_\lambda(u,v) \tag{3.11}$$

and

$$\Sigma_\lambda(S_0u, \sigma\cdot v) = \sigma\cdot\Sigma_\lambda(u,v)\,, \tag{3.12}$$

Moreover, one has that $v^*(0,\lambda) = 0$, for $\lambda \in \mathbb{R}^m$ near 0, and $D_uv^*(0,0) = 0$. Then, uniqueness of the solution and (3.12) imply that $v^*(S_0u,\lambda) = \sigma \cdot v^*(u,\lambda)$.
Substituting the solution $v^*_\lambda(u)$ into the equation (3.9)(a) gives the *determining equation*

$$S_0u = \Phi_{r,\lambda}(u) \tag{3.13}$$

where the *reduced map* $\Phi_r : U \times \mathbb{R}^m \to U$ is defined by

$$\Phi_{r,\lambda}(u) := \Psi_\lambda(u, v^*_\lambda(u))\,. \tag{3.14}$$

The following lemma summarizes some basic properties of the reduced map.

Lemma 3.4. *The map* $\Phi_{r,\lambda}$ *defined by* (3.14), is such that

(i) $\Phi_{r,0}(0) = 0$, $\lambda \in \mathbb{R}^m$;

(ii) $D_u\Phi_{r,\lambda}(0) = A_0|_U$;

(iii) $\Phi_{r,\lambda}$ is $\mathbb{Z}_q$-equivariant: $\Phi_{r,\lambda}(S_0u) = S_0\Phi_{r,\lambda}(u)$ for all $(u,\lambda) \in U \times \mathbb{R}^m$.

We still have to prove the reversibility of the reduced map $\Phi_{r,\lambda}$. To this purpose, observe that reducing equation (3.4) to the system (3.9) and then to the equation (3.13) proves that: *for* $u_1, u_2 \in U$, $v \in \operatorname{Im}\left(\widehat{S}_0 - \sigma\right)$, *the equation* $\widehat{\Phi}_\lambda\left(\xi\left(u_1\right)+v\right)=\xi\left(u_2\right)+\sigma v$ *holds if and only if* $v=v_\lambda^*\left(u_1\right)$ *and* $u_2=\Phi_{r,\lambda}\left(u_1\right)$.

Lemma 3.5. *The map* $\Phi_r : U \times \mathbb{R}^m \to U$ *defined by* (3.14) is R-reversible.

Proof. For each $(u,\lambda) \in U \times \mathbb{R}^m$, we may write

$$\begin{aligned}\widehat{\Phi}_\lambda\left(\xi\left(u\right)+v_\lambda^*\left(u\right)\right) &= \xi\left(\Psi_\lambda\left(u, v_\lambda^*\left(u\right)\right)\right)+\Sigma_\lambda\left(u, v_\lambda^*\left(u\right)\right)\\ &= \xi\left(\Phi_{r,\lambda}\left(u\right)\right)+\sigma\cdot v_\lambda^*\left(u\right)\end{aligned}$$

Acting with the linear reversal operator $\gamma \in \mathcal{L}\left(Y_q\right)$ on both sides of this equation, one obtains

$$\gamma\widehat{\Phi}_\lambda\left(\xi\left(u\right)+v_\lambda^*\left(u\right)\right)=\gamma\xi\left(\Phi_{r,\lambda}\left(u\right)\right)+\gamma\cdot\sigma\cdot v_\lambda^*\left(u\right).$$

Because of the reversibility of $\widehat{\Phi}_\lambda$ (see Lemma (3.2)-(ii)) and the properties (3.5), this implies that

$$\widehat{\Phi}_\lambda^{-1}\left(\xi\left(Ru\right)+\gamma v_\lambda^*\left(u\right)\right)=\gamma\xi\left(\Phi_{r,\lambda}\left(u\right)\right)+\gamma\cdot\sigma\cdot v^*\left(u,\lambda\right).$$

Applying $\widehat{\Phi}_\lambda$ to both sides gives

$$\xi\left(Ru\right)+\left(\sigma\cdot\sigma^{-1}\right)\cdot\gamma\cdot v_\lambda^*\left(u\right)=\widehat{\Phi}_\lambda\left(\xi\left(R\Phi_{r,\lambda}\left(u\right)\right)+\sigma^{-1}\cdot\gamma\cdot v_\lambda^*\left(u\right)\right).$$

From the remark above it follows that $\sigma^{-1}\cdot\gamma\cdot v_\lambda^*\left(u\right)=v_\lambda^*\left(R\Phi_{r,\lambda}\left(u\right)\right)$ and $Ru=\Phi_{r,\lambda}\left(R\Phi_{r,\lambda}\left(u\right)\right)$. Hence, $R\cdot\Phi_{r,\lambda}\cdot R=\Phi_{r,\lambda}^{-1}$. □

The $\mathbb{Z}_q$-equivariance of Φ_r, see Lemma 3.4-(iii), implies that for each solution $u \in U$ of the determining equation also the other points $S_0u, S_0^2u, \ldots, S_0^{q-1}u$ on the $\mathbb{Z}_q$-orbit of u solve this equation. We can summarize our reduction up to this point as follows.

Proposition 3.6. *Assume* (H1) *and* (R). *Let* $A_0 = S_0 + N_0$ *be the SN-decomposition of* $A_0 := D_x\Phi_0(0)$, *and define* U *as in* (1.9). *Also, fix some* $q \geq 1$ *and define* Y_q, σ, γ, $\widehat{\Phi}$ *and* ξ *by respectively* (1.2), (1.4), (1.7), (1.3), (3.8). *Then a sufficiently small* $(y,\lambda) \in Y_q \times \mathbb{R}^m$ *satisfies* (1.5) *if and only if* $y = \xi(u)+v^*(u,\lambda)$ *for some small* $u \in U$ *satisfying the determining equation* (3.13). *The maps* v^* *and* Φ_r *appearing in this statement and* (3.13) *have the properties given above. In particular,* Φ_r *is* $\mathbb{Z}_q$*-equivariant and* R*-reversible.*

Now, if we consider the special case where the mapping Φ is also S_0-equivariant such as $\Phi_{r,\lambda}$ is, i.e.,

$$\Phi\left(S_0 x, \lambda\right) = S_0 \Phi\left(x, \lambda\right), \quad \text{for all } (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m, \tag{3.15}$$

then, for $(u, \lambda) \in U \times \mathbb{R}^m$,

$$\Phi\left(u, \lambda\right) = \Phi\left(S_0^q u, \lambda\right) = S_0^q \Phi\left(u, \lambda\right),, \tag{3.16}$$

which means $\Phi\left(u, \lambda\right) \in U$ and $\widehat{\Phi}_\lambda\left(\xi\left(u\right)\right) \in \widehat{U}$. Comparing with (3.10) it follows that

$$\Psi(u, 0, \lambda) = \Phi_\lambda(u) \quad \text{and} \quad \Sigma(u, 0, \lambda) = 0.$$

Therefore,[1]

$$v^*(u, \lambda) = 0 \quad \text{and} \quad \Phi_{r,\lambda}(u) = \Psi_\lambda(u, v^*(u, \lambda)) = \Psi_\lambda(u, 0) = \Phi_\lambda(u).$$

In combination with (3.4) this proves the following.

Lemma 3.7. *Under the foregoing assumptions, assume also that* (3.15) *holds. Then, for each* $(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m$ *sufficiently small,* x *is a* q*-periodic point of* $\Phi(\cdot, \lambda)$ *if and only if* x *belongs to* U *and satisfies the equation* $S_0 x = \Phi\left(x, \lambda\right)$.

Application of Lemma 3.7 to $\Phi_r : U \times \mathbb{R}^m \to U$ implies that all small q-periodic orbits of $\Phi_{r,\lambda}\left(\cdot\right)$ are necessarily $\mathbb{Z}_q$-orbits determined by S_0.

Corollary 3.8. *For* (u, λ) *sufficiently small,* $u \in U$ *is a* q*-periodic point of* $\Phi_{r,\lambda}\left(\cdot\right)$ *if and only if* $S_0 u = \Phi_{r,\lambda}\left(u\right)$.

As a consequence, the Reversible GLS Reduction Theorem 1 now is easily proved.

Proof of the Reversible GLS *Reduction Theorem 1.* Let $x_\lambda^*(u) = u + v_{1,\lambda}^*(u) \in \mathbb{R}^n$, where $v_{1,\lambda}^*$ is the first component of $v_\lambda^*(u) \in Y_q$. Then, the properties (i), (ii) and (v) follow from the properties of v_λ^* discussed before. Item (iii) and (iv) are proved by taking for Φ_r the map defined by (3.14). To prove (vii) proceed as follows. Suppose $(u, \lambda) \in U$ is a solution of the equation (3.13), then $u = S_0^{-1} \Phi_{r,\lambda}\left(u\right)$ and $\Phi_{r,\lambda}^{-1}\left(S_0 u\right) = S_0 \Phi_{r,\lambda}^{-1}(u)$. Hence,

$$\mathcal{B}\left(u, \lambda\right) = 0. \tag{3.17}$$

[1] We proved that $v = v^*(u, \lambda)$ is the unique solution of $\sigma \cdot v = \Sigma(u, v, \lambda)$. But also $\Sigma(u, v, \lambda) = 0 = \sigma \cdot 0$; i.e. $\underline{0}$ is a solution. Then by uniqueness of solutions $v = v^*\left(u, \lambda\right) = 0$.

Conversely, assume $\mathcal{B}(u,\lambda) = 0$. Then, $S_0^{-1}\Phi_{r,\lambda}(u) = \Phi_r^{-1}(S_0 u,\lambda)$. This gives

$$\Phi_{r,\lambda}(u) = S_0\Phi_r^{-1}(S_0 u,\lambda) = \Phi_r^{-1}\left(S_0^2 u,\lambda\right). \tag{3.18}$$

Application of Φ_r to both sides of this equation gives

$$\Phi_r^2(u,\lambda) = S_0^2 u \tag{3.19}$$

In view of Theorem 1-(vi), we have to distinguish two cases: q even and q odd. First suppose that $q = 2k$ (even). From (3.19) it follows that

$$\Phi_{r,\lambda}^{2k}(u) = S_0^{2k}u = I \cdot u = u \tag{3.20}$$

(remember that on U $S_0^q = I$), hence $\Phi_r^q(u,\lambda) = u$, i.e., u is a q-periodic point of the reduced diffeomorphism $\Phi_r(\cdot,\lambda)$. Therefore equation (3.13) holds.
Similarly, if $q = 2k+1$ (odd), equation (3.19) implies that u is a $2q$-periodic point of $\Phi_r(\cdot,\lambda)$, i.e.

$$\Phi_{r,\lambda}^{2q}(u) = S_0^{2q}u = I \cdot u = u, \tag{3.21}$$

By the reduction method we know that $2q$-periodic points of $\Phi_r(\cdot,\lambda)$ are the solutions of the equation

$$S_0|_U u = \widetilde{\Phi}_{r,\lambda}(u), \tag{3.22}$$

with $u \in \ker\left(S_0|_U^{2q} - I\right)$ and $\widetilde{\Phi}_{r,\lambda}(\cdot)$ the reduced map obtained by applying the reduction to the equation $\Phi_{r,\lambda}^{2q}(u) = u$. But the space $\ker\left(S_0|_U^{2q} - I\right)$ is the full space U, indeed, since $S_0^q = \mathrm{I}$ in U, $S_0^{2q} = S_0^q$ in U. Hence, the reduced map $\widetilde{\Phi}_{r,\lambda}(\cdot)$ coincides with the map $\Phi_{r,\lambda}(\cdot)$. Therefore we conclude that u is a $2q$-periodic point of $\Phi_{r,\lambda}(\cdot)$ if and only if

$$S_0 u = \Phi_{r,\lambda}(u)$$

or equivalently if and only if u is a q-periodic point of $\Phi_{r,\lambda}(\cdot)$, that is if and only if (3.13) holds. Hence, (vii) follows. Now, using the $\mathbb{Z}_q$-equivariance and the R-reversibility of $\Phi_{r,\lambda}$ it is straightforward that equation (3.13) is $\mathbb{D}_q$-equivariant, i.e. (viii) holds. □

Remarks

1- The Reversible GLS Reduction Theorem 1 tells us that there is a one-to-one relation between the small q-periodic orbits of the map Φ_λ and those of the *reduced map* $\Phi_{r,\lambda}$ which lives on a *reduced phase space* $U = \ker(S_0^q - \mathrm{I})$. The reduced map is both reversible and $\mathbb{Z}_q$-equivariant, and its small q-periodic orbits coincide with its small $\mathbb{Z}_q$-orbits, which leads to the determining equation (3.13). To get a full $\mathbb{D}_q$-equivariance one has to go yet one step further and reformulate the problem as in (vii). Indeed, by (viii), the *(reduced) branching function* $\mathcal{B}(\cdot,\lambda)$ is $\mathbb{D}_q$-equivariant with respect to the $\mathbb{D}_q$-action generated by S_0 and R on U ($S_0^q = R^2 = I$ and $S_0^{-1}R = RS_0$ on U).

2- Note that the GLS reduction, as described above, can be worked out such that (reasonable) additional structures of Φ_λ are preserved. For example, if Φ_λ is symplectic, then the reduced map will be symplectic and $\mathbb{Z}_q$-equivariant. We refer for the results in this setting to [32], see also [25].

3.2.2 Reduced Map via Normal Form

We conclude this chapter by proving Corollary 1.2, which provides a way to approximate the reduced map $\Phi_{r,\lambda}(u)$ up to any dersired order.

Proof of Corollary 1.2. By Theorem 2 we have that, given $k \geq 1$, we can without loss of generality assume that $\Phi_\lambda(x)$ has the form

$$\Phi_\lambda(x) = \Phi_\lambda^{NF}(x) + R_{k+1}(x,\lambda), \tag{3.23}$$

where the normal form part commutes with S_0:

$$\Phi_\lambda^{NF}(S_0 x) = S_0 \Phi_\lambda^{NF}(x), \quad \forall (x,\lambda) \in \mathbb{R}^n \times \mathbb{R}^m, \tag{3.24}$$

and where $R_{k+1}(x,\lambda) = O(||x||^{k+1})$ as $x \to 0$ uniformly for λ in some neighbourhood ω_k of the origin in $\mathbb{R}^m$. Now, going through the reduction as we did to prove Lemma 3.7, we find that

$$\begin{aligned} &\Phi_\lambda^{NF}(u) \in U, \quad \forall (u,\lambda) \in U \times \omega_k, \\ &\Psi(u,0,\lambda) = \Phi_\lambda^{NF}(u) + O(|x|^{k+1}) \quad \text{and} \quad \Sigma(u,0,\lambda) = O(|u|^{k+1}). \end{aligned}$$

Also, $v^*(u,\lambda) = O(|x|^{k+1})$, eventually for λ in a smaller neighbourhood. $\square$

Remark From (3.23) and (1.20) it follows that if the linear part of Φ is in normal form then $D_u\Phi_{r,\lambda}(0) = D_x\Phi_\lambda^{NF}(0)|_U = D_x\Phi_\lambda(0)|_U = A_\lambda|_U$. It follows that the eigenvalues of $D_u\Phi_{r,\lambda}(0)$ coincide with the eigenvalues of A_λ which are close to qth roots of unity.

4

Parametrized Reversible Normal Forms for Maps

This chapter roughly consists of two parts. To begin with, a number of subsections is devoted to formal normal form results near fixed points of general diffeomorphisms with parameters. Finally, the reversible case is analysed and Theorem 2 is proved.

We recall from Section 1.3 that our aim is to *simplify* the Taylor series at the origin of a given smooth map $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, satisfying (H1). This is done stepwise, using coordinates changes generated by homogeneous vector fields of the appropriate order. We start by explaining what the word 'simple' means in our context.

4.1 'Simple' in Terms of Adjoint Action

Given is the local map $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, with $\Phi_\lambda(0) = 0$, and $D\Phi_0(0)$ invertible, having SU decomposition $D\Phi_0(0) = S_0 \exp(\mathcal{N}_0)$. We consider the Adjoint action of the group $\mathrm{Diff}_0(\mathbb{R}^n)$ on itself,

$$\mathrm{Ad} : \mathrm{Diff}_0(\mathbb{R}^n) \times \mathrm{Diff}_0(\mathbb{R}^n) \to \mathrm{Diff}_0(\mathbb{R}^n), \qquad \mathrm{Ad}(\Psi)\Phi = \Psi^{-1}\Phi\Psi.$$

Goal is to find a smooth map Ψ_λ so that the Taylor expansion at 0 of $\mathrm{Ad}(\Psi_\lambda)\Phi_\lambda$ is as simple as possible for λ in a neighbourhood of the origin. Let $\mathcal{H}_k = \mathcal{H}_k(\mathbb{R}^n)$ be the space of polynomial maps homogeneous of degree k, note that $\mathcal{H}_1 = gl(n, \mathbb{R})$. Then the Taylor series of $\Phi \in \mathrm{Diff}_0(\mathbb{R}^n)$ is an element of the space of formal power series $\prod_{k=1}^{\infty} \mathcal{H}_k$. Also, considering $A_0 = D\Phi_0(0) \in \mathcal{L}(\mathbb{R}^n)$, it directly follows from the definitions that $\mathrm{Ad}(A_0)$ induces a linear map $\mathcal{H}_k \to \mathcal{H}_k$ to be denoted by $\mathrm{Ad}_k A_0$. Let $B_k := \mathrm{Im}\,(\mathrm{Ad}_k A_0 - \mathrm{I})$ be the image of the map $(\mathrm{Ad}_k A_0 - \mathrm{I})$ in $\mathcal{H}_k$. Then, for any complement C_k of B_k in $\mathcal{H}_k$, i.e.,

$$\mathcal{H}_k = B_k \oplus C_k,$$

we define the notion of 'simpleness' by requiring the homogeneous part of degree k to be in C_k. The choice of C_k is not unique. However, one general way to choose C_k is the following, see e.g. [19, 75, 74],

$$C_k = \ker\left(\mathrm{Ad}_k A_0^T - \mathrm{I}\right).$$

Here, A_0^T is the transposed of A_0 defined by $\langle A_0^T x, y\rangle = \langle x, A_0 y\rangle$, with $\langle \cdot,\cdot\rangle$ a suitable inner product in $\mathbb{R}^n$. So, the normal form can be interpreted in terms of symmetry with respect to the group generated by A_0^T. Note that if A_0 is semisimple, meaning that it is complex diagonalisable, one can choose

$$\mathcal{H}_k = \ker\left(\mathrm{Ad}_k A_0 - \mathrm{I}\right) \oplus \mathrm{Im}\left(\mathrm{Ad}_k A_0 - \mathrm{I}\right)$$

and the normalized part commutes with A_0.
For the validity of this set up when a given structure has to be preserved, one has to study in how far the grading

$$\prod_{k=1}^{\infty} \mathcal{H}_k$$

of the formal power series, as well as the splittings

$$B_k \oplus C_k = \mathcal{H}_k$$

are compatible with the structure at hand, [7]. This is the case when we deal with reversibility, see sections 2.2 and 2.3 for more details.

4.2 Normal Form for General Diffeomorphisms

The following proposition is a generalization of [74].

Proposition 4.1. *Let $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be a smooth local map satisfying* (H1). *Let $A_0 := D_x\Phi_{\lambda=0}(0) = S_0 \exp(\mathcal{N}_0)$ be the SU-decomposition of A_0 and let $\langle\cdot,\cdot\rangle$ be a scalar product as in Lemma 2.9. Then, for each $k \geq 1$ there exist a neighbourhood ω_k of the origin in $\mathbb{R}^m$ and a S_0-equivariant transformation $\Psi_k : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ satisfying $\Psi_{k.\lambda}(0) = 0$ and $D\Psi_{k,\lambda=0}(0) = \mathrm{I}$, such that*

$$\mathrm{Ad}(\Psi_{k,\lambda})\Phi_\lambda = S_0 e^{\mathcal{N}_0 + Z_\lambda} + R_{k+1}, \tag{4.1}$$

with $R_{k+1}(x,\lambda) = O\left(\|x\|^{k+1}\right)$ uniformly for $\lambda \in \omega_k$. Moreover, the smooth vector field Z_λ is such that

$$Z_\lambda(0) = 0, \quad DZ_{\lambda=0}(0) = 0, \tag{4.2}$$

$$S_0 \circ Z_\lambda = Z_\lambda \circ S_0, \tag{4.3}$$

$$DZ_\lambda(x)\mathcal{N}_0^T x = \mathcal{N}_0^T Z_\lambda(x), \forall x \in \mathbb{R}^n. \tag{4.4}$$

Although the proof is similar to the one in [74], see also [70], we include it here since it provides the basis for our further considerations and because we want to emphasize property (4.4). The proof proceeds by induction on k and uses a number of technical lemmas which we consider first. We refer to (4.1) as to the *nilpotent (general) normal form (of order k)* meaning that the vector field Z_λ commutes with $e^{t\mathcal{N}_0^T}$, i.e., it satisfies (4.4).

4.2.1 Basic Tools

Given a vector field $X \in \mathcal{X}_0$, let $(t,x) \mapsto e^{tX}(x)$ be the associated flow. The following lemma establishes the relation between the terms of the Taylor expansion

$$X = X_1 + \cdots + X_k \ \operatorname{mod}\mathcal{X}_0^k, \qquad \text{with } X_j \in \mathcal{H}_j \tag{4.5}$$

of X and those of the Taylor expansion

$$e^{tX} = \Psi_1(t) + \cdots + \Psi_k(t) \ \operatorname{mod}\mathcal{X}_0^k, \quad \text{with } \Psi_j(t) \in \mathcal{H}_j \tag{4.6}$$

of e^{tX}.

Lemma 4.2. *Consider a vector field $X \in \mathcal{X}_0(\mathbb{R}^n)$ with Taylor expansion* (4.5)*, and let $\dot{x} = X(x)$ be the corresponding ODE with associated flow e^{tX}. Then, for each $k \geq 1$,*

$$e^{tX} = \sum_{j=1}^{k} \Psi_j(t) \ \operatorname{mod}\mathcal{X}_0^k, \tag{4.7}$$

with

$$\Psi_1(t) = e^{tX_1}, \quad k = 1$$

$$\Psi_k(t) = \int_0^t e^{(t-s)X_1} X_k\left(e^{sX_1}\right) ds + \int_0^t e^{(t-s)X_1} B_k(s) ds, \quad \text{for all } k \geq 2,$$

where $B_k \in \mathcal{H}_k$ is uniquely determined by $X_1, \dots, X_{k-1}, \Psi_1, \dots, \Psi_{k-1}$.

Proof. Consider the projection

$$\pi_j : \mathcal{X}_0 \to \mathcal{H}_j, \quad (\pi_j X)(x) := \frac{1}{j!} D^j X(0)\, x^j,$$

which associates to X its j-th Taylor component, and set $\Psi_k(t) := \pi_k\left(e^{tX}\right)$. Consider the system

$$\begin{cases} \frac{d}{dt}(e^{tX}) & = & X \cdot e^{tX} \\ e^{tX}|_{t=0} & = & \mathrm{I}, \end{cases}$$

then replacing each map by its Taylor expansion gives:

(i) for $k=1$: $\dot{\Psi}_1 = X_1 \cdot \Psi_1 \operatorname{mod}\mathcal{X}_0^1$, with $\Psi_1(0) = \mathrm{I}$,

(ii) for $k \geq 2$: $\dot{\Psi}_k(t) = X_1 \cdot \Psi_k(t) + X_k(\Psi_1(t)) + B_k \operatorname{mod}\mathcal{X}_0^k$, with $\Psi_k(0) = 0$ and where $B_k = \sum_{j=2}^{k-1} X_j(\Psi_1 + \cdots + \Psi_{k-1}) - \sum_{j=1}^{k-1} X_j(\Psi_1) - \sum_{j=1}^{k-1} B_j \operatorname{mod}\mathcal{X}_0^k$.

Therefore,

$$\Psi_1(t) = e^{tX_1},$$

and

$$\Psi_k(t) = \int_0^t e^{(t-s)X_1} X_k(e^{sX_1})ds + \int_0^t e^{(t-s)X_1} B_k(s)ds, \quad k \geq 2,$$

which proves the proposition. □

The following two corollaries of Lemma 4.2 allow us to prove a weaker version of Proposition 4.1 (see Lemma 4.5); 'weaker' in the sense that we do not require the vector field $X_\lambda = \mathcal{N}_0 + Z_\lambda$ to commute with S_0. A weaker form of the Campbell-Hausdorff formula for vector fields [56] will then be used to get Proposition 4.1.

Define the operator $C_k : \mathcal{H}_1 \to \mathcal{L}(\mathcal{H}_k)$, for all $X_1 \in \mathcal{H}_1$, $X_k \in \mathcal{H}_k$, by

$$C_k(X_1)\, X_k := \int_0^1 e^{-sX_1} X_k e^{sX_1} ds. \tag{4.8}$$

Note that (4.8) can be rewritten as

$$C_k(X_1) = \int_0^1 Ad_k(e^{-sX_1})ds = \int_0^1 e^{-s\,\mathrm{ad}_k(X_1)}ds.$$

Observe also that if $X_1 = 0$, then

$$C_k(0) = \mathrm{I}_{\mathcal{H}_k}. \tag{4.9}$$

For simplicity of notation we denote Ad_k by Ad from now on.

Lemma 4.3. *Consider* $X \in \mathcal{X}_0$ *with Taylor expansion* (4.5) *and* $Z_k \in \mathcal{X}_0^k$ *. Then*

$$e^{X+Z_k} = e^X + e^{X_1} C_k\left(X_1\right) Z_k \operatorname{mod} \mathcal{X}_0^k. \tag{4.10}$$

Proof. Let $e^{t(X+Z_k)}$ be the flow of the system $\dot{x} = X(x) + Z_k(x)$. Suppose that the Taylor expansion of such flow is given by

$$e^{t(X+Z_k)} = \widetilde{\Psi}_1(t) + \cdots + \widetilde{\Psi}_k(t) \operatorname{mod} \mathcal{X}_0^k.$$

Then by Lemma 4.2 we have that, for all $t \in \mathbb{R}$,

$$\begin{aligned} \widetilde{\Psi}_j(t) &= \Psi_j(t) \operatorname{mod} \mathcal{X}_0^j, \quad 1 \le j \le k-1, \\ \widetilde{\Psi}_k(t,x) &= \Psi_k(t) + \int\limits_0^t e^{(t-s)X_1} Z_k\left(e^{sX_1}\right) ds \operatorname{mod} \mathcal{X}_0^k. \end{aligned} \tag{4.11}$$

Therefore, $e^{t(X+Z_k)} = e^{tX} + \int_0^t e^{(t-s)X_1} Z_k e^{sX_1} ds \operatorname{mod} \mathcal{X}_0^{\mathrm{k}}$. Hence, by setting $t = 1$ the result follows. □

Note that if $Z_k \in \mathcal{H}_k$, then

$$e^{Z_k} = \mathrm{I} + Z_k \operatorname{mod} \mathcal{X}_0^k. \tag{4.12}$$

For later use, we discuss two properties of the operator $C_k(X_1)$ defined in (4.8).

Lemma 4.4. *Let* $C_k : \mathcal{H}_1 \to \mathcal{L}(\mathcal{H}_k)$ *be defined as in* (4.8)*. The set* U_k *of all* $X_1 \in \mathcal{H}_1$ *such that* $C_k(X_1)$ *is invertible is open and contains all nilpotent* $N \in \mathcal{H}_1$.

Proof. Recall that $\mathcal{H}_1 = \mathcal{L}(\mathbb{R}^n)$. Observe that U_k is nothing but the inverse image of $GL(\mathcal{H}_k, \mathbb{R})$ under C_k; i.e., $U_k = C_k^{-1}(GL(\mathcal{H}_k, \mathbb{R}))$. Hence, by continuity of the map C_k, U_k is open. It remains to prove that

$$N \in \mathcal{L}\left(\mathbb{R}^n\right) \text{ nilpotent} \Longrightarrow C_k(N) \in \mathcal{L}\left(\mathcal{H}_k\right) \text{ invertible.}$$

By definition, we have that $C_k(N) = \int\limits_0^1 e^{-s(\mathrm{ad}_k N)} ds$. Now, the nilpotency of N implies that of $ad_k(N)$, and therefore

$$e^{-s(\mathrm{ad}_k(N))} = \mathrm{I}_{\mathcal{H}_k} + \widetilde{N}_{k,s}, \tag{4.13}$$

where $\widetilde{N}_{k,s}$ is nilpotent in $\mathcal{H}_k$, since it is a polynomial expression in s with nilpotent coefficients of the form $(-1)^j(\mathrm{ad}_k(N))^j$, $j \geq 0$. Thus, $e^{-s(\mathrm{ad}_k(N))}$ is invertible and so is $C_k(N)$. □

Let $\mathcal{P}_k$ be the space of all polynomial maps $p : \mathbb{R}^n \to \mathbb{R}^n$, $p(0) = 0$, of degree less or equal to k. According to our previous notation, $\mathcal{P}_k = \mathcal{X}_0/\mathcal{X}_0^k$, $k \geq 1$. Set

$$X^{[k]} := X_1 + \cdots + X_k \in \mathcal{P}_k, \quad \text{with each } X_j \in \mathcal{H}_j.$$

Lemma 4.5. *Let $\Phi_\lambda : \mathbb{R}^n \to \mathbb{R}^n$, $\lambda \in \mathbb{R}^m$ be a smooth map satisfying* (H1) *and let $A_0 = S_0 e^{\mathcal{N}_0}$ be the SU-decomposition of $A_0 := D_x\Phi_0(0)$. Then, for each $k \geq 1$, there exist a neighbourhood ω_k of the origin in $\mathbb{R}^m$ and a unique smooth $X_\lambda^{[k]} : \omega_k \to \mathcal{P}_k$, depending on λ, such that*

$$\Phi_\lambda = S_0 e^{X_\lambda^{[k]}} \,\mathrm{mod}\mathcal{X}_0^k, \quad \lambda \in \omega_k, \quad \textit{with} \quad X_{1,\lambda=0} = \mathcal{N}_0. \tag{4.14}$$

Proof. We proceed by induction on k . The basis of the induction is the linear case $k = 1$.

Set $A_\lambda := D\Phi_\lambda(0) = S_0 B_\lambda$, with $B_0 = e^{\mathcal{N}_0}$. The linear operator $X_{1,\lambda} := \log B_\lambda \in \mathcal{H}_1$ is well defined in some neighbourhood $\omega_1 \subseteq \mathbb{R}^m$ of $\lambda = 0$. Obviously

$$X_{1,\lambda=0} = \mathcal{N}_0, \quad \text{and} \quad \Phi_\lambda = S_0 e^{X_{1,\lambda}} \,\mathrm{mod}\mathcal{X}_0^1, \quad \forall \lambda \in \omega_1. \tag{4.15}$$

The uniqueness follows because the exponential map is a local diffeomorphism.

In the inductive step we suppose that $\Phi_\lambda = S_0 e^{X_\lambda^{[k-1]}} \mathrm{mod}\mathcal{X}_0^{k-1}$, where $X_\lambda^{[k-1]} \in \mathcal{P}_{k-1}$ and $X_\lambda^{[k-1]} \mathrm{mod}\mathcal{X}_0^1$ coincides with $X_{1,\lambda}$ found at step $k = 1$. If $k > 1$, we can always write

$$\Phi_\lambda = S_0 e^{X_\lambda^{[k-1]}} \left(I + Y_{k,\lambda}\right) \,\mathrm{mod}\mathcal{X}_0^k, \tag{4.16}$$

with $Y_{k,\lambda} := \left(e^{-X_\lambda^{[k-1]}} S_0^{-1} \Phi_\lambda - I\right) \,\mathrm{mod}\mathcal{X}_0^k \in \mathcal{H}_k$. In particular Y_k is zero up to order $k - 1$. Therefore, (4.16) implies

$$\Phi_\lambda = S_0 e^{X_\lambda^{[k-1]}} + S_0 e^{X_{1,\lambda}} Y_{k,\lambda} \,\mathrm{mod}\mathcal{X}_0^k. \tag{4.17}$$

So, we look for $Z_{k,\lambda} \in \mathcal{H}_k$ such that

$$\Phi_\lambda = S_0 e^{X_\lambda^{[k]}} \operatorname{mod}\mathcal{X}_0^k, \quad \text{with } X_\lambda^{[k]} := X_\lambda^{[k-1]} + Z_{k,\lambda}.$$

By Corollary 4.3 one has

$$\Phi_\lambda = S_0 e^{X_\lambda^{[k-1]}} + S_0 e^{X_{1,\lambda}} C_k\left(X_{1,\lambda}\right) Z_{k,\lambda} \operatorname{mod}\mathcal{X}_0^k,$$

and comparing with (4.17) gives

$$C_k\left(X_{1,\lambda}\right) Z_{k,\lambda} = Y_{k,\lambda}. \tag{4.18}$$

Hence Lemma 4.4 implies that there exists a neighborhood $\omega_k \subset \omega_{k-1}$ of $\lambda = 0$ such that $C_k\left(X_{1,\lambda}\right)$ is invertible, so

$$Z_{k,\lambda} = C_k\left(X_{1,\lambda}\right)^{-1} Y_{k,\lambda} \in \mathcal{H}_k(\mathbb{R}^n), \tag{4.19}$$

which completes the proof. □

The following result provides a weaker form of the Campbell-Hausdorff formula [56].

Lemma 4.6. *Given $X \in \mathcal{X}_0$, let $X_1 := X \operatorname{mod}\mathcal{X}_0^1$ such that $C_k(X_1)$ is invertible. Then, for any $Y_k \in \mathcal{H}_k$,*

(i) $e^X e^{Y_k} = e^{X + C_k(X_1)^{-1} Y_k} \operatorname{mod}\mathcal{X}_0^k,$

(ii) $e^{Y_k} e^X = e^{X + C_k(-X_1)^{-1} Y_k} \operatorname{mod}\mathcal{X}_0^k.$

Proof. On behalf of (i) note that $e^X e^{Y_k} = e^X\left(I + Y_k\right) \operatorname{mod}\mathcal{X}^k = e^X + e^{X_1} Y_k \operatorname{mod}\mathcal{X}_0^k$ by Taylor expansion. Corollary 4.3 implies that $e^{X + C(X_1)^{-1} Y_k} = e^X + e^{X_1} C_k\left(X_1\right)\left(C_k\left(X_1\right)^{-1} Y_k\right) = e^X + e^{X_1} Y_k$, hence (i) follows. To obtain (ii), we first rewrite $e^{Y_k} e^X$:

$$\begin{aligned} e^{Y_k} e^X = \left(e^X e^{-X}\right) e^{Y_k} e^X &= e^X e^{\operatorname{Ad}\left(e^{-X}\right) \cdot Y_k} \operatorname{mod}\mathcal{X}_0^k \\ &= e^{X + C_k(X_1)^{-1} \operatorname{Ad}\left(e^{-X}\right) \cdot Y_k} \operatorname{mod}\mathcal{X}_0^k. \end{aligned}$$

Now, (ii) follows if we prove that

$$C_k(-X_1)\left(C_k(X_1)^{-1} \operatorname{Ad}(e^{-X_1}) Y_k\right) = Y_k.$$

Now,

$$\begin{aligned} C_k(X_1)\left(C_k(X_1)^{-1}\mathrm{Ad}(e^{-X_1})Y_k\right) \\ &= \int_0^1 e^{-sX_1}\left(C_k(X_1)^{-1}\mathrm{Ad}(e^{-X_1})Y_k\right)e^{sX_1}ds \\ &= \mathrm{Ad}(e^{-X_1})Y_k, \end{aligned}$$

therefore,

$$\begin{aligned} Y_k &= \int_0^1 e^{(1-s)X_1}\left(C_k(X_1)^{-1}\mathrm{Ad}(e^{-X_1})Y_k\right)e^{-(1-s)X_1}ds \\ &= \int_0^1 e^{sX_1}\left(C_k(X_1)^{-1}\mathrm{Ad}(e^{-X_1})Y_k\right)e^{-sX_1}xds \\ &= C_k(-X_1)\left(C_k(X_1)^{-1}\mathrm{Ad}(e^{-X_1})Y_k\right)(x), \end{aligned}$$

hence $C_k(-X_1)^{-1}Y_k = C_k(X_1)^{-1}\mathrm{Ad}(e^{-X_1})Y_k$, which proves (ii). □

Remarks

1- Let $X \in \mathcal{X}_0$ be such that $X_1 = 0$ then $C_k(X_1) = \mathrm{I}_{\mathcal{H}_k}$ and $e^{Y_k}e^X = e^X e^{Y_k} = e^{X+Y_k}$.

2- Let $X \in \mathcal{X}_0$ and $\Psi \in \mathrm{Diff}_0(\mathbb{R}^n)$, then

$$\mathrm{Ad}(\Psi)e^X = e^{\Psi^* X}, \tag{4.20}$$

where we recall that $\Psi^* X \in \mathcal{X}_0$ is the pushforward of X under Ψ, i.e., $(\Psi^* X)(\Psi(x)) = D\Psi(x)X(x)$. If $\Psi = e^{\Psi_k}$, with $\Psi_k \in \mathcal{H}_k$ $(k > 1)$, then, setting $X_1 := DX(0)$ we have that

$$(\Psi^* X)(e^{\Psi_k}x) = X(x) + D\Psi_k(x)X_1(x) \mod \mathcal{X}_0^k$$

and

$$\begin{aligned} (\Psi^* X)(x) &= X(x - \Psi_k(x)) + D\Psi_k(x)X_1(x) \mod \mathcal{X}_0^k \\ &= X(x) - X_1\Psi_k(x) + D\Psi_k(x)X_1(x) \mod \mathcal{X}_0^k. \end{aligned}$$

Hence

$$\mathrm{Ad}(e^{\Psi_k})e^X = e^{X - \mathrm{ad}_k(X_1)\Psi_k} \mod \mathcal{X}_0^k. \tag{4.21}$$

4.2.2 Semisimple Normal Form

We now have all the ingredients to show that if $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ satisfies (H1), with $A_0 = D\Phi_0(0) = S_0 \exp(\mathcal{N}_0)$ (SU-decomposition), then for each $k \geq 1$ there exists a S_0-equivariant polynomial vector field $X_\lambda^{[k]} \in \mathcal{P}_k$ such that Φ_λ can be written as $\Phi_\lambda = A_0 e^{X_\lambda^{[k]}} \mod \mathcal{X}_0^k$. More precisely we prove the following proposition.

Proposition 4.7. *Let $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be a smooth family of local diffeomorphisms satisfying* (H1) *and let $A_0 := D\Phi_0(0) = S_0 \exp(\mathcal{N}_0) \in GL(n, \mathbb{R})$ be the SU-decomposition of A_0. Then, for each $k \geq 1$ there exists a neighbourhood ω_k of the origin in $\mathbb{R}^m$ and a parameter dependent near-identity transformation $\Psi_{k,\lambda} : \mathbb{R}^n \to \mathbb{R}^n$ ($\lambda \in \mathbb{R}^m$) with $\Psi_{k,\lambda}(0) = 0$ and $D\Psi_{k,0}(0) = \mathrm{I}$, such that*

$$\mathrm{Ad}(\Psi_{k,\lambda})\Phi_\lambda = A_0 e^{X_\lambda^{[k]}} \mod \mathcal{X}_0^k, \quad \forall \lambda \in \omega_k \tag{4.22}$$

with $X_\lambda^{[k]} \in \ker(\mathrm{Ad}(S_0) - \mathrm{I}) \subset \mathcal{P}_k$ and $X_{\lambda=0}^{[k]} = 0 \mod \mathcal{X}_0^1$.

We call (4.22) the *semisimple (general) normal form (of order k)* meaning that the vector field $X_\lambda^{[k]}$ is equivariant with respect to the semisimple part S_0 of $D\Phi_0(0)$. Note that Proposition 4.1 is an improvement of this result in the sense that the vector field Z_λ in (4.1) is not only S_0-equivariant but also belongs to $\ker(\mathrm{ad}(\mathcal{N}_0^T))$.

The proof of Proposition 4.7 proceeds by induction on k and the basis of induction is given by the following lemma.

Lemma 4.8 (Linear Semisimple Normal Form). *Consider $A_0 \in GL(n, \mathbb{R})$ and let $A_0 = S_0 e^{\mathcal{N}_0}$ be its SU-decomposition. Then, there exists a smooth map $\widetilde{\Psi} : GL(n, \mathbb{R}) \to gl(n, \mathbb{R})$, with $\widetilde{\Psi}(A_0) = 0$, such that for all $A \in GL(n, \mathbb{R})$ close to A_0*

$$\mathrm{Ad}\left(e^{\widetilde{\Psi}(A)}\right) \cdot A = A_0 e^{B(A)}, \tag{4.23}$$

with $B(A) \in \ker(\mathrm{Ad}(S_0) - \mathrm{I}) \subset gl(n, \mathbb{R})$ and $B(A_0) = 0$.

Proof. Define $f : gl(n, \mathbb{R}) \times GL(n, \mathbb{R}) \to gl(n, \mathbb{R})$ by the relation

$$\mathrm{Ad}(e^{\Psi})A := A_0 e^{f(\Psi, A)}, \tag{4.24}$$

with

$$f(\Psi, A) := \log(h(\Psi, A)), \tag{4.25}$$
$$h(\Psi, A) := A_0^{-1} e^{\Psi} A e^{-\Psi}. \tag{4.26}$$

Note that $h(0, A_0) = \mathrm{I}$ and hence the log in (4.25) is well defined and smooth near $(0, A_0)$. Now,

$$f(0, A_0) = 0$$

and differentiating (4.24) with respect to Ψ at $(0, A_0)$ one obtains

$$D_\Psi f(0, A_0) = \mathrm{Ad}(A_0^{-1}) - \mathrm{I} \in \mathcal{L}(gl(n, \mathbb{R})).$$

Let $\pi \in \mathcal{L}(gl(n, \mathbb{R}))$ be the projection of $gl(n, \mathbb{R})$ on $\mathrm{Im}(\mathrm{Ad}_1(S_0^{-1}) - \mathrm{I})$ parallel to $\ker(\mathrm{Ad}_1(S_0^{-1}) - \mathrm{I})$. Observe that $\mathrm{Ad}_1(A_0^{-1}) - \mathrm{I}$ leaves $\mathrm{Im}(\mathrm{Ad}_1(S_0^{-1}) - \mathrm{I})$ invariant and it is invertible on $\mathrm{Im}(\mathrm{Ad}_1(S_0^{-1}) - \mathrm{I})$. Now, define $g := \pi \cdot f|_{\mathrm{Im}(\mathrm{Ad}(S_0^{-1}) - \mathrm{I}) \times GL(n, \mathbb{R})}$. Then

$$g(0, A_0) = 0 \quad \text{and} \quad D_\psi g(0, A_0) \in \mathcal{L}(\mathrm{Im}(\mathrm{Ad}_1(S_0^{-1}) - \mathrm{I})) \text{ is invertible.}$$

It follows by the Implicit Function Theorem that there exists a map $\tilde{\Psi} : GL(n, \mathbb{R}) \to gl(n, \mathbb{R})$ with $\tilde{\Psi}(A_0) = 0$ such that

$$g(\Psi, A) = 0$$

near $(0, A_0)$ if and only if

$$\Psi = \tilde{\Psi}(A).$$

Hence, setting

$$B(A) := f(\tilde{\Psi}(A), A) \in \ker(\mathrm{Ad}_1(S_0^{-1}) - \mathrm{I})$$

proves the result. Note that we used $\ker(\mathrm{Ad}_1(S_0^{-1}) - \mathrm{I}) = \ker(\mathrm{Ad}_1(S_0) - \mathrm{I})$. □

We now proof Proposition 4.7.

Proof of Proposition 4.7. We use induction on k. For $k=1$ the result follows from Proposition 4.7 by taking

$$\Psi_{1,\lambda} := e^{\tilde{\Psi}(A_\lambda)} \quad \text{and} \quad X_\lambda^{[1]} = B(A_\lambda) \in \mathcal{P}_1 = gl(n, \mathbb{R}).$$

Now, let $k > 1$ and assume the result true for $(k-1)$. Denoting $\mathrm{Ad}(\Psi_{k-1,\lambda})\Phi$ again by Φ_λ, this means that

$$\Phi_\lambda = A_0 e^{X_\lambda^{[k-1]}} \mod \mathcal{X}_0^{k-1},$$

with $X_\lambda^{[k-1]} \in \ker(\mathrm{Ad}(S_0) - \mathrm{I}) \subset \mathcal{P}_{k-1}$ and $X_{\lambda=0}^{[k-1]} = 0 \mod \mathcal{X}_0^1$. Then,

$$\Phi_\lambda = A_0 e^{X_\lambda^{[k-1]} + Z_{k,\lambda}} \mod \mathcal{X}_0^k,$$

for some $Z_{k,\lambda} \in \mathcal{H}_k$. Our aim is to show that we can transform Φ_λ further so that $Z_{k,\lambda}$ is in $\ker(\mathrm{Ad}(S_0) - \mathrm{I})$. To this purpose, consider a transformation $\Psi = e^{\Psi_k}$, with $\Psi_k \in \mathcal{H}_k$ and define $f_k : \mathcal{H}_k \times \mathbb{R}^m \to \mathcal{H}_k$ by the relation

$$\mathrm{Ad}(e^{\Psi_k})\Phi_\lambda = A_0 e^{X_\lambda^{[k-1]} + f_k(\Psi_k, \lambda)} \mod \mathcal{X}_0^k.$$

The mapping $f_k(\Psi_k, \lambda)$ is well defined and smooth for (Ψ_k, λ) near $(0, 0)$, and $f_k(0, \lambda) = Z_{k,\lambda}$. An explicit form for $f_k(\Psi_k, \lambda)$ is found as follows.

$$\begin{aligned}
\mathrm{Ad}(e^{\Psi_k})\Phi_\lambda &= e^{\Psi_k} A_0 e^{X_\lambda^{[k-1]} + Z_{k,\lambda}} e^{-\Psi_k} \mod \mathcal{X}_0^k \\
&= A_0 e^{\mathrm{Ad}(A_0^{-1})\Psi_k} e^{X_\lambda^{[k-1]} + Z_{k,\lambda}} e^{-\Psi_k} \mod \mathcal{X}_0^k \\
&= A_0 e^{X_\lambda^{[k-1]} + Z_{k,\lambda} + C_k(-X_{1,\lambda})^{-1}\mathrm{Ad}(A_0^{-1})\Psi_k - C_k(X_{1,\lambda})^{-1}\Psi_k} \mod \mathcal{X}_0^k,
\end{aligned}$$

where $X_{1,\lambda} = DX_\lambda^{[k-1]}(0) \in gl(n, \mathbb{R})$. It follows that

$$f_k(\Psi_k, \lambda) = Z_{k,\lambda} + C_k(-X_{1,\lambda})^{-1}\mathrm{Ad}(A_0^{-1})\Psi_k - C_k(X_1, \lambda)^{-1}\Psi_k.$$

So, $f_k(\Psi_k, \lambda)$ has the form

$$f(\Psi_k, \lambda) = Z_{k,\lambda} + D_k(\lambda)\Psi_k,$$

with $D_k(\lambda) \in \mathcal{L}(\mathcal{H}_k)$ given by

$$D_k(\lambda) := C_k(-X_{1,\lambda})^{-1}\mathrm{Ad}(A_0^{-1}) - C_k(X_1, \lambda)^{-1}.$$

Note that for $\lambda = 0$ we have

$$X_{1,0} = 0 \quad \text{and} \quad D_k(0) = \mathrm{Ad}(A_0^{-1}) - \mathrm{I}.$$

Now, recall that $\mathcal{H}_k = \mathrm{Im}(\mathrm{Ad}_k(S_0^{-1}) - \mathrm{I}) \oplus \ker(\mathrm{Ad}_k(S_0^{-1}) - \mathrm{I})$ and that the operator $\mathrm{Ad}(A_0^{-1}) - \mathrm{I}|_{\mathrm{Im}(\mathrm{Ad}(S_0^{-1})-\mathrm{I})}$ is an isomorphism of $\mathrm{Im}(\mathrm{Ad}(S_0^{-1}) - \mathrm{I})$ into itself. Let $\pi \in \mathcal{L}(\mathcal{H}_k)$ be the projection of $\mathcal{H}_k$ on $\mathrm{Im}(\mathrm{Ad}_k(S_0^{-1}) - \mathrm{I})$ parallel to $\ker(\mathrm{Ad}_k(S_0^{-1}) - \mathrm{I})$ and define $g_k := \pi_k \circ f_k|_{\mathrm{Im}(\mathrm{Ad}_k(S_0^{-1})-\mathrm{I})\times\mathbb{R}^m}$. Now, since the operator $\pi_k D_k(0)|_{\mathrm{Im}(\mathrm{Ad}_k(S_0^{-1})-\mathrm{I})} \in \mathcal{L}(\mathrm{Im}(\mathrm{Ad}(S_0^{-1}) - \mathrm{I}))$ is an isomorphism then so is $\pi_k D_k(\lambda)|_{\mathrm{Im}(\mathrm{Ad}(S_0^{-1})-\mathrm{I})}$ for λ in a sufficiently small neighbourhood ω_k of the origin in $\mathbb{R}^m$. For such λ the equation $g_k(\Psi_k, \lambda) = 0$ has a unique solution $\Psi = \Psi^*_{k,\lambda} \in \mathrm{Im}(\mathrm{Ad}_k(S_0^{-1}) - \mathrm{I})$. Then $f_k(\Psi^*_{k,\lambda}, \lambda) \in \ker(\mathrm{Ad}_k(S_0^{-1}) - \mathrm{I}) = \ker(\mathrm{Ad}_k(S_0^{-1}) - \mathrm{I})$. The result follows by setting

$$\Psi_{k,\lambda} = e^{\Psi^*_{k,\lambda}}\Psi_{k-1,\lambda} \quad \text{and} \quad X_\lambda^{[k]} = X_\lambda^{[k-1]} + f_k(\Psi^*_{k,\lambda}, \lambda).$$

□

4.2.3 Nilpotent Normal Form

Goal of this section is to prove Proposition 4.1 by using Proposition 4.7. That is, using the semisimple normal form (4.22) and the SU-decomposition $A_0 = S_0 e^{\mathcal{N}_0}$, we can assume that (for each $k \geq 1$) Φ_λ is put in the form

$$\Phi_\lambda = S_0 e^{\mathcal{N}_0 + \hat{X}_\lambda^{[k]}} \mod \mathcal{X}_0^k,$$

with $\hat{X}_\lambda^{[k]}$ S_0-equivariant, i.e., $\hat{X}_\lambda^{[k]} \in \ker(\mathrm{Ad}(S_0) - \mathrm{I}) \subset \mathcal{P}_k$, and satisfying $\hat{X}_\lambda^{[k]}(0) = 0$ and $D\hat{X}_{\lambda=0}^{[k]}(0) = 0$. We show that it is possible to transform Φ_λ further so that the (approximate) S_0-equivariance is preserved and $\hat{X}_\lambda^{[k]}$ is simplified (i.e. it also commutes with $e^{t\mathcal{N}_0^T}$). We start with the linear case $k = 1$.

Lemma 4.9 (Linear Nilpotent Normal Form). *Let $\Phi_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ satisfy the hypotheses of Proposition 4.1. Then, there exists a parameter dependent S_0-equivariant linear transformation $\Psi_{1,\lambda} \in \mathcal{L}(GL(n, \mathbb{R}))$ with $\Psi_{1,0} = \mathrm{I}$ such that*

$$\mathrm{Ad}(\Psi_{1,\lambda})\Phi_\lambda = S_0 e^{\mathcal{N}_0 + \hat{Y}_\lambda^{[1]}} \mod \mathcal{X}_0^1, \tag{4.27}$$

with $\hat{Y}_\lambda^{[1]} \in \ker(\mathrm{Ad}_1(S_0) - \mathrm{I}) \cap \ker(\mathrm{ad}_1(\mathcal{N}_0^T))$ for all $\lambda \in \mathbb{R}^m$ in a sufficiently small neighbourhood ω_1 of the origin in $\mathbb{R}^m$. Also, $\hat{Y}_0^{[1]} = 0$.

Proof. By Lemma 4.8 we can assume that $\Phi_\lambda = S_0 e^{\mathcal{N}_0 + \hat{X}_\lambda^{[1]}} \mod \mathcal{X}_0^1$ with $\hat{X}_\lambda^{[1]} \in \ker(\mathrm{Ad}_1(S_0) - \mathrm{I})$. Consider a transformation $\Psi \in \mathcal{L}(GL(n,\mathbb{R}))$ of the form $\Psi := e^{\Psi_1}$ with $\Psi_1 \in \ker(\mathrm{Ad}_1(S_0) - \mathrm{I}) \subset gl(n,\mathbb{R})$. Then,

$$\mathrm{Ad}(e^{\Psi_1})\Phi_\lambda = S_0 e^{\mathcal{N}_0 + f_1(\Psi_1,\lambda)} \mod \mathcal{X}_0^1, \tag{4.28}$$

with $f_1 : \ker(\mathrm{Ad}_1(S_0) - \mathrm{I}) \times \mathbb{R}^m \to \ker(\mathrm{Ad}_1(S_0) - \mathrm{I})$ well defined and smooth for (Ψ_1, λ) near $(0,0)$ and such that $f_1(0,0) = 0$. Using the S_0-equivariance of Ψ_1 one obtains from (4.28) the explicit form of $f_1(\Psi_1, \lambda)$:

$$f_1(\Psi_1, \lambda) = \mathrm{Ad}(e^{\Psi_1})(\mathcal{N}_0 + X_\lambda^{[1]}) - \mathcal{N}_0.$$

Now, for $\lambda = 0$,

$$\begin{aligned} &f_1(\Psi_1, 0) = \mathrm{Ad}(e^{\Psi_1})\mathcal{N}_0 - \mathcal{N}_0, \\ &D_{\Psi_1} f_1(0,0) = -\mathrm{ad}(\mathcal{N}_0)|_{\ker(\mathrm{Ad}_1(S_0)-\mathrm{I})} \in \mathcal{L}(\ker(\mathrm{Ad}_1(S_0) - \mathrm{I})). \end{aligned}$$

Due to our choice of scalar products in $\mathbb{R}^n$ and $gl(n,\mathbb{R})$ (see Lemma 2.9) the operators $\mathrm{ad}_1(\mathcal{N}_0)$ and $\mathrm{ad}_1(\mathcal{N}_0^T)$ leave the complementary subspaces $\ker(\mathrm{Ad}_1(S_0) - \mathrm{I})$ and $\mathrm{Im}(\mathrm{Ad}_1(S_0) - \mathrm{I})$ of $gl(n,\mathbb{R})$ invariant and $\left(\mathrm{ad}_1(\mathcal{N}_0)|_{\ker(\mathrm{Ad}_1(S_0)-\mathrm{I})}\right)^T = \mathrm{ad}_1(\mathcal{N}_0^T)|_{\ker(\mathrm{Ad}_1(S_0)-\mathrm{I})}$. Hence,

$$\begin{aligned} \ker(\mathrm{Ad}_1(S_0) - \mathrm{I}) = {} & [\ker(\mathrm{Ad}_1(S_0) - \mathrm{I}) \cap \ker(\mathrm{ad}_1(\mathcal{N}_0))] \\ & \oplus \left[\ker(\mathrm{Ad}_1(S_0) - \mathrm{I}) \cap \mathrm{Im}(\mathrm{ad}_1(\mathcal{N}_0^T))\right] \end{aligned}$$

and

$$\begin{aligned} \ker(\mathrm{Ad}_1(S_0) - \mathrm{I}) = {} & [\ker(\mathrm{Ad}_1(S_0) - \mathrm{I}) \cap \mathrm{Im}(\mathrm{ad}_1(\mathcal{N}_0))] \\ & \oplus \left[\ker(\mathrm{Ad}_1(S_0) - \mathrm{I}) \cap \ker(\mathrm{ad}_1(\mathcal{N}_0^T))\right]. \end{aligned} \tag{4.29}$$

Moreover, $\mathrm{ad}_1(\mathcal{N}_0)$ is an isomorphism from $\ker(\mathrm{Ad}_1(S_0) - \mathrm{I}) \cap \mathrm{Im}(\mathrm{ad}_1(\mathcal{N}_0^T))$ onto the subspace $\ker(\mathrm{Ad}_1(S_0) - \mathrm{I}) \cap \mathrm{Im}(\mathrm{ad}_1(\mathcal{N}_0))$. Let $\pi : \ker(\mathrm{Ad}_1(S_0) - \mathrm{I}) \to [\ker(\mathrm{Ad}_1(S_0) - \mathrm{I}) \cap \mathrm{Im}(\mathrm{ad}_1(\mathcal{N}_0))]$ be the projection in $\ker(\mathrm{Ad}_1(S_0) - \mathrm{I})$ associated to the splitting (4.29). Define

$$\begin{aligned} g_1 : {} & \left[\ker(\mathrm{Ad}_1(S_0) - \mathrm{I}) \cap \mathrm{Im}(\mathrm{ad}_1(\mathcal{N}_0^T)) \times \mathbb{R}^m\right] \\ & \to [\ker(\mathrm{Ad}_1(S_0) - \mathrm{I}) \cap \mathrm{Im}(\mathrm{ad}_1(\mathcal{N}_0))] \end{aligned}$$

by

$$g_1 := \pi \circ f_1|_{\left[\ker(\mathrm{Ad}_1(S_0)-\mathrm{I}) \cap \mathrm{Im}(\mathrm{ad}_1(\mathcal{N}_0^T))\right] \times \mathbb{R}^m}.$$

Then, $g_1(0,0)=0$ and $D_{\Psi_1}g_1(0,0)$ is an isomorphism from $[\ker(\mathrm{Ad}_1(S_0)-\mathrm{I})\cap \mathrm{Im}(\mathrm{ad}_1(\mathcal{N}_0^T))]$ onto $[\ker(\mathrm{Ad}_1(S_0)-\mathrm{I})\cap\mathrm{Im}(\mathrm{ad}_1(\mathcal{N}_0))]$. It follows by the Implicit Function Theorem that for (Ψ_1,λ) near $(0,0)$

$$g_1(\Psi_1,\lambda)=0$$

if and only if $\Psi_1=\Psi_1^*(\lambda)$ with $\Psi_1^*(\lambda)\in\left[\ker(\mathrm{Ad}_1(S_0)-\mathrm{I})\cap\mathrm{Im}(\mathrm{ad}_1(\mathcal{N}_0^T))\right]$ and $\Psi_1^*(0)=0$. Then, $f_1(\Psi_1^*(\lambda),\lambda)\in\left[\ker(\mathrm{Ad}_1(S_0)-\mathrm{I})\cap\mathrm{Im}(\mathrm{ad}_1(\mathcal{N}_0^T))\right]$. Hence the result follows by setting $\Psi_{1,\lambda}=e^{\Psi_1^*(\lambda)}$ and $\hat{Y}_\lambda^{[1]}=f_1(\Psi_1^*(\lambda),\lambda)$. $\square$

We now prove Proposition 4.1.

Proof of Proposition 4.1. We use induction on k. The case $k=1$ is given by Lemma 4.9. Let $k>1$, and suppose the result true for $k-1$, i.e., denoting $\mathrm{Ad}(\Psi_\lambda^{[k-1]})\Phi_\lambda$ again by Φ_λ we have that

$$\Phi_\lambda = S_0 e^{\mathcal{N}_0+\hat{Y}_\lambda^{[k-1]}} \mod \mathcal{X}_0^{k-1},$$

with $\hat{Y}_\lambda^{[k-1]}\in\ker(\mathrm{Ad}(S_0)-\mathrm{I})\cap\ker(\mathrm{ad}(\mathcal{N}_0^T))\subset\mathcal{P}_{k-1}$. Now,

$$\Phi_\lambda = S_0 e^{\mathcal{N}_0+\hat{Y}_\lambda^{[k-1]}+Z_{k,\lambda}} \mod \mathcal{X}_0^{k},$$

with $Z_{k,\lambda}\in\ker(\mathrm{Ad}_k(S_0)-\mathrm{I})\subset\mathcal{H}_k$. Our goal is to bring the term $Z_{k,\lambda}$ in $\ker(\mathrm{Ad}_k(S_0)-\mathrm{I})\cap\ker(\mathrm{ad}_k(\mathcal{N}_0^T))$. To do so, we consider a transformation $\Psi=e^{\Psi_k}$ with $\Psi_k\in\ker(\mathrm{Ad}_k(S_0)-\mathrm{I})$. Then $\mathrm{Ad}(\Psi)\Phi_\lambda$ remains S_0-equivariant and

$$\mathrm{Ad}(e^{\Psi_k})\Phi_\lambda = S_0 e^{\mathcal{N}_0+\hat{Y}_\lambda^{[k-1]}+f_k(\Psi_k,\lambda)} \mod \mathcal{X}_0^k$$

for some $f_k:\ker(\mathrm{Ad}_k(S_0)-\mathrm{I})\times\mathbb{R}^m\to\ker(\mathrm{Ad}_k(S_0)-\mathrm{I})$ well defined and smooth, with $f_k(0,\lambda)=Z_{k,\lambda}$. More explicitly

$$\begin{aligned}
\mathrm{Ad}(e^{\Psi_{k,\lambda}})\Phi_\lambda &= \mathrm{Ad}(e^{\Psi_{k,\lambda}})S_0 e^{\mathcal{N}_0+\hat{Y}_\lambda^{[k-1]}+Z_{k,\lambda}} \mod \mathcal{X}_0^k\\
&= S_0\mathrm{Ad}(e^{\Psi_{k,\lambda}})e^{\mathcal{N}_0+\hat{Y}_\lambda^{[k-1]}+Z_{k,\lambda}} \mod \mathcal{X}_0^k\\
&= S_0 e^{\mathcal{N}_0+\hat{Y}_\lambda^{[k-1]}+Z_{k,\lambda}-\mathrm{ad}(\mathcal{N}_0+\hat{Y}_\lambda^{[1]})\Psi_{k,\lambda}} \mod \mathcal{X}_0^k,
\end{aligned}$$

where $\hat{Y}_\lambda^{[1]}=D\hat{Y}_\lambda^{[k-1]}(0)$. It follows that

$$f_{k,\lambda}=Z_{k,\lambda}-\mathrm{ad}(\mathcal{N}_0+\hat{Y}_\lambda^{[1]})\Psi_{k,\lambda}=Z_{k,\lambda}-D_k(\lambda)\Psi_{k,\lambda},$$

with $D_k(\lambda) \in \mathcal{L}(\ker(\mathrm{Ad}_k(S_0) - \mathrm{I}))$ given by

$$D_k(\lambda) := \mathrm{ad}(\mathcal{N}_0 + \hat{Y}_\lambda^{[1]})|_{\ker(\ \mathrm{Ad}_k(S_0) - \mathrm{I})} \ .$$

Note that for $\lambda = 0$ we have that $\hat{Y}_0^{[1]} = 0$ and $D_k(0) = \mathrm{ad}(\mathcal{N}_0)|_{\ker(\mathrm{Ad}_k(S_0)-\mathrm{I})}$. By our choice of scalar product, both $\mathrm{ad}_k(\mathcal{N}_0)$ and $\mathrm{ad}_k(\mathcal{N}_0^T)$ leave the complementary subspaces $\ker(\mathrm{Ad}_k(S_0) - \mathrm{I})$ and $\mathrm{Im}(\mathrm{Ad}_k(\ S_0) - \mathrm{I})$ of $\mathcal{H}_k$ invariant and $\left(\mathrm{ad}_k(\mathcal{N}_0)|_{\ker(\mathrm{Ad}_k(S_0)-\mathrm{I})}\right)^T = \mathrm{ad}_k(\mathcal{N}_0^T)|_{\ker(\mathrm{Ad}_k(S_0)-\mathrm{I})}$.

It follows that in $\mathcal{H}_k$ it holds

$$\begin{aligned}\ker(\mathrm{Ad}_k(S_0) - \mathrm{I}) = [&\ker(\mathrm{Ad}_k(S_0) - \mathrm{I}) \cap \ker(\mathrm{ad}_k(\mathcal{N}_0))] \\ &\oplus \left[\ker(\mathrm{Ad}_k(S_0) - \mathrm{I}) \cap \mathrm{Im}(\mathrm{ad}_k(\mathcal{N}_0^T))\right]\end{aligned}$$

and

$$\begin{aligned}\ker(\mathrm{Ad}_k(S_0) - \mathrm{I}) = [&\ker(\mathrm{Ad}_k(S_0) - \mathrm{I}) \cap \mathrm{Im}(\mathrm{ad}_k(\mathcal{N}_0))] \\ &\oplus \left[\ker(\mathrm{Ad}_k(S_0) - \mathrm{I}) \cap \ker(\mathrm{ad}_k(\mathcal{N}_0^T))\right] . \end{aligned} \tag{4.30}$$

Moreover, $\mathrm{ad}_k(\mathcal{N}_0)$ is an isomorphism from $\ker(\mathrm{Ad}_k(S_0) - \mathrm{I}) \cap \mathrm{Im}(\mathrm{ad}_k(\mathcal{N}_0^T))$ onto $\ker(\mathrm{Ad}_k(S_0) - \mathrm{I}) \cap \mathrm{Im}(\mathrm{ad}_k(\mathcal{N}_0))$.

Let $\pi : \ker(\mathrm{Ad}_k(S_0) - \mathrm{I}) \to [\ker(\mathrm{Ad}_k(S_0) - \mathrm{I}) \cap \mathrm{Im}(\mathrm{ad}_k(\mathcal{N}_0))]$ be the projection in $\ker(\mathrm{Ad}_k(S_0) - \mathrm{I})$ associated to the splitting (4.30). Define

$$\begin{aligned}g_k : &\left[\ker(\mathrm{Ad}_k(S_0) - \mathrm{I}) \cap \mathrm{Im}(\mathrm{ad}_k(\mathcal{N}_0^T))\right] \\ &\qquad \to [\ker(\mathrm{Ad}_k(S_0) - \mathrm{I}) \cap \mathrm{Im}(\mathrm{ad}_k(\mathcal{N}_0))]\end{aligned}$$

by

$$g_k := \pi \circ f_k|_{\left[\ker(\mathrm{Ad}_k(S_0)-\mathrm{I}) \cap \mathrm{Im}(\mathrm{ad}_k(\mathcal{N}_0^T))\right] \times \mathbb{R}^m} .$$

Then, $g_k(\Psi_{k,\lambda}) = \pi_k Z_{k,\lambda} - \widetilde{D}_k(\lambda)\Psi_k$, where $\widetilde{D}_k(\lambda) \in \mathcal{L}\left(\left[\ker(\mathrm{Ad}_k(S_0) - \mathrm{I}) \cap \mathrm{Im}(\mathrm{ad}_k(\mathcal{N}_0^T))\right], [\ker(\mathrm{Ad}_k(S_0) - \mathrm{I}) \cap \mathrm{Im}(\mathrm{ad}_k(\mathcal{N}_0))]\right)$ depends smoothly on λ and is such that $\widetilde{D}_k(0)$ is an isomorphism. Therefore, $\widetilde{D}_k(\lambda)$ is an isomorphism for all λ in a sufficiently small neighbourhood ω_k of the origin in $\mathbb{R}^m$. For all such λ the equation $g(\Psi_k, \lambda) = 0$ has a unique solution $\Psi_k = \Psi_k^*(\lambda) = \widetilde{D}_k(\lambda)^{-1} \pi Z_{k\lambda} \in \left[\ker(\mathrm{Ad}_k(S_0) - \mathrm{I}) \cap \mathrm{Im}(\mathrm{ad}_k(\mathcal{N}_0^T))\right]$. Then

$$f_k(\Psi_k^*(\lambda), \lambda) \in \left[\ker(\mathrm{Ad}_k(S_0) - \mathrm{I}) \cap \mathrm{Im}(\mathrm{ad}_k(\mathcal{N}_0^T))\right]$$

and the result follows by taking $\Psi_{k,\lambda} = e^{\Psi_k^*(\lambda)} \Psi_{k-1,\lambda}$ and $Z_\lambda = \hat{Y}_\lambda^{[k]} = \hat{Y}_\lambda^{[k-1]} + f_k(\Psi_k^*(\lambda), \lambda)$. $\square$

4.3 Reversible Parametrized Normal Form (RPNF)

Goal of this section is to prove Theorem 2. The proof is divided into two parts. We first show that a map Φ_λ satisfying the hypotheses (H1) and (R) admits a normal form of the type 'A_0 times the time-one map of an S_0-equivariant R-reversible vector field' (*reversible semisimple normal form*). At a second stage, we prove that the transformation can be chosen so that the properties (1.16)– (1.19) hold (*reversible nilpotent normal form*). Recall that we restrict to R-equivariant transformations in order to preserve R-reversibility and proceed by induction.

4.3.1 Technicalities

Summarizing from section 4.2, let $\Phi_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ be such that $\Phi_\lambda(0) = 0$ and let $A_0 = S_0 e^{\mathcal{N}_0}$ be the SU-decomposition of $A_0 := D\Phi_0(0) \in GL(n, \mathbb{R})$. Then, for each $k \geq 1$ and for all sufficiently small $\lambda \in \mathbb{R}^m$ there exist polynomial vector fields $X_\lambda^{[k]}, \widehat{X}_\lambda^{[k]} \in \mathcal{P}_k$ such that

$$\begin{aligned} \Phi_\lambda &= A_0 e^{X_\lambda^{[k]}} \mod \mathcal{X}_0^k \\ &= S_0 e^{\mathcal{N}_0 + \widehat{X}_\lambda^{[k]}} \mod \mathcal{X}_0^k . \end{aligned}$$

Now, if Φ_λ is R-reversible, then

$$(RA_0) X_\lambda^{[k]} (RA_0) = -X_\lambda^{[k]} \quad \text{and} \quad (RS_0) \widehat{X}_\lambda^{[k]} (RS_0) = -\widehat{X}_\lambda^{[k]};$$

i.e. $X_\lambda^{[k]}$ is RA_0-reversible and $\widehat{X}_\lambda^{[k]}$ is RS_0-reversible. If moreover Φ_λ is S_0-equivariant (up to order k) then $\mathrm{Ad}(S_0)\Phi_\lambda = \Phi_\lambda \mod \mathcal{X}_0^k$ and $X_\lambda^{[k]}$, $\widehat{X}_\lambda^{[k]} \in \ker(\mathrm{Ad}(S_0) - \mathrm{I}) \subset \mathcal{P}_k$. Our aim is to show that it is possible to transform Φ_λ further so that $X_\lambda^{[k]}$, $\widehat{X}_\lambda^{[k]}$ are also R-reversible.

Note that the reversibility of A_0 (S_0) with respect to the involution R implies that RA_0 (RS_0) is also an involution, i.e., $(RA_0)^2 = \mathrm{I}$ ($(RS_0)^2 = \mathrm{I}$).

Lemma 4.10. *Let $R \in GL(n, \mathbb{R})$ be such that $R^2 = \mathrm{I}$ and let $A_0 \in GL_{-R}(n, \mathbb{R})$ have SU-decomposition $A_0 = S_0 e^{\mathcal{N}_0}$. Define the projections $\pi_{\pm R} : \mathcal{X}_0 \to \mathcal{X}_0^{\pm R}$ and $\pi_{\pm RA_0} : \mathcal{X}_0 \to \mathcal{X}_0^{\pm RA_0}$ by*

$$\pi_{\pm R} := \frac{1}{2}(\mathrm{I} + \mathrm{Ad}(R)) \quad \textit{and} \quad \pi_{\pm RA_0} := \frac{1}{2}(\mathrm{I} + \mathrm{Ad}(RA_0)). \tag{4.31}$$

Then,

$$\pi_{\pm R}\left(\mathrm{Ad}(A_0)-\mathrm{I}\right)=\left(\mathrm{Ad}(A_0)-\mathrm{I}\right)\pi_{\mp RA_0} \tag{4.32}$$

$$\pi_{\pm RA_0}\left(\mathrm{Ad}(A_0^{-1})-\mathrm{I}\right)=\left(\mathrm{Ad}(A_0^{-1})-\mathrm{I}\right)\pi_{\mp R}. \tag{4.33}$$

Also, $\pi_{\pm R}$ *maps* $\ker\left(\mathrm{Ad}(S_0)-\mathrm{I}\right)$ *into itself and*

$$\begin{aligned}\ker\left(\mathrm{Ad}_k(S_0)-\mathrm{I}\right)=&\left[\ker\left(\mathrm{Ad}_k(S_0)-\mathrm{I}\right)\cap\mathrm{Im}(\pi_{+R})\right]\\ &\oplus\left[\ker\left(\mathrm{Ad}_k(S_0)-\mathrm{I}\right)\cap\mathrm{Im}(\pi_{-R})\right].\end{aligned} \tag{4.34}$$

Proof. To prove (4.32), we calculate

$$\begin{aligned}\pi_{\pm R}\left(\mathrm{Ad}(A_0)-\mathrm{I}\right)&=\frac{1}{2}\left(\mathrm{I}\pm\mathrm{Ad}(R)\right)\left(\mathrm{Ad}(A_0)-\mathrm{I}\right)\\ &=\frac{1}{2}\mathrm{Ad}(A_0)-\frac{1}{2}\mathrm{I}\pm\frac{1}{2}\mathrm{Ad}(RA_0)\mp\frac{1}{2}\mathrm{Ad}(R)\\ &=\frac{1}{2}\mathrm{Ad}(A_0)-\frac{1}{2}\mathrm{I}\pm\frac{1}{2}\mathrm{Ad}(RA_0)\mp\frac{1}{2}\mathrm{Ad}(A_0)\mathrm{Ad}(RA_0)\\ &=\left(\mathrm{Ad}(A_0)-\mathrm{I}\right)\frac{1}{2}(\mathrm{I}\mp\mathrm{Ad}(RA_0))\\ &=\left(\mathrm{Ad}(A_0)-\mathrm{I}\right)\pi_{\mp RA_0}.\end{aligned}$$

The relation (4.33) follows by replacing R by RA_0 and A_0 by A_0^{-1} in the calculation above. Observe that $(RA_0)^2=\mathrm{I}$ and $RA_0R=A_0^{-1}$ imply that $(RA_0)A_0^{-1}(RA_0)=A_0$.

The splitting (4.34) follows from the fact that $\pi_{\pm R}$ maps $\ker\left(\mathrm{Ad}(S_0)-\mathrm{I}\right)$ into itself. Indeed, if $\Psi\in\ker\left(\mathrm{Ad}(S_0)-\mathrm{I}\right)$ then

(i) $\mathrm{Ad}(S_0)\Psi=\Psi$ if and only if $\mathrm{Ad}(S_0^{-1})\Psi=\Psi$;

(ii) $\mathrm{Ad}(S_0)\pi_{\pm R}\Psi=\pi_{\pm R}\Psi$.

□

4.3.2 Reversible Semisimple Normal Form

We prove the reversible version of Lemma 4.8. The proof consists of three main ideas. The first is to construct an isomorphism between two appropriate splittings of $\mathrm{Im}\left(\mathrm{Ad}(\mathrm{S}_0^{-1})-\mathrm{I}\right)$, the second is to take a suitable direct sum decomposition of $gl_{-RA_0}(n,\mathbb{R})$. The final one is merely an application of the Implicit Function Theorem.

More in detail, we consider the following lemma.

Lemma 4.11. *Let $A_0 = S_0 e^{\mathcal{N}_0}$ be the SU-decomposition of $A_0 \in GL_{-R}(n,\mathbb{R})$. Then,*

(i) *the following diagram holds,*

$$\begin{tikzcd} & U_1 \oplus U_2 \arrow[d, "\mathrm{Ad}(A_0^{-1})-\mathrm{I}"] \\ \mathrm{Im}(\mathrm{Ad}(\mathrm{S}_0)-\mathrm{I}) \arrow[ur, "\cong"] \arrow[r, "\cong"'] & V_2 \oplus V_1 \end{tikzcd} \tag{4.35}$$

where

$$\begin{aligned} U_1 &= \mathrm{Im}\left(\mathrm{Ad}\left(S_0^{-1}\right) - \mathrm{I}\right) \cap gl_{+R}(n,\mathbb{R}), \\ U_2 &= \mathrm{Im}\left(\mathrm{Ad}\left(S_0^{-1}\right) - \mathrm{I}\right) \cap gl_{-R}(n,\mathbb{R}) \\ V_1 &= \mathrm{Im}\left(\mathrm{Ad}\left(S_0^{-1}\right) - \mathrm{I}\right) \cap gl_{+RA_0}(n,\mathbb{R}) \\ V_2 &= \mathrm{Im}\left(\mathrm{Ad}\left(S_0^{-1}\right) - \mathrm{I}\right) \cap gl_{-RA_0}(n,\mathbb{R}), \end{aligned}$$

(ii) $gl_{-RA_0}(n,\mathbb{R}) = \left[gl_{-RA_0}(n,\mathbb{R}) \cap \mathrm{Im}(\mathrm{Ad}(S_0^{-1}) - \mathrm{I})\right] \oplus \left[gl_{-RA_0}(n,\mathbb{R}) \cap \ker\left(\mathrm{Ad}(S_0^{-1}) - \mathrm{I}\right)\right].$

Proof. Item (i) is a consequence of the invertibility of $\mathrm{Im}(\mathrm{Ad}(A_0) - \mathrm{I})$ on $\mathrm{Im}(\mathrm{Ad}(S_0) - \mathrm{I})$ and of the fact that it maps U_1 in V_2 and U_2 in V_1 injectively. However, we show for completeness that $\mathrm{Ad}(A_0^{-1}) - \mathrm{I}$ is an isomorphism from U_1 onto V_2. Now, $\mathrm{Ad}(A_0^{-1}) - \mathrm{I}$ maps $\mathrm{Im}(\mathrm{Ad}(S_0^{-1}) - \mathrm{I})$ bijectively into itself and as it follows from Lemma 4.10 it also maps $gl_{+R}(n,\mathbb{R})$ into $gl_{-RA_0}(n,\mathbb{R})$. So, it is sufficient to prove surjectivity. To this purpose, let $\Psi \in gl_{-RA_0}(n,\mathbb{R}) \cap \mathrm{Im}(\mathrm{Ad}(S_0^{-1}) - \mathrm{I})$ and let $\widetilde{\Psi} \in \mathrm{Im}(\mathrm{Ad}(S_0^{-1}) - \mathrm{I})$ be such that $(\mathrm{Ad}(A_0^{-1}) - \mathrm{I})\widetilde{\Psi} = \Psi$ ($\widetilde{\Psi}$ is uniquely determined). Then, by Lemma 4.10,

$$\Psi = \pi_{-RA_0}\Psi = \pi_{-RA_0}(\mathrm{Ad}(A_0^{-1}) - \mathrm{I})\widetilde{\Psi} = (\mathrm{Ad}(A_0^{-1}) - \mathrm{I})\pi_{+R}\widetilde{\Psi}.$$

If we can show that $\pi_{+R}\widetilde{\Psi} \in \mathrm{Im}(\mathrm{Ad}(S_0^{-1}) - \mathrm{I})$ then $\pi_{+R}\widetilde{\Psi} = \widetilde{\Psi}$ by uniqueness, and $\widetilde{\Psi} \in gl_{+R}(n,R) \cap \mathrm{Im}(\mathrm{Ad}(S_0^{-1}) - \mathrm{I})$, which proves the surjectivity. Since $\widetilde{\Psi} \in \mathrm{Im}(\mathrm{Ad}(S_0^{-1}) - \mathrm{I})$ there exists some $\widehat{\Psi} \in gl(n,\mathbb{R})$ such that $\widetilde{\Psi} = (\mathrm{Ad}(S_0^{-1}) - \mathrm{I})\widehat{\Psi}$. Hence, using Lemma 4.10 and the fact that $S_0^{-1} \in GL_{-R}(n, \mathbb{R})$ one obtains that

$$\begin{aligned} \pi_{+R}\widetilde{\Psi} &= \pi_{+R}(\mathrm{Ad}(S_0^{-1}) - \mathrm{I})\widehat{\Psi} \\ &= (\mathrm{Ad}(S_0^{-1}) - \mathrm{I})\pi_{-S_0 R}\widehat{\Psi} \in \mathrm{Im}(\mathrm{Ad}(S_0^{-1}) - \mathrm{I}). \end{aligned}$$

Item (ii) is proved as follows. Since $gl(n, \mathbb{R}) = \mathrm{Im}(\mathrm{Ad}(S_0^{-1}) - \mathrm{I}) \oplus \ker(\mathrm{Ad}(S_0^{-1}) - \mathrm{I})$, each $\Psi \in gl_{-RA_0}(n, \mathbb{R})$ has a unique decomposition $\Psi = \widetilde{\Psi} + \widehat{\Psi}$, with $\widetilde{\Psi} \in \mathrm{Im}(\mathrm{Ad}(S_0^{-1}) - \mathrm{I})$ and $\widehat{\Psi} \in \ker(\mathrm{Ad}(S_0^{-1}) - \mathrm{I})$. Recall that $\ker(\mathrm{Ad}(S_0^{-1}) - \mathrm{I}) = \ker(\mathrm{Ad}(S_0) - \mathrm{I})$. Then $\Psi = \pi_{-RA_0}\Psi = \pi_{-RA_0}\widetilde{\Psi} + \pi_{-RA_0}\widehat{\Psi}$. Now, $S_0^{-1} \in GL_{-RA_0}(n, \mathbb{R})$ and by Lemma 4.10 it follows that

$$\pi_{-RA_0}(\mathrm{Ad}(S_0^{-1}) - \mathrm{I}) = (\mathrm{Ad}(S_0^{-1}) - \mathrm{I})\pi_{RA_0 S_0^{-1}},$$

and

$$\pi_{RA_0 S_0^{-1}}(\mathrm{Ad}(S_0) - \mathrm{I}) = (\mathrm{Ad}(S_0) - \mathrm{I})\pi_{-RA_0}.$$

This implies that

$$\pi_{-RA_0}\widetilde{\Psi} \in \mathrm{Im}(\mathrm{Ad}(S_0^{-1}) - \mathrm{I}) \quad \text{and} \quad \pi_{-RA_0}\widehat{\Psi} \in \ker(\mathrm{Ad}(S_0) - \mathrm{I}).$$

The uniqueness of the splitting of Ψ then implies that $\widetilde{\Psi} = \pi_{-RA_0}\widetilde{\Psi}$ and $\widehat{\Psi} = \pi_{-RA_0}\widehat{\Psi}$. Hence, $\widetilde{\Psi} \in gl_{-RA_0} \cap \mathrm{Im}(\mathrm{Ad}(S_0^{-1}) - \mathrm{I})$ and $\widehat{\Psi} \in gl_{-RA_0} \cap \ker(\mathrm{Ad}(S_0^{-1}) - \mathrm{I})$, which proves the result. □

Lemma 4.12 (Reversible Linear Semisimple Normal Form). *Given $A_0 \in GL_{-R}(n, \mathbb{R})$, let $A_0 = S_0 e^{\mathcal{N}_0}$ be its SU-decomposition. Then there exist a neighbourhood Ω of A_0 in $GL_{-R}(n, \mathbb{R})$ and a map $\Psi : \Omega \to gl_{+R}(n, \mathbb{R})$ such that $\Psi(A_0) = 0$ and*

$$\mathrm{Ad}\left(e^{\Psi(A)}\right) \cdot A = A_0 e^{B(A)}$$

for some $B \in \ker\,(Ad(S_0) - I) \cap gl_{-RA_0}(n, \mathbb{R})$ satisfying $B(A_0) = 0$.

Proof. Define $f : gl_{+R}(n, \mathbb{R}) \times GL_{-R}(n, \mathbb{R}) \to gl_{-RA_0}(n, \mathbb{R})$ by the relation

$$\mathrm{Ad}(e^{\Psi})A = A_0 e^{f(\Psi(A))}. \tag{4.36}$$

That is, $f(\Psi, A) := \log(g(\Psi, A))$ with $g(\Psi, A) := A_0^{-1} e^{\Psi} A e^{-\Psi}$. The map f is well-defined and smooth for (Ψ, A) near $(0, A_0)$ since $g(0, A_0) = \mathrm{I}$. Also, $\mathrm{Ad}(e^{\Psi})A \in GL_{-R}(n, \mathbb{R})$, $f(0, A_0) = 0$ and

$$D_{\Psi} f(0, A_0) = (\mathrm{Ad}(A_0^{-1}) - \mathrm{I})|_{gl_{+R}(n,\mathbb{R})} \in \mathcal{L}(gl_{+R}(n, \mathbb{R}), gl_{-RA_0}(n, \mathbb{R})).$$

The remainder of the proof is then analogous to the proof of Lemma 4.8, where we use the projection π corresponding to the splitting (ii) in Lemma 4.11 and the fact that $D_{\Psi} f(0, A_0)$ is an isomorphism between $\mathrm{Im}(\mathrm{Ad}(S_0^{-1}) - \mathrm{I}) \cap gl_{-RA_0}(n, \mathbb{R})$ and $\mathrm{Im}(\mathrm{Ad}(S_0^{-1}) - \mathrm{I}) \cap gl_{+R}(n, \mathbb{R})$ (cf. Lemma 4.11-(i)). □

Proposition 4.13. *Let* $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *be a smooth family of local diffeomorphisms satisfying* (H1) *and* (R). *Let* $A_0 := D\Phi_0(0) = S_0 \exp(\mathcal{N}_0) \in GL_{-R}(n, \mathbb{R})$ *be the SU-decomposition of* A_0. *Then, for each* $k \geq 1$ *there exists a neighbourhood* ω_k *of the origin in* $\mathbb{R}^m$ *and a parameter dependent near-identity R-equivariant transformation* $\Psi_{k,\lambda} : \mathbb{R}^n \to \mathbb{R}^n$ *(*$\lambda \in \mathbb{R}^m$*) with* $\Psi_{k,\lambda}(0) = 0$ *and* $D\Psi_{k,0}(0) = \mathrm{I}$, *such that*

$$\mathrm{Ad}(\Psi_{k,\lambda})\Phi_{\lambda} = A_0 e^{X_{\lambda}^{[k]}} \mod \mathcal{X}_0^k, \quad \forall \lambda \in \omega_k, \tag{4.37}$$

with $X_{\lambda}^{[k]} \in \ker(\mathrm{Ad}(S_0) - \mathrm{I}) \cap \mathrm{Im}(\pi_{-RA_0}) \subset \mathcal{P}_k$ *and* $X_{\lambda=0}^{[k]} = 0 \mod \mathcal{X}_0^1$.

Proof. We use induction on k. For $k = 1$ the result follows from Lemma 4.12. The induction step is proved by a similar argument as in Proposition 4.7 with the following refinements:

(i) $X_{\lambda}^{[k-1]} \in \ker(\mathrm{Ad}(S_0) - \mathrm{I}) \cap \mathrm{Im}(\pi_{-RA_0}) \subset \mathcal{P}_{k-1}$;

(ii) $Z_{k,\lambda} \in \mathcal{H}_k^{-RA_0} := \mathcal{H}_k \cap \mathrm{Im}(\pi_{-RA_0})$;

(iii) we use R-equivariant transformations, i.e. $\Psi_k \in \mathcal{H}_k^{+R} := \mathcal{H}_k \cap \mathrm{Im}(\pi_{+R})$.

Then, $\mathrm{Ad}(e^{\Psi_k})\Phi_{\lambda}$ is R-reversible, and $f_k(\Psi_k, \lambda) \in \mathcal{H}_k^{-RA_0}$. Using the same argument as in Lemma 4.11 (see also Lemma 4.12) one shows that $D_k(0) = \mathrm{Ad}_k(A_0^{-1}) - \mathrm{I}$ is an isomorphism from $\mathcal{H}_k^{+R} \cap \mathrm{Im}(\mathrm{Ad}_k(S_0^{-1}) - \mathrm{I})$ onto $\mathcal{H}_k^{-RA_0} \cap \mathrm{Im}(\mathrm{Ad}_k(S_0^{-1}) - \mathrm{I})$ and that

$$\mathcal{H}_k^{-RA_0} = \left[\mathcal{H}_k^{-RA_0} \cap \mathrm{Im}(\mathrm{Ad}_k(S_0^{-1}) - \mathrm{I})\right] \oplus \left[\mathcal{H}_k^{-RA_0} \cap \ker(\mathrm{Ad}_k(S_0^{-1}) - \mathrm{I})\right].$$

The rest of the proof is then analogous to that of Proposition 4.7, using the projection π on the first component of the splitting above. □

4.3.3 Proof of the RPNF Theorem 2

We now have all the ingredients to prove the RPNF Theorem 2. We use the reversible semisimple normal form (Proposition 4.13). That is, for each $k \geq 1$ the R-reversible family Φ_λ can be put in the form $\Phi_\lambda = S_0 e^{\mathcal{N}_0 + \widehat{X}_\lambda^{[k]}} \mod \mathcal{X}_0^k$ where $A_0 = S_0 e^{\mathcal{N}_0}$ is the SU-decomposition of A_0, $\widehat{X}_\lambda^{[k]} \in \ker(\mathrm{Ad}(S_0) - \mathrm{I}) \cap \mathrm{Im}(\pi_{-RS_0}) = \ker(\mathrm{Ad}(S_0) - \mathrm{I}) \cap \mathrm{Im}(\pi_{-R})$ and $\widehat{X}_\lambda^{[k]}(0) = 0$ and $D\widehat{X}_{\lambda=0}^{[k]}(0) = 0$. Our aim is to apply further transformations which preserve the approximate S_0-equivariance and reversibility, and further simplify $\widehat{X}_\lambda^{[k]}$.

We start from the linear case.

Lemma 4.14 (Reversible Linear nilpotent Normal Form). *Let $\Phi_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ satisfy the hypotheses of Proposition 4.13. Then, there exists a parameter dependent S_0-equivariant linear transformation $\Psi_{1,\lambda} \in \mathcal{L}(GL(n,\mathbb{R}))$ with $\Psi_{1,0} = \mathrm{I}$ and $R\Psi R = \Psi$ such that*

$$\mathrm{Ad}(\Psi_{1,\lambda})\Phi_\lambda = S_0 e^{\mathcal{N}_0 + \hat{Y}_\lambda^{[1]}} \mod \mathcal{X}_0^1, \tag{4.38}$$

with $\hat{Y}_\lambda^{[1]} \in \ker(\mathrm{Ad}_1(S_0) - \mathrm{I}) \cap \ker(\mathrm{ad}_1(\mathcal{N}_0^T)) \cap gl_{-R}(n,\mathbb{R})$ for all $\lambda \in \mathbb{R}^m$ in a sufficiently small neighbourhood ω_1 of the origin in $\mathbb{R}^m$. Also, $\hat{Y}_0^{[1]} = 0$.

Proof. Proceed as in Lemma 4.9 with the following changes

(i) $\widehat{X}_\lambda^{[1]} \in \ker(\mathrm{Ad}_1(S_0) - \mathrm{I}) \cap gl_{-R}(n,\mathbb{R})$;

(ii) $\Psi \in \ker(\mathrm{Ad}_1(S_0) - \mathrm{I}) \cap gl_{+R}(n,\mathbb{R})$, i.e., we restrict to consider R-equivariant transformations.

Then $f(\Psi,\lambda) \in \ker(\mathrm{Ad}(S_0) - \mathrm{I}) \cap gl_{-R}(n,\mathbb{R})$, that is,

$$f : [\ker(\mathrm{Ad}_1(S_0) - \mathrm{I}) \cap gl_{+R}(n,\mathbb{R})] \times \mathbb{R}^m \\ \to [\ker(\mathrm{Ad}_1(S_0) - \mathrm{I}) \cap gl_{-R}(n,\mathbb{R})].$$

Note that the derivative $D_\Psi(f(0,0)) = -\mathrm{ad}(\mathcal{N}_0)|_{\ker(\mathrm{Ad}_1(S_0)-\mathrm{I}) \cap gl_{+R}(n,\mathbb{R})}$ belongs to
$\mathcal{L}\big(\ker(\mathrm{Ad}_1(S_0) - \mathrm{I}) \cap gl_{+R}(n,\mathbb{R}), \ker(\mathrm{Ad}_1(S_0) - \mathrm{I}) \cap gl_{-R}(n,\mathbb{R})\big)$ and that

by the choice of scalar product (see section 2.3) both $\mathrm{ad}_1(\mathcal{N}_0)$ and $\mathrm{ad}_1(\mathcal{N}_0^T)$ map $gl_{\pm R}(n,\mathbb{R})$ into $gl_{\mp R}(n,\mathbb{R})$. Consider the splitting

$$\begin{aligned}
&\ker\left(\mathrm{Ad}_1(S_0)-\mathrm{I}\right)\cap gl_{+R}(n,\mathbb{R})\\
&\quad=\left[\ker\left(\mathrm{Ad}_1(S_0)-\mathrm{I}\right)\cap\ker\left(\mathrm{ad}_1(\mathcal{N}_0)\right)\cap gl_{+R}(n,\mathbb{R})\right]\\
&\qquad\qquad\oplus\left[\ker\left(\mathrm{Ad}_1(S_0)-\mathrm{I}\right)\cap\mathrm{Im}(\mathrm{ad}_1(\mathcal{N}_0^T))\cap gl_{+R}(n,\mathbb{R})\right]
\end{aligned}\tag{4.39}$$

for the domain of $D_\Psi f(0,0)$, and the splitting

$$\begin{aligned}
&\ker\left(\mathrm{Ad}_1(S_0)-\mathrm{I}\right)\cap gl_{+R}(n,\mathbb{R})\\
&\quad=\left[\ker\left(\mathrm{Ad}_1(S_0)-\mathrm{I}\right)\cap\mathrm{Im}(\mathrm{ad}_1(\mathcal{N}_0))\cap gl_{-R}(n,\mathbb{R})\right]\\
&\qquad\qquad\oplus\left[\ker\left(\mathrm{Ad}_1(S_0)-\mathrm{I}\right)\cap\ker\left(\mathrm{ad}_1(\mathcal{N}_0^T)\right)\cap gl_{-R}(n,\mathbb{R})\right]
\end{aligned}\tag{4.40}$$

for the range of $D_\Psi f(0,0)$. Observe that $D_\Psi f(0,0)$ is an isomorphism from the second component of (4.39) onto the first component of (4.40). The remainder of the proof is analogous of that of Lemma 4.9 when one define $g(\cdot,\lambda)$ as the projection of the restriction of $f(\cdot,\lambda)$ to the second component of (4.39) in the first component of (4.40). □

Proof of the RPNF *Theorem 2.* We proceed by induction on k. The case $k=1$ is given by Lemma 4.14. The induction argument is analogous to that of Proposition 4.1 with the following changes:

(i) $\widehat{Y}_\lambda^{[k-1]}\in\ker\left(\mathrm{Ad}(S_0)-\mathrm{I}\right)\cap\ker\left(\mathrm{ad}(\mathcal{N}_0^T)\right)\cap\mathrm{Im}(\pi_{-R})$;

(ii) $Z_{k,\lambda}\in\ker\left(\mathrm{Ad}_k(S_0)-\mathrm{I}\right)\cap\mathrm{Im}(\pi_{-R})$;

(iii) we restrict to R-equivariant transformations, i.e., $\Psi_k\in\ker\left(\mathrm{Ad}_k(S_0)-\mathrm{I}\right)\cap\mathrm{Im}(\pi_{+R})$.

It follows that

$$\begin{aligned}
f_k:\left[\ker\left(\mathrm{Ad}_k(S_0)-\mathrm{I}\right)\cap\mathrm{Im}(\pi_{+R})\right]\times\mathbb{R}^m&\\
\to\left[\ker\left(\mathrm{Ad}_k(S_0)-\mathrm{I}\right)\cap\mathrm{Im}(\pi_{-R})\right]&,
\end{aligned}$$

with

$$f_k(\Psi_k,\lambda)=Z_{k,\lambda}-D_k(\lambda)\Psi_k,$$

where now

$$D_k(\lambda)\in\mathcal{L}\left(\ker\left(\mathrm{Ad}_k(S_0)-\mathrm{I}\right)\cap\mathrm{Im}(\pi_{+R}),\ker\left(\mathrm{Ad}_k(S_0)-\mathrm{I}\right)\cap\mathrm{Im}(\pi_{-R})\right).$$

Since $\pi_{\pm R}$ maps $\ker(\mathrm{Ad}_k(S_0) - \mathrm{I})$ into itself and both $\mathrm{ad}(\mathcal{N}_0)$, $\mathrm{ad}(\mathcal{N}_0^T)$ map $\mathrm{Im}(\pi_{\pm R})$ into $\mathrm{Im}(\pi_{\mp R})$ (because of the choice of the scalar product), one has that the domain of $D_k(\lambda)$ admits the splitting

$$\begin{aligned}
&\ker(\mathrm{Ad}_k(S_0) - \mathrm{I}) \cap \mathrm{Im}(\pi_{+R}) \\
&\quad = [\ker(\mathrm{Ad}_k(S_0) - \mathrm{I}) \cap \ker(\mathrm{ad}_k(\mathcal{N}_0)) \cap \mathrm{Im}(\pi_{+R})] \\
&\qquad\qquad \oplus \left[\ker(\mathrm{Ad}_k(S_0) - \mathrm{I}) \cap \mathrm{Im}(\mathrm{ad}_k(\mathcal{N}_0^T)) \cap \mathrm{Im}(\pi_{+R})\right], \qquad (4.41)
\end{aligned}$$

while the range of $D_k(\lambda)$ can be written as

$$\begin{aligned}
&\ker(\mathrm{Ad}_k(S_0) - \mathrm{I}) \cap \mathrm{Im}(\pi_{-R}) \\
&\quad = [\ker(\mathrm{Ad}_k(S_0) - \mathrm{I}) \cap \mathrm{Im}(\mathrm{ad}_k(\mathcal{N}_0)) \cap \mathrm{Im}(\pi_{-R})] \\
&\qquad\qquad \oplus \left[\ker(\mathrm{Ad}_k(S_0) - \mathrm{I}) \cap \ker(\mathrm{ad}_k(\mathcal{N}_0^T)) \cap \mathrm{Im}(\pi_{-R})\right]. \qquad (4.42)
\end{aligned}$$

Now, $D_k(0) = \mathrm{ad}_k(\mathcal{N}_0)|_{\ker(\mathrm{Ad}_k(S_0)-\mathrm{I})\cap\mathrm{Im}(\pi_{+R})}$ is an isomorphism from the second component of (4.41) onto the first component of (4.42). The remainder of the proof proceeding is analogous to that of Proposition 4.1. □

5

Stability of Periodic Points

The goal of this chapter is to obtain information on the stability properties of possible solutions of problem (P) proving Proposition 1.3.

Referring to section 1.4 recall that for all small $(u,\lambda) \in U \times \mathbb{R}^m$ satisfying the determining equation $\Phi_{r,\lambda}(u) = S_0 u$, the (linear) stability of a bifurcating periodic solution $x = x^*(u,\lambda)$ as given in Theorem 1 is determined by the eigenvalues of

$$\mathcal{D}(u,\lambda) := D\Phi_\lambda^q\left(x^*\left(u,\lambda\right)\right). \tag{5.1}$$

Moreover, if (u,λ) solves the determining equation (1.10) or equivalently (1.12) and u is *symmetric*, i.e., the $\mathbb{Z}_q$-orbit through u is R-invariant: $Ru = S_0^j u$ for some $j \in \mathbb{Z}$, then the corresponding q-periodic orbit of Φ_λ is also symmetric; i.e. $Ru = S_0^j u$ and $Rx^*(u,\lambda) = \Phi_\lambda^j(x^*(u,\lambda))$ for some $j \in \mathbb{Z}$. A straightforward calculation shows the following result for such symmetric orbits.

Lemma 5.1. *Let $\Phi_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ satisfy (H1), and (R). Suppose that $x \in \mathbb{R}^n$ generates symmetric q-periodic orbit of Φ_λ. Then there exists an involution $\Psi \in \mathcal{L}(\mathbb{R}^n)$ such that*

$$D_x\Phi_\lambda^q(x)\Psi = \Psi\left(D_x\Phi_\lambda^q(x)\right)^{-1}.$$

It follows that if $(u,\lambda) \in U \times \mathbb{R}^m$ is symmetric, then there exists an involution $\tilde{\Psi}(u,\lambda)$ such that $\tilde{\Psi}\mathcal{D}(u,\lambda)\tilde{\Psi} = \mathcal{D}(u,\lambda)^{-1}$. Therefore, if $\mu \in \mathbb{C}$ is an eigenvalue of $\mathcal{D}(u,\lambda)$ and u is symmetric then also μ^{-1} is an eigenvalue. Consequently, we can only have a weak form of stability for symmetric periodic orbits. Namely, or all the eigenvalues of $\mathcal{D}(u,\lambda)$ are on the unit circle and the orbit is stable, or there are eigenvalues both inside and outside the unit circle, in which case the orbit is unstable.

Proof of Lemma 5.1. We distinguish two cases: (i) $x = Rx$, or, (ii) $Rx = \Phi_\lambda^j(x)$ for some $j \in \mathbb{Z}$.

Case (i). Suppose that $x = Rx$. Then from $\Phi_\lambda^q(Rx) = R\left(\Phi_\lambda^{-q}\right)(x)$ and the q-periodicity of x, it follows by the chain rule that

$$D\Phi_\lambda^q(x)R = D\Phi_\lambda^q(Rx)R = R\left(D\Phi_\lambda^q\left(\Phi_\lambda^{-q}(x)\right)\right)^{-1} = R\left(D\Phi_\lambda^q(x)\right)^{-1}, \tag{5.2}$$

Hence, the result taking $\Psi = R$. Note that if $x \neq Rx$ we can only write $D\Phi_\lambda^q(Rx) = R\left(D\Phi_\lambda^q(x)\right)^{-1} R$.

Case (ii). Suppose that the whole orbit $\{x, \Phi_\lambda(x), \ldots, \Phi_\lambda^{q-1}(x)\}$ generated by $x \in \mathbb{R}$ is R-invariant; i.e. $Rx = \Phi_\lambda^j(x), \quad$ for some j. This is equivalent to say that $x = R\Phi_\lambda^j(x)$ for some j. Observe that $R\Phi_\lambda^j$ is not linear and that $(R\Phi_\lambda^j)^2 = \mathrm{I}$ (i.e. $(R\Phi_\lambda^j)$ is an involution). So, the lemma follows if we can prove that Φ_λ is $R\Phi_\lambda^j$-reversible, i.e.,

$$\Phi_\lambda\left(R\Phi_\lambda^j\right) = \left(R\Phi_\lambda^j\right)\Phi_\lambda^{-1}. \tag{5.3}$$

This is readily proved as follows. For simplicity of notation set $\Psi = R\Phi_\lambda^j$, then

$$\begin{aligned}\Phi_\lambda\left(\Psi(x)\right) = \Phi_\lambda\left(R\Phi_\lambda^j(x)\right) &= R\Phi_\lambda^{-1}(\Phi_\lambda^j(x)) = R\Phi_\lambda^{j-1}(x)\\ &= R\Phi_\lambda^j\left(\Phi_\lambda^{-1}(x)\right) = \Psi\left(\Phi_\lambda^{-1}(x)\right),\end{aligned}$$

therefore

$$\Phi_\lambda^q\left(\Psi(x)\right) = \Psi\left(\Phi_\lambda^{-q}(x)\right). \tag{5.4}$$

Differentiating both sides yields

$$D\Phi_\lambda^q\left(\Psi(x)\right)D\Psi(x) = D\Psi\left(\Phi_\lambda^{-q}(x)\right)D\Phi_\lambda^{-q}(x).$$

Since $x = \Phi_\lambda^q(x) = \Phi_\lambda^{-q}(x)$ (x is q-periodic) and $x = \Psi(x)$ (the orbit of x is R-invariant), then

$$\begin{aligned}D\Phi_\lambda^q(x)D\Psi(x) = D\Phi_\lambda^q\left(\Psi(x)\right)D\Psi(x) &= D\Psi\left(\Phi_\lambda^{-q}(x)\right)\Phi_\lambda^{-q}(x)\\ &= D\Psi(x)\left(D\Phi_\lambda^q(x)\right)^{-1}.\end{aligned} \tag{5.5}$$

Hence, the lemma follows. Note that $D\Psi(x)$ is a linear operator of which the square is also a linear operator. In particular, $D\Psi(x)D\Psi(x) = \mathrm{I}$. $\square$

If $A_0 := D\Phi_0(0)$, then $\mathcal{D}(0,0) = A_0^q$, which implies that 1 is an eigenvalue of $\mathcal{D}(0,0)$ with algebraic multiplicity equal to dim(U), and with geometric multiplicity equal to the sum of the geometric multiplicities of the resonant eigenvalues of A_0. Recall that the resonant eigenvalues are those μ that are qth roots of unity. Now, the eigenvalues of $\mathcal{D}(u,\lambda)$ are close to those of $\mathcal{D}(0,0)$ for small $(u,\lambda) \in U \times \mathbb{R}^n$. We assume that

$$\boxed{\begin{array}{c}\text{all non-resonant eigenvalues } \mu \text{ of } A_0 \text{ are simple and}\\ \text{on the unit circle.}\end{array}} \tag{S}$$

So, if (S) holds, then the eigenvalues of $\mathcal{D}(u,\lambda)$ which are not close to $\mu = 1$ will be simple. Moreover, if u is symmetric then the eigenvalues of $\mathcal{D}(u,\lambda)$ not close to 1 will be simple and on the unit circle. Therefore the stability of the bifurcating periodic orbits is determined by the eigenvalues of $\mathcal{D}(u,\lambda)$ close to 1. We call these eigenvalues *critical.* To calculate them one can put $\mathcal{D}(u,\lambda)$ in block form using the splitting $\mathbb{R}^n = U \oplus V = \ker(S_0^q - \mathrm{I}) \oplus \mathrm{Im}(S_0^q - \mathrm{I})$, and prove the existence of a similarity transformation which makes this block form triangular; the critical eigenvalues are then eigenvalues of the block corresponding to the subspace U. This diagonalisation procedure is relatively easy to work out when Φ_λ is in normal form up to a sufficiently high order, namely, when $\Phi_\lambda = S_0 \Psi_\lambda^{NF} + O(||\cdot||^{k+1})$ with $\Psi_\lambda^{NF} = S_0^{-1}\Phi_\lambda^{NF}$ such that $S_0 \circ \Psi_\lambda^{NF} = \Psi_\lambda^{NF} \circ S_0$ and $R \circ \Psi_\lambda^{NF} \circ R = \left(\Psi_\lambda^{NF}\right)^{-1}$, (see PRNF Theorem 2). In this case the outlined procedure gives the following result.

Proposition 5.2. *Assume that Φ_λ satisfies* (H1) *and is in normal form up to order k as given in the* PRNF *Theorem 2. Let $\Psi_\lambda^{NF} := S_0^{-1}\Phi_\lambda^{NF}$ and define U as in* (1.9)*. Consider the direct sum splitting $\mathbb{R}^n = U \oplus \mathrm{Im}(S_0^q - \mathrm{I})$ and let π_U and $\pi_{\mathrm{Im}(S_0^q-\mathrm{I})}$ be the projections of $\mathbb{R}^n$ onto respectively U and $\mathrm{Im}(S_0^q - \mathrm{I})$. Then, there exists a linear mapping $T_\lambda \in \mathcal{L}(\mathbb{R}^n)$, $\lambda \in \mathbb{R}^m$, such that for $(u,\lambda) \in U \times \mathbb{R}^m$*

$$T_\lambda(u)^{-1}\mathcal{D}(u,\lambda)\,T_\lambda(u) = \begin{pmatrix} A(u,\lambda) & B(u,\lambda) \\ 0 & D(u,\lambda) \end{pmatrix}, \tag{5.6}$$

where

$$A(u,\lambda) = \left(\pi_U D\left(\Psi_\lambda^{NF}(u)\right)|_U\right)^q + O\left(\|u\|^k\right) \tag{5.7}$$

and

$$D(u,\lambda) = S_0^q\left(\pi_{\mathrm{Im}\left(S_0^q-\mathrm{I}\right)} D\left(\Psi_\lambda^{NF}(u)\right)|_{\mathrm{Im}\left(S_0^q-\mathrm{I}\right)}\right)^q + O\left(\|u\|^k\right). \tag{5.8}$$

Proof. By the RPNF Theorem 2, it follows that

$$D\Psi_\lambda^{NF}\left(S_0^j u\right) S_0^j = S_0^j D\Psi_\lambda^{NF}(u), \qquad 1 \le j \le q,$$

and

$$x_\lambda^*(u) = u + O\left(\|u\|^{k+1}\right).$$

Recall that $x^*(0,\lambda) = 0$, for all $\lambda \in \mathbb{R}^m$, and $D\Phi_0^q(x^*(0,0)) = A_0^q = S_0^q e^{q\mathcal{N}_0}$. Hence,

$$\mathcal{D}(u,\lambda) = S_0^q \left(D\Psi_\lambda^{NF}(u)\right)^q + O\left(\|u\|^k\right), \qquad (u,\lambda) \in U \times \mathbb{R}^m. \quad (5.9)$$

The first term of the right handside of (5.9) is triangular with respect to the splitting $\mathbb{R}^n = U \oplus \mathrm{Im}\,(S_0^q - \mathrm{I})$ as we explain below. Given $x \in \mathbb{R}^n$ we can always write $x = u + v$, with $u \in U$ and $v \in \mathrm{Im}(S_0^q - \mathrm{I})$. The invariance of U and $\mathrm{Im}\,(S_0^q - \mathrm{I})$ under Ψ_λ^{NF} implies then that

$$\Psi_\lambda^{NF}(x) = \Psi_\lambda^{NF}(u+v) = \widetilde{\Psi}_\lambda^{NF}(u,v) + \widehat{\Psi}_\lambda^{NF}(u,v),$$

where

$$\begin{array}{rcl} \widetilde{\Psi}_\lambda^{NF}(u,v) & := & \pi_U \Psi_\lambda^{NF}(x) \in U \\ \widehat{\Psi}_\lambda^{NF}(u,v) & := & \pi_{\mathrm{Im}(S_0^q - \mathrm{I})} \Psi_\lambda^{NF}(x) \in \mathrm{Im}(S_0^q - \mathrm{I}), \end{array}$$

with $\widehat{\Psi}_\lambda^{NF}(u,0) = 0$ for all $u \in U$. Hence,

$$S_0^q D\Psi_\lambda^{NF}(u) = \begin{pmatrix} \left(D_u \widetilde{\Psi}_\lambda^{NF}(u,0)\right)^q & \spadesuit \\ 0 & S_0^q|_{\mathrm{Im}(S_0^q - \mathrm{I})} \left(D_v \widehat{\Psi}_\lambda^{NF}(u,0)\right)^q \end{pmatrix}, \quad (5.10)$$

where we used $S_0^q|_U = \mathrm{I}_U$. It follows that

$$\mathcal{D}(u,\lambda) = \begin{pmatrix} \widehat{A}_\lambda(u) & \widehat{B}_\lambda(u) \\ \widehat{C}_\lambda(u) & \widehat{D}_\lambda(u) \end{pmatrix}, \quad (5.11)$$

with

$$\widehat{A}_\lambda(u) := \left(D_u \widetilde{\Psi}\lambda^{NF}(u,0)\right)^q + \left(O\,\|u\|^k\right), \quad (5.12)$$

$$\widehat{B}_\lambda(u) := \spadesuit + O\left(\|u\|^k\right), \tag{5.13}$$

$$\widehat{C}_\lambda(u) := O\left(\|u\|^k\right), \tag{5.14}$$

and

$$\widehat{D}_\lambda(u) := S_0^q|_{\mathrm{Im}(S_0^q-\mathrm{I})}\left(D_v\widehat{\Psi}_\lambda^{NF}(u,0)\right)^q + O\left(|u|^k\right). \tag{5.15}$$

Notice that $\mathcal{D}(0,0) = A_0^q$ implies that

(i) $\widehat{A}_0(0) = e^{q\mathcal{N}_0|_U}$ is unipotent,

(ii) $\widehat{B}_0(0) = \widehat{C}_0(0) = 0$,

(iii) $\widehat{D}_0(0) = \left(S_0^q e^{q\mathcal{N}_0}\right)\big|_{\mathrm{Im}\left(S_0^q-I\right)} = A_0^q|_{\mathrm{Im}\left(S_0^q-I\right)}$,

(iv) $\widehat{D}_0(0) - \mathrm{I}|_{\mathrm{Im}\left(S_0^q-\mathrm{I}\right)}$ is invertible.

The idea now is to apply an appropriate similarity transformation to (5.11) such that the resulting operator is triangular. Such transformation is found by the Implicit Function Theorem as follows. Let $T(u,\lambda) \in \mathcal{L}(\mathbb{R}^n)$ be of the form

$$T_\lambda(u) = \begin{pmatrix} I_U & 0 \\ E_\lambda(u) & I_{\mathrm{Im}\left(S_0^q-\mathrm{I}\right)} \end{pmatrix},$$

with $E_\lambda(u)$ some linear operator whose existence is yet to prove. Then,

$$T_\lambda^{-1}(u)\mathcal{D}_\lambda(u)T_\lambda(u) = \begin{pmatrix} \widehat{A} + \widehat{B}E & \widehat{B} \\ -E\widehat{A} + \widehat{C} - E\widehat{B}E + \widehat{D}E & -E\widehat{B} + \widehat{D} \end{pmatrix}.$$

Define then the operator $F : U \times \mathbb{R}^m \times \mathcal{L}(U, \mathrm{Im}(S_0^q - \mathrm{I})) \to \mathcal{L}(U, \mathrm{Im}(S_0^q - \mathrm{I}))$ by

$$F(u,\lambda;E) := -E\widehat{A}(u,\lambda) + \widehat{C}(u,\lambda) - E\widehat{B}(u,\lambda)E + \widehat{D}(u,\lambda)E. \tag{5.16}$$

Since $\widehat{C}_0(0) = 0$, we have that $F(0,0,0) = 0$ and also

$$\begin{aligned} D_E F(0,0,0)\cdot\widetilde{E} &= \widehat{D}(0,0)\cdot\widetilde{E} - \widetilde{E}\widehat{A}(0,0) \\ &= \left(\widehat{D}(0,0) - I_{\mathrm{Im}\left(S_0^q-\mathrm{I}\right)}\right)\cdot\widetilde{E} - \widetilde{E}\left(\widehat{A}(0,0) - \mathrm{I}_U\right). \end{aligned}$$

The operator $\left(\widehat{D}\,(0,0) - I_{\mathrm{Im}\left(S_0^q - I\right)}\right)$ is invertible on $\mathrm{Im}\,(S_0^q - \mathrm{I})$, while the operator $\left(\widehat{A}\,(0,0) - \mathrm{I}_U\right)$ is nilpotent on U. Hence, $D_E F\,(0,0,0) \in \mathcal{L}\,(U,$ $\mathrm{Im}\,(S_0^q - \mathrm{I}))$ is invertible. Then, there exists, by the Implicit Function Theorem, $\widetilde{E}\,(u,\lambda) \in \mathcal{L}\,(U, \mathrm{Im}\,(S_0^q - \mathrm{I}))$ such that for all $(u,\lambda) \in U \times \mathbb{R}^m$ close to $(0,0)$,

$$F\left(u,\lambda;\widetilde{E}\,(u,\lambda)\right) = 0 \quad \text{and} \quad \widetilde{E}(u,\lambda) = O(||u||^k).$$

So, setting

$$T\,(u,\lambda) = \begin{pmatrix} I_U & 0 \\ \widetilde{E}\,(u,\lambda) & I_{\mathrm{Im}\left(S_0^q - I\right)} \end{pmatrix}$$

yields

$$T_\lambda^{-1}\,(u)\,\mathcal{D}_\lambda(u) T_\lambda(u) = \begin{pmatrix} \widehat{A}\,(u,\lambda) + O(||u||^k) & \widehat{B}\,(u,\lambda) \\ 0 & \widehat{D}\,(u,\lambda) + O(||u||^k), \end{pmatrix} \tag{5.17}$$

which completes the proof. □

Corollary 5.3. *All the information about the critical eigenvalues of the operator* $\mathcal{D}\,(u,\lambda)$ *is contained in the block* $\mathcal{A}\,(u,\lambda)$ *of* (5.6)*, for each* (u,λ) $\in U \times \mathbb{R}^m$ *close to* $(0,0)$.

Proof. Since similar matrices have the same eigenvalues, the result follows from the following observations.

(i) For $(u,\lambda) = (0,0) \in U \times \mathbb{R}^m$, (5.6) implies that

$$T_0(0)^{-1}\mathcal{D}\,(0,0)\,T_0(0) = \begin{pmatrix} e^{q\mathcal{N}_0|_U} & 0 \\ 0 & S_0^q e^{q\mathcal{N}_o}\big|_{\mathrm{Im}\left(S_0^q - I\right)} \end{pmatrix},$$

(ii) all the eigenvalues of $e^{q\mathcal{N}_0|_U}$ are equal to 1,

(iii) the spectrum of the operator $S_0^q e^{q\mathcal{N}_o}\big|_{\mathrm{Im}\left(S_0^q - I\right)}$ is away from 1.

□

Proposition 1.3 then is a direct consequence of Proposition 5.2 and Corollary 5.3.

The fact that $\Psi_\lambda^{NF} = \exp(\mathcal{N}_0 + Z_\lambda)$ (see RPNF Theorem 2) can also be used. Set $X_\lambda := \mathcal{N}_0 + Z_\lambda$.

Proposition 5.4. *Suppose that $\Phi_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the hypotheses of Theorem 1 and let $\gamma = \left\{ \left(S_0^j \widetilde{u}(\rho), \widetilde{\lambda}(\rho) \right) \middle| \quad 0 < \rho < \rho_0,\ 0 \le j \le q-1 \right\}$ be a bifurcating branch of periodic orbits of Φ_r; i.e., $\tilde{u}(\rho)$ and $\tilde{\lambda}(\rho)$ are smooth maps satisfying $\tilde{u}(0) = 0$, $\tilde{\lambda}(0) = 0$ and $\Phi_r(\tilde{u}(\rho), \tilde{\lambda}(\rho)) = S_0 \tilde{u}(\rho)$. Assume also that Φ_λ is in normal form up to some order $k \ge 1$. Then,*

$$X_{\widetilde{\lambda}(\rho)}\left(\widetilde{u}(\rho)\right) = O\left(\rho^{k+1}\right) \tag{5.18}$$

and

$$\widetilde{\mathcal{D}}\left(\widetilde{u}(\rho), \widetilde{\lambda}(\rho)\right) = \exp\left(DX_{\widetilde{\lambda}(\rho)}\left(\widetilde{u}(\rho)\right)\right) + O\left(\rho^k\right). \tag{5.19}$$

The proof of (5.18) proceeds by induction on k and it is based on the following lemma.

Lemma 5.5. *Let*

$$\dot{x} = N_0 x + b, \quad x, b \in \mathbb{R}^n \tag{5.20}$$

be an autonomous system in $\mathbb{R}^n$, with $N_0 \in \mathcal{L}(\mathbb{R}^n)$ nilpotent. Then (5.20) *has no other periodic orbits than equilibria.*

Proof. We prove that one-periodic orbits are equilibria, the same proof applies for all other periodic orbits.
Consider the homogeneous counterpart $\dot{x} = N_0 x$ of (5.20) and denote by V the (Banach) space of all one-periodic C^1-functions, i.e.,

$$V := \left\{ x : \mathbb{R}^n \to \mathbb{R}^n \middle| \ x \text{ is } C^1 \text{ and one-periodic} \right\}.$$

Let $L : V \to V$ given by $(Lx)(\cdot) := \frac{d}{dt}x(\cdot) - N_0 x(\cdot)$ be the Fredholm operator associated with system (5.20) [81]. Consider the Cauchy initial value problem

$$\begin{cases} \dot{x} & = N_0 x \\ x(0) & = x_0. \end{cases} \tag{5.21}$$

The solution of (5.21) exists and is unique, therefore by the nilpotency of N_0 one proves that

$$\ker(L) = \{\text{constant functions on } V\}. \tag{5.22}$$

Namely, $\ker(L)$ is the space of all one-periodic solution of the homogeneous counterpart of (5.20) and the general solution of the Cauchy problem (5.21) can be written as

$$x(t) = e^{N_0 t} x_0.$$

Now, if $x(t)$ is one-periodic, then $x(1) = x(0)$. Therefore, $\left(e^{N_0} - I\right) x_0 = 0$ for such solutions. But

$$\left(e^{N_0} - I\right) = \left(I + \sum_{i=1}^{k-2} \frac{1}{(i+1)!} N_0^i\right) N_0,$$

for some $k \in \mathbb{N}$, since N_0 is nilpotent. Being $\left(I + \sum_{i=1}^{k-2} \frac{1}{(i+1)!} N_0^i\right)$ invertible, it follows that $N_0 x_0 = 0$, that in turn implies $x(t) = x_0$, for all $t \in \mathbb{R}$. Hence, (5.22) follows.
The map L is S^1-equivariant. In fact, for each $\varphi \in S^1$ V is invariant under the phase shift $S_\varphi \in \mathcal{L}(V)$ given by $(S_\varphi x)(t) := x(t+\varphi)$, for all $x \in V$, for all $t \in \mathbb{R}$, and

$$L S_\varphi x = S_\varphi L x. \tag{5.23}$$

Note that $\ker(L)$ is also S^1-invariant and that if x is a solution of (5.20) then so is $S_\varphi x$. Hence, there exists a S^1-invariant topological complement Y of $\ker(L)$ in V such that

$$V = \ker(L) \oplus Y. \tag{5.24}$$

Returning to our problem, we see that if $b \notin \operatorname{Im}(L)$, then there are no one-periodic solutions of (5.20). If $b \in \operatorname{Im}(L)$, by Fredholm operator theory based on the splitting (5.24), we obtain the existence of a unique one-periodic solution $x^* \in Y$ of (5.20). Since $S_\varphi x^*$ is also a solution of (5.20) and $S_\varphi x^* \in Y$, then $x^* = S_\varphi x^*$, which on the other hand implies that x^* is an equilibrium of $\dot{x} = N_0 x$. In other words, *all solutions belonging to* $x^* + N(L)$ *are equilibria.* □

Proof of Proposition 5.4. We start with some general considerations and then prove (5.18) by induction on $1 \leq j \leq k$.

Let

$$\widetilde{u}(\rho) = \sum_{i=1}^{k} \rho^i a_i + O\left(\rho^{k+1}\right), \qquad a_i \in \mathbb{R}^n \tag{5.25}$$

be the Taylor expansion of $\widetilde{u}(\rho)$, and consider the system

$$\begin{cases} \dot{u} = & X_{\widetilde{\lambda}(\rho)}(u) \\ u(0) = & \widetilde{u}(\rho) \end{cases}, \tag{5.26}$$

where we recall that $X_0(0) = 0$, and $DX_0(0) = \mathcal{N}_0$, (cf. RPNF Theorem 2). Let $\varphi(t, \rho)$ be a solution of (5.26), then

$$\varphi(1, \rho) = \widetilde{\Psi}^{NF}_{\widetilde{\lambda}(\rho)}(\widetilde{u}(\rho)) = \widetilde{u}(\rho) + O\left(\rho^{k+1}\right). \tag{5.27}$$

If

$$\varphi(t, \rho) = \sum_{i=1}^{k} \varphi_i(t)\rho^i + O\left(\rho^{k+1}\right) \tag{5.28}$$

is the Taylor expansion of $\varphi(t, \rho)$, then

$$X_{\widetilde{\lambda}(\rho)}(\varphi(t, \rho)) = \sum_{i=1}^{k} \widetilde{\varphi}_i(t)\rho^i + O\left(\rho^{k+1}\right), \tag{5.29}$$

where

$$\widetilde{\varphi}_i(t) = \mathcal{N}_0\varphi_i(t) + b_i(t),$$

where $b_i(t)$ is a polynomial expression in $\varphi_1(t), \ldots, \varphi_{i-1}(t)$ depending also on the Taylor coefficients of $\widetilde{\lambda}(\rho)$. Observe that if $\varphi_1(t), \ldots, \varphi_{i-1}(t)$ are proved to be constant, then so is b_i.

We now describe the induction procedure which proves the proposition. Let $j = 1$, then from (5.25), (5.26) and (5.28) it follows that

$$\begin{cases} \dot{\varphi}_1(t) & = \mathcal{N}_0\varphi_1(t) \\ \varphi_1(0) & = a_1. \end{cases} \tag{5.30}$$

Relation (5.28) also implies that $\varphi_1(1) = a_1$, hence by Lemma 5.5 $\varphi_1(t)$ is a one-periodic solution of (5.30), i.e.,

$$\varphi_1(t) = a_1, \text{ for all } t \in \mathbb{R}.$$

Note also that $\mathcal{N}_0 a_1 = 0$. It follows that

$$\varphi(\rho, t) = a_1 \rho + O\left(\rho^2\right) \quad \text{and} \quad X_{\widetilde{\lambda}(\rho)}\left(\widetilde{u}(\rho) + O\left(\rho^2\right)\right) = 0 + O(\rho^2)$$

which proves the basis of the induction.

Let now $1 < j < k$ and assume the result true for $j+1$, i.e.,

$$\begin{aligned} &\varphi(t, \rho) = \sum_{i=1}^{j} a_i \rho^i + O\left(\rho^{j+1}\right), \\ &\mathcal{N}_0 a_i + b_i = 0, \quad i = 1, ..., j, \\ &X_{\widetilde{\lambda}(\rho)}\left(\widetilde{u}(\rho)\right) = O\left(\rho^{j+1}\right). \end{aligned} \tag{5.31}$$

Writing $\varphi(t, \rho) = \sum_{i=1}^{j} a_i \rho^i + \varphi_{j+1}(t) \rho^{j+1} + O\left(\rho^{j+2}\right)$, it follows from (5.29) that

$$\begin{cases} \dot{\varphi}_{j+1}(t) & = \mathcal{N}_0 \varphi_{j+1}(t) + b_{j+1} \\ \varphi_{j+1}(0) & = a_{j+1} \end{cases}. \tag{5.32}$$

Relation (5.27) implies that $\varphi_{j+1}(1) = a_{j+1}$, hence $\varphi_{j+1}(t)$ is a one-periodic solution of (5.32). This in turn implies that $\varphi_{j+1}(t) = a_{j+1}$, for all t. Therefore,

$$\varphi(t, \rho) = \sum_{i=1}^{j+1} a_i \rho^i + + O\left(\rho^{j+2}\right),$$

$$X_{\widetilde{\lambda}(\rho)}\left(\widetilde{u}(\rho)\right) = O\left(\rho^{j+2}\right), \tag{5.33}$$

which completes the proof. □

Remarks

1- If dim $U = 2$, and if μ and $-\mu$ are both eigenvalues of $DX_\lambda(u)$, it follows that one has stability when $\det\left(DX_\lambda(u)\right) > 2$ and instability when $\det\left(DX_\lambda(u)\right) < 2$. In fact, $\exp(\mu)$ and $\exp(-\mu)$ are the eigenvalues of $D\Psi_\lambda^{NF}(u)$ and

$$\begin{aligned} \operatorname{tr}\left(D\Psi_\lambda^{NF}|_U(u)\right) &= \exp(\mu) + \exp(-\mu) = 2 + \mu^2 + O(\mu^3) \\ &= 2 - \det DX_\lambda(u) + O(\mu^3). \end{aligned} \tag{5.34}$$

2- It should be stressed that when we apply these results, even in simple cases, explicit calculations needed for the stability analysis might be involved. Nevertheless, the presence of symmetries may simplifies things a lot.

6

Proofs of the Theorems 3–7

Goal of this chapter is to prove the bifurcations Theorems 3–7.

To simplify notations, set $\chi_q := \exp(i\theta_0) \in \mathbb{C}$, with $\theta_0 := 2\pi ip/q$, $0 < p < q$ and $\gcd(\mathrm{p}, \mathrm{q}) = 1$.

6.1 Proof of the SRU Theorem 3

In this section we prove the SRU Theorem 3, i.e., we solve problem (P) under the hypotheses (H1), (R), (H2), (H2a) and (T1). That is, we want to determine the (small) q-periodic points near the origin of a one-parameter family of maps Φ_λ (i.e., $\lambda \in \mathbb{R}$) satisfying (H1), (R) and such that for fixed $q \geq 3$ and $0 < p < q$ with $\gcd(\mathrm{p}, \mathrm{q}) = 1$, $A_0 := D\Phi_0(0)$ has a pair of simple eigenvalues χ_q, $\overline{\chi}_q$ and no other eigenvalues $\mu \in \mathbb{C}$ such that $\mu^q = 1$, cf. section 1.5.1. The further assumption (T1) gives transversality along the unit circle.

We first illustrate the main ideas of the proof and then prove the technical details in a series of lemmas afterwards.

Sketch of proof. Using the Reduction Theorem 1 problem (P) reduces to solving the branching equation (1.12). It follows from (H2) and (H2a) that $\dim U = 2$. Therefore, we can identify U with $\mathbb{C}$, where we shall show that S_0 acts as a multiplication by $\exp(2\pi p/q)$ while R acts as $z \mapsto \overline{z}$. Adapting [32] concerning the normal form of $\mathbb{D}_q$-equivariant functions, from (1.13) it follows that the branching function $\mathcal{B}$ has the form

$$\mathcal{B}(z, \lambda) = i\theta_1(z, \lambda)z + i\theta_2(z, \lambda)\overline{z}^{q-1}, \tag{6.1}$$

with $\theta_i : \mathbb{C} \times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ smooth real-valued $\mathbb{D}_q$-equivariant functions. In particular

$$\theta_1(z, \lambda) = 2\sin(\beta_q(\lambda)) + \sum_{j=1}^{\nu} b_j(\lambda)|z|^{2j} + O(|z|^q), \quad \nu := \left[\frac{q-1}{2}\right],$$

$$(6.2)$$

with $b_j : \mathbb{R} \to \mathbb{R}$ $(1 \le j \le \nu)$ smooth functions. Using polar coordinates $z = \rho e^{i\varphi} \neq 0$, the bifurcation equation $\mathcal{B}(z, \lambda) = 0$ yields the equations

$$\theta_2(\rho e^{i\varphi}, \lambda)\sin(q\varphi) = 0, \tag{6.3}$$

$$h(z, \varphi, \lambda) := \theta_1(\rho e^{i\varphi}, \lambda) + \rho^{q-2}\theta_2(\rho e^{i\varphi}, \lambda)\cos(q\varphi) = 0. \tag{6.4}$$

One can solve (6.3) by $\varphi_1 = 0 \,(\text{mod } 2\pi/q)$ or $\varphi_2 = \pi/q \,(\text{mod } 2\pi/q)$. Substituting in (6.4) then gives equations for λ as a function of ρ. These can be solved by the Implicit Function Theorem, so to obtain two branches of non-trivial q-periodic orbits of the form

$$\gamma_i = \left\{ \left(\exp(j2\pi p/q)\tilde{z}_i(\rho), \tilde{\lambda}_i(\rho) \right) \mid 0 < \rho < \rho_0,\ 0 \le j \le q-1 \right\}, \tag{6.5}$$

$i = 1, 2$, with $\tilde{z}_1(\rho) := \rho$, $\tilde{z}_2(\rho) := \rho e^{i\pi/q}$, and $\lambda = \tilde{\lambda}_i(\rho)$ solution of the equation $h(\tilde{z}_i, \varphi_i, \lambda) = 0$, $i = 1, 2$. For $q \ge 5$, the stability of the periodic orbits corresponding to $(\tilde{z}_i(\rho), \tilde{\lambda}_i(\rho))$ is determined by the number

$$\tau_i(\rho) := \operatorname{tr}\, \tilde{D}\left(\tilde{z}_i(\rho), \tilde{\lambda}_i(\rho) \right), \tag{6.6}$$

see Proposition 1.3. Using then the RPNF Theorem 2 one can explicitly calculate $\tau_i(\rho)$ up to order $O(\rho^{q+1})$ and conclude that one of the branches is stable and the other unstable. □

The rest of this section is devoted to the proof of the SRU Theorem 3.

Lemma 6.1. *Assume* (H1), (R), (H2) *and* (T1). *Let* $w_0 = w_0^1 + iw_0^2$ *be an eigenvector of* A_0 *corresponding to* χ_q. *Then*

(i) $U = span\{w_0^1, w_0^2\}$,

(ii) *the map* $\varphi : \mathbb{C} \to U$ *defined by*

$$\varphi(z) := \operatorname{Re}(zw_0) \tag{6.7}$$

is an isomorphism.

Proof. Item (i) follows from (H2)-(H2a). Indeed, χ_q is a simple eigenvalue of S_0 and $U = \ker(S_0^q - \mathrm{I}) = \ker(S_0 - \chi_q \mathrm{I}) = span\{w_0^1, w_0^2\}$, dim$U$=2. The statement (ii) is trivial. □

From now on we use the complex number z to parametrise the elements of U. Observe that the w_0 of Lemma 6.1 can always be chosen such that $Rw_0 = \overline{w}_0$. In fact, $\overline{w}_0$ is the A_0-eigenvector of $\overline{\chi}_q$, then $R\overline{w}_0$ belongs to $\ker(S_0 - \chi_q \mathrm{I})$. Hence, $R\overline{w}_0 = \beta w_0$ for some $\beta \in \mathbb{C}$. Using $R^2 = \mathrm{I}$ one obtains that $|\beta| = 1$, i.e., $\beta = \exp(2i\phi)$ for some $\phi \in \mathbb{R}$. Replacing w_0 by $\exp(i\phi)w_0$ we can choose w_0 so that $Rw_0 = \overline{w}_0$. So, one has that $R|_U$ is given by

$$z \mapsto \overline{z},$$

and by definition also that $S_0|_U$ is given by

$$z \mapsto \chi_q z.$$

Furthermore, it holds that

$$\varphi(\chi_q z) = S_0 \varphi(z) \quad \text{and} \quad R\varphi(z) = \varphi(\overline{z}).$$

Now, returning to the he branching equation (1.12), that we want to solve, it is clear that it takes the form

$$\mathcal{B}_\lambda(z) = (\chi_q)^{-1}\Phi_{r,\lambda}(z) - \chi_q \Phi_{r,\lambda}^{-1}(z) = 0, \tag{6.8}$$

and the $\mathbb{D}_q$-equivariance properties (1.13) of $\mathcal{B}_\lambda$ read as

$$\mathcal{B}_\lambda\left(\chi_q z\right) = \chi_q \mathcal{B}_\lambda\left(z\right), \quad \mathcal{B}_\lambda\left(\overline{z}\right) = -\overline{\mathcal{B}_\lambda\left(z\right)}, \quad \forall\left(z,\lambda\right) \in \mathbb{C}\times\mathbb{R}.$$

The following lemma gives a normal form for smooth $\mathbb{D}_q$-equivariant functions.

Lemma 6.2 (see e.g. [71, 25]). *Fix some integer $q \geq 3$. Let Λ be a Banach space, and let $f : \mathbb{C}\times\Lambda \to \mathbb{C}$ be a smooth*[1] *map such that*

$$f\left(\chi_q z, \lambda\right) = \chi_q f\left(z,\lambda\right) \qquad f\left(\overline{z},\lambda\right) = -\overline{f\left(z,\lambda\right)}, \quad \left(z,\lambda\right) \in \mathbb{C}\times\Lambda. \tag{6.9}$$

Then, there exist unique smooth maps $\theta_1 : \mathbb{C}\times\Lambda \to \mathbb{R}$, $\theta_2 : \mathbb{C}\times\Lambda \to \mathbb{R}$ such that

(i) $f\left(z,\lambda\right) = iz\theta_1\left(z,\lambda\right) + i\overline{z}^{q-1}\theta_2\left(z,\lambda\right)$*;*

[1] When, in this setting, we talk about smooth maps from $\mathbb{C}$ into itself, we mean maps that are C^∞ when considering both the domain $\mathbb{C}$ and the range $\mathbb{C}$ as two-dimensional real vector space.

(ii) $\theta_i\left(\chi_q z, \lambda\right) = \theta_i\left(z, \lambda\right) = \theta_i\left(\overline{z}, \lambda\right),\ i = 1, 2;$

(iii) $\theta_1\left(0, 0\right) = 0$

It follows that (6.8) takes the form

$$iz\theta_1\left(z, \lambda\right) + i\overline{z}^{q-1}\theta_2\left(z, \lambda\right) = 0, \tag{6.10}$$

where $\theta_1 : \mathbb{C}\times\mathbb{R} \to \mathbb{R}$, $\theta_2 : \mathbb{C}\times\mathbb{R} \to \mathbb{R}$ are smooth functions satisfying the conditions (ii)-(iii) of Lemma 6.2. Recall that the RPNF Theorem 2 implies that if the linearization of Φ_λ at the origin is in normal form then $D_u\Phi_{r,\lambda}(0) = A_\lambda|_U$ and therefore

$$\theta_1(0, \lambda) = \exp(i\beta_q(\lambda)) - \exp(-i\beta_q(\lambda)), \tag{6.11}$$

where $\exp(i\beta_q(\lambda))\chi_q$ is the eigenvalue of A_λ continuation of the eigenvalue χ_q of A_0, cf. section 1.5.1. Now, writing polar coordinates $z = \rho e^{i\varphi}$ and setting $\widetilde{\theta}_k\left(\rho, \varphi, \lambda\right) := \theta_k\left(\rho e^{i\varphi}, \lambda\right)$ in (6.10) $(k = 1, 2)$ give

$$i\rho e^{i\varphi}\widetilde{\theta}_1\left(\rho, \varphi, \lambda\right) + i\widetilde{\theta}_2\left(\rho, \varphi, \lambda\right)\rho^{q-1}e^{-i(q-1)\varphi} = 0. \tag{6.12}$$

Dropping the tildes for simplicity, it is clear that for non-trivial solutions we can multiply by $\overline{z}$ and divide by ρ^2, yielding

$$\theta_1\left(\rho, \varphi, \lambda\right) + \theta_2\left(\rho, \varphi, \lambda\right)\rho^{q-2}e^{-iq\varphi} = 0,$$

which by splitting into real and imaginary part corresponds to the system

$$\theta_1\left(\rho, \varphi, \lambda\right) + \theta_2\left(\rho, \varphi, \lambda\right)\rho^{q-2}\cos\left(q\varphi\right) = 0, \tag{6.13}$$
$$\theta_2\left(\rho, \varphi, \lambda\right)\sin\left(q\varphi\right) = 0. \tag{6.14}$$

Assuming $\theta_2\left(0, 0\right) \neq 0$, equation (6.14) has the only solutions

$$\varphi_1 = j\frac{2\pi}{q}, \quad \varphi_2 = \frac{\pi}{q} + j\frac{2\pi}{q}, \qquad j = 0, 1, \ldots, q-1. \tag{6.15}$$

For $\varphi = \varphi_k$ $(k = 1, 2)$ equation (6.13) reduces to

$$h\left(\rho, \varphi_k, \lambda\right) := \theta_1\left(\rho, \varphi_k, \lambda\right) + \theta_2\left(\rho, \varphi_k, \lambda\right)\rho^{q-2}\cos\left(q\varphi_k\right) = 0 \tag{6.16}$$

By Lemma 6.2-(iii), (6.16) doesn't depend on j. Also, $h\left(-\rho, \varphi_k + \pi, \lambda\right) = h\left(\rho, \varphi_k, \lambda\right)$ and $h\left(0, \theta, 0\right) = \theta_1\left(0, 0\right) = 0$. Since $(\rho, \lambda) = (0, 0)$ solves (6.16) and $D_\lambda\theta_1\left(0, 0\right) \neq 0$ because of (T1) (see (6.11)), then we can invoke the

Implicit Function Theorem to solve (6.16). That is, there exists a unique solution $\lambda = \tilde{\lambda}^k(\rho)$ $(k = 1, 2)$ of (6.16), with $\tilde{\lambda}^k : \mathbb{R} \to \mathbb{R}$ smooth near the origin and such that $\tilde{\lambda}^k(0) = 0$. This proves the SRU Theorem 3.

Remarks

1- Using (6.16), straightforward calculations show that $\left|\widetilde{\lambda}^{(1)}(\rho) - \widetilde{\lambda}^{(2)}(\rho)\right|$ $= O\left(\rho^{\frac{q-2}{2}}\right)$.

2- The periodic orbits we found are R-symmetric, i.e., they are invariant under R.

6.1.1 Normal Form Approximation of the Branching Equation

Using normal forms near the origin it is possible to approximate the map $\mathcal{B}_\lambda(\cdot)$ up to any desired order. By Proposition 1.2, the reduced diffeomorphism $\Phi_{r,\lambda}$ can be approximated by first putting Φ_λ in normal form and then restricting the normal form to the reduced phase space U. Therefore, setting $\Psi_\lambda^{NF} := S_0^{-1}\Phi_\lambda^{NF} = \exp(X_\lambda)$, $X_\lambda := \mathcal{N}_0 + Z_\lambda$ (cf. RPNF Theorem 2), one has

$$\Phi_{r,\lambda}(u) = S_0\Psi_\lambda^{NF}(u) + O(||u||^{k+1}). \tag{6.17}$$

Note that here $\mathcal{N}_0 = 0$, because of (H2) and (H2a). Up to terms of order k, then, the determining equation takes the form

$$\Psi_\lambda^{NF}(u) = u,$$

which for (u, λ) small enough is equivalent to

$$X_\lambda(u) = 0.$$

In fact, on the one hand if $X_\lambda(u) = 0$ then by definition of flow $\Psi_\lambda^{NF}(u) = u$. On the other hand, if $\Psi_\lambda^{NF}(u) = u$ then u generates a one-periodic solution of the equation $\dot{u} = X_\lambda(u)$, which can only be an equilibrium since $DX_0(0) = 0$. This means that the q-periodic points of $\Phi_{r,\lambda}$ can be approximated by the equilibria $u \in U$ of the normal form vector field X_λ. In particular one has that X_λ is reversible and $\mathbb{Z}_q$-equivariant (see Theorem 2-(1.17)), therefore using a result similar to Lemma 6.2

$$X_\lambda(z) = ig_1(z, \lambda)\, z + ig_2(z, \lambda)\, \overline{z}^{q-1}, \tag{6.18}$$

(up to some order q) with $g_i(z,\lambda)$ real-valued functions, $i = 1,2$, and $g_1(0,0) = 0$. This means that, up to order $k = q-1$,

$$X_\lambda(z) = i\left(\sum_{j=0}^{\nu} b_j(\lambda)\,|z|^{2j}\right) z + ia(\lambda)\,\overline{z}^{q-1} + O(|z|^q), \quad \nu := \left[\frac{q-2}{2}\right], \tag{6.19}$$

with $a(\lambda), b_j(\lambda) \in \mathbb{R}$. Note that $b_0(0) = 0$, since $DX_0(0) = 0$. Assume that

$$b_0'(0) \neq 0, \qquad \text{and} \qquad a(0) \neq 0. \tag{6.20}$$

For the flow of the vector field X_λ the following holds.

Lemma 6.3. *A solution $z(t)$ of the initial value problem*

$$\begin{cases} \dot{z} & = X_\lambda(z) \\ z(0) & = z_0 \end{cases} \tag{6.21}$$

has the form

$$z(t) = z_0 \exp\Big(i\left(\sum_{j=0}^{\nu} b_j(\lambda)\,|z_0|^{2j}\right) t\Big) + ic_\lambda(t)\,\overline{z}_0^{q-1} + O(|z_0|^q) \tag{6.22}$$

with $c_\lambda(t)$ a complex-valued function of λ and t, such that $c_\lambda(0) = 0$.

Proof. We start verifying that $z(t)$ of the form (6.22) satisfies

$$|z(t)| = \left|z_0^2\right| + O(|z_0|^q). \tag{6.23}$$

So, define

$$\widetilde{z}(t) := e^{i\left(\sum_{j=0}^{\nu} b_j(\lambda)|z|^{2j}\right)t} z_0 + ic_\lambda(t)\,\overline{z}_0^{q-1}, \tag{6.24}$$

then

$$|z(t)|^2 = |\widetilde{z}(t)|^2 + O(|z_0|^q) = \widetilde{z}(t)\,\overline{\widetilde{z}}(t) + O(|z_0|^q).$$

Now, since $\overline{\widetilde{z}}(t) = e^{-i\left(\sum_{j=0}^{\nu} b_j(\lambda)|z|^{2j}\right)t}\overline{z}_0 - i\overline{c}_\lambda(t)\,z_0^{q-1}$, it follows that

$$\widetilde{z}(t)\,\overline{\widetilde{z}}(t) = |z_0|^2 + O(|z_0|^q),$$

which proves (6.23).
We now prove the existence of a function $c_\lambda(t)$ such that (6.22) holds. From (6.19) it follows that

$$\frac{d}{dt}z(t) = i\left(\sum_{j=0}^{\nu} b_j(\lambda)\,|z(t)|^{2j}\right) z(t) + ia(\lambda)\overline{z}(t)^{q-1} + O(|z|^q),$$

therefore $z(t)$ of the form (6.22) is a solution of (6.21) if and only if

$$\begin{aligned} z_0 i\left(\sum_{j=0}^{\nu} b_j(\lambda)|z_0|^{2j}\right) & e^{i\left(\sum_{j=0}^{\nu} b_j(\lambda)|z_0|^{2j}\right)t} + i\dot{c}_\lambda(t)\overline{z}_0^{q-1} + O(|z_0|^q) \\ &= i\left(\sum_{j=0}^{\nu} b_j(\lambda)|z_0|^{2j}\right)\left(e^{i\left(\sum_{j=0}^{\nu} b_j(\lambda)|z_0|^{2j}\right)t} z_0 + ic_\lambda(t)\overline{z}_0^{q-1}\right) \\ &\qquad + ia(\lambda)e^{-ib_0(\lambda)(q-1)t}\overline{z}_0^{q-1} + O(|z_0|^q). \end{aligned}$$

That is, if and only if there exists a function $c = c_\lambda(t)$ such that

$$\begin{cases} i\dot{c}_\lambda & = -c_\lambda(t)\,b_0(\lambda) + ia(\lambda)\,e^{-i(q-1)tb_0(\lambda)} \\ c_\lambda(0) & = 0 \end{cases}.$$

The solution of such system exists and is unique, see e.g. [2]:

$$c_\lambda(t) = a(\lambda)\,e^{ib_0(\lambda)t}\int_0^t e^{-ib_0(\lambda)qs}ds. \tag{6.25}$$

Hence, the proposition is proved. □

Remark The function $c_\lambda(t)$ is such that

$$c_\lambda(-t) = -\overline{c_\lambda(t)}. \tag{6.26}$$

Therefore,

$$c_\lambda(1) - c_\lambda(-1) = c_\lambda(1) + \overline{c_\lambda(1)} = 2\mathrm{Re}\,(c_\lambda(1)) \in \mathbb{R}. \tag{6.27}$$

Using (6.22) one shows that

$$\begin{aligned}
\mathcal{B}_\lambda(z) &= \exp(X_\lambda(z)) - \exp(X_\lambda(z)) + O(\|z\|^q) \\
&= z(1) - z(-1) + O(\|z\|^q) \\
&= e^{i\left(\sum_{j=0}^{\nu} b_j(\lambda)|z|^{2j}\right)} z + ic_\lambda(1)\,\overline{z}^{q-1} \\
&\quad - \left\{ e^{-i\left(\sum_{j=0}^{\nu} b_j(\lambda)|z|^{2j}\right)} z + ic_\lambda(-1)\,\overline{z}^{q-1} \right\} + O(\|z\|^q).
\end{aligned}$$

Since $e^{iz} - e^{-iz} = 2i\sin z$, it follows that

$$\begin{aligned}
\mathcal{B}_\lambda(z) &= 2i\sin\left(\sum_{j=0}^{\nu} b_j(\lambda)|z|^{2j}\right) z + i\left(c_\lambda(1) - c_\lambda(-1)\right)\overline{z}^{q-1} \\
&\quad + O(\|z\|^q).
\end{aligned} \tag{6.28}$$

Comparing with (6.12) gives

$$\theta_1(z,\lambda) = 2\sin\left(\sum_{j=0}^{\nu} b_j(\lambda)|z|^{2j}\right) + O\left(\|z\|^{q-1}\right); \tag{6.29}$$

$$\theta_2(z,\lambda) = c_\lambda(1) - c_\lambda(-1) + O(\|z\|). \tag{6.30}$$

Thus, the coefficients in (6.28) satisfy the following properties:

(i) $\theta_1(0,0) = 0 \quad \Leftrightarrow \quad b_0(0) = 0.$
Indeed, $\theta_1(0,0) = 2\sin b_0(0)$;

(ii) $\theta_2(0,0) \neq 0 \quad \Leftrightarrow \quad a(0) \neq 0.$
Indeed, by (6.25) and (6.27) one has that

$$\theta_2(0,0) = 2\mathrm{Re}(c_0(1)) = 2\mathrm{Re}\left(a(0)\int_0^1 ds\right) = 2a(0).$$

Note that we used $b_0(0) = 0$;

(iii) $\dfrac{\partial}{\partial\lambda}\theta_1(0,0) \neq 0 \quad \Leftrightarrow \quad b_0'(0) \neq 0$ (transversality condition).
Indeed, $\dfrac{\partial}{\partial\lambda}\theta_1(0,0) = 2b_0'(0)\cos(b_0(0))$ and $b_0(0) = 0$.

6.1.2 Stability of the SRU Branching Solutions

Our goal is here to show that one of the two branches in (3) is stable, the other unstable, using Proposition 1.3. We recall the following: $\dim(U) = 2$, the two bifurcating branches are R-symmetric, and $\mathcal{D}(0,0) = \mathrm{I}_U$, where $\mathcal{D}$ is the operator defined in (5.1). If not otherwise specified, the map Φ_λ is assumed to be in normal form up to order $k \geq q+1$ with $q \geq 5$ throughout.

Corollary 5.4 implies that the stability of the periodic orbit corresponding to $\left(\widetilde{z}_i(\rho), \widetilde{\lambda}_i(\rho)\right)$ (with $i = 1, 2$ and $0 < \rho < \rho_0$) is determined by the number

$$\tau_i(\rho) := \operatorname{tr}\widetilde{\mathcal{D}}\left(\widetilde{z}_i(\rho), \widetilde{\lambda}_i(\rho)\right), \tag{6.31}$$

compare also with (5.34). More precisely, the operator $\widetilde{\mathcal{D}}\left(\widetilde{z}_i(\rho), \widetilde{\lambda}_i(\rho)\right)$ will have two eigenvalues μ, μ^{-1}, with

(i) $|\mu| = 1$ (**stability**), if $\tau_i(\rho) \leq 2$,

(ii) $|\mu| \neq 1$ (**instability**), if $\tau_i(\rho) > 2$.

Setting $J^{(i)}(\rho) := DX_{\widetilde{\lambda}_i(\rho)}(\widetilde{z}_i(\rho))$, the reversibility of the (normal form) vector filed X implies that

$$J^{(i)}(\rho)R = -RJ^{(i)}(\rho) \tag{6.32}$$

Therefore, if $\mu_i(\rho) \in \mathbb{C}$ is an eigenvalue of $J^{(i)}(\rho)$, then so is $-\mu_i(\rho)$. Moreover, (5.19) implies that

$$\tau_i(\rho) = \exp(\mu_i(\rho)) + \exp(-\mu_i(\rho)) + O\left(\rho^{q+1}\right). \tag{6.33}$$

The relations (5.18) and (6.33) then lead to the following.

Lemma 6.4. *Assume $q \geq 5$, (H1), (R), (H2), (6.19), (6.20) and $b_1(0) \neq 0$. Then there exists a $\rho_0 > 0$ such that for $i = 1, 2$ and $0 < \rho < \rho_0$ the operator $J^{(i)}(\rho)$ has two simple eigenvalues $\pm\mu_i(\rho)$, with*

$$-(\mu_i(\rho))^2 = (-1)^i 2q\rho^q a(0)\, b_1(0) + O\left(\rho^{q+1}\right), \tag{6.34}$$

and $\mu_i(\rho) = O\left(\rho^{\frac{q}{2}}\right)$.

Proof. As we observed before if $\left(\widetilde{z}(\rho), \widetilde{\lambda}(\rho)\right)$ is a solution branch of the bifurcation equation then $\left(\widetilde{z}(\rho), \widetilde{\lambda}(\rho)\right)$ is (up to higher order terms) an equlibrium of the autonomous equation

$$\dot{z} = X_\lambda (z) . \tag{6.35}$$

Introducing polar coordinates $z = \rho e^{i\varphi}$ ($\rho \neq 0$), by (6.18) equation (6.35) is equivalent to

$$\left\{ \begin{array}{ll} \dot{\rho} & = \rho^{q-1} \widetilde{g}_2 (\rho, \varphi, \lambda) \sin (q\varphi) \\ \dot{\varphi} & = \widetilde{h} (\rho, \varphi, \lambda) , \end{array} \right. \tag{6.36}$$

where $\widetilde{h}(\rho, \varphi, \lambda) := \widetilde{g}_1(\rho, \varphi, \lambda) + \rho^{q-2} \cos(q\varphi)\, \widetilde{g}_2(\rho, \varphi, \lambda)$, and $\widetilde{g}_i(\rho, \varphi, \lambda) = g_i\left(\rho e^{i\varphi}, \lambda\right)$ $(i = 1, 2)$. Observe further that

$$g_1 (z, \lambda) = \sum_{j=0}^{m} b_j (\lambda) |z|^{2j} + O\left(\rho^{q-1}\right) \quad \text{and} \quad g_2 (0, \lambda) = a (\lambda) ,$$

see (6.19). Under the given assumptions the vector field X_λ has got, for arbitrary (ρ, λ) (small), two branches of equlibria of the form

$$\left\{ \left(z^{(i)} (\rho) , \lambda_*^{(i)} (\rho) \right) \middle| \ \rho \in (0, \rho_0) , \ i = 1, 2 \right\} , \tag{6.37}$$

with $z^{(j)}(\rho) = \rho e^{i\varphi^{(j)}}$, $(j = 1, 2)$, and $\lambda^{(j)} = \lambda_*^{(j)}(\rho)$ solution of the equation

$$\widetilde{h}\left(\rho, \varphi^{(j)}, \lambda\right) = 0, \qquad j = 1, 2,$$

where $\varphi^{(1)} = k\frac{2\pi}{q}$ and $\varphi^{(2)} = \frac{\pi}{q} + k\frac{2\pi}{q}$, $k = 0, \ldots, q-1$. One also verifies that the Jacobian is such that

$$J\left(\widetilde{z}_i(\rho), \widetilde{\lambda}_i(\rho)\right) = J\left(\rho, \varphi^{(i)}, \lambda^{(i)}\right) + O\left(\rho^{q+1}\right).$$

Knowing that $\cos\left(q\varphi^{(i)}\right) = (-1)^i$ and $\sin q\varphi^{(i)} = 0$, one calculates

$$\begin{aligned} & J\left(\rho, \varphi^{(i)}, \lambda^{(i)}\right) \\ & \quad = \det \begin{pmatrix} 0 & q\rho^{q-1}\widetilde{g}_2\left(\rho, \varphi^{(i)}, \lambda^{(i)}\right)(-1)^i \\ \frac{\partial}{\partial\rho}\widetilde{h}\left(\rho, \varphi^{(i)}, \lambda^{(i)}\right) & \frac{\partial}{\partial\varphi}\widetilde{h}\left(\rho, \varphi^{(i)}, \lambda^{(i)}\right) \end{pmatrix} \\ & \quad = -(-1)^i q\rho^{q-1}\widetilde{g}_2\left(\rho, \varphi^{(i)}, \lambda^{(i)}\right) \frac{\partial}{\partial\rho}\widetilde{h}\left(\rho, \varphi^{(i)}, \lambda^{(i)}\right). \end{aligned} \tag{6.38}$$

It is then starightforward to calculate the first non-zero term in the Taylor expansion of this expression:

$$J\left(\rho, \varphi^{(s)}, \lambda^{(s)}\right) = (-1)^s\, 2q\rho^q \widetilde{g}_2\,(0,0)\, b_1\,(0) + O\left(\rho^{q+1}\right), \tag{6.39}$$

$s = 1, 2$. This proves the lemma. □

Corollary 6.5. *Under the assumptions of Lemma 6.4, one has that if $i = 1$ the bifurcating solutions given by Theorem 3 are stable if $a\,(0)\, b_1\,(0) < 0$, and unstable if $a\,(0)\, b_1\,(0) > 0$. For the solutions along the q branches with $i = 2$ the opposite holds.*

Proof. Combine (6.33), (6.34) to obtain

$$\tau_i\,(\rho) = 2 - (-1)^i\, 4q\rho^q a\,(0)\, b_1\,(0) + O\left(\rho^{q+1}\right), \qquad i = 1, 2. \tag{6.40}$$

□

Remark The coefficients $a\,(0)$ and $b_1\,(0)$, which determine the direction of the bifurcation and the satbility properties of the bifurcating solutions, can be determined from the Taylor expansion of $X_\lambda\,(z)$.

6.2 Proof of RRU Theorem 4

Goal of this section is to prove the RRU Theorem 4, i.e., for some fixed $q \geq 3$ we want to solve (P) under the assumptions (H1), (R), (H3) and (H3-a). We start with some linear algebra considerations to solve then in section 6.2.1 the branching equation by mean of the Implicit Function Theorem.

The reduced phase space $U = \ker\,(S_0^q - \mathrm{I})$ coincides with the generalized eigenspace of the eigenvalue χ_q, i.e.,

$$U := \ker\left(\left((A_0 - \cos\theta_0 \mathrm{I})^2 + \mathrm{I}\sin^2\theta_0\right)^2\right), \quad \text{where } \theta_0 := 2\pi p/q. \tag{6.41}$$

Also, U can be written as $U = \ker\left((S_0 - \cos\theta_0 \mathrm{I})^2 + \mathrm{I}\sin^2\theta_0\right)$, where $\dim_{\mathbb{R}} U = 4$. The eigenspace U_q of the eigenvalues χ_q is the 2-dimensional subspace of U given by

$$U_q := \ker\left((A_0 - \cos\theta_0 \mathrm{I})^2 + \mathrm{I}\sin^2\theta_0\right) \subseteq U, \tag{6.42}$$

which is invariant under the involution R and under $J_0 \in \mathcal{L}(U)$, where

$$J_0 := \frac{1}{\sin\theta_0}\,(S_0 - \mathrm{I}\cos\theta_0) \in \mathcal{L}(U) \quad \text{with } \sin\theta_0 \neq 0.$$

Observe that $J_0^2|_U = -\mathrm{I}|_U$ and that $RJ_0 = \frac{1}{\sin\theta_0}\left(S_0^{-1} - \mathrm{I}\cos\theta_0\right)$, since S_0 is R-reversible. Note that one can write

$$S_0 = \exp(\theta_0 J_0) = \mathrm{I}\cos\theta_0 + J_0\sin\theta_0,$$

and similarly $S_0^{-1} = \exp(-\theta_0 J_0) = \mathrm{I}\cos\theta_0 - J_0\sin\theta_0$. Therefore, one says that J_0 and R generate a $S^1 \ltimes \mathbb{Z}_2(\mathbb{R})$-action on U, leaving U_q invariant.

Lemma 6.6. *Under the hypotheses of Theorem 4, let $A_0 = S_0 e^{\mathcal{N}_0}$ be the SU-decomposition of $A_0 := D\Phi_0(0)$. There exists a basis $B_U = \{e_1, e_2, f_1, f_2\}$ of U such that*

$$\begin{aligned}
&\mathcal{N}_0 e_i = 0, \quad \mathcal{N}_0 f_i = e_i, \quad i = 1, 2\\
&J_0 e_1 = e_2, \quad J_0 e_2 = -e_1,\\
&J_0 f_1 = f_2, \quad J_0 f_2 = -f_1,\\
&R f_1 = f_1, \quad R f_2 = -f_2,\\
&R e_1 = -e_1, \quad R e_2 = e_2.
\end{aligned} \tag{6.43}$$

Proof. As observed before, J_0 and R generate a $S^1 \ltimes \mathbb{Z}_2(\mathbb{R})$-action on U, and U_q is $S^1 \ltimes \mathbb{Z}_2(\mathbb{R})$-invariant, therefore there exists a $S^1 \ltimes \mathbb{Z}_2(\mathbb{R})$-invariant complement $\mathcal{F}$ of U_q in U; i.e. $U = U_q \oplus \mathcal{F}$, with $R : \mathcal{F} \to \mathcal{F}$, $J_0 : \mathcal{F} \to \mathcal{F}$ and $\dim_{\mathbb{R}}\mathcal{F} = 2$. So the lemma is proved if we can find a basis of $\{e_1, e_2\}$ of U_q and a basis $\{f_1, f_2\}$ of $\mathcal{F}$ satisfying (6.43).

Let $f_1 \in \mathcal{F}$ be an eigenvector of R, i.e., $Rf_1 = \delta f_1$, with $\delta = \pm 1$. Setting $f_2 := J_0 f_1$ we find

$$Rf_2 = \frac{1}{\sin\theta_0}(S_0^{-1} - \mathrm{I}\cos\theta_0)Rf_1 = \frac{-\delta}{\sin\theta_0}(S_0 - \mathrm{I}\cos\theta_0)f_1 = -\delta f_2.$$

So we can assume $\delta = 1$ (interchange f_1 and f_2 in the other case), resulting in a basis $\{f_1, f_2\}$ of $\mathcal{F}$ such that $Rf_1 = f_1$ and $Rf_2 = -f_2$.

From (H3)-(H3a) it follows that $\mathcal{N}_0$ is nilpotent with height 2, i.e., $\mathcal{N}_0^2 = O$. For a basis of U_q choose then $\{e_1, e_2\}$ with $e_1 = \mathcal{N}_0 f_1$ and $e_2 = \mathcal{N}_0 f_2$. It follows that $\mathcal{N}_0 e_i = 0$, $i = 1, 2$, and

$$Re_1 = R\mathcal{N}_0 f_1 = -\mathcal{N}_0 R f_1 = -\mathcal{N}_0 f_1 = -e_1$$

while similarly

$$Re_2 = e_2.$$

Note also that

$$J_0 e_1 = J_0 \mathcal{N}_0 f_1 = \mathcal{N}_0 J_0 f_1 = \mathcal{N}_0 f_2 = e_2$$

and similarly $J_0 e_2 = -e_1$. □

Using the basis $B_U = \{e_1, e_2, f_1, f_2\}$, we may identify U with $\mathbb{R}^4$ and get the following explicit representations

$$J_0 = \begin{pmatrix} J_2 & 0 \\ 0 & J_2 \end{pmatrix} \in gl_{-R}(4, \mathbb{R}), \quad \text{with } J_2 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{6.44}$$

$$R = \begin{pmatrix} -R_1 & 0 \\ 0 & R_1 \end{pmatrix} \in gl(4, \mathbb{R}), \quad \text{with } R_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6.45}$$

$$\mathcal{N}_0 = \begin{pmatrix} 0 & \mathrm{I}_2 \\ 0 & 0 \end{pmatrix} \in gl_{-R}(4, \mathbb{R}), \tag{6.46}$$

$$S_0 = \exp(\theta_0 J_0) = \begin{pmatrix} R(\theta_0) & 0 \\ 0 & R(\theta_0) \end{pmatrix} \in Gl_{-R}(4, \mathbb{R}), \tag{6.47}$$

where

$$R(\theta_0) := \begin{pmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{pmatrix} = \exp(\theta_0 J_2) = I\cos\theta_0 + J_2 \sin\theta_0, \tag{6.48}$$

and I_2 is the 2×2 identity matrix.

Using the explicit form of $A_0 = S_0 \exp(\mathcal{N}_0)$ with respect to the basis B_U one directly proves the following.

Lemma 6.7. *Let $A_0 \in GL_{-R}(4, \mathbb{R})$ be of the form $S_0 \exp(\mathcal{N}_0)$, with $S_0, \mathcal{N}_0$ as in* (6.47), (6.46) *and* $\sin(\theta_0) \neq 0$. *Then*

$$\ker\left(\mathrm{ad}(\mathcal{N}_0^T)\right) \cap \ker\left(\mathrm{ad}(S_0)\right) \cap gl_{-R}(4, \mathbb{R}) = \{B(\vartheta, \sigma) \mid \vartheta, \sigma \in \mathbb{R}\}, \tag{6.49}$$

with

$$B(\vartheta, \sigma) := \begin{pmatrix} \vartheta J_2 & \mathrm{O} \\ \sigma \mathrm{I} & \vartheta J_2 \end{pmatrix}. \tag{6.50}$$

Combining Lemma 6.7 with the RPNF Theorem 2 shows that on U

$$\begin{aligned} D\Phi(0,\lambda) = A(\lambda) &= S_0 \exp(\mathcal{N}_0 + B(\vartheta(\lambda), \sigma(\lambda))) \\ &= S_0 \exp(\vartheta(\lambda) J_0) \exp(\mathcal{N}_0 + \sigma(\lambda)\mathcal{M}_0) \end{aligned} \tag{6.51}$$

where

$$\mathcal{N}_0 := \begin{pmatrix} 0 & \mathrm{I}_2 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{M}_0 := \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}, \quad \text{and} \quad \vartheta(0) = \sigma(0) = 0. \tag{6.52}$$

Notice that $\mathcal{M}_0\mathcal{N}_0 \neq \mathcal{N}_0\mathcal{M}_0$. In order to determine the eigenvalues of $A(\lambda) \in Gl_{-R}(4, \mathbb{R})$ close to A_0 we must calculate the eigenvalues of (6.51) for λ close to 0. We can write $A(\lambda)|_U = R(\theta_0 + \vartheta(\lambda))\Sigma(\lambda)$, where $\Sigma(\lambda) := \exp\left(\begin{pmatrix} O & \mathrm{I} \\ \sigma(\lambda)\mathrm{I} & O \end{pmatrix}\right)$, and $R(\theta_0+\vartheta(\lambda))$ is the rotation on the angle $\theta_0+\vartheta(\lambda)$. The eigenvalues of $\Sigma(\lambda)$ are $\exp(\pm\sqrt{\sigma(\lambda)})$ if $\sigma(\lambda) \geq 0$ or $\exp(\pm i\sqrt{\tilde{\sigma}(\lambda)})$ if $\sigma(\lambda) = -\tilde{\sigma}(\lambda)$ with $\tilde{\sigma}(\lambda) > 0$. It follows that $A(\lambda)$ has either

(i) a pair of double eigenvalues on the unit circle

$$\exp\left(\pm i(\theta_0 + \vartheta(\lambda))\right), \quad \text{if } \sigma(\lambda) = 0; \tag{6.53}$$

or

(ii) a quadruplet of simple complex eigenvalues off the unit circle

$$\exp\left(\pm i(\theta_0 + \vartheta(\lambda))\right) \exp\left(\pm\sqrt{\sigma(\lambda)}\right), \quad \text{if } \sigma(\lambda) > 0; \tag{6.54}$$

or

(iii) a quadruplet of simple complex eigenvalues on the unit circle

$$\exp\left(\pm i(\theta_0 + \vartheta(\lambda))\right) \exp\left(\pm i\sqrt{\tilde{\sigma}(\lambda)}\right), \quad \text{if } \sigma(\lambda) < 0; \tag{6.55}$$

So we see that as $\sigma(\lambda)$ increases through zero the eigenvalues of $A(\lambda)$ (λ small) move towards each other on the unit circle, collide at $\sigma(\lambda) = 0$, and splitt off the unit circle for $\sigma(\lambda) > 0$. The question is what happens to the periodic solutions corresponding to these eigenvalues.

Transversality condition Recall from section 1.5.2 that we assumed (T2), i.e.

$$\frac{\partial(\sigma, \vartheta)}{\partial(\lambda_1, \lambda_2)} \neq 0.$$

Hence we can take $(\lambda_1, \lambda_2) := (\sigma, \vartheta)$ as new parameters. So, (6.50) reads

$$B(\vartheta(\lambda), \sigma(\lambda)) = \begin{pmatrix} \lambda_1 J_2 & O \\ \lambda_2 \mathrm{I} & \lambda_1 J_2 \end{pmatrix} = B(\lambda). \tag{6.56}$$

6.2.1 Solution of the Bifurcation Equation

By Theorem 1 proving the RRU Theorem 4 reduces to solving the corresponding branching equation $\mathcal{B}(u, \lambda) = 0$, $(u, \lambda) \in U \times \mathbb{R}^2$. To do so, let $e := e_1 + ie_2$ and $f := f_1 + if_2$ and identify U with $\mathbb{C} \times \mathbb{C}$ via the maps

$$\phi : \mathbb{C} \to U_q, \quad z \mapsto \phi(z) := \mathrm{Re}(z(e_1 + ie_2)), \tag{6.57}$$
$$\psi : \mathbb{C} \to \mathcal{F}, \quad w \mapsto \psi(w) := \mathrm{Re}(w(f_1 + if_2)). \tag{6.58}$$

Then,

$$R(z, w) = (\overline{z}, -\overline{w}), \quad \text{and} \quad S_0(z, w) = e^{i\theta_0}(z, w).$$

Now, the branching function and its $\mathbb{D}_q$-equivariance properties read as follows

$$\mathcal{B} : \mathbb{C} \times \mathbb{C} \times \mathbb{R}^m \to \mathbb{C} \times \mathbb{C} \times \mathbb{R}^m, \qquad (z, w, \lambda) \mapsto \mathcal{B}_\lambda(z, w) = (\mathcal{B}_{1,\lambda}, \mathcal{B}_{2,\lambda})$$

with

$$\overline{\mathcal{B}_{1,\lambda}(z, w)} = -\mathcal{B}_{1,\lambda}(\overline{z}, -\overline{w}), \ \mathcal{B}_{1,\lambda}(e^{i\theta_0} z, e^{i\theta_0} w) = e^{i\theta_0} \mathcal{B}_{1,\lambda}(z, w), \tag{6.59}$$
$$\overline{\mathcal{B}_{2,\lambda}(z, w)} = \mathcal{B}_{2,\lambda}(\overline{z}, -\overline{w}), \quad \mathcal{B}_{2,\lambda}(e^{i\theta_0} z, e^{i\theta_0} w) = e^{i\theta_0} \mathcal{B}_{2,\lambda}(z, w). \tag{6.60}$$

Our goal is to solve the system of two equatios

$$\mathcal{B}_{j,\lambda}(z, w) = 0, \quad (z, w) \in \mathbb{C} \times \mathbb{C}, \quad j = 1, 2. \tag{6.61}$$

The first is solved by direct application of the Implicit Function Theorem, the second equation is first transformed in a more convenient form to be then split in two real equations, one of which is solved by the Implicit Function

Theorem. We start solving the equation $\mathcal{B}_1(z,v,\lambda) = 0$. Consider the linearization at zero of $\mathcal{B}(z,w,\lambda)=0$, i.e.,

$$D_u\mathcal{B}(0,0,\lambda)\tilde{u} = S_0A(\lambda_1,\lambda_2)\tilde{u} - S_0A^{-1}(\lambda_1,\lambda_2)\tilde{u} = 0,$$

where $\tilde{u} := (\tilde{z},\tilde{w}) \in \mathbb{C}\times\mathbb{C}$. Referring to (6.51), this equation is equivalent to

$$\mathcal{A}(\lambda_1,\lambda_2)\tilde{u} = 0, \tag{6.62}$$

where

$$\begin{aligned}\mathcal{A}(\lambda_1,\lambda_2) :=& \exp(\lambda_1 J_0)\exp(\mathcal{N}_0+\lambda_2\mathcal{M}_0)\\ &- \exp(-\lambda_1 J_0)\exp(-(\mathcal{N}_0+\lambda_2\mathcal{M}_0)).\end{aligned} \tag{6.63}$$

So, $D_{(z,w)}\mathcal{B}(0,0,0) = 2\mathcal{N}_0$ and therefore

$$D_w\mathcal{B}_1(0,0,0)\cdot\tilde{w} = 2\mathcal{N}_0\tilde{w}. \tag{6.64}$$

Since $\mathcal{N}_0$ is an isomorphism of $\mathcal{F}$ onto U_q, it follows by the Implicit Function Theorem that for small values of (z,λ) there exists $w = w^*(z,\lambda)$ such that

$$\mathcal{B}_1(z,w^*(z,\lambda),\lambda) = 0$$

with $w^* : \mathbb{C}\times\mathbb{R}^2 \to \mathbb{C}$ satisfying $w^*(0,\lambda)=0$, $w^*(S_0z,\lambda) = S_0w^*(z,\lambda)$ and $-w^*(\overline{z},\lambda) = \overline{w^*(z,\lambda)}$, and $\mathcal{B}_2(\overline{z},w^*(\overline{z},\lambda),\lambda) = \overline{\mathcal{B}_2(z,w^*(z,\lambda),\lambda)}$.

Observe also that for $\lambda_1 = 0$ we have that

$$w^*(z,\lambda_1=0,\lambda_2) = 0 + \text{h.o.t} \tag{6.65}$$

since (6.62) implies that

$$\mathcal{B}(z,w,0,\lambda_2) = 2\begin{pmatrix} O & \mathrm{I}\\ \lambda_2\mathrm{I} & O\end{pmatrix}\begin{pmatrix} z\\ w\end{pmatrix} + \text{h.o.t.}$$

and hence

$$\mathcal{B}_1(z,w,0,\lambda_2) = 2w + \text{h.o.t.}$$

Replacing w by $w^*(z,\lambda)$ in the second equation of (6.61) gives the complex $\mathbb{D}_q$-equivariant equation

$$\mathcal{B}_3(z,\lambda) := \mathcal{B}_2(z,w^*(z,\lambda),\lambda) = 0 \tag{6.66}$$

where $\mathcal{B}_3 : \mathbb{C} \times \mathbb{R}^2 \to \mathbb{C}$. Using the $\mathbb{D}_q$-equivariance of $\mathcal{B}_3$ (see (6.60)), we may write

$$\mathcal{B}_3(z, \lambda) = z\theta_1(z, \lambda) + \overline{z}^{q-1}\theta_2(z, \lambda),$$

with $\theta_i : \mathbb{C} \times \mathbb{R}^2 \to \mathbb{R}$ such that

$$\theta_i\left(e^{i\theta_0}z, \lambda\right) = \theta_i(z, \lambda) = \theta_i(\overline{z}, \lambda),\ i = 1, 2, \quad \text{and} \quad \theta_1(0, 0) = 0,$$

compare with Lemma 6.2. Equation (6.66) therefore yields

$$z\theta_1(z, \lambda) + \overline{z}^{q-1}\theta_2(z, \lambda) = 0, \quad z \in \mathbb{C},\ \lambda \in \mathbb{R}^2.$$

Note that

$$\theta_1(0, \lambda) = D_z\mathcal{B}_3(0, \lambda). \tag{6.67}$$

Setting $z = \rho e^{i\varphi}$ and $\hat{\theta}_i(\rho, \varphi, \lambda) := \theta_i(\rho e^{i\varphi}, \lambda)$, gives

$$\rho e^{i\varphi}\hat{\theta}_1(\rho, \varphi, \lambda) + \rho^{q-1}e^{-i(q-1)\varphi}\hat{\theta}_2(\rho, \varphi, \lambda) = 0.$$

On the one hand, dropping the hats, multiplication by $\overline{z}$ and division by ρ^2 imply that the non-trivial solutions have to satisfy

$$\theta_1(\rho, \varphi, \lambda) + \theta_2(\rho, \varphi, \lambda)\rho^{q-2}e^{-iq\varphi} = 0. \tag{6.68}$$

On the other hand (6.68) is equivalent to the system

$$\theta_1(\rho, \varphi, \lambda) + \theta_2\rho^{q-2}\cos(q\varphi) = 0, \tag{6.69}$$
$$\theta_2(\rho, \varphi, \lambda)\sin(q\varphi) = 0. \tag{6.70}$$

Suppose that $\theta_2(0, \varphi, 0) \neq 0$, then (6.70) has solutions

$$\varphi_1 = j\frac{2\pi}{q}, \quad \varphi_2 = \frac{\pi}{q} + j\frac{2\pi}{q} \quad j = 0, \ldots, q-1.$$

Substituting $\varphi = \varphi_1$ or $\varphi = \varphi_2$ in (6.69) yields then

$$h_{j,k}(\rho, \lambda) := \theta_1(\rho, \varphi_{j,k}, \lambda) + \theta_2(\rho, \varphi_{j,k}, \lambda)\rho^{(q-2)}\cos(q\varphi_{j,k}) = 0, \tag{6.71}$$

$k = 1, 2$, $j = 0, \ldots, q-1$, which does not depend on j. We suppress the indeces j, k. Our aim is to solve (6.71) by applying the Implicit Function Theorem. It is straightforward to verify that

$$h(0, 0) = \theta_1(0, 0, 0) = 0 \quad \text{and} \quad \frac{\partial}{\partial\lambda_2}h(0, 0) = \frac{\partial}{\partial\lambda_2}\theta_1(0, 0, 0). \tag{6.72}$$

Now, one verifies that

$$\mathcal{B}_3(z, \lambda_1 = 0, \lambda_2) = 2\lambda_2 z + \text{h.o.t.} \tag{6.73}$$

using (6.65) and (6.63) where we set $\lambda_1 = 0$. Combining (6.73) with (6.67) it follows that

$$\theta_1(z = 0, \lambda_1 = 0, \lambda_2) = 2\lambda_2 + \text{h.o.t.}$$

and therefore

$$\frac{\partial \theta_1}{\partial \lambda_2}(0, 0, 0) = 2 \neq 0.$$

Hence, we can invoke the Implicit Function Theorem to get the existence of a solution $\lambda_2 = \lambda_2^*(\rho, \lambda_1)$ of the equation (6.71). Note that $\lambda_2^*(0, 0) = 0$ and that $\lambda_2^*(\rho, \lambda_1)$ is quadratic in ρ. This concludes the proof of the RRU Theorem 4. That is, for each (sufficiently small) λ_1 two branches of (R-symmetric) q-periodic orbits of Φ_λ bifurcate at some parameter value $\lambda_2 = \widehat{\lambda}_2(\lambda_1)$ near zero, where $\widehat{\lambda}_2(\lambda_1) = \lambda_2^*(0, \lambda_1)$, i.e. the bifurcation point is $\rho = 0$.

6.3 Proof of the Primary Branch Theorem 5

Goal is to prove the PB Theorem 5, i.e, we want to solve (P) for $q = 1$ under the assumptions (H1), (R) and (H4), cf. section 1.6.

Again by Theorem 1 we have to solve the branching equation $\mathcal{B}(u, \lambda) = 0$ on $U = \ker(S_0 - \mathrm{I})$, compare with sections 6.1 and 6.2. Recall from section 1.6 that $U = \ker(S_0 - \mathrm{I}) = \{\alpha e_0 | \, \alpha \in \mathbb{R}\}$ with $e_0 \in U$ an eigenvector of the eigenvalue 1 of S_0 such that $Re_0 = e_0$ and $\mathcal{N}_0 e_0 = 0$. Obviously all elements of U are R-symmetric; i.e., Ru=u for all $u \in U$. In combination with (1.13) this implies that

$$\mathcal{B}(u, \lambda) = \mathcal{B}(Ru, \lambda) = -R\mathcal{B}(u, \lambda) \; = -\mathcal{B}(u, \lambda)$$

hence

$$\mathcal{B}(u, \lambda) = 0, \quad \forall (u, \lambda) \in U \times \mathbb{R}^m.$$

That is, each sufficiently small $u \in U$ solves the branching equation $\mathcal{B}(u, \lambda) = 0$. For the original mapping Φ_λ the conclusion is that: *under the generic assumption* (H4) *for the multiplier* 1, *the fixed point* $x = 0$ *belongs to a one-parameter family of symmetric fixed points.* This proves Theorem 5. In view of what follows we call this family of fixed point the *primary branch*.

Choice of coordinate on the primary branch. Consider the direct sum decomposition $\mathbb{R}^{2n-1} = \ker\,(S_0 - \mathrm{I}) \oplus \mathrm{Im}\,(S_0 - \mathrm{I})$. Recall that $\ker\,(S_0 - \mathrm{I}) = \mathbb{R}e_0$, with $S_0 e_0 = e_0$ and $Re_0 = e_0$; therefore, $x = \alpha e_0 + v$, for some $v \in \mathrm{Im}(S_0 - \mathrm{I})$ and for all $x \in \mathbb{R}^{2n-1}$. Also, from Theorem 5, we have that

$$x^*_\lambda(u) = \alpha e_0 + v_\lambda(\alpha), \quad \text{for some } v_\lambda(\alpha) \in \mathrm{Im}\,(S_0 - \mathrm{I})\,, \tag{6.74}$$

where $Re_0 = e_0$ and $Rv_\lambda(\alpha) = v_\lambda(\alpha)$ (see Theorem 5-(ii)).

Lemma 6.8. *It is possible to find a R-equivariant transformation $T : \mathbb{R}^{2n-1} \times \mathbb{R}^m \to \mathbb{R}^{2n-1}$ such that $x^*_\lambda(\alpha) = \alpha e_0$, (i.e. with the corresponding $v_\lambda(\alpha) = 0$) and*

$$T_\lambda^{-1}\Phi_\lambda T_\lambda(\alpha e_0) = \alpha e_0. \tag{6.75}$$

Proof. Set

$$T_\lambda(\alpha e_0 + \hat{v}) = \alpha e_0 + \hat{v} + v_\lambda(\alpha),$$

then

$$T_\lambda(\alpha e_0) = x^*_\lambda(\alpha). \tag{6.76}$$

Observe that $T_\lambda^{-1}(\alpha e_0 + \hat{v}) = \alpha e_0 + \hat{v} - v_\lambda(\alpha)$, and $T_\lambda^{-1} R T_\lambda = R$. Indeed, $R\hat{v} + v_\lambda(\alpha) \in \mathrm{Im}(S_0 - \mathrm{I})$ and

$$\begin{aligned} &T_\lambda^{-1} R T_\lambda(\alpha e_0 + \hat{v}) = \\ &T_\lambda^{-1} R(\alpha e_0 + \hat{v} + v_\lambda(\alpha)) = T_\lambda^{-1}(\alpha e_0 + R\hat{v} + v_\lambda(\alpha)) \\ &\qquad\qquad\qquad\qquad\qquad = \alpha e_0 + R\hat{v} = R(\alpha e_0 + \hat{v}). \end{aligned}$$

Define now, $\tilde{\Phi}_\lambda := T_\lambda^{-1}\Phi_\lambda T_\lambda$. Obviously $\tilde{\Phi}_\lambda$ is R-reversible and $\tilde{\Phi}_\lambda(\alpha e_0) = \alpha e_0$. Calculating the linearization at $(\lambda, x) = (0, 0)$ one obtains that

$$DT_0(0) = \mathrm{I}, \quad D\tilde{\Phi}_0(0) = D\Phi_0(0). \tag{6.77}$$

In fact, $DT_0(0)(\beta e_0 + v') = \beta e_0 + v' + \dfrac{dv_0(0)}{d\alpha} \cdot \beta = \beta e_0 + v'$. This means that the transformation T does not change the properties of the linearization at $(0, 0)$ of the original local diffeomorphism Φ. $\square$

Dropping the tilda's from now on, we have a reversible map Φ_λ with spectrum properties as before and satisfying the additional property

$$\Phi_\lambda(\alpha e_0) = \alpha e_0. \tag{6.78}$$

Define $A_{\lambda,\alpha} := D\Phi_\lambda(\alpha e_0)$. Then $A_{\lambda,\alpha}$ is R-reversible and if Φ_λ is in normal form up to order k one also has that $A_{\lambda,\alpha} S_0 = S_0 A_{\lambda,\alpha}$.

6.3.1 Digression on the Reversible GLS Reduction and Reversible Normal Form

Suppose that the family $\{\Phi_\lambda\}_{\lambda\in\mathbb{R}^m}$ satisfying (H1) and (R) satisfies

$$\Phi_\lambda(\alpha e_0) = \alpha e_0, , \tag{6.79}$$

where $e_0 \in \mathrm{Fix}R$, $e_0 \neq 0$ and $\alpha \in \mathbb{R}$. Then, $D\Phi_\lambda(\alpha e_0)\cdot e_0 = e_0$, which means that 1 is an eigenvalue of the linearization of Φ_λ at αe_0 for all λ. In particular 1 is an eigenvalue of A_0.

Lemma 6.9. *Under the hypotheses of the* GLS *Theorem 1 assume also that*

(i) $A_0 := D\Phi_0(0)$ *has simple eigenvalue* 1 *with corresponding eigenvector* e_0*;*

(ii) Φ_λ *satisfies (6.79) for all* $\alpha \in \mathbb{R}$ *and all* $\lambda \in \mathbb{R}^m$*.*

Then, for all $\alpha \in \mathbb{R}$ *and all* $\lambda \in \mathbb{R}^m$*,*

$$\Phi_{r,\lambda}(\alpha e_0) = \alpha e_0, \quad \forall\lambda \in \mathbb{R}^m. \tag{6.80}$$

Proof. Note that $\ker(S_0 - \mathrm{I}) = \mathbb{R}e_0 \subseteq U$. It is straightforward to verify that $\widehat{R}\hat{e}_0 = \hat{e}_0$, $\widehat{S}_0\hat{e}_0 = \hat{e}_0$, $\widehat{A}_0\hat{e}_0 = \hat{e}_0$ and that $\widehat{\Phi}_\lambda(\hat{e}_0) = \hat{e}_0$, where $\hat{e}_0 \in \widehat{U} \subseteq Y_q$ is the lift of e_0 to Y_q. Now, recalling that $\widehat{U} \cong U$ and that $\widehat{\Phi}_\lambda : U \oplus \mathrm{Im}(\widehat{S}_0 - \sigma) \to U \oplus \mathrm{Im}(\widehat{S}_0 - \sigma)$, $(u,v) \mapsto \widehat{\Phi}_\lambda(u,v) := (\Psi_\lambda(u,v), \Sigma_\lambda(u,v))$, one obtains that

$$\Psi_\lambda(\alpha e_0, 0) = \alpha e_0, \quad \text{and} \quad \Sigma_\lambda(\alpha e_0, 0) = 0. \tag{6.81}$$

As before, solving problem (P) reduces to solving the system of two equations $S_0 u = \Psi_\lambda(u,v)$ and $\sigma v = \Sigma_\lambda(u,v)$. Now, $D_v\Sigma_\lambda(\alpha e_0, 0) = \widehat{A}_{\lambda,\alpha}|_{\mathrm{Im}(\hat{S}_0-\sigma)}$, and in particular $\widehat{A}_0|_{\mathrm{Im}(\hat{S}_0-\sigma)}$ is invertible. It follows by the Implicit Function Theorem that there exists a solution $v = v^*(u,\lambda)$ of $\sigma v = \Sigma_\lambda(u,v)$, such that $v^*(0,0) = 0$, $v^*(S_0 u, \lambda) = \sigma v^*(u,\lambda)$. Also, by uniqueness of solution,

$$v^*(\alpha e_0, \lambda) = 0. \tag{6.82}$$

It follows that $\Phi_{r,\lambda}(\alpha e_0) := \Psi_\lambda(\alpha e_0, v^*_\lambda(\alpha e_0)) = \Psi_\lambda(\alpha e_0, 0) = \alpha e_0$. □

One can also show that if (6.79) holds, then Φ_λ can be brought in normal form up to any order $k \geq 1$: $\Phi_\lambda(x) = \Phi_\lambda^{NF}(x) + O(||x||^{k+1})$, so that also

$$\Phi_\lambda^{NF}(\alpha e_0) = \alpha e_0, \quad \forall \alpha \in \mathbb{R}, \forall \lambda \in \mathbb{R}. \tag{6.83}$$

The proof is similar to that of the RPNF Theorem 2. The basic observation here is that if $\Psi \in GL_{+R}(n, \mathbb{R})$ satsfies $\Psi(\alpha e_0) = \alpha e_0$ then $\mathrm{Ad}(\Psi) \cdot \Phi_\lambda$ is R-reversible and is also such that $\mathrm{Ad}(\Psi) \cdot (\Phi_\lambda)(\alpha e_0) = \alpha e_0$. For completeness, we provide the details of the proof in the linear case only.

Define $GL^*_{\pm R}(n, \mathbb{R}) \subset GL_{\pm R}(n, \mathbb{R})$ and $gl^*_{\pm}(n, \mathbb{R}) \subset gl_{\pm}(n, \mathbb{R})$ by

$$GL^*_{\pm}(n, \mathbb{R}) := \left\{ \Psi \in GL_{\pm}(n, \mathbb{R}) | \, \Psi(\alpha e_0) = \alpha e_0, \, \forall \alpha \in \mathbb{R}, , e_0 \in \mathrm{Fix} R \right\} \tag{6.84}$$

$$gl^*_{\pm}(n, \mathbb{R}) := \left\{ \Psi \in gl_{\pm}(n, \mathbb{R}) | \, \Psi(\alpha e_0) = 0, \, \forall \alpha \in \mathbb{R}, , e_0 \in \mathrm{Fix} R \right\}. \tag{6.85}$$

Note that (6.79) implies $\mathcal{N}_0 \in gl^*_{-R}(n, \mathbb{R})$.

Lemma 6.10. *Let $A_0 = S_0 \exp(\mathcal{N}_0)$, with $S_0 e_0 = e_0$ be the SU-decomposition of $A_0 \in Gl^*_{-R}(n, \mathbb{R})$. Then there exist a neighbourhood $U \subset Gl^*_{-R}(n, \mathbb{R})$ of A_0 and a mapping $\Psi : U \to Gl_{+R}(n, \mathbb{R})$ such that $\Psi(\alpha e_0) = \alpha e_0$, $\quad \Psi(A_0) = \mathrm{I}$, and*

$$\mathrm{Ad}\,(\Psi(A)) \cdot A = A_0 \exp(C(A)), \tag{6.86}$$

for some $C : U \to \ker\,(\mathrm{Ad}(S_0) - \mathrm{I}) \cap gl^*_{-R}(n, \mathbb{R})$, with $C(A_0) = 0$.

Proof. Let $\Psi \in Gl^*_{+R}(n, \mathbb{R})$ be such that $\Psi = e^\psi$, with $\psi \in gl^*_{+R}(n, \mathbb{R})$, i.e. $\psi(\alpha e_0) = 0$ and $R \psi R = \psi$. Define $F : gl^*_{+R}(n, \mathbb{R}) \times Gl^*_{-R}(n, \mathbb{R}) \to Gl^*_{-R}(n, \mathbb{R})$ by $F(\psi, A) := \mathrm{Ad}(e^\psi) \cdot A$. Write then

$$\mathrm{Ad}(e^\psi) \cdot A = A_0 e^{f(\psi A)}, \tag{6.87}$$

with $f : gl^*_{+R}(n, \mathbb{R}) \times Gl^*_{-R}(n, \mathbb{R}) \to gl^*_{-RA_0}(n, \mathbb{R})$ defined by

$$f(\psi, A) := \log(g(\psi, A)), \quad \text{where } g(\psi, A) := e^{\mathrm{Ad}(A_0^{-1})\psi} (A_0^{-1} A) e^{-\psi}.$$

Now, $f(0, A_0) = 0$ and $D_\psi f(0, A_0) = \mathrm{Ad}(A_0^{-1}) - \mathrm{I}$. Considering the projection $\pi : gl(n, \mathbb{R}) \to \mathrm{Im}\,(\mathrm{Ad}(S_0) - \mathrm{I})$ one verifies that that

$$\pi \left(gl^*_{-RA_0}(n, \mathbb{R}) \right) = \mathrm{Im}\,(\mathrm{Ad}(S_0) - \mathrm{I}) \cap gl^*_{-RA_0}(n, \mathbb{R}), \tag{6.88}$$

compare with Lemma 4.11. Hence, we have to prove the existence of $\psi \in gl^*_{+R}(n,\mathbb{R})$ and $U \subseteq Gl^*_{-R}(n,\mathbb{R})$ so that

$$\pi\,(\psi, A) = 0, \quad \forall A \in U.$$

The map

$$\pi\,(f(\cdot,\cdot)) : \big(gl^*_{+R}(n,\mathbb{R}) \cap \mathrm{Im}\,(\mathrm{Ad}(S_0) - \mathrm{I})\big) \times Gl^*_{-R}(n,\mathbb{R}) \to \pi(gl^*_{-RS_0}(n,\mathbb{R}))$$

is such that $\pi(f(0, A_0)) = 0$ and $D_\psi\,(\pi(f(0, A_0)))$ is surjective on $gl^*_{+R}(n,\mathbb{R})$ $\cap \mathrm{Im}\,(\mathrm{Ad}(S_0) - \mathrm{I})$. So, by the Implicit Function Theorem there exist a neighbourhood $U \subset Gl^*_{-R}(n,\mathbb{R})$ of A_0 and $\psi : U \to gl^*_{+R}(n,\mathbb{R})$ such that $\psi(A_0) = 0$, $\pi(f(\psi(A), A)) = 0$ for all $A \in U$ and also $f(\psi(A), A) \cdot \alpha e_0 = 0$. Setting $C(A) := \pi(f(\psi(A), A))$ then proves the proposition. □

We conclude this section by observing that if Φ_λ verifies (6.79) and it is in normal form up to order k, then

$$A_{\lambda,\alpha} := D\Phi_\lambda(\alpha e_0) = D\Phi^{NF}_\lambda(\alpha e_0) + O(\alpha^k), \tag{6.89}$$

where in particular $D\Phi^{NF}_\lambda(\alpha e_0)$ commutes with S_0.

6.4 Period-doubling

We now analyse problem (P) when $q = 2$ under the assumptions (H1), (R) and (H4). To obtain bifurcating solution branches we use the normal form part of the equations, neglecting the higher order terms. A more careful analysis shows that the results obtained in this way persist when the neglected terms are taken into account.

We denote by u_0 the eigenvector corresponding to the eigenvalue 1 of A_0 such that $Ru_0 = u_0$. We also assume that

$$\boxed{\begin{array}{c} -1 \text{ is a non-semisimple eigenvalue of } A_0 \text{ with} \\ \text{algebraic multiplicity equal to } 2 \end{array}} \tag{6.90}$$

Let v_0 be the corresponding eigenvector, then $S_0 v_0 = -v_0$, $\mathcal{N}_0 v_0 = 0$, and since $\ker\,(A_0 + \mathrm{I})$ is R-invariant (by the reversibility) it holds that either $Rv_0 = v_0$ or $Rv_0 = -v_0$. Without loss of generality, we assume that $Rv_0 =$

v_0, since in the other case ($Rv_0 = -v_0$) it holds $RS_0v_0 = v_0$, and replacing R by RS_0 in what follows the same analysis goes through. Both $\ker(A_0+\mathrm{I})$ and $\ker((A_0+\mathrm{I})^2)$ are R-invariant and there exists a one dimensional complement W_0 of $\mathbb{R}v_0$ in $\ker((A_0+\mathrm{I})^2)$ which is also R-invariant. It exists a unique element $w_0 \in W_0$ such that $A_0w_0 = -w_0 - v_0$, from which it follows that $A_0Rw_0 = -Rw_0 + v_0$, and hence $Rw_0 = -w_0$. The reduced phase space U is then given by

$$U = \{\alpha u_0 + \beta v_0 + \gamma w_0 | \, \alpha, \beta, \gamma \in \mathbb{R}\}, \tag{6.91}$$

and the restrictions of $S_0, \mathcal{N}_0$ and R to U by

$$\begin{aligned} S_0(\alpha u_0 + \beta v_0 + \gamma w_0) &= \alpha u_0 - \beta v_0 - \gamma w_0, \\ \mathcal{N}_0(\alpha u_0 + \beta v_0 + \gamma w_0) &= \gamma v_0, \\ R(\alpha u_0 + \beta v_0 + \gamma w_0) &= \alpha u_0 + \beta v_0 - \gamma w_0. \end{aligned}$$

The normal form vector filed Z, (see Theorem 2), restricted to U, can be written as

$$Z(\alpha u_0 + \beta v_0 + \gamma w_0) = g(\alpha, \beta, \gamma)u_0 + h_1(\alpha, \beta, \gamma)v_0 + h_2(\alpha, \beta, \gamma)w_0, \tag{6.92}$$

with functions g, h_1, h_2 which are of second order in the origin (since $Z(0) = 0$) and $DZ(0) = 0$, and such that

$$g(\alpha, -\beta, -\gamma) = g(\alpha, \beta, \gamma), \quad g(\alpha, \beta, -\gamma) = -g(\alpha, \beta, \gamma) \tag{6.93}$$
$$h_1(\alpha, -\beta, -\gamma) = -h_1(\alpha, \beta, \gamma), \quad h_1(\alpha, \beta, -\gamma) = -h_1(\alpha, \beta, \gamma) \tag{6.94}$$
$$h_2(\alpha, -\beta, -\gamma) = -h_2(\alpha, \beta, \gamma) \quad h_2(\alpha, \beta, -\gamma) = h_2(\alpha, \beta, \gamma) \tag{6.95}$$

(these correspond to the conditions that Z commutes with S_0 and anti-commutes with R). In order to impose the additional constraint of $DZ_\lambda(x) \cdot \mathcal{N}_0^T x = \mathcal{N}_0^T Z_\lambda(x)$ we can use a scalar product on U such that the basis $\{u_0, v_0, w_0\}$ of U is orthonormal (compare with Lemma 1.1). Then $\mathcal{N}_0^T(\alpha u_0 + \beta v_0 + \gamma w_0) = \beta w_0$, and the constraint takes the form

$$\beta\frac{\partial g}{\partial \gamma}(\alpha, \beta, \gamma) = 0, \quad \beta\frac{\partial h_1}{\partial \gamma}(\alpha, \beta, \gamma) = 0, \quad \beta\frac{\partial h_2}{\partial \gamma}(\alpha, \beta, \gamma) = h_1(\alpha, \beta, \gamma).$$

It follows that

$$g(\alpha, \beta, \gamma) = 0, \quad h_1(\alpha, \beta, \gamma) = 0, \quad h_2(\alpha, \beta, \gamma) = \beta\varphi(\alpha, \beta^2), \tag{6.96}$$

with $\varphi : \mathbb{R}^2 \to \mathbb{R}$ smooth and such that $\varphi(0,0) = 0$. The solutions of $\mathcal{N}_0(u) + Z_\lambda(u) = 0$ are given by either $(\alpha,\beta,\gamma) = (\alpha,0,0)$ with $\alpha \in \mathbb{R}$ small (since $S_0\alpha u_0 = \alpha u_0$ these correspond to the primary branch), or by $(\alpha,\beta,\gamma) = (\alpha,\beta,0)$ with $(\alpha,\beta) \in \mathbb{R}^2$ such that

$$\varphi(\alpha,\beta^2) = 0. \tag{6.97}$$

If we also assume the *transversality condition*

$$\frac{\partial\varphi}{\partial\alpha}(0,0) \neq 0, \tag{6.98}$$

then one verifies that the linear operator $\mathcal{N}_0 + Z_\lambda(\alpha u_0)|_U$ has eigenvalues 0 and $\pm\sqrt{\varphi(\alpha,0)}$, corresponding respectively to the multipliers 1 and $-\exp(\pm\sqrt{\varphi(\alpha,0)})$ along the primary branch. Th transversality condition then means that as one moves along the primary branch two complex conjugate eigenvalues move along the unit circle and with non-zero speed towards -1, and after colliding split off the unit circle into a pair of real multipliers moving away from -1 along the real axis, one inside and one outside the unit circle. The condition (6.98) allows us to solve (6.97) for $\alpha = \alpha^*(\beta^2)$, giving a solution branch of the determining equation. For fixed $\beta \neq 0$ the solutions $(\alpha^*(\beta^2), \pm\beta, 0)$ correspond to the two points of a symmetric 2-periodic orbit of Φ. For the original system this means that a single branch of symmetric periodic orbits bifurcates from the primary branch; the limiting period along this branch is $2T_0$ and so we have period-doubling.

Remark We can also determine the (linear) stability of these bifurcating solutions. Writing $\varphi(\alpha,\beta^2) = C(\alpha) + D(\alpha)\beta^2 + O(\beta^4)$, one finds that the eigenvalues of $\big(\mathcal{N}_0 + DZ(\alpha^*(\beta^2)u_0 + \beta v_0)\big)|_U$ are given by 0 and $\pm|\beta|\sqrt{2D(0)} + O(\beta^2)$. Taking the exponential gives two critical multipliers along the bifurcating branch: these are real and off the unit circle if $D(0) > 0$ (*instability*), and they lie on the unit circle if $D(0) < 0$ (*stability*). So the stability is determined by the sign of the third order coefficient in the normal form.

6.5 Proof of SBSRU Theorem 6

Assume (H1), (R) and (H5), our aim is to solve problem (P) for $q \geq 3$ and so prove the SBSRU Theorem 6.

By Theorem 1 we have to solve the branching equation $\mathcal{B}(u,\lambda)=0$ on the reduced phase space U. To this purpose, we first solve the truncated normal form of the branching equations and then, using techniques similar to those in section 6.1, we solve in section 6.5.2 the exact (branching) equation.

The reduced space U is 3-dimensional and can be written as the direct sum of the one-dimensional $U_0 := \ker(S_0 - \mathrm{I})$ and the two-dimensional subspace $U_q := \ker\left((A_0-\cos\theta_0\mathrm{I})^2+\sin^2\theta_0\mathrm{I}\right)$, which is R-invariant. Recall that $\theta_0 := 2\pi p/q$. Let now $\xi_0 \in U_q$ be an eigenvector of R: $R\xi_0=\epsilon\xi_0$, with $\epsilon=\pm1$. Setting $\xi_1 := (\sin\theta_0)^{-1}(A_0-\cos\theta_0 I)\,\xi_0$ one has

$$R\xi_1 = (\sin\theta_0)^{-1}\left(A_0^{-1}-\cos\theta_0\mathrm{I}\right)\xi_0, \quad R\xi_0=-\epsilon\xi_1. \tag{6.99}$$

So we can assume that $\epsilon=1$ (interchange ξ_0 and ξ_1 in other case), giving the basis $\{e_0,\xi_0,\xi_1\}$ of $U=U_0\oplus U_q$ such that $Re_0=e_0$, $R\xi_0=\xi_0$, $R\xi_1=-\xi_1$ and $\mathcal{N}_0e_0=0$, $\mathcal{N}_0\xi_i=0$, $i=0,1$, $S_0e_0=e_0$ and

$$S_0\xi_0=\cos\theta_0\xi_0+\sin\theta_0\xi_1, \quad S_0\xi_1=-\sin\theta_0\xi_0+\cos\theta_0\xi_1. \tag{6.100}$$

Bifurcating q-periodic points can be approximated by determining the equilibria of the normal form system on U, $\dot u = Z(u)$, where the normal form vector field $Z(u)$ commutes with S_0 and anti-commutes with R (see Theorem 2). To find the form of $Z(u)$ we identify the 3-dimensional reduced space U with $\mathbb{R}\times\mathbb{C}$ via the mapping $\varphi:\mathbb{R}\times\mathbb{C}\to U$ given by

$$\varphi(\alpha,z) := \alpha e_0 + \mathrm{Re}\left(z(\xi_0-i\xi_1)\right); \tag{6.101}$$

then

$$S_0(\alpha,z)=(\alpha,\exp(i\theta_0)z) \quad \text{and} \quad R(\alpha,z)=(\alpha,\overline{z}).$$

The system $\dot u = Z(u)$ then takes the form

$$\frac{d}{dt}(\alpha e_0+\mathrm{Re}(z\xi)) = Z(\alpha e_0+\mathrm{Re}(z\xi)) = f_0(\alpha,z)+\mathrm{Re}(g_0(\alpha,z)\xi)$$

where $f_0:\mathbb{R}\times\mathbb{C}\to\mathbb{R}$ and $g_0:\mathbb{R}\times\mathbb{C}\to\mathbb{C}$ are such that

$$\begin{aligned} &f_0(0,0)=0, \quad f_0(\alpha,\chi_q z)=f_0(\alpha,z), \quad f_0(\alpha,\overline{z})=-f_0(\alpha,z)\\ &g_0(0,0)=0, \quad g_0(\alpha,\chi_q z)=\chi_q g_0(\alpha,z), \quad g_0(\alpha,\overline{z})=-\overline{g_0(\alpha,z)}. \end{aligned}$$

By Lemma 6.2 we can write

$$g_0(\alpha,z)=ig_1(\alpha,z)+ig_2(\alpha,z)\overline{z}^{q-1}$$

with $g_i : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ $(i = 1, 2)$ S_0- and R-invariant, and $g_1(0, 0) = 0$. While by [25](appendix), we can write

$$f_0(\alpha, z) = f(\alpha, z)\mathrm{Im}(z^q)$$

with $f : \mathbb{R}\times\mathbb{C} \to \mathbb{R}$ S_0- and R-invariant. It follows that the system $\dot{u} = Z(u)$ takes the explicit form

$$\begin{aligned} \dot{\alpha} &= f(\alpha, z)\mathrm{Im}(z^q) \\ \dot{z} &= ig_1(\alpha, z) + ig_2(\alpha, z)\overline{z}^{q-1}, \end{aligned}$$

where the functions $f : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ and $g_i : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ $(i = 1, 2)$ are S_0- and R-invariant, and $g_1(0, 0) = 0$. We also assume that

$$g_2(0, 0) \neq 0 \tag{6.102}$$

Setting $z = r \exp i\theta$ the above system is equivalent to

$$\begin{aligned} \dot{\alpha} &= r^q f(\alpha, \exp(i\theta) \sin(q\theta)), \\ \dot{r} &= r^{q-1} g_2(\alpha, r \exp(i\theta)) \sin(q\theta), \\ \dot{\theta} &= g_1(\alpha, r \exp(i\theta)) + r^{q-2} g_2(\alpha, r \exp(i\theta)) \cos(q\theta). \end{aligned} \tag{6.103}$$

Now, the α-axis forms a line of equilibria, corresponding to the primary branch and $DZ(\alpha e_0)|_U$ has eigenvalues 0 and $\pm g_1(\alpha, 0)$ corresponding to the eigenvalues 1 and $\pm i(\theta_0 + g_1(\alpha, 0))$ along the primary branch. Assume that

$$\frac{\partial g_1}{\partial \alpha}(0, 0) \neq 0; \tag{6.104}$$

this *transversality condition* means that as we move along the primary branch a pair of simple eigenvalues moves with non-zero speed along the unit circle, passing through the root of unity $\exp(\pm i\theta_0)$ for $\alpha = 0$.

From (6.103) and (6.102) it follows that non-trivial solution (i.e. with $r \neq 0$) are such that $\sin(q\theta) = 0$, i.e.

$$\theta = 0 \bmod \theta_0, \quad \text{and} \quad \theta = \frac{\pi}{q} \bmod \theta_0. \tag{6.105}$$

For $\theta = 0$ (mod θ_0) the bifurcation equation reduces to

$$g_1(\alpha, r) + r^{q-2} g_2(\alpha, r) = 0, \tag{6.106}$$

and for $\theta = \frac{\pi}{q} \pmod{\theta_0}$ to

$$g_1(\alpha, r\exp(i\pi/q)) - r^{q-2} g_2(\alpha, r\exp(i\pi/q)) = 0. \tag{6.107}$$

Under the transversality condition, both equations can be solved for α as a function of r, giving respectively $\alpha = \alpha_1^*(r)$ for (6.106) and $\alpha = \alpha_2^*(r)$ for (6.107). Then, the solution set is given by the union of

$$\gamma_1 := \{(\alpha, z) = (\alpha_1^*(r), r\exp(ik\theta_0)) |\, r > 0, 0 \leq k \leq q-1\} \tag{6.108}$$

and

$$\gamma_2 := \{(\alpha, z) = (\alpha_2^*(r), r\exp(i(\pi/q + k\theta_0))) |\, r > 0, 0 \leq k \leq q-1\}. \tag{6.109}$$

It is important to note that for each fixed $r_0 > 0$ the intersection of γ_1 with $r = r_0$ is S_0- and R-invariant; therefore, it corresponds to a symmetric q-periodic orbit of Φ. The same holds for γ_2, and since $\alpha_0^*(1) = \alpha_2^*(0) = 0$, we have found two symmetric branches bifurcating from the primary branch.

Remark Observe that the two branches are *close* to one another for high value of q; in fact, it holds $\alpha_2^*(r) = \alpha_1^* + O(r^{q-2})$ as $r \to 0$.

6.5.1 Stability of the Subharmonics

To determine the stability of these solutions, we linearize (6.103) at the points of γ_1 and γ_2 and calculate the eigenvalues of this linearization.

Direct calculations give that next to the trivial eigenvalue 0, there are a pair of non-trivial eigenvalues $\pm\sqrt{\mu_1(r)}$ along γ_1, and another pair $\pm\sqrt{\mu_2(r)}$ along γ_2. To simplify the expression of $\mu_1(r)$ and $\mu_2(r)$, we set

$$\delta := g_2(0,0), \quad \tau := \frac{\partial g_1}{\partial \alpha}(0,0), \quad \gamma := f(0,0).$$

Also, expanding $g_1(\alpha, z) = g_1(\alpha, 0) + \tilde{g}_1(\alpha) r^2 + O(r^3)$, we set $\nu := \tilde{g}_1(0)$. Then, one finds

$$\alpha_{1,2}^*(r) = -\frac{\nu}{\tau} r^2 \mp \frac{\delta}{\tau} r^{q-2} + O(r^3) \tag{6.110}$$

and

$$\mu_{1,2}(r) = \pm q(\gamma\tau + 2\nu q)r^q + q(q-2)\delta^2 r^(2q-4) + O(r^{q+1}). \quad (6.111)$$

For $q = 3$ both branches will be unstable, for $q = 4$ the sign of $\mu_{1,2}$ depends on all the constants involved, but for $q \geq 5$ we have

$$\alpha^*_{1,2}(r) = -\frac{\nu}{\tau}r^2 + O(r^3) \quad \text{and} \quad \mu_{1,2}(r) = \pm q(\gamma\tau + 2\nu\delta)r^q +)(r^{q+1});$$

hence, assuming $\gamma\tau + 2\nu\delta \neq 0$ implies that the branches γ_1 and γ_2 have opposite stability properties: one is stable, the other is unstable.

6.5.2 SBSRU Continued

Inspired by the methods used in subsection 6.1, we show that the results obtained in section 6.5 persist when the neglected (higher order) terms are taken into account. We start with some linear algebra considerations and then solve the branching equation $\mathcal{B}(u, \lambda) = 0$, see (1.12).

In the hypotheses of section 6.5, we identify the 3-dimensional reduced space U with $\mathbb{R} \times \mathbb{C}$ again via the mapping $\varphi : \mathbb{R} \times \mathbb{C} \to U$ given by

$$\varphi(\alpha, z) := \alpha e_0 + \operatorname{Re}\left(z(\xi_0 - i\xi_1)\right);$$

so that

$$S_0(\alpha, z) = (\alpha, \exp(i\theta_0)z) \quad \text{and} \quad R(\alpha, z) = (\alpha, \overline{z}),$$

compare with (6.101). Let A_α be the linearization of Φ along the primary branch, i.e. $A_\alpha := D\Phi(\alpha e_0)$, $(\alpha \in \mathbb{R})$ and recall that $\Phi(\alpha e_0) = \alpha e_0$. Using the normal form Theorem 2 (the linear version of it), one calculates that the eigenvalues of $A_\alpha|_U$ are

$$1 \quad \text{and} \quad \exp\left(\pm i(\theta_0 + \beta_q(\alpha))\right), \quad \text{with } \beta_q(0) = 0, \quad (6.112)$$

recall that $\theta_0 := 2\pi p/q$. Assume the *transversality condition*

$$\beta_q'(0) \neq 0. \quad (6.113)$$

For notational convenience denote by $\xi = \xi_0 - i\xi_1$, then the branching function (1.11) as the form

$$\mathcal{B}(\alpha e_0 + \operatorname{Re}(z\xi)) = b_0(\alpha, z)e_0 + \operatorname{Re}(b_1(\alpha, z)\xi), \quad (6.114)$$

where $b_0 : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ and $b_1 : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ satisfy

$$b_0(0,0) = 0, \quad b_0(\alpha, \chi_q z) = b_0(\alpha, z), \quad b_0(\alpha, \overline{z}) = -b_0(\alpha, z) \tag{6.115}$$

$$b_1(0,0) = 0, \quad b_1(\alpha, \chi_q z) = \chi_q b_1(\alpha, z),\ b_1(\alpha, \overline{z}) = -\overline{b_1(\alpha, z)}, \tag{6.116}$$

the solutions of the branching equation $\mathcal{B}(\alpha, z) = 0$ then must satisfy

$$b_0(\alpha, z) = 0 \quad \text{and} \quad b_1(\alpha, z) = 0. \tag{6.117}$$

By Lemma 6.2 we can write

$$b_1(\alpha, z) := iz\theta_1(z, \alpha) + i\overline{z}^{q-1}\theta_2(z, \alpha),$$

with $\theta_i(\alpha, z) = \theta_i(\alpha, \overline{z}) = \theta_i(\alpha, \chi_q z)$, $i = 1, 2$, and $\theta_1(0,0) = 0$. The relations (6.115) imply the existence of $\tilde{b} : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ with the same properties of the θ_i and such that

$$b_0(\alpha, z) = \operatorname{Im}(\overline{z}^q)\tilde{b}(\alpha, z),$$

see [25] (appendix) for a proof. Hence, we are reduced to solve the system of two equations

$$\operatorname{Im}(\overline{z}^q)\tilde{b}(\alpha, z) = 0, \tag{6.118}$$

$$iz\theta_1(\alpha, z) + i\overline{z}^{q-1}\theta_2(z, \alpha) = 0, . \tag{6.119}$$

Remarks

1- Note that $z = 0$ is a trivial solution corresponding with the the solutions along the primary branch.

2- Note that if $\tilde{b}(0,0) \neq 0$ then the non-trivial solutions of (6.118) must satisfy $\operatorname{Im}(z^q) = 0$, i.e., $\sin(q\varphi) = 0$ when $z = \rho \exp(i\varphi)$ in polar coordinates.

3- Note that $z = \rho \exp(i\varphi)$ $(z \neq 0)$ with $\varphi = \dfrac{k\pi}{q}$, for some $k \in \mathbb{Z}$, $0 \leq k \leq q-1$, gives two possibilities: $z = \rho \exp(i2j\frac{\pi}{q})$, $0 \leq j \leq q-1$, or $z = \rho \exp\left(i(\frac{\pi}{q} + 2j\frac{\pi}{q})\right)$, $0 \leq j \leq q-1$.

Returning to equation (6.119), assume $\rho > 0$, and take $z = \rho \exp(i\varphi)$. Multiplication by $\overline{z}$ and division by ρ^2 then give

$$\theta_1(\alpha, z) + \theta_2(\alpha, z)\rho^{q-2} \exp(-iq\varphi) = 0.$$

Splitting into real and imaginary part yields

$$\theta_1(\alpha,z)+\theta_2(\alpha,z)\rho^{q-2}\cos(q\varphi)=0 \tag{6.120}$$
$$\theta_2(\alpha,z)\sin(q\varphi)=0. \tag{6.121}$$

If $\theta_2(0,0)\neq 0$, equation (6.121) gives the lines $\mathrm{Im}(z^q)=0$. We are therefore reduced to solve the equation (6.120) along the lines $\mathrm{Im}(z^q)=0$. That is, we have to solve the two scalar equations (independent on j)

$$\theta_1(\alpha,\rho)+\theta_2(\alpha,\rho)\rho^{q-2}=0,$$
$$\theta_1(\alpha,\rho\exp(ip/q))-\theta_2(\alpha,\rho\exp(ip/q))\rho^{q-2}=0.$$

In the hypothesis that $D_\alpha\theta_1(0,0)\neq 0$, knowing that $\theta_1(0,0)=0$, these equations can both be solved by the Implicit Function Theorem for α as a function of ρ. Hence, the same conclusion as in section 6.5 holds and Theorem 6 is proved.

The fact that $D_\alpha\theta_1(0,0)\neq 0$ follows from the transversality condition (6.113) by calculations similar to those in section 6.1 (compare with (6.11)). Indeed, $\theta_1(0,\alpha)=D_z b_1(0,\alpha)$ and from (6.112) it follows that $\theta_1(0,\alpha)=e^{i\beta_q(\alpha)}-e^{-i\beta_q(\alpha)}$.

Note that in the hypotheses that $\tilde{b}(0,0)\neq 0$ and $\theta_2(0,0)\neq 0$ the solutions we found are the only solutions of the problem.

6.6 Proof of SBRRU Theorem 7

In this section we solve problem (P) for a family of reversible mappings satisfying (H1), (H4), (H6) and (H6-a).

Application of the reduction Theorem 1 as before shows that we are left with a 5-dimensional problem on $U=\ker(S_0^q-\mathrm{I})$, $\dim U=5$. We proceed stepwise as in section 6.2 and solve the branching equation by means of the Implicit Function Theorem in section 6.6.1.

The reduced space U is the direct sum of the one-dimensional $U_0:=\ker(S_0-\mathrm{I})$ and the four-dimensional subspace

$$U_q:=\ker\left(\left((A_0-\cos\theta_0\mathrm{I})^2+\mathrm{I}\sin^2\theta_0\right)^2\right),$$

which is R-invariant; here $\theta_0:=2\pi p/q$. Compare also with section 6.5.

Now, the space U_0 can be identified with $\{\alpha e_0 | \, \alpha \in \mathbb{R}\}$, where e_0 is the eigenvector corresponding to the eigenvalue 1 of S_0 with $Re_0 = e_0$ and $\mathcal{N}_0 e_0 = 0$. For the 4-dimensional subspace U_q of U we can find a basis $\mathcal{B}_{Uq} = \{e_1, e_2, f_1, f_2\}$ where

$$\begin{aligned}
&\mathcal{N}_0 e_i = 0, \quad \mathcal{N}_0 f_i = e_i, \quad i = 1,2\\
&J_0 e_1 = e_2, \quad J_0 e_2 = -e_1,\\
&J_0 f_1 = f_2, \quad J_0 f_2 = -f_1,\\
&R f_1 = f_1, \quad R f_2 = -f_2,\\
&R e_1 = -e_1, \quad R e_2 = e_2,
\end{aligned}$$

with $J_0 := \sin(\theta_0)^{-1}(S_0 - \mathrm{I}\cos(\theta_0))$, compare with Lemma 6.6. Using the basis $B_U = \{e_0, e_1, e_2, f_1, f_2\}$, we may identify U with $\mathbb{R}^5$ and get the following explicit representations

$$R \;=\; \begin{pmatrix} 1 & 0\;0 & 0\;0 \\ 0 & & \\ 0 & -R_1 & \mathrm{O} \\ 0 & & \\ 0 & \mathrm{O} & R_1 \end{pmatrix} \in gl(5, \mathbb{R}), \tag{6.122}$$

$$\mathcal{N}_0 \;=\; \begin{pmatrix} 0 & 0\;0 & 0\;0 \\ 0 & & \\ 0 & \mathrm{O} & \mathrm{I}_2 \\ 0 & & \\ 0 & \mathrm{O} & \mathrm{O} \end{pmatrix} \in gl_{-R}(5, \mathbb{R}), \tag{6.123}$$

$$S_0 \;=\; \begin{pmatrix} 1 & 0\;0 & 0\;0 \\ 0 & & \\ 0 & R(\theta_0) & \mathrm{O} \\ 0 & & \\ 0 & \mathrm{O} & R(\theta_0) \end{pmatrix} \in Gl_{-R}(4, \mathbb{R}), \tag{6.124}$$

where $R_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and

$$R(\theta_0) := \begin{pmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{pmatrix} = \exp(\theta_0 J_2) = I\cos\theta_0 + J_2 \sin\theta_0,$$

with $J_2 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Using the explicit form of $A_0 = S_0 \exp(\mathcal{N}_0)$ with respect to the basis B_U one directly shows that

$$\ker\left(\mathrm{ad}(\mathcal{N}_0^T)\right) \cap \ker\left(\mathrm{ad}(S_0)\right) \cap gl_{-R}(5,\mathbb{R}) = \{B(\vartheta,\sigma) \,|\, \vartheta,\sigma \in \mathbb{R}\}, \tag{6.125}$$

with

$$B(\vartheta,\sigma) := \begin{pmatrix} 0 & 0 \;\; 0 & 0 \;\; 0 \\ \begin{matrix} 0 \\ 0 \end{matrix} & \vartheta J_2 & \mathrm{O} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \sigma \mathrm{I}_2 & \vartheta J_2 \end{pmatrix}. \tag{6.126}$$

This in combination with the RPNF Theorem 2 yields

$$A(\alpha,\lambda)|_U = S_0 \exp(\mathcal{N}_0 + B(\vartheta(\alpha,\lambda),\sigma(\alpha,\lambda))) \tag{6.127}$$

with $\vartheta : \mathbb{R}^2 \to \mathbb{R}$, $\sigma : \mathbb{R}^2 \to \mathbb{R}$ such that $\vartheta(0,0) = 0$ and $\sigma(0,0) = 0$. Note that $\alpha \in \mathbb{R}$ is the coordinate on the primary branch, and it plays the role of an internal (or specified) parameter. One calculates that $A(\alpha,\lambda)|_U$ has next to the simple eigenvalue 1, either

(i) a pair of double eigenvalues on the unit circle

$$\exp\left(\pm i(\theta_0 + \vartheta(\alpha,\lambda))\right), \quad \text{if } \sigma(\alpha,\lambda) = 0; \tag{6.128}$$

or

(ii) a quadruplet of simple complex eigenvalues off the unit circle

$$\exp\left(\pm i(\theta_0 + \vartheta(\alpha,\lambda))\right) \exp\left(\pm\sqrt{\sigma(\alpha,\lambda)}\right), \quad \text{if } \sigma(\alpha,\lambda) > 0; \tag{6.129}$$

or

(iii) a quadruplet of simple complex eigenvalues on the unit circle

$$\exp\left(\pm i(\theta_0 + \vartheta(\alpha,\lambda))\right) \exp\left(\pm i\sqrt{\tilde{\sigma}(\alpha,\lambda)}\right), \tag{6.130}$$

if $\sigma(\alpha,\lambda) = -\tilde{\sigma}(\alpha,\lambda)$ with $\tilde{\sigma}(\alpha,\lambda) > 0$.

Transversality condition Assume

$$\frac{\partial(\sigma,\vartheta)}{\partial(\alpha,\lambda)} \neq 0. \tag{6.131}$$

Hence we can take $(\alpha,\lambda) := (\sigma,\vartheta)$ as new parameters and write

$$B(\vartheta(\alpha,\lambda),\sigma(\alpha,\lambda)) = \begin{pmatrix} 0 & 0\ \ 0 & 0\ \ 0 \\ 0 & \multirow{2}{*}{αJ_2} & \multirow{2}{*}{O} \\ 0 & & \\ 0 & \multirow{2}{*}{$\lambda \mathrm{I}_2$} & \multirow{2}{*}{αJ_2} \\ 0 & & \end{pmatrix} = B(\alpha,\lambda). \tag{6.132}$$

6.6.1 Solution of the Branching Equation

By Theorem 1 proving the SBRRU Theorem 7 reduces to solving the corresponding branching equation $\mathcal{B}(u,\lambda) = 0$, $(u,\lambda) \in U \times \mathbb{R}$. To do so, let $e := e_1 + ie_2$ and $f := f_1 + if_2$ and identify U_q with $\mathbb{C} \times \mathbb{C}$ by the mapping

$$(\phi,\psi) : U \to \mathbb{C} \times \mathbb{C}, \quad (z,w) \mapsto (\phi(z),\psi(w)), \tag{6.133}$$

where

$$\phi(z) := \mathrm{Re}(z(e_1 + ie_2)) \quad \text{and} \quad \psi(w) := \mathrm{Re}(w(f_1 + if_2)). \tag{6.134}$$

So, for all $(\alpha,z,w) \in U$ we have that

$$R(\alpha,z,w) = (\alpha,\overline{z},-\overline{w}) \quad \text{and} \quad S_0(\alpha,z,w) = (\alpha,\exp(i\theta_0)z,\exp(i\theta_0)w).$$

Hence, the branching function (1.11) $\mathcal{B}_\lambda : \mathbb{R} \times \mathbb{C} \times \mathbb{C} \to \mathbb{R} \times \mathbb{C} \times \mathbb{C}$ is such that

$$(\alpha,z,w,\lambda) \quad \mapsto \quad (\mathcal{B}_{0,\lambda}(\alpha,z,w,\lambda),\mathcal{B}_{1,\lambda}(\alpha,z,w,\lambda),\mathcal{B}_{2,\lambda}(\alpha,z,w,\lambda))$$

with

$$\begin{aligned} \mathcal{B}_2\,(\alpha,\overline{z},-\overline{v},\lambda) &= \overline{\mathcal{B}_2(\alpha,z,v,\lambda)}, \\ \mathcal{B}_j\,(\alpha,\overline{z},-\overline{v},\lambda) &= -\overline{\mathcal{B}_j(\alpha,z,v,\lambda)} \end{aligned} \tag{6.135}$$

for $j = 0,1$ and

$$\begin{aligned} \mathcal{B}_0(\alpha,e^{i\theta_0}z,e^{i\theta_0}w,\lambda) &= \mathcal{B}_0(\alpha,z,w,\lambda), \\ \mathcal{B}_k(\alpha,e^{i\theta_0}z,e^{i\theta_0}w,\lambda) &= e^{i\theta_0}\mathcal{B}_k(\alpha,z,w,\lambda), \end{aligned} \tag{6.136}$$

with $k = 1, 2$. Our goal is to solve the equation $\mathcal{B}_\lambda(\alpha, z, w) = 0$ or equivalently the system of three equations (one real and two complex):

$$\mathcal{B}_{j,\lambda}(\alpha, z, w) = 0, \quad j = 0, 1, 2.$$

Observe that along the primary branch (i.e. for $(\alpha, z, v) = (\alpha, 0, 0)$ with $\alpha \in \mathbb{R}$ small but arbitrary) the real equation $\mathcal{B}_{0,\lambda}(\alpha, z, v) = 0$ is automatically fulfilled.

We focus first on the two complex equations. Consider

$$\mathcal{B}_j(\alpha, z, v, \lambda) = 0, \quad j = 1, 2$$

where $\alpha \in \mathbb{R}$ is taken into account as parameter, $\lambda \in \mathbb{R}$ and $(z, v) \in \mathbb{C} \times \mathbb{C}$. Note that (6.127) implies that

$$\mathcal{B}(z, v, \alpha, 0) = 2 \begin{pmatrix} 0 & 0\ 0 & 0\ 0 \\ 0 & & \\ 0 & \alpha J_2 & \mathrm{I}_2 \\ 0 & & \\ 0 & \mathrm{O} & \alpha J_2 \end{pmatrix} \begin{pmatrix} \alpha \\ z \\ v \end{pmatrix} + \text{h.o.t.} \tag{6.137}$$

and similarly

$$\mathcal{B}(z, v, 0, \lambda) = 2 \begin{pmatrix} 0 & 0\ 0 & 0\ 0 \\ 0 & & \\ 0 & \mathrm{O} & \mathrm{I}_2 \\ 0 & & \\ 0 & \lambda \mathrm{I}_2 & \mathrm{O} \end{pmatrix} \begin{pmatrix} \alpha \\ z \\ v \end{pmatrix} + \text{h.o.t.} \tag{6.138}$$

Observe that $\mathcal{B}_j(0, 0, \lambda, \alpha) = 0$ $(j = 1, 2)$ and $D_{(\alpha,z,v)}\mathcal{B}(0, 0, 0, 0) = 2\mathcal{N}_0$. Now, $\mathcal{N}_0 \begin{pmatrix} \alpha \\ z \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix}$, then $D_v\mathcal{B}_1(0, 0, 0, 0) \cdot \tilde{v} = 2\mathcal{N}_0\tilde{v}$ and hence the equation $\mathcal{B}_1(z, v, \lambda, \alpha) = 0$ can be solved for v by the Implicit Function Theorem. That is, there exists a solution $v = v^*(z, \lambda, \alpha)$, such that the mapping $v^* : \mathbb{C} \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ satisfies

$$\begin{aligned} v^*(0, \lambda, \alpha) &= 0, \\ v^*(S_0 e, \lambda, \alpha) &= S_0 v^*(z, \lambda, \alpha), \\ v^*(\overline{z}, \lambda, \alpha) &= -\overline{v^*(z, \lambda, \alpha)}. \end{aligned} \tag{6.139}$$

Note also that $v^*(z,\alpha,0) = -\alpha J_2 z + \text{h.o.t.}$

Remark Observe that $v = v^*(z,\lambda)$ solution of $\mathcal{B}_1(z,v,\lambda) = 0$ implies that

$$\mathcal{B}_1\left(\overline{z}, -\overline{v^*(z,\lambda)}, \lambda\right) = 0.$$

From which it follows that $v^*(\overline{z},\lambda) = -\overline{v^*(z,\lambda)}$. Substituting in $\mathcal{B}_2(z,v,\lambda)$ gives $\mathcal{B}_2(z,\lambda) := \mathcal{B}_2(z,v^*(z,\lambda),\lambda)$ such that

$$\begin{aligned}
\mathcal{B}_2(\overline{z},\lambda) &= \mathcal{B}_2\left(\overline{z}, v^*(\overline{z},\lambda),\lambda\right) \\
&= \mathcal{B}_2\left(\overline{z}, -\overline{v^*(z,\lambda)}, \lambda\right) = \overline{\mathcal{B}_2\left(z, v^*(z,\lambda),\lambda\right)} \\
&= \overline{\mathcal{B}_2(z,\lambda)}
\end{aligned} \tag{6.140}$$

Replacing v by $v^*(z,\lambda,\alpha)$ in $\mathcal{B}_2(z,v,\lambda,\alpha)$ yields the complex equation

$$\mathcal{B}_2(z,\lambda,\alpha) := \mathcal{B}_2(z,v^*(z,\lambda,\alpha),\lambda) = 0.$$

The properties (6.135)-(6.136) (and (6.140)) imply that

$$\mathcal{B}_2(z,\lambda,\alpha) = z\theta_1(z,\lambda,\alpha) + \overline{z}^{q-1}\theta_2(z,\lambda,\alpha),$$

with $\theta_i : \mathbb{C}\times\mathbb{R}^2 \to \mathbb{R}$ such that

$$\theta_i\left(e^{i\theta_0}z,\lambda,\alpha\right) = \theta_i(z,\lambda,\alpha) = \theta_i(\overline{z},\lambda,\alpha), \quad i = 1,2,$$

and

$$\theta_1(0,0,0) = 0.$$

The problem is thus to solve the equation

$$z\theta_1(z,\lambda,\alpha) + \overline{z}^{q-1}\theta_2(z,\lambda,\alpha) = 0, \quad z\in\mathbb{C},\ \lambda,\alpha\in\mathbb{R}.$$

Note that the linear part of this equation is $\theta_1(0,\lambda,\alpha)$. Setting $z = \rho e^{i\varphi}$ and $\overline{\theta}_i(\rho,\varphi,\lambda,\alpha) := \theta_i(\rho e^{i\varphi},\lambda,\alpha)$ gives

$$\rho e^{i\varphi}\overline{\theta}_1(\rho,\varphi,\alpha,\lambda) + \rho^{q-1}e^{-i(q-1)}\overline{\theta}_2(\rho,\varphi,\alpha,\lambda) = 0.$$

Again, this is equivalent to the system of real equations

$$\overline{\theta}_1(\rho,\varphi,\lambda,\alpha) + \overline{\theta}_2(\rho,\varphi,\lambda,\alpha)\rho^{q-2}\cos(q\varphi) = 0 \tag{6.141}$$

$$\overline{\theta}_2(\rho,\varphi,\lambda,\alpha)\sin(q\varphi) = 0 \tag{6.142}$$

Suppose that $\overline{\theta}_2(0, \varphi, 0, 0) \neq 0$, then (6.142) is automatically fulfilled along the lines

$$\varphi_1 = j\frac{2\pi}{q}, \quad \varphi_2 = \frac{\pi}{q} + j\frac{2\pi}{q} \quad j = 0, \dots, q-1. \tag{6.143}$$

For φ fixed equal to either φ_1 or φ_2, equation (6.141) reads

$$h(\rho, \lambda, \alpha) := \overline{\theta}_1(\rho, \varphi, \lambda, \alpha) + \overline{\theta}_2(\rho, \varphi, \lambda, \alpha)\rho^{(q-2)} \cos(q\varphi) = 0. \tag{6.144}$$

For simplicity of notation, we denote $\overline{\theta_i}$ again by θ_i. Then one verifies that

$$\theta_1(0, \lambda, \alpha) \cdot \mathrm{I} = D_z\mathcal{B}_2(0, \lambda, \alpha), \quad h(0,0,0) = \theta_1(0,0,0) = 0$$

and

$$\frac{\partial}{\partial\lambda}h(0,0,0) = \frac{\partial}{\partial\lambda}\theta_1(0,0,0) = D_\lambda D_z\mathcal{B}_2(0,0,0).$$

From (6.138) it follows that

$$\frac{\partial}{\partial\lambda}\theta_1(0,0,0) \neq 0, \tag{6.145}$$

hence one can solve (6.144) by the Implicit Function Theorem. That is, there exists a solution $\lambda = \lambda^*(\rho, \alpha)$ such that $h(\rho, \alpha, \lambda^*(\rho, \alpha)) = 0$ and with $\lambda^*(0, 0) = 0$.

So, we are left with the real equation $\mathcal{B}_0(\alpha, z, v, \lambda) = 0$. Now, observe that if $v = v^*(z, \lambda, \alpha)$ then (6.135)-(6.136) in combination with (6.139) imply that

$$\mathcal{B}_{0,\lambda}(\alpha, z) = -\mathcal{B}_{0,\lambda}(\alpha, \overline{z}) \quad \text{and} \quad \mathcal{B}_{0,\lambda}(\alpha, z) = \mathcal{B}_{0,\lambda}(\alpha, \chi_q z)$$

where we denoted $\mathcal{B}_{0,\lambda}(\alpha, z, v^*(z, \lambda, \alpha)) := \mathcal{B}_{0,\lambda}(\alpha, z)$. Hence by [25] (appendix) there exists $\tilde{B}_{0,\lambda} : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$, with the same properties of the θ_i above, such that

$$\mathcal{B}_{0,\lambda}(\alpha, z) = \mathrm{Im}(\overline{z}^q)\tilde{B}_{0,\lambda}(\alpha, z). \tag{6.146}$$

Assuming that

$$\tilde{B}_{0,0}(0,0) \neq 0 \tag{6.147}$$

then the non-trivial solutions of $\mathcal{B}_0(\alpha, z, v, \lambda) = 0$ must satisfy $\mathrm{Im}(z^q) = 0$, i.e., $\sin(q\varphi) = 0$ when $z = \rho\exp(i\varphi)$ in polar coordinates. Therefore, this equation is automatically fulfilled along the lines (6.143).

We conclude by underlying that the solutions we found are the only ones under the assumptions $\tilde{B}_{0,0}(0,0) \neq 0$ and $\overline{\theta}_2(0, \varphi, 0, 0) \neq 0$.

Part II

Persistence of quasi periodic orbits in families of reversible systems with a 1:1 resonance

7

Persistence Problem

We consider quasi periodic tori in reversible systems, where the normal linear part has a 1:1 resonance, both the generic and the semisimple case. The corresponding quasi periodic bifurcation involves the linear centralizer unfolding of the corresponding infinitesimal reversible matrix.

7.1 Introduction

The main issue of the Kolmogorov Arnold Moser (KAM) theory is the persistence of quasi-periodic invariant tori in integrable systems for small near-integrable perturbations. Here we are interested in the occurrence of quasi periodicity in the class of reversible systems. The term *integrable* refers to a toroidal symmetry of the system, which implies that the invariant tori in the integrable approximation are of Floquet type: the normal linear part is constant over the tori. By a simple scaling device, this perturbation problem can be translated to the case where 'integrable' is replaced by 'linear and integrable' (i.e. of Floquet type) and where the perturbation is of general reversible form, compare [14]. Several authors, e.g. [3, 58, 59, 60, 64, 51, 12], addressed the problem in the case where the eigenvalues of the normal linear part (*normal eigenvalues*) are simple. Also see [14] and many references therein, as well as [66]. The main purpose is here to generalize their results to the 1:1 resonance case, i.e. when the normal eigenvalues coincide in one complex conjugate pair on the imaginary axis.

We mention a few relevant issues of our approach. One element is the presence of parameters. Indeed, as in [51, 12], we consider families of vector fields where in the integrable approximation the frequencies of the invariant tori vary with the parameters. This property is part of a wider nondegeneracy condition of Kolmogorov type, involving the whole nonlinear part. Already in [51, 14, 12] it turned out that a central part of the nondegeneracy condition is that the matrices in the normal linear part should form a versal unfolding in the sense of [3]. To be more specific, an extensive use could be made of the so–called linear centralizer unfolding. Another element is the construction of a conjugacy between the integrable approximating family and its perturbation, restricted to a foliation of invariant tori, parametrized over a 'Cantor set' of positive measure, cf., e.g., [51]. The Cantor set is defined by *Diophantine conditions*, necessary to compensate for small divisors. Following [64], both foliation and conjugacy are smooth in the sense of Whitney, meaning that they can be extended as smooth maps of a full neighbourhood. Since the conjugacy also is close to the identity map, this implies that the perturbed system inherits a Cantor foliation of invariant tori of positive measure. The existence of such a conjugacy can be viewed as a kind of structural stability, for this occasion called *quasi-periodic stability*. Also, the persistent tori are Floquet and the normal linear part is preserved.

The invariant foliations generally live in the product of phase space and parameter space. In a few cases all parameters can be 'compensated' by phase space variables, but not in general. In [51, 12, 13] the matrix of the normal linear part has only simple eigenvalues and the linear centralizer unfolding is parametrized by these same eigenvalues, also see [14]. The general flavour of the KAM results in these settings is persistence of Cantor foliations of quasi periodic tori, parametrized by these eigenvalues.

Presently, these eigenvalues are no longer simple. Indeed, we focus on the normal $1 : 1 : \ldots : 1$ resonance, and show that the above approach of, e.g., [51, 12] still works to a large extent. First we study the linear centralizer unfolding of the normal matrix, both in the generic and in the semisimple case. In the normal linear theory this already leads to bifurcations, where both normal ellipticity and hyperbolicity do occur, see Fig. 7.1. Next we develop KAM theory that again leads to Cantor foliations of tori, associated to the corresponding parametrisation.

Our subsequent interest is with branching off of invariant tori of 1 dimension higher in the elliptic parameter region. This excitation of normal modes is suggested by a normal form approximation, which leads to a *reversible quasi periodic Hopf bifurcation* in turn. For similar, but simpler bifurcations of this type compare [16]. For Hamiltonian analogues see [45, 35, 20]

7.2 Preliminaries

In this section we describe the framework for our further analysis and give precise definitions of a number of terms introduced in the previous section.

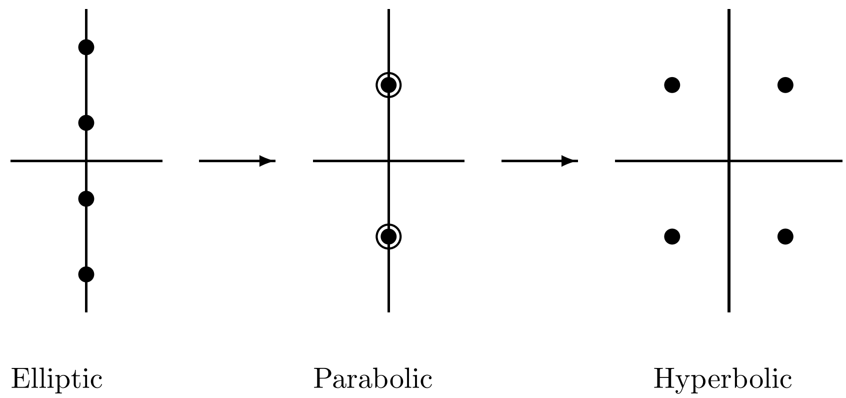

Figure 7.1: *Position of the eigenvalues in a generic* $1:1$ *resonance. A dot denotes a single eigenvalue, a circle-dot a double eigenvalue.*

We start with a suitable family of integrable reversible vector fields which is then perturbed by a small non-integrable but still reversible perturbation. Using a localization procedure such as in [51, 14] , the unperturbed vector field can be brought into an appropriate Floquet-like form having an invariant *zero-torus*. The main goal is to study the persistence of this invariant torus under the non-integrable perturbation.

7.2.1 Framework

We work throughout with the phase space $M = \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2p}$, where $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ is the n-torus with coordinates $x = (x_1, \ldots, x_n) \pmod{2\pi}$, while on $\mathbb{R}^m$ and $\mathbb{R}^{2p}$ the coordinates are respectively $y = (y_1, \ldots, y_m)$ and $z = (z_1, \ldots, z_{2p})$. A vector field on M takes the form

$$\dot{x} = f(x, y, z), \qquad \dot{y} = g(x, y, z), \qquad \dot{z} = h(x, y, z),$$

or in shorthand notation:

$$X(x, y, z) = f(x, y, z)\partial_x + g(x, y, z)\partial_y + h(x, y, z)\partial_z. \tag{7.1}$$

We assume that the vector field X depends analytically on all variables, including possible parameters which we suppress for the moment. Referring to [64, 51, 14], note that our results remain valid when ‘analyticity’ is replaced by ‘a sufficiently high degree of differentiability’.

To define reversibility, consider an involution $G : M \to M$ on M of the form

$$G(x, y, z) = (-x, y, Rz), \tag{7.2}$$

with $R \in \mathcal{L}(\mathbb{R}^{2p})$ a linear involution on $\mathbb{R}^{2p}$ (i.e. $R^2 = I$) such that

$$\dim \operatorname{Fix}(R) = p. \tag{7.3}$$

The vector field X is called *G-reversible* (or reversible for short) if

$$G_*(X) = -X.$$

Using (7.1) this reversibility condition takes the explicit form

$$\begin{aligned} f(-x, y, Rz) &= f(x, y, z), \\ g(-x, y, Rz) &= -g(x, y, z), \\ h(-x, y, Rz) &= -Rh(x, y, z), \end{aligned} \tag{7.4}$$

for all $(x, y, z) \in M$.

Following [51, 14, 12] the vector field X is called *integrable* if it is equivariant with respect to the natural action $(x_0, (x, y, z)) \in \mathbb{T}^n \times M \mapsto (x + x_0, y, z) \in M$ of $\mathbb{T}^n$ on M, or in other words, if the functions f, g and h in (7.1) are independent of the x-variable. Such integrable vector field

$$X(x, y, z) = f(y, z)\partial_x + g(y, z)\partial_y + h(y, z)\partial_z \tag{7.5}$$

is reversible if

$$f(y, Rz) = f(y, z), \;\; g(y, Rz) = -g(y, z), \;\; h(y, Rz) = -Rh(y, z) \tag{7.6}$$

for all $(y, z) \in \mathbb{R}^m \times \mathbb{R}^{2p}$. This implies $g(y, z) = 0$ for all $(y, z) \in \mathbb{R}^m \times \operatorname{Fix}(R)$.

Now suppose that $h(y_0, z_0) = 0$ for some $(y_0, z_0) \in \mathbb{R}^m \times \operatorname{Fix}(R)$; then the n-torus $\mathbb{T}^n \times \{y_0\} \times \{z_0\}$ is invariant under the flow of the vector field X. If moreover the derivative $D_z h(y_0, z_0) \in \mathcal{L}(\mathbb{R}^{2p})$ is invertible then by the Implicit Function Theorem the equation $h(y, z) = 0$ has for each $y \in \mathbb{R}^m$ close to y_0 a unique solution $z = \tilde{z}(y) \in \mathbb{R}^{2p}$ close to z_0. The uniqueness of this solution together with the reversibility condition (7.6) implies that $\tilde{z}(y) \in \operatorname{Fix}(R)$. As a consequence, not only $h(y, \tilde{z}(y)) = 0$ but also $g(y, \tilde{z}(y)) = 0$ for all $y \in \mathbb{R}^m$ near y_0, meaning that for all such y the n-torus $T_y := \mathbb{T}^n \times \{y\} \times \{\tilde{z}(y)\}$ is invariant under the flow of the vector field X. This shows that *under the assumption that $D_z h(y_0, z_0)$ is non-singular the existence of a single X-invariant torus implies the existence of an m-parameter family of such invariant tori.* Observe that the last reversibility condition in (7.6) implies that $D_z h(y, 0) \in \mathcal{L}(\mathbb{R}^{2p})$ maps $\mathrm{Fix}(R)$ into $\mathrm{Fix}(-R)$, and $\mathrm{Fix}(-R)$ into $\mathrm{Fix}(R)$; hence the invertibility of $D_z h(y_0, z_0)$ can only be satisfied if $\dim \mathrm{Fix}(R) = \dim \mathrm{Fix}(-R)$, i.e. if $\dim \mathrm{Fix}(R) = p$, cf. (7.3). This was the reason for introducing this condition in the first place.

The family of invariant tori $\{T_y := \mathbb{T}^n \times \{y\} \times \{\tilde{z}(y)\} \mid y \in \mathbb{R}^m\}$ can be brought in a more convenient form by using the diffeomorphism

$$\Psi : M \longrightarrow M, \quad (x, y, z) \longmapsto \Psi(x, y, z) := (x, y, \tilde{z}(y) + z)$$

which is G-equivariant and also commutes with the $\mathbb{T}^n$-action on M. Therefore the pull-back $\Psi^*(X)$ is still G-reversible and integrable, while $\Psi^{-1}(T_y) = \mathbb{T}^n \times \{y\} \times \{0\}$ is for each $y \in \mathbb{R}^m$ near y_0 a $\Psi^*(X)$-invariant n-torus. Modulo this transformation and denoting $\Psi^*(X)$ again by X, we can without loss of generality assume that $h(y, 0) = 0$ for all y in some open subset of $\mathbb{R}^m$, such that $T_y := \mathbb{T}^n \times \{y\} \times \{0\}$ is an X-invariant n-torus for all y in the same subset.

Our goal is to determine which of the invariant tori T_y can be continued into an invariant torus for small non-integrable (but still reversible) perturbations of X.

When trying to answer the persistence problem it is convenient to focus on (a sufficiently small neighborhood of) each of the invariant tori T_ν ($\nu \in \mathbb{R}^m$) separately, considering the label $\nu \in \mathbb{R}^m$ of the chosen torus as a parameter. Formally this can be done by a localizing transformation, setting

$$y = \nu + y_{loc} \qquad \text{and} \qquad X_{loc}(x, y_{loc}, z; \nu) := X(x, \nu + y_{loc}, z).$$

So we get a parametrized family of reversible and integrable vector fields, still on the same state space M. In this localized situation the focus is on the persistence in a small neighborhood of the invariant torus T_0, corresponding to $(y_{loc}, z) = (0, 0)$. For simplicity the additional parameter ν is absorbed with other possible parameters (which were suppressed until now), and we also drop the subscript "*loc*".

After localization the persistence problem takes the following form: given an analytic family

$$X(x, y, z, \lambda) = f(y, z, \lambda)\partial_x + g(y, z, \lambda)\partial_y + h(y, z, \lambda)\partial_z \tag{7.7}$$

of reversible and integrable vector fields on M, with parameter λ belonging to an open subset $P \subset \mathbb{R}^q$ and such that, for all $(y, \lambda) \in \mathbb{R}^m \times P$,

$$h(y, 0, \lambda) = 0 \quad \text{and} \quad D_z h(y, 0, \lambda) \in \mathcal{L}(\mathbb{R}^{2p}) \text{ is invertible,} \tag{7.8}$$

which of the X-invariant tori $V_\lambda := T_0 \times \{\lambda\}$ $(\lambda \in P)$ will persist under an appropriately small reversible perturbation $\widetilde{X}$ of X which is not necessarily integrable?

7.2.2 Normal Linearity

In this subsection we perform one further transformation which allows us to restrict to the case where the unperturbed (integrable and reversible) vector field X is *Floquet normal linear form*, meaning that the flow of X is linear in the z-direction normal to the family T_y $(y \in \mathbb{R}^m)$ of invariant tori (see (7.10) for the precise expression). The approach used here is inspired by the treatment in [51]. We start with a general observation.

Suppose that a vector field X on M (say given by (7.1)) leaves the torus T_0 invariant; then it can be written in the form

$$\begin{aligned} X(x, y, z) = f(x, y, z)\partial_x &+ [g_1(x, y, z)y + g_2(x, y, z)z]\,\partial_y \\ &+ [h_1(x, y, z)y + h_2(x, y, z)z]\,\partial_z, \end{aligned}$$

with g_1, g_2, h_1 and h_2 appropriate linear operators. Let $T_{T_0}(M)$ be the restriction of the tangent bundle $T(M)$ of M to T_0, $T(T_0)$ the tangent bundle of T_0, and $N(T_0) := T_{T_0}(M)/T(T_0)$; we call $N(T_0)$ the *normal bundle* of T_0 in M. The coordinates (x, y, z) on M can also be used as coordinates on $N(T_0)$, and the vector field X induces a vector field on $N(T_0)$ given by

$$\begin{aligned} N(X)(x, y, z) = f(x, 0, 0)\partial_x &+ [g_1(x, 0, 0)y + g_2(x, 0, 0)z]\,\partial_y \\ &+ [h_1(x, 0, 0)y + h_2(x, 0, 0)z]\,\partial_z; \end{aligned}$$

we call $N(X)$ the *normal linear vector field* induced by X on $N(T_0)$. Clearly

$$g_1(x, 0, 0) = \frac{\partial g}{\partial y}(x, 0, 0) \qquad \text{and} \qquad g_2(x, 0, 0) = \frac{\partial g}{\partial z}(x, 0, 0),$$

while similar expressions hold for h_1 and h_2. Therefore $N(X)$ can be seen as the normal linearization of X with respect to T_0.

Returning to the case of the reversible and integrable vector field (7.7), consider for each $\epsilon > 0$ the scaling operator

$$\mathcal{D}_\epsilon : M \longrightarrow M,\ (x,y,z) \longmapsto \left(x, \frac{y}{\epsilon}, \frac{z}{\epsilon^2}\right); \tag{7.9}$$

this operator commutes with G and with the T^n-action on M, and hence preserves reversibility and integrability. Using (7.5) and the linearity of $\mathcal{D}_\epsilon$ the push-forward $(\mathcal{D}_\epsilon)_* X$ of X under $\mathcal{D}_\epsilon$ takes the form

$$\begin{aligned}(\mathcal{D}_\epsilon)_* X(x,y,z,\lambda) &= \mathcal{D}_\epsilon \left(X\big(\mathcal{D}_\epsilon^{-1}(x,y,z),\lambda\big)\right)\\ &= f(\epsilon y, \epsilon^2 z, \lambda)\partial_x + \frac{1}{\epsilon} g(\epsilon y, \epsilon^2 z, \lambda)\partial_y\\ &\quad + \frac{1}{\epsilon^2} h(\epsilon y, \epsilon^2 z, \lambda)\partial_z.\end{aligned}$$

By (7.6) and (7.8) then $N_0(X) := \lim_{\epsilon\to 0} (\mathcal{D}_\epsilon)_* X$ is given by

$$N_0(X)(x,y,z,\lambda) = \omega(\lambda)\partial_x + \Omega(\lambda)\, z\, \partial_z, \tag{7.10}$$

with

$$\omega(\lambda) = f(0,0,\lambda) \quad \text{and} \quad \Omega(\lambda) = D_z h(0,0,\lambda), \qquad \forall \lambda \in P. \tag{7.11}$$

The vector field $N_0(X)$ is again reversible and integrable. For a fixed value of λ, it is characterized by the frequency vector $\omega(\lambda) = (\omega_1(\lambda), \ldots, \omega_n(\lambda)) \in \mathbb{R}^n$ which describes the flow along the invariant tori $T_y = \mathbb{T}^n \times \{y\} \times \{0\}$ ($y \in \mathbb{R}^m$), and by the matrix $\Omega(\lambda) \in gl(2p;\mathbb{R}) = \mathcal{L}(\mathbb{R}^{2p})$ which determines the linear flow in the z-direction normal to the family of invariant tori.

The Floquet matrix $\Omega(\lambda)$ appearing in (7.10) is not arbitrary, since, it follows from the reversibility of the vector field X that $R\Omega(\lambda) = -\Omega(\lambda)R$. So, $\Omega(\lambda)$ is *infinitesimally R-reversible.* The subspace of such infinitesimally reversible linear operators on $\mathbb{R}^{2p}$ is denoted by $gl_{-R}(2p;\mathbb{R})$. In a similar way $gl_{+R}(2p;\mathbb{R})$ the subspace of all R-equivariant linear operators on $\mathbb{R}^{2p}$, i.e.

$$gl_{\pm R}(2p;\mathbb{R}) := \{\Omega \in gl(2p;\mathbb{R}) \mid \Omega R = \pm R\Omega\}, \tag{7.12}$$

see also Chapter 2.

Observe that if $\mu \in \mathbb{C}$ is an eigenvalue of any $\Omega \in gl_{-R}(2p;\mathbb{R})$ then so is $-\mu$. Moreover, if $\Omega \in gl_{-R}(2p;\mathbb{R})$ is invertible (which by (7.8) is the case for the $\Omega(\lambda)$ appearing in (7.10)) then the eigenvalues of Ω can be grouped into either complex quadruples which are symmetric with respect to the real and the imaginary axis, or in conjugate purely imaginary pairs, or in symmetric real pairs. All eigenvalues within each such group have the same eigenspace structure. The family X given by (7.7) has a $1:1:\ldots:1$-resonance (with 1 appearing $r \leq p$ times) at $\lambda = \lambda_0$ when $\Omega(\lambda_0)$ has a pair of purely imaginary eigenvalues $\pm i\kappa$ $(\kappa > 0)$ with algebraic multiplicity equal to r. We mainly concentrate on the case of a $1:1$-resonance.

The normal linear vector field $N_0(X)$ plays a central role in obtaining the persistence results we are aiming for as follows. Under appropriate conditions and using an appropriate topology in a suitable space of reversible vector fields the normal linear vector field $N_0(X)$ has a neighborhood $\mathcal{U}$ consisting of vector fields which are conjugate to $N_0(X)$ on a set of the form $T_0 \times \mathcal{C}$, where $\mathcal{C} \subset P$ is a Cantor subset of the parameter space P. This implies that we obtain a "Cantor set of $N_0(X)$-invariant tori" (namely $\{T_0 \times \{\lambda\} \mid \lambda \in \mathcal{C}\}$) which persist under sufficiently small perturbations of $N_0(X)$. In order to obtain a similar result for the original vector field X one chooses a sufficiently small $\epsilon > 0$ such that $(\mathcal{D}_\epsilon)_* X \in \mathcal{U}$; then a conjugacy such as just described also exists between $(\mathcal{D}_\epsilon)_* X$ and all vector fields in $\mathcal{U}$. Finally by applying the pull-back $(\mathcal{D}_\epsilon)^*$ one concludes that all vector fields in the neighborhood $\mathcal{U}_\epsilon := (\mathcal{D}_\epsilon)^* (\mathcal{U})$ of X are conjugate on an appropriate Cantor set of X-invariant tori to X itself, i.e., the X-invariant tori in this Cantor set will persist under small perturbations.

Hence it is sufficient to prove our persistence results for vector fields on M which are in a Floquet normal linear form such as (7.10), see Fig. 7.2.2. For simplicity we replace the notation $N_0(X)$ by X, i.e., we consider a family of reversible and integrable vector fields of the form

$$X(x, y, z, \lambda) = \omega(\lambda)\partial_x + \Omega(\lambda) z\, \partial_z, \tag{7.13}$$

with $(x, y, z) \in M = \mathbb{T} \times \mathbb{R}^m \times \mathbb{R}^{2p}$ and $\lambda \in P \subset \mathbb{R}^q$, and where the mappings $\omega : P \to \mathbb{R}^n, \lambda \mapsto \omega(\lambda)$ and $\Omega : P \to gl_{-R}(2p; \mathbb{R}), \lambda \mapsto \Omega(\lambda)$ are assumed to be analytical. We also assume that $\det \Omega(\lambda) \neq 0$ for all $\lambda \in P$.

7.2.3 Non-degeneracy and Diophantine Conditions

In this subsection a non-degeneracy condition for the unperturbed vector field X is introduced, as well as the Diophantine conditions which determine the Cantor set $\mathcal{C} \subset P$ mentioned before. Non-degeneracy and diophantine

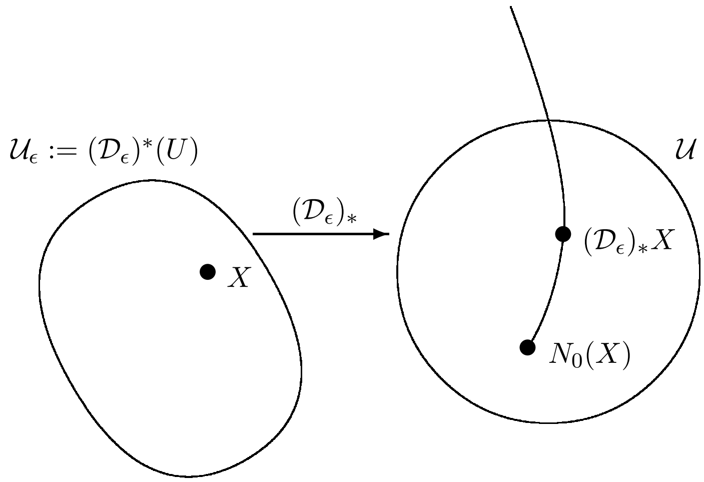

Figure 7.2: Transfer of the perturbation problem to the normal bundle.

conditions form two central themes of KAM theory, and we refer to [51, 14, 12, 13] for more background also about the approach followed here.

To define the non-degeneracy of the family (7.13) we first recall some facts from Chapter 2 and [3]. Consider the subspaces $gl_{+R}(2p;\mathbb{R})$ and $gl_{-R}(2p;\mathbb{R})$ of $gl(2p;\mathbb{R})$. The set $gl_{+R}(2p;\mathbb{R})$ forms a subalgebra of $gl(2p;\mathbb{R})$, corresponding to the subgroup $Gl_{+R}(2p;\mathbb{R}) := GL(2p;\mathbb{R}) \cap gl_{+R}(2p;\mathbb{R})$ of the Lie group $GL(2p;\mathbb{R})$. The Adjoint action of $GL(2p;\mathbb{R})$ on $gl(2p;\mathbb{R})$ is defined by

$$\begin{aligned}\mathrm{Ad} : GL(2p;\mathbb{R}) \times gl(2p;\mathbb{R}) &\longrightarrow gl(2p;\mathbb{R}),\\ (A,\Omega) &\longmapsto \mathrm{Ad}(A)\cdot\Omega := A\,\Omega\,A^{-1}; \end{aligned} \tag{7.14}$$

where both $gl_{+R}(2p;\mathbb{R})$ and $gl_{-R}(2p;\mathbb{R})$ are invariant under $\mathrm{Ad}(A)$ if $A \in GL_{+R}(2p;\mathbb{R})$. Consequently we can consider the adjoint action of $GL_{+R}(2p;\mathbb{R})$ on $gl_{-}(2p;\mathbb{R})$, and the orbit $\mathcal{O}(\Omega_0) := \{\mathrm{Ad}(A)\cdot\Omega_0 \mid A \in GL_{+R}(2p;\mathbb{R})\}$ of $\Omega_0 \in gl_{-}(2p;\mathbb{R})$ under this action. Since $GL_{+R}(2p;\mathbb{R})$ is algebraic it follows that $\mathcal{O}(\Omega_0)$ is a smooth submanifold of $gl_{-R}(2p;\mathbb{R})$. The tangent space at Ω_0 to this orbit is given by

$$\begin{aligned} T_{\Omega_0}\mathcal{O}(\Omega_0) &= \{\mathrm{ad}(A)\cdot\Omega_0 = A\Omega_0 - \Omega_0 A \mid A \in gl_{+R}(2p;\mathbb{R})\}\\ &= \mathrm{ad}(\Omega_0)\left(gl_{+R}(2p;\mathbb{R})\right), \end{aligned}$$

where we have used $\mathrm{ad}(A)\cdot\Omega = -\mathrm{ad}(\Omega)\cdot A$ for all $A,\Omega \in gl(2p;\mathbb{R})$.

An *unfolding* of Ω_0 is a smooth (analytic) mapping $\Omega : \mathbb{R}^s \to gl_{-R}(2p;\mathbb{R})$, $\mu \mapsto \Omega(\mu)$ such that $\Omega(0) = \Omega_0$. An unfolding is *versal* if it is transversal to $\mathcal{O}(\Omega_0)$ at $\mu = 0$, which requires that $s \geq \operatorname{codim} \mathcal{O}(\Omega_0)$. A versal unfolding with the minimal number of parameters (i.e. with s equal to the codimension of $\mathcal{O}(\Omega_0)$ in $gl_{-R}(2p;\mathbb{R})$) is called *universal.*

Using the Implicit Function Theorem one shows that given a universal unfolding $\Omega : \mathbb{R}^s \to gl_{-R}(2p;\mathbb{R})$ of $\Omega_0 \in gl_{-R}(2p;\mathbb{R})$, each $\widetilde{\Omega} \in gl_{-R}(2p;\mathbb{R})$ near Ω_0 can be written in the form $\widetilde{\Omega} = \operatorname{Ad}(A) \cdot \Omega(\mu)$ for some $(A, \mu) \in gl_{+R}(2p;\mathbb{R}) \times \mathbb{R}^s$ close to $(I, 0)$ and depending smoothly on $\widetilde{\Omega}$. For more details on versal and universal unfoldings we refer to [3, 39].

Definition 1. The parametrized vector field $X(x, y, z, \lambda) = \omega(\lambda)\partial_x + \Omega(\lambda)z\partial_z$ is *non-degenerate* at $\lambda = \lambda_0 \in \mathbb{R}^q$ if the mapping $\omega \times \Omega : P \to \mathbb{R}^n \times gl_{-R}(2p;\mathbb{R}), \lambda \mapsto (\omega(\lambda), \Omega(\lambda))$ is at $\lambda = \lambda_0$ transversal to $\{\omega(\lambda_0)\} \times \mathcal{O}\left(\Omega(\lambda_0)\right)$.

Such non-degeneracy requires that $q \geq n + \operatorname{codim} \mathcal{O}\left(\Omega(\lambda_0)\right)$. If all parameters originate from a localisation procedure as explained in subsection 7.2.1 this means that we should have $m \geq n + \operatorname{codim} \mathcal{O}\left(\Omega(\lambda_0)\right)$.

Assume now that $X(x, y, z, \lambda)$ is non-degenerate at $\lambda_0 \in \mathbb{R}^q$, and let (ω_0, Ω_0) $:= (\omega(\lambda_0), \Omega(\lambda_0))$. Using the results mentioned in the preceding paragraph together with a reparametrisation and a parameter-dependent linear transformation in the z-space, without loss of generality we assume that the parameter λ takes the form $\lambda = (\omega, \mu, \tilde{\mu})$ and belongs to a neighborhood P of $\lambda_0 := (\omega_0, 0, 0)$ in $\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^{q-n-s}$, while

$$X(x, y, z, \omega, \mu, \tilde{\mu}) = \omega\partial_x + \Omega(\mu)z\partial_z, \tag{7.15}$$

where $\Omega : \mathbb{R}^s \to gl_{-R}(2p;\mathbb{R})$ is a given universal unfolding of Ω_0. The $\tilde{\mu}$-part of the parameter does not appear in this expression for the (unperturbed) vector field X. Although it might appear explicitly in the perturbations. However it turns out that $\tilde{\mu}$ plays no role at all in the further analysis. Therefore from now on it is suppressed and we just keep the essential parameters (ω, μ) and set $P := \mathbb{R}^n \times \mathbb{R}^s$, with $s = \operatorname{codim} \mathcal{O}(\Omega(0))$. The question how to make a convenient particular choice for the universal unfolding $\Omega(\mu)$ appearing in (7.15) is handled later on.

The diophantine conditions we introduce involve the frequency vector $\omega \in \mathbb{R}^n$ as well as the imaginary parts of the eigenvalues of $\Omega(\mu) \in gl_{-R}(2p;\mathbb{R})$. Given $\Omega_0 \in gl_{-R}(2p;\mathbb{R})$ one can choose a *normal frequency mapping* ω^N : $gl_{-R}(2p;\mathbb{R}) \to \mathbb{R}^{2p}$ which is continuous in the neighbourhood of Ω_0 and such that the components of $\omega^N(\Omega)$ (with $\Omega \in gl_{-R}(2p;\mathbb{R})$ close to Ω_0) are equal to the imaginary parts of the eigenvalues of $\Omega \in gl_{-R}(2p;\mathbb{R})$. Higher multiplicities are taken into account by repeating eigenvalues as many times as necessary.

Definition 2. A pair $(\omega, \Omega) \in \mathbb{R}^n \times gl_{-R}(2p;\mathbb{R})$ is said to satisfy a *Diophantine condition* if there exists constants $\tau > n-1$ and $\gamma > 0$ such that

$$|\langle \omega, k\rangle + \langle \omega^N(\Omega), \ell\rangle| \geq \gamma |k|^{-\tau}, \tag{7.16}$$

for all $k \in \mathbb{Z}^n \setminus \{0\}$ and for all $\ell \in \mathbb{Z}^{2p}$ with $|\ell| \leq 2$. Here $\langle \omega, k\rangle = \sum_{j=1}^n \omega_j k_j$ and $|k| = \sum_{j=1}^n |k_j|$, with similar expressions for $\langle \omega^N, \ell\rangle$ and $|\ell|$.

The remarks which follow aim to clarify this definition.

Remarks

1- When applying the condition (7.16), the constant $\tau > n-1$ will be fixed and γ will have the role of a parameter which can be adjusted whenever necessary.

2- For small $\gamma > 0$ the diophantine subset

$$(\mathbb{R}^n \times \mathbb{R}^{2p})_\gamma := \{(\omega, \omega^N) \in \mathbb{R}^n \times \mathbb{R}^{2p} \mid |\langle \omega, k\rangle + \langle \omega^N, \ell\rangle| \geq \gamma |k|^{-\tau}, \\ \forall k \in \mathbb{Z}^n \setminus \{0\}, \forall \ell \in \mathbb{Z}^{2p} : |\ell| \leq 2\} \tag{7.17}$$

forms a nowhere dense subset of $\mathbb{R}^n \times \mathbb{R}^{2p}$ of large measure (see [58, 59, 14]). The same remains true when in this statement $\mathbb{R}^{2p}$ is replaced by any subspace V of $\mathbb{R}^{2p}$ which is defined by a finite number of equations of the form $\omega_i^N = 0$, $\omega_i^N = \omega_j^N$ or $\omega_i^N = -\omega_j^N$ $(1 \leq i, j \leq 2p,\ i \neq j)$.

3- The condition (7.16) is independent of the order of the components of $\omega^N(\Omega)$. Also, if (ω, Ω) satisfies (7.16) then the same is true for all $(\omega, \widetilde{\Omega})$ with $\widetilde{\Omega} \in \mathcal{O}(\Omega)$. Moreover, note that whenever $(\omega, \omega^N) \in (\mathbb{R}^n \times \mathbb{R}^{2p})_\gamma$ and $\lambda \geq 1$, then also $(\lambda\omega, \lambda\omega^N) \in (\mathbb{R}^n \times \mathbb{R}^{2p})_\gamma$.

4- Given a universal unfolding $\Omega(\mu)$ of $\Omega_0 = \Omega(0)$ (such as in (7.15)), $\omega^N(\mu)$ denotes the normal frequency vector $\omega^N(\Omega(\mu))$. The parameter values $\mu \in \mathbb{R}^s$ for which all eigenvalues of $\Omega(\mu)$ are simple form an open and dense subset of $\mathbb{R}^s$. The map $\mu \mapsto \omega^N(\mu)$ is at such parameter values a smooth submersion of $\mathbb{R}^s$ onto an appropriate subspace V of $\mathbb{R}^{2p}$ of the form mentioned in Remark 2- (remember the eigenvalue structure of $\Omega \in gl_{-R}(2p;\mathbb{R})$).

5- Combining the remarks 2- and 3- it follows that there is an open and dense subset of the parameter space $P = \mathbb{R}^n \times \mathbb{R}^s$ where the set

$$P_\gamma := \{(\omega, \mu) \in \mathbb{R}^n \times \mathbb{R}^s \mid (\omega, \omega^N(\mu)) \in (\mathbb{R}^n \times \mathbb{R}^{2p})_\gamma\} \qquad (7.18)$$

is nowhere dense but still of large measure. Of course this measure will increase by taking smaller values of γ. For simplicity we say that P_γ is a 'Cantor set'.

The last ingredient to complete the material needed for the formulation of the main results is that of the real analytic topology on spaces of real analytic families of reversible vector fields, which is the compact open topology on holomorphic extensions. See, e.g., [14, 13, 51].

7.3 Main Results

We treat the persistence problem for the case where Ω_0 has a $1:1$-resonance, i.e., assuming that $p = 2$ and that $\Omega_0 \in gl_{-R}(4; \mathbb{R})$ has a pair of purely imaginary eigenvalues $\pm i\kappa$ ($\kappa > 0$) with algebraic multiplicity two. At several points in the analysis two cases have to be distinguished: the generic (non-semisimple) case where the geometric multiplicity is equal to one, and the non-generic (semisimple) case where the geometric multiplicity equals two. In [12] a persistence result was established for the case where $\Omega_0 \in gl_{-R}(2p; \mathbb{R})$ has only simple eigenvalues (which implies that $\operatorname{codim} \mathcal{O}(\Omega_0) = p$) and where $m \geq n + p$. The present situation forms the simplest case where the simpleness assumption of [12] is not satisfied. Our approach already suggests how to obtain a persistence result which does not depend on the eigenvalue structure of $\Omega_0 \in gl_{-R}(2p; \mathbb{R})$, compare with [47] (see [6] for similar steps towards such a general persistence result). In this section we formulate the main results; proofs are postponed to the subsequent sections.

Since we have chosen $p = 2$ both R and $\Omega(\mu)$ will be represented by 4×4-matrices. Throughout we use block form, using (most of the time) the particular 2×2-matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, R_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and}$$

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (7.19)$$

as building blocks. Observe that $J^2 = -I$ and $JR_0 = -R_0 J$.

Our first result follows from some simple algebra. In the statement we assume that Ω_0 has $\pm i$ as eigenvalues with algebraic multiplicity two, i.e., we take $\kappa = 1$; this is no restriction, since a simple time rescale will multiply both ω and $\Omega(\mu)$ in (7.15) with the same scaling constant. Compare also with [66, 68, 13].

Proposition 7.1. *Assume that $\Omega_0 \in gl_{-R}(4;\mathbb{R})$ has $\pm i$ as eigenvalues with algebraic multiplicity two. Then there exists a basis of $\mathbb{R}^4$ with respect to which R has the form*

$$R = \begin{pmatrix} R_0 & O \\ O & R_0 \end{pmatrix}, \tag{7.20}$$

while Ω_0 takes the form

$$\Omega_0 = \begin{pmatrix} J & J \\ O & J \end{pmatrix} \tag{7.21}$$

when the geometric multiplicity is equal to 1*, and the form*

$$\Omega_0 = \begin{pmatrix} J & O \\ O & J \end{pmatrix} \tag{7.22}$$

when the geometric multiplicity is equal to 2*.*

We refer to the case (7.21) as the *generic* one (case (**G**) for short), and to (7.22) as the *semisimple* case (denoted by (**S**)).

Proof. We start proving (7.22). Let $U := \ker(\Omega_0^2 - \mathrm{I})$ and consider its direct sum splitting $U = U_+ \oplus U_-$, where $U_+ := \mathrm{Fix}(R)$ and $U_- := \mathrm{Fix}(-R)$. Let $B_{U_+} = \{u_i \in U_+ | \, 1 \leq i \leq k\}$ be a basis of U_+, so $Ru_i = u_i$. Define $v_i := -\Omega_0 u_i$, $1 \leq i \leq k$. Then,

$$Rv_i = -R\Omega_0 u_i = \Omega_0 R u_i = \Omega u_i = -v_i,$$

therefore, $v_i \in U_-$, and $\Omega_0 v_i = u_i$, $1 \leq i \leq k$. It follows that $\dim U_- \geq \dim U_+$. Now, interchanging the role of U_- and U_+ in the reasoning above, one gets $\dim U_+ \geq \dim U_-$. Hence, $\dim U_- = \dim U_+$ and

$$B_U := \{u_i, v_i \in U | \, 1 \leq i \leq k\}$$

is a basis of U with

$$\Omega_0 u_i = -v_i, \quad \Omega_0 v_i = u_i, \quad Ru_i = u_i, \quad Rv_i = -v_i. \tag{7.23}$$

With respect to this basis, Ω_0 and R take respectively the form (7.22) and (7.20).

To prove (7.21), let $U := \ker(\Omega_0^2 + \mathrm{I})^2$ and $U_1 := \ker(\Omega_0^2 + \mathrm{I})$. Note that $\dim U = 4$ and $\dim U_1 = 2$. Now, U_1 is R-invariant since $R(\Omega_0^2 + \mathrm{I}) = (\Omega_0^2 + \mathrm{I})R$. Also, $\Omega_0|_{U_1}$ is semisimple and therefore there exists a basis of U_1, $B_{U_1} = \{u_1, v_1\}$, such that

$$\Omega_0 u_1 = -v_1, \quad \Omega_0 v_1 = u_1, \quad R u_1 = u_1, \quad R v_1 = -v_1. \tag{7.24}$$

Now, let U_2 be an R-invariant complement of U_1 in U. We have that $(\Omega_0^2 + \mathrm{I})U_2 = U_1$ and $\ker(\Omega_0^2 + \mathrm{I})|U_2 = \{0\}$, hence $\Omega_0^2 + \mathrm{I}$ is an isomorphism of U_2 onto U_1. Choose the $u_2, v_2 \in U_2$ such that

$$(\Omega_0^2 + \mathrm{I})u_2 = 2\Omega_0 u_1 = -2v_1 \quad \text{and} \quad (\Omega_0^2 + \mathrm{I})v_2 = 2\Omega_0 v_1 = 2u_1.$$

Moreover, $Ru_2 = -u_2$ and $Rv_2 = v_2$, since

$$(\Omega_0^2 + \mathrm{I})Ru_2 = R(\Omega_0^2 + \mathrm{I})u_2 = -2Rv_1 = 2v_1$$

and

$$(\Omega_0^2 + \mathrm{I})Rv_2 = R(\Omega_0^2 + \mathrm{I})v_2 = 2Ru_1 = 2u_1.$$

Now, let $\Omega_0 u_2 = w_1 + w_2$, with $w_1 \in U_1$ and $w_2 \in U_2$. Then, $(\Omega_0^2 + \mathrm{I})\Omega_0 u_2 = (\Omega_0^2 + \mathrm{I})w_2 = \Omega_0(\Omega_0^2 + \mathrm{I})u_2 = -2\Omega_0 v_1 = -2u_1$, hence $w_2 = -v_2$. So, $\Omega_0 u_2 = -v_2 + w_1$ and similarly $\Omega_0 v_2 = u_2 + \widetilde{w}_1$, for some $w_1, \widetilde{w}_1 \in U_1$ to be determined. Now, from $R\Omega_0 u_2 = \Omega_0 u_2$ it follows that $Rw_1 = w_1$ and therefore $w_1 = \alpha u_1$, for some $\alpha \in \mathbb{R}$. Similarly, $R\widetilde{w}_1 = -\widetilde{w}_1$ and therefore $\widetilde{w}_1 = \beta v_1$, for some $\beta \in \mathbb{R}$. Since $(\Omega_0^2 + \mathrm{I})u_2 = -(\alpha + \beta)v_1 = -v_1$ and $(\Omega_0^2 + \mathrm{I})v_2 = (\alpha + \beta)u_1 = 2u_1$ then $\alpha + \beta = 2$. It follows that $B = \{u_1, v_1, u_2, v_2\}$ is a basis of U with

$$\begin{aligned}
&\Omega_0 u_1 = -v_1, \quad \Omega_0 v_1 = u_1,\\
&\Omega_0 u_2 = -v_2 + \alpha u_1,\\
&\Omega_0 v_2 = u_2 + \beta v_1,\\
&R u_1 = u_1, \quad R u_2 = -u_2,\\
&R v_1 = -v_1, \quad R v_2 = v_2.
\end{aligned} \tag{7.25}$$

To obtain (7.21) we have to go yet a step further. Let $\tilde{u}_2, \tilde{v}_2 \in U$ be such that $\tilde{u}_2 = u_2 + \gamma v_1$ and $\tilde{v}_2 = v_2 + \delta u_1$ for some γ, δ, with $R\tilde{u}_2 = -\tilde{u}_2$,

$R\tilde{v_2} = \tilde{v_2}$. Then,

$$\begin{aligned}\Omega_0 \tilde{u}_2 = -v_2 + \alpha u_1 + \gamma u_1 = -(v_2 + \delta u_1) + (\alpha + \gamma + \delta) u_1 \\ = -\tilde{v_2} + (\alpha + \gamma + \delta) u_1,\end{aligned}$$

and

$$\begin{aligned}\Omega_0 \tilde{v_2} = u_2 + \beta v_1 - \delta v_1 = (u_2 + \gamma v_1) + (\beta - \delta - \gamma) v_1 \\ = \tilde{u}_2 + (\beta - delta - \gamma) v_1.\end{aligned}$$

We require that

$$\begin{cases} \alpha + \gamma + \delta & = & 1 \\ \beta - \gamma - \delta & = & 1 \end{cases} \Longrightarrow \gamma + \delta = \frac{\beta - \alpha}{2}, \tag{7.26}$$

take for example $\gamma = \beta/2$ and $\delta = -\alpha/2$. It follows that

$$\begin{aligned}&\Omega_0 u_1 = -v_1, \quad \Omega_0 v_1 = u_1, \\ &\Omega_0 \tilde{u}_2 = -\tilde{v_2} + u_1, \\ &\Omega_0 \tilde{v_2} = \tilde{u}_2 + v_1, \\ &R u_1 = u_1, \quad R\tilde{u}_2 = -\tilde{u}_2, \\ &R v_1 = -v_1, \quad R\tilde{v_2} = \tilde{v_2}.\end{aligned} \tag{7.27}$$

So, (7.21) holds with respect to the basis $\{u_1, v_1, \tilde{u}_2, \tilde{v_2}\}$. □

From now on, fix a basis of $\mathbb{R}^4$ such that (7.20) and (7.21) (respectively (7.20) and (7.22)) hold. The next proposition gives a particular universal unfolding of Ω_0.

Proposition 7.2. *Under the assumptions of Proposition 7.1 the operator Ω_0 has codimension two in the generic case* (**G**)*, and codimension four in the semisimple case* (**S**)*. A universal unfolding of Ω_0 is given by*

$$\Omega(\mu) = \Omega_0 + \begin{pmatrix} \mu_1 J & O \\ \mu_2 J & \mu_1 J \end{pmatrix}, \qquad \mu = (\mu_1, \mu_2) \in \mathbb{R}^2, \tag{7.28}$$

in the generic case (**G**)*, and by*

$$\Omega(\mu) = \Omega_0 + \begin{pmatrix} \mu_1 J & \mu_3 J \\ \mu_2 J & \mu_4 J \end{pmatrix}, \qquad \mu = (\mu_1, \mu_2, \mu_3, \mu_4) \in \mathbb{R}^4, \tag{7.29}$$

in the semisimple case (**S**).

Observe that in both cases $\Omega(\mu) - \Omega_0$ depends linearly on the parameter μ and commutes with the semisimple part $\mathcal{S}_0$ of Ω_0, which is given by

$$\mathcal{S}_0 = \begin{pmatrix} J & O \\ O & J \end{pmatrix}. \tag{7.30}$$

In the generic case $\Omega(\mu) - \Omega_0$ also commutes with the transposed of the nilpotent part $\mathcal{N}_0$ of Ω_0, given by

$$\mathcal{N}_0 = \begin{pmatrix} O & J \\ O & O \end{pmatrix} \quad \Longrightarrow \quad \mathcal{N}_0^T = \begin{pmatrix} O & O \\ -J & O \end{pmatrix}. \tag{7.31}$$

These properties characterize the so-called *linear centralizer unfolding* (LCU for short) of Ω_0 (see section 7.4 for the precise definition and further details).

Using (7.28) or (7.29) one directly computes the eigenvalues of $\Omega(\mu)$ and so obtains explicit expressions for the normal frequency map $\omega^N : \mathbb{R}^s \to \mathbb{R}^4$ (with $s = 2$ in case (**G**) and $s = 4$ in case (**S**)). The results of these calculations and more details on the sets P_γ (see (7.18)) will be given in section 7.4.

Now, consider the unperturbed parametrized vector field

$$X(x, y, z, \omega, \mu) = \omega \partial_x + \Omega(\mu) z \partial_z, \tag{7.32}$$

with $\Omega(\mu)$ given by (7.28) in case (**G**) and by (7.29) in case (**S**). Also, consider perturbations $\widetilde{X}(x, y, z, \omega, \mu)$ of X, of the form

$$\begin{aligned} \widetilde{X}(x, y, z, \omega, \mu) = \Big[\omega + \tilde{f}(x, y, z, \omega, \mu)\Big]\, \partial_x + \tilde{g}(x, y, z, \omega, \mu)\, \partial_y \\ + \Big[\Omega(\mu) z + \tilde{h}(x, y, z, \omega, \mu)\Big]\, \partial_z. \end{aligned} \tag{7.33}$$

The perturbed vector field $\widetilde{X}$ is assumed to be real analytic in all space variables and parameters, as well as G-reversible. The latter leads to conditions on $\tilde{f}$, $\tilde{g}$ and $\tilde{h}$ as in (7.4). As before denote $M = \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^4$ and $P = \mathbb{R}^n \times \mathbb{R}^2$ or $P = \mathbb{R}^n \times \mathbb{R}^4$, depending on the case.

Theorem 10 (KAM). *Fix some $\omega_0 \in \mathbb{R}^n$ and $\gamma > 0$ sufficiently small. Then there exist a neighbourhood Γ of $(\omega_0, 0)$ in P, neighbourhoods $\mathcal{Y}$ and $\mathcal{Z}$ of the origin in respectively $\mathbb{R}^m$ and $\mathbb{R}^4$, and a neighbourhood $\mathcal{U}$ of X in the compact-open topology on the space of reversible analytic vector fields*

$\widetilde{X} : M \times P \to TM$ *such that for each* $\widetilde{X} \in \mathcal{U}$ *one can find a mapping* $\Phi : \mathbb{T}^n \times \mathcal{Y} \times \mathcal{Z} \times \Gamma \to M \times P$ *of the form*

$$\Phi(x, y, z, \omega, \mu) = \Big(x + \widetilde{U}(x, \omega, \mu), y + \widetilde{V}(x, y, \omega, \mu), z + \widetilde{W}(x, y, z, \omega, \mu), \omega + \widetilde{\Lambda}_1(\omega, \mu), \mu + \widetilde{\Lambda}_2(\omega, \mu)\Big) \quad (7.34)$$

for which the following holds:

(i) *the mapping* Φ *is* G*-equivariant, real-analytic in the* x*-variable and normally affine in the* y *and* z *variables;*

(ii) Φ *is* C^∞*-close to the identity map and is a* C^∞*-diffeomorphism onto its image;*

(iii) *the restriction of* Φ *to the Cantor set* $\mathbb{T}^n \times \{(0,0)\} \times (P_\gamma \cap \Gamma)$ *of diophantine* X*-invariant tori conjugates* X *to* $\widetilde{X}$*; the restriction of* Φ *to* $\mathbb{T}^n \times \mathcal{Y} \times \mathcal{Z} \times (P_\gamma \cap \Gamma)$ *also preserves the normal linear behaviour to these invariant tori.*

Remarks

1- The neighbourhood Γ depends on the choice of γ, the neighbourhoods $\mathcal{Y}$ and $\mathcal{Z}$ depend on γ and Γ, and the neighbourhood $\mathcal{U}$ depends on γ, Γ, $\mathcal{Y}$ and $\mathcal{Z}$.

2- By considering different choices for ω_0 in the foregoing statement one can replace 'a neighbourhood Γ of $(\omega_0, 0)$ in P' by 'a neighbourhood Γ of $K \times \{0\}$ in P', where $K \subset \mathbb{R}^n$ is any chosen bounded subset.

3- The condition that Φ is a full conjugacy from X to $\widetilde{X}$ means that

$$\Phi_*(X) = \widetilde{X} \qquad \Longleftrightarrow \qquad \left(\Phi^{-1}\right)_*(\widetilde{X}) = X. \quad (7.35)$$

What we will actually prove is the existence of a local diffeomorphism Φ such that

$$\left(\Phi^{-1}\right)_*(\widetilde{X})(x, y, x, \omega, \mu) = X(x, y, z, \omega, \mu) + O(|y|, |z|)\partial_x + O(|y|, |z|^2)\partial_y + O(|y|, |z|^2)\partial_z \quad (7.36)$$

for all $(\omega,\mu) \in P_\gamma$ which are sufficiently close to $(\omega_0, 0)$. The property (7.36) implies that for all parameter values (ω,μ) in the indicated Cantor set the X-invariant torus $\mathbb{T}^n \times \{0\} \times \{0\}$ is mapped by Φ into an $\widetilde{X}$-invariant torus on which the $\widetilde{X}$-flow is conjugate to the constant flow $\omega\partial_x$ on $\mathbb{T}^n$. This means that a Cantor subset of large measure of the family $\mathbb{T}^n \times \{(0,0)\} \times P$ of X-invariant tori survives the perturbation to $\widetilde{X}$.

4- Theorem 10 complements the main result of [12] where a similar persistence result was obtained under the condition that all eigenvalues of Ω_0 are simple.

In order to apply the result of Theorem 10 to small perturbations of a family $X(x,y,z,\lambda)$ in Floquet form (7.13) we have to restrict to parameter values where X is non-degenerate, and also replace the set P_γ given by (7.18) by its pullback in the λ-space $\mathbb{R}^q$. More precisely, assume that $\omega(\lambda)$ and $\Omega(\lambda)$ are defined on an open subset $P \subset \mathbb{R}^q$. Using the normal frequency mapping ω^N introduced in subsection 7.2.3 and the Diophantine subset $(\mathbb{R}^n \times \mathbb{R}^{2p})_\gamma$ given by (7.17), define for each $\Gamma \subset P$ the associated Diophantine subset

$$\Gamma_\gamma := \{\lambda \in \Gamma \mid (\omega(\lambda), \omega^N(\Omega(\lambda))) \in (\mathbb{R}^n \times \mathbb{R}^{2p})_\gamma \ \}. \tag{7.37}$$

When Γ is a neighbourhood of $\lambda_0 \in P$ where X is non-degenerate, then Γ_γ is nowhere dense but with large measure (if γ is small). Again we say that Γ_γ is a 'Cantor set'. An application of Theorem 10 then yields

Corollary 7.3. *Consider a real analytic reversible family $X(x,y,z,\lambda)$ in Floquet normal linear form* (7.13) *(with $p=2$), and let $\lambda_0 \in P$ be such that*

(i) *X is non-degenerate at $\lambda = \lambda_0$;*

(ii) *$\Omega(\lambda_0) \in gl_{-R}(4;\mathbb{R})$ has $\pm i$ as eigenvalues with algebraic multiplicity two.*

Then, for sufficiently small $\gamma > 0$ the following holds. There exists a neighbourhood Γ of λ_0 in $\mathbb{R}^q$ such that for all real analytic reversible families $\widetilde{X}$ sufficiently close to X: there exists an $\widetilde{X}$-invariant "Cantor set" $\widetilde{V} \subset M \times P$ which is a C^∞-near-identity diffeomorphic image of the foliation $\mathbb{T}^n \times \{(0,0)\} \times \Gamma_\gamma$ of n-tori. In the tori this diffeomorphism is an analytic conjugacy from X to $\widetilde{X}$, and it also preserves the normal linear behaviour at these tori.

This corollary can be combined with the scaling argument of subsection 7.2.2 to deduce a persistence result for the invariant tori of a family of integrable and reversible vector fields such as (7.7). To formulate this combined result we consider a family X of the form (7.7), subject to the conditions (7.8) and with $p = 2$. Together with X we also consider its normal linearization $N_0(X)$ given by (7.10) and (7.11). Finally, for each $\Gamma \subset P$ and each $\gamma > 0$ the Diophantine subset Γ_γ is given by (7.37). Then the following holds.

Corollary 7.4. *Under the foregoing conditions, let $\lambda_0 \in P$ be such that*

(i) $N_0(X)$ *is non-degenerate at* $\lambda = \lambda_0$*;*

(ii) $\Omega(\lambda_0) \in gl_{-R}(4;\mathbb{R})$ *has* $\pm i$ *as eigenvalues with algebraic multiplicity two.*

Then, for sufficiently small $\gamma > 0$ the following holds. There exists a neighbourhood Γ of λ_0 in $\mathbb{R}^q$ such that for all real analytic reversible families $\widetilde{X}$ sufficiently close to X: there exists an $\widetilde{X}$-invariant "Cantor set" $\widetilde{V} \subset M \times P$ which is a C^∞-near-identity diffeomorphic image of the foliation $\mathbb{T}^n \times \{(0,0)\} \times \Gamma_\gamma$ of n-tori. More in particular, there exists an X-invariant Cantor set $V \subset M \times P$ and a conjugacy from V onto $\widetilde{V}$; this conjugacy preserves the projection on the parameter space and the normal linear behaviour to the invariant tori.

The sections which follow are devoted to the proofs of the foregoing propositions and of the KAM theorem. We start with the universal unfolding of the 1 : 1-resonance.

7.4 Unfolding the Reversible 1 : 1-Resonance

In this section we present universal unfoldings of the 1 : 1 resonance in both semisimple and generic case when $p = 2$ (i.e., when $\Omega(\mu_0) \in gl_{-R}(4, \mathbb{R})$). In section 8.5 (Theorem 12) we prove that, under appropriate assumptions, the following constructive method to obtain such unfoldings holds: *let* $\Omega_0 \in gl_{-R}(2p, \mathbb{R})$*. Then, the map* $\Omega : \ker\left(\mathrm{ad}\left(\Omega_0^T\right)\right) \cap gl_{-R}(2p, \mathbb{R}) \to gl_{-R}(2p, \mathbb{R})$, *given by*

$$A \mapsto \Omega(A) := \Omega_0 + A \tag{7.38}$$

determines a universal unfolding of Ω_0. The unfolding (7.38) is called *the linear centralizer unfolding* (LCU), [3, 39]. We now describe the space

$gl_{-R}(4,\mathbb{R})$ and its stratification by parabolic, elliptic and hyperbolic matrices near a $1:1$ resonance. Recall that the involution $R \in gl(\mathbb{R},4)$ is fixed by (7.20).

Consider the 16-dimensional linear space $gl(4,\mathbb{R})$ of general 4×4 matrices and observe that for the linear subspace $gl_{-R}(4,\mathbb{R})$ one has $\operatorname{cod} gl_{-R}(4,\mathbb{R}) = \dim gl_{-R}(4,\mathbb{R}) = 8$. Indeed, generally $A \in gl(4,\mathbb{R})$ in block form reads

$$A = \begin{pmatrix} A_1 & A_3 \\ A_2 & A_4 \end{pmatrix}, \quad \text{with } A_i \in gl(2,\mathbb{R}),\ (i = 1,\dots,4),$$

and $A \in gl_{-R}(4,\mathbb{R})$ if and only if

$$A_i = \begin{pmatrix} 0 & b_i \\ a_i & 0 \end{pmatrix},\ a_i, b_i \in \mathbb{R}, \tag{7.39}$$

which proves our assertion. Inside $gl_{-R}(4,\mathbb{R})$ both the subsets of elliptic and hyperbolic matrices form open strata.

Proposition 7.5. *The subset of parabolic matrices in $gl_{-R}(4,\mathbb{R})$ forms a codimension 1 stratum separating the open elliptic and hyperbolic strata. Moreover, the generic (non-semisimple) parabolic sub-case forms a codimension 0 substratum, i.e. an open subset of the parabolic stratum, while the semisimple parabolic sub-case forms a codimension 2 substratum of the parabolic stratum. See Fig.* 7.3.

Proof. Consider the parabolic matrices in $gl_{-R}(4,\mathbb{R})$, i.e., matrices with a pair of double purely imaginary eigenvalues: $\pm i\kappa$, for some $\kappa > 0$. For any $A \in gl_{-R}(4,\mathbb{R})$ as before, define the symmetric polynomials

$$ch_0(A) := (a_1a_2 - a_2a_3)b_1b_4 + (-a_1a_4 + a_2a_3)b_2b_3,$$

and

$$ch_2(A) := a_1b_1 + a_2b_3 + a_3b_2 + a_4b_4,$$

where $\Delta := ch_2(A)^2 - 4ch_0(A)$. Then A is parabolic if and only if

$$\Delta = 0 \quad \text{and} \quad ch_2(A) < 0.$$

From this the first assertion follows. The second and third claims are direct consequences of (7.38) as we explain below.

Indeed, let

$$\Omega_0 = \kappa\mathcal{S}_0 + \mathcal{N}_0 \in gl_{-R}(4,\mathbb{R}), \tag{7.40}$$

$\mathcal{S}_0$ and $\mathcal{N}_0$ as in (7.30) and (7.31). It is quickly verified that 2 unfolding parameters are necessary to describe the LCU of (7.40) given by (7.28), i.e.,

$$\Omega(\mu) := \Omega(\mu_1,\mu_2) = \Omega_0 + \begin{pmatrix} \mu_1 J & O \\ \mu_2 J & \mu_1 J \end{pmatrix} \in gl_{-R}(4,\mathbb{R}),\ \ \mu_1,\mu_2 \in \mathbb{R}.$$

The eigenvalues of $\Omega(\mu)$ in this case are

$$\lambda_j(\mu) = \pm i[(\kappa+\mu_1) \pm \sqrt{\mu_2}], \quad j = 1,2,3,4. \tag{7.41}$$

As μ_2 crosses zero, two pairs of purely imaginary eigenvalues of $\Omega(\mu)$ collide and split off the imaginary axis. It follows that the parabolic matrices belonging to the LCU of Ω_0 have the generic (non-semisimple) form

$$\Omega(\mu) = \begin{pmatrix} (\kappa+\mu_1)J & 0 \\ 0 & (\kappa+\mu_1)J \end{pmatrix} + \begin{pmatrix} O & J \\ O & O \end{pmatrix}, \quad \mu_1 \in \mathbb{R} \text{ close to } 0.$$

This proves the second assertion.

On the other hand, let

$$\Omega_0 = \kappa\mathcal{S}_0 \in gl_{-R}(4,\mathbb{R}). \tag{7.42}$$

be the matrix representation of the 1 : 1 resonance in the semisimple case. Then, the LCU of (7.42) is given by (7.29), i.e.,

$$\Omega(\mu) := \Omega(\mu_1,\ldots,\mu_4) = \Omega_0 + \begin{pmatrix} \mu_1 J & \mu_3 J \\ \mu_4 J & \mu_2 J \end{pmatrix} \in gl_{-R}(4,\mathbb{R}),$$

$(\mu_1,\ldots,\mu_4) \in \mathbb{R}^4$. Here 4 unfolding parameters are necessary and the eigenvalues of $\Omega(\mu)$ are

$$\lambda_j(\mu) = \pm\frac{1}{2}\left(i(\mu_2+\mu_1+2\kappa) \pm i\sqrt{\Delta}\right), \qquad j = 1,\ldots,4, \tag{7.43}$$

where $\Delta = \Delta(\mu) := (\mu_2-\mu_1)^2 + 4\mu_3\mu_4$. It follows that $\Omega(\mu)$ is parabolic if and only if $\Delta = 0$. The general (not necessarily semisimple) form of such parabolic matrices is

$$\Omega(\mu) = \begin{pmatrix} (\kappa+\frac{1}{2}(\mu_2+\mu_1))J & O \\ O & (\kappa+\frac{1}{2}(\mu_2+\mu_1))J \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}(\mu_2-\mu_1)J & \mu_3 J \\ \mu_4 J & \frac{1}{2}(\mu_2-\mu_1)J \end{pmatrix}. \tag{7.44}$$

Any non-semisimple parabolic matrix of the form (7.44) can be reduced to the form (7.40), by first applying a time-scaling and then a similarity transformation. In the case $\kappa = 1$, the time-scaling transforms (7.44) into

$$\Omega(\mu) = \mathcal{J} + \frac{1}{1 + \frac{\mu_1+\mu_2}{2}} \begin{pmatrix} -\frac{\mu_2-\mu_1}{2} J & \mu_3 J \\ \mu_4 J & \frac{\mu_2-\mu_1}{2} J \end{pmatrix}. \tag{7.45}$$

Then, the similarity transformation $T^{-1}\Omega(\mu)T$ transforms (7.45) into (7.40), where

$$T := \begin{pmatrix} I & O \\ -\frac{a}{b} I & \frac{1}{b} I \end{pmatrix},$$

with $a = \frac{\mu_2-\mu_1}{2+\mu_2+\mu_1}$ en $b = 2\,\frac{\mu_3}{2+\mu_2+\mu_1}$, ($\mu_1, \ldots, \mu_4$ satisfying $\Delta = 0$).

In Fig. 7.3 we have sketched this stratification of $gl_{-R}(4, \mathbb{R})$ in terms of the different eigenvalue configurations. The parabolic variety $\mathcal{C}$ is a 7-dimensional cone in $gl(4, \mathbb{R})$ consisting of two strata: the vertex v which represents the semi-simple parabolic substratum, the open stratum $\mathcal{C} \setminus \{v\}$ of non-semisimple parabolic matrices. The inner and outer part of $\mathcal{C}$ correspond to the hyperbolic and elliptic strata respectively. $\square$

Remarks

1- Straightforward generalization to the case $p \geq 2$ is possible. Fig. 7.3 does not hold any longer, but can be generalized by using Theorem 12 (cf. section 8.5) in combination with the linear algebra developed in [54].

2- In the Hamiltonian setting one distinguishes between the $1:1$ and the $1:-1$ resonance, compare [Jon99] and [Mee85]. In the former case the eigenvalues remain on the imaginary axis, while in the latter they can come off and form quadruples (gyrostatic destabilization). In the present reversible context this difference does not occur, compare [Hov96].

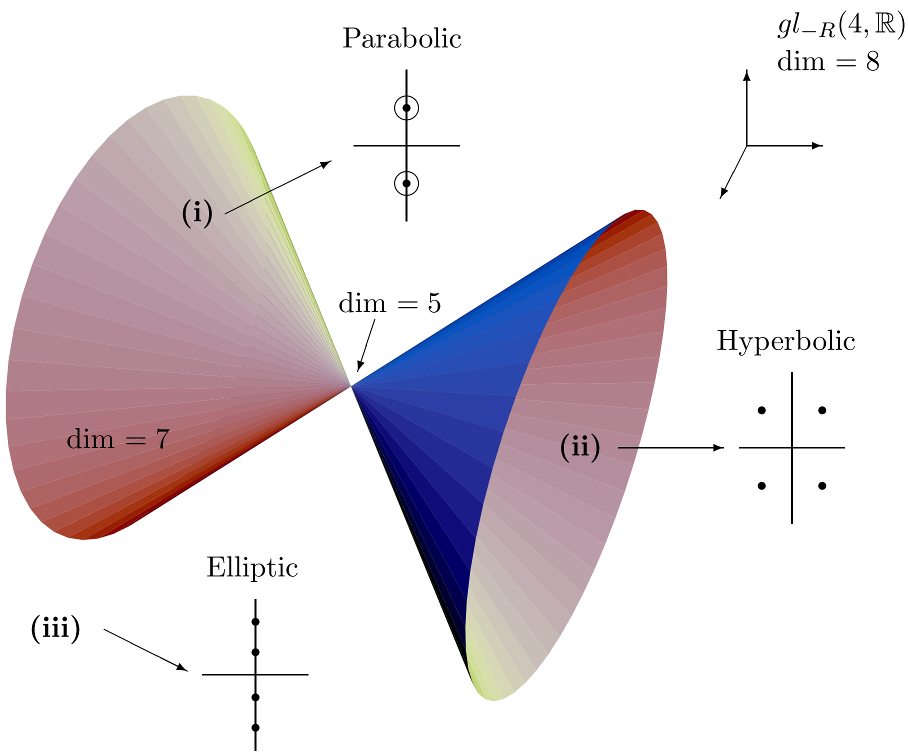

Figure 7.3: *Stratification of $gl_{-R}(4,\mathbb{R})$.* **(i)** *The surface of the cone is the parabolic stratum and the vertex represents the codimension 2 semisimple parabolic substratum.* **(ii)** *Inside the parabolic cone lies the hyperbolic open stratum.* **(iii)** *Outside the cone one has the open elliptic stratum. The numbers $8, 7, 5$ denote dimensions in $gl(4,\mathbb{R})$.*

7.5 Bifurcational Aspects

Let $R, \Omega_0 \in gl(4,\mathbb{R})$ be fixed as in (7.20) and (7.21) and let M be a phase space as introduced before. Let $\Omega_0 = \mathcal{S}_0 + \mathcal{N}_0$ be the SN decomposition of $\Omega_0 \in gl_{-R}(4,\mathbb{R})$ and let $\Omega(\mu) \in gl_{-R}(4,\mathbb{R})$ given by (7.28) be its LCU. Consider the P-parametrized (R-) reversible vector field $X_\mu(x,y,z) = \omega\partial_x + \Omega(\mu)z\partial_z$+h.o.t with $X_0 = \omega\partial_x + \Omega_0 z\partial_z$+h.o.t.. If we say that X_μ is put in (R-reversible) normal form (up to some order $k \geq 1$) when we can find a (R-equivariant) transformation such that $X_\mu \mod \mathcal{X}_0^k$ commutes with $\mathcal{S}_0$ and $\mathcal{N}_0^T$, then by a standard procedure, see e.g. [42], one shows that X_μ can be transformed into the system $\widehat{X}_\mu = \omega\partial_x + (\Omega(\mu)z + Z)\partial_z$+h.o.t. with $Z = Z(z_1,\ldots,z_4) \in \mathcal{X}_{-R}$ given by

$$\begin{aligned}
\dot z_1 &= az_2(z_1^2+z_2^2) + cz_2(z_1z_3+z_2z_4)\\
\dot z_2 &= -az_1(z_1^2+z_2^2) - cz_1(z_1z_3+z_2z_4)\\
\dot z_3 &= az_1(-z_2z_3+z_1z_4) + bz_2(z_1z_3+z_2z_4)\\
&\quad +cz_4(z_1z_3+z_2z_4) - dz_2(z_1^2+z_2^2)\\
\dot z_4 &= az_1(-z_2z_3+z_1z_4) - bz_1(z_1z_3+z_2z_4)\\
&\quad -cz_3(z_1z_3+z_2z_4) - dz_1(z_1^2+z_2^2)
\end{aligned} \tag{7.46}$$

for suitable constants a,b,c,d.

Now, to work out an example, identify for simplicity $\mathbb{R}^4$ with $\mathbb{C}^2$ by $(z_1,z_2,z_3,z_4) \mapsto (z_1 := z_1 - iz_2, z_2 := z_3 - iz_4)$. Fix then $a=b=c=0$ and $d=\pm1$ in (7.46) and consider the reversible and integrable system

$$\left\{\begin{aligned}
\dot x &= \omega(y)\\
\dot y &= 0\\
\dot z_1 &= i(1+\mu_1)z_1 + iz_2\\
\dot z_2 &= i\mu_2 z_1 + i(1+\mu_1)z_2 \pm i|z_1|^2 z_1
\end{aligned}\right. . \tag{7.47}$$

This system has solutions of the form

$$(x(t),y(t),z(t)) = \left(x_0 + \omega(y_0)t, y_0, \rho e^{(1+\mu_1+\sigma)t}, \sigma\rho e^{(1+\mu_1+\sigma)t}\right), \tag{7.48}$$

for each (ρ,σ) satisfying

$$\sigma^2 = \mu_2 \pm \rho^2. \tag{7.49}$$

For fixed (y_0, ρ, σ) this solution generates an invariant $(n+1)$-torus. Depending on the sign, equation (7.49) describes two different scenario's on how these invariant $(n+1)$-tori bifurcates from the invariant n-torus T_{y_0}. One question is: what happens to these scenario's under small reversible but not necessarily integrable perturbation? Now, the normal linear part of (7.47) satisfies the hypothesis of Corollary 7.3, therefore we expect (local) quasi periodic stability. A more careful bifurcation analysis is under current investigation.

7.6 KAM Theory

We sketch the idea behind the proof the KAM Theorem 10 and provide some necessary ingredients.

Towards a Newtonian iteration, structure properties In a first attempt to prove the KAM Theorem 10, one looks for a G-equivariant family of transformations $\Phi : M \times P \to M \times P$ conjugating X to $\tilde{X}$, i.e., such that

$$\Phi_*(X) = \tilde{X}. \tag{7.50}$$

This is a nonlinear equation in Φ, to be solved, as far as possible, by a Newtonian iteration process. Here the well-known small divisor problem is met, to be overcome by Diophantine conditions, cf. e.g. [13]. For further details on this general approach to KAM theory, we refer to, e.g., [58, 59, 64], and [51, 14, 12, 13]. To be more specific, when writing

$$\Phi(\xi, \eta, \zeta, \sigma, \nu) = \left(\xi + \tilde{U}, \eta + \tilde{V}, \zeta + \tilde{W}, \sigma + \tilde{\Lambda}_1(\sigma, \nu), \nu + \tilde{\Lambda}_2(\sigma, \nu)\right), \tag{7.51}$$

where $\tilde{U} = \tilde{U}(\xi, \sigma, \nu)$, $\tilde{V} = \tilde{V}(\xi, \eta, \zeta, \sigma, \nu)$ and $\tilde{W} = \tilde{W}(\xi, \eta, \zeta, \sigma, \nu)$, the conjugacy equation (7.50) translates to the system of non-linear equations

$$\begin{aligned}
&\frac{\partial \tilde{U}}{\partial \xi}(\sigma + \tilde{f}) + \tilde{f} = \tilde{\Lambda}_1(\sigma, \nu), \\
&\frac{\partial \tilde{V}}{\partial \xi}(\sigma + \tilde{f}) + \left(I + \frac{\partial \tilde{V}}{\partial \eta}\right)\tilde{g} + \frac{\partial \tilde{V}}{\partial \zeta}\left(\Omega(\nu)\zeta + \tilde{h}\right) = 0, \\
&\frac{\partial \tilde{W}}{\partial \xi}(\sigma + \tilde{f}) + \frac{\partial \tilde{W}}{\partial \eta}\tilde{g} + \frac{\partial \tilde{W}}{\partial \zeta}\left(\Omega(\nu)\zeta + \tilde{h}\right) + \tilde{h} = A(\tilde{\Lambda}_2)\zeta \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \Omega\left(\nu + \tilde{\Lambda}_2\right)\tilde{W},
\end{aligned}$$

where everything is expressed in $(\xi, \eta, \zeta, \sigma, \nu)$. A standard linearization procedure then leads to the homological equation:

$$\begin{aligned}
&\frac{\partial \tilde{U}}{\partial \xi}\sigma = \tilde{\Lambda}_1 - \tilde{f} \\
&\frac{\partial \tilde{V}}{\partial \xi}\sigma + \tilde{g} + \frac{\partial \tilde{V}}{\partial \zeta}\Omega(\nu)\zeta = 0 \\
&\frac{\partial \tilde{W}}{\partial \xi}\sigma + \frac{\partial \tilde{W}}{\partial \zeta}\Omega(\nu)\zeta + \tilde{h} = A(\tilde{\Lambda}_2)\zeta + \Omega(\nu)\tilde{W}.
\end{aligned} \tag{7.52}$$

In the Newtonian iteration process, at each (KAM) step, we solve equations as (7.52). Indeed, (7.52) will be adapted by suitable truncation in a Taylor-Fourier series. We will respect the G-structure (reversibility, equivariance) at each step.

Freely quoting from [59, 51, 12], as a counterpart of the adjoint action at the level of matrices, in the space of vector fields we consider the adjoint (infinitesimal) action $\mathrm{ad}(X)$: $Y \mapsto [X, Y]$. It is easy to check that the operator $\mathrm{ad}\, X$ leaves the set of all normal linear vector fields invariant. Moreover, for a given $X \in \mathcal{X}_{-G}$, the map $\mathrm{ad}(X)$ interchanges the properties reversible and equivariant. Therefore, we may restrict to

$$\mathrm{ad}(X) : \mathcal{L}_{\pm G} \to \mathcal{L}_{\mp G},$$

where $\mathcal{L}_{+G}$ stands for all linear G-equivariant vector fields and $\mathcal{L}_{-G}$ for all linear reversible vector fields. The following lemma provides properties related to the class of vector fields $\mathcal{X}_{-G}$. For an element $Y \in \mathcal{X}$, in the coordinates (x, y, z) as introduced before, we write (suppressing the parameters)

$$\begin{aligned} Y(x,y,z) &= \sum_{j=1}^{n} F_j(x,y,z)\frac{\partial}{\partial x_j} + \sum_{l=1}^{m} G_l(x,y,z)\frac{\partial}{\partial y_l} \\ &\quad + \sum_{t=1}^{2p} H_t(x,y,z)\frac{\partial}{\partial z_t} \\ &= F(x,y,z)\frac{\partial}{\partial x} + G(x,y,z)\frac{\partial}{\partial y} + H(x,y,z)\frac{\partial}{\partial z}. \end{aligned}$$

Consider the following truncations of Y:

$$\begin{aligned} Y_{lin}(x,y,z) &:= F(x,0,0)\frac{\partial}{\partial x} + \{G(x,0,0) + G_y(x,0,0)y \\ &\quad + G_z(x,0,0)z\}\frac{\partial}{\partial y} + \{H(x,0,0) + H_y(x,0,0)y \\ &\quad + H_z(x,0,0)z\}\frac{\partial}{\partial z}, \\ Y_d(x,y,z) &:= \sum_{|k|\le d}\Big(F_k(y,z)e^{\langle x,k\rangle}\frac{\partial}{\partial x} + G_k(y,z)e^{\langle x,k\rangle}\frac{\partial}{\partial y} \\ &\quad + H_k(y,z)e^{\langle x,k\rangle}\frac{\partial}{\partial z}\Big) \end{aligned}$$

where $d \ge 0$ is an arbitrary integer and F_k, G_k, H_k for $k \in \mathbb{Z}^n$ are the Fourier coefficients of F, G, H. By $\mathcal{Y}_{lin}$ and $\mathcal{Y}_d$ we denote the sets of these truncations and in particular $\mathcal{L}_d$ denotes $\mathcal{Y}_{lin,d} \subset \mathcal{Y}_{lin}$.

Lemma 7.6. *If* $Y \in \mathcal{X}_{-G}$ *then* $Y_{lin}, Y_d \in \mathcal{X}_{-G}$, *for all* $d \in \mathbb{N}$. *Moreover, the following direct sum splitting holds:*

$$\mathcal{L}_d = \mathcal{L}_{d,-G} \oplus \mathcal{L}_{d,G}. \tag{7.53}$$

Frequency map. Distinguishing between the generic and the semisimple cases, we explore the frequency map

$$\mathcal{F}_s : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^n \times \mathbb{R}^{2p}, \quad (\omega,\mu) \mapsto \mathcal{F}_s(\omega,\mu) := (\omega, \omega^N(\mu)) \tag{7.54}$$

such that for each $\mu \in \mathbb{R}^s$ the components $\omega_j^N(\mu)$ $(1 \le j \le 2p)$ are the imaginary parts of the eigenvalues of $\Omega(\mu)$. For the case (**G**) we have that $(p = 2, s = 2)$

(i) when $\mu_2 > 0$,

$$\omega_1^N(\mu_1,\mu_2) = -\omega_2^N(\mu_1,\mu_2) = 1+\mu_1+\sqrt{\mu_2} \tag{7.55}$$
$$\omega_3^N(\mu_1,\mu_2) = -\omega_4^N(\mu_1,\mu_2) = 1+\mu_1-\sqrt{\mu_2} \tag{7.56}$$

(ii) when $\mu_2 \le 0$

$$\omega_1^N(\mu_1,\mu_2) = -\omega_2^N(\mu_1,\mu_2) = 1+\mu_1 \tag{7.57}$$
$$\omega_3^N(\mu_1,\mu_2) = -\omega_4^N(\mu_1,\mu_2) = 1+\mu_1. \tag{7.58}$$

Observe that for $\mu_2 > 0$ the map $\omega^N : \mathbb{R}^2 \to \mathbb{R}^4$ is a local submersion on the subspace $V_> := \{\omega^N \in \mathbb{R}^4 | \, \omega_1^N = -\omega_2^N, \, \omega_3^N = -\omega_4^N\}$, while for $\mu_2 < 0$ we obtain a local submersion onto the subspace $V_< := \{\omega^N \in \mathbb{R}^4 | \, \omega_1^N = -\omega_2^N = \omega_3^N = -\omega_4^N\}$. It follows that $\widetilde{P} = \{(\omega,\mu) \in \mathbb{R}^n \times \mathbb{R}^2 | \, \mu_2 \neq 0\}$ is an open and dense subset of $(\mathbb{R}^n \times \mathbb{R}^s)$ such that $\widetilde{P} \cap (\mathbb{R}^n \times \mathbb{R}^s)_\gamma$ is nowhere dense and of large measure (full measure as $\gamma \to 0_+$). We restrict our considerations to the case (**G**) throughout. Note that in case (**S**) ($p = 2, s = 4$)

$$\mathcal{F}_4 = \mathcal{F}_2 \circ \mathcal{T},$$

with $\mathcal{T} : \mathbb{R}^4 \to \mathbb{R}^2$ given by

$$\mathcal{T}(\mu_1,\ldots,\mu_4) := (\tfrac{1}{2}(\mu_1+\mu_2), (\mu_1-\mu_2)^2 + 4\mu_3\mu_4).$$

In what follows it will be convenient to take the frequencies (ω,ω^N) as parameters (instead of (ω,μ)), therefore it is suitable to introduce the piecewise diffeomorphism $\mathcal{G} : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}^n \times \mathbb{R}^2$ given by

$$(\omega,\mu_1,\mu_2) \mapsto \begin{cases} \big(\omega, \omega_1^N(\mu_1,\mu_2), \omega_3^N(\mu_1,\mu_2)\big) & \text{if } \mu_2 > 0 \\ \big(\omega, \omega_1^N(\mu_1,\mu_2), 0\big) & \text{if } \mu_2 = 0 \\ \big(\omega, \omega_1^N(\mu_1,\mu_2), \mu_2\big) & \text{if } \mu_2 < 0 \end{cases} \tag{7.59}$$

Then, if Γ denotes an open neighbourhood of the origin in the parameter space, one has

$$\mathcal{G}(\Gamma) = \mathcal{G}_< \cup \mathcal{G}_= \cup \mathcal{G}_>, \tag{7.60}$$

where the subscript indicates the sign of μ_2. Setting $\mathcal{G}_\le = \mathcal{G}_= \cup \mathcal{G}_<$, the inverse $\mathcal{G}^{-1} : \mathcal{G}_\le \cup \mathcal{G}_> \to \Gamma$ is given by

$$(\omega,\omega^N,\mu_2) \mapsto (\omega, \mu_1(\omega^N), \mu_2(\omega^N)), \quad \text{if } \mu_2 \le 0,$$

with

$$\mu_1(\omega_N) = \omega_N - 1, \quad \mu_2(\omega_N) = \mu_2,$$

and

$$(\omega, \omega_1^N, \omega_3^N) \mapsto (\omega, \mu_1(\omega^N), \mu_2(\omega^N)), \quad \text{if } \mu_2 > 0,$$

with

$$\mu_1\left(\omega^N\right) = \tfrac{1}{2}\left(\omega_1^N + \omega_3^N - 2\right), \quad \mu_2\left(\omega^N\right) = \tfrac{1}{4}\left(\omega_1^N - \omega_3^N\right)^2.$$

In Fig. 7.4 we sketch the Cantor sets: $\mathcal{G}_{<,\gamma}$, $\mathcal{G}_{=,\gamma}$, and $\mathcal{G}_{>,\gamma}$. Recall that the subindex γ indicates that the diophantine conditions are satisfied.

Also, define

$$\mathcal{G}'_{\leq,\gamma} := \left\{(\omega, \omega_N, \mu_2) \in \mathcal{G}_{\leq} \,|\, \operatorname{dist}\big((\omega, \omega_N), \partial(\mathcal{G}_{\leq})\big) \geq \gamma\right\},$$

and $\mathcal{G}'_{>,\gamma}$ accordingly.

Analytic neighbourhoods Let us specify the form of a neighbourhood $\mathcal{A}$ of X as in (7.32). For given $S \subseteq \mathbb{R}^k$ and $\rho > 0$ let

$$S + \rho := \cup_{s \in S}\left\{z \in \mathbb{C}^k |\, 1 \leq j \leq k : |z_j - s_j| \leq \rho\right\}.$$

Consider a compact neighbourhood, $\mathcal{O}$, of $T^n \times \{0\} \times \{0\} \times \mathcal{G}(\Gamma)$ in $\mathbb{C}^n/(2\pi\mathbb{Z})^n \times \mathbb{C}^m \times \mathbb{C}^{2p} \times \mathcal{G}(\Gamma)$ such that X has a complex analytic extension to it. We take $\mathcal{O}$ of the form

$$\mathcal{O} := (\mathbb{T}^n + \kappa) \times \Upsilon \times Z \times (\mathcal{G}(\Gamma) + r^*), \tag{7.61}$$

where $\mathcal{Y} := \{y \in \mathbb{C}^m |\, |y| < \epsilon^\star\}$, $\mathcal{Z} := \{z \in \mathbb{C}^{2p} |\, |z| < \rho^\star\}$ are compact neighbourhoods of the origin, and κ, $\epsilon^\star$, $\rho^\star > 0$ and $0 < r^* \leq 1$ are given constants. Now, $\mathcal{A}$ is a compact open neighbourhood of X (intersected with the space of real analytic extensions) determined by $\mathcal{O}$, γ, and a constant $\delta > 0$ that will be specified later. A vector field $\tilde{X} \in \mathcal{A}$ has the form

$$\tilde{X} = X + \tilde{f}\partial_x + \tilde{g}\partial_y + \tilde{h}\partial_z, \tag{7.62}$$

with real analytic $\tilde{f}, \tilde{g}, \tilde{h}$ defined in $\mathcal{O}$ and such that they are small in the supremum norm on $\mathcal{O}$.

In the above notation, Theorem 10 is reformulated as follows.

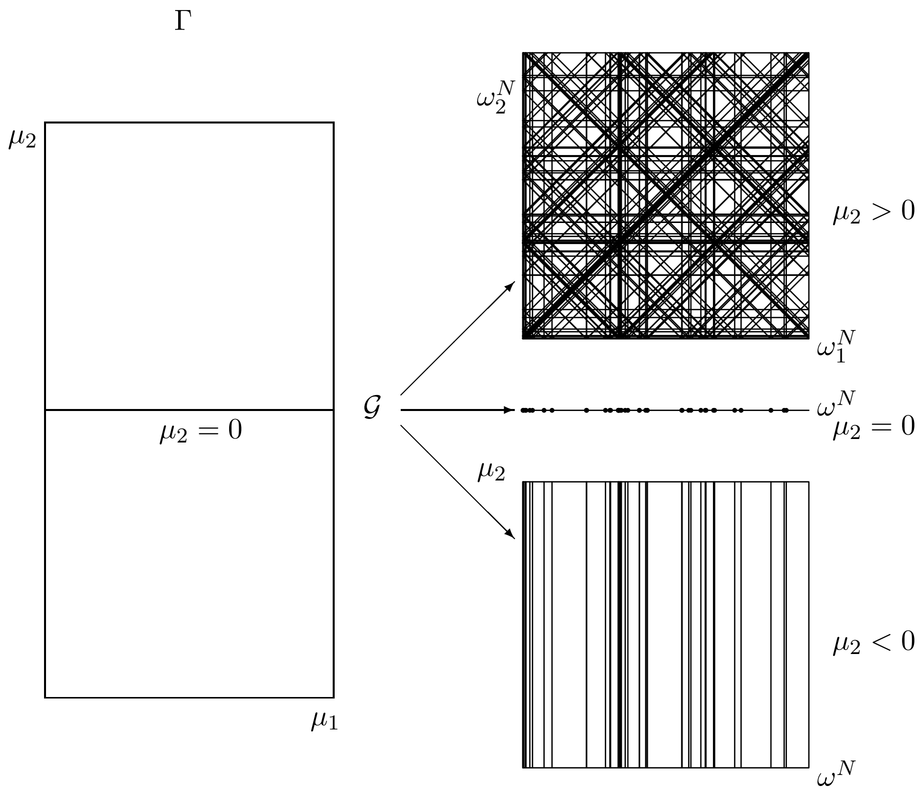

Figure 7.4: *Cantor Sets. Bottom: hyperbolic case. Center: parabolic case. Top: elliptic case*

Theorem 11. *Let* $X_{\omega,\omega_N,\mu_2}(x,y,z) = \omega\partial_x + \Omega(\omega_N,\mu_2)z\partial_z$ *be an analytic family of reversible vector fields as before. Then, for any* $\gamma \geq 0$, $\epsilon^\star$, $\rho^\star, > 0$, *there exists* $\delta^\star > 0$ *such that for any analytic family* $\tilde{X} = X + \tilde{f}\partial_x + \tilde{g}\partial_y + \tilde{h}\partial_z$, *with*

$$|\tilde{f}|_{\mathcal{O}} < \gamma\delta, \quad |\tilde{g}|_{\mathcal{O}} < \gamma\delta^2, \quad |\tilde{h}|_{\mathcal{O}} < \gamma\delta^2. \tag{7.63}$$

on a fixed neighbourhood $\mathcal{O}$ *of the form* (7.61), *there exists a* C^∞*-mapping* $\Phi : \mathbb{T}^n \times \mathcal{Y} \times \mathcal{Z} \times \mathcal{G}(\Gamma) \to M \times \mathbb{R}^n \times \mathbb{R}^3$, *with the following properties.*

(i) *For each* (ω,ω_N,μ_2) *belonging to one of the boxes* $\mathcal{G}(\Gamma)_{\leq,\gamma}$ *or* $\mathcal{G}(\Gamma)_{>,\gamma}$ *one has*

$$\begin{aligned}&(\Phi)_*^{-1}\tilde{X}\left(x,y,z,\omega+\Lambda_1(\omega,\omega_N),\omega^N+\Lambda_2(\omega,\omega_N),\mu_2\right) = \\ &X\left(\omega,\omega^N,\mu_2\right) + O(|y|,|z|)\partial_x + O(|y|,|z|^2)\partial_y + O(|y|,|z|^2)\partial_z.\end{aligned}$$

(ii) *There exists a constant* $b > 0$ *such that*

$$\|\Phi - id\|_{C^j} \approx O(\delta_*^b), \quad \text{as } \delta^* \downarrow 0 \tag{7.64}$$

on $\mathbb{T}^n \times \mathcal{Y} \times \mathcal{Z} \times \mathcal{G}(\Gamma)'$, *for all* $j \in \mathbb{N}$ *in the* C^j*-norm.*

Remark Note that Theorem 11 implies Theorem 10 and that it is sufficient to prove it for the case $\gamma = \gamma_0$, where γ_0 is some positive constant. Indeed, by linearity of the unfolding we can rescale time t to $\frac{\gamma}{\gamma_0}t$ and stretch the parameters by a factor $\frac{\gamma}{\gamma_0}$. From now on we take $\gamma = 1$.

8

Proof of the KAM Theorem 10

Aim of this chapter is to prove Theorem 11. The proof is divided in four parts. In the first part we sketch the (Newtonian) iteration process to approximate the 'conjugacy' Φ, then we deal with one iteration step and derive the linearized homological equation and its formal solvability conditions. Third, the estimates for the iteration step are given, and finally, the convergence of the process is proved [12, 13, 14, 15].

Recall that $p = 2$, i.e., our phase space reads $M = \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^4$ with coordinates (x, y, z).

8.1 The Iteration Process

Given a small perturbation $\tilde{X}$ of X, the conjugacy Φ solving the conjugacy equation will be obtained as a C^∞-limit of a sequence $\{\Phi_j\}_{j=0}^{\infty}$ of (G-equivariant) analytic transformations, defined on complex neighbourhoods D_j^e of $\mathbb{T}^n \times \{0\} \times \{0\} \times \mathcal{G}(\Gamma)'_\gamma$ to be specified later. The Φ_j will be constructed inductively (iteratively), starting with $\Phi_0 = \mathrm{I}$. For $j \geq 0$, whenever Φ_j is defined, let $x_j = (x_{j_1}, \ldots, x_{j_n})$, $y_j = (y_{j_1} \ldots, y_{j_m})$, $z_j = (z_{j_1}, \ldots, z_{j_4})$ and $\omega_j = (\omega_{j_1}, \ldots, \omega_{j_n})$, $\mu = (\omega_{j_1}^N, \ldots, \omega_{j_r}^N, \mu_2)$ be the component functions of the inverse Φ_j^{-1} and define

$$\tilde{X}_j := \Phi_j^*(\tilde{X}), \;\; (\Phi_j^* = (\Phi_j^{-1})_*). \tag{8.1}$$

Assume that both X and $\tilde{X}$ have complex analytic extensions to a set $\mathcal{O}$ (see (7.61)). Then both Φ_j and $\tilde{X}_j$ will have complex analytic extensions to complex neighbourhoods D_j^e of the (Cantor) set $\mathbb{T}^2 \times \{0\} \times \{0\} \times \mathcal{G}(\Gamma)'_\gamma$, with $D_0^e \subseteq \mathcal{O}$ and which for $j \to \infty$ shrink in an appropriate (geometric) way. It will be shown that $|\tilde{f}|, |\tilde{g}|, |\tilde{h}| \to 0$ rapidly as $j \to 0$. Application of the Inverse Approximation Lemma and Whitney Extension Theorem [13, 31, 14, 64] then gives limits Φ_∞ and $\tilde{X}_\infty$, where $\tilde{X}_\infty := \Phi_\infty^*(\tilde{X})$ and $x_\infty, \ldots, \mu_\infty$

are defined accordingly. In these coordinates

$$\tilde{X}^{\omega_\infty,\mu_\infty}(x_\infty, y_\infty, z_\infty) = X^{\omega_\infty,\mu_\infty}(x_\infty, y_\infty, z_\infty) + O(|y_\infty|, |z_\infty|)\partial_{x_\infty} + O(|y_\infty|, |z_\infty|^2)\partial_{y_\infty} + O(|y_\infty|, |z_\infty|^2)\partial_{z_\infty} \quad (8.2)$$

where the parameter $(\omega_\infty, \mu_\infty)$ is restricted to the 'Cantor set' defined by $\mathcal{G}(\Gamma)_\gamma$. Granted the details, the Whitney Extension Theorem ([15, 13, 14]) applied to Φ_∞, provides the map Φ.

8.2 The Iteration Step

We describe one step of the induction or KAM-iteration. Suppose $\Phi_0 = \mathrm{I}$ and Φ_j given, then $\Phi_{j+1} = \Phi_j \circ \Psi_j$ (for $j \geq 0$), where $\Psi_j : D^e_{j+1} \to D^e_j$, $j \in \mathbb{Z}_+$, is appropriately constructed. More explicitly:

$$(x_{j+1}, y_{j+1}, z_{j+1}, \omega_{j+1}, \mu_{j+1}) \overset{\Psi_j}{\mapsto} (x_j, \ldots, \mu_j) \overset{\Phi_j}{\mapsto} (x, \ldots, \mu),$$

$$(x_{j+1}, \ldots, \mu_{j+1}) \overset{\Phi_{j+1}}{\longmapsto} (x, \ldots, \mu).$$

Then for all $j \in \mathbb{Z}_+$,

$$\Phi_{j+1} = \Psi_0 \circ \cdots \circ \Psi_j \quad \text{and} \quad \tilde{\mathrm{X}}_{\mathrm{j+1}} = \Psi^*_{\mathrm{j}}(\tilde{\mathrm{X}}_{\mathrm{j}}),$$

where $\Psi^* := (\Psi_j^{-1})_*$. Since $\Phi_0 = \mathrm{I}$, we have $\tilde{X}_0 = \tilde{X}$.
To simplify notations, introduce the + notation: suppress the index j and write (x, y, z, ω, μ) and $(\xi, \eta, \zeta, \sigma, \nu)$ instead of $(x_j, y_j, z_j, \omega_j, \mu_j)$ and $(x_{j+1}, y_{j+1}, z_{j+1}, \omega_{j+1}, \mu_{j+1})$ respectively. Also replace $\tilde{f}^j$ by $\tilde{f}$ and $\tilde{f}^{j+1}$ by $\tilde{f}^+$, D_j by D, etc. The map Ψ, whenever defined, will be taken of the form

$$(\xi, \eta, \zeta, \sigma, \nu) \mapsto (\xi + U, \eta + V, \zeta + W, \sigma + \Lambda_1, \nu + \Lambda_2),$$

where $U = U(\xi, \sigma, \nu)$, $V = V(\xi, \eta, \zeta, \sigma, \nu)$ and $W = W(\xi, \eta, \zeta, \sigma, \nu)$. The parameter shifts

$$\sigma \mapsto \sigma + \Lambda_1(\sigma, \nu), \quad \nu \mapsto \nu + \Lambda_2(\sigma, \nu) \quad (8.3)$$

should guarantee that the perturbed and unperturbed tori have the same normal linear parts. The aim is to construct Ψ stepwise such that the perturbations $\tilde{f}^+, \tilde{g}^+, \tilde{h}^+$ are smaller than $\tilde{f}, \tilde{g}, \tilde{h}$ (when excluding a small set of parameters). Our wish is also that in the limit $j \to \infty$ the functions $\tilde{f}^j, \tilde{g}^j, \tilde{h}^j$ tend to zero very fast.

To specify Ψ further, take

$$\Psi = \exp(\overline{\Psi}), \quad \text{for some } \overline{\Psi} \in \mathcal{L}_G. \tag{8.4}$$

Here $\overline{\Psi}$ is chosen of the form

$$\overline{\Psi}(\xi,\eta,\zeta,\sigma,\nu,) = \overline{U}(\xi,\sigma,\nu,)\partial_\xi + \overline{V}(\xi,\eta,\zeta,\sigma,\nu,)\partial_\eta + \overline{W}(\xi,\eta,\zeta,\sigma,\nu,)\partial_\zeta, \tag{8.5}$$

with

$$\begin{aligned} \overline{V}(\xi,\eta,\zeta,\sigma,\nu,) &= \overline{V}_0(\xi,\sigma,\nu,) + \overline{V}_1(\xi,\sigma,\nu,)\eta + \overline{V}_2(\xi,\sigma,\nu,)\zeta, \\ \overline{W}(\xi,\eta,\zeta,\sigma,\nu,) &= \overline{W}_0(\xi,\sigma,\nu,) + \overline{W}_1(\xi,\sigma,\nu,)\eta + \overline{W}_2(\xi,\sigma,\nu,)\zeta. \end{aligned}$$

To make the iteration work, the unknowns $\overline{U}, \overline{V}, \overline{W}, \Lambda_1, \Lambda_2$ will be determined by the following *truncation* of system (7.52), which have to be solved in terms of the *perturbation* $\tilde{f}, \tilde{g}, \tilde{h}$ (compare with [59, 13, 51]),

$$\operatorname{ad} X(\overline{\Psi}) = L + \mathcal{N}, \tag{8.6}$$

where

$$L(\xi,\eta,\zeta,\sigma,\nu) = (\tilde{X} - X)_{lin,d}(\xi,\eta,\zeta,\sigma,\nu) \tag{8.7}$$

and

$$\mathcal{N}(\xi,\eta,\zeta,\sigma,\nu) = \Lambda_1(\sigma,\nu)\partial_\xi + A(\Lambda_2(\sigma,\nu))\zeta\partial_\zeta.$$

The integer d in (8.7) indicates an appropriate order of truncation in the Fourier series with respect to the ξ, to be determined later. Equation (8.6) is also called (*linearized*) *homological equation.* Observe that by the reversibility

$$\tilde{g}(0,\eta,\zeta,\sigma,\nu) = 0, \qquad \partial_\eta \tilde{g}(0,0,0,\sigma,\nu) = 0$$

and

$$\partial_\zeta \tilde{g}(0,0,0) = 0, \qquad \tilde{h}_1(0,0,0,) = 0,$$

where $\tilde{h} = (\tilde{h}_1, \tilde{h}_2)$ is in accordance with the splitting of $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$ in eigenspaces of R.
The strategy is to solve (8.6) in $\overline{\Psi}$ and $\mathcal{N}$, where the parameters vary in a neighbourhood of $\mathcal{G}(\Gamma)_\gamma$, by finding $\overline{\Psi} \in \mathcal{L}_G$ and $\mathcal{N}$ such that the sum $\mathcal{N} + L$ belongs to $\operatorname{ad} X(\mathcal{L}_G)$.

8.3 The Formal Solution

Fourier expanding in ξ gives for the given perturbation $\tilde{X}$:

$$\begin{aligned}
\tilde{f}(\xi,0,0,\sigma,\nu) &= \sum_{|k|\le d} \tilde{f}_k(\sigma,\nu)\exp(\langle k,\xi\rangle),\\
\tilde{g}(\xi,0,0,\sigma,\nu) &= \sum_{|k|\le d} \tilde{g}_k(\sigma,\nu)\exp(\langle k,\xi\rangle),\\
\tilde{h}(\xi,0,0,\sigma,\nu) &= \sum_{|k|\le d} \tilde{h}_k(\sigma,\nu)\exp(\langle k,\xi\rangle).
\end{aligned}$$

For the unknown transformation $\overline{\Psi}$ we set

$$\begin{aligned}
\overline{U} &= \sum_{|k|\le d} \overline{U}_k(\sigma,\nu)\exp(i\langle k,\xi\rangle),\\
\overline{V}_j &= \sum_{|k|\le d} \overline{V}_{j,k}(\sigma,\nu)\exp(i\langle k,\sigma\rangle),\ \ j=0,1,2,\\
\overline{W}_j &= \sum_{|k|\le d} \overline{W}_{j,k}(\sigma,\nu)\exp(i\langle k,\xi\rangle),\ \ j=0,1,2.
\end{aligned}\tag{8.8}$$

Comparing the coefficients of $\exp(i\langle k,\xi\rangle)$ in the truncated system (8.6) gives:

$$\begin{aligned}
&\Lambda_1(\sigma,\nu) = -\tilde{f}_0(\sigma,\nu),\\
&i\langle k,\sigma\rangle\overline{U}_k = \tilde{f}_k(\sigma,\nu), \qquad k\neq 0,\\
&i\langle k,\sigma\rangle\overline{V}_{0,k} = \tilde{g}_k(\sigma,\nu), \qquad k\neq 0,\\
&i\langle k,\sigma\rangle\overline{V}_{1,k} = (\tilde{g}_\eta)_k(\sigma,\nu), \qquad k\neq 0,\\
&[i\langle k,\sigma\rangle I + \Omega(\nu)]\,\overline{V}_{2,k} = (\tilde{g}_\zeta)_k, \quad\ k\neq 0,\\
&\Omega(\nu)\overline{V}_{2,0} = (\tilde{g}_\eta)_0,
\end{aligned}\tag{8.9}$$

moreover

$$\begin{aligned}
&[i\langle k,\sigma\rangle I - \Omega(\nu)]\,\overline{W}_{0,k} = \tilde{h}_k(\sigma,\nu), \qquad k\neq 0,\\
&[i\langle k,\sigma\rangle I - \Omega(\nu)]\overline{W}_{1,k} = (\tilde{h}_\eta)_k(\sigma,\nu), \qquad k\neq 0,\\
&\overline{W}_{1,0} = \Omega(\nu)^{-1}(\tilde{h}_\eta)_0(\sigma,\nu),\\
&\overline{W}_{0,0} = \Omega(\nu)^{-1}\tilde{h}_0(\sigma,\nu),
\end{aligned}\tag{8.10}$$

and finally

$$[i\langle k,\sigma\rangle I - ad\Omega(\nu)]\overline{W}_{2,k} = \tilde{h}_{\zeta,k}(\sigma,\nu,) + A(\Lambda_2(\sigma,\nu)). \tag{8.11}$$

The functions $\overline{U}_0(\sigma,\nu)$, $\overline{V}_{0,0}(\sigma,\nu)$ and $\overline{V}_{1,0}(\sigma,\nu)$ are arbitrary (because of the G-equivariance structure). For simplicity, and for obtaining a unique solution, they are fixed equal to zero. Recalling that $\lambda_q = \mathrm{Re}\,\lambda_q + \mathrm{Im}\,\lambda_q$, $(q = 1,\cdots,4)$ are the eigenvalues of the matrix $\Omega(\nu)$, equations (8.9)-(8.10) can be solved if and only if

$$\begin{aligned} \langle k,\sigma\rangle &\neq 0,\ \Omega\,\text{invertible}, \\ i\langle k,\sigma\rangle &\neq \lambda_q,\ \ (1\leq q\leq 4), \\ i\langle k,\sigma\rangle &\neq -\lambda_q,\ \ (1\leq q\leq 4). \end{aligned} \tag{8.12}$$

It therefore remains to solve equation (8.11). The cases $k\neq 0$ and $k=0$ are treated separately.

If $k\neq 0$, equation (8.11) in both generic and semisimple case admits the solution

$$\overline{W}_{2,k} = [i\langle k,\sigma\rangle \mathrm{I} - \mathrm{ad}\,\Omega(\nu)]^{-1}(\tilde{h}_\zeta)_k(\sigma,\nu), \tag{8.13}$$

if and only if the operator $[i\langle k,\sigma\rangle\mathrm{I} - ad\,\Omega(\nu)]$ is invertible. That is, if all the eigenvalues of $\mathrm{ad}\,\Omega(\nu)$ are unequal to $i\langle k,\sigma\rangle$. Therefore we turn to the spectrum of $\mathrm{ad}\,\Omega$, which is nothing but the spectrum of its semisimple part. From Lemma 2.5 it follows that the solvability condition in this case is

$$i\langle k,\sigma\rangle \neq \lambda_q - \lambda_p. \tag{8.14}$$

If $k=0$ equation (8.11) reads

$$-\mathrm{ad}\,\Omega(\nu)\,\overline{W}_{2,0} = \tilde{h}_{\zeta,0} + A(\Lambda_2(\sigma,\nu)), \tag{8.15}$$

and a distinction between the two settings (**S**) and (**G**) is necessary. In particular, the existence (and uniqueness) of solution follows from the lemma below.

Lemma 8.1. *Consider a linear reversible operator* $\Omega_0 \in gl_{-R}(4,\mathbb{R})$ *with SN-decomposition* $\Omega_0 = S_0 + N_0$, *and let* $\Omega : \mu\in\mathbb{R}^c \mapsto \Omega(\mu) := \Omega_0 + A(\mu) \in gl_{-R}(4,\mathbb{R})$ *be its* LCU *as in Theorem 12. Then, if* $h\in gl_{-R}(2p,\mathbb{R})$ *is a given matrix, the equation*

$$-\mathrm{ad}\,\Omega(\mu)W = h + A(\Lambda) \tag{8.16}$$

admits a (unique) solution $\Lambda = \Lambda(\mu,h)\in\mathbb{R}^c$ *and* $W = W(\mu,h)\in gl_{-R}(2p,\mathbb{R})$.

Proof. Consider the direct sum splittings

$$gl_{-R}(2p,\mathbb{R}) = \mathrm{ad}\,(\Omega_0)\,\big(gl_{+R}(2p,\mathbb{R})\big) \oplus C_-(\Omega_0),$$
$$gl_{-R}(2p,\mathbb{R}) = \big(\ker\,(\mathrm{ad}\,(\Omega_0)) \cap gl_{-R}(2p,\mathbb{R})\big) \oplus Y_0,$$

with Y_0 appropriately chosen. By π denote the projection of $gl_{-R}(2p,\mathbb{R})$ onto the subspace $\mathrm{ad}\,(\Omega_0)\,\big(gl_{+R}(2p,\mathbb{R})\big)$ parallel to $C_-(\Omega_0)$. Solving (8.16) then is equivalent to solving the system

$$-\pi\big(\mathrm{ad}\,(\Omega(\mu))\,W\big) = \pi(h) \tag{8.17}$$
$$-(I-\pi)\big(\mathrm{ad}\,(\Omega(\mu))\,W\big) = (I-\pi)(h) + A(\Lambda). \tag{8.18}$$

We first solve (8.17) that does not depend on A. The map $\mathrm{ad}\,(\Omega_0)\,|_{Y_0} : Y_0 \to \mathrm{ad}\,(\Omega_0)\,\big(gl_{+R}(2p,\mathbb{R})\big)$ is an isomorphism, which implies that

$$-\pi\,(\mathrm{ad}\,(\Omega(\mu)))\,|_{Y_0} : Y_0 \to \mathrm{ad}\,(\Omega_0)\,\big(gl_{+R}(2p,\mathbb{R})\big)$$

is an isomorphism for small μ ($\mu \approx 0$). Hence, by the Implicit Function Theorem there exist a neighbourhood $\mathcal{Q}$ of the origin in $\mathbb{R}^c$ and a map $W : \mathcal{Q} \to Y_0$, $\mu \mapsto W(\mu)$, such that

$$-\pi\big(\mathrm{ad}\,(\Omega(\mu))\,W(\mu)\big) = \pi(h)$$

holds true for all $\mu \in \mathcal{Q}$. Replacing W by $W(\mu)$ in (8.18), we find the equation

$$-(I-\pi)\big(\mathrm{ad}\,(\Omega(\mu))\,W(\mu)\big) = (I-\pi)(h) + A(\Lambda).$$

Since A is an isomorphism, this equation always admits a solution

$$\Lambda = \Lambda(\mu) := A^{-1}\Big(-(I-\pi)\big(\mathrm{ad}\,(\Omega(\mu))\,W(\mu) - (h)\big)\Big). \tag{8.19}$$

Last but not least observe that Y_0 can always be chosen as in (8.37) below. Observe that if Ω_0 is semisimple, i.e., if $N_0 = O$, then the choice of the splittings is trivial. We just take $Y_0 = \mathrm{ad}\,\Omega_0(gl_{+R}(2p,\mathbb{R}))$. □

One verifies that the system of formal solvability conditions (8.12) and (8.14) is implied by

$$\det\Omega \neq 0$$
$$\langle k,\sigma\rangle \neq \langle \omega^N, l\rangle \qquad \forall l \in \mathbb{Z}^r,\ |l| \le 2. \tag{8.20}$$

Observe that the non-resonance conditions (8.12) and (8.20) are implied by the Diophantine conditions, which also imply the convergence of the formal series.

8.4 Estimates for the Iteration Step

As mentioned before, the solution constructed in section 8 is going to be used in the Inverse Approximation Lemma (compare [13], chapter 6). This means we have to determine a geometric sequence $\{r_j\}_{j\geq 0}$ such that for each $j \geq 1$, for any $\beta \notin \mathbb{N}$ and for some constant M, the inequality $|\Phi^+ - \Phi|_{D^+} \leq Mr^\beta$ holds. This is needed to guarantee the Whitney-differentiability of the conjugacy Φ ([64, 13, 51, 14], etc.). For this purpose, the functions U, V, W need at least be well defined on D^e_+. Estimates for $|\tilde{f}^+|_{D^e}$, $|\tilde{g}^+|_{D^e}$, $|\tilde{h}^+|_{D^e}$ in terms of $|\tilde{f}|_{D^e}$, $|\tilde{g}|_{D^e}$, $|\tilde{h}|_{D^e}$ are needed, as well as for the deviation of the map $\Psi = \exp \overline{\Psi}$ from the identity. The complex domains D^e and the order of truncation d are specified in this section.

Proposition 8.2 and Lemma 8.4 below conclude our considerations by establishing the claim that the (convergence) proof of [14] ([13, 15]) applies here too. A different aspect is the extra dimension y and the construction of $\overline{W}_{2,0}$. We recall that the main tools are the Paley-Wiener Lemma ([13]) on the exponential decay of Fourier coefficients of analytic functions, the Cauchy Integral Formula ([13]) and Gronwall's inequality.

8.4.1 Preliminaries

Analytic maps will be estimated by the supremum-norm on (appropriate) complex domains of definition. In the case of matrices the operator-norm is used.

Let $(s^j)_{j\geq 0}$ be any geometric sequence of positive numbers with ratio less than $\frac{1}{2}$ and next define the geometric sequence $(r^j)_{j\geq 0}$ by

$$r_j = s_j^{2\tau+2},\ j \geq 0.$$

Let $(\varepsilon_j)_{j\geq 0}$ and $(\rho_j)_{j\geq 0}$ be sequences of positive numbers such that for all $j \geq 0$

$$\varepsilon_{j+1} < \tfrac{1}{2}\varepsilon_j, \qquad \rho_{j+1} < \tfrac{1}{2}\rho_j.$$

We anticipate that ε_j, ρ_j will be fixed as exponential series. Then, referring to section 7.6 for the notation, define

$$D_j := \left(\mathbb{T}^2 + \frac{1}{2}\kappa + s_j\right) \times \left(\mathcal{G}(\Gamma)'_\gamma + r_j\right), \tag{8.21}$$

and

$$D_j^e := D_j \times \mathcal{Y}_j \times \mathcal{Z}_j. \tag{8.22}$$

To ensure $D_0^e \subset \mathcal{O}$ assume that

$$0 < s_0 < \min\{\tfrac{1}{2}\kappa, \frac{1}{2}\}, \quad \varepsilon_0 < \frac{1}{2}, \quad \text{and } \rho_0 \text{ small enough.}$$

In the last part of the proof, (see section 8.4.3), the sequences $(s_j)_{j\geq 0}$, $(\varepsilon_j)_{j\geq 0}$, $(\rho_j)_{j\geq 0}$ will be fixed in such a way that the iteration process of section 8 converges. If no confusion is possible, we again adopt the +notation. We introduce intermediate sets between D_+^e and D^e as follows. Define the numbers

$$\begin{aligned}
s_* &= \tfrac{1}{2}(s + s_+) & s_{**} &= \tfrac{1}{3}(2s + s_+) \\
r_* &= \tfrac{1}{2}(r + r_+) & r_{**} &= \tfrac{1}{3}(2r + r_+) \\
\varepsilon_* &= \tfrac{1}{2}(\varepsilon + \varepsilon_+) & \varepsilon_{**} &= \tfrac{1}{3}(2\varepsilon + \varepsilon_+) \\
\rho_* &= \tfrac{1}{2}(\rho + \rho_+) & \rho_{**} &= \tfrac{1}{3}(2\rho + \rho_+).
\end{aligned} \tag{8.23}$$

and with these the set $D_* = (\mathbb{T}^n + \frac{1}{2}\kappa + s_*) \times \left(\mathcal{G}(\Gamma)'_\gamma + r_*\right)$. Define D_*^e, D_{**}^e, $\mathcal{Y}_*$, $\mathcal{Y}_{**}$, $\mathcal{Z}_*$, $\mathcal{Z}_{**}$ accordingly.

Remark Throughout the proof positive constants appear depending only on n, m, τ, κ. If we need not remember these we shall neglect them. The corresponding inequality sign will be denoted by $\leq c$, meaning that the estimates are true up to some constant that we do not specify. A subscript is used for constants that have to be remembered for later use.

8.4.2 Estimates for the KAM Step

Estimates on $\overline{\Psi}$ and some of its derivatives are given. The definition of $\overline{\Psi}$ uses an order of truncation d in the Fourier series, which is now specified as

$$d := [s^{-2}]$$

where $[\cdot]$ denotes the integral part. To guarantee convergence of the Fourier series, the denominators in the expression (8.8) of $\overline{\Psi}$ are estimated as follows.

Proposition 8.2. *For all $(\sigma, \nu) \in \mathcal{G}(\Gamma)'_\gamma + r$, for all $0 < |k| < d$ and all eigenvalues λ_q $(q = 1, \ldots, 4)$ of $\Omega(\nu)$ we have that $|\, i < k, \sigma >|$, $|\, i < k, \sigma > - \lambda_q|$, $|\, i < k, \sigma > - \lambda_p + \lambda_q|$ are bounded from below by $\frac{1}{4}|k|^{-\tau}$.*

Proof. Set $r := 4s^{2\tau+2}$, then $d \approx r^{-\frac{1}{\tau+1}}$, and both $r, d \to 0$ as $j \to \infty$.
(i) By definition $d \leq s^{-2} = \frac{1}{4}r^{-(\tau+1)^{-1}}$, therefore, for $0 \leq |k| \leq d$, it holds

$$r|k| \leq \tfrac{1}{4}|k|^{-\tau}.$$

For $(\sigma, \nu) \in \mathcal{G}(\Gamma)'_\gamma + r_j$ there exists a pair $(\sigma^*, \nu^*) \in (I \times \Gamma)'_\gamma$ such that for all k $(0 \leq |k| \leq d)$ one has

$$\begin{aligned} |<\sigma, k>| &\geq |<\sigma^*, k>| - |\sigma - \sigma^*|\,|k| \geq |k|^{-\tau} - r\,|k| \\ &\geq |k|^{-\tau} - \tfrac{1}{4}\,|k|^{-\tau} = \tfrac{3}{4}|k|^{-\tau} > \tfrac{1}{4}|k|^{-\tau}. \end{aligned}$$

(ii) To prove the last inequality note that for any $z \in \mathbb{C}$ we have that $|z| \geq |\operatorname{Im} \mathrm{z}|$. For $(\sigma, \nu) \in \mathcal{G}(\Gamma)'_\gamma + r$ there exists (σ^*, ν^*) such that $|\sigma - \sigma^*| \leq r$ and $|\nu - \nu^*| \leq r$. It follows that

$$\begin{aligned} &|\,i<k,\sigma> - \lambda_p + \lambda_q| \geq |<k,\sigma> - \nu_p + \nu_q| \\ &\geq \left|<k,\sigma^*> - \nu_p^* + \nu_q^*\right| - |<k,\sigma-\sigma^*>| - \left|\nu_p - \nu_p^*\right| - \left|\nu_q - \nu_q^*\right| \\ &\qquad \geq |k|^{-\tau} - r|k| - 2r \geq |k|^{-\tau} - \tfrac{1}{4}\,|k|^{-\tau} - \tfrac{1}{2}|k|^{-\tau} = \tfrac{1}{4}|k|^{-\tau} \end{aligned}$$

□

Proposition 8.3. *Assume the diophantine conditions. Then, the vector field* $\overline{\Psi} = \overline{U}\partial_x + \overline{V}\partial_y + \overline{W}\partial_z$ *is real analytic on* D. *For the maps* $\overline{U}, \overline{V}, \overline{W}$ *we have:*

(i) $s^{2\tau}|\overline{U}|_{D_{**}}, s^{2\tau+1}|\frac{\partial\overline{U}}{\partial\xi}|_{D_{**}} \leq c_0|\tilde{f}|_{D^e}$;

(ii) $s^{2\tau}|\overline{V}|_{D_{**}}, s^{2\tau+1}|\frac{\partial\overline{V}}{\partial\xi}|_{D_{**}}, \varepsilon s^{2\tau}|\frac{\partial\overline{V}}{\partial\eta}|_{D_{**}}, \rho s^{2\tau}|\frac{\partial\overline{V}}{\partial\zeta}|_{D_{**}} \leq c_0|\tilde{g}|_{D^e}$;

(iii) $\rho s^{2\tau}|\overline{W}_{2,0}|_{D_{**}} \leq c_0|\tilde{h}|_{D_e}$ *and* $\rho s^{2\tau+1}|\overline{W}_{2,k}|^{k\geq 1}_{D_{**}} < c|\tilde{h}|_{D^e}$;

(iv) $s^{2\tau}|\overline{W}|_{D_{**}}, s^{2\tau+1}|\frac{\partial\overline{W}}{\partial\xi}|_{D_{**}}, \varepsilon s^{2\tau}|\frac{\partial\overline{W}}{\partial\eta}|_{D_{**}}, \rho s^{2\tau}|\frac{\partial\overline{W}}{\partial\zeta}|_{D_{**}} \leq c_0|\tilde{h}|_{D^e}$.

The inequality (iii) follows directly from the formula (8.11). All other inequalities are analogous to those in [14] and the proofs follow the scheme in [13] (chapter 6), and [15], (Prop. 5.3). We will omit them here.

For the estimates of the errors $|\tilde{f}^+|$, $|\tilde{g}^+|$, $|\tilde{h}^+|$ on D^e_+ and of the components of $\Psi = \exp(\overline{\Psi})$ proceed as follows. Consider first the deviation of the map Ψ from the identity and introduce the notation:

$$\exp(t\overline{\Psi}) = (x(t,\xi), y(t,\xi,\eta,\zeta), z(t,\xi,\eta,\zeta)), \quad t \in \mathbb{R},\ (\xi,\eta,\zeta) \in D^e_*.$$

We suppressed the parameters (σ, ν).

Lemma 8.4. *Assume that*

$$|\tilde{f}|_{D^e} \leq \frac{1}{12c_0} s^{2\tau+1}, \qquad \max\left(|\tilde{g}|_{D^e}, |\tilde{h}|_{D^e}\right) \leq \frac{1}{48c_0} \max(\varepsilon,\rho) s^{2\tau}. \tag{8.24}$$

Then $(x(t,\xi), y(t,\xi,\eta,\zeta), z(t,\xi,\eta,\zeta)) \in D^e_{**}$, *if* $(\xi,\eta,\zeta) \in D^e_*$ *and* $0 \leq t \leq 1$. *Furthermore,*

$$\begin{array}{rcl} x(t,\xi) & = & \xi + U(t,\xi), \\ \begin{pmatrix} y(t,\xi,\eta,\zeta) \\ z(t,\xi,\eta,\zeta) \end{pmatrix} & = & \begin{pmatrix} \eta \\ \zeta \end{pmatrix} + \begin{pmatrix} V_0(t,\xi) \\ W_0(t,\xi) \end{pmatrix} \\ & & + \begin{pmatrix} V_1(t,\xi) & V_2(t,\xi) \\ W_1(t,\xi) & W_2(t,\xi) \end{pmatrix} \begin{pmatrix} \eta \\ \zeta \end{pmatrix}. \end{array}$$

For $0 \leq t \leq 1$ *and* $\xi \in D_*$, *it holds*

(i) $|U(t,\xi)| \leq |\overline{U}|_{D_{**}}$,

(ii) $\max|\left(V_0(t,\xi), W_0(t,\xi)\right)| \leq 2\max\left(|\overline{V}_0|, |\overline{W}_0|\right)_{D_{**}}$,

(iii) $\max|\left(V_{1,2}(t,\xi), W_{1,2}(t,\xi)\right)| \leq 2\max\left(|\overline{V}_{1,2}|, |\overline{W}_{1,2}|\right)_{D_{**}}$.

Proof. Set $s := (y,z) \in \mathbb{R}^m \times \mathbb{R}^4$, and

$$\overline{S}_0 := \begin{pmatrix} V_0(t,\xi) \\ W_0(t,\xi) \end{pmatrix}, \quad \overline{S}_1 := \begin{pmatrix} V_1 & V_2 \\ W_1 & W_2 \end{pmatrix} \begin{pmatrix} \eta \\ \zeta \end{pmatrix}.$$

The maps $t \mapsto x(t,\xi)$, $t \mapsto s(t,\xi,(\eta,\zeta))$ satisfy the Cauchy problem

$$\begin{aligned} &\dot{x} = \overline{U}(x), \\ &\dot{s} = \overline{S}_0(x) + \overline{S}_1(x)s, \\ &x(0,\xi) = \xi \\ &s(0,\xi,(\eta,\zeta)) = (\eta,\zeta). \end{aligned}$$

The maps $\overline{U}, \overline{S}_i$ are bounded. So, for $t \approx 0$ there exist unique solution of the system above. Write these solution as

$$x(t,\xi) = \xi + U(t,\xi)$$
$$s(t,\xi,(\eta,\zeta)) = (\eta,\zeta) + S_0(t,\xi) + S_1(t,\xi)(\eta,\zeta);$$

note that $S_0(t,\xi) = s(t,\xi,0)$. We need that for $(\xi,(\eta,\zeta)) \in D^e_*$ and $t \in [0,1]$ the vector $(x,s) \in D^e_{**}$. This guarantees that $\Psi : D^e_* \to D^e_{**}$.
Since the estimate for $|U|$ is completely analogous to that in [14](pg. 64), it will be omitted here. For the other component, one obtains

$$\dot{S}_0 = \dot{\overline{S}}_0(x(t,\xi)) + \overline{S}_1(x(t,\xi))S_0, \qquad S_0(0,\xi) = 0.$$

It follows that, for $t \in [0,1]$,

$$S_0(t,\xi) = \int_0^t [\overline{S}_0(x(r,\xi)) + \overline{S}_1(x(r,\xi))S_0(r,\xi)]dr,$$

and by Gronwall's inequality

$$|S_0(t,\xi)|_{D_{**}} \leq |\overline{S}_0|_{D_{**}} \exp |\overline{S}_1|_{D_{**}},$$

so,

$$|\overline{S}_1|_{D_{**}} = \max(|\overline{V}_{1,2}|, |\overline{W}_{1,2}|)_{D_{**}} \leq c_0 \max(\varepsilon^{-1}, \rho^{-1})|(\tilde{g}, \tilde{h})|_{D_*} \leq \frac{1}{48}.$$

Therefore,

$$|S_0(t,\xi)|_{D_{**}} \leq 2|\overline{S}_0|_{D_{**}} \leq \frac{1}{24}\max(\varepsilon,\rho).$$

Similarly,

$$|S_1(t,\xi)|_{D_{**}} \leq 2|\overline{S}_1|_{D_{**}} \leq \frac{1}{24}.$$

Hence, $|s(t,\xi,(\eta,\zeta)) - (\eta,\zeta)| \leq |\overline{S}_0(t,\xi)| + |\overline{S}_1(t,\xi)(\eta,\zeta)| \leq \frac{1}{12}\max(\varepsilon,\rho)$, which is smaller than $[\max(\varepsilon_{**},\rho_{**}) - \max(\varepsilon_*,\rho_*)]$. Therefore, for $t \in [0,1]$, $s(t,\xi,(\eta,\zeta)) \in \mathcal{Y}_{**} \times \mathcal{Z}_{**}$. □

By definition the map $\Psi : D^e_* \to D^e_{**}$ is obtained by taking $t = 1$ in Lemma 8.4. Let us write

$$\Psi(\xi,\eta,\zeta,\sigma,\nu) = (\xi + U, \eta + V, \zeta + W, \sigma + \Lambda_1, \nu + \Lambda_2).$$

The linearity of the unfolding, Cauchy's inequality, and Lemma 8.4 then imply the following estimates.

Corollary 8.5. *In the assumptions of Lemma 8.4 the following holds.*

(i) $|U|_{D_*} \le c_1\, s^{-2\tau}|\tilde f|_{D^e}$,

(ii) $\max(|V|_{D_*}, |W|_{D_*}) \le c_1\, 2s^{-2\tau}\max(|\tilde g|_{D^e}, |\tilde h|_{D^e})$,

(iii) $\max(|\frac{\partial V}{\partial \xi}|_{D_*}, |\frac{\partial W}{\partial \xi}|_{D_*}) \le c_1\, s^{-2\tau+1}\max(|\tilde g|_{D^e}, |\tilde h|_{D^e})$,

(iv) $\max(|\frac{\partial V}{\partial \eta}|_{D_*}, |\frac{\partial W}{\partial \eta}|_{D_*}) \le c_1\, \varepsilon^{-1}s^{-2\tau}\max(|\tilde g|_{D^e}, |\tilde h|_{D^e})$,

(v) $\max(|\frac{\partial V}{\partial \zeta}|_{D_*}, |\frac{\partial W}{\partial \zeta}|_{D_*}) \le c_1\, \rho^{-1}s^{-2\tau}\max(|\tilde g|_{D^e}, |\tilde h|_{D^e})$,

(vi) $|\Lambda_1|_{D_*},\, r|\frac{\partial \Lambda_1}{\partial \sigma}|_{D_{**}},\, r|\frac{\partial \Lambda_1}{\partial \nu}|_{D_{**}} \le c_1\, |\tilde f|_{D^e}$,

(vii) $\rho|\Lambda_2|_{D_*},\, r\rho|\frac{\partial \Lambda_2}{\partial \sigma}|_{D_{**}},\, r\rho|\frac{\partial \Lambda_2}{\partial \nu}|_{D_{**}} \le c_1\, |\tilde h|_{D^e}$.

To proceed estimating the terms $\tilde f^+$, $\tilde g^+$, $\tilde h^+$, observe that (8.6) yields

$$\begin{pmatrix} 1+\frac{\partial U}{\partial \xi} & 0 & 0 \\ \frac{\partial V}{\partial \xi} & 1+\frac{\partial V}{\partial \eta} & \frac{\partial V}{\partial \zeta} \\ \frac{\partial W}{\partial \xi} & \frac{\partial W}{\partial \eta} & 1+\frac{\partial W}{\partial \zeta} \end{pmatrix} \begin{pmatrix} \tilde f^+ \\ \tilde g^+ \\ \tilde h^+ \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} \tag{8.25}$$

with

$$\begin{aligned} R_1 &= \tilde f(\xi+U, \eta+V, \zeta+W, \sigma+\Lambda_1, \nu+\Lambda_2) - \tilde f_d(\xi, 0, 0, \sigma, \nu) \\ &\quad +\mathcal{D}_1(\overline{U}-U)(\xi, \sigma, 0), \\ R_2 &= \tilde g(\xi+U, \eta+V, \zeta+W, \sigma+\Lambda_1, \nu+\Lambda_2) \\ &\quad +\mathcal{D}_2(\overline{V}-V)(\xi, \sigma, 0) \\ &\quad -\{\tilde g(\xi,0,0,\sigma,\nu) + \frac{\partial}{\partial \eta}\tilde g(\xi,0,0,\sigma,\nu)\eta + \frac{\partial}{\partial \zeta}\tilde g(\xi,0,0,\sigma,\nu)\zeta\}_d, \\ R_3 &= \tilde h(\xi+U, \eta+V, \zeta+W, \sigma+\Lambda_1, \nu+\Lambda_2) \\ &\quad +\mathcal{D}_3(\overline{W}-W)(\xi, \eta, \zeta, \sigma, \nu) \\ &\quad -\{\tilde h(\xi,0,0,\sigma,\nu) + \frac{\partial}{\partial \eta}\tilde h(\xi,0,0,\sigma,\nu)\eta + \frac{\partial}{\partial \zeta}\tilde h(\xi,0,0,\sigma,\nu)\zeta\}_d. \end{aligned} \tag{8.26}$$

We denoted $\mathcal{D}_i$ $(i=1,2,3)$:

$$\begin{aligned} \mathcal{D}_1(\overline{U}-U) &= \frac{\partial \overline{U}}{\partial \xi}\sigma - \frac{\partial U}{\partial \xi}\sigma, \\ \mathcal{D}_2(\overline{V}-V) &= \frac{\partial(\overline{V}-V)}{\partial \xi}\sigma + \frac{\partial(\overline{V}-V)}{\partial \zeta}\Omega(\nu)\zeta, \end{aligned}$$

and

$$\mathcal{D}_3(\overline{W}-W)=\frac{\partial(\overline{W}-W)}{\partial\xi}\sigma+\frac{\partial(\overline{W}-W)}{\partial\zeta}\Omega(\nu)\zeta-\Omega(\nu)(\overline{W}-W).$$

It follows that

$$\begin{aligned}\tilde{f}^+&=(1+\frac{U}{\partial\xi})^{-1}R_1,\\ \tilde{g}^+&=\mathcal{Q}^{-1}[(1+\frac{\partial W}{\partial\zeta})R_2-\frac{\partial V}{\partial\xi}\tilde{f}^+-\frac{\partial V}{\partial\zeta}(R_3-\frac{\partial W}{\partial\xi}\tilde{f}^+)],\\ \tilde{h}^+&=\mathcal{Q}^{-1}[-\frac{\partial W}{\partial\eta}(R_2-\frac{\partial V}{\partial\xi}\tilde{f}^+)+(1+\frac{\partial V}{\partial\eta})(R_3-\frac{\partial W}{\partial\xi}\tilde{f}^+)],\end{aligned}\tag{8.27}$$

where $\mathcal{Q}:=(1+\frac{\partial V}{\partial\eta})(1+\frac{\partial W}{\partial\zeta})-\frac{\partial W}{\partial\eta}\frac{\partial V}{\partial\xi}$. The estimates of R_i, $i=1,2,3$, need be independent of γ and are obtained using (7.52). These estimates are similar to those of R_1, R_2 in [14] (pg. 71-72), compare also with Proposition 6.6 in [13] and the proof of Proposition 5.7 in [15]. We omit them here. Estimating term by term (8.26) and (8.27) allows then to bound the terms $|\tilde{f}^+|_{D^e}$, $|\tilde{g}^+|_{D^e}$, $|\tilde{h}^+|_{D^e}$ from below.

Corollary 8.6. *Under the assumptions of Lemma 8.4, if $ds>2n$ and moreover $\Psi(D^e_+)\subseteq D^e_*$, then $|\tilde{f}^+|_{D^e_+}$, $|\tilde{g}^+|_{D^e_+}$, $|\tilde{h}^+|_{D^e_+}$ are bounded from below. Set $\mathrm{M}_{\tilde{g},\tilde{h}}:=\max\Big(|\tilde{g}|_{D^e},|\tilde{h}|_{D^e}\Big)$. Then it holds:*

(i) $|\tilde{f}^+|_{D^e_+}\le c\,(I+II+III+IV)$, *where*

$$\begin{aligned}I&=|f|_{D^e}d^ne^{-\frac{1}{2}ds},\\ II&=\max(\varepsilon^{-1}\varepsilon^+,\rho^{-1}\rho^+)|f|_{D^e},\\ III&=\max\{\max(\varepsilon^{-1}\varepsilon^+,\rho^{-1}\rho^+),d^ne^{-\frac{1}{2}ds}\}|f|_{D^e},\\ IV&=\max\{s^{-(2\tau+2)}|f|^2_{D^e},\\ &\qquad s^{-(2\tau+2)}\max(\varepsilon^{-1},\rho^{-1})|f|_{D^e}|g|_{D^e}|h|_{D^e}\};\end{aligned}$$

(ii) $|\tilde{g}^+|_{D^e_+}\le c\,\{V+VI\}$, *where*

$$\begin{aligned}V&=\frac{s^{-2\tau}}{\varepsilon}\mathrm{M}_{\tilde{g},\tilde{h}}\,|R_2-R_3|_{D^e_+}\\ VI&=-s^{-(2\tau+1)}\mathrm{M}_{\tilde{g},\tilde{h}}\,|\tilde{f}^+|_{D^e_+}+|R_2|_{D^e_+}\end{aligned}$$

The estimates of $|R_2-R_3|_{D^e_+}$ and $|R_2|_{D^e_+}$ are given below;

(iii) $|\tilde{h}^+|_{D^e_+} \le c\{VII + VIII\}$, *where*

$$VII = \frac{s^{-2\tau}}{\varepsilon} \mathrm{M}_{\tilde{g},\tilde{h}}\, |R_3 - R_2|_{D^e_+}$$
$$VIII = -s^{-(2\tau+1)} \mathrm{M}_{\tilde{g},\tilde{h}}\, |\tilde{f}^+|_{D^e_+} + |R_3|_{D^e_+}.$$

The estimates of $|R_3 - R_2|_{D^e_+}$ *and* $|R_3|_{D^e_+}$ *are given below.*

Proof. The proof of (i) is analogous to that of [15] (Prop.5.7), and is omitted. Note that the assumption $ds > 2n$ is needed for example to get I, cf. [13]. For the other inequalities proceed as follows. To estimate $|\tilde{g}^+|_{D^e}$ and $|\tilde{h}^+|_{D^e}$, we need estimates for R_2, R_3. Indeed, (8.27) implies

$$\begin{aligned} |\tilde{g}|_{D^e_+} \quad \le \quad & c\Big\{\varepsilon^{-1} s^{-2\tau}\, \mathrm{M}_{\tilde{g},\tilde{h}}\, |R_2 - R_3|_{D^e_+} \\ & \qquad - s^{-(2\tau+1)}\, \mathrm{M}_{\tilde{g},\tilde{h}}\, |\tilde{f}^+|_{D^e_+} + |R_2|_{D^e_+}\Big\} \end{aligned}$$

and

$$\begin{aligned} |\tilde{h}|_{D^e_+} \quad \le \quad & c\Big\{\varepsilon^{-1} s^{-2\tau}\, \mathrm{M}_{\tilde{g},\tilde{h}}\, |R_3 - R_2|_{D^e_+} \\ & \qquad - s^{-(2\tau+1)}\, \mathrm{M}_{\tilde{g},\tilde{h}}\, |\tilde{f}^+|_{D^e_+} + |R_3|_{D^e_+}\Big\}. \end{aligned}$$

Estimating term by term the right hand side of (8.26) gives

$$\begin{aligned} |R_2|_{D^e_+} \quad \le \quad & \big|\mathcal{D}_2(\overline{V} - V)(\xi, \sigma, 0)\big|_{D^e_+} \\ & + |\tilde{g}(\xi + U, \dots, \nu + \Lambda_2) - \tilde{g}(\xi, \dots, \nu)|_{D^e_+} \\ & + \Bigg|\tilde{g}(\xi, \dots, \nu) - \{\tilde{g}(\xi, 0, 0, \sigma, \nu) + \frac{\partial}{\partial \eta}\tilde{g}(\xi, 0, 0, \sigma, \nu)\eta \\ & \qquad + \frac{\partial}{\partial \zeta}\tilde{g}(\xi, 0, 0, \sigma, \nu)\zeta\}_d\Bigg|. \end{aligned}$$

Therefore,

$$\begin{aligned} |R_2|_{D^e_+} \le c\big\{ & (r_+ s^{-(2\tau+1)} + \rho_+ \rho^{-1} s^{-2\tau}) \left(|\tilde{g}|_{D^e} - \mathrm{M}_{\tilde{g},\tilde{h}}\right) + \max\big(\\ & s^{-(2\tau+2)}|\tilde{g}||\tilde{f}|, 2s^{-2\tau}\max(\varepsilon^{-1}, \rho^{-1})\mathrm{M}_{\tilde{g},\tilde{h}}\, |\tilde{g}|_{D^e}, s^{-(2\tau+1)}\rho^{-1}|\tilde{g}||\tilde{h}|\big) \\ & + |D_2\tilde{g}(\xi, 0, 0, \sigma, \nu)|\,(|\eta|^2 + |\zeta|^2) + \Bigg|\sum_{|j| \ge d} \tilde{g}_j(\sigma, \nu)e^{i<j,\xi>}\Bigg| \big\}. \end{aligned}$$

Similarly for R_3,

$$
\begin{aligned}
|R_3|_{D^e_+} \;\leq\; & c\Big\{(r_+ s^{-(2\tau+1)} + \rho^+\rho^{-1}s^{-2\tau})\left(|\tilde{h}|_{D^e} - \mathrm{M}_{\tilde{g},\tilde{h}}\right) \\
& + s^{-2\tau}\left(2\mathrm{M}_{\tilde{g},\tilde{h}} - |\tilde{h}|_{D^e}\right) + \max\Big\{s^{-(2\tau+2)}|\tilde{h}|_{D^e}|\tilde{f}|_{D^e}, \\
& 2s^{-2\tau}\max(\varepsilon^{-1},\rho^{-1})\mathrm{M}_{\tilde{g},\tilde{h}}\,|\tilde{h}|_{D^e}, s^{-(2\tau+1)}\rho^{-1}|\tilde{h}|^2_{D^e}\Big\} \\
+ \;& \left|D_2\tilde{h}(\xi,0,0,\sigma,\nu)\right|\left|(|\eta|^2+|\zeta|^2)\right| + \left|\sum_{j\geq d}\tilde{h}_j(\sigma,\nu)e^{i<j,\xi>}\right|\Big\}.
\end{aligned}
$$

To complete the proof observe that:

$$
\begin{aligned}
\left|\sum_{j\geq d}\tilde{g}_j(\sigma,\nu)e^{(i<j,\xi>)}\right|_{D^e_+} \;&\leq\; c\,|\tilde{g}|_{D^e}\sum_{|j|>d} e^{|Im\xi|-(\frac{1}{2}\kappa+s)|j|} \\
&\leq\; c\,|\tilde{g}|_{D^e}d^n e^{-\frac{1}{2}sd},
\end{aligned}
$$

we used $|Im\xi| \leq \frac{1}{2}\kappa + s_+$, $s - s_+ > \frac{1}{2}s$ and $ds > 2n$; see [13] page 150 for more details. An analogue estimate holds for $\left|\sum_{|j|\geq d}\tilde{h}_j(\sigma,\nu)e^{i<j,\xi>}\right|$ and by Cauchy inequalities we get

$$
\left|D_2\tilde{h}(\xi,0,0,\sigma,\nu)\right|(|\eta|^2+|\zeta|^2) \leq c\,2|\tilde{h}|_{D^e}(\varepsilon_+^2+\rho_+^2)(\varepsilon^{-2}+\rho^{-2}).
$$

The same holds if we change $\tilde{h}$ by $\tilde{g}$. This complete the proof. □

Estimates of $|\Phi_+ - \Phi|_{D^e_+}$ To apply the Inverse Approximation Lemma, the difference $|\Phi_+ - \Phi|_{D^e_+}$ has to be estimated. Since $\Phi_+ = \Phi \circ \Psi$, setting $\Psi = id + \varphi$, the mean value theorem implies

$$
\begin{aligned}
|\Phi_+ - \Phi|_{D^e_+} \;&\leq\; |D\Phi|_{D_*}|\Psi|_{D^e_+} \leq |D\Phi|_{D_*}|\varphi|_{D^e_+}, \\
|\Phi_+|_{D^e_+} \;&=\; |\Phi\circ\Psi - \Phi + \Phi| \\
&\leq\; \max(|\Phi|_{D_*}, |D\Phi|_{D_*})(1+|\varphi|_{D_+}), \\
|D\Phi_+|_{D^e_+} \;&\leq\; |D\Phi|_{D_*}(1+|D\varphi|_{D_+}).
\end{aligned}
\tag{8.28}
$$

From the previous estimates on U, V, W, it follows that

$$\begin{aligned} |\varphi|_{D^e_+} &\leq c \quad \max\Big\{ s^{-2\tau} |\tilde{f}|_{D^e}, \\ &\qquad s^{-2\tau} \max(\rho^{-1}, \varepsilon^{-1}) \max(|\tilde{g}|_{D^e}|, \tilde{h}|_{D^e}) \Big\}, \\ |D\varphi|_{D^e_+} &\leq c \quad \max\Big\{ s^{-(4\tau+2)} |\tilde{f}|_{D^e}, s^{-(2\tau+2)} \rho^{-1} |\tilde{h}|_{D^e}, \\ &\qquad s^{-(4\tau+2)} \max(\varepsilon^{-1}, \rho^{-1}) \max(|\tilde{g}|_{D^e}|, \tilde{h}|_{D^e}) \Big\}. \end{aligned}$$

8.4.3 Convergence

To complete the proof of Theorem 11, it remains to ensure the convergence of the iteration process. This is achieved by a proper choice of the sequences $(s_j)_{j\geq 0}, (\varepsilon_j)_{j\geq 0}$, and $(\rho_j)_{j\geq 0}$. The domains D_j in the (σ, ν)-directions will shrink geometrically, as $j \to \infty$, while the errors $|\tilde{f}_j|_{D^e_j}, |\tilde{g}_j|_{D^e_j}, |\tilde{h}_j|_{D^e_j}$ as well as the coefficent functions of $\Phi_j - \Phi_{j+1}$ on D_{j+1} will decay exponentially as $j \to \infty$.
First, we introduce two exponential sequences

$$\delta_{j+1} = \delta_j^{1+p} \quad \text{and} \quad \delta_j^q = \max\{\varepsilon_j, \rho_j\}, \quad j \geq 0,\ p, q > 0. \tag{8.29}$$

Proposition 8.7. *Suppose p, q are fixed with $1 < q < 2$, $0 < p < 1 - \frac{q}{2}$. Then for $\delta_0 > 0$ small enough, $\exists s_0 \in (0, min\{\frac{1}{2}\kappa, \frac{1}{2}\})$ such that if $|\tilde{f}^0| < \delta_0$, $max\{|\tilde{g}^0|, |\tilde{h}^0|\} \leq \delta_0^2$, then for all j*

(i) *the assumptions of Lemma 8.4 are satisfied;*

(ii) $\left|\tilde{f}^j\right|_{D^e_j} \leq \delta_j$ *and* $\left|\max(\tilde{g}^j, \tilde{h}^j)\right|_{D^e_j} \leq \delta_j^2$;

(iii) *the sequence $|\Phi^{j+1} - \Phi^j|_{D^e_{j+1}}$ decreases exponentially as $j \to \infty$, with initial terms tending to zero as $\delta_0 \to 0$.*

The proof uses Lemma 8.4 and is analogue to that in [51] (Proposition 8.10), see also [15, 14, 13].
The proof of the KAM Theorem 11 is now completed by taking $\delta = \delta_0$ and applying the Inverse Approximation Lemma to the Φ_j on the domains D_j $(j \geq 0)$.

Remark It has not escaped our attention that the KAM theorem we proved for $p = 2$ has straightforward generalization for any $p > 2$. Our approach suggests the crucial role of the unfolding theory, which we already give in a general. So, the question is now how to generalize Fig. 7.3 describing the underlying geometry. To this purpose the lemmas in [54] may be useful.

8.5 Linear Centralizer Unfolding

We now turn to the problem of finding an efficient method to construct universal unfoldings of reversible linear operators. We prove (7.38). Define

$$C_{-}(\Omega) := \ker \operatorname{ad}(\Omega^T) \cap gl_{-R}(2p, \mathbb{R}) \tag{8.30}$$

as the *linear centralizer* of an operator $\Omega \in gl_{-R}(2p, \mathbb{R})$. If Ω_0 is a given general (i.e. not necessarily semisimple) reversible linear operator, we will prove that every matrix in its linear centralizer unfolding (LCU) is given by the map $\Omega : \mu \in \mathbb{R}^c \to \Omega_0 + A(\mu) \in gl_{-R}(n, \mathbb{R})$, with $A : \mu \in \mathbb{R}^c \mapsto A(\mu) \in C_{-}(\Omega_0)$ linear isomorphism.

Remark Note that if $\Omega_0 \in gl_{-R}(n, \mathbb{R})$ is semisimple then the following direct sum decomposition holds:

$$gl_{-R}(n, \mathbb{R}) = \operatorname{ad}\Omega_0\big(gl_{+R}(n, \mathbb{R})\big) \oplus C_{-}\left(\Omega_0\right), \tag{8.31}$$

where $C_{-}\left(\Omega_0\right) := \ker\left(ad\left(\Omega_0\right)\right) \cap gl_{-R}(n, \mathbb{R})$. This in turn implies $c_{\Omega_0} = \dim C_{-}(\Omega_0)$. Recalling the one-to-one relation between universal and mini-transversal unfoldings, see [39] (chapter 3), this implies that we can use the kernel of the adjoint operator to construct a versal unfolding of a semisimple matrix in $gl_{-R}(n, \mathbb{R})$. Basically this is all we need to construct the LCU of a given semisimple linear operator. But, for the sake of generality, the result in the semisimple case is given as a particular case of the generic one (cf. Corollary 8.10).

We start with a technical result, the proof is omitted because it is analogous to that of Lemma 1.1, see also [54].

Lemma 8.8. *Let $S_0 \in gl_{-R}(n, \mathbb{R})$ be semisimple (when considered as an element of $gl(n, \mathbb{R})$). Then there exists a scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ such that*

(i) *the involution* $R \in gl(n, \mathbb{R})$ *is orthogonal, i.e.* $R^T R = \mathrm{I}$;

(ii) $\ker\left(\mathrm{ad}\left(S_0^T\right)\right) = \ker\left(\mathrm{ad}\left(S_0\right)\right)$.

Observe that (i) implies that together with A also A^T belongs to $gl_{-R}(n, \mathbb{R})$.

Lemma 8.9. *Let* $\Omega_0 = S_0 + N_0$ *be the SN-decomposition of* $\Omega_0 \in gl_{-R}(n, \mathbb{R})$. *Then* S_0, N_0 *belong to* $gl_{-R}(n, \mathbb{R})$. *Moreover, if* $\langle \cdot, \cdot \rangle$ *is a scalar product as in Lemma 8.8, then*

(i) Ω_0^T, S_0^T *and* $N_0^T \in gl_{-R}(n, \mathbb{R})$;

(ii) $\ker\left(\mathrm{ad}\left(\Omega_0^T\right)\right) = \ker\left(\mathrm{ad}\left(S_0\right)\right) \cap \ker\left(\mathrm{ad}\left(N_0^T\right)\right)$;

(iii) $gl_{-R}(n, \mathbb{R}) = \mathrm{ad}\left(\Omega_0\right)\left(gl_{+R}(n, \mathbb{R})\right) \oplus \Big(\ker\left(\mathrm{ad}\left(S_0\right)\right) \cap \ker\left(\mathrm{ad}\left(N_0^T\right)\right) \cap gl_{-R}(n, \mathbb{R})\Big)$.

Proof. We only prove (ii) and (iii). To show (ii), let $A = S + N$ be the SN-decomposition of $A \in gl(n, \mathbb{R})$, then

$$\ker\left(\mathrm{ad}\left(A\right)\right) = \ker\left(\mathrm{ad}\left(S\right)\right) \cap \ker\left(\mathrm{ad}\left(N\right)\right).$$

Indeed, the inclusion $\ker\left(\mathrm{ad}\left(S\right)\right) \cap \ker\left(\mathrm{ad}\left(N\right)\right) \subseteq \ker\left(\mathrm{ad}\left(\mathrm{A}\right)\right)$ is obvious, while the opposite inclusion follows from the fact that S and N can be written as polynomial expressions in A.
Applying this result to $A = \Omega_0^T$ (with SN-decomposition $\Omega_0^T = S_0^T + N_0^T$), shows that

$$\ker\left(\mathrm{ad}\left(\Omega_0^{\mathrm{T}}\right)\right) = \ker\left(\mathrm{ad}\left(\mathrm{S}_0^{\mathrm{T}}\right)\right) \cap \ker\left(\mathrm{ad}\left(\mathrm{N}_0^{\mathrm{T}}\right)\right),$$

which in turn implies (ii), since $\ker\left(\mathrm{ad}\left(S_0^T\right)\right) = \ker\left(\mathrm{ad}\left(S_0\right)\right)$ by the choice of the scalar product.

To prove (iii) proceed as follows. Define a scalar product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ on $gl(n, \mathbb{R})$ by

$$\langle\!\langle A, B \rangle\!\rangle := \mathrm{trace}\,(A^T B),\ \forall\, A, B \in gl(n, \mathbb{R}), \tag{8.32}$$

and denote by A^* the transpose of the linear operator $A \in gl(n, \mathbb{R})$ with respect to this scalar product. Fixing some $\Omega_0 \in gl_{-R}(n, \mathbb{R})$ the decomposition (iii) is then an immediate consequence of (ii) and of the relations

$$\begin{aligned} gl_{-R}(n, \mathbb{R}) &= \mathrm{ad}\left(\Omega_0\right)\left(gl_{+R}(n, \mathbb{R})\right) \\ &\quad \oplus \Big(\ker\left(\left(\mathrm{ad}\left(\Omega_0\right)\right)^*\right) \cap gl_{-R}(n, \mathbb{R})\Big), \qquad (8.33) \\ \left(\mathrm{ad}\left(\Omega_0\right)\right)^* &= \mathrm{ad}\left(\Omega_0^T\right). \qquad (8.34) \end{aligned}$$

One obtains (8.33)-(8.34) from the following. Denote by $\langle\!\langle \cdot,\cdot \rangle\!\rangle_-$ and $\langle\!\langle \cdot,\cdot \rangle\!\rangle_+$ the induced scalar products on the subspaces $gl_{-R}(n,\mathbb{R})$ and $gl_{+R}(n,\mathbb{R})$ respectively. Considering $\mathrm{ad}\,(\Omega_0)$ as a linear operator from the linear space $gl_{-R}(n,\mathbb{R})$ into $gl_{+R}(n,\mathbb{R})$ we have then by a classical result that

$$gl_{-R}(n,\mathbb{R}) = \mathrm{ad}\,(\Omega_0)\,\big(gl_{+R}(n,\mathbb{R})\big) \oplus \big(\ker\left((\mathrm{ad}\,(\Omega))^*\right) \cap gl_{-R}(n,\mathbb{R})\big),$$

where $(\mathrm{ad}\,(\Omega_0))^* \in \mathcal{L}\big(gl_{-R}(n,\mathbb{R}), gl_{+R}(n,\mathbb{R})\big)$ is uniquely defined by

$$\langle\!\langle ad\,(\Omega_0)\cdot A, B\rangle\!\rangle_- = \langle\!\langle A, (\mathrm{ad}(\Omega_0))^*\cdot B\rangle\!\rangle_+, \tag{8.35}$$

for all $(A,B) \in gl_{-R}(n,\mathbb{R}) \times gl_{+R}(n,\mathbb{R})$. A direct calculation shows that for any $(A,B) \in gl_{-R}(n,\mathbb{R}) \times gl_{+R}(n,\mathbb{R})$ we have that

$$\begin{aligned}
\langle\!\langle \mathrm{ad}\,(\Omega_0)\cdot A, B\rangle\!\rangle_- &= \langle\!\langle A\cdot\Omega_0 - \Omega_0\cdot A, B\rangle\!\rangle \\
&= \mathrm{trace}((A\cdot\Omega_0 - \Omega_0\cdot A)^T B) \\
&= \mathrm{trace}(\Omega_0^T A^T B - A^T \Omega_0^T B) \\
&= \mathrm{trace}(A^T(B\Omega_0^T - \Omega_0^T B)) \\
&= \langle\!\langle A, \mathrm{ad}\,\big(\Omega_0^T\big)\cdot B\rangle\!\rangle_+.
\end{aligned} \tag{8.36}$$

Comparing (8.36) with (8.35) gives (8.34), which in combination with (8.33) and (ii) proves (iii). □

Remark From the commutativity properties $N_0^T S_0^T = S_0^T N_0^T$ and $S_0 N_0 = N_0 S_0$ it follows that a complement of $\ker\,(\mathrm{ad}\,(\Omega_0)) \cap gl_{-R}(n,\mathbb{R})$ in $gl_{-R}(n,\mathbb{R})$ is given by

$$\begin{aligned}
Y_0 \;=\; & \mathrm{ad}\,(S_0)\,\big(gl_{+R}(n,\mathbb{R})\big) \\
& \oplus\Big(\ker\,(\mathrm{ad}\,(S_0)) \cap \mathrm{ad}\,\big(N_0^T\big)\,\big(gl_{+R}(n,\mathbb{R})\big) \cap gl_{-R}(n,\mathbb{R})\Big).
\end{aligned} \tag{8.37}$$

Lemma 8.9-(iii) is together with the Implicit Function Theorem the main ingredient in the proof of our main result on miniversal unfolding of a R-reversible linear operator $\Omega_0 \in gl_{-R}(n,\mathbb{R})$, of which Proposition 7.2 is an obvious consequence.

Theorem 12. *Let $\Omega_0 = S_0 + N_0$ be the SN-decomposition of $\Omega_0 \in gl_{-R}(n,\mathbb{R})$, and let $< \cdot,\cdot >$ be a scalar product as in Lemma 8.8. Then there exist* *a neighbourhood* $\mathcal{O}$ *of* Ω_0 *in* $gl_{-R}(n, \mathbb{R})$ *and a smooth mapping* $\tilde{\Psi} : \mathcal{O} \to Gl_+(n, \mathbb{R})$ *with* $\tilde{\Psi}(\Omega_0) = \mathrm{I}$ *such that for all* $\Omega \in \mathcal{O}$

$$\mathrm{Ad}\left(\tilde{\Psi}(\Omega)\right)\cdot\Omega - \Omega_0 \in \ker\left(\mathrm{ad}\left(S_0\right)\right) \cap \ker\left(ad\left(N_0^T\right)\right) \cap gl_{-R}(n, \mathbb{R}). \quad (8.38)$$

Proof. Define $G : GL_+(n, \mathbb{R}) \times gl_{-R}(n, \mathbb{R}) \to gl_{-R}(n, \mathbb{R})$

$$G(\Psi) \cdot \Omega := \mathrm{Ad}(\Psi) \cdot \Omega = \Psi^{-1}\Omega\Psi.$$

Calculate $G(\mathrm{I}, \Omega_0)$ and $D_\Psi G(\mathrm{I}, \Omega_0)\cdot\Sigma = \Omega_0\Sigma - \Sigma\Omega_0 = \mathrm{ad}(\Sigma)\cdot\Omega_0 = -\mathrm{ad}\,(\Omega_0)\cdot \Sigma$, i.e.

$$D_\Psi G(\mathrm{I}, \Omega_0) = -\mathrm{ad}\,(\Omega_0) \in \mathcal{L}\big(gl_{-R}(n, \mathbb{R}), gl_{+R}(n, \mathbb{R})\big).$$

Let $\pi : gl_{-R}(n, \mathbb{R}) \to \mathrm{ad}\,(\Omega_0)\,\big(gl_{+R}(n, \mathbb{R})\big)$ be the projection of $gl_{-R}(n, \mathbb{R})$ on the first factor of the direct sum splitting

$$gl_{-R}(n, \mathbb{R}) = \mathrm{ad}\,(\Omega_0)\,\big(gl_{+R}(n, \mathbb{R})\big) \oplus \Big(\ker\big(\mathrm{ad}\,\big(\Omega_0^T\big)\big) \cap gl_{-R}(n, \mathbb{R})\Big).$$

Define a smooth mapping F from an open neighbourhood U of (I, Ω_0) in $GL_{+R}(n, \mathbb{R}) \times gl_{-R}(n, \mathbb{R})$ to $\mathrm{ad}\,(\Omega_0)\,\big(gl_{+R}(n, \mathbb{R})\big)$ by:

$$F(\Psi, \Omega) := \pi\big(G(\Psi, \Omega) - \Omega_0\big), \qquad \forall(\Psi, \Omega) \in U.$$

By the definitions it follows: $\pi(\mathrm{I}, \Omega_0) = 0$, while $D_\Psi F(\mathrm{I}, \Omega_0)$ surjective onto the space $\mathrm{ad}\,(\Omega_0)\,\big(gl_{+R}(n, \mathbb{R})\big)$. By the Implicit Function Theorem it then follows that there exist a neighbourhood $\mathcal{O}$ of Ω_0 in $gl_{-R}(n, \mathbb{R})$ and a smooth mapping $\tilde{\Psi} : \mathcal{O} \to Gl_+(n, \mathbb{R})$ such that

$$\tilde{\Psi}(\Omega_0) = \mathrm{I} \quad \text{and} \quad F(\tilde{\Psi}(\Omega), \Omega) = 0, \quad \text{for all}\, \Omega \in \mathcal{O}.$$

By definition of F then

$$\mathrm{Ad}\left(\tilde{\Psi}(\Omega)\right) \cdot \Omega - \Omega_0 \in \ker\left(\mathrm{ad}\left(\Omega_0^T\right)\right), \qquad \text{for all}\, \Omega \in \mathcal{O},$$

which in combination with Lemma 8.9-(ii) proves the theorem. □

An immediate consequence of Theorem 12 is the following Corollary.

Corollary 8.10. *Let* $\Omega_0 \in gl_{-R}\,(n, \mathbb{R})$ *be semisimple. Then the lcu of* Ω_0 *given by*

$$\Omega :\ A \in C_-(\Omega_0) \mapsto \Omega_0 + A \in gl_{-R}(n, \mathbb{R}) \quad (8.39)$$

is minitransversal. The $\mathrm{cod}\,\Omega_0 = \dim C_-(\Omega_0)$.

Remarks

- Note that Corollary 8.10 can also be proved independently from Theorem 12. Indeed, it is a straightforward consequence of the Implicit Function Theorem and the direct sum splitting (8.31).

- Note that an operator Ω belonging to the LCU of a semisimple Ω_0 is not necessarily semisimple since we do not assume the eigenvalues of Ω_0 to be simple. Corollary 8.10 in this context is more general than the result given in [12] on the miniversal unfolding of semisimple linear reversible operators.

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Index

Short List of Symbols

Ad,	Adjoint
ad,	adjoint
$\mathrm{Diff}(\mathbb{R}^n)$,	space of diffeomorphisms on $\mathbb{R}^n$
$\mathrm{Diff}_0(\mathbb{R}^n)$,	diffeomorphisms on $\mathbb{R}^n$ with a fixed point at the origin
$gl(n,\mathbb{R})$,	$\mathcal{L}(\mathbb{R}^n)$, linear operators on $\mathbb{R}^n$ (i.e. $n \times n$ matrices)
$GL(n,\mathbb{R})$,	invertible linear operators on $\mathbb{R}^n$
$GL_{\pm R}(n,\mathbb{R})$,	R-equivariant/reversible (invertible) linear operators
$gl_{\pm R}(n,\mathbb{R})$,	R-equivariant/(infinitesimally) reversible linear operators
$\mathcal{H}_k$,	space of ploynomials maps homogeneous of degree k
$\mathcal{L}(X,Y)$,	continuos linear operators from X to Y
$\mathcal{L}(X)$,	$\mathcal{L}(X,X)$
$\mathcal{P}_k$,	$\mathcal{P}_k = \mathcal{X}_0/\mathcal{X}_0^k$, $k \geq 1$
R,	$R \in GL(n,\mathbb{R})$ involution, i.e. $R^2 = \mathrm{I}$
$\mathcal{X} = \mathcal{X}(\mathbb{R}^n)$,	set of all smooth vector fields on $\mathbb{R}^n$
$\mathcal{X}_0$,	subset of vector fields with a fixed point at the origin
$\mathcal{X}_0^k$,	$\mathcal{X}_0^k := \{X \in \mathcal{X}_0 \mid D^j X(0) = 0,\ 1 \leq j \leq k\}$
$\mathcal{X}^{\pm R}$,	R-equivariant/reversible vector fields
$\Psi_* X$,	pushforward
$\Psi^* X$,	pullback
$\mathbb{R}$, $\mathbb{C}$,	real numbers, complex numbers

$\mathbb{T}^n$, n-torus

$\mathbb{D}_q$, $\mathbb{Z}_q$, dihedral group, cyclic group

Samenvatting

Deze thesis bevat twee delen. Het doel van het eerste deel is het ontwerpen van een algemeen kader voor de studie van bifurcatie van q-periodieke punten uit een vast punt van een familie van reversibele diffeomorphismes. In het tweede deel wordt de omkadering verbreed en is de focus op families van reversibele vectorvelden aan een 1:1 resonantie. Vanuit de KAM theorie vragen we ons af welke types dynamische karakteristieken (invariante tori in ons geval) onder (reversibele) perturbaties blijven bestaan.

Het eerste deel heeft zijn wortels in een serie artikels, [52, 53, 54, 82], waar een techniek voor de studie van periodieke (subharmonische) oplossingen in de buurt van equilibria (periodische oplossingen) van conservatieve/ equivariante/ reversibele systemen ontwikkeld werd. Deze techniek combineert in zekere zin twee min of meer standaard benaderingen tot bifurcatieproblemen, de Lyapunov-Schmidt (LS) reductie en de normale vorm reductie. De LS reductie methode concentreert (exclusief) op periodieke oplossingen, ander dynamisch gedrag verwaarlozend. Ze wordt normaal gezien toegepast om existentie problemen voor periodieke oplossingen van een gegeven systeem te reduceren naar het oplossen van algebraïsche vergelijkingen op een lager dimensionale ruimte [81]. Een belangrijke eigenschap van deze benadering is dat ze leidt tot vergelijkingen die een expliciete cirkelsymmetrie (S^1-symmetrie) vertonen. Daartegenover houdt de normale vorm benadering rekening met de volledige dynamica, en bestaat ze essentieel uit het uitvoeren van transformaties die een gegeven vergelijking in een 'simpelere' vorm brengen [43]. Zie ook [41]. Parallel met deze twee theorieën voor vectorvelden bestaat er een overeenkomstige aanpak voor diffeomorphismes. Vaak zal men, in plaats van direct te werken met (subharmonische) bifurcaties van de periodieke oplossingen, de vaste punten (periodieke punten) van de corresponderende Poincaré afbeelding bestuderen. Als de differentiaalvergelijkingen een additionele structuur bewaren, wordt dit weerspiegeld in de structuureigenschappen van de Poincaré afbeelding. Merk op dat subharmonische periodieke oplossingen corresponderen met periodieke punten van deze afbeelding. In [74] werd een gelijkaardig resultaat als in [82] bewezen voor de bifurcatie van periodieke punten uit een vast punt van een familie van diffeomorphismes. Een vanzelfsprekende vraag is of het resultaat van [74] kan herformuleerd worden op een structuur bewarende manier. In [32] wordt het geval van een familie van symplectische diffeomorphismes geanalyseerd. In deze thesis ligt de focus op de reversibele diffeomorphismes. We beschouwen een C^∞-gladde lokale afbeelding $\Phi : (\mathbb{R}^n, 0) \times (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$, $(x, \lambda) \mapsto \Phi_\lambda(x) = \Phi(x, \lambda)$ die aan de volgende hypothese voldoet: voor alle $\lambda \in \mathbb{R}^m$ in een omgeving van nul,

$$\Phi_\lambda(0) = 0 \text{ en } D_x\Phi_\lambda(0) \in \mathcal{L}(\mathbb{R}^n) \text{ is inverteerbaar,} \tag{H1}$$

als ook de *reversibiliteits voorwaarde*

$$R \circ \Phi_\lambda \circ R = \Phi_\lambda^{-1}, \tag{R}$$

ten opzichte van een lineaire involutie $R \in \mathcal{L}(\mathbb{R}^n)$. Gegeven een geheel getal $q \geq 1$, ligt onze interesse bij alle kleine q-periodieke punten van Φ_λ, met λ dichtbij 0. Met andere woorden, we willen alle oplossingen (x, λ) bij $(0, 0)$ bepalen van de vergelijking

$$x = \Phi_\lambda^q(x), \tag{P}$$

waar $\Phi_\lambda^q := \Phi_\lambda \circ \cdots \circ \Phi_\lambda$ (q keer). In Theorema 1 (zie sectie 1.2 en hoofdstuk 3 voor het bewijs zelf) tonen we aan dat het probleem herleid kan worden tot een gelijkaardig probleem voor een gereduceerde familie van diffeomorphismes, $\Phi_{r,\lambda}$, die zelf reversibel is, maar ook een extra $\mathbb{Z}_q$-symmetrie heeft. De reversibiliteit in combinatie met de $\mathbb{Z}_q$-symmetrie vertaalt zich in een $\mathbb{D}_q$-symmetrie van het probleem, wat toelaat om de bifurcatie vergelijkingen neer te schrijven. Het bewijs combineert de LS reductie methode en het herhaaldelijk gebruik van de Impliciete Functie Stelling.

Om nu het reductie resultaat (Theorema 1) toe te passen op concrete voorbeelden heeft men een methode nodig om het gereduceerde diffeomorphisme $\Phi_{r,\lambda}$ te berekenen of te benaderen. We benaderen $\Phi_{r,\lambda}$ door een normale vorm van Φ_λ. Vooraleer dit verder te onderzoeken leggen we eerst uit wat 'normale vorm' in de uiteengezette context betekent. Ons hoofddoel is het vereenvoudigen van de Taylor serie in het vaste punt van het diffeomorphisme Φ_λ. 'Vereenvoudiging' betekent hier symmetrie: het genormaliseerde deel is invariant onder zekere lineaire transformaties en, in het bijzonder, is reversibel. Daarom herorganiseren en veralgemenen we in hoofdstuk 4 een serie resultaten over normale vorm theorie voor families van diffeomorphismes met geen a priori structuur (algemene diffeomorphismes) [69, 70, 18, 8, 84, 9, 72], zodat uit de bewijzen duidelijk wordt welke moeilijkheden moeten overkomen worden om de theorie te doen passen in een structuur behoudende opzet. De idee is om het behoud van de structuur uit te drukken in termen van Lie algebras en groepen. Maar, gezien reversibele afbeeldingen of reversibele vectorvelden geen Lie groep of algebra vormen is dit niet rechtstreeks mogelijk. Dit lossen we op door gebruik te maken van R-equivariante transformaties die de reversibele structuur respecteren.

Alhoewel wij concentreren op reversibele normale vormen, dienen we toch op te merken dat het algemene resultaat kan aangepast worden aan andere structuren. Zie bv. [32] voor het symplectische geval. Merk ook op dat het probleem van structuur behoudende normale vormen voor vectorvelden nabij equilibria reeds eerder door Broer [8, 9] in termen van gegradeerde en gefilterde Lie algebras werd bestudeerd, en daarna uitgebreid tot het geval van kiemen van diffeomorphismes in [18].

In detail, Theorema 2 bewijst dat onder geschikte aannames, voor elk geheel getal $k \geq 1$, men de familie $\{\Phi_\lambda\}$ kan transformeren zodat Φ_λ commuteert met het semisimpele deel van $D\Phi_0(0)$ tot en met termen van de orde $k+1$, en, in tegenstelling tot bekende resultaten, hebben we dat de normale vorm aan extra voorwaarden voldoet tengevolge van het feit dat we het nilpotente deel van $D\Phi_0(0)$ in rekening brengen. In het bijzonder wordt de reversibiliteit bewaard in de normale vorm reductie. Een ander belangrijk punt om in achting te nemen is dat onze reductie naar normale vorm op een gladde manier afhankelijk is van de parameters λ en ze geldig is in een omgeving van $\lambda = 0$. Merk op dat deze omgeving van k afhangt en kan inkrimpen tot $\{0\}$ als $k \to \infty$.

Keren we terug naar de gereduceerde afbeelding $\Phi_{r,\lambda}$. Het blijkt dat de restrictie van de normale vorm van Φ_λ (zie (1.15)) tot de gereduceerde faseruimte een goede benadering oplevert van $\Phi_{r,\lambda}$, zie Corollarium 1.2.

In hoofdstuk 6 (zie ook hoofdstuk 1 secties 1.5.1 en 1.5.2) analyseren we, als een toepassing op de ontwikkelde theorieën, eerst de afsplitsing van periodieke punten nabij een vast punt in een familie van reversibele afbeeldingen als voor een kritische waarde van de parameters de linearisatie aan het vaste punt, ofwel een paar simpele zuiver imaginaire eigenwaarden heeft die eenheidswortel zijn (*bifurcatie aan een simpele eenheidswortel*, het SRU geval) ofwel een paar niet-semisimpele zuiver imaginaire eigenwaarden heeft die eenheidswortel zijn met algebraïsche multipliciteit 2 en geometrische multipliciteit 1 (*bifurcatie aan een resonante eenheidswortel*, het RRU geval). In beide gevallen bewijzen we dat de familie Φ_λ een bifurcatie ondergaat vanuit het vaste punt van twee families van q-periodieke banen die aan bepaalde symmetrievoorwaarden voldoen. Zie Theorema's 3 en 4 voor de precieze formulering van de resultaten. We concentreren ons daarna op de subharmonische bifurcatiesverschijnselen voor reversibele vectorvelden, zie hoofdstuk 1 en hoofdstuk 6. Hierin beschouwen we het geval dat de afbeelding Φ de Poincaré afbeelding is van een $2k$-dimensionaal autonoom tijds-reversibel vectorveld met een niet constante R-symmetrische periodieke oplossing γ_0 met periode T_0. Het blijkt dat in dit geval 1 altijd een eigenwaarde van $D\Phi(0)$ is. Dit geeft de existentie aan van een één-dimensionale tak van R-symmetrische vaste punten, dat is, een één-parameter familie van periodieke oplossingen met periode dicht bij de periode van het originele systeem. Deze aftakking wordt de *primaire tak* genoemd. Nemen we een coördinaat op de primaire tak als parameter, dan bekomen we voor geïsoleerde waarden van de parameter symmetrische vaste punten waar de afgeleide van Φ eigenwaarden heeft die q-eenheidswortel zijn, $q \geq 3$. Dan rijst opnieuw de vraag of dit leidt tot de aftakking van q-periodieke punten, oftewel, subharmonische bifurcaties van het originele systeem. We zijn geïnteresseerd in twee gevallen: *subharmonische bifurcatie aan een simpele eenheidswortel* (SBSRU geval) en *subharmonische bifurcatie aan een resonante eenheidswortel* (SBRRU geval). In het eerste geval is de hoofdaanname dat de linearisatie van Φ in 0, naast de eigenwaarde 1, een paar simpele eigenwaarden heeft die $q$de eenheidswortel zijn. In het laatste geval is dat dat de linearisatie van Φ in 0 eigenwaarde 1 heeft en een paar niet-semisimpele eigenwaarden, met algebraïsche multipliciteit 2 en geometrische multipliciteit 1, die $q$de eenheidswortel zijn. We verkrijgen in beide gevallen de existentie van twee families van subharmonische oplossingen die van de primaire tak afsplitsen, zie Theorema's 6 en 7 voor de precieze formulering van de resultaten.

In hoofdstuk 5 beschrijven we hoe we informatie over de stabiliteit van aftakkende periodieke banen kunnen verkrijgen door zowel de resultaten van de normale vorm als het reductieresultaat te gebruiken.

Het tweede deel van deze thesis past in de zogeheten KAM (Kolmogorov Arnold Moser) theorie. Hoofdinteresse van de KAM theorie is de persistentie van quasi-periodieke invariante tori in integreerbare stelsels onder kleine quasi-integreerbare perturbaties. De term *integreerbaar* verwijst naar een toroïdale symmetrie van het stelsel. Via een simpele herschaling kan dit perturbatieprobleem herleid worden tot het geval waar 'integreerbaar' vervangen wordt door 'lineair en integreerbaar' (dwz. van Floquet type).

Hierbij zijn wij geïnteresseerd in het voorkomen van quasi periodiciteit in de klasse van reversibele stelsels. In het bijzonder ligt de focus op quasi periodieke tori in reversibele stelsels, waar het normale lineaire deel een 1:1 resonantie heeft. Dat is, in tegenstelling met de eerder bestudeerde semisimpele gevallen, bv. [3, 58, 59, 60, 64, 51, 12], wanneer de eigenwaarden van het normale lineaire deel samenvallen in een complex toegevoegd paar op de imaginaire as. In dit opzicht is ons doel het veralgemenen van [12] tot het 1:1 resonante geval. We tonen aan dat ook in dit geval de aanpak van [51, 12] in grote mate blijft functioneren.

In grote lijnen bestaat ons KAM resultaat uit de persistentie van Cantor foliaties van quasi periodieke tori, geparametriseerd door de eigenwaarden van het normale lineaire deel van het stelsel.

We vermelden enkele relevante punten van onze aanpak. Een element is de aanwezigheid van parameters. Inderdaad, zoals in [51, 12] beschouwen we families van vectorvelden waar in de integreerbare benadering de frequenties van de invariante tori kunnen variëren met de parameters. Deze eigenschap is deel van een bredere niet-degeneratieve voorwaarde van het Kolmogorov type, dat het gehele niet lineaire deel omvat. Reeds in [51, 14, 12] werd duidelijk dat het belangrijk is dat de matrices in het normale lineaire deel een versale ontvouwing in de betekenis van [3] moeten vormen. Bijgevolg worden ontvouwingen in detail bestudeerd, zowel in het generische als in het semisimpele geval, zie secties 7.3, 7.4 en 8.5. Een ander element is de constructie van een conjugatie tussen de integreerbare onverstoorde familie en zijn perturbatie, beperkt tot een foliatie van invariante tori geparametriseerd over een ‘Cantor verzameling’ met positieve maat, cf., bv., [51]. De Cantor verzameling is gedefinieerd door *Diophantiene condities*, noodzakelijk om te compenseren voor kleine noemers. In analogie met [64] zijn zowel de foliatie en de conjugatie Whitney glad, waarmee we bedoelen dat ze kunnen uitgebreid worden tot gladde afbeeldingen over een volledige omgeving. Gezien de conjugatie ook dicht bij de identiteit ligt, impliceert dit dat het verstoorde stelsel een Cantor foliatie van invariante tori met positieve maat erft. De existentie van zo een conjugatie kan gezien worden als een soort structurele stabiliteit, die we hier *quasi-periodieke stabiliteit* noemen.

In meer detail gaan we als volgt te werk. We starten met een geschikte familie van integreerbare reversibele vectorvelden die dan verstoord wordt door een kleine, niet-integreerbare, maar nog steeds reversibele, perturbatie. Gebruik makende van een lokalisatie procedure zoals in [51, 14] bekomen we dat het onverstoorde vectorveld in een gepaste Floquet achtige vorm gebracht kan worden, beschikkende over een invariante *zero-torus*. We willen dan de persistentie van deze invariante torus bestuderen onder de niet-integreerbare perturbatie.

We werken in deel twee altijd in de faseruimte $M = \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2p}$, waar

$\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ de n-torus is met coördinaten $x = (x_1, \dots, x_n) \pmod{2\pi}$, terwijl in $\mathbb{R}^m$ en $\mathbb{R}^{2p}$ de coördinaten respectievelijk $y = (y_1, \dots, y_m)$ en $z = (z_1, \dots, z_{2p})$ zijn. We zijn geïnteresseerd in het volgende probleem. Gegeven een analytische familie

$$X(x, y, z, \lambda) = f(y, z, \lambda)\partial_x + g(y, z, \lambda)\partial_y + h(y, z, \lambda)\partial_z$$

van reversibele en integreerbare vectorvelden op M, met parameter λ behorende tot een open deelverzameling $P \subset \mathbb{R}^q$ en zo dat, voor alle $(y, \lambda) \in \mathbb{R}^m \times P$,

$$h(y, 0, \lambda) = 0 \quad \text{en} \quad D_z h(y, 0, \lambda) \in \mathcal{L}(\mathbb{R}^{2p}) \text{ inverteerbaar is.}$$

Welke van de X-invariante tori $V_\lambda := \mathbb{T}^n \times \{0\} \times \{0\} \times \{\lambda\}$ $(\lambda \in P)$ zullen blijven bestaan onder een kleine reversibele storing $\widetilde{X}$ van X die niet noodzakelijk integreerbaar is?

We tonen aan dat het volstaat te bewijzen dat het persistentie resultaat geldt in het geval waar het onverstoord (integreerbaar en reversibel) vectorveld X in *Floquet normale lineaire vorm* is. Daarmee bedoelen we dat de stroming van X lineair is in de z-richting normaal t.o.v. de familie van invariante tori. We hebben aldus,

$$X(x, y, z, \lambda) = \omega(\lambda)\partial_x + \Omega(\lambda)z\,\partial_z,$$

met $(x, y, z) \in M = \mathbb{T} \times \mathbb{R}^m \times \mathbb{R}^{2p}$ en $\lambda \in P \subset \mathbb{R}^q$, en waar de afbeeldingen $\omega : P \to \mathbb{R}^n, \lambda \mapsto \omega(\lambda)$ en $\Omega : P \to gl_{-R}(2p; \mathbb{R}), \lambda \mapsto \Omega(\lambda)$ analytisch ondersteld zijn. We onderstellen ook dat $\det \Omega(\lambda) \neq 0$ voor alle $\lambda \in P$. Hier is $gl_{-R}(2p; \mathbb{R})$ de verzameling van alle R-reversibele matrices; dus

$$gl_{-R}(2p; \mathbb{R}) := \{\Omega \in gl(2p; \mathbb{R}) \mid \Omega R = -R\Omega\},$$

met $R \in gl(2p, \mathbb{R})$ en $R^2 = \mathrm{I}$. De familie X heeft een 1:1 resonantie in $\lambda = \lambda_0$ als $\Omega(\lambda_0)$ een paar zuiver imaginaire eigenwaarden $\pm i\kappa$ $(\kappa > 0)$ met algebraïsche multipliciteit gelijk aan 2 heeft. In de veronderstelling dat het vectorveld *niet-gedegenereerd* is (zie Definitie 1 in hoofdstuk 7) bewijzen we, in het geval van 1:1 resonantie, de persistentie van een foliatie van invariante tori geparametriseerd over een 'Cantor verzameling'. We verwijzen naar Theorema 10 (zie ook Theorema 11) voor de exacte formulering van het resultaat.

Hoofdstuk 8 is dan volledig gewijd aan het bewijs van Theorema 10. We dienen hierbij te onderlijnen dat net zoals in [12] een centrale rol in het bewijs gespeeld wordt door de *lineaire gecentraliseerde ontvouwing* van het normale lineaire deel $\Omega(\mu)$.