# Geometrische resolutie van ruimte-tijd-singulariteiten

# Geometrical Resolution of Spacetime Singularities

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# Geometrical Resolution of Spacetime Singularities

General relativity posits that spacetime is not a fixed structure but that it is represented by a dynamical metric field. The gravitational attraction between matter appears as follows: the dynamics of the metric field is related to the distribution of matter throughout the universe, and the propagation of matter through spacetime is influenced by the metric field that represents the universe. After the discovery that matter particles and microscopic forces obey quantum mechanical laws, it has thus become necessary to quantize the metric field as well. One of the research lines for a consistent theory of quantum gravity has led to string theory. String theory describes the gravitational interaction in terms of gravitons, which are the quanta of the gravitational force. In string theory it is assumed that at the smallest scales elementary particles have a stringlike nature instead of a pointlike nature. But the behaviour of strings on time-dependent backgrounds such as our expanding universe is not well understood yet.

General relativity predicts the existence of gravitational singularities at the classical level: our universe started with the big bang, and massive stars can collapse into black holes. A theory that describes quantum gravitational effects should elucidate our understanding of these singularities. The existence of these singularities also raises the question whether propagation of quantum fields through a singularity is possible (and how it should be formulated). String theory can already deal with some timelike singularities but not yet with spacelike singularities like the big bang. Near singularities, strings often interact strongly. A formulation of string theory that allows to take strong interactions between strings into account is given by matrix theory. Matrix theory models that describe singularities often have a dual translation in terms of a quantum field theory that is defined on a singular background spacetime.

In this dissertation we investigate these issues. We use a geometric regularization prescription to define the evolution of a free scalar field and of a free string through a singularity in an unambiguous manner. Remarkably, this geometric regularization seems to reveal there is a certain feature of discreteness related to the evolution across the singularity. We also consider an important class of time-dependent backgrounds that can be investigated in string theory. This class is called gravitational plane waves. These plane waves can be used to investigate the strong curvature effects related to a singularity. Our study shows that it is necessary to take into account that the strings can interact strongly near the singularity. In order to obtain a better understanding of matrix theory on a plane wave background we investigate solutions that describes D-branes in plane wave backgrounds. D-branes are objects that appear in string theory besides strings, and that are important for the formulation of matrix theory.

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*"Symphony No. 5 in B flat Major, Fourth Movement," Anton Bruckner*

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# Prologue

## Motivation

> *"Without gravity you wouldn't have a ~~sport~~ research,"*
>
> ~~*Matthew Ketterling*~~

Why doing research in theoretical physics? And why writing a thesis? Or more precisely, since the writing almost inevitably follows the research, why writing this thesis in the way it has been written? Let me first address the latter question. The public I mainly have in mind when presenting these results are not only the professional researchers in string theory because the material present in this PhD thesis has already been presented in scientific papers. On the other hand, scientific knowledge possesses a cultural merit (a fact often underappreciated, I think, or at least not likely to be stressed) which shouldn't remain privileged (even undeliberately) to the small circles where it is investigated, and which has the right to diffuse among a wider community. Therefore I would like to present this work as a readable account (or at least as an attempt thereto). The aim is that whoever has an understanding of quantum mechanics and special relativity, can use this thesis to understand the derivation of my results that have appeared in the scientific literature, and the motivation why they were derived.

Then, why did I choose to do research in quantum gravity? I must say that I think (although the process of remembering probably alters my own memories) I became first interested in the subject of quantum gravity while reading Pauli's account on general relativity [13]. Despite the fact that its content has become dated, Pauli's treatise is important from the historical point of view. It contains a critical survey of several old theories (like Weyl's unification of gravitation and electrodynamics), which in a modern book would never be mentioned because they have since then been surpassed by experimental counterevidence.

## Acknowledgements

*In der Beschränkung zeigt sich erst der Meister*

*"Natur und Kunst," J. W. Goethe*

Actually, I would really like to thank the Research Foundation Flanders - FWO for providing me with financial support (under the title "Aspirant") to investigate this subject. Then, concerning the more "personal" acknowledgements, in the first place I want mention my PhD advisor Ben Craps, whom I would immediately recommend as an advisor for students interested in doing research in string theory. Meanwhile I should certainly not forget my collaborators Oleg and Federico, not in the least for some interesting discussions within or outside the scope of our research. For administrative matters Bernadette at VUB and Margot at UGent have always been very helpful. Additional thanks to my colleagues Alice, Wieland... Though not immediately connected to the research, except for checking the spelling, I also feel grateful towards my parents, siblings, family and friends for their support. And to you reader, thanks for reading my thesis!

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# Quantum gravity

*"Raffiniert ist der Herr Gott, aber boshaft ist er nicht."*

*Albert Einstein*

As indicated by its title, this thesis deals with spacetime singularities. These are predicted by general relativity, and it is expected (or hoped) that a quantum theory of gravity will shed light on their nature. At a first glance, one may suggest that the development of a consistent quantum mechanical version of general relativity is merely a philosophical question, because it is possible to write an effective quantum field theory for the gravitational interaction that is valid at energies much below the Planck scale [27, 28]. However, the construction of such an effective quantum theory breaks down near the Planck scale, where we expect (or hope) the quantum effects to become important in order to resolve classical spacetime singularities. Still, one might comment that we cannot directly observe the Planck scale where quantum corrections would perhaps significantly alter the classical results, and for the energies we can observe the standard model of elementary particle physics provides a very good effective description. Nevertheless (and even putting aesthetical problems like the unification of forces aside) because of the fact that the matter and energy that sources the gravitational field is quantized, the gravitational force has to be quantized too out of consistency, as has already been remarked in [13] when the quantum nature of matter became first apparent. In addition, the diffeomorphism invariance behind general relativity is not incorporated in the standard model either, which is formulated with respect to flat Minkowski spacetime. For a generally covariant formulation of the standard model, quantum field theory on arbitrarily curved (but still classical) spacetimes would suffice [22]. But in this approach the degrees of freedom of the gravitational field (or the degrees of freedom that lead to the gravitational field) aren't taken into account yet.

The research presented in this thesis is in the context of only one approach to quantum gravity, more precisely string theory. Perhaps because of the difficulty to compare quantum gravity theories with experiment, there are a number of alternative and largely independent approaches to the problem. First of all, it is in principle possible that the renormalization group equations for the gravitational force simply possess a nontrivial fixed point. This scenario is sometimes called "asymptotic safety". For a review see e.g. [29]. An alternative, called loop quantum gravity [17, 20], is to consider that the Planck scale offers a natural cutoff, somewhat reminiscent of the cutoff in solid state field theory from the underlying atom lattice. In this approach one primarily tries to reconcile the foundational concepts of quantum mechanics and general relativity, without immediately aiming to unify gravity with the standard model forces. There are also numerous other approaches to quantum gravity, which

I will not list here. For a brief overview, see for example the introduction of [20]. Historically speaking, also higher derivative theories (adding higher order curvature invariants to the Einstein-Hilbert action) and supergravity theories have been investigated in the context of quantum gravity, but some of these have been incorporated as low-energy limits of string theory.

Nevertheless, before we might become entangled in our attempt to describe and understand (aspects of) quantum gravity, we should already be aware of how profoundly our traditional view of the Newtonian gravitational attraction already has to be changed by the combination of general relativity and recent cosmological measurements. General relativity posits that every field with energy-momentum couples to the spacetime structure. While the contribution of the standard model fields to the energy-momentum tensor is well understood, cosmological observations have confronted us that most of the energy-momentum tensor in the universe consists of "dark matter" and "dark energy" of which we do not know the exact nature yet.

Furthermore, the masses that appear in Newton's fourth law (which, up till now, has been tested down to the millimeter scale) have a more complicated origin than we would perhaps implicitly assume. The main contribution to the mass of the known matter is due to hadrons (e.g. neutrons, protons) but their large effective mass is actually generated by quantum-chromodynamics (through the gluon fields and sea quarks that dress the valence quarks in the hadrons). In addition, the rest mass of the standard model fermions (e.g. 511 keV for an electron) is generated by interactions between the fermions and the Brout-Englert-Higgs field.

# Conclusions?

*All the time you're saying to yourself "I could do that, but I won't" - which is just another way of saying that you can't.*

*R. P. Feynman*

In fact, the working title of my thesis was "The big bang in string theory", and notwithstanding its ambitious sound, I am (and I guess I am not alone) still far from a thorough understanding of gravitational quantum physics, let alone a satisfactory understanding of the big bang singularity and the beginning of the universe. But, I suppose that science benefits from all its contributions, whether great or small, so allow me to quote an observation from ecology which states that (in general) "ants are indispensable for the well-functioning of an ecosystem", so if this work were only "a small step for the ecosystem" it was nevertheless still "a great step for the ant". And now I've already mentioned ants, let me introduce another analogy. Because there is a wonderful relation between the ability of a nest of ants to construct the shortest path between two points and the "intuitive" concept behind path integrals (I cannot help wondering if Feynman came up with his idea after having been busy to keep ants out of his fridge), maybe I could rather compare scientific progress with a path integral and consider my "ant" work as a (quantum) fluctuation that ultimately contributes to the (semiclassical) path that leads to an understanding of quantum gravity. So then I must hope the phase factor of my contribution does not oscillate too rapidly ;-) And if it would, let me at least hope (to carry the comparison a little further) that the road to quantum gravity is determined by a Gaussian fixed point, even while quantum gravity itself isn't, so that future scientific developments will finally shed light on a fundamental problem that is already open since (neglecting a decade or two) the beginning of the previous century.

# Outline

*When Dirac visited Princeton in 1928 he gave a seminar report on his paper showing the connection of exchange energy with the spin variables of the electron. In the discussion following the report, Weyl protested that Dirac had said that he would derive the results without the use of group theory, but, as Weyl said, all of Dirac's arguments were really applications of group theory. Dirac replied, "I said I would obtain the results without previous knowledge of group theory."*

*"Theory of Atomic spectra," Condon and Shortly*

The thesis "Geometrical resolution of spacetime singularities" deals with some special cases of string theory (and field theory) in lightcone time-dependent and singular spacetimes. Because it is a joint thesis between two different faculties, I have to anticipate that readers can have a very different background. In addition, precisely because the research results presented here do not require the full mathematical machinery of string theory, it is an opportunity to make the whole thesis more accessible for non-experts. Therefore the thesis opens with a large introduction, and is afterwards divided into several parts, aimed at different levels of specialisation. Each of the parts consists of several chapters which are relatively self-contained. Especially the chapters of the second and the third part can be largely read in the order preferred by the reader: cross-references point to relevant material discussed elsewhere. The expert readers can immediately read the third part "Research".

In the first part "Foundations" I present some preliminaries concerning cosmology, general relativity, string theory and field theory in curved spacetimes, aimed at a researcher in physics without expert knowledge in string theory. Because it is impossible to give a self-contained discussion in only a few chapters, I will primarily illustrate some important ideas. As already mentioned in the Prologue, I assume that the reader is acquainted with special relativity, quantum mechanics (and basics of quantum field theory).

In the second part "Background" I discuss more specific topics that appeared at various points during my research. Some of the latter topics (concerning plane waves, singularities and matrix theory) are perhaps also useful for researchers in string theory specialized in a different direction. For the specialists it may appear that the second part is written at a pedestrian level, this is in order to remain clear for a wider public (I hope).

The third part "Research" deals most directly with the topics I have investigated. The first two chapters deal with geometrical resolutions, the third with the construction of an important class of singular time-dependent backgrounds. For convenience I have also added a chapter with conclusions that summarize the work.

The material that did not fit into one of the three previous parts, but which is perhaps useful for a subset of the readers, has been moved to the appendices, in which I have also included a Dutch summary.

# Chapter 1

# Introduction

*The desire of children for visualizability is reasonable and healthy, still such a desire in physics can never be an argument for maintaining a particular system of concepts.*

*Wolfgang Pauli*

## 1.1 Gravitational singularities

After the discovery in 1929 by Edwin Hubble that the light we observe from other galaxies is redshifted in proportion to their distance, it was interpreted that all galaxies move away from each other. This observation strongly supported the Friedmann-Lemaître model of an expanding universe. Assuming that there is no special "center" of the universe, the current expansion of our universe leads to the conclusion that in the past all galaxies must have been closer together, when the universe was smaller. If we apply general relativity to derive the evolution of our spacetime, and we make some very reasonable assumptions about the matter content of the universe, we have to conclude that $13,7$ billion years ago the universe began with a "big bang" singularity. At the big bang singularity the matter density in the universe was infinite and distance between (spacelike separated) points of the spacetime was zero. Around the seventies the singularity theorems of Penrose and Hawking made the prediction of an initial singularity even stronger, because they proved that the existence of a "big bang" in a general relativistic spacetime does not rely on a homogeneous distribution of matter.

In 1964 Penzias and Wilson observed the presence of an isotropic electromagnetic background at microwave frequencies. The signal was characterized by a blackbody spectrum at a temperature of $2,725\,\mathrm{K}$, and seemed to be present throughout the whole universe. This "cosmic microwave background" is even stronger support for the standard big bang cosmology because the origin of the cosmic microwave background is straightforwardly clarified by the big bang cosmological model. The high densities of matter in the early stages of the universe were accompanied by high particle energies. At such high densities photons were continuously scattered by charged particles. While the universe was expanding, the matter density and average particle energy dropped. And after 380 000 years, when the energy had dropped below $0,25\,\text{eV}$, electrons and protons had recombined to form neutral hydrogen. At that moment the universe became transparent for photons (atomic electrons interact much more weakly with photons) and at that energy the free photons decoupled from matter. Because the universe was still expanding afterwards, the photon wavelengths became redshifted. The cosmic microwave background observed today is then the remnant of the high energies of these photons at the moment of decoupling, indicating that the universe has indeed expanded from a more dense and energetic state. But if we extrapolate the temperature of the universe towards earlier times, when the universe was even smaller, the particle energies grow so high that we expect that general relativity breaks down as a classical theory. It is expected that, like all other known forces, the gravitational force has a quantum mechanical nature, but gravitational quantum effects only become important at very high energies around the Planck scale. The Planck scale lies at an energy of $1,22 \cdot 10^{28}\,\text{eV}$. It is determined by a combination of the strength of the gravitational interaction (Newton's constant $G_N$), the speed of light $c$, and the reduced Planck's constant $\hbar$, which are, as a matter of fact, our only natural units to construct the units of mass and length. This energy corresponds to a Planck time, at $10^{-43}$ seconds "after" the big bang. What has happened before this time lies outside the realm of (classical) general relativity. Therefore the question what was the origin of the universe is a question for a quantum theory of gravity. Although, strictly speaking, we can only safely trace back the evolution of the early universe till $10^{-12}$ seconds after the big bang, because this time corresponds to the energies we have already investigated in particle accelerators.

Because gravitation tends to increase spatial differences in density, small irregularities in the matter distribution in the early universe have lead to the formation of structure. Matter has become collected into superclusters, clusters, galaxies and stars. In stars the matter distribution reaches high densities again, and hydrogen nuclei become sufficiently hot for a chain of nuclear reactions to take place. During the star's main sequence the outward pressure of these nuclear reactions balances out the gravitational attraction inside the star. Stars like our sun simply burn out their nuclear fuel, but the stellar nucleus of very massive stars collapses under its own weight. If the star was heavy enough, its nucleus will shrink and become more and more dense and the original stellar matter will collapse into a point of infinite density, a "black hole" singularity, which is another example of a gravitational singularity. Again, we expect that gravitational quantum effects will become important once the stellar matter is contained in a region with a size of about the Planck length. In the case of black holes, general relativity predicts the appearance of a “horizon”. The horizon surface is the boundary of the region in which all light curves end in the black hole singularity. The horizon prevents the outside world to receive information from the inside of the black hole, so the singularity cannot be detected. According to general relativity very massive stars collapse into a gravitational singularity. The precise nature of the spacetime structure inside the black hole horizon awaits an answer in a quantum theory of gravity. Both big bang spacetimes and Schwarzschild black holes have spacelike singularities, which means that the singularity is reached when a timelike coordinate approaches a singular value, or more physically, that the singularity is present in a certain region of spacetime at a specific value of the time coordinate.

A quantum theory of gravity is also important out of theoretical considerations. Two of the intellectual breakthroughs of early $20^{\text{th}}$ century physics are general relativity and quantum mechanics. At first sight they seem to describe different phenomena: general relativity describes relativistic effects in gravitational physics (like the bending of light rays in a gravitational field) while quantum mechanics describes the behaviour of microscopic particles. Quantum mechanics, together with special relativity, has led to quantum field theory. Quantum field theory is the basis of the standard model of particle physics which gives a very precise description of all known forces except gravitation. There is only one particle in the standard model that hasn’t been observed yet: the Brout-Englert-Higgs particle, which is a remnant of the symmetry breaking of the electroweak force. This mechanism is necessary to explain why the particles that transmit the weak nuclear force have finite range (the weak force is transmitted by massive bosonic particles, and the Brout-Englert-Higgs-Guralnik-Hagen-Kibble mechanism explains the origin of the masses of these bosons).

As discovered by Einstein, general relativity modifies Newtonian gravity, and thereby predicts that spacetime is not static but that it is (classically) described by a metric field. The metric field is determined (up to an equivalence class) by the matter distribution in spacetime. Physical theories like general relativity and the standard model are important to shape our understanding of nature, but they are also relevant technologically. For example, general relativity is necessary to make the GPS system operate and quantum electrodynamics, one of the quantum field theories behind the standard model, is necessary to study laser fields of relativistic power [31]. Yet, general relativity and quantum field theory appear to be mutually inconsistent. First, the usual methods to subtract the divergences that arise in quantum field theory when one takes quantum effects into account, do not work for general relativity. Second, the standard formulation of quantum field theory is worked out in a flat background without curvature, called Minkowski spacetime (Minkowski spacetime is reminiscent of Euclidean space). Still, through Einstein's equations, the spacetime structure of our universe is influenced by its matter content, and this matter consists of particles (or fields) that obey quantum principles, so the metric field in general relativity has to be quantized too. Likewise, the structure of spacetime influences the behaviour of matter fields, so one should be able to formulate quantum field theories in curved backgrounds. But the question how to formulate gravity at the quantum mechanical level is of a more fundamental nature than describing propagation of quantum fields in curved spacetime. The quantisation of the metric field has turned out to be much harder than the research of quantum effects in curved spacetimes. Since it is expected that curved spacetimes are describable by macroscopic states in a quantized theory of gravity, the formulation of quantum fields in curved spacetimes should arise from a semiclassical description of a quantum theory of gravity coupled to matter particles.

The presence of an initial singularity in our universe, and of final singularities in black holes, also raises the question whether propagation of matter through a singularity is possible and how it should be formulated in a rigorous theoretical manner. Such evolution across a singularity is important for certain cosmological models. The study of how singularities affect the evolution of fields near (or "through") a singularity requires a theoretical framework to take care of quantum gravitational effects.

Ideally, a quantum theory of gravity should give an explanation of these cosmological singularities. Several attempts have been made to construct a quantum theory of gravity that reproduces general relativity at energies much lower than the Planck scale. Yet it turned out that string theory, which was originally constructed as an attempt to formulate the strong nuclear force, could describe the exchange of gravitons, the quanta of the gravitational force, in a consistent manner. But in some sense, some of the older ideas to formulate a quantum theory of gravity, supergravity and higher derivative theories, are incorporated in string theory.

## 1.2 String theory as a quantum gravity theory

The physical description of a certain system differs when we study it at a "macroscopic" level versus a "microscopic" level. The length scales that macroscopic and microscopic refer to, of course depend on the physical context. Although we are looking at the same physical system, we have to use different physical quantities to describe the phenomena that happen at different length scales in an effective way. For example, a gas can be characterized by its temperature and its pressure at the macroscopic level, but at the molecular level there are colliding gas molecules with certain chemical properties instead. At a first glance the "low energy excitations" at the macroscopic level appear largely independent of the "high-energetic phenomena" that appear at much smaller length scales. For example, the Navier-Stokes equations describe the behaviour of fluids in the continuum limit, but in general these equations are independent of the type of molecules in the fluid. Yet the Navier-Stokes equations are an effective description that follows from the chemical interaction between the fluid molecules. At a deeper level, the microscopic molecular interactions do leave their trace in macroscopic parameters like the viscosity. The Navier-Stokes equations cannot be used to describe molecular interactions and in general we cannot expect that we can use low-energy effective interactions to give a consistent description of high-energetic phenomena. Coming back to the gravitational force (or the interactions transmitted by the metric field), we do not expect that general relativity can adequately describe the high-energetic physics close to the big bang, where quantum gravitational effects are expected to appear. String theory is capable of describing the exchange of gravitons, which are the quanta of the gravitational force, and general relativity can be obtained as the (classical) low-energy limit of the gravitational interaction in string theory.

To achieve a well-defined quantum theory that describes the interaction of gravitons, one has to assume that string theory (in its usual perturbative formulation reminiscent of the expansion in terms of Feynman diagrams that is used in quantum field theory) is described by the propagation of extended string-like objects and their interactions. Another prediction is that the number of spacetime dimensions is not arbitrary but is required to be ten for critical superstrings. For noncritical strings the number of dimensions can be different, but this leads to a strong curvature of the spacetime geometry of the order of the inverse Planck area, which cannot correspond to our universe. The necessity of having ten dimensions for critical superstrings means that at the low energies we are used to observe, we can only observe four dimensions that have decompactified (this is the standard interpretation, alternative views propose the existence of large extra dimensions). String theory is said to unify general relativity and the standard model because it is not only a quantum theory of gravity, but also offers a framework capable of incorporating the non-gravitational interactions of the standard model. But in order to include oscillations in the spectrum of superstring theory that have a fermionic character to be able to describe the matter in the universe, one has to assume that at high energies there exists a symmetry that relates the bosons (up till now the force-carriers) and the fermions (up till now the matter particles) in the universe. This relation is called supersymmetry and originally it was proposed independently of strings. What the precise mechanism is that breaks this symmetry at low energies, will be investigated in the Large Hadron Collider at

CERN.

String theory owns its name because originally its fundamental degrees of freedom were considered to be stringlike (not pointlike). Physical effects can be calculated by means of a perturbation expansion into strings that split and join on a fixed spacetime background (strictly speaking there are five ways to define a consistent superstring theory that includes fermions). But after subsequent research it has turned out that string theory also includes pointlike and higher-dimensional objects (coined "$p$-branes", a 0-brane is pointlike, a 1-brane is a stringlike object, a 2-brane is a membrane etc.) which can become the fundamental degrees of freedom in regimes where the usual perturbative expansion of string theory in terms of the original strings breaks down. As a matter of fact, none of these objects can be called more fundamental than the others. Depending on the circumstances, each of these objects can be more appropriate to work out a perturbation expansion to describe the physical effects of string theory. In order to distinguish the underlying theory that is behind all these perturbative expansions, compared to the original perturbation series in terms of strings, one often hears the name "M-theory" [34]. M-theory is used to describe the research towards a nonperturbative, background-independent theory of gravity, with the earlier five string theories as approximations when the string coupling is small, but in this thesis I will simply use the name string theory.

In brief, research in string theory is an important approach to investigate gravitational quantum effects. However, these quantum effects are expected to become only important near singularities or at length scales around the Planck length. If we take a $20^{\text{th}}$ century point of view on scientific progress and we look at the research field of quantum gravity with Karl Popper in mind, the difference of magnitude between the Planck scale and the TeV scale is unfortunate because it means that (barring unexpected future developments) one cannot directly compare theoretical predictions with experiments. But since quantum gravitational effects were important in the early universe, they will have influenced the earliest structure in the universe. Therefore they may lead to observable effects in cosmological measurements, for example in the higher order fluctuations of the cosmic microwave background. Last year the satellite PLANCK has been launched to investigate these issues in more detail than its precessor WMAP. Furthermore, at the fundamental level a theory of quantum gravity is needed for theoretical consistency. As a clear experimental verification is (at present) not possible, the provision of a theoretical backbone for well-established theories is an important result. For example, a calculation by Stephen Hawking in the context of quantum fields on curved spacetime showed that black holes emit a blackbody radiation [30]. Basically, the origin of this so-called "Hawking radiation" is that black holes curve spacetime so strongly that particles are produced. Because of the blackbody spectrum it is possible to assign a temperature and an entropy to the black hole. Hawking's computation showed that the entropy is related to the horizon area of a black hole. In statistical mechanics the entropy is related to the number of states of a system. The question how the number of internal states of a black hole can be represented by its horizon area size was elucidated by Vafa and Strominger [33], who presented an important derivation in string theory. Their construction (with some technical assumptions on the class of black holes) allowed them to calculate the entropy out of the number of string states that make up the black hole. They found exactly the same relation between entropy and horizon area and reproduced the precise numerical coefficient. Another important result in string theory is the realisation of certain aspects of the "holographic principle". The holographic principle, introduced by Gerard 't Hooft [32] in 1993, is based on the behaviour of gravity at high energies and posits (roughly stated) that gravitational phenomena in a spacetime region can be described in an equivalent (or dual) "holographic" manner by a quantum field theory in a smaller amount of dimensions. The principle is apparent in Maldacena's conjecture [127] about the equivalence between a certain superstring theory on a specific spacetime with a negative cosmological constant (which is called "Anti-deSitter"), and a conformal quantum field theory that is defined on flat Minkowski spacetime, which is the boundary of that Anti-deSitter spacetime. Another example of the holographic principle is matrix theory [98], that gives a non-perturbative formulation of string theory in certain regimes. In matrix theory, the matrix model of a spacetime gives a description of gravity in terms of quantum mechanical matrices. These examples of holography show how string theory can elucidate previously unexplained relations between quantum field theory and gravity.

Mathematically, the perturbation expansion of string theory is formulated on a background spacetime. This construction is well understood if the background spacetime is static. But according to standard cosmology, our universe is not static but expanding and it has an initial singularity. Currently string theory cannot deal well with time-dependent backgrounds, nor with spacelike singularities like the big bang. To understand whether our universe can be described by a consistent solution of string theory, it is therefore crucial to develop techniques that allow to describe string theory in time-dependent backgrounds. Near singularities strings can become highly excited, which would lead to backreaction effects of the string on the metric through Einstein's equations (or an appropriate quantum-mechanical generalization thereof in string theory). Often the interaction between strings grows strong and it becomes necessary to investigate non-perturbative formulations of string theory that can deal with strong coupling between strings. Therefore, non-perturbative formulations of string theory, among which matrix theory, have attracted a lot of attention in the study of cosmological singularities. These non-perturbative formulations were already known for a restricted class of usually static backgrounds. Recent work consists of extending these non-perturbative descriptions to certain time-dependent backgrounds with cosmological singularities, for example the matrix big bang model by Craps, Sethi and Verlinde [107]. Matrix models for spacetimes that include a big bang type of singularity typically involve a quantum field theory with singular features (one might say that the spacetime singularity has been mapped to another singularity because of the intrinsic holographic nature of the matrix models). The singularities in the dual field theories may appear as time-dependent terms in the Hamiltonian, or the field theory may be defined on a singular (auxiliary) spacetime. The formulation of matrix models in arbitrary spacetimes remains to be further developed.

## 1.3 Matrix theory and the emergence of spacetime

In the previous section I have already mentioned some aspects of matrix theory, but without specifying what it exactly describes. I will give a very brief introduction to matrix theory, which gives a description of spacetime at the quantum level when strings strongly interact.

Matrix theory is a conjecture by Banks *et al* [98] to capture, in certain regimes, physical phenomena associated to strongly interacting strings. Matrix theory gives a nonperturbative formulation of superstring theory in terms of a dual Super-Yang-Mills field theory of quantum-mechanical matrices. Roughly speaking, the dual theory is derived from Yang-Mills fields that propagate on D0-branes. These D0-branes are very massive objects when superstrings interact weakly, but become the fundamental degrees of freedom at strong coupling between strings. The original matrix theory model describes asymptotically flat eleven-dimensional spacetime, but during consequent work matrix models have been derived for other spacetimes as well, for example the matrix big bang model of Craps *et al* [107] that is a toy-model for a big-bang like singularity.

In matrix theory spacetime is an emergent concept. Let me briefly illustrate this, the bosonic part of the matrix theory action is given by

$$S = \int \mathrm{d}t \mathrm{Tr} \left\{ \left(D_0 X^i\right)^2 + ([X^i, X^j])^2 \right\} , \tag{1.1}$$

where the $X^i$ ($i = 1 \ldots 9$) are $N \times N$ matrices (for the matrix theory description of string theory the limit $N \to \infty$ has to be taken). For clarity I have neglected some prefactors and I have omitted the fermionic partners of the spacetime fields (these fermionic partners are present because the theory is supersymmetric). The symbol "Tr" is the trace over the matrices $X^i$. The eigenvalues of the $X^i$ are the position vectors of $N$ D0-branes, and there is a commutator potential

$$V = -([X^i, X^j])^2 \,. \tag{1.2}$$

When the D0-branes are far apart, the potential would behave as a quartic potential in generic directions and contribute strongly to the energy. Therefore in the low-energy configurations the matrices $X^i$ have to commute, which means that their diagonal elements give a good indication of the position in spacetime. On the other hand, when the D0-branes move closer, the off-diagonal modes of the matrices $X^i$ become important, and the position of the D0-branes becomes fuzzy, which is interpreted that spacetime becomes non-commutative at small distances.

## 1.4 Aim and scope of my thesis

The aim of the research is the study of cosmological singularities, and string theory provides a theoretical framework. Nevertheless, some results can be understood independently of string theory, in the context of quantum field theory on singular spacetimes. In addition, to understand the other results it is not necessary to be skilled in the full computational machinery of string theory.

My research is divided into two main topics: the use of geometric resolutions to define evolution across singularities, and the usefulness of singular plane wave models for studying the behaviour of string theory on singular and time-dependent backgrounds. Somewhat lurking in the background is the matrix theory description of singularities that gives rise to field theory models on singular spacetimes.

The definition of field propagation through singularities suffers from ambiguities. The geometric resolution prescription allows us to investigate the evolution across a singular spacetime in a concise manner. That is, once a specific geometric resolution is chosen, the procedure leads to well-defined results.

In the following sections I introduce gravitational plane waves and I illustrate the geometric regularization procedure. We use a geometric resolution to investigate the propagation of a scalar field on a singular spacetime and of a string on a singular plane wave background. Finally, in order to develop a better understanding of matrix models for plane waves, we investigate the formulation of $p$-branes in an asymptotically plane wave background.

My research about propagation through singularities is in some sense a "double" toy model, if we consider it as a direct attack to investigate singular spacetimes. I use the name "double toy model" because we study the propagation of "test fields" on a "fixed background" spacetime. I believe it is worth clarifying both concepts because it will elucidate the reasoning behind the specific conclusions of my research projects that I will discuss in the next sections. In a complete quantum gravitational setting we would expect that quantum fluctuations of the spacetime can appear (hence, no fixed background) and that there is a backreaction of the propagated field on the spacetime. However, at present such a complete framework would be hopelessly intractable (and not well-defined yet). Therefore we have to resort to simplifications that allow us to derive conclusions. But in a more indirect manner, as I already mentioned above and as I will clarify shortly afterwards, my research is also important for a matrix model resolution of singular spacetimes.

We can motivate the choice for a fixed background spacetime because we expect that in a semiclassical limit the interference of the quantum fluctuations in the metric field (more precisely the stringlike "gravitons") should lead to a classical background (which, loosely speaking, should exist as a kind of coherent state in the quantum theory) accompanied by a phase factor. Then, in an appropriate approximation scheme around this classical background, the quantum fluctuations can be considered as perturbations with respect to this classical background. Therefore, at the zeroth order of the approximation we can limit ourselves to considering the classical background only.

The second simplification consists of the introduction of test fields (particles or strings) that propagate on the classical background. This is usually called the probe limit approximation. By definition "test" fields would mean the fields do not interact with the spacetime background, but in a relativistic setting both energy and mass couple to the geometry of the spacetime, and even massless test fields carry energy. Therefore we should assume that both the mass and the energy of the "test fields" remains small enough such that it doesn't disturb the background spacetime. But because of Heisenberg's uncertainty principle in quantum mechanics low energies of a field mode are related to long wavelengths (for the sake of simplicity I present an argument for a massless field that travels at the speed of light). Therefore test fields with small energies only permit us to probe the slow variations in the spacetime. In other words, we can fairly reasonably assume that the introduction of test fields is permitted in a background with weak curvature, where the spacetime geometry varies slowly. But singular spacetimes typically produce a very strong curvature of the classical metric near the singularity, and singularities with a diverging curvature often tend to blueshift the energy of a string oscillation mode which leads to backreaction of the field on the spacetime geometry.

If the probe limit approximation (for example, loosely interacting strings on singular spacetimes) is invalid, we can turn our attention to matrix models of the singular spacetimes. There is a double advantage to matrix models: they allow to study the quantum behaviour of a spacetime that reduces to a classical background far away from the singularity, and they also take the backreaction of strings on the spacetime background into account. But the behaviour of matrix models immediately leads back to the investigation of how quantum fields propagate on singular spacetimes. It turns out that, due to their holographic nature, matrix models for spacetimes that include a big bang type of singularity typically involve quantum field theories that possess various types of singularities, for example, the field theory that describes the matrix models may be defined on a singular spacetime. So in addition to the usefulness of my results as a "double toy model" to investigate singularities, the propagation of quantum fields on a singular spacetime is very useful for the interpretation of matrix models of singular spacetimes. Of course, these auxiliary singular spacetimes on which a matrix model is constructed are still toy models for a real cosmological spacetime with a big bang singularity, but the consideration of a quantum field theory on a singular background is strongly motivated by the observation that they appear in the formulation of a matrix (toy) model for a spacetime singularity.

## 1.5 Gravitational plane waves in string theory

There is an important class of spacetime structures, called gravitational plane waves, which constitute an interesting background in string theory. First, the plane wave spacetimes can be time-dependent and singular, depending on the chosen plane wave profile. Plane waves possess a special metric structure. In lightcone gauge the equations of motion for the oscillation modes of a string are exactly solvable. In addition, the metrical structure of a plane wave guarantees there are no $\alpha'$ corrections[1] in the low-energy effective action for string theory and that plane waves are exact string theory solutions. The $\alpha'$ corrections are related to the extended length of strings thus their absence in the low-energy effective action guarantees that in plane waves "stringlike" effects are not directly visible (though the consideration of strings is of course still necessary to guarantee a consistent perturbative approach to study the propagation of gravitons on a background spacetime). Therefore in singular plane waves it becomes possible to primarily concentrate on the effects due to the divergent curvature near the singularity.

In addition, plane waves are naturally associated to cosmological singularities. The "Penrose limit", a certain procedure developed by Roger Penrose,

[1] Two important parameters in string theory are the string coupling $g_s$ and $\alpha'$. The latter is related to the string length $\ell_s$ by $\sqrt{\alpha'} = \ell_s$. If one regards string theory as a quantum theory of gravity (say, in ten dimensions) the string length $\ell_s$ has to be of the order of the Planck length $\ell_P$ according to

$$\ell_P \approx g_s^{1/4} \ell_s. \tag{1.3}$$

In the formulation of string theory as a perturbation series in the string coupling $g_s$, factors of $g_s$ appear in front of quantum corrections, somewhat like the perturbation expansion of quantum electrodynamics in terms of the fine structure constant $\alpha = e^2/(4\pi\epsilon_0\hbar c)$. On the other hand, $\alpha'$ corrections are related to the fact that a string has a finite length compared to a point particle. The $\alpha'$ corrections appear at low energies where string theory is approximated by an effective action, in which the higher derivative terms are suppressed by powers of $\alpha'$.

associates a plane wave to any spacetime. This is done by "zooming in" onto the original spacetime near a lightlike geodesic. The resulting plane wave geometry is simpler to study but it captures essential information of the spacetime in the neighbourhood of the geodesic. If the Penrose limit is performed along a geodesic that hits a singularity, one obtains a singular plane wave. One of the possible characteristics of a singularity (but one that is physically very intuitive) is a divergent tidal force (that is due to the divergent curvature of the spacetime metric) is encoded in the plane wave profile. For typical cosmological singularities Blau *et al* [70] have shown that under a mild technical, physically very reasonable assumption, a Penrose limit leads to scale-invariant plane waves. The scale-invariance means that it is possible to rescale the two coordinates that characterize the directions on the light cone, while keeping the metric that characterizes the spacetime invariant.

## 1.6 Geometrical regularizations

In this section I will illustrate the main idea behind the geometrical resolution procedure. The problem is to find an unambiguous way to define the propagation of a field across a spacetime singularity[2] and a major step in the procedure is the construction of a class of geometrically regularized metrics related to the singular metric. To illustrate the idea, let us consider a simple system that consists of a massive scalar field on a curved background that is characterized by a metric tensor $g_{\mu\nu}$. Taking the "mainly plus" convention for the metric, the field evolution is determined by the action

$$S = -\frac{1}{2} \int \sqrt{-g} \left( g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} m^2 \phi^2 \right) \mathrm{d}^D x, \tag{1.4}$$

which is invariant under coordinate reparametrizations $x \to x'(x)$ as required by general relativity. $\partial_\mu$ denotes the ordinary partial derivative $\partial / \partial x^\mu$. The "determinant of the metric" $g = \det g_{\mu\nu}$ and the inverse metric tensor $\mathrm{g}^{\mu\nu}$ take into account that the scalar field propagates on a curved spacetime. There are no interaction terms higher than the mass term $m^2\phi^2$, so the scalar field is called "free", and the metric field $g_{\mu\nu}$ is considered to be non-dynamical. In other words, to excite the metric field with respect to its classical background value requires such high energies compared to the excitations of the scalar field, that as long as we are interested in configurations with low energies, we can safely neglect the kinetic term of the metric in the action ($\sqrt{-g}R$, as it appears general relativity). We then demand that the action is stationary with respect

[2]In order to define a gravitational singularity I would have to introduce a few elements from differential geometry. I advise the interested reader to take an immediate look at appendix A which gives an explanation at an introductory level.

to infinitesimal variations in the field $\phi \to \phi + \delta\phi$, in order to determine the (Klein-Gordon) equation of motion

$$\left(-\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right) + m^2\right)\phi = 0\,. \tag{1.5}$$

In the case of a free scalar field, we can find the full solution by decomposing it into field modes (with wave vector $k$) to which creation operators $a_k^\dagger$ and annihilation operators $a_k$ are associated,

$$\phi = \sum_k f_k a_k + f_k^* a_k^\dagger\,, \tag{1.6}$$

and the mode functions $f_k$ are classical solutions to the wave equation (1.4).

In order to proceed with the illustration of the geometrical resolution of a free field, let us consider a specific example. Suppose we are investigating the propagation of a free massive scalar field across the singularity of a four-dimensional singular plane wave metric of the form

$$ds^2 = 2dudv - \frac{\omega_0^2}{u^2}\left(x^2 - y^2\right)du^2 + dx^2 + dy^2\,. \tag{1.7}$$

Here $u$ and $v$ are "lightcone coordinates" defined by $u = (z-t)/\sqrt{2}$ and $v = (z+t)/\sqrt{2}$ with the plane wave propagating in the direction $z$. By means of the action (1.4) we obtain the Klein-Gordon wave equation

$$-\partial_u\partial_v\phi - \frac{1}{2}\partial_x^2\phi - \frac{1}{2}\partial_y^2\phi - \frac{\omega_0^2}{2u^2}\left(x^2 - y^2\right)\partial_v^2\phi + \frac{m^2}{2}\phi = 0. \tag{1.8}$$

A Fourier transform with respect to $v$, given by

$$\phi(u,v,x,y) = \frac{1}{\sqrt{2\pi}}\int \mathrm{d}k_v\hat{\phi}_{k_v}(u,x,y)e^{ik_v v}, \tag{1.9}$$

puts the Klein-Gordon equation in a Schrödinger-like form (in units where $\hbar = 1$)

$$i\frac{\partial}{\partial t}\hat{\phi}_{k_v}(t,x,y) = \left\{\frac{\omega_0^2 k_v}{2t^2}\left(x^2 - y^2\right) - \frac{\partial_x^2}{2k_v} - \frac{\partial_y^2}{2k_v} + \frac{m^2}{2k_v}\right\}\hat{\phi}_{k_v}(t,x,y). \tag{1.10}$$

For clarity we have performed the substitution $u \to t$. We can recognize this as a Schrödinger equation,

$$i\frac{\partial}{\partial t}|\Psi\rangle = \mathcal{H}|\Psi\rangle, \tag{1.11}$$

with $\hat{\phi}_{k_v}(t,x,y) = \langle x,y|\Psi\rangle$. In principle can find the exact solution $\hat{\phi}$ with semiclassical methods, because the (auxiliary) Hamiltonian associated to (1.10) is quadratic in positions and conjugate momenta,

$$\mathcal{H} = \frac{1}{2k_v}\left(p_x^2 + p_y^2\right) + \frac{\omega_0^2 k_v}{2t^2}\left(x^2 - y^2\right) + \frac{m^2}{2k_v}. \tag{1.12}$$

The semiclassical method of Wentzel-Kramers-Brillouin (WKB) prescribes to write the wavefunction in the form of an exponential times a time-dependent prefactor

$$\Psi(t) = \mathcal{A}(t,t_0)\exp\left(iS_{cl}(x,y,t|x_0,y_0,t_0)\right), \tag{1.13}$$

with $S_{cl}$ the classical action associated to the auxiliary Hamiltonian (1.12) and $x_0$ and $y_0$ the initial conditions at $t = t_0$. However, there is a singularity in the Schrödinger equation at $t = 0$, as a consequence of the singularity in the metric. Moving back temporarily to a more abstract notation, the Hamiltonian (1.12) consists of the following operator structure

$$\mathcal{H}(t) = \sum_i f_i(t)\mathcal{O}_i, \tag{1.14}$$

where the operators are time-independent but some of the time-dependent prefactors $f_i(t)$ become singular at $t = 0$ such that the expression

$$\lim_{t\to 0}\mathcal{H}(t) \tag{1.15}$$

is ill-defined. In principle it is straightforward to solve equation (1.10) away from $t = 0$. We could call the solutions $\Psi_+(t)$ and $\Psi_-(t)$ (for $t > 0$ and $t < 0$ respectively). However, how to match $\Psi_+$ and $\Psi_-$ across the singularity? The evolution is ambiguous. In the example considered above the divergent functions are $f_x(t) = -f_y(t) = \omega_0^2/\left(2t^2\right)$.

To proceed we introduce a regularization parameter $\epsilon$ and we define a family of regularized Hamiltonians $\mathcal{H}_\epsilon$ such that their behaviour near the singular point $t = 0$,

$$\lim_{t\to 0}\mathcal{H}_\epsilon(t) = \mathcal{H}_\epsilon \tag{1.16}$$

is now well-defined and we retrieve the original Hamiltonian in the singular limit ($\epsilon \to 0$)

$$\lim_{\epsilon\to 0}\mathcal{H}_\epsilon(t) = \mathcal{H}(t). \tag{1.17}$$

We then solve Schrödinger equation of the regularized system

$$i\hbar\frac{\partial}{\partial t}\Psi_\epsilon = \mathcal{H}_\epsilon\Psi_\epsilon, \tag{1.18}$$

for $\Psi_\epsilon$. The class of resolved metrics, labeled by the parameter $\epsilon$, is perfectly regular. As a consequence, the Schrödinger equation related to the evolution of a field on the resolved spacetime, will be regular as well. Finally we investigate the singular limit of the wavefunction

$$\lim_{\epsilon\to 0} \Psi_\epsilon \stackrel{?}{=} \Psi \,. \tag{1.19}$$

If this limit exists, we consider this expression as the solution to the singular problem. Up till now we haven't explained what is specifically "geometric" about our regularization. It means that we introduce a regularization parameter $\epsilon$ in the metric. But we have to impose appropriate conditions to remove the arbitrariness of the procedure such as that the resolved spacetime still makes sense physically. For example, in one of the coefficients of the metric tensor we could easily replace a singular function such as $1/t^2$ as follows:

$$\frac{1}{t^2} \quad \to \quad \frac{1}{t^2+\epsilon^2} \,. \tag{1.20}$$

But this would be completely arbitrary without further restrictions that relate the metric coefficients. We impose the condition that our class of "resolved" metrics should still satisfy Einstein's equations (in a sense to be specified below) for all values of the regularization parameter $\epsilon$. If it does, we call it a geometrical resolution.

In fact, for the simple example considered above, the regularization (1.20) is already geometrical (the singular plane wave is written in Brinkmann form and the function $f_x(t) = -f_y(t)$ can be modified arbitrarily) and the regularized Hamiltonian simply becomes

$$\mathcal{H}_\epsilon = \frac{1}{2k_v}\left(p_x^2+p_y^2\right) + \frac{\omega_0^2}{2(t^2+\epsilon^2)}\left(x^2-y^2\right) + \frac{m^2}{2k_v}. \tag{1.21}$$

But such a simple structure of the resolved Hamiltonian is rather accidental, and as we will argue during chapter 10, nothing forbids that in a more general case the geometrical resolution would lead to a resolved Hamiltonian of a structure where several operators are involved in the resolution by the parameter $\epsilon$,

$$\mathcal{H}_\epsilon = \sum_i f_i(t,\epsilon)\mathcal{O}_i + \sum_j g_j(t,\epsilon)\mathcal{Q}_{j(\epsilon)}, \tag{1.22}$$

and where the additional operators $\mathrm{Q}_{j(\epsilon)}$ represent the possibility that the operator structure of the Hamiltonian is modified during the geometric regularization (the $\mathrm{Q}_{j(\epsilon)}$ will have to disappear in the singular limit).

Notwithstanding the fact that the geometrical resolution prescription is a reasonable way to investigate the propagation of fields across spacetime singularities (the regularized spacetime does admit a geometrical interpretation), it certainly does not remove the large amount of ambiguity related to the propagation of fields through the singularity. One may prefer another regularization prescription.

Another remark is that the demand for the regularized spacetimes to “solve” Einstein’s equation (without specification of the energy-momentum tensor) is not necessarily a physical criterion, because any spacetime can be considered as a solution to Einstein’s equation as long as the matter content of the universe is not specified (though there are some reasonable physical restrictions on the type of matter that can be allowed). In the case of singular plane waves in the context of string theory, the analogue of Einstein’s equations are the background consistency conditions of string theory that relate the metric to other fields. It is therefore natural to demand that these consistency conditions remain valid at the level of the class of regularized spacetimes. In case our regularization parameter introduces additional “fictitious” matter in the class of regularized spacetimes (the fictitious $\epsilon$-dependent matter would of course have to disappear in the singular limit) it would be prefarable to have a physical argument for such fictitious matter.

## 1.7 Free scalar field on the parabolic orbifold

Motivated by the role of quantum fields in a singular spacetime associated to matrix models, in my first collaboration I have investigated the appearance of singular time-dependent terms in the Hamiltonian. In an earlier project, my collaborators Ben Craps and Oleg Evnin had considered how to regularize those Hamiltonians by means of the most conservative approach that would allow them to define a unitary evolution across the singularity [94]. This approach, which they called “minimal subtraction”, consists of modifying the singular time dependences in the Hamiltonian to become distributions while keeping the operator structure of the Hamiltonian unchanged (this approach is relevant if the transition through the singularity is dominated by a single term in the Hamiltonian). The cancellation of the divergence is essentially due to the negative contributions in the distributions. We found that this prescription was different from a geometric regularization because the negative function values associated to the distributions conflict with a geometrical interpretation. For a geometrical resolution of dynamics on a singular spacetime background, one generally needs to relax the specifications of the “minimal subtraction” approach, and permit modifications in the operator structure of the Hamiltonian, as well as modifications in its time dependence, in the vicinity of the singular region.

As a specific example we considered the propagation of a massive scalar field in a singular spacetime. We investigated the so-called parabolic orbifold that appears by making an identification along one of the two directions on the lightcone (the other direction will be interpreted as the time direction) in flat Minkowski spacetime. Because of this identification a singularity is created that provides a toy model to investigate singularities (compare for simplicity the singular tip of a cone that appears by folding a flat sheet). The parabolic orbifold can be viewed as the singular limit of the (regular) nullbrane, which is a four-dimensional orbifold with one free parameter $R$. In the limit $R \to 0$, the nullbrane geometry reduces to the parabolic orbifold geometry times the real line, and in this sense the nullbrane is a geometrical regularization of the parabolic orbifold. Earlier, Liu *et al* [84, 86] had investigated the parabolic orbifold in the context of string perturbation theory.

According to our geometric resolution prescription, we first analysed the evolution of a free scalar field on the regular nullbrane, before taking the singular limit. To be able to investigate the singular limit, we introduced a new set of coordinates on the nullbrane that is globally defined and has a well-defined singular limit. We also considered a generalized nullbrane metric (which is essentially the nullbrane enlarged with two free parameters). The essential step towards the solution was to translate the quantum mechanical evolution on the nullbrane into known evolution equations of a dynamical group (in this case the two photon group from quantum optics). Notwithstanding the apparently strongly singular behaviour (the singular terms cannot even be written as distributions) of the limiting Hamiltonian, the quantum mechanical evolution across the singularity is well defined. The commutation properties of the different operator terms in the Hamiltonian exactly compensate the singular behaviour. But we find that the singular limit only exists for a discrete subset of the possible parameter values within the family of generalized nullbrane geometries. We can label the subset by one integer number. As could have been expected, the original nullbrane falls into this subset.

The evolution of the modes of the scalar field is completely charactised by its mode functions (the same mode functions also appear in the description of strings on the nullbrane). If we compare our results with Liu *et al* [84, 86] we find the same mode functions except for the exponential that characterizes the wave travelling in the $X$-direction, where there is a sign$(t)$ factor in front of the coordinate $X$. Its effect is that the position and velocity in the $X$-direction for all particles are reflected as they pass through the singularity. The difference is due to our new coordinate system that does not fail at the origin $t = 0$.

If we look at the discrete subset of generalized nullbranes for which the singular limit exists, we find that their mode functions are equivalent up to a (global) phase jump across the singularity, which appears because at the origin the field modes cross a number of focal points that is proportional to the integer that characterizes the subset.

## 1.8 String modes in singular plane waves

I have already mentioned that plane waves provide an analytically solvable background in string theory. In collaboration with Ben Craps and Oleg Evnin

I have investigated the evolution of a free string on a singular plane wave background [120]. The approximation of a free string can be seen as a preliminary consideration before investigating perturbation theory on such a background. We have concentrated on plane waves that exhibit a scale-invariant profile,

$$ds^2 = -2dx^+dx^- - \lambda \sum_i \left(\frac{x^i}{x^+}\right)^2 (dx^+)^2 + \sum_i \left(dx^i\right)^2 , \qquad (1.23)$$

as these naturally appear from a Penrose limit of cosmological singularities. The coordinates transverse to the lightcone directions $x^+$ and $x^-$ are denoted by $x^i$ and we use $x^+$ to indicate lightcone time. The scale-invariance means that the plane wave metric is invariant under rescalings in the lightcone coordinates $(x^+, x^-) \to (Cx^+, x^-/C)$. Because of the scale-invariance of the singular profile it is natural to impose that our class of resolved metrics be scale-invariant as well, in addition to the constraint that our resolved metrics satisfy the generalization of Einstein's equations in string theory (the "background consistency conditions"). We resolve the scale-invariant profile as

$$\lambda \sum_i \left(\frac{x^i}{x^+}\right)^2 \quad \to \quad \frac{\lambda}{\epsilon^2}\Omega(x^+/\epsilon) \sum_i \left(x^i\right)^2 , \quad \lim_{\eta \to \pm\infty} \Omega(\eta) \sim \frac{1}{\eta^2} , \qquad (1.24)$$

where we call $\lambda$ the "normalisation" of the wave profile, $\epsilon$ the unique resolution parameter and $\Omega$ the resolved profile. Because of the scale-invariance there is only one dimensionful parameter (i.e. $\epsilon$) that can appear in the resolved metric, which is the resolution parameter that we will remove in the singular limit.

To satisfy the background consistency conditions in string theory we add a dilaton field. The dilaton is an oscillation mode of the string, like the graviton, but it also determines the string coupling. The background consistency conditions relate the curvature of the spacetime metric to the spacetime variation of the dilaton. Naturally, we also demand that the dilaton can propagate across the singularity and prove that it is possible.

In lightcone gauge the Schrödinger equation for the string is determined by a Hamiltonian that can be separated as a sum of quadratic Hamiltonians with a time-dependent frequency, each determining the behaviour of one oscillation mode of the string. Therefore we can initially consider all the string modes separately. When the resolution parameter is removed, the frequencies diverge at $t = 0$. Due to the quadratic dependence of these Hamiltonians on the position and momentum operators, we can obtain an exact solution to the Schrödinger equation by the semiclassical approximation. This means that the string wavefunction is completely determined by solutions to the classical equations of motion with appropriate boundary conditions. We notice that the equations of motion for the oscillations of the string are related to the propagation of the center-of-mass mode (or zero mode) of the string. In fact, the only difference between the equations for the excited string modes compared to the center-of-mass mode is the square of the mode number that contributes to the time-dependent frequency in the Hamiltonian. But the mode number is a finite term compared to the (diverging) time-dependent frequency and we can rigorously prove it doesn't change the existence of the singular limit. We consider the mode number as a small perturbation and we derive a bound on how much the solutions for the excited modes can differ with respect to the zero mode. In the singular limit the difference between the solutions disappears and we can prove that the excited modes can propagate through the singularity whenever the zero mode propagates. In an earlier publication Evnin and Nguyen [119] had already shown that the zero mode can propagate through the singularity, leading to as discrete spectrum for the parameter $\lambda$ in the wave profile.

So, just as in the case of the free scalar field on the parabolic orbifold, also here we find a discrete spectrum related to the propagation across the singularity. The zero mode (and the excited modes) can only cross the plane wave singularity for (generically) a discrete set of $\lambda$. The precise spectrum of $\lambda$ is determined by the shape of the resolved profile $\Omega(\eta)$. But the scale-invariance of the resolution has permitted us to derive the propagation through the singularity without any further specification of the resolution profile $\Omega(\eta)$ except for its asymptotics (1.24).

We have found that all the different modes of the string can propagate through the singularity separately, but in order for the string to propagate through the singularity as a whole, we have to demand that the excitation energy of the string remains finite during the transition. We find that this is only the case if the "normalisation" $\lambda$ of the plane wave profile satisfies the condition

$$\lambda = \frac{1}{4} - \left(N + \frac{1}{2}\right)^2, \tag{1.25}$$

where $N$ is an natural number ($N = 0$ corresponds to Minkowski spacetime or the lightlike reflector plane of [95]). But, for $\lambda < 0$, the dilaton diverges near the singularity and the string coupling becomes strong without bound, thereby invalidating perturbative string theory. Thus it is impossible that the total excitation energy remains finite under the assumption the string is free (out of consistency the consideration of a free string requires that the interaction between strings is small). Since perturbative string theory becomes invalid near the singularity, this further motivates us to investigate matrix models of singular plane waves that can deal with strong interaction between strings.

## 1.9 Supergravity D$p$-brane solutions

Matrix models that describe the strong coupling limit of string theory, are formulated in terms of the effective action of D0-branes (or D1-branes). Therefore, if we want to better investigate the properties of matrix models of singular plane waves (for example the matrix big bang model of Craps, Sethi and Verlinde [107] or the plane wave matrix models of Blau and O'Loughlin [112]) we should study the formulation of D-branes in an asymptotically plane wave background. D-branes are a special class of branes that represent degrees of freedom characterized by strings with specific boundary conditions. They play an important role as the effective degrees of freedom in matrix theory. The branes that appear in string theory are dynamical objects, but they can also be described as classical solutions in supergravity. Supergravity is an extension of general relativity that also includes fermions in its spectrum (all the matter we know consists of fermions). Supergravity is a low-energy approximation to string theory, in other words it is valid when there is insufficient energy to excite the higher oscillations of the string. Therefore at the classical level, where the D-branes are massive, they can be described by a metric, a dilaton and a field potential (the field potential appears because the D-brane is charged). These supergravity brane solutions can also be used to formulate a duality between (lightcone) time-dependent bulk and boundary theories *a la* AdS/CFT correspondence [128, 75]. The standard matrix model Hamiltonian (1.1) describes eleven-dimensional static Minkowski spacetime. The low-energy description of the matrix model Hamiltonian is given by string theory in a supergravity background of D0-branes [105]. The matrix big bang [107] which is a time-dependent model, is formulated in terms of D1-branes in a time-dependent plane wave background. Therefore we are interested in the classical supergravity solutions that describe time-dependent D-branes in an asymptotically plane wave background.

This means we are looking for a spacetime solution that will resemble a plane wave at the radial asymptotics transverse to the brane, but the presence of the D-branes at the origin will alter the metric for finite distances. In the context of matrix theory models for singular plane waves we are primarily interested in the question how to formulate a system of D1-branes if the spatial dimension of the branes is perpendicular to the lightcone. A easier problem is the formulation of D1-branes that are aligned with the lightcone (in other words, the "world-volume" of the brane is parallel to the propagation direction of the brane). In collaboration with Ben Craps, Oleg Evnin and Federico Galli, I have derived the metric that describes extremal D1-branes in asymptotically plane wave backgrounds, and we have extended these supergravity solutions to higher dimensional D$p$-branes (with $p \geq 1$) [136]. The extension of these $p$-brane solutions to a configuration of D0-branes in a dilaton-gravity plane wave is under study. In this case there is no worldvolume of the brane to align the propagation direction of the wave with.

## 1.10 Summary

The large scale structure of our universe is very well described by general relativity, but general relativity predicts the existence of gravitational singularities such as the big bang or black holes. The appearance of singularities indicates where a theory loses its predictive value. In this sense, the theoretical framework of general relativity breaks down near spacetime singularities, where the gravitational tidal forces may become infinite and the mathematical description becomes meaningless. String theory is an extension of general relativity, that allows to describe the exchange of gravitons, the quanta of the gravitational force. It is expected that quantum effects influence the spacetime behaviour near a singularity. In addition, string theory offers the possibility of a unified description of gravity with the electromagnetic and nuclear forces.

The research to extend the original perturbative expansion of string theory (reminiscent of the expansion of quantum field theory in terms of Feynman diagrams) towards a fully non-perturbative theory has led, among other, to matrix theory. In matrix theory spacetime is described in terms of quantum-mechanical matrices. Because of its intrinsic non-perturbative nature matrix models are an appropriate method to describe spacetimes with singularities where the interaction between strings becomes large and the usual formulation of string theory as a perturbation expansion in function of the interaction between strings becomes invalid. Matrix models for spacetimes that include a big bang type of singularity typically involve quantum field theories that possess various types of singularities. The singularities may appear as time-dependent terms in the Hamiltonian, or the field theory may be defined on a singular spacetime. A question related to the existence of singularities, is whether a dynamical transition across the singularity can be defined.

In the context of cosmological singularities it is natural to investigate (singular) plane wave spacetimes, because they capture an essential characteristic of a singularity, namely the diverging tidal forces, and they can provide a solvable background for a string theoretical analysis. Other examples of spacetimes that provide useful toy models for a singularity are orbifolds. Orbifolds are spacetimes that are obtained by making discrete identifications in a spacetime. If we want to investigate the evolution of fields across spacetime singularities, we find that there is a certain amount of ambiguity related to the question. We have applied a "geometrical resolution" prescription that demands that the resolved spacetime satisfies Einstein's equation, such that it still makes sense physically as a spacetime. We have applied the geometrical resolution prescription to the propagation of free fields across the spacetime singularity.

We have first investigated the case of a free scalar field on a two-parameter generalization of the nullbrane spacetime [95], which itself is a geometrical resolution of the parabolic orbifold. We find that the singular limit of free scalar field evolution exists for a discrete subset of the possible values of the two parameters. The coordinates we introduce in this study reveal a peculiar reflection property of the scalar field propagation on the generalized (as well as the original) nullbrane. We have also investigated the propagation of a free string on a scale-invariant singular plane wave [120]. Scale-invariant plane waves are related to typical cosmological singularities by a so-called Penrose limit. Again we use a geometric resolution prescription and we consider a smooth class of resolved, scale-invariant plane waves. We prove that the existence of the singular limit for the string oscillation modes is determined by the center-of-mass mode. In particular, the demand that a free string can propagate through the singularity yields conditions on the plane wave profile. But for such plane wave profiles the associated dilaton leads to a blowup of the string coupling near the singularity. This indicates that free strings are not a realistic physical approximation near the singularity and it encourages us to look at matrix models. In order to be able to study matrix models for plane wave singularities into more detail, we have investigated the formulation of D$p$-branes in an asymptotically plane wave background. We have derived a family of ten-dimensional supergravity solutions of extremal $p$-branes that are embedded into dilaton-gravity plane waves, with the brane world-volume parallel to the propagation direction of the wave. Our solutions [136] are time-dependent and supersymmetric and the freedom in the wave profile allows for the presence of a singularity in the spacetime.

# Part I

# Foundations

# Chapter 2

# The standard big bang cosmology

*El universo (que otros llaman la Biblioteca) se compone de un número indefinido, y tal vez infinito, de galerías hexagonales.*

*El primero: La Biblioteca existe* ab aeterno.

*El segundo:* El número de símbolos ortográficos es veinticinco.

*"La Biblioteca de Babel," Jorge Luis Borges*

The main theme of this thesis is the investigation of spacetime singularities by means of geometrical resolutions. And although the final aim of our study of spacetime singularities would be to resolve cosmological singularities, this thesis does not directly involve cosmology. Nevertheless, for general knowledge, and perhaps as a warm-up for the later chapters which are more mathematically involved, it may be useful to summarize some basic cosmology.

The standard big bang cosmology states that our universe is expanding, and that it had a beginning at the big bang. The big bang model that gives the best accordance with modern data, such as the recent WMAP measurements of the cosmic microwave background [35], is called the "concordance model". Currently, the concordance model is the so-called $\Lambda$ CDM model, which includes cold dark matter (CDM) and a cosmological constant $\Lambda$. In this model the universe is $13,7$ billion years old and made up of 4% baryonic matter, 23% dark matter and 73% dark energy. The Hubble constant for this model is $71 \cdot 10^3\,\mathrm{m/(s \cdot Mpc)}$ and the universe is very close to spatial flatness. The dark energy is best modeled by a cosmological constant.

Recent observations in cosmology have introduced physical questions that do require an explanation, most notably the cosmological constant and the cold dark matter. Of course, these observations would not necessarily require an explanation in terms of a quantum theory of gravity. For example, a neat explanation for dark matter is given by supersymmetry. But in order to relate these phenomenological models (with all respect for the models, of course) to a smaller set of underlying physical principles, it may be worth trying to derive them from of a consistent quantum theory of gravity, especially because the dynamics of spacetime (in some sense the large scale cosmology) is very well described by the classical theory of gravity, i.e. general relativity. In addition, precision cosmology is very probably the most direct method we have at our disposal to discover experimental signatures of quantum gravity, for example the primordial density fluctuations in the cosmic microwave background. The cosmic microwave background is the strongest evidence that the universe has been in a very dense state initially and it confirms the expansion of the universe. The expansion of the universe is the natural conclusion when the experimental redshift of galaxies discovered by Edwin Hubble, is combined with the cosmological principle.

As an extension of the Copernican principle, the cosmological principle states that (on a large scale) all spatial positions in the universe are equivalent. It means that at a large scale our universe looks similar in the three spatial directions (it is isotropic) from every point (so it is also homogeneous). As expressed by the Hubble law, the universe is also observed to be expanding. Together, the cosmological principle and the expansion of the universe lead to the standard big bang model. Extrapolating the expansion of the universe with general relativity, the initial universe must have been denser and hotter. The big bang model asserts that our universe expanded out of a very hot and dense initial state. The laws of nuclear and particle physics then allow an analysis of the early universe. The predictions are in perfect agreement with the abundancy of elements in the universe. The big bang model also provides an answer to the Olbers paradox (if the universe is infinite, why isn't it infinitely bright) because the light of very distant stars hasn't reached us yet.

In short, the big bang model is well confirmed. General relativity predicts that the universe would have started with a beginning in time at the "big bang" singularity. A resolution of the nature of the big bang singularity itself most likely awaits an answer in a theory of quantum gravity because the energies involved are near the Planck scale. Some theories predict that time did not begin at the big bang singularity but that the universe went through a "bounce", separating our universe from a previous universe [36], or that new universes are created from black holes singularities [37]. For cosmological theories of this kind there is the immediate question whether quantum fields can propagate through the singularity. In that case the initial matter anisotropies in the universe may be related to the structure of the universe before the bounce. Hence it appears important to investigate if it is possible for fields to propagate through a singularity. A first question is then how to describe propagation through a singularity. As a leading candidate for quantum gravity, string theory offers a natural framework to investigate these issues.

At cosmological scales, the evolution of the universe is determined by the Friedmann equations. The Friedmann equations can be derived from the Einstein equations for a Robertson-Walker metric, see formula (3.28) in the next chapter. They express the time-evolution of a scale factor $a(t)$ in function of the matter and energy content of the universe. If we restrict our attention to a universe filled with an isotropic, perfect fluid, the matter content is expressed by an energy density $\rho$ and a pressure $p$. More generally, if there are different matter contributions as in the concordance model, we can write $\rho = \sum_i \rho_i$ and $p = \sum_i p_i$. It is common to express a relation between $\rho_i$ and $p_i$ by means of the equation of state $p_i = w_i \rho_i$. Radiation (e.g. photons) and "hot matter" (moving at relativistic speed with negligible rest mass) have $w = 1/3$, whereas (cold) matter has $w = 0$.

The Robertson-Walker universe possesses a parameter $k$ that expresses the spatial curvature: $k = 0$ implies spatial flatness, $k > 0$ represents a closed universe and $k < 0$ an open universe. We add the cosmological constant $\Lambda$ to the left-hand-side of Einstein's equation. The Friedmann equations are:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2}\,, \tag{2.1}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}\left(\rho + 3p\right) + \frac{\Lambda}{3}\,. \tag{2.2}$$

Here $G_N$ is Newton's constant and we work in units where $c = 1$. The Hubble parameter $H$ is defined in function of the time evolution of the scale factor of the Robertson-Walker universe as

$$H = \frac{\dot{a}}{a}. \tag{2.3}$$

The two Friedmann equations can be combined to yield an equation that expresses energy conservation

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + p\right) = 0. \tag{2.4}$$

As mentioned above, the parameter $k$ is related to the spatial curvature of the universe, but our universe is extremely close to spatial flatness which means that $k = 0$ (spatial flatness is one of the predictions of the inflationary model). It is common to rewrite the first Friedmann equation as

$$1 = \Omega_m + \Omega_\Lambda + \Omega_k\,, \tag{2.5}$$

with

$$\Omega_m = \frac{8\pi G_N}{3H^2}\rho\,, \quad \Omega_\Lambda = \frac{\Lambda}{3H^2}\,, \quad \Omega_k = -\frac{k}{a^2 H^2}\,. \tag{2.6}$$

The concordance model states that

$$\Omega_m + \Omega_\Lambda \approx 1. \tag{2.7}$$

In the concordance model the presence of the cosmological constant $\Lambda$ is confirmed as the most likely explanation for the accelerated expansion. The acceleration can be written as

$$\ddot{a} \propto -(1+3w) \times (1-\Omega_k)\ . \tag{2.8}$$

In fact, the cosmological constant can also be viewed as part of the matter content of the universe, for which $w = -1$, and with a density given by

$$\rho_\Lambda = \frac{\Lambda}{8\pi G_N}. \tag{2.9}$$

I would like to remark that the cosmological constant that is naively obtained from quantum field theory is a factor of 120 orders of magnitude too large (but see also [38] for recent comments on $\Lambda$). There do exist alternative explanations for the accelerated expansion of the universe. For example, if $k > -a^2H^2$ for all $t$, the presence of a scalar field for which $w < -1/3$ (called “quintessence”) suffices.

Although the measured cosmological constant is tiny, it will be the dominant contribution for the future evolution of the universe. In analogy with the term big bang, several names have been suggested for the possible fates of the universe (with $k \approx 0$). A “big crunch” appears when the gravitational attraction of the matter content of the universe is large enough to make the scale factor $a(t)$ grow smaller ($\dot{a} < 0$). Then the universe contracts again and finally collapses. With the current value of the cosmological constant this will be impossible. A more likely scenario is the “big freeze”: the universe keeps expanding, $\dot{a} \geq 0$, but $\dot{H} \leq 0$ and the Hubble parameter decreases in time. A third scenario is the “big rip” (which can only happen if $w < -1$, such a material is called phantom energy) in which case $\dot{H} > 0$. In the case of a big rip there is a future singularity at

$$t_r = \frac{2}{3|1+w|}\frac{1}{H_0}, \quad H_0 = H(t=0)\,. \tag{2.10}$$

If we write $t = t_r + \delta t$, then the scale factor near a big rip singularity (infinite-expansion singularity) behaves as

$$a \propto (\delta t)^{-2/3|1+w|}\,. \tag{2.11}$$

On the other hand, near a big crunch or big bang singularity (infinite-contraction singularity) the scale factor behaves as

$$a \propto (\delta t)^h\,, \quad h > 0\,. \tag{2.12}$$

# Chapter 3

# Gravitation in general relativity

*And you run and you run to catch up with the sun, but it's sinking*
*And racing around to come up behind you again*
*The sun is the same in the relative way, but you're older*
*Shorter of breath and one day closer to death.*

*"Time," Pink Floyd*

In this chapter I will give a very brief introduction to a few aspects of general relativity, to make the rest of the thesis more accessible for general readers. The scope is very limited, therefore I only want to give some necessary background such that the following chapters can be understood. Classic textbooks on the subject are [21, 24], both offering a different point of view. In [21] the geometrical interpretation of the gravitational interaction is stressed. On the other hand, [24] remarks that the interpretation of the gravitational interaction in terms of geometry may be very effective to describe the universe, but that it is not necessarily fundamental and that it may delude the relation between the gravitational force and the standard model forces.

In the beginning of the chapter I will introduce the equivalence principle. I discuss the Riemann tensor in section 3.2 and the Weyl tensor in section 3.3, adding a few notions concerning spacetime classification useful for chapter 9. In section 3.4, I present the Einstein-Hilbert action that describes the dynamics of the gravitational field and I end the chapter with some examples of metrics that exhibit gravitational singularities.

I will give some mathematical preliminaries that allow to describe generally covariant systems in appendix B. I will introduce manifolds, vectors, the metric tensor and forms in appendix B.1. A very important concept is covariant differentiation, summarized in appendix B.2. For later convenience, especially in chapter 9, the concept of Killing vector is briefly described in appendix B.3. In the course of this thesis I mainly use the older formulation of general relativity in terms of a metric. A more modern formalism, illustrated in appendix B.4, makes use of the vielbein. This formulation is necessary in order to be able to describe fermions in a curved spacetime. But most results of this work can just as well be obtained with the older formalism, which is easier to communicate.

## 3.1 The equivalence principle

In the framework of special relativity, Einstein combined Galileo's relativistic principle (speed is relative) with the experimental observation that the speed of light, which appears in Maxwell's equations for electrodynamics, is constant for any observer. However, in special relativity the notion of simultaneity becomes observer-dependent and interactions in spacetime happen at a certain "event", instead of at a certain time and a certain place. The place and the time that correspond to the event depend on the observer. In special relativity only local theories can be causal. Newton's gravitational law for the attraction between two masses becomes acausal because it is expressed in terms of "action at a distance" between the two masses. The equality between inertial mass and gravitational mass in Newton's laws led Einstein to his equivalence principle.

The (strong) equivalence principle states that, in an infinitely small region, it is always possible to find a coordinate system in which gravitation has no influence on the motion of particles or on any other physical process [13]. In other words, every physical law can be brought (at least locally) to the form that it takes in special-relativity by transforming away the gravitational field (by going to a coordinate system in "free fall"). The "transforming away" of the gravitational field is only possible because the gravitational field has the fundamental property that it imparts the same acceleration on all bodies (i.e. the gravitational mass is equal to the inertial mass).

General relativity also incorporates some of Mach's principles, for example that the inertia of a certain body is related to the distribution of the other masses in the universe. This is achieved because the dynamics of the spacetime geometry is related to the distribution of matter in the universe through the equality between the spacetime curvature and the energy-momentum tensor expressed by Einstein's equation. Nevertheless, in four (or higher) dimensions the gravitational field will also have its own degrees of freedom, for example there can be gravitational waves in regions that are void of matter.

To incorporate the strong equivalence principle mathematically, the most elegant way is to formulate all physical laws in a generally covariant manner. General covariance means that equations are written in such a way that they have tensorial transformation properties under the diffeomorphism group. In this way the results obtained from the equations that express the physical laws will be invariant with respect to coordinate changes and they can always brought to the form (locally) that they take in special relativity. In general relativity spacetime will be identified with a four-dimensional Lorentzian manifold (see appendix B.1). A point at the manifold can be identified with a spacetime event where an interaction takes place and the coordinate changes then reflect the different ways in which the manifold can be parametrized. The mathematical incorporation of the principle of equivalence by means of general covariance does not mean that everything that is written in a covariant manner automatically satisfies the principle of equivalence: scalar-tensor gravitational theories like Brans-Dicke theory violate the principle of equivalence. In some sense the principle of equivalence implicitly also assumes that the spacetime is dynamical in the presence of gravitation (or that there is a dynamical gravitational field) because if the spacetime were not dynamical, the laws of physics would be in their special-relativistic formulation everywhere.

The representation of our spacetime as a manifold is valid in general relativity which is a classical theory, with which I mean that spacetime in a quantum theory of gravity is not expected to correspond to a manifold, it may be that the manifold representation only emerges in the classical limit. To some extent (at least in classical general relativity) it also depends on someone's point of view whether one sees the metric tensor that determines the properties of the manifold as more fundamental (and the gravitational force determined by the properties of the manifold) or whether one looks at the spacetime manifold as a computational scheme that is a convenient representation of the gravitational force between two objects. The general relativistic description of gravity can also be reformulated in a manner that is more reminiscent of the electromagnetic and strong and weak force, which are forces that appear because of local symmetries. In this formulation gravity is related to local Lorentz symmetries. Whatever the point of view, general relativity is a very effective way to describe our universe, as has been verified again very recently by investigation of the gravitational lensing effect [39].

To recapitulate, general relativity is a classical theory for the gravitational force and it is based on the (strong) equivalence principle: physical interactions are relative between the interacting objects and do not depend on the motion of the observer. Mathematically the principle of equivalence can be incorporated by writing the physical laws as tensor equations in a generally covariant manner. The principle of equivalence then means that physical observations are invariant with respect to diffeomorphisms on the manifold that corresponds to our spacetime. The gravitational attraction between objects is incorporated in the dynamics of the spacetime geometry, which influences the movement of matter along geodesics on the spacetime. The dynamics of the spacetime geometry is reflected by Einstein's equation that expresses a relation between the spacetime curvature and the presence of matter in the spacetime, which we will derive in section 3.4. The mathematical expression for the curvature of the spacetime also has an immediate physical interpretation. As we will see in the following section, the spacetime curvature gives the deviation of two nearby point particles that move along geodesics in a gravitational field. Or, equivalently, the spacetime curvature expresses the distortion forces on a macroscopic object in a gravitational field.

## 3.2 Spacetime curvature and geodesic deviation

The Riemann curvature tensor expresses to what extent the covariant derivatives of a vector don't commute (at a certain point on the manifold). The commutator of two covariant derivatives $D_\mu$ of a dual vector field $\omega$ only depends on the value of this dual vector field, and thereby defines a tensor of rank (1,3) $R_{\mu\nu\gamma}{}^{\delta}$, called the Riemann tensor, by

$$R_{\mu\nu\gamma}{}^{\delta}\omega_\delta = [D_\mu, D_\nu]\omega_\gamma. \tag{3.1}$$

Using a coordinate expression for the covariant derivatives, the curvature tensor can be evaluated as

$$R_{\mu\alpha\nu}{}^{\beta} = \partial_\alpha \Gamma_{\mu\nu}{}^{\beta} - \partial_\mu \Gamma_{\alpha\nu}{}^{\beta} + \Gamma_{\mu\nu}{}^{\epsilon}\,\Gamma_{\epsilon\alpha}{}^{\beta} + \Gamma_{\alpha\nu}{}^{\epsilon}\,\Gamma_{\epsilon\mu}{}^{\beta}. \tag{3.2}$$

We have used the convention for the curvature tensor as in [21]. Other authors frequently prefer a different order of indices or add a minus sign. For instance this happens if one defines the curvature tensor alternatively by

$$[D_a, D_b]V^c = R_{ab}{}^{c}{}_{d}V^d. \tag{3.3}$$

Curvature is an intrinsic property of a manifold and it does not depend on how we would visualize the spacetime manifold as a higher-dimensional surface that is embedded in a flat space. Therefore it does not correspond to our intuitive notion of curvature of a surface (the latter is described by the "extrinsic curvature" of a surface). For example, the (intrinsic) curvature of any line is zero, no matter how curved it might appear.

In a curved spacetime, freely falling (point) particles follow trajectories that maximize their proper time, which are geodesics. Of course, macroscopic objects will occupy a volume intersected by many geodesics, and in a gravitational field the macroscopic object will be distorted. Let us therefore consider a one-parameter family of geodesics $\gamma_s(\tau)$ where $s$ labels the family of the geodesics and $\tau$ represents an an affine parameter or the eigentime along a geodesic of the family. If we define the vector field $T^\mu = (\partial/\partial\tau)^\mu$ as the tangent vector to the geodesic, the equation for a geodesic becomes

$$T^\nu D_\nu T^\mu = 0. \tag{3.4}$$

In a coordinate representation of the geodesic by $x^\mu(\tau)$, equation (3.4) becomes equal to expression (A.5) of appendix A:

$$\ddot{x}^\mu + \Gamma_{\nu\sigma}{}^\mu \dot{x}^\nu \dot{x}^\sigma = 0\,, \tag{3.5}$$

and the dot denotes the derivative with respect to $\tau$. We define a vector $X^\mu = (\partial/\partial s)^\mu$ that reflects the displacement between two (infinitesimally) nearby geodesics. $X^\mu$ is usually called the deviation vector, and coordinate freedom allows us to set $X^\mu T_\mu = 0$. The relative rate of change of the displacement vector between nearby geodesics is defined by the parallel transport (see appendix B.2) of $X^\mu$ along the geodesic,

$$v^\mu = T^\nu D_\nu X^\mu, \tag{3.6}$$

and is called the deviation velocity $v^\mu$, as it gives the relative velocity of nearby geodesics. Next, the parallel transport of the deviation velocity gives the relative acceleration of (infinitesimally) nearby geodesics, by

$$a^\mu = T^\nu D_\nu v^\mu. \tag{3.7}$$

By rearrangement of the covariant derivatives one can then show that the acceleration of nearby geodesics can be expressed in terms of the Riemann tensor, contracted with the deviation vector and the tangent vector along the geodesic, as

$$a^\sigma = -R_{\mu\nu\pi}{}^\sigma X^\nu T^\mu T^\pi. \tag{3.8}$$

Formula (3.8) now clarifies the meaning of curvature tensor: the Riemann curvature tensor determines the "tidal forces" that govern the relative acceleration of nearby geodesics. Nearby geodesics tend to diverge or converge from each other under the influence of gravity, and they remain parallel ($a^\mu = 0$) if the curvature tensor[1] is zero, i.e. if the gravitational field is zero. For manifolds with dimension $D \geq 3$ the Riemann tensor can be decomposed in terms of the Weyl tensor, Ricci tensor, and curvature scalar as

$$R_{\mu\nu\pi\sigma} = C_{\mu\nu\pi\sigma} + \frac{2}{n-2}\left(g_{\mu[\pi}R_{\sigma]\nu} - g_{\nu[\pi}R_{\sigma]\mu}\right) - \frac{2}{(n-1)(n-2)}R g_{\mu[\pi}g_{\sigma]\nu}. \tag{3.9}$$

The brackets over the indices denote a totally antisymmetrized product with prefactor $1/(n!)$. The Ricci tensor and the curvature scalar $R$ of equation (3.9) are given by

$$R_{\mu\nu} = R_{\mu\alpha\nu}{}^\alpha, \qquad R = g^{\mu\nu}R_{\mu\nu}. \tag{3.10}$$

[1] Geodesics of a given congruence can also remain parallel in a curved spacetime if only the magnetic component of the curvature tensor with respect to the given congruence is nonzero, because the vanishing of the tidal force only gives information about the curvature's electric part with respect to the given congruence.

Let us consider a macroscopic object moving along a geodesic. With respect to the deviation of nearby geodesics expressed in (3.8), the Ricci curvature tensor only contains the information about how the volume of the object changes in the presence of tidal forces. The Weyl tensor expresses how the shape of the body is distorted by the tidal force (without reference to its volume).

## 3.3 The Weyl tensor and its classification

In general relativity, the Weyl curvature is the only part of the curvature that exists in free space if the metric is a solution of Einstein's euqation in vacuum. The Weyl curvature tensor governs the propagation of gravitational radiation through regions of space devoid of matter. In two and three dimensions the Weyl tensor vanishes identically, it appears first in four dimensions. This is in accordance with the fact that gravitons first appear in four dimensions, see appendix D. Three-dimensional gravity is topological. Spacetime becomes first dynamical in four dimensions, which corresponds (roughly speaking) to the existence of gravitational solutions in regions devoid of matter. If the Weyl tensor vanishes, then the metric is locally conformally flat: in that case there exists a local coordinate system in which the metric tensor is proportional to a constant tensor. On the other hand, manifolds which have $R_{\mu\nu} = 0$ are called Ricci-flat, they only possess Weyl curvature.

The Weyl tensor is the traceless component of the Riemann tensor. It is a tensor that has the same symmetries as the Riemann tensor, i.e.

$$C_{\mu\nu\pi\sigma} = C_{[\mu\nu][\pi\sigma]} \,, \quad C_{\mu\nu\pi\sigma} = C_{\pi\sigma\mu\nu} \,, \quad C_{[\mu\nu\pi]}{}^{\sigma} = 0 \,, \tag{3.11}$$

with the extra condition that the Weyl tensor is also trace-free (the metric contraction on any pair of indices of the Weyl tensor yields zero). Because of these symmetries, the Weyl tensor can be viewed as a linear map from bivectors to bivectors, which are tensors for which $A_{\mu\nu} = A_{[\mu\nu]}$ (or "two-forms", see appendix B.1.4). The analysis of the eigenvalue problem

$$C_{\mu\nu}{}^{\alpha\beta} A_{\alpha\beta} = \lambda A_{\mu\nu} \,, \tag{3.12}$$

then leads to the Petrov classification according to the algebraic structure of the Weyl tensor at a spacetime event (this classification is purely algebraic and does not consider the matter content of the spacetime). The most important conclusion of the Petrov analysis is that there exist, in general, four principal null directions $k^\mu$ defined by

$$k^\nu k^\pi k_{[\alpha} C_{\mu]\nu\pi[\sigma} k_{\beta]} = 0 \,. \tag{3.13}$$

It is understood that $k^\mu$ is a null vector. These principal null directions are related to the eigenbivectors $A_{\mu\nu}$ that satisfy (3.12). The proof of (3.13) is most conveniently given in terms of spinor methods (see e.g. [21]). Although the Petrov classification is not of great importance in this thesis, it appears in chapter 9 in the discussion on gravitational plane waves. The Petrov classification states there are six possible types of algebraic symmetry depending on the multiplicities of the principal null directions. The Weyl tensor at a spacetime event can have the types I, II, D, III, N and O. A spacetime of Petrov type I is defined by the general formula (3.13) with four distinct null directions. In case one or more of these principal null directions become aligned ($k^a = C\tilde{k}^a$ with $k^a$ and $\tilde{k}^a$ two solutions to (3.13)), stronger conditions hold on the Weyl tensor. In the case there is one pair of principal null directions that coincides, we have an event of Petrov type II. For the coinciding null direction the condition (3.13) becomes

$$k^\nu k^\pi C_{\mu\nu\pi[\sigma}k_{\alpha]} = 0\,. \tag{3.14}$$

When two pairs of principal null directions coincide, we have a spacetime event of type D and two solutions of the condition (3.14). A spacetime event that has Petrov type III has three coinciding null directions, and the Weyl tensor satisfies

$$k^\pi C_{\mu\nu\pi[\sigma}k_{\beta]} = 0. \tag{3.15}$$

If the spacetime has regions where the Weyl tensor is of Petrov type III, the gravitational field in these regions is related to longitudinal gravitational radiation, which decays like $O(r^{-2})$, with $r$ the characteristic distance to the source of the gravitational radiation. Then, in an event of type N all four principal null directions coincide and

$$k^\pi C_{\mu\nu\pi\sigma} = 0. \tag{3.16}$$

Regions in spacetime where the Weyl tensor is of type N, can be associated to transverse gravitational radiation. In this case the (quadruple) null vector $k^\pi$ corresponds to the wave vector that describes the propagation direction of the radiation. Transverse gravitational radiation decays as $O(r^{-1})$, and therefore the long-range fields away from sources of the energy-momentum tensor are of Petrov type N. Finally, for Petrov type O the Weyl tensor is zero and the spacetime is conformally flat (the Riemann curvature tensor is then also zero in regions devoid of matter). An example of conformally flat spacetimes are the Robertson-Walker cosmological models, which will appear a little further.

## 3.4 Einstein's equation

We now derive Einstein's equation which relates the dynamical evolution of the spacetime metric to the matter content that is distributed on the spacetime. The quickest way to do this is from an action principle. We have to include a kinetic term from the metric $R$, a gravitational "self-energy" term $\Lambda$ (the cosmological constant) and, because the gravitational field is coupled to matter, we also consider a matter action $S_M$. The action is

$$S = S_{EH} + S_M\,, \quad S_{EH} = \frac{1}{16\pi G_N}\int \mathrm{d}^D x\,\sqrt{-g}\,(R - 2\Lambda)\,, \quad S_M = \int \mathrm{d}^D x\,\mathcal{L}_M. \tag{3.17}$$

The scalar curvature $R$ was defined in (3.10) and we will discuss the matter Lagrangian $S_M$ below. We write out the variation of the Einstein-Hilbert action with respect to the inverse metric, and we impose that at spatial infinity the variation of the field becomes zero. Then

$$\delta S_{EH} = \frac{1}{16\pi G_N}\int \mathrm{d}^D x\,\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}\right)\delta g^{\mu\nu}, \tag{3.18}$$

where we have made use of the two expressions

$$\delta\left(\sqrt{-g}\right) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\,, \quad \det\left[g_{ij}\right] = e^{\mathrm{Tr}[\log\, g_{ij}]}. \tag{3.19}$$

The demand that the action be stationary with respect to variations in the metric yields Einstein's equation (or the Einstein equations, if one rather wants to consider the equations for each set of indices separately):

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}. \tag{3.20}$$

In the formula above the energy-momentum tensor $T_{\mu\nu}$ is defined in terms of the matter action by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{3.21}$$

In the limit of a weak gravitational field, Einstein's equation reduces to Newton's equation for the gravitational interaction, written in the form of a Poisson equation. The energy-momentum tensor is conserved. It is the Noether charge associated to the diffeomorphism invariance.

Any metric can be a solution to Einstein's equation if the energy-momentum is determined according to (3.20) but the related matter action (3.21) would not necessarily represent a physically realistic situation. In order to determine the precise influence of known matter on the dynamics of the metric we have to specify the matter Lagrangian $\mathcal{L}_M$. It turns out that the matter Lagrangian has precisely the same form as in Minkowski space (perhaps up to a total derivative or a multiplicative constant) but made "generally covariant", that is, with replacement of the ordinary partial derivatives to covariant derivatives. For example, the matter Lagrangian of a (classical) massive free Klein-Gordon scalar field becomes

$$\mathcal{L}_{KG} = -\frac{\sqrt{-g}}{2}\left(g^{\mu\nu}D_\mu\phi D_\nu\phi + m^2\phi^2\right). \tag{3.22}$$

In general it is possible to add other terms such as $R\phi^2$ but we have chosen a "minimally coupled" matter Lagrangian. The matter Lagrangian for the (classical) electromagnetic field is

$$\mathcal{L}_{EM} = -\frac{\sqrt{-g}}{4\pi} g^{\mu\pi} g^{\nu\sigma} D_{[\mu} A_{\nu]} D_{[\pi} A_{\sigma]} . \tag{3.23}$$

The factor $\sqrt{-g}$ ensures the covariant transformation of the matter Lagrangian under diffeomorphisms.

If the energy-momentum tensor and the cosmological constant are zero, one of the vacuum solutions of Einstein's equation is simply the flat-space Minkowski metric

$$ds^2 = -dt^2 + (dx^i)^2 . \tag{3.24}$$

Another representation of this metric is given by advanced $(v)$ and retarded $(u)$ null coordinates. More specifically, rewriting $v = (t+r)/\sqrt{2}$ and $u = (t-r)/\sqrt{2}$ with $r$ a radial coordinate $(0 < r < \infty)$, the Minkowski metric becomes

$$\begin{aligned} ds^2 &= -2dudv + r^2 \left( d\theta^2 + sin^2\theta d\phi^2 \right) ; \\ u, v &= -\infty \dots \infty,\ \theta = 0 \dots \pi,\ \phi = 0 \dots 2\pi . \end{aligned} \tag{3.25}$$

These coordinates can be thought of as incoming $(v)$ and outgoing $(u)$ spherical waves travelling at light speed [8]. The coordinates $u$ and $v$ are often called lightcone coordinates and they will be used throughout this thesis, though often in the form where they correspond to plane waves propagating in a particular direction $x$ (instead of spherical waves propagating radially). One of the lightlike coordinates $u$ (or $v$) can be chosen to formally replace time [41].

## 3.5 Singular spacetimes

In appendix A, I illustrate the concept of a spacetime singularity in terms of the incompleteness of causal geodesics. A more rigorous approach is to consider singular boundary points of a spacetime. An introduction to such a boundary construction is given in appendix C. This section will simply deal with some examples of singular spacetimes: among which a big bang singularity, black hole singularities, extremal $p$-brane solutions and Szekeres-Iyer singularities. To determine whether a singularity is of spacelike, lightlike or timelike nature, one can draw a Penrose diagram[2] of (an embedding of) the singular spacetime and analyze its boundary structure.

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[2]The Penrose-Carter diagram, see e.g. [8], is a way to visualize in two dimensions the structure of infinity of a spherically symmetric spacetime. It is obtained by performing a conformal transformation on the spacetime metric

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} , \tag{3.26}$$

The metric of a spherically-symmetric Robertson-Walker spacetime[3] is determined by the scale factor $a(t)$ and spatial curvature $k$:

$$ds^2 = -dt^2 + a^2(t)\left(\frac{1}{1-kr^2}dr^2 + r^2 d\Omega_2^2\right) . \tag{3.28}$$

Here $d\Omega_2^2$ represents the metric on the (two-)sphere

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\phi^2. \tag{3.29}$$

The metric (3.28) is singular in case the scale-factor vanishes at a certain time $t^*$. If $a(t^*)$ vanishes, it is a big bang or a big crunch singularity.

Another example of a gravitational singularity is that of the static Schwarzschild black hole. The Schwarzschild spacetime

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \tag{3.30}$$

represents the gravitational field of a point mass $M$. The Schwarzschild solution can be generalized to include electromagnetic fields, in which case one obtains the Reissner-Nordström spacetime

$$ds^2 = -\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2 + \left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \tag{3.31}$$

where $Q$ is the charge of the point charge located at the origin $r = 0$. The Schwarzschild and Reissner-Nordström spacetimes possess horizons. Although the Schwarzschild spacetime appears to be time-independent, it does not possess a globally defined timelike Killing vector. Inside the horizon, the Killing vector $\partial_t$ becomes spacelike, the Schwarzschild solution is only static for an observer outside of the horizon. This is also the reason why the singularity at $r = 0$ is called spacelike. The singularity of the Reissner-Nordström spacetime is timelike. In the limiting case of $M^2 = Q^2$, the Reissner-Nordström black hole is called *extremal* and it does possess a globally defined timelike Killing vector.

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with the conformal factor $\Omega$ appropriately chosen such that the (infinite) spacetime is represented by a diagram of finite size. The diagram captures the causal relations between different points in spacetime. Radial null geodesics in the spacetime will correspond to straight lines at angles of $\pm\pi/4$ on the diagram.

[3]The energy-momentum tensor associated to the Robertson-Walker spacetime can be taken as

$$T_{\mu\nu} = \rho u_\mu u_\nu + p\left(g_{\mu\nu} + u_\mu u_\nu\right) . \tag{3.27}$$

It represents a perfect fluid with density $\rho$, pressure $p$ and four-velocity $u^\mu$. The Einstein equations for the Robertson-Walker solution (3.27-3.28) lead to the Friedmann equations (2.1-2.2).

There are generalizations of a pointlike black hole, which are called black branes. These are $(p+1)$-dimensional extended objects, embedded in a $D$-dimensional spacetime. The spatial dimensions are then split into two groups: $p$ worldvolume coordinates $y^\alpha$ (with $\alpha = 1 \dots p$) and $D-p-1$ transverse coordinates $x^a$ (with $a = 1 \dots D-p-1$). Along the worldvolume the solution looks like flat space, with respect to the transverse dimensions the solution has the appearance of a black hole. For example we can write the metric of a (static) extremal $p$-brane in ten dimensions as

$$\begin{aligned} ds^2 &= H(r)^{(p-7)/8}\left(-dt^2 + (dy^\alpha)^2\right) + H(r)^{(p+1)/8}\left(dr^2 + r^2 d\Omega^2_{D-p-2}\right) , \\ H(r) &= 1 + \frac{R^{7-p}}{r^{7-p}} . \end{aligned} \tag{3.32}$$

The radial coordinate is isotropic with respect to the transverse coordinates ($r = \sqrt{x^a x^a}$) and $d\Omega^2_{D-p-2}$ represents the metric on the $(D-p-2)$-sphere. I want to remark that the metric (3.32) is not a vacuum solution, the energy-momentum tensor related to (3.32) receives contributions from a dilaton (see e.g. chapter 4) and from a field strength, related to the charge of the brane (see chapter 8). The Reissner-Nordström and Robertson-Walker spacetimes aren't vacuum solutions either. Examples of vacuum solutions, i.e. spacetimes with a zero energy-momentum tensor, are the Schwarzschild spacetime and, in the presence of a cosmological constant $\Lambda = R/4$, deSitter spacetime ($\Lambda > 0$) and Anti-deSitter spacetime ($\Lambda < 0$).

The Szekeres-Iyer [40] metric

$$ds^2 = -2e^U du dv + e^V d\Omega_2^2 , \tag{3.33}$$

describes *spherically symmetric* power-law singularities. This includes the Tolman Bondi dust collapses that have the energy-momentum tensor $T_{\mu\nu} = \rho u_\mu u_\nu$ (with $u^\mu = (1,0,0,0)$ in the coordinate system $\{t, r, \theta, \phi\}$). In (3.33) $U$ and $V$ are functions of the spherical lightcone coordinates $u$ and $v$ with $u = (t-r)/\sqrt{2}$ and $v = (t+r)/\sqrt{2}$ and $r$ a radial coordinate. In the vicinity of the singularity the metric (3.33) is approximated by

$$ds^2 = -2(ku + lv)^p du dv + (ku + lv)^q d\Omega_2^2 . \tag{3.34}$$

If $k \cdot l = 0$, the singularity at $ku = -lv$ is lightlike. If $k \cdot l = 1$, the singularity is spacelike. The exponents $p$ and $q$ are called the Kasner exponents and these characterize the behaviour near the singularity. For a discussion of this kind of power-law singularities in the context of string theory, see e.g. [42].

# Chapter 4

# From particles to strings

*"Muss es sein? Es muss sein!"*

*String Quartet No. 16 in F Major, Op. 135, Ludwig van Beethoven*

In this chapter I will introduce some basic elements of (bosonic) string theory, while focusing on the quantum description of gravity. The aim is to show where gravitons appear in the spectrum of string theory. Meanwhile I will also describe the lightcone quantisation of a string. This is certainly very standard material, but it does appear as the starting point of one of my projects (see chapter 13) where we will consider a string propagating in a curved and singular spacetime. There are two other necessary (though basic) elements of string theory that I need to introduce to support the non-experts to follow my work, which are on the one hand the background consistency conditions of string theory that lead to the supergravity equations of motion, and on the other hand D-branes. But I will consider those topics in chapter 7 and chapter 8.

## 4.1 Perturbative approach to quantum gravity

Let me first motivate why to consider strings instead of particles. The general relativistic description of gravity, when dealt with in a perturbative manner, possesses small field excitations which are called gravitons (see appendix D). From the quantum field theoretical point of view these gravitons are massless spin-two particles and, when coupled to matter, they are subject to a gauge principle. However, the gauge force they transmit turns out to be non-renormalisable [43], which means, roughly stated, that the quantum field theory related the gravitational force (gravitons exchanged in Minkowski spacetime) has to be regarded as an effective theory, which is only valid at energies much lower than the Planck scale. For a more expanded argument, see e.g. [44].

Therefore it is very likely that in order to formulate a gravitational theory that remains valid at the energies near the Planck scale, new physical degrees of freedom have to be taken into account. For example, because of the effective nature of the gravitational interaction in general relativity (of the form $\sqrt{g}R$), additional high-energetical degrees of freedom can be present in the underlying theory, as is the case in string theory. The high-energetic degrees of freedom that are necessary for a consistent theoretical description at the Planck scale can be integrated out at lower energies to yield a low-energy effective action that agrees with general relativity (see e.g. chapter 7) which is our best description of gravity and spacetime at large distance scales. A closed string contains graviton-like oscillation modes and thus it is likely that these new degrees of freedom at the Planck scale are stringlike (or related to strings by dualities like D-brane degrees of freedom), especially because string theory yields finite scattering amplitudes (hence, no conceptual problem with divergences in quantum loops) and its theoretical description allows to deal with the gravitational force in a consistent perturbative manner. A recent review of the status of string theory as a quantum theory of gravity is given by e.g. [45].

One approach to cure the divergences related to the perturbation theory of the quantized Einstein-Hilbert action is to modify the gravitational interaction such that the interaction becomes "smeared out" over a small region instead of concentrated into one point. Intuitively, one already feels that this idea requires the introduction of extended objects instead of point particles, and one arrives at bosonic strings.

In principle, closed bosonic strings[1] would suffice to describe the behaviour of the gravitational force at Planck-scale energies. Open strings allow to include gauge groups reminiscent of the other forces of nature that appear in the standard model of elementary particle physics and thus it is believed that interactions between strings allow to describe all the forces of nature. In that case, it is natural to expect that also the matter content of our universe is described by excitations of strings. But all matter in the universe discovered so far is fermionic, and fermionic excitations are not described by bosonic string theory. This has lead to the development of superstrings, which allow to describe fermionic oscillations of strings, under the assumption of a strong symmetry relation between bosons and fermions that is called supersymmetry.

## 4.2 The Nambu-Goto string

In appendix A, I have introduced the action of a point particle in terms of its worldline, which yields the equation of a geodesic at the classical level. In fact,

[1]There is a tachyon (a particle with negative mass) in the spectrum of bosonic strings, but it is likely that the tachyon signals an unstability of the vacuum with respect to which bosonic string theory is perturbatively defined.

it is possible to build a relativistic quantum theory based on the worldline action of a particle. This is called a first-quantized approach with respect to the (usual) second-quantized formalism of quantum field theory. Non-relativistic quantum mechanics treats time and space differently, in the Heisenberg picture space is an operator $X(t)$ that depends on time. In the worldline formalism time and space (the position of the particle) are promoted to operators $T(\tau)$, $X(\tau)$, which depend on the eigentime along the worldline. Interactions are given at the splittings of different worldlines. Instead, in the second-quantized formulation of quantum field theory one introduces operator fields (e.g. $\phi(t,x)$) which are a function of both space and time. The operator fields are then expanded in creation and annihilation operators and the interactions are determined by an interaction Hamiltonian. Though the first-quantized worldline theory for particles is a more cumbersome formalism than the second-quantized theory, it can be generalized from a particle worldline to a string worldsheet, with the position of the field in spacetime $X^\mu(\sigma,\tau)$ now depending on two worldsheet variables $\tau$ (temporal) and $\sigma$ (spatial). The action for a point particle is given by the length of the worldline. The action for a string is given by the area of its worldsheet, which leads to the Nambu-Goto action

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int_{\mathcal{M}} \left[-\det\left(g_{\mu\nu}\partial_a X^\mu \partial_b X^\mu\right)\right]^{1/2} d\sigma d\tau. \tag{4.1}$$

Here $\mathcal{M}$ denotes the worldsheet. More abstractly, we can look upon $X^\mu(\tau,\sigma)$ as a map from the worldsheet to a "target space". The curved "Greek" indices are lowered and raised by the target space metric $g_{\mu\nu}$, which we currently restrict to flat Minkowski spacetime $\eta_{\mu\nu}$. This will be extended in chapter 7. The determinant is on the worldsheet indices $(a,b)$. The action is invariant under Poincaré transformations in the target space (because the target space is Minkowski spacetime), and it is also invariant under diffeomorphisms on the worldsheet (any theory formulated on a surface should not depend on the parametrization of the surface).

## 4.3 The Polyakov string

The Nambu-Goto action is nonlinear in the spacetime coordinates but it can be simplified by introducing an auxiliary worldsheet metric $\gamma_{ab}$, to yield the Brink-Di Vecchia-Howe-Deser-Zumino action [46]

$$S_P = -\frac{1}{4\pi\alpha'}\int_{\mathcal{M}} \sqrt{-\gamma}\left(\gamma^{ab}\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu\right) d\sigma d\tau\,, \tag{4.2}$$

which is often called the Polyakov or string action. The string action is classically equivalent to the Nambu-Goto action (4.1), but because it is bilinear in the spacetime coordinates it can be quantized by conventional methods (with a Fock space representation). Because of historical reasons, it is also called a two-dimensional (because of the two worldsheet dimensions) sigma model of $D$ scalar particles. The $X^\mu$ are scalars with respect to the worldsheet, but they are vectors in the target space. In the action (4.2) for a free string there is only one free parameter, $\alpha'$, which is related to the string length $\ell_s$ by $\ell_s = \sqrt{\alpha'}$, and it is also related to the tension of the string

$$T_s = (2\pi\alpha')^{-1}\,, \tag{4.3}$$

which is the prefactor in front of the string action (4.2). In fact, $\alpha'$ sets the units of the dimension of area, and it is naturally related to the Planck length in $D$ dimensions:

$$\sqrt{\alpha'} \approx g_s^{-2/(D-2)} \ell_P^{(D)}\,. \tag{4.4}$$

The reason for this is that string theory naturally incorporates the gravitational interaction, and the quantum effects of the gravitational interaction should appear at the Planck scale, determined by the Planck length $\ell_P$. As we will see, there are essentially two arguments that relate string theory to gravity: the first is the appearance of a massless spin two particle (the "graviton") as one of the oscillation states of the string, the second reason is that strings can only propagate on a background spacetime that satisfies Einstein's equation.

The string action has an additional symmetry compared to the Nambu-Goto action. Varying the action with respect to $\gamma$ leads to the constraint

$$\gamma_{ab} = \exp\left(2\omega(\sigma,\tau)\right) \partial_a X^\mu \partial_b X_\mu, \tag{4.5}$$

where an arbitrary conformal factor has been inserted (it drops out if one plugs the solution for $\gamma_{ab}$ into the string action to obtain the Nambu-Goto action). The string action is invariant under conformal (or Weyl) transformations,

$$\gamma'_{ab}(\sigma,\tau) = \exp\left(2\omega(\sigma\tau)\right) \gamma_{ab}(\sigma,\tau)\,, \quad X'^\mu(\sigma,\tau) = X^\mu(\sigma,\tau)\,, \tag{4.6}$$

while keeping the coordinates $X^\mu$ fixed. Conformal transformations are local transformations because of their dependence on the worldsheet coordinates. It is not possible to simply categorize the conformal (or Weyl) transformations as a subgroup of the two-dimensional diffeomorphism group, because the spacetime coordinates do change under the worldsheet diffeomorphisms $(\sigma'(\sigma,\tau), \tau'(\sigma,\tau))$,

$$\gamma'_{ab}(\sigma',\tau') = \frac{\sigma^c}{\sigma'^a}\frac{\sigma^d}{\sigma'^b}\gamma_{cd}(\sigma,\tau)\,, \quad X'^\mu(\sigma',\tau') = X^\mu(\sigma,\tau)\,, \tag{4.7}$$

where $(a,b)$ run over $(\sigma,\tau)$. The Weyl invariance of the string theory sigma model is very important, because it allows us to remove the third component of the worldsheet metric tensor. On a two-dimensional worldsheet we would expect only two gauge conditions (we can freely redefine the coordinates $\tau$ and $\sigma$) but because there is a (classical) invariance with respect to Weyl transformations we can impose a third gauge condition. This is fortunate, because otherwise we might have introduced an unphysical degree of freedom. Strictly speaking, there is no kinetic term for the worldsheet metric in the string action (4.2) because it does not contain derivatives of the worldsheet metric, for example the curvature scalar of the worldsheet $R^{(\gamma)}$. So in some sense we could still consider the worldsheet metric to be non-dynamical. Nevertheless, heuristically speaking, interactions between strings are related to the two-dimensional topology (the "handles" on the worldsheet) which is related to the curvature scalar of the worldsheet through the Gauss-Bonnet theorem (see also equation (4.28) at the end of this chapter).

The worldsheet energy-momentum tensor is determined by

$$T^{ab}(\sigma,\tau) = -\frac{4\pi}{\sqrt{-\gamma}}\frac{\delta S_P}{\delta\gamma_{ab}}\,, \tag{4.8}$$

and the invariance under Weyl transformations leads to a traceless tensor $T^a{}_a = 0$, at least at the classical level. Precisely because we will use the Weyl invariance to remove an unphysical degree of freedom (namely, one of the components of the auxiliary worldsheet metric), it will be important to demand that the Weyl invariance of the string sigma model still holds at the quantum level.

## 4.4 Graviton-like oscillations of the closed string

In the context of this thesis, string theory is used as a description of quantum gravity, therefore I will focus on closed strings. In this section, largely based on [14], we will derive how one of the excitations of a closed bosonic string can be identified with the graviton. The classical equations of motion related to the action (4.2) are,

$$T_{ab} = 0\,, \tag{4.9}$$

$$\partial_a\left(\sqrt{-\gamma}\gamma^{ab}\partial_b X^\mu\right) = 0\,. \tag{4.10}$$

For a closed string we will consider the coordinate region $-\infty \leq \tau \leq \infty$ and $0 \leq \sigma \leq 2\pi$. The fields $X^\mu(\sigma,\tau)$ and $\gamma_{ab}(\sigma,\tau)$ are periodic in $\sigma$ with period $2\pi$. To quantize, let us choose lightcone gauge which allows us to impose three conditions. For the proof that this gauge is possible, see [14]. In lightcone gauge we align the worldsheet time with the lightcone time of Minkowski spacetime and restrict two of the metric components as

$$X^+ = \alpha' p^+ \tau \,, \tag{4.11}$$

$$\partial_\sigma \gamma_{\sigma\sigma} = 0 \,, \tag{4.12}$$

$$\det \gamma_{ab} = -1 \,. \tag{4.13}$$

We now solve for $g_{\tau\tau}$ and obtain the Lagrangian which is still function of $g_{\sigma\sigma}(\tau)$ and $g_{\sigma\tau}(\sigma, \tau)$,

$$L = -\frac{1}{4\pi\alpha'} \int_0^{2\pi} \mathrm{d}\sigma \Bigg( 2\gamma_{\sigma\sigma} p^+ \alpha' \partial_\tau X^- - 2\gamma_{\tau\sigma} \left( \alpha' p^+ \partial_\sigma X^- - \partial_\tau X^i \partial_\sigma X^i \right) - \gamma_{\sigma\sigma} \sum_{i=1}^{8} (\partial_\tau X^i)^2 + \gamma_{\sigma\sigma}^{-1} (1 - \gamma_{\tau\sigma}^2) \sum_{i=1}^{8} (\partial_\sigma X^i)^2 \Bigg) . \tag{4.14}$$

We can split the oscillator $X^-$ in a part that depends on $\sigma$ and a part independent of $\sigma$ by

$$X^-(\sigma, \tau) = x^-(\tau) + \hat{X}^-(\sigma, \tau) \,, \quad x^-(\tau) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\sigma X^-(\tau, \sigma) \,. \tag{4.15}$$

The $\sigma$-dependent part of the oscillator $X^-$ is non-dynamical: $\hat{X}^-$ enforces $\gamma_{\tau\sigma} = 0$. The $\sigma$-independent part of the oscillator $X^-$ can be eliminated as a constraint that sets $\gamma_{\sigma\sigma} = 1$ (for more information about constraints see appendix F). We can write the following worldsheet Hamiltonian:

$$H = \frac{1}{4\pi\alpha'} \int \mathrm{d}\sigma \sum_{i=1}^{d} \left( \pi^2 (P_i)^2 + \left( \partial_\sigma X^i \right)^2 \right) . \tag{4.16}$$

Here $P_i$ are the momenta conjugate to $X^i$,

$$P_i = \frac{\delta L}{\delta(\partial_\tau X^i)} \,. \tag{4.17}$$

The Hamiltonian (4.16) implies a wave equation for the coordinates transverse to the propagation direction of the string,

$$\partial_\tau^2 X^i = (2\pi\alpha')^{-2} \partial_\sigma^2 X^i \,. \tag{4.18}$$

A solution for the string coordinates $X^i$ is given by

$$X^i = i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{m=\infty \\ m\neq 0}}^{+\infty} \frac{1}{m} \left\{ \alpha_m^i \exp\left( -im(\sigma + \frac{1}{2\pi\alpha'}\tau) \right) + \tilde{\alpha}_m^i \exp\left( im(\sigma - \frac{1}{2\pi\alpha'}\tau) \right) \right\} . \tag{4.19}$$

The $\alpha^i_m$ and $\tilde{\alpha}^i_m$ are oscillators for left-moving and right-moving waves along the string, with mode number $m \neq 0$. In principle there are additional terms in the solution for $X^i$ (4.19) due to the string's center-of-mass position $x^i$ and center-of-mass momentum $p^i$. The center-of-mass momentum $p^i$ is given in terms of the zero modes $\alpha^i_0$ and $\tilde{\alpha}^i_0$. For the closed string periodicity in $\sigma$ forces the zero mode oscillators to satisfy $\alpha^i_0 = \tilde{\alpha}^i_0$, at least if the $i$'th dimension is non-compact. We will come back to this in chapter 8. But these center-of-mass terms are not important for the discussion below.

The degrees of freedom are the oscillators of the string modes (and in principle also the center-of-mass variables, but we will concentrate on the oscillators) which are promoted to operators in the quantum theory for which we can write the commutation algebra as

$$[\alpha^i_m, \alpha^j_n] = m\delta^{ij}\delta_{m,-n}\,, \quad [\tilde{\alpha}^i_m, \tilde{\alpha}^i_n] = m\delta^{ij}\delta_{m,-n}\,. \tag{4.20}$$

We can now build a general state on top of a state $|0;k\rangle$ characterized by its center-of-mass momentum $k^\mu$ and annihilated by all the oscillators $\alpha^i_m$ and $\tilde{\alpha}^i_m$ with $m > 0$. This general state is written as

$$|N', \tilde{N}'; k\rangle = \left[\prod_{i=2}^{D-1}\prod_{n=1}^{\infty} \frac{(\alpha^i_{-n})^{N^i_n}(\tilde{\alpha}^i_{-n})^{\tilde{N}^i_n}}{\sqrt{n}^{N^i_n \tilde{N}^i_n}\left(N^i_n!\,\tilde{N}^i_n!\right)^{1/2}}\right]|0;k\rangle\,, \tag{4.21}$$

where the primed occupation numbers $N'$ and $\tilde{N}'$ are shorthand for the whole set of $N^i_n$ and $\tilde{N}^i_n$ that specify the occupation numbers at each mode level and for each transverse dimension (hence the products over the mode levels and the dimensions). The factors of $\sqrt{n}$ appear in the denominator because of the specific normalization of the commutation relations of the oscillators in (4.21).

For the closed string the total number of left moving oscillations must be equal to the number of right moving oscillations, due to invariance under translations of the worldsheet coordinate $\sigma$. This means that the expectation value of the total number operators on a physical state must satisfy $N = \tilde{N}$, with $N$ and $\tilde{N}$ the total left-moving and right-moving number operators respectively. These are defined by the sum over all oscillation modes over all transverse dimensions:

$$N = \sum_{n=1}^{\infty}\sum_{i=2}^{D-1} N^i_n = \sum_{n=1}^{\infty}\sum_{i=2}^{D-1} \alpha^i_{-n}\alpha^i_{-n}\,, \tag{4.22}$$

$$\tilde{N} = \sum_{n=1}^{\infty}\sum_{i=2}^{D-1} \tilde{N}^i_n = \sum_{n=1}^{\infty}\sum_{i=2}^{D-1} \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_{-n}\,. \tag{4.23}$$

Perhaps it is appropriate to comment that the worldsheet approach yields a first-quantized theory from the viewpoint of the target-space (i.e. from the point of view of the background spacetime), because a state in the Hilbert space spanned by the states (4.21) will correspond to a single string in spacetime.

The mass of a string which is in a state with $N$ oscillations (with respect to the state $|0;k\rangle$) is given by the occupation number of the oscillations modes of the string

$$m^2 = \frac{4}{\alpha'}\left(N + \frac{D-2}{2}\sum_{n=1}^{\infty} n\right), \tag{4.24}$$

where the second contribution is given by the sum of the zero-point energies, which has to be renormalized carefully, taking care of Lorentz invariance. The infinite sum over $n$ can be written in terms of the Riemann zeta function and in this way the factor $\zeta(-1) = -1/12$ appears.

At the first excited level of the string we find a state

$$\alpha^i_{-1}\alpha^j_{-1}|0,0;k\rangle\,, \quad m^2 = \frac{26-D}{6\alpha'}\,, \tag{4.25}$$

which transforms as a tensor under $SO(D-2)$ rotations. Out of consistency this state must be massless because a massive state would fill out a representation of $SO(D-1)$ (for the representation theory of a massless particle one cannot single out a preferred rest frame). This condition fixes the dimension of the target space to be $D = 26$, at least for the critical bosonic string. In the case of non-critical strings additional terms are added to the string action, which may change the dimension in which the string propagates (but these additional terms induce a curvature of the target space of the order of $1/\alpha'$). So it is the demand for Lorentz invariance[2] that will fix the dimension of the spacetime in which a quantized string can propagate consistently.

The state (4.25) transforms as a full two-tensor under the rotation group of the transverse dimensions $SO(D-2)$. We can decompose this two-tensor into a traceless symmetric tensor (with the number of degrees of freedom of a graviton), a scalar (the trace) and an antisymmetric tensor, because these components do not mix under the rotations in $SO(D-2)$,

$$e^{ij} = \frac{1}{2}\left(e^{ji} + e^{ij} - \frac{2}{D-2}\delta^{ij}e^{kk}\right) + \frac{1}{2}\left(e^{ij} - e^{ji}\right) + \frac{1}{D-2}\delta^{ij}e^{kk}. \tag{4.26}$$

The traceless symmetric tensor is now identified with the graviton (see also appendix D), the massless field corresponding to the antisymmetric tensor is called the Kalb-Ramond field and the masless scalar field corresponding to the trace of the two-tensor is called the dilaton. In principle, all these massless fields

[2] If the determination of the dimension in the lightcone quantisation procedure outlined above does not fully convince the reader, I refer to the literature for a more rigorous treatment, for example the "old covariant" quantisation approach or the more modern Becchi-Rouet-Stora-Tyutin quantisation.

can be added[3] to the worldsheet sigma model, which will be shown in chapter 7. Even more remarkably, we will see that Einstein's equation (more precisely its generalization in supergravity) follows from the demand of conformal invariance of the nonlinear string sigma model in which the graviton, Kalb-Ramond and dilaton fields are included.

So far I have (briefly) described the spectrum of a free bosonic string. Interactions between strings are described in terms of the splitting and joining of string worldsheets, the strength of which is captured by the parameter $g_s$, the string coupling. The interaction between strings can be written in terms of a perturbation expansion with respect to the topology (roughly, the number of "handles" that appear when strings join or split) of the worldsheet, which is called the genus expansion. One then finds that at each order of the worldsheet topology the total amplitude remains finite, which is essentially due to the strong symmetries of the string theory worldsheet sigma model (modular invariance), which restrict the possibility that divergences appear. So we find that strings define finite quantum interactions (including the gravitational interaction). Of course, it can very well be that the total sum of all the finite perturbative terms is still infinite, but this issue is rather related to the perturbative approach and is just as much a problem for the perturbative expansion of ordinary quantum field theories as it is a problem of string theory. When the coupling between strings becomes large one has to use a dual description in terms of other effective degrees of freedom, for example a description in terms of D-branes. The latter are introduced in chapter 8.

As mentioned above, the scalar particle that corresponds to the trace of the $SO(D-2)$ tensor in the state (4.25) is called the dilaton $\phi$. It turns out that the strength of the string coupling is directly related to the expectation value of the dilaton by $g_s = e^{\phi}$. If one writes the string action for the free string, it is natural to include an additional term given by the scalar dilaton coupled to the worldsheet curvature scalar $R^{(\gamma)}$, because such a term respects all the symmetries of the string action. With $S_P$ defined by (4.2) the total string action becomes

$$S = S_P + \frac{1}{4\pi}\int_{\mathcal{M}} \sqrt{-\gamma}R^{(\gamma)}\phi \, \mathrm{d}\sigma\mathrm{d}\tau \,. \tag{4.27}$$

The dilaton is defined up to a constant $\phi_0$. The constant $\phi_0$ gives the Euler number contribution to (4.27). In the case of closed strings there are no boundaries of the worldsheet and the Euler number, which characterizes the topology of the worldsheet, is written as

$$\chi = \frac{1}{4\pi}\int_{\mathcal{M}} \sqrt{-\gamma}R^{(\gamma)}\mathrm{d}\sigma\mathrm{d}\tau \,. \tag{4.28}$$

[3]To be complete, the Kalb-Ramond field is absent if the string is unoriented, but we will later focus on type IIA and type IIB superstring theories and these are oriented.

In terms of the worldsheet topology we can write the Euler number as

$$\chi = 2 - 2h\,, \tag{4.29}$$

where $h$ is the number of "handles" which determines the "genus" of the worldsheet. Therefore the term $\phi_0\chi$ in (4.27) contributes a factor $e^{2\phi_0}$ for each handle that is added to the worldsheet. Such a handle corresponds e.g. to the absorption and re-emission of a closed string (or it appears during a virtual process when a closed string is emitted and re-absorbed). Therefore for each interaction between closed strings (governed by $g_s$) a factor $e^{\phi}$ appears.

In chapter 7 we will generalize the string action in Minkowski spacetime (4.27) to curved space.

# Chapter 5

# Supersymmetry and superstrings

*Spieglein, Spieglein an der Wand,*
*Wer ist die Schönste im ganzen Land?*

*"Kinder- und Hausmärchen," Jakob und Wilhelm Grimm.*

In the case of the bosonic string, all states in the spectrum are bosonic. To be able to describe states with a fermionic character, superstrings are introduced, for which the critical dimension is $D_c = 10$. In my thesis I have (usually somewhat implicitly) worked in the context of type IIA and type IIB superstring theory, because these contain closed strings which can describe gravitons. Superstring theories are string theories that respect an additional symmetry principle, called supersymmetry. An important argument for supersymmetry from experimental cosmology is that supersymmetry gives an elegant explanation for dark matter. The lightest supersymmetric particle is stable (it cannot decay into any lighter particle because of the conserved supersymmetry charge), which makes it a natural candidate for cold dark matter if it interacts very weakly with the standard model particles.

Supersymmetry is one of the key concepts in superstring theory, and deserves to be introduced, even though it is not essential to understand the results of chapters 12 and 13. It appears explicitly only for a very brief moment in chapter 14. Therefore, the sole aim of the present chapter is to illustrate (and hopefully clarify a little) the concept of supersymmetry in this specific context. For more information about superstrings I refer the reader to [5] and the second volume of [14]. For more information about supersymmetry I refer to [25] and the third volume of [23].

## 5.1 Supersymmetry

Supersymmetry is a symmetry that that extends the Poincaré algebra of translation, rotation and boost operators, by relating bosonic and fermionic fields through a symmetry which is generated by anticommuting generators $Q_\alpha$. Here $\alpha$ is a Weyl spinor index, the fermionic generator $Q_\alpha$ has definite chirality. Its Hermitian conjugate of the opposite chirality is denoted as $Q^\dagger_{\dot\alpha}$. I'm essentially using the conventions of [18] because these don't conflict with my convention for the signature of the metric. In the present section, we assume that the spacetime metric $g_{\mu\nu}$ is equal to the Minkowski metric $\eta_{\mu\nu}$.

In the simplest case where there is only one set of supersymmetry generators $Q_\alpha$ and $Q^\dagger_{\dot\alpha}$, which is called $\mathcal{N} = 1$ supersymmetry, the anticommutation relations between the supersymmetry generators become,

$$\{Q_\alpha, Q^\dagger_{\dot\beta}\} = -2\sigma^\mu{}_{\alpha\dot\alpha} P_\mu \,, \tag{5.1a}$$

$$\{Q_\alpha, Q_\beta\} = 0\,. \tag{5.1b}$$

$Q_\alpha$ and $Q^\dagger_{\dot\alpha}$ commute with the four-momentum $P_\mu$ and are the conserved (super)charges. The matrices $\sigma_\mu$ are the two-dimensional identity matrix and Pauli matrices,

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.2}$$

A representation of the supersymmetry algebra (5.1) is given by a supermultiplet, which consists of a collection of particles that are called superpartners. In a $\mathcal{N} = 1$ supersymmetric theory as in (5.1), each boson is accompanied by one fermionic superpartner, and vice versa. But the supermultiplets can be larger in the case of extended supersymmetry algebras. Because (at least up till the energy scales we could already observe) no superpartners of standard model particles have been observed yet, supersymmetry cannot be an exact symmetry of our universe and has to be broken at a certain energy scale.

In the case of multiple supersymmetry generators (labeled by $r, s, \ldots$), the extended supersymmetry algebra becomes,

$$\{Q_{\alpha r}, Q^\dagger_{\dot\alpha s}\} = -2\delta_{rs}\sigma^\mu{}_{\alpha\dot\alpha} P_\mu \,, \tag{5.3a}$$

$$\{Q_{\alpha r}, Q_{\beta s}\} = \epsilon_{\alpha\beta} Z_{rs}\,. \tag{5.3b}$$

For the antisymmetric tensor $\epsilon_{\alpha\beta}$ we use the matrix representation

$$\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{5.4}$$

The matrix $Z$ is antisymmetric. Its elements $Z_{rs}$ are "central charges" of the supersymmetry algebra. They commute with themselves and with the supersymmetry generators $Q_{\alpha r}$ and $Q^\dagger_{\dot\alpha s}$.

## 5.2 BPS condition in supersymmetry

As I already mentioned, the only place where supersymmetry appears more concretely is in chapter 14, where we check the extremality of certain extended objects. The extremality means that the objects preserve (some fraction of) the supersymmetry charges, and are therefore called BPS objects. The abbreviation BPS stands for Bogomol'ny-Prasad-Sommerfield, by analogy with their derivation of a lower bound on the mass of magnetic monopoles. In the context of an extended supersymmetry of $\mathcal{N}$ generators, the BPS bound (which is derived by acting with a positive operator constructed from the supersymmetry generators on a massive multiplet and using polar decomposition for the square matrix $Z$) can be written as

$$M \geq \frac{1}{2\mathcal{N}} \mathrm{Tr}\left(Z^H Z\right), \tag{5.5}$$

with $M$ the mass of the multiplet and $Z^H$ the Hermitian conjugate of $Z$. If the bound (5.5) is saturated, the object is called BPS. In particular, massless states must be neutral. In the case $\mathcal{N} = 2$, the matrix $Z$ is determined by one complex number (say, $Z_{12}$) and the BPS inequality becomes $M \geq |Z_{12}|/2$. When $M = |Z_{12}|/2$, the helicity content of the massive $\mathcal{N} = 2$ supermultiplets are the same as those for zero mass, and they are called "short" supermultiplets.

## 5.3 Supergravity

The supersymmetry algebra presented in the (anti)commutation relations (5.3) can be understood as the algebra of a global symmetry. But in the context of gravity the supersymmetry is naturally promoted to a local symmetry, precisely because the anticommutator of two supersymmetry generators yields the momentum operator, which is the vector field that generates diffeomorphisms in general relativity. When supersymmetry is promoted to a gauge symmetry, it leads to supergravity theories. These are gravity theories in which there is a precise set of fields in addition to the metric field of Einstein's general relativity. Together with the metric (or rather the vielbein) the additional supergravity fields are grouped in a supergravity multiplet (the fields in the supergravity multiplet are related to each other by supersymmetry transformations).

An important particle in supergravity theories is the gravitino, which is a supersymmetric partner of the graviton. The gravitino is a fermion of spin-3/2 and therefore it obeys the Rarita-Schwinger equation. A massless spin-3/2 particle as the gravitino has to mediate a gauge symmetry (like the graviton does for diffeomorphisms or the photon for the $U(1)$ phase associated to electromagnetism). The gauge symmetry associated to the gravitino is precisely the local supersymmetry transformation of supergravity.

For example, in eleven dimensional supergravity (from which all other supergravities can be derived) the (massless) supergraviton multiplet consists of 256 states: there are 128 bosonic components and 128 fermionic components. Denoting eleven dimensional indices by Roman capitals, the bosonic components can be grouped in a vielbein field $e_I^a$ with 44 components (which corresponds to a symmetric traceless transverse tensor in eleven dimensions), supplemented by a three-form potential $A_{IJK}$ with 84 components. The fermionic degrees of freedom correspond to a Rarita-Schwinger (spin-3/2) field in eleven dimensions.

Not every spacetime that is a solution to Einstein's equations is automatically a solution in supergravity. The spacetime has to admit a Rarita-Schwinger field which describes the spin-3/2 gravitino. Of course, to allow for describing spin-1/2 fermions also puts restrictions on the spacetime manifold: it has to be orientable.

## 5.4 Superstring theory

Supergravity has a better high-energy behaviour than the "ordinary" gravity described by general relativity (because of the supersymmetry several Feynman diagrams cancel: diagrams with internal fermion loops cancel diagrams with internal boson loops). Notwithstanding the virtues of supersymmetry with respect to cancellations of quantum loops, a quantum theory of supergravity also needs an infinite number of counterterms to cancel all the divergent Feynman diagrams (though the issue seems to be still open for $\mathcal{N} = 8$ supergravity in four dimensions, see e.g. [47]). On the other hand, it has turned out that supergravity is the low-energy effective limit of superstring theory.

There are several ways to formulate superstrings with spacetime supersymmetry: the Ramond-Neveu-Schwarz formalism, the Green-Schwarz formalism and the pure spinor formalism. In the Ramond-Neveu-Schwarz formulation the bosonic string worldsheet action is extended with worldsheet fermions to become supersymmetric on the worldsheet. However, it is cumbersome to show the explicit target space supersymmetry of the Ramond-Neveu-Schwarz formalism. In the Green-Schwarz formalism supersymmetry in the target space is manifest by adding anticommuting degrees of freedom to the action which already transform as spacetime spinors (but as worldsheet vectors). In the action for the Green-Schwarz superstring the supersymmetry on the worldsheet is not the standard supersymmetry, but it is the so-called $\kappa$-symmetry. The pure spinor formalism [48] was conceived in order to quantize the superstring covariantly (which is not possible in the Green-Schwarz formalism) while being manifestly spacetime supersymmetric. I will not go into further details and refer the interested reader to the literature [5, 14, 48].

There are a few superstring theories, among which the type IIA and type

IIB superstrings. The low energy limit of these theories is given by type IIA and type IIB supergravity theory respectively. The latter are supergravity theories in ten dimensions with thirty-two supersymmetry generators. These theories have $\mathcal{N} = 2$ supersymmetry in ten dimensions: the supersymmetry generators can be written as two sixteen-dimensional Weyl spinors. Each of these sixteen-dimensional spinors can be chosen as Majorana-Weyl (see e.g. appendix I) with a definite chirality while satisfying a (Majorana) reality condition. In type IIA supergravity theory the two supersymmetry charges have opposite chirality, while they have the same chirality in type IIB supergravity theory. In addition to the graviton, dilaton and Kalb-Ramond field, the type IIA and type IIB supergravity theories have additional bosonic Ramond-Ramond fields that are sourced by extended objects called $p$-branes.

In order to be able to formulate a strong coupling prescription of superstring theory like matrix theory, supersymmetry appears to be a necessary tool [104] (up till now we do not know if a strong coupling limit of bosonic string theory exists). Nevertheless, the possible existence of supersymmetry at energies near the Planck scale is largely independent of its possible existence at the TeV scale. Supersymmetry has not been observed at the energy scales we have already probed with particle accelerators, therefore it must be broken. One scenario is the spontaneous breaking mechanism of supersymmetry, to which the so-called Goldstino fermion is associated. The Goldstino fermion can be absorbed by the gravitino of supergravity to explain why the Goldstino and the gravitino are invisible at low-energies (somewhat like the Brout-Englert-Higgs mechanism that makes the $W$ and $Z$ bosons massive in the electroweak theory).

# Chapter 6

# Particle creation in time-dependent spacetimes

**Question:** *Why do only three generations of particles exist?*

**Answer:** *God suddenly created a "stop".*

*F.D.R.*

During one of my projects, I will discuss the phenomenon of mode creation when a first-quantized string propagates across a singularity. In general, when particles or strings propagate on a time-dependent spacetime, particle or string creation may occur. But according to [49], which had a rather strong influence on this chapter, it could just as well be called "field excitation". It is already sufficient to have a simple time-dependence in the metric of the spacetime to allow for particle creation. Essentially the phenomenon of particle creation can be already be illustrated by a simple harmonic oscillator with a time-dependent frequency.

In this chapter, I will first describe how a free scalar field on a time-dependent spacetime leads to a collection of time-dependent harmonic oscillators. Afterwards I will show that the field excitation appears because the creation and annihilation operators that are defined at asymptotic ("in" and "out") times are different. The example of a scalar field allows to refresh some notions from field theory, which may be useful for some readers in the light of later chapters.

## 6.1 Scalar field in a time-dependent spacetime

In quantum field theory, a particle is defined as an excited state with respect to a vacuum. Although there is a natural notion of vacuum in Minkowski spacetime, it is not possible to define a unique vacuum in a general curved spacetime. This leads, for example, to the Unruh and the Hawking effects. In the Unruh effect, an accelerating observer perceives the Minkowski vacuum as a thermal state due to his non-intertial motion. The Hawking effect near a black hole horizon states that black holes emits radiation with a blackbody spectrum at a temperature inversely proportional to the mass of the black hole. The Hawking effect appears because the vacuum of the observer associated to the asymptotically flat spacetime is different from the vacuum of an observer in free fall.

The quantum field theory formalism in Minkowski background can be extended to include quantum fields in curved backgrounds, see for example [2, 22]. To achieve sufficient generality it often becomes necessary to use the algebraic approach to quantum field theory [6] where the algebra is considered separately from the representation. Nevertheless, in this chapter it will be sufficient to limit ourselves to the more conventional (and more concrete) representation approach. I will illustrate the concept of particle creation with a free scalar field, for which the action reads

$$S = -\frac{1}{2} \int \mathrm{d}^D x \, \sqrt{-g} \left( g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \left( m^2 + \xi R \right) \phi^2 \right), \tag{6.1}$$

and it leads to the wave equation

$$\left( -\frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu + m^2 + \xi R \right) \phi = 0. \tag{6.2}$$

The factor $\xi$ expresses the coupling of the field to the curvature scalar of the spacetime background. Let us now specify the line-element of a generic (but spatially flat) Robertson-Walker spacetime,

$$ds^2 = -dt^2 + a^2(t) dx^i dx^i, \tag{6.3}$$

where $a(t)$ is the scale factor. If we decompose the scalar field in Fourier modes,

$$\phi(t, \vec{x}) = \frac{1}{\sqrt{(2\pi)^{D-1}}} \int d^{D-1} \vec{x} \, \chi_{\vec{k}}(t) \exp(i \vec{k} \cdot \vec{x}), \tag{6.4}$$

the equation of motion (6.2) yields

$$\left[ \partial_t^2 + (D-1) \frac{\dot{a}}{a} \partial_t + \frac{\vec{k}^2}{a^2} + m^2 + 2(D-1)\xi \left( \frac{\ddot{a}}{a} + \frac{D-2}{2} \frac{\dot{a}^2}{a^2} \right) \right] \chi_{\vec{k}} = 0. \tag{6.5}$$

In conformal time $\eta$ (obtained by $\dot{\eta} = a^{-1}(t)$) the Robertson-Walker metric takes the (conformally flat) form

$$ds^2 = a^2(\eta) \left[ dx^i dx^i - d\eta^2 \right]. \tag{6.6}$$

Then, through the substitution

$$\chi_{\vec{k}} = a^{1-D/2} \zeta_{\vec{k}}, \tag{6.7}$$

the field equation (6.5) can be written as a time-dependent harmonic oscillator equation

$$\frac{\partial^2}{\partial \eta^2} \zeta_{\vec{k}} + \omega^2(\eta) \zeta_{\vec{k}} = 0. \tag{6.8}$$

The time-dependent frequency squared is given by

$$\omega^2(\eta) = \vec{k}^2 + m^2 a^2 + \left( \frac{(D-4)}{a^2} \left( \frac{\partial a}{\partial \eta} \right)^2 + \frac{2}{a} \frac{\partial^2 a}{\partial \eta^2} \right) \left( (D-1)\xi - \frac{(D-2)}{4} \right). \tag{6.9}$$

Sometimes the value $\xi = (D-2)/4(D-1)$ is chosen to achieve "conformal coupling", but in the remainder of the thesis we will assume that $\xi = 0$.

## 6.2 Spacetimes without particle creation

Now that we have seen how a free scalar field in a time-dependent spacetime leads to a harmonic oscillator equation with time-dependent frequency, I will describe how this time-dependent frequency leads to particle creation. However, I want to remark that if one can define a conserved frequency because the spacetime metric admits a covariantly constant null vector or a globally defined timelike Killing vector, there will be no particle creation [56] nor string creation [58] for an observer in free fall. However for strings there may still be excitation of the string, i.e. creation of string modes. This is because while propagating in a time-dependent spacetime the quantized string may start to oscillate because the frequency is not necessarily conserved on the worldsheet. Let us clarify this difference with the example of a (lightcone) time-dependent plane wave. There is a covariantly constant null vector, hence no particle creation nor string creation. But in lightcone gauge the timelike direction on the worldsheet becomes aligned with the lightcone time, and the plane wave profile induces a time-dependent mass term for the string modes, very much like an ordinary time-dependence in the metric would lead to a time-dependent mass for a scalar field as in (6.5).

The difference between string mode creation and string or particle creation is of relevance in this thesis. Both in chapter 12 and in chapter 13 the background metric depends on lightcone time but is independent of another null coordinate. But while there is no particle creation for the scalar field that we investigate in chapter 12, there is generically string mode creation in chapter 13.

## 6.3 Mode decomposition of the free scalar field

For later convenience and to stress the ambiguous notion of particle in a general curved spacetime, I will first recapitulate some basic aspects of scalar field theory. For a free field the quantum evolution is completely determined by classical solutions to the wave equation. The action depends on the Lagrangian density $\mathcal{L}$ by

$$S = \int dx^0 \int d^{D-1}\vec{x}\ \mathcal{L}(\phi, \dot{\phi}, t), \tag{6.10}$$

where $x^0$ is a timelike coordinate. Upon canonical quantisation, the field $\phi$ and its conjugate momentum $\pi$, which is given by

$$\pi = \delta\mathcal{L}/\delta\partial_0\phi, \tag{6.11}$$

are promoted to Hermitian operators, which satisfy the (equal-time) canonical commutation relations

$$[\phi(x^0, \vec{x}), \pi(x^0, \vec{y})] = i\hbar\delta^{D-1}(\vec{x} - \vec{y}). \tag{6.12}$$

We define the Klein-Gordon inner product according to [49, 66] (a different convention from the one used by [2, 94]) between two complex solutions to the wave equation (6.2)

$$\langle f, g\rangle_{KG} = -\frac{i}{\hbar}\int_\Sigma \mathrm{d}\Sigma^\mu\ \sqrt{-g}W_\mu[f^*, g]\,, \quad W_\mu[f, g] = (f\partial_\mu g - g\partial_\mu f)\,, \tag{6.13}$$

where $\mathrm{d}\Sigma^\mu$ the future directed volume element in the spacelike hypersurface $\Sigma$. We can write $\mathrm{d}\Sigma^\mu = n^\mu \mathrm{d}\Sigma$ with $\mathrm{d}\Sigma$ the volume element in $\Sigma$ and $n^\mu$ a future directed unit vector orthogonal to the hypersurface $\Sigma$ (thus $n^\mu$ lies in the future half of the lightcone at each spacetime event, we assume that the manifold is time orientable). I also recall that the definition for the signs of the metric is mostly plus. Because $f$ and $g$ are complex solutions to the wave equation (6.2), the Klein-Gordon product is independent of the surface $\Sigma$. It defines an inner product that is conserved during the field evolution.

For a free field, the field operator $\phi$ can be expanded in modes, to which annihilation and creation operators are associated,

$$\phi(\vec{x}, x^0) = \sum_{\vec{k}} f_{\vec{k}} a_{\vec{k}} + f^*_{\vec{k}} a^\dagger_{\vec{k}}\,. \tag{6.14}$$

A mode decomposition doesn't really have any fundamental meaning because it amounts to choosing a basis in the solution space. Nevertheless, for practical reasons one set of modes may appear more natural in a specific problem (for example, positive and negative frequency solutions with respect to time translations in Minkowski spacetime). The annihilation operator associated to the complex classical solution $f$ is defined as the Klein-Gordon product between the solution $f$ and the field operator (see e.g. [18])

$$a(f) = \langle f, \phi \rangle_{KG}. \tag{6.15}$$

By assumption $f$ satisfies the wave equation, as does the field operator $\phi$, so the annihilation operator $a(f)$ associated to $f$ is well-defined. The Hermitian conjugate of the annihilation operator is the creation operator,

$$a^{\dagger}(f) = -a(f^{*}) \tag{6.16}$$

The commutators between annihilation and creation operators are given by

$$[a(f), a^{\dagger}(g)] =< f, g >_{KG}, \tag{6.17}$$
$$[a(f), a(g)] =< f, g^{*} >_{KG}, \tag{6.18}$$
$$[a^{\dagger}(f), a^{\dagger}(g)] = - < f^{*}, g >_{KG} . \tag{6.19}$$

If $f$ is a solution that has unit norm $\langle f, f \rangle_{KG} = 1$, then the annihilation and creation operators satisfy the usual commutation relation $[a(f), a^{\dagger}(f)] = 1$.

We now specify a quantum state $|\Psi\rangle$ that satisfies the normalization condition

$$a(f)|\Psi >= 0. \tag{6.20}$$

The Fock space that contains the wavepackets with $n$-particle excitations above the state $|\Psi\rangle$ is given by the span of all quantum states of the type

$$|n(f), \Psi\rangle = \frac{1}{\sqrt{n!}}(a^{\dagger}(f))^{n}|\Psi\rangle. \tag{6.21}$$

With respect the number operator $N(f) = a^{\dagger}(f)a(f)$ (associated to the solution $f$) the excitations $|n(f), \Psi\rangle$ are normalized with eigenvalue $n$. The space $\mathcal{S}$ of complex solutions to the wave equation is given by a direct product between a positive subspace $\mathcal{S}_p$ and its complex conjugate $\mathcal{S}_p^{*}$,

$$\mathcal{S} = \mathcal{S}_p \oplus \mathcal{S}_p^{*}, \tag{6.22}$$

with the conditions on the (positive) subspace $\mathcal{S}_p$,

$$\langle f, f \rangle_{KG} > 0, \qquad \forall f \in \mathcal{S}_p, \tag{6.23}$$
$$\langle f, g^{*} \rangle_{KG} = 0 \qquad \forall f, g \in \mathcal{S}_p. \tag{6.24}$$

The full Hilbert space of the field theory can be defined as the space of finite norm sums of (possibly infinitely many) states of the form

$$a^\dagger(f_1)\cdots a^\dagger(f_n)|0\rangle, \qquad \{f_1,\ldots,f_n\}\in\mathcal{S}_p, \tag{6.25}$$

where the state $|0\rangle$ is defined by

$$a(f)|0\rangle = 0, \quad \forall f\in\mathcal{S}_p. \tag{6.26}$$

The state $|0\rangle$ defined in (6.26), is called a Fock vacuum. It depends on the specific decomposition of the solution space and is in general not the ground state of the system. The ground state is only well defined if the background metric is globally static.

In flat spacetime, a natural decomposition of the solution space $\mathcal{S}$ is given by positive and negative frequency solutions (with respect to Minkowski time translations): the positive frequency modes

$$f^+_{\vec{k}}(t,\vec{x}) = \frac{1}{2}\sqrt{\frac{\hbar}{\pi^3}\omega(k)}e^{-i\omega(k)t}e^{i\vec{k}\cdot\vec{x}}, \tag{6.27}$$

and the negative frequency modes

$$f^-_{\vec{k}}(t,\vec{x}) = \frac{1}{2}\sqrt{\frac{\hbar}{\pi^3}\omega(k)}e^{i\omega(k)t}e^{i\vec{k}\cdot\vec{x}}. \tag{6.28}$$

The field operator has the expansion

$$\phi = \sum_{\vec{k}} f^+_{\vec{k}}a_{\vec{k}} + f^{+*}_{\vec{k}}a^\dagger_{\vec{k}}. \tag{6.29}$$

In a general curved spacetime, there is no analog of the Minkowski vacuum. The notion of particle is ambiguous, and states characterized by the field observables. Nevertheless, if the wavevector and the frequency of the field mode are high enough compared to the inverse radius of curvature (that is, the frequency is high compared to the square root of the curvature scalar $R$), then we can locally introduce an analog notion of the Minkowski vacuum and particle states can be defined approximately [49].

## 6.4 Time-dependent harmonic oscillator

Let us now consider the equation of motion of a time-dependent harmonic oscillator

$$\ddot{x} + \omega^2(t)x = 0. \tag{6.30}$$

We assume that the frequency is asymptotically constant, with the asymptotes

$$\lim_{t\to\mp\infty} \omega(t) = \omega_{\mp}, \tag{6.31}$$

for "incoming" (-) and "outgoing" (+) states respectively. In the Schrödinger picture, we assume that the oscillator is asymptotically in the state

$$\lim_{t\to-\infty} |\psi_S(t)\rangle = |0_-\rangle\,, \tag{6.32}$$

with $|O_-\rangle$ the ground state of the "incoming" Hilbert space related to the frequency $\omega_-$ . Then we would like to know what

$$\lim_{t\to\infty} |\psi_S(t)\rangle \tag{6.33}$$

will correspond to. In general (6.33) will not correspond to $|0_+\rangle$, the ground state of the Hilbert space related to $\omega_+$. But to stress the analogy between the quantum oscillator and the free scalar field, we will keep on working in the Heisenberg picture. In the Heisenberg picture the state is time-independent, so if it starts as $|\psi_H\rangle = |0_-\rangle$, it will remain in $|0_-\rangle$ for all times. The question is now how the state $|0_-\rangle$ is expressed as a Fock state in the "outgoing" Hilbert space that is related to $\omega_+$. In other words, we want to find the operator $F(a_+^\dagger, a_+)$ that is defined as,

$$|0_-\rangle \stackrel{?}{=} F(a_+^\dagger, a_+)|0_+\rangle. \tag{6.34}$$

We write out solutions $f^-(t)$ and $f^+(t)$ to equation (6.30), which are normalized according to $< f^-, f^- >_{KG} = < f^+, f^+ >_{KG} = 1$, and which have the following asymptotic behaviour

$$\lim_{t\to\pm\infty} f^{\pm} \propto \sqrt{\frac{\hbar}{\omega_\pm}} \exp\left(-i\omega_\pm t\right). \tag{6.35}$$

Because (6.30) is a second order differential equation, it possesses a two-parameter family of solutions, which implies there is a complex relation between $f^+$ and $f^-$ that involves two complex constants $\alpha$ and $\beta$ such that

$$f^+ = \alpha f^- + \beta f^{-*}, \qquad |\alpha|^2 - |\beta|^2 = 1. \tag{6.36}$$

In analogy with (6.15), the "outgoing" annihilation operator can then be written as

$$a_+ = \langle f^+, x\rangle_{KG} \tag{6.37}$$

$$= \alpha a_- - \beta a_-^\dagger \tag{6.38}$$

Such a linear relation between two sets of annihilation and creation operators is called a Bogoliubov (or Bogoliubov-Valatin) transformation. The coefficients $\alpha$ and $\beta$ are called the Bogoliubov coefficients. We then find that the expectation value of the "outgoing number operator" $N_+ = a_+^\dagger a_+$ in the "incoming ground state" $|0_-\rangle$ is nonzero,

$$\langle 0_- | N_+ | 0_- \rangle = |\beta^2|, \tag{6.39}$$

and the number $|\beta|^2$ characterizes the excitation number, or the "particle creation". The more general relation between the incoming vacuum and the outgoing vacuum defined in (6.34), is given by means of a *squeeze* operator $S$,

$$|0_-\rangle = S^\dagger |0_+\rangle, \tag{6.40}$$

which only depends on the exponential of the "two-particle" creation and annihilation operators $a^\dagger a^\dagger$ and $aa$.

Summarizing the underlying theoretical mechanism behind the phenomenon particle creation: the mode decomposition at early times (with respect to "incoming modes") is different from the mode decomposition at late times (in terms of"outgoing modes"), so a quantum state $|\psi_H\rangle$ that is identified as the vacuum state $|0_-\rangle$ with respect to the basis of incoming modes will be described by an excited state in the basis of outgoing modes.

# Part II

# Background

# Chapter 7

# Background consistency in string theory

*The first law of attribution:*

*"Everything that is discovered, is named after someone else."*

*V. I. Arnold*

*(Of course Arnold's law is self-referential.)*

In chapter 4, I have shown that the oscillation modes of a string, propagating in Minkowski spacetime, contain a massless particle that can be identified with the graviton. Now I will illustrate that the demand for a consistent quantisation of the sigma model on the string worldsheet, determined by the conformal invariance of the string theory sigma model, forces a general spacetime background to satisfy (a generalization of) Einstein's equation. Sometimes, as in chapter 13, these equations are called "background consistency conditions", because they constrain the spacetime backgrounds on which strings can propagate. In the context of superstring theory the background consistency conditions concur with the supergravity equations of motion, and the latter will appear in chapter 14. Some elements of this chapter have been based on [14, 52, 53].

## 7.1 Polyakov action in curved spacetime

We would like to extend the formalism of a string propagating in Minkowski spacetime to a string that moves in an more general curved spacetime. Or more precisely, we want to consider a string that propagates in a background of other strings, because according to general relativity the curved spacetime describes the gravitational interaction, and according to string theory the gravitational interaction is described by interactions between strings. Instead of writing the action (4.2) of a linear sigma model, we now embed the two-dimensional worldsheet of the string in a background spacetime, which should be viewed as a kind of coherent state of the *massless* string modes [14]. Among the massless string oscillations is of course the graviton (a coherent state of gravitons leads to the spacetime metric $g_{\mu\nu}$), but also the dilaton $\phi$ and the Kalb-Ramond field $B_{\mu\nu}$, the latter being an antisymmetric tensor field that is sometimes also called the B-field. For the sake of simplicity, we will write the string action in a general curved spacetime for bosonic string theory. This is given by

$$S = \frac{1}{4\pi\alpha'} \int_M \mathrm{d}^2\sigma\sqrt{\gamma} \left[ \left( \gamma^{ab} g_{\mu\nu}(X) + i\epsilon^{ab} B_{\mu\nu}(X) \right) \partial_a X^\mu \partial_b X^\nu + \alpha' R^{(\gamma)} \phi(X) \right] . \tag{7.1}$$

Because of the appearance of the spacetime metric $g_{\mu\nu}$ in front of the worldsheet derivatives of the string coordinates, this action represents a nonlinear sigma model, or an interacting two-dimensional quantum field theory. As mentioned, we will assume that only the massless states of the string are excited. We can neglect the excitation of the massive string modes when the typical radius of curvature $\rho$ of the background spacetime is much larger than the characteristic length scale of the string $\ell_s = \sqrt{\alpha'}$, or $\sqrt{\alpha'}\rho^{-1} \ll 1$. In fact, without this condition perturbation theory in the sigma model is not a useful approximation neither. In (7.1), the dilaton couples through the worldsheet curvature $R^{(\gamma)}$ and as we will see in the next section, it ensures that the two-dimensional sigma model is conformal, or that the worldsheet energy-momentum tensor is traceless at the quantum level [50] (for example, in lightcone gauge the dilaton cancels the contribution of the transverse string coordinates to the conformal or Weyl anomaly [66]).

## 7.2 Conformal invariance

As illustrated in section 4.3, the sigma model on the string worldsheet is invariant under local conformal transformations at the classical level. These local symmetries should not be broken by quantum corrections, because the formulation of the worldsheet sigma model depends on the conformal invariance and otherwise the theory would become inconsistent. Therefore we should check that quantum corrections do not break the conformal symmetry of the string theoretical sigma model.

Quantum corrections that add to the trace of the worldsheet energy-momentum tensor may break the conformal symmetry. They contribute to the so-called conformal or Weyl anomaly, which can be written schematically as

$$T^a{}_a = -\frac{1}{2\alpha'}\beta^g_{\mu\nu}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu - \frac{i}{2\alpha'}\beta^B_{\mu\nu}\epsilon_{ab}\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\beta^\phi R^{(\gamma)}, \tag{7.2}$$

where the prefactors $\beta$ are the "beta functions" of the sigma model, and they depend on the background fields $g_{\mu\nu}$, $B_{\mu\nu}$ and $\phi$. Then, the demand that there is no Weyl anomaly forces the prefactors $\beta^g_{\mu\nu}$, $\beta^B_{\mu\nu}$ and $\beta^\phi$ to be zero. Thus the cancellation of the Weyl anomaly leads to the following equations,

$$\beta^g_{\mu\nu} = 0 \quad \Rightarrow \quad \alpha' \left( R_{\mu\nu} + 2D_\mu D_\nu \phi - \frac{1}{4}, H_{\mu\lambda\omega} H_\nu{}^{\lambda\omega} \right) + O(\alpha'^2) = 0, \tag{7.3a}$$

$$\beta^B_{\mu\nu} = 0 \quad \Rightarrow \quad \alpha' \left( D^\omega \phi H_{\omega\mu\nu} - \frac{1}{2} D^\omega H_{\omega\mu\nu} \right) + O(\alpha'^2) = 0 \tag{7.3b}$$

$$\beta^\phi = 0 \quad \Rightarrow \quad \frac{D - D_c}{6} + \alpha' \left( D_\omega \phi D^\omega \phi - \frac{1}{2} D^\omega D_\omega \phi - \frac{1}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right) + O(\alpha'^2) = 0\,. \tag{7.3c}$$

(7.3) determines the conformal invariance at the quantum level. The three-form field strength $H_{\mu\nu\kappa}$ is derived from the Kalb-Ramond field $B_{\mu\nu}$ by

$$H_{\mu\nu\kappa} = \partial_{[\mu} B_{\nu\kappa]}\,. \tag{7.4}$$

At the order of $\alpha'$ there are at most two derivatives in (7.3). Terms with more than two derivatives only appear with the higher order terms in $\alpha'$, which involve for example contractions of the products of the Riemann tensor. At energies that are low compared to $1/\sqrt{\alpha'}$, spacetimes that are vacuum solutions to Einstein's equation are also approximate backgrounds in string theory, by adding a constant dilaton and a zero B-field. In the next chapter I will describe a class of exact classical solutions in string theory that do not involve $\alpha'$ corrections.

In the equations (7.3) I have written the symbol $D_c$ for generality: in bosonic string theory we have $D_c = 26$. Nevertheless, in critical superstring theory in ten dimensions, the fermionic superpartners of the ten spacetime coordinates give a quantum contribution to the trace of the worldsheet energy-momentum tensor that is equivalent (in the context of the cancellation of the Weyl anomaly) to sixteen spacetime dimensions. So the formulas (7.3) above are also valid in superstring theory for $D_c = 10$ if we consider bosonic background fields. In fact, up till now we have only considered the so-called "Neveu-Schwarz-Neveu-Schwarz" background fields, because there may appear additional bosonic terms in the equations (7.3) due to so-called "Ramond-Ramond" fields, which will briefly appear in the next chapter.

So to summarize some material of chapter 4 and the material presented here, we have seen that in order to be able to quantize the string, we needed to resort to the action (4.2). It possessed an auxiliary worldsheet metric that should not introduce unphysical degrees of freedom. Classically, the Polyakov action is invariant under local Weyl (conformal) transformations which permit to remove the dependence of the string theory sigma model on the auxiliary worldsheet metric. For theoretical consistency, the invariance under local Weyl transformations should hold at the quantum level as well. The condition for the absence of the conformal anomaly now leads to (a generalization of) Einstein's equation. So general relativity appears to be a consequence of the Weyl invariance of the worldsheet sigma model that describes the propagation of the string.

## 7.3 Low-energy effective actions

The field equations for the metric, Kalb-Ramond field and dilaton that guarantee that there is no Weyl anomaly in the quantum theory of the worldsheet sigma model, can be derived from an effective spacetime action

$$S_{st}^{eff} = \frac{1}{16\pi\mathcal{G}} \int \mathrm{d}^D x \sqrt{-g} e^{-2\phi} \left[ -\frac{2(D-D_c)}{3\alpha'} + R - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right. \\ \left. + 4\partial_\mu\phi\partial^\mu\phi + O(\alpha') \right] . \quad (7.5)$$

This is called the effective action in string frame because of the dilaton-dependent prefactor in front of the spacetime curvature scalar. As a consequence the metric that is obtained by solving its equation of motion is called the string frame metric. In fact we have a kind of "generalized spacetime" because of the presence of the dilaton and the Kalb-Ramond field in addition to the metric, somewhat like the Brans-Dicke theory of gravitation (e.g. [24]) that generalizes Einstein's description of gravity by adding a scalar field, in order to construct a theory that was even more in accordance with Mach's principle than Einstein's general relativity.

In the low energy limit where this action gives a valid effective description of string theory, the string length is small and the stringlike nature of particles can be neglected. If we then consider the propagation of point particles in the spacetime of which the dynamics is described by the action (7.5), they don't follow geodesics because the dilaton exerts a force on them. In this sense, the dilaton breaks the principle of equivalence even at the classical level, even though the action is still generally covariant and does have a dynamical metric. On the other hand, there are mechanisms to make the dilaton become massive[1], so the principle of equivalence still holds (up to a certain accuracy) at energies below the mass scale of the dilaton.

Because point particles don't follow geodesics, there is no preferred definition of a metric. It is sometimes convenient to rescale the string metric by a

[1] For example, in [5] it is argued how the mass of the dilaton could be associated to a cosmological constant, at a time that it was generally believed that the cosmological constant was zero.

factor of the dilaton, in order to remove the exponential prefactor before the curvature scalar. This prefactor is not important in the classical theory. It would certainly influence the quantum theory, but for a consistent quantisation of gravity the full string sigma model is needed and this effective action is not expected to be valid at energies near the Planck scale anyway. So we will rescale the metric by a dilaton-dependent prefactor (with $\phi_0$ the average value of the dilaton) according to

$$\tilde{g}_{\mu\nu} = \exp\left(-4\tilde{\phi}/(D-2)\right) g_{\mu\nu} \,, \quad \tilde{\phi} = \phi - \phi_0 \,. \tag{7.6}$$

In that case we pass to the effective action in Einstein frame:

$$S_E^{eff} = \frac{1}{16\pi G_N} \int \mathrm{d}^D x \sqrt{-\tilde{g}} \left[ -\frac{2(D-D_c)}{3\alpha'} \exp\left(\frac{4\tilde{\phi}}{D-2}\right) + \tilde{R} - \frac{4}{D-2} \tilde{g}^{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} \right] . \tag{7.7}$$

I have omitted the field strength $H_{\mu\nu\kappa}$ because it will not reappear later in this thesis. The metric obtained by solving the equation of motions from the action (7.7) is called the Einstein frame metric. For critical superstrings $D = D_c = 10$ the Einstein frame metric is related to the string frame metric by

$$ds_E^2 = e^{-\phi/2} ds_{st}^2 \,, \tag{7.8}$$

which will reappear a few times in this thesis.

At an energy scale far below the mass of the dilaton, (7.7) is the appropriate action to compare with the Einstein-Hilbert action of general relativity. And it does indeed resemble the Einstein-Hilbert action more clearly than (7.5). The prefactor $G_N$ in (7.7) is Newton's constant and differs from the prefactor $\mathcal{G}$ in (7.5) by a factor of the dilaton:

$$G_N^{(D)} = \left[g_s^{(0)}\right]^2 \mathcal{G} \,, \quad g_s^{(0)} = e^{\phi_0} \,. \tag{7.9}$$

Here the superscript "$(D)$" indicates that we are considering Newton's constant in a target space with $D$ dimensions. From dimensional analysis, the Planck length is related to Newton's constant by

$$G_N^{(D)} \approx \left[\ell_P^{(D)}\right]^{D-2} \,. \tag{7.10}$$

The prefactor $\mathcal{G}$ in the string frame effective action is naturally determined in function of the string length as

$$\mathcal{G} \approx \left[\ell_s\right]^{D-2} \,, \tag{7.11}$$

which leads us to the relation between Planck length and string length:

$$\ell_P^{(D)} \approx g_s^{2/(D-2)} \ell_s \,. \tag{7.12}$$

This relation has already appeared in the introduction and in chapter 4.

The action (7.7) also agrees with (the bosonic part of) the supergravity action, at least for the bosonic fields in the Neveu-Schwarz-Neveu-Schwarz sector. There is also another bosonic sector called the Ramond-Ramond sector, as I already mentioned. In the following chapter we will see that there exist extended objects in string theory that are called D$p$-branes. These D$p$-branes are sources for the Ramond-Ramond gauge fields. These fields are higher-dimensional generalisations of electromagnetic fields. If D$p$-branes are present, they lead to additional terms in the effective action (see for instance, the supergravity action in chapter 14) which were not written above.

# Chapter 8

# D-branes in string theory


*Allegro molto alla "Notte e giorno faticar" di Mozart, Ludwig van Beethoven*

In this chapter I will introduce the subject of D$p$-branes. D$p$-branes are solitons of string theory. They can also be seen as a specific kind of $p$-branes, which already appeared as gravity solutions in section 3.5. The reason I have to make some comments about D-branes in the context of my thesis is twofold: D-branes are important for the formulation of matrix theory in chapter 11, and they also appear in my last research project in chapter 14. I will not delve too deep into the theory of D-branes, because this would lead us to far from the main line of the thesis. Therefore the readers who are primarily interested in the research topics that concern the geometrical resolution of singularities, can easily skip this chapter.

To recapitulate, string theory is formulated as a two-dimensional (conformal) quantum field theory on string worldsheets with signature $(-1, 1)$. The (1+1)-dimensional quantum field theory describes $D$ scalar bosons $X^\mu$, which describe the position of the string in $D$-dimensional spacetime (plus their supersymmetric extension in superstring theory). The quantum propagation of strings involves an expansion in the string coupling, the interaction parameter between strings, which corresponds to a sum over string worldsheets with different topologies. The worldsheet sigma model of the string is a first-quantized approach because it deals with the evolution of a single string. To deal with strings propagating in the presence of other strings would require a second-quantized formulation of strings as quantum fields on spacetime, called "string field theory". Nevertheless, if the number of backgrounds strings is large enough that a semiclassical limit is justified, the traditional first-quantized approach appears sufficient. The problem of a string propagating in a coherent state of other strings is understood as a string propagating in a curved spacetime. When the string propagates in a nontrivial spacetime background, the world-sheet sigma model has to be adapted: the spacetime metric appears in front of the kinetic terms of the fields $X^\mu$.

The formulation of string theory in curved spacetime backgrounds allows us to investigate the behaviour of strings in relation to certain configurations of other strings. But string theory also contains different objects of other dimensionalities which are more directly related to classical spacetime backgrounds. These objects are generically called $p$-branes, where the dimensionality of the "brane" is $p+1$: there are $p$ spatial dimensions and there is one timelike dimension along the worldvolume of the brane. There is one exception, the D(-1)-brane, which is an "instanton", one spacetime event. The worldvolume is the collection of points in spacetime that are swept out by the movement of the brane, a generalization of the worldline of a particle and the worldsheet of a string. The remaining $(D-p-1)$ dimensions are sometimes called "transverse". The notation $D$ both for Dirichlet-brane and also for the total number of spacetime dimensions (and the covariant derivative) is a little bit unfortunate, but it should be clear from the context. All the $p$-branes can be described as solutions in a gravitational theory, with a certain mass or charge distributed on the worldvolume. In this way, a Schwarzschild black hole can also be viewed as a 0-brane, and a string as a "fundamental" 1-brane or F1-brane (these are not D-branes).

## 8.1 What are D-branes?

To illustrate how string theory offers a quantized description of gravity, I have commented on the massless spectrum of a closed string in section 4.4. There are certain objects in closed string theory, called D-branes, which are static at weak string coupling (they have infinite mass), but whose dynamics becomes important at strong string coupling. The theoretical formulation of closed string theory does not necessarily have to involve open strings as well (I remark it is not possible to formulate open strings without including closed strings), at weak string coupling D-branes appear most naturally in the formulation of open string theory, so I will introduce them that way.

In the case of an open string, the variation of the Polyakov action yields a boundary term at the endpoints of the string (it is standard to take $\sigma = 0, \pi$). As a consequence one has to impose a condition on the movement of the string to make this boundary term vanish. In analogy with the theory of partial differential equations on a compact region, the conditions are called Neumann and Dirichlet conditions. Neumann boundary conditions specify that there is no momentum flowing out of the string (that is, the endpoints can move freely)

$$\frac{\partial X^\mu}{\partial \sigma}\bigg|_{\sigma=0,\pi} = 0, \tag{8.1}$$

and Dirichlet boundary conditions specify that the endpoints of the string do not move,

$$\frac{\partial X^\mu}{\partial \tau}\bigg|_{\sigma=0,\pi} = 0. \tag{8.2}$$

More specifically, let us choose (ten-dimensional) Minkowski spacetime in light-cone coordinates and write the metric as

$$ds^2 = -2dudv + \sum_{\alpha=1}^{p-1} y^\alpha y^\alpha + \sum_{i=1}^{9-p} dx^i dx^i, \tag{8.3}$$

We now impose Neumann conditions for the $(p+1)$ string coordinates $U(\sigma,\tau)$, $V(\sigma,\tau)$ and $Y^\alpha(\sigma,\tau)$, but Dirichlet conditions for the $(9-p)$ coordinates $X^i(\sigma,\tau)$. In other words, the string endpoints can move freely on the $(p+1)$-dimensional subspace $\Sigma^{p+1}$ along the worldvolume dimensions $\{u,\, v,\, y^\alpha\}$, but they are fixed in the transverse dimensions $x^i$. These subspaces to which the endpoints of open strings are attached, are called D-branes. A $\mathrm{D}_{D-1}$-brane fills the whole space and corresponds to a open string that can move freely, with only Neumann conditions at the endpoints.

At weak string coupling the D-branes can be considered as rigid objects. In 1995 it was understood by Polchinski [51] that the stable Dirichlet-branes of superstring theory carry Ramond-Ramond charges. As a consequence, D-branes can be represented by (stable) supergravity solutions in the low-energy limit of string theory. These solutions describe extremal black $p$-branes, which I introduced in section 3.5. Black $p$-branes are higher dimensional generalizations of black holes, and "extremal" signifies that there is an equality between the mass and the charge of the $p$-brane, generalizing the extremal Reissner-Nordström black hole (3.31). In bosonic string theory D-branes carry no conserved charges and are not associated with gravity solutions, but in this thesis it is tacitly understood that I'm referring to superstring theory.

To recapitulate, there are two complementary ways to look upon D-branes in superstring theory. The first way is to describe D-branes as subspaces on which strings have Dirichlet boundary conditions: it is possible to consider open strings with Dirichlet boundary conditions on some number $(9-p)$ of the spatial string coordinates $X^i$. The locus of points defined by the chosen Dirichlet boundary conditions defines a $(p+1)$-dimensional subspace $\Sigma^{p+1}$ in the ten-dimensional spacetime. The second way is to interprete D-branes as extremal brane solutions of supergravity that carry conserved charges.

Concerning the conserved charges carried by the D$p$-brane, the ten-dimensional type IIA and IIB supergravity theories each have a set of $(p+1)$-form gauge potentials $A^{(p+1)}$. These gauge potentials are a generalization of the electromagnetic potential which is a one-form field. They are the Ramond-Ramond fields in the massless superstring spectrum. For type IIA supergravity $p$ is even and for type IIB supergravity $p$ is odd. The $(p+1)$-form gauge potentials $A^{(p+1)}$ lead to a $(p+2)$-form field strength, and the charge that couples to the gauge fields is carried by the $p$-brane. For each of the gauge potentials $A^{(p+1)}$ there is a solution of the supergravity equations of motion (i.e. Einstein's equation generalized to supergravity), that is invariant under Lorentz transformations along the brane worldvolume, and which has the form of an extremal black hole solution in the $9-p$ spatial directions that are not affected by the fields $A^{(p+1)}$. These extremal $p$-brane solution are so-called BPS states in the supergravity theory that preserve half of the supersymmetries of the vacuum supergravity theory (see section 5.2). The supergravity solution associated to the $p$-brane expresses the functional shape of the gravitational field and gauge field strength that are sourced by the charged $p$-brane, in a way similar to how the Reissner-Nordström solution represents the electro-gravitational field of a massive point charge.

## 8.2 T-duality and D-branes

When in closed bosonic string theory one dimension (say, for example, $X^c$) is compactified on a circle with radius $R$, symbolically written as

$$X^c \sim X^c + 2\pi n R, \tag{8.4}$$

the behaviour of the theory turns out to be equivalent to a compactification on a circle of radius $\alpha'/R$. More specifically, invariance of the closed string under periodic shifts of the worldsheet coordinate $\sigma \to \sigma + 2\pi$ does not require that the left-moving and right-moving zero mode creation operators $\alpha_0^i$ and $\tilde{\alpha}_0^i$ be equal as for the string coordinate in an uncompactified spacetime direction, but it only requires

$$\sqrt{\frac{\alpha'}{2}}(\alpha_0^c - \tilde{\alpha}_0^c) = mR\,, \quad m \in \mathbb{Z}\,. \tag{8.5}$$

On the other hand, through the compactification the momentum along the compact dimension must be quantized as $p^c = n/R$ and the operators that create the string oscillations become

$$\alpha_0^c = \sqrt{\frac{\alpha'}{2}}\left(\frac{n}{R} + m\frac{R}{\alpha'}\right)\,, \quad \tilde{\alpha}_0^c = \sqrt{\frac{\alpha'}{2}}\left(\frac{n}{R} - m\frac{R}{\alpha'}\right). \tag{8.6}$$

However, because the winding modes (mode number $m$) and the momentum modes (mode number $n$) do not appear independently in the spectrum of the theory, it is not possible to distinguish them physically. For example the mass of the string states has a term proportional to

$$\Delta M = (\alpha_0^c)^2 + (\tilde{\alpha}_0^c)^2, \tag{8.7}$$

which is invariant under the changes $R \to \alpha'/R$ and $n \leftrightarrow m$ and therefore the physical predictions remain the same.

When two different theoretical descriptions lead to the same physical behaviour the equivalence between them is often called a "duality". In some sense, on could view a duality as a complicated relation between two mathematical representations of the same physical problem. In the case described above one speaks about T-duality. The "T" stands for toroidal compactification, which is a more general type of compactification that the compactification along a circle described in (8.4). Under the T-duality the field $X^c$ is replaced by a dual field $\hat{X}^c$.

The also exists a T-duality between the two closed $\mathcal{N} = 2$ superstring theories: type IIA superstring theory defined on a background with a compactified dimension of a radius $R$, is T-dual to type IIB superstring theory defined on the same background but with compactification radius $\alpha'/R$. But if in addition to the closed superstrings, also open superstrings are considered, then the D-branes have to be taken into account explicitly when the T-duality is performed. Under the T-duality a D$p$-brane is mapped into a D$(p \pm 1)$-brane.

## 8.3 D-branes as dynamical objects

When $p$ is even in type IIA superstring theory (or odd in type IIB superstring theory) the quantum spectrum of the string theory contains a set of massless gauge fields $A_m$, with $m = \{u, v, \alpha = 1 \ldots p-1\}$ and fields $X^i$, $i = 1, \ldots, (9-p)$. These gauge fields $A_m$ are defined on the worldvolume of the D$p$-brane. They should not be confused with the Ramond-Ramond gauge potentials $A^{(p+1)}$ of section 8.1 because the latter are present throughout the whole spacetime. The gauge fields $A_m$ can be identified as a set of degrees of freedom that describe the dynamical fluctuations of the D$p$-brane in the transverse directions $x^i$. To clarify this, suppose we start with a critical open superstring with only Neumann conditions, which can move freely on the D9-brane that is spacetime. The open string can couple to a $U(1)$ gauge field $A_\mu$ ($\mu = 0 \ldots 9$) present on D9-brane, so the gauge field is present in the whole spacetime. The components of the gauge field are derived from the vertex operators of the open string that have the structure $V^{(\mu)} = \partial_t X^\mu$. Upon an T-duality, we will obtain a D8-brane instead of a D9-brane, and the $A_9$ component of the gauge field will be transformed into a field that describes the transverse fluctuations of the brane.

The reason is that upon T-duality the vertex operator $\partial_t X^9$ is transformed into the operator $\partial_n \hat{X}^9$ where $\partial_n$ is the normal derivative transverse to the brane and $\hat{X}^9$ is the T-dual field of $X^9$. The quantum fluctuations of the open string describe a fluctuating hypersurface[1] through the operator $\partial_n \hat{X}^9$ and therefore the D8-brane is not static.

## 8.4 Effective action for D-branes

To determine the effective action of a system that contains D-branes, we will again demand conformal invariance as in the previous chapter. In the presence of D-branes, the boundary terms in the action will lead to additional terms in the background consistency conditions (7.3). The additional terms can be derived from an effective action which we will write down here, because it will turn out to be important for the formulation of matrix theory in chapter 11. Let us consider a collection of $N$ D-branes. For $N$ non-coinciding D-branes the massless vector states in the string spectrum only come from strings with both ends on the same D-brane. For clarity I have kept the Kalb-Ramond field $B_{\mu\nu} = 0$ and also Ramond-Ramond $p$-form fields are omitted. I will first write out the effective action for a single (bosonic) D$p$-brane, which describes the oscillations of a brane with an equilibrium position at the origin,

$$S_{DBI} = -\frac{\alpha'^{-(p+1)/2}}{(2\pi)^p} \int d^{p+1}\xi e^{-\phi} \left[ \det \left( g_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^m} \frac{\partial X^\nu}{\partial \xi^n} + 2\pi\alpha' F_{mn} \right) \right]^{1/2}, \tag{8.8}$$

with the worldvolume parametrized by $\xi^m$. For historical reasons this action is called the Dirac-Born-Infeld action (DBI). $F_{mn}$ is the field strength of the gauge field $A_m$ present on the brane worldvolume (this is not to be confused with the Ramond-Ramond gauge field that is sourced by the brane and present throughout the whole spacetime). In the case of a single brane it is simply (we can restrict to the dimensions along the brane)

$$F_{mn} = \partial_m A_n - \partial_n A_m \,. \tag{8.9}$$

In the case of Minkowski spacetime background, up to lowest order in the gauge field strength, and ignoring the coordinates $X^\mu$ transverse to the brane, the effective action (8.8) for a single D-brane can be rewritten as

$$S_{1\mathrm{D}p} \approx -\frac{\alpha'^{-(p+1)/2}}{g_s (2\pi)^p} \int d^{p+1}\xi \quad - \frac{1}{4g_s} \frac{\alpha'^{(3-p)/2}}{(2\pi)^{p-2}} \int \mathrm{d}^{p+1}\xi F_{mn} F^{mn}. \tag{8.10}$$

[1] A hypersurface is a subspace of codimension one, i.e. it has one dimension less than the spacetime.

For $N > 1$ there are additional terms because of the interactions between the branes. We neglect the constant (infinite) term in formula (8.10). The bosonic part of the effective action for $N$ coinciding D$p$-branes at the origin becomes

$$S_{N\mathrm{D}p} \approx -\frac{1}{4g_s}\frac{\alpha'^{(3-p)/2}}{(2\pi)^{p-2}}\int \mathrm{d}^{p+1}\xi F_{mn}^2 + \frac{\alpha'^{-(p+1)/2}}{g_s(2\pi)^p}\int \mathrm{d}^{p+1}\xi \mathrm{Tr}\left(-\frac{1}{2}(D_m X^i)^2 + \frac{1}{16\pi^2\alpha'^2}([X^i, X^j])^2\right), \tag{8.11}$$

with the non-abelian gauge field strength

$$F_{mn} = \partial_m A_n - \partial_n A_m + i[A_m, A_n], \tag{8.12}$$

and the covariant derivative

$$D_m X_j = \partial_m X_j + i[A_m, X_j]. \tag{8.13}$$

The trace in the action (8.11) is over the $U(N)$ matrices. For a single D-brane, the gauge field $A_\mu$ that is associated to the D-brane surface is in the adjoint representation of the gauge group $U(1)$. An open string that has a Dirichlet condition with respect to this brane will couple to the gauge field $A_m$ on this brane. For $N$ non-coinciding D-branes, there is one $U(1)$ factor for each D-brane. So for $N$ non-coinciding D$p$-branes the gauge group becomes $U(1)^N$, though there are also massive modes related to the open strings stretching between different D-branes. For coinciding D-branes the full gauge group is restored to $U(N)$ because open strings stretching between the different D-branes are just as massless as strings with both endpoints on only one brane. Fields in the adjoint representation of $U(N)$ are represented by $N \times N$ unitary matrices. Because the fluctuations of the D-branes around their equilibrium position is (by T-duality) directly related to these gauge field $A^\mu$, their representation will have to written as $U(N)$ matrices as well. I will illustrate in chapter 11 that the equilibrium position of the branes is given by the diagonal matrix-elements, at least for widely separated branes. For widely separated branes, the string modes between different branes are very massive and the gauge group $U(N)$ breaks down to $U(1)^N$, which corresponds to a $N \times N$ diagonal unitary matrix.

For a general D$p$-brane, the tension $\tau_p$ of the brane is of the order of the prefactor of the kinetic term for the matrices $X^i$ in the effective action (8.11),

$$\tau_p \propto g_s^{-1}\alpha'^{-(p+1)/2}\,. \tag{8.14}$$

The tension is related to a mass scale:

$$\tau_p \approx (M_{\mathrm{D}p})^{p+1}\,. \tag{8.15}$$

The mass can be derived from the central charge of the supergravity superalgebra, because D-branes are extremal objects for which the mass is equal to the charge. We will come back to the expressions (8.15) and (8.11) in chapter 11.

# Chapter 9

# Gravitational plane waves

*The second law of attribution:*

*(prompted by the observation that the sequence of antecedents under Arnold's law seems endless)*

*"Nothing is ever discovered for the first time,"*

*Michael Berry*

In section D.1 of appendix D gravitational waves are introduced as solutions to the perturbed Einstein equations on a Minkowski background. These gravitational waves are derived as perturbations and out of consistency their amplitude is therefore assumed to be small. Nevertheless, it is possible to write down metrics that represent gravitational waves in general relativity that can become arbitrarily strong, because they satisfy the full nonlinear Einstein equation. The existence of a nonlinear plane wave solution makes it clear that the linear gravitational waves are certainly not coordinate artifacts. They really represent ripples in spacetime that can transport energy. So far, gravitational waves have not been directly observed, but it is the aim of high precision experiments such as LIGO, GEO600, VIRGO, MiniGrail and LISA to detect them experimentally.

I will introduce the nonlinear gravitational plane waves and also a more general class of spacetimes that they belong to, which are called pp-waves. I will describe why plane waves are an interesting background for studying toy models in string theory. In this thesis, singular plane waves will be used as spacetime backgrounds for a free string propagating across a singularity in chapter 13. Plane waves will also appear as an asymptotical spacetime for a class of D-brane solutions in chapter 14. Some material of this chapter is based on [19] and I have recapitulated some useful elementary material of general relativity in chapter 3 and appendix B.

## 9.1 pp-waves

In four dimensions, Lorentzian spacetimes that admit a covariantly constant null vector field $k$ ($k$ satisfies equation (B.19) and $k_\mu k^\mu = 0$) are called "plane-fronted gravitational waves with parallel rays", or pp-waves in short. Historically, pp-waves were first introduced by Hans Brinkmann in 1925 [54], but the term "pp" was only introduced in 1962 by Jürgen Ehlers and Wolfgang Kundt [55]. In between, pp-waves were rediscovered many times, for example by Albert Einstein and Nathan Rosen in 1937. The term pp-waves is purely related to the mathematical properties of a manifold, irrespective of the physical matter present in the spacetime.

Because of the requirement of the covariantly constant null vector field (we will denote it by $k = \partial_v$), electromagnetic non-null fields, perfect fluids and solutions with a cosmological constant cannot occur and the metric of the vacuum pp-wave, or generalized with Einstein-Maxwell null fields and pure radiation fields can be written as,

$$ds^2 = -2dudv + dx_1 dx_1 + dx_2 dx_2 - F(u, x_1, x_2)du^2. \tag{9.1}$$

In higher dimensions the characterisation of a pp-wave is a little bit more complicated: pp-waves are not simply identical to the class of Lorentzian spacetimes that admit a covariantly constant null vector field. When pp-waves are mentioned in the literature one is usually referring to the *generalized* pp-waves, in contrast to the *vacuum* plane waves. Higher dimensional vacuum pp-waves are the Ricci flat Lorentzian spacetimes of Petrov type $N$ that admit a covariantly constant null vector field. Generalized pp-waves are not Ricci flat and these spacetimes have to be equipped with additional matter fields to satisfy Einstein's equation, but the generalized pp-waves are also of Petrov type $N$ and admit a covariantly constant null vector. Petrov type $N$ refers to an algebraic classification of the Weyl tensor as illustrated in section 3.3). It describes the possible algebraic symmetries of the Weyl tensor at each event in a spacetime manifold. In the remainder of this thesis I will make no particular distinction between (vacuum) pp-waves and generalized pp-waves and use the name pp-waves for both.

We will write the metric for the $D$-dimensional pp-wave in Brinkmann coordinates as

$$ds^2 = -2dudv + \sum_{i=1}^{d} dx^i dx^i - F(u, x^i)du^2, \tag{9.2}$$

where the wave profile $F(u, x^i)$ only depends on the lightcone time $u$ and the transverse coordinates $x^i$. The index $i$ runs over the $d = D - 2$ transverse coordinates. The Riemann tensor can be easily characterized in terms of the wave profile. The only nonzero term is

$$R_{uau}{}^{b} = \partial_a \partial_b F(u, x^i)/2, \tag{9.3}$$

and those related by symmetries like $R_{uab}{}^{v}$. The Ricci tensor is determined by the single component

$$R_{uu} = \sum_{a=1}^{d} \partial_a^2 F(u, x^i)/2. \tag{9.4}$$

So in vacuum $F(u, x^i)$ is a solution to the (transverse) Laplace equation.

# 9.2 Plane waves

## 9.2.1 Brinkmann coordinates

Plane waves[1] are pp-waves that have a wave profile $F(u, x^i)$ that depends quadratically on the transverse coordinates:

$$F(u, x^i) = f_{ab}(u) x^a x^b \,. \tag{9.5}$$

Here we write the metric

$$ds^2 = -2dudv - f_{ab}(u) x^a x^b du^2 + \delta_{ab} dx^a dx^b \tag{9.6}$$

in Brinkmann coordinates, which are global coordinates. Any linear dependence of $F(u, x^i)$ on $x^i$ can be removed by a coordinate transformation. $f_{ab}$ is sometimes called the plane wave profile matrix. The Ricci tensor is then immediately given by the trace of the plane wave profile matrix $R_{uu} = f_{ii}(u)$. In vacuum these plane waves have a traceless profile matrix and they have the same amount of degrees of freedom as a graviton.

To anticipate the appearance of singular plane waves in chapter 13, we will briefly investigate the geodesics related to the spacetime (9.6), which are given by the equations,

$$\ddot{u} = 0, \tag{9.7}$$

$$\ddot{x}^i + f_{ai}(u) x^a \dot{u}^2 = 0, \tag{9.8}$$

$$\ddot{v} + \frac{1}{2} \frac{\partial f_{ab}(u)}{\partial u} x^a x^b \dot{u}^2 + 2 f_{ab}(u) x^a \dot{x}^b \dot{u} = 0, \tag{9.9}$$

where the dot denotes the derivative with respect to the affine parameter $\tau$. In case one of the functions $f_{ab}(u)$ diverges at $u = 0$ there is a singularity because timelike geodesics have $u = p^u \tau$ $(p \neq 0)$ and reach the origin $u = 0$ in finite proper time. Furthermore, in the case of a diverging $f_{ab}(u)$ all timelike geodesics will be incomplete, so the singularity in such a singular plane wave

[1] These are also sometimes called “exact” plane waves, e.g. in [60].

will be analogous to a cosmological singularity. The geodesic deviation equation yields,

$$\ddot{y}^i + (p^u)^2 f_{ij} y^j = 0, \tag{9.10}$$

with $y^i$ the transverse separation between nearby geodesics that have the same momentum $p^u$. For a vacuum plane wave the trace $f_{aa} = 0$ and the tidal forces will be attractive in some directions, and repulsive in some others.

Brinkmann coordinates have a few special properties [73]. They provide a measure of the invariant geodesic distance in spacetime with respect to the null geodesic $u = p^u \tau$ (in that respect they are reminiscent of Riemann coordinates). Another property of Brinkmann coordinates is that the curvature is directly related to the profile of the wave.

### 9.2.2 Rosen coordinates

Another useful coordinate system is given by Rosen coordinates, in which the metric only depends on the lightcone time

$$ds^2 = -2y^+ y^- + \mu_{ij}(y^+) dx^i dx^j, \tag{9.11}$$

and it is manifest that each spacetime that is conformal to another plane wave with the conformal factor only dependeng on lightcone time $x^+$, is also a plane wave. However, Rosen coordinates suffer from coordinate singularities (e.g. the metric is not invertible when $\mu_{ij} = 0$ for a certain value of $x^+$).

The coordinate transformation between Brinkmann and Rosen coordinates for general wave profile can be expressed most easily in terms of a vielbein field $e_i^a$ (see section B.4 where "i" takes the role of curved index and "a" the role of Lorentz index) that only depends on lightcone time and that is restricted to the directions transverse to the wave propagation direction. The vielbein field satisfies (with $\delta_{ab}$ instead of the Minkowski tensor because of the restriction to the transverse directions)

$$g_{ij} = e_i^a e_j^b \delta_{ab}. \tag{9.12}$$

The coordinate transformation between Brinkmann and Rosen coordinates is then given by

$$y^+ = u\,, \quad , y^- = v - \frac{1}{2} \dot{e}_{ai} e_b^i x^a x^b\,, \quad y^i = e_a^i x^a, \tag{9.13}$$

and $e_b^i$ is the inverse vielbein. To exclude that $dy^+ dy^i$ terms would appear in the metric, the vielbein has to be chosen such that it satisfies the symmetry condition $\dot{e}_{ai} e_b^i = \dot{e}_{bi} e_a^i$. The plane wave profile is now given by

$$f_{ab} = -\ddot{e}_{ai}(u) e_b^i(u). \tag{9.14}$$

In the special case of a diagonal plane wave the coordinate transformation is simply given by

$$\begin{cases} u & = y^+ \\ v & = y^- + \frac{1}{4}\sum_i \dot{\mu}_i (y^i)^2, \\ x^a & = \sum_i \sqrt{\mu_{ii}} y^i \, \delta_i^a, \end{cases} \tag{9.15}$$

and the plane wave profile matrix $f_{ab}$ is related to the polarization matrix $\mu_{ij}$ by

$$f_{ab}(u) = -\frac{1}{\sqrt{\mu_{aa}}} \frac{\partial^2 \sqrt{\mu_{aa}}}{\partial y^+}\bigg|_{y^+=u} \delta_{ab}. \tag{9.16}$$

### 9.2.3 Homogeneous plane waves

I have already mentioned the Killing vector $k$, but an arbitrary plane wave with profile $f_{ab}$ possesses additional Killing vectors (see e.g. [66, 67]). In case the plane wave possesses another *null* Killing vector in addition to $k = \partial_v$ that generates the lightcone-time translations, then the plane wave is called homogeneous. All homogeneous plane waves were given in [67]. An important time-dependent class of homogeneous plane waves are the scale-invariant plane waves that have a specific dependence of the plane wave profile on the lightcone time, given by

$$f_{ab}(u) = \frac{\Lambda_{ab}}{u^2}. \tag{9.17}$$

These scale-invariant plane waves possess the additional null Killing vector

$$\ell = u\partial_u - v\partial_v \tag{9.18}$$

that is related to the boost isometry $(u, v, x^i) \to (Cu, v/C, x^i)$. After diagonalisation of the profile, the scale-invariant plane wave can be written in Rosen coordinates as

$$ds^2 = -2y^+y^- + (y^+)^{2m_i} dx^i dx^i, \tag{9.19}$$

with $m_i$ the “Kasner exponents”. In Brinkmann coordinates,

$$ds^2 = -2dudv - \sum_a \lambda_a \frac{x^a x^a}{u^2} du^2 + \delta_{ab} dx^a dx^b, \tag{9.20}$$

these Kasner exponents are related to the normalizations $\lambda_a$ of the eigenvalues of the matrix $\Lambda_{ab}$ by,

$$\lambda_a = m_a(1 - m_a), \tag{9.21}$$

and it is clear that the transformation $m_i \to 1 - m_i$ of the Kasner exponents leaves the metric invariant. In the specific case of an isotropic plane wave profile that we will study in chapter 13, we will describe $\lambda$ as the “overall normalization" of the plane wave profile. The scale-invariant plane wave has a singularity at $u = 0$ which is at finite distance (the lightcone time $u$ serves as affine parameter). In principle we should only consider the spacetime from $u = ]0, +\infty[$, but in our study of transition of fields through singularities we will extend the spacetime to the coordinate range $u = ]-\infty, +\infty[$. In this case, we should interprete the full spacetime as a spacetime with a bounce singularity: the spacetime first contracts to a big crunch singularity and then expands from a big bang singularity (at least for positive $\lambda$). In addition, out of general considerations we will assume that it is possible to have a different wave profile before the singularity compared to after the singularity. In chapter 13 we will denote this difference as $\lambda k_-$ before the singularity and $\lambda k_+$ after the singularity, instead of the previous $\lambda$.

### 9.2.4 Plane waves from Penrose limits

Roger Penrose observed in 1976 that, near a null geodesic, every Lorentzian spacetime locally looks like a plane wave [57]. A certain procedure, called the Penrose limit, puts this phrase in a mathematical form and associates a plane wave geometry to a null geodesic of a certain spacetime. For a modern exposition that gives a covariant characterisation of the Penrose limit I refer the reader to [69]. Their results show that the plane wave profile matrix $f_{ab}$ of (9.6) is directly related to the curvature of the original spacetime along the null geodesic through

$$f_{ab}(u) = \tilde{R}_{aubu}|_{\tilde{\gamma}(u)} \tag{9.22}$$

Here $\tilde{\gamma}(u)$ represents the null geodesic along which the Penrose limit is taken, and the $\tilde{R}_{aubu}$ components of the Riemann tensor of the original spacetime are evaluated in a parallel-transported frame along the null geodesic.

We will focus on Penrose limits of singular spacetimes that are taken along a null geodesic that hits the singularity. In this case, we will obtain singular plane waves. Thus plane waves arise naturally in study of cosmological singularities. Furthermore, singular plane waves are not simply a toy model for a singularity. Through the Penrose limit the Brinkmann profile matrix contains the components of the Riemann tensor along the null geodesic that hits the singularity. An important feature of certain cosmological singularities are diverging tidal forces. The tidal forces are encoded in the Riemann tensor, and lead to the deviation of nearby geodesics by equation (3.8). Because the singular plane wave profile contains all the information about the Riemann tensor along the geodesic that hits the singularity, it contains the essential information of the geodesic deviation equation around that geodesic. Therefore the Penrose limit captures essential features of the original singularity, promoting the singular plane wave to a real first approximation for a spacetime singularity. As an approximation, singular plane waves are simpler to study than the original singular spacetime, essentially because one only focuses on the spacetime structure near one geodesic.

In [70] it was proven that the Penrose limits of metrics with singularities of an arbitrary power-law type, show a universal scale-invariant $1/u^2$ behaviour near the singularity in the plane wave, under the condition that the dominant energy condition [8, 21] is satisfied and not saturated. This means that the energy-momentum tensor $T_{\mu\nu}$ satisfies two conditions: for every timelike vector $v^\mu$ its contraction with the energy momentum tensor is strictly positive $T_{\mu\nu}v^\mu v^\nu > 0$, and the vector $T^\mu_\nu v^\nu$ is non-spacelike. In other words, for any observer the local energy density is positive and the energy flux is causal.

## 9.3 Exact backgrounds in string theory

The pp-waves are solutions to the Einstein equation in vacuum and therefore we may expect that they are related to certain string theory backgrounds. We recall that the background consistency conditions in string theory modify Einstein's equation at the classical level by adding an infinite number of higher order terms, which involve higher powers and derivatives of the curvature. The classical equation of motion for the metric in string theory (the background consistency condition) is the condition for conformal invariance of the sigma model on the two-dimensional string worldsheet,

$$R_{\mu\nu} + \frac{\alpha'}{2} R_{\mu\pi\rho\sigma} R_\nu{}^{\pi\rho\sigma} + \ldots = 0, \tag{9.23}$$

were I have written only the first curvature terms that are related to the metric and I have omitted other fields like the dilaton or the axion. If the curvature of the spacetime is small compared to the Planck scale, then the higher order corrections are small and a spacetime that is a solution to Einstein's equation in vacuum will be an approximative solution in string theory too.

However, due to their metrical structure, vacuum pp-waves guarantee that all the higher order terms (the $\alpha'$ corrections) are zero, so they are classical solutions of the background consistency conditions up to all order in the worldsheet sigma model perturbation theory [58, 59]. But also generalized pp-waves satisfy the background consistency conditions in string theory and as a consequence several generalized pp-wave solutions that are equipped with Ramond-Ramond fields or with a time-dependent dilaton have been constructred, all of which are exact classical solutions as string theory backgrounds, up to all orders in $\alpha'$. In the case of plane waves, where the dependence of the plane wave profile on the transverse coordinates is quadratic, the two-dimensional sigma model on the string worldsheet can be solved exactly [58].

In order to further elucidate the usefulness of pp-waves as backgrounds in string theory, we should comment on certain algebraic properties of spacetimes.

The first is the existence of a covariantly constant null vector field. In case the spacetime possesses such field, the lightcone gauge can be imposed on the worldsheet sigma model, which simplifies the formulation of string theory. Furthermore, such a field also permits the definition of a conserved frequency, implying that there is no particle nor string creation in the spacetime, as was considered in chapter 6. A less stringent condition, the existence of a lightlike Killing vector, is necessary for the formulation of matrix theories, which will be studied in chapter 11. In order for the spacetime background to preserve supersymmetry, it has to admit a Killing spinor. A Killing spinor implies the existence of a null or timelike Killing vector. More precisely, the derivation of a null or a timelike Killing vector from a Killing spinor has been proved for a number of supergravity theories and is widely believed to hold for all supergravity theories (see e.g. [74]). Therefore the existence of a lightlike Killing vector is a necessary (but not sufficient) condition to preserve some supersymmetry.

Another important algebraic property of a spacetime is when it has vanishing curvature invariants. This means that all the curvature scalars that can be constructed from the Riemann tensor and the metric are zero. Such spacetimes are important because the structure of the higher order terms of the background consistency condition for the metric simplifies considerably. Because spacetimes with vanishing curvature invariants are exact classical backgrounds in (bosonic) string theory, it is interesting to know which of these spacetimes preserve supersymmetry. In [74] it was proven that for spacetimes that have vanishing scalar curvature invariants, they cannot preserve supersymmetry if they do not possess a covariantly constant null vector. Thus exact classical solutions in string theory that preserve some supersymmetry must possess a covariantly constant null vector. In four dimensions all spacetimes that admit a covariantly constant null vector also have vanishing curvature invariants (see e.g. [65]). In higher dimensions there do exist spacetimes with a covariantly constant null vector, but which have non-vanishing curvature invariants nonetheless (see e.g. [71]). All higher-dimensional spacetimes that possess both vanishing curvature invariants and a covariantly constant null vector are given by [72]. These spacetimes are interesting backgrounds for string theory $\sigma$-models.

Plane waves also appear in the formulation of certain time-dependent generalizations of the AdS/CFT correspondence [75], but since this would distract us from the main topics of this thesis I will not discuss this any further.

# Chapter 10

# Resolution of singularities

*"Man who says it cannot be done*
*should not interrupt man doing it,"*

*Gábor Vattay (and probably many other persons)*

In this thesis, a geometrical resolution prescription is applied to investigate free field propagation across singularities. The geometrical resolution prescription involves the regularization of the singular spacetimes into a class of regular spacetimes with specified metric components. The regularized metric allows to define the field transition across the singularity and to obtain a solution for the field evolution (at least in principle). This is achieved by solving the field evolution on the class of regularized spacetimes. Finally the limit should be taken in which the metric on the regularized spacetimes reduces to the original singular metric, and we are interested in the singular limit of the solution for the (free) field propagation. If such a limiting solution exists, we will say it defines the field evolution on the singular spacetime according to the geometrical resolution prescription.

In this chapter we will also look at some comments about the nature of singularities that can appear at the classical level of string theory, and what approaches are used to resolve them. I will also give an overview of how we can study the transition of (free) fields through a singularity. In between I will introduce orbifolds, which is an important class of toy models for singularities in string theory. Some comments about the operator structure of Hamiltonians obtained by a geometrical resolution are mentioned in the final section of the chapter.

## 10.1 Geometrical regularization prescription

The geometrical resolution prescription consists of two steps: the regularization of a singular spacetime into a class of regular spacetimes, and the taking of the singular limit. We start with a singular spacetime

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{10.1}$$

with a metric $g_{\mu\nu}$ given in a certain coordinate system. We give an explicit expression for the "regularized metric components" $g_{\mu\nu}(\epsilon)$ in terms of $\epsilon$, such that they are regular for $\epsilon \neq 0$. A geometrical regularization of the original singular spacetime is then given by the class of regularized spacetimes (labeled by $\epsilon$) with the line element

$$ds^2(\epsilon) = g^{(\epsilon)}_{\mu\nu} dx^\mu dx^\nu \, . \tag{10.2}$$

The regularized spacetimes (10.2) are perfectly regular for $\epsilon \neq 0$. We demand that these spacetimes, possibly equipped with additional massless string fields, allow for consistent propagation of quantum strings. Therefore they have to satisfy the generalization of Einstein's equation in string theory, i.e. the background consistency conditions of chapter 7. The singular metric components in (10.1) have been replaced by regularized metric components in (10.2). Of course, in the singular limit the latter limit to the former:

$$\lim_{\epsilon \to 0} g^{(\epsilon)}_{\mu\nu} = g_{\mu\nu}. \tag{10.3}$$

The geometrical resolution prescription now prescribes that we derive the field evolution on the class of regularized spacetimes. Finally, we take the singular limit in which the metric on the class of regularized spacetimes limits to the metric on the singular spacetime. Notice that when we take the singular limit of the regularized metric components in (10.3), we make use of the coordinate system in which we defined $g_{\mu\nu}(\epsilon)$.

Because (10.2) represents a class of spacetimes one could perhaps wonder what its limit would be without specification of the coordinate system. However, in that case the limit of a class of spacetimes is not well-defined by the limiting value of $\epsilon$ only. In the next section we will comment on a coordinate-free treatment of the limits of spacetimes. But this problem has no influence on our geometrical resolution prescription outlined above.

We are considering the limit of the class of spacetimes (10.2) in the coordinate system in which we proposed the explicit expressions for the $g_{\mu\nu}(\epsilon)$. Our procedure is well-defined in as much as the coordinate system in which we defined the $g_{\mu\nu}(\epsilon)$ is well-defined. For example, in chapter 13 we construct a class of regularized plane waves, for which we use the globally well-behaved Brinkmann coordinates. The Brinkmann coordinate system completely covers the spacetime manifold [55].

## 10.2 Limits of spacetimes

Without further specification "the limit of a spacetime" is not a well-defined concept. This issue has already been raised by Geroch in [76]. Let us consider the Schwarzschild spacetime as an example. With respect to the metric of (3.30) there is no limit of the spacetime when $M \to \infty$. However, let us apply the following $M$-dependent coordinate transformation:

$$\hat{r} = M^{-1/3} r\,, \quad \hat{t} = M^{1/3} t\,, \quad \hat{\psi} = M^{1/3}\theta\,. \tag{10.4}$$

The line element (3.30) now becomes

$$\begin{aligned} ds^2 = &- \left(\frac{1}{M^{2/3}} - \frac{2}{\hat{r}}\right) d\hat{t}^2 + \left(\frac{1}{M^{2/3}} - \frac{2}{\hat{r}}\right)^{-1} d\hat{r}^2 \\ &+ \hat{r}^2 \left(d\hat{\psi}^2 + M^{2/3}\sin^2(M^{-1/3}\hat{\psi}) d\phi^2\right)\,. \end{aligned} \tag{10.5}$$

In the coordinates (10.4) the limit of the Schwarzschild spacetime exists. The line element of the limiting spacetime becomes

$$ds^2 = \frac{2}{\hat{r}} d\hat{t}^2 - \frac{\hat{r}}{2} d\hat{r}^2 + \hat{r}^2 \left(d\hat{\psi}^2 + \hat{\psi}^2 d\phi^2\right)\,. \tag{10.6}$$

It is obtained by taking $M \to \infty$ in the metric components of (10.5).

So how to define a unique singular limit without specifying the coordinate system? In the case of the possible limits of the Schwarzschild spacetime, a coordinate independent description of the limits of a class of spacetimes was introduced by Paiva *et al* [81]. This coordinate independent description is based on the Karlhede procedure. The Karlhede procedure [77] consists of an algorithm to compute the Cartan scalars. The Cartan scalars are the set of frame components of the covariant derivatives of the Riemann tensor. These are scalars under coordinate transformations and contain all the local information about the metric (however, they do change under a change of tangent frame). The Karlhede procedure leads to a coordinate free characterization of spacetimes. By making use of the Karlhede procedure it is now possible to specify a unique limit of a spacetime in a coordinate-independent manner. The limiting procedure now consists of specifying the limiting value of $\epsilon$ *and* the limiting values for all of the (algebraically independent) Cartan scalars. The freedom of choosing different limiting values for the Cartan scalars reflects the ambiguity of the $\epsilon \to 0$ limit for the class of spacetimes $(\mathcal{M}_\epsilon, g_{\mu\nu}(\epsilon))$ when different coordinates are kept fixed in the limit. Of course, the singular limit can also be well-defined by specifying a particular coordinate system.

## 10.3 Singularities in string theory

For a review of time-dependent solutions and spacelike singularities in string theory, see e.g. [93] and references therein. Their focus is complementary to the geometrical resolutions procedure advocated in my thesis. For a discussion of models in string theory to investigate spacetime singularities see [111]. The research of singularities in string theory is focused on the resolution of lightlike singularities. Real cosmological singularities are spacelike, but they are much harder to investigate.

Let us come back to our definition of singularities in appendix A. Spacetimes can be called singular in general relativity when they are geodesically incomplete, meaning that classical test particles cannot evolve for an infinite time. In string theory, the situation is different. Because the center-of-mass motion of a classical string reduces to the null geodesic equation, any spacetime that is singular in general relativity will also be singular in string theory at the classical level. However, there do exist spacetimes that are singular in the sense of general relativity, but the propagation of a first-quantized string is nevertheless well defined (these spacetimes belong to a class called orbifolds, see the following section). Therefore we should rather consider quantized test strings in order to determine whether a spacetime in string theory is singular. In order to determine the singular nature of a spacetime, the usual first-quantized approach to string theory will be sufficient in time-independent spacetimes, but in general time-dependent backgrounds there will be string creation. An important exception are the spacetimes that possess a covariantly constant null vector like pp-waves, where the first-quantized approach will suffice (see the previous chapter).

In general, we should call a string theory solution singular if the expectation value of a certain physical observable associated to the test string diverges [60]. Because the propagation of a first-quantized string may involve other background fields in addition to the spacetime metric (e.g. dilaton or axion fields), it may be that a certain string theory solution is singular while the spacetime metric is nonsingular nonetheless. We do not consider such theoretical cases, because it is quite possible that these are essentially unphysical examples, unless these cases would be related by some kind of duality to a *real* singular spacetime with a singular metric. Also, in some cases singular spacetimes are related to smooth solutions under dualities.

From the perspective of resolving spacetime singularities in general relativity by means of string theory, the classification of a string theory solution as singular or non-singular can still be quite subtle, as I will illustrate here with an example. If we consider flat spacetime to which we add a linear dilaton field, then we have an exact solution to the string equations of motion [79]. In Einstein frame, the spacetime corresponds to an expanding Robertson-Walker solution with a singularity [80]. However, it might seem that this would provide a regular solution in string theory, e.g. [59]. Nevertheless, even in string frame the solution is singular because the string coupling blows up [107]. I will discuss this model in more detail in section 11.6.

## 10.4 Orbifolds

In a purely mathematical context an orbifold (from "orbit-manifold") is a generalization of a manifold ; it allows the presence of points whose neighbourhood is diffeomorphic to a quotient of $\mathbb{R}^n$ by a finite group, i.e. $\mathbb{R}^n/G$. In a physical context like in string theory, the name orbifold usually describes an object that can be globally written as an orbit space $M/G$ where $M$ is a manifold, and $G$ is a group of some its isometries and/or discrete symmetries. In an orbit space $M/G$ (also called a quotient space or coset space) all the elements of $G$ are identified with the identity element.

An example of a quotient space is the periodic segment $\mathbb{R}/T(R)$ where $T(R)$ is a translation that maps $x \in \mathbb{R}$ into $x + 2\pi R$, also written as,

$$x \sim x + 2\pi nR\,, \quad n \in \mathbb{Z}. \tag{10.7}$$

The integer $n$ is usually not written out explicitly when it appears linearly, but rather implicitly understood in the symbol "$\sim$" that signifies equality under the identification group. Still, I have chosen to write the integer $n$ explicitly, in order to stress the similarity between the general formula and more complicated realisations that involve higher powers of $n$, as in formula (10.13). We have already encountered this simple example of compactification along a circle in section 8.2. Actually, in this simple example the orbifold produces a regular compactified space (the real line is mapped onto a circle of radius $R$) that is invariant under discrete coordinate shifts by $2\pi R$ (which are in this example the elements of the discrete group $G$), and all physical observables should also be invariant under these shifts.

If the group $G$ has fixed points, then the orbifold $M/G$ will have singular points. We will be interested in such orbifolds with singular points because they provide a toy-model for a singular spacetime. In the case of a so-called "conical" singularity, the curvature is not defined at the singular point, and geodesics end after a finite eigentime. The singularities appear, for example, when in a spacetime manifold points are identified by combining a periodic identification and a reflection. Let us specify a coordinate $X^r$, and consider the identification of points under the reflection

$$X^r \sim -X^r. \tag{10.8}$$

If such a reflection is simultaneously combined with a periodic identification,

$$X^r \sim X^r + 2\pi nR, \quad n \in \mathbb{Z}, \tag{10.9}$$

then we obtain a singular spacetime, which possesses a compactified direction $X^r$, with the fundamental region $0 \leq X^r \leq \pi R$, and both $X^r = 0$ and $X^r = \pi R$ are fixed points under the orbifold group. In the remainder of this thesis, I will use the name orbifold for these singular orbit-manifolds.

Orbifolds provide toy-models of spacetime singularities in string theory. The conical singularity of an orbifold is sometimes viewed as a "mild" singularity because it can be removed by going to the covering space of the orbifold. Because of the identification under a reflection, orbifolds lead to the appearance of "twisted closed strings". Loosely speaking, these twisted strings are wound around the conical singularity. Timelike orbifold singularities are resolved in string theory precisely because of the twisted states. Although these orbifolds are geodesically incomplete, it has been shown [78] that the propagation of a first-quantized string is well defined. There is no such resolution for point particles. On the other hand, for lightlike orbifold singularities string theory is not well-behaved on the orbifold because of backreaction effects [87]. One example of a lightlike singularity is the parabolic orbifold, which I will discuss into more detail because of its importance in chapter 12.

### 10.4.1 Parabolic orbifold

In the literature the parabolic orbifold is also known as the null orbifold. In Rosen coordinates, the metric on the parabolic orbifold [61, 84] is given by

$$ds^2 = -2dy^+dy^- + (y^+)^2dy^2\,, \qquad (10.10)$$

where the following identification is understood

$$\begin{pmatrix} y^+ \\ y^- \\ y \end{pmatrix} \sim \begin{pmatrix} y^+ \\ y^- \\ y + 2\pi n \end{pmatrix}, \qquad n \in \mathbb{Z}\,. \qquad (10.11)$$

There is no reflection identification as in the example (10.8), but because of the combination of the periodic identification in the $y$ direction and the metric coefficient $g_{yy} = (y^+)^2$, the parabolic orbifold is singular nevertheless.

The parabolic orbifold is in fact simply (three-dimensional) Minkowski spacetime where a lightlike identification has been made. That the metric becomes Minkowski spacetime (albeit compactified) written in lightcone coordinates $ds^2 = -2dx^+dx^- + dx^2$, can be seen by making a coordinate transformation

$$x^+ = y^+\,, \quad x^- = y^- - \frac{1}{2}y^+y^2\,, \quad x = y^+y. \qquad (10.12)$$

In the Minkowski lightcone-coordinates the idenfication for Rosen coordinates

(10.11) becomes

$$\begin{pmatrix} x^+ \\ x \\ x^- \end{pmatrix} \sim \exp(2\pi n \mathcal{J}) \begin{pmatrix} x^+ \\ x \\ x^- \end{pmatrix}, \quad n \in \mathbb{Z}, \quad \mathcal{J} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{10.13}$$

In fact, the identification group for the parabolic orbifold is a lightlike Lorentz boost generated by the Killing vector $x^+\partial_x + x\partial_{x^-}$. The Lorentz group in $1+2$ dimensions has three classes:

- A Killing vector from the elliptic class (spacelike rotations) generates the $\mathbb{Z}_n$ orbifolds.
- A Killing vector from the hyperbolic class (a Lorentz boost) generates the boost orbifold (also called Misner space in [93]). The three-dimensional boost orbifold is related to the Milne spacetime, which is defined as two-dimensional Minkowski spacetime under a boost identification. The Milne spacetime has four regions: the future and past cones and two "whisker" regions with closed timelike curves. I do not give the metric here because it will appear shortly in equation (10.25). We will also consider the future cone of the Milne orbifold in section 11.6 in the context of the "matrix big bang" toy model.
- A Killing vector from the parabolic class generates the parabolic orbifold. The parabolic orbifold is the only orbifold that does not break supersymmetry.

For $y^+ \to 0$ there is a singularity in the Rosen metric (10.10), which is not a coordinate artifact because of the identification (10.11). Indeed, for $x^+ = 0$, all the points $x^-$ and $x$ are identified with $x^- = 0$ and $x = 0$ respectively, so the parabolic orbifold has a (lightlike) singularity occurring at zero lightcone time $x^+ = 0$, reminiscent of a "big crunch" followed by a "big bang". The parabolic orbifold can be visualized as two cones (parametrized by $y$ and $y^+$) with a common tip at $y^+ = 0$, times the real line labeled by $y^-$. $y$ plays the role of angular variable of the cones and $y^+$ of radial coordinate, in addition to its role as time variable.

Although the parabolic orbifold has Minkowski spacetime as its covering space, and Minkowski spacetime is certainly regular, the covering space should by no means be confused with a geometrical regularization of the parabolic orbifold like the nullbrane, discussed in the following section.

### 10.4.2 Nullbrane

The nullbrane spacetime was originally introduced in [82] and can be considered as a geometrical regularization of the parabolic orbifold (actually the review [93] that I referred to before also uses the name "nullbrane" to specify the parabolic orbifold itself). The nullbrane was studied in the context of perturbative string theory in [86, 88]; a matrix theory description was provided in [109].

We will consider four-dimensional Minkowski spacetime in lightcone coordinates $ds^2 = -2dx^+dx^- + dx^2 + dz^2$. The nullbrane is obtained by identifying

$$\begin{pmatrix} x^+ \\ x \\ x^- \\ z \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2\pi n & 1 & 0 & 0 \\ 2\pi^2 n^2 & 2\pi n & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^+ \\ x \\ x^- \\ z \end{pmatrix} + 2\pi n \begin{pmatrix} 0 \\ 0 \\ 0 \\ R \end{pmatrix}, \quad n \in \mathbb{Z}. \tag{10.14}$$

In the $R \to 0$ limit the nullbrane reduces to the parabolic orbifold times the real line labeled by $z$. In this sense, the nullbrane is a geometrical regularization of the parabolic orbifold. In the $R \to \infty$ limit the nullbrane reduces to Minkowski space.

In [86], the nullbrane geometry was discussed in two coordinate systems (to facilitate comparison we use their notation for the coordinates),

$$ds^2 = -2dy^+dy^- + du^2 + (R^2 + (y^+)^2)dy^2 + 2Rdydu; \tag{10.15}$$

$$ds^2 = -2d\tilde{x}^+d\tilde{x}^- + d\tilde{x}^2 + (R^2 + \tilde{x}^2)d\theta^2 + 2(\tilde{x}^+d\tilde{x} - \tilde{x}d\tilde{x}^+)d\theta, \tag{10.16}$$

related to Minkowski coordinates by

$$x^+ = y^+,\ x = yy^+,\ x^- = y^- + \tfrac{1}{2}y^+y^2,\ u = z - Ry; \tag{10.17}$$

$$x^+ = \tilde{x}^+,\ x = \tilde{x} + \theta\tilde{x}^+,\ x^- = \tilde{x}^- + \theta\tilde{x} + \tfrac{1}{2}\theta^2\tilde{x}^+,\ \theta = \tfrac{z}{R}. \tag{10.18}$$

Unfortunately, neither coordinate system is fully satisfactory for studying the $R \to 0$ limit of dynamics on the nullbrane. To study the singular limit, we would like to phrase the dynamics in terms of a Hamiltonian (see expression (10.39)) that has the structure $H = \sum_i f_i(t,R)H_i$ in which $f_i(t,R)$ is regular in $t$ for $R \neq 0$ and regular away from $t = 0$ for $R = 0$. The terms that appear in the Hamiltonian can be easily deduced from the inverse metric. On the one hand, the $y$-coordinates are not globally defined since they are singular at $y^+ = 0$ for any $R$. On the other hand, the $\tilde{x}$-coordinates, which are nonsingular for $R \neq 0$, do not have an $R \to 0$ limit even away from the parabolic orbifold singularity, as the determinant of the metric is $-R^2$ everywhere.

Therefore, in our publication [95] we introduced new coordinates that interpolate between the $\tilde{x}$-coordinates (for small $\tilde{x}^+$) and the $y$-coordinates (for large $y^+$):

$$\begin{aligned}
X^+ &= \tilde{x}^+ = y^+; && (10.19)\\
X^- &= \tilde{x}^- - \frac{1}{2}\frac{\tilde{x}}{\tilde{x}^+}\left(1 - \frac{R^4}{\left(R^2 + (\tilde{x}^+)^2\right)^2}\right) = y^- + \frac{R^2}{2}\frac{y^+ u^2}{\left(R^2 + (y^+)^2\right)^2};\\
X &= -\frac{R\tilde{x}}{\sqrt{R^2 + (\tilde{x}^+)^2}} = \frac{y^+ u}{\sqrt{R^2 + (y^+)^2}};\\
\Theta &= \theta + \frac{\tilde{x}}{\tilde{x}^+}\left(1 - \frac{R^2}{R^2 + (\tilde{x}^+)^2}\right) = y + \frac{Ru}{R^2 + (y^+)^2}.
\end{aligned}$$

In this coordinate system, the metric has determinant

$$\det\left[g_{\mu\nu}\right] = -(R^2 + (X^+)^2), \tag{10.20}$$

which is regular for all $X^+$. Despite the capital notation, the spacetime coordinates should not be confused with string coordinates. We will use this coordinate system in chapter 12 to derive the evolution of a scalar field (hence, confusion with string coordinates is unlikely).

### 10.4.3 The generalized nullbrane

The metric of the nullbrane spacetime, written in our new coordinates, can be naturally generalized to a two-parameter family of metrics, which we will call "generalized nullbrane". The family is labeled by two parameters $\alpha$ and $\beta$, and the original nullbrane corresponds to $\alpha = 3, \beta = 2$:

$$\begin{aligned}
ds^2 =& \frac{R^2 X^2\left(\beta^2 - \alpha\right)}{\left(R^2 + (X^+)^2\right)^2}\left(dX^+\right)^2 - 2dX^+dX^- + \frac{2\beta R X}{\sqrt{R^2 + (X^+)^2}}dX^+ d\Theta\\
&+ \left(R^2 + (X^+)^2\right)d\Theta^2 + dX^2.
\end{aligned} \tag{10.21}$$

For the reader's future convenience, the inverse metric has the following components:

$$g^{\mu\nu} = \begin{bmatrix} 0 & -1 & 0 & 0\\ -1 & \dfrac{\alpha R^2 X^2}{\left(R^2 + (X^+)^2\right)^2} & \dfrac{\beta R X}{\left(R^2 + (X^+)^2\right)^{3/2}} & 0\\ 0 & \dfrac{\beta R X}{\left(R^2 + (X^+)^2\right)^{3/2}} & \dfrac{1}{R^2 + (X^+)^2} & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{10.22}$$

with $X^\mu \in [X^+, X^-, \Theta, X]$.

As a matter of fact, the generalized nullbrane is only a solution of Einstein's equation in vacuum when

$$\alpha = 1 + \beta^2/2\,. \tag{10.23}$$

Therefore, unless we could provide a reasonable physical explanation that justifies the nonzero value of the Ricci tensor of the generalized nullbrane

$$R_{X^+X^+} = -\frac{R^2\left(2+\beta^2-2\alpha\right)}{2\left(R^2+(X^+)^2\right)^2}\,, \tag{10.24}$$

it would seem more appropriate to only consider the one-parameter subfamily of the generalized nullbrane determined by (10.23). Nevertheless, in chapter 12 where we consider the geometrical resolution of a scalar field on the parabolic orbifold we will keep the values of both parameters $\alpha$ and $\beta$ arbitrary and consider the whole parameter space of the generalized nullbrane. This allows us to discover an interesting phenomenon: if the singular limit of the scalar field (propagating on the generalized nullbrane specified by the coordinates (10.21)) is to exist, the parameters $\alpha$ and $\beta$ are forced to be discrete. In fact, the only discrete value that corresponds to a vacuum solution is precisely the original nullbrane.

## 10.5 Transition through singularities

The presence of singularities in the spacetime that describes our universe, and the proposal of cosmological models that possess a bounce, where (part of) the universe collapses into a singularity and then re-expands, raises the question how transition of matter through the singularity should be described at the theoretical level. String theory, as a unified formalism of matter and gravitation, is a natural framework to investigate these issues. According to string theory, matter particles are contained in the excitation spectrum of strings, but geometry is contained as well (through gravitons), and we may expect that the issue of matter propagating through a singularity in the spacetime geometry should be describable in string theory.

Nevertheless, defining dynamical transitions through spacetime singularities entails a very large amount of ambiguity, both technically and conceptually, and a deeper and more systematic understanding of gravitational physics is needed in order to address the issue with full legitimacy. Nevertheless, even in the absence of such understanding, it appears desirable to explore the range of possibilities presented by the problem of evolution across singularities.

In the context of string theory, holography can be understood as a duality between a gravitational theory and a quantum field theory. For details about holography, I refer the reader to the review [83]. In several string theory approaches, holography maps the study of cosmological singularities to the study of quantum mechanics or quantum field theory with certain singular features. For example, the couplings may develop a singularity as a function of time, or the quantum field theory may be defined on a singular spacetime background. This is the case in the "matrix big bang model" of [107, 110] where the gravitational phenomena near the singularity are described by a quantum field theory of matrices on (the future cone of) the compactified Milne spacetime. To be able to illustrate this model, we first have to introduce certain aspects of matrix theory, so we will consider this in more detail in the next chapter. A similar model studied in [109] involves quantum field theory of matrices on the nullbrane spacetime and its singular limit, the parabolic orbifold. Another model [112] involves quantum field theory of matrices on singular homogeneous plane waves. The cosmological models presented in [108] are quantum mechanical models of matrices instead. All these related approaches of field theory or quantum mechanics of matrices are collectively called "matrix models" (though the term matrix models can also be used more widely than in the matrix theory context described here).

An important question is what happens in these matrix models when the background spacetime on which they propagate, develops a singularity. In such a case, a geometrical resolution of the field evolution can be called an appropriate regularization prescription, at least if we maintain a classical view on the background spacetime. In the classical picture the singular spacetime background does permit a geometrical interpretation and we can, at least in principle, relate the singular metric to a class of regular metrics. The geometrical resolution of the background spacetime will then translate in a resolution of the (time-dependent) couplings of the matrix model. As mentioned above, in some cases the singular time-dependences of the couplings can be absorbed into the worldsheet metric. Instead of regularizing the original singular spacetime metric one could then also opt to regularize the worldsheet metric more directly, for example by devising a kind of geometrical regularization of the singular worldsheet metric. Nevertheless, the (singular) worldsheet metric is an auxiliary concept, so a geometrical resolution cannot be advocated in the same manner as in the case of a singularity of a real spacetime. The nontrivial worldsheet metrics appearing in some matrix models should not be considered as a real spacetime background with a geometrical interpretation. Still, for the both kinds of matrix models described above, be it field theory defined on a singular background or field theory with a singular coupling, the singular behaviour leads to singular Hamiltonians. While certain subtle questions related to the large $N$ limit ($N$ being the size of the matrices) have not been fully addressed yet, these matrix models clearly motivate the study of field theory with (near-)singular Hamiltonians.

As an initial step in the study of singularities by means of a geometrical resolution prescription, I have investigated if a (free) scalar field or a (free) string can propagate through the singularity. The spacetime metric affects the evolution of the field, but in principle the field will "backreact" and influence the spacetime metric through its energy-momentum tensor. However, this is in the case we are considering a gravitational theory such as string theory on a singular spacetime background, or if we would consider the propagation a scalar field on a *real* spacetime. In that case we will consider the "probe limit approximation", in which we neglect the backreaction of the field on the metric. In the case we are considering a *holographically dual* quantum field theory on an *auxiliary* singular spacetime, it is understood that the duality has translated the gravitational phenomena related to the singularity of the *real* spacetime in a *dual* field theory language on an *auxiliary* spacetime, at least for some of the matrix models described in the previous paragraph (other holographic models may map gravitational theories onto field theories defined on the boundary of the real spacetime). The formulation of the dual field theory on the fixed auxiliary background includes all the backreaction effects in principle. How all the complicated physical effects near the singularity are captured by the dual field theory in practice, is a major open task that will demand substantial research in the future.

In chapter 13 we study the evolution of a scalar field on a singular spacetime. On the one hand, this could be viewed as the propagation of a *real* field on a *real* spacetime with a singularity in the probe limit where we neglect backreaction. On the other hand, we can also look upon this model as a dual formulation of the gravity. From the latter point of view the gravitational physics has been translated into a *dual* field theory, defined on an *auxiliary* spacetime with a singularity, where we should not consider *auxiliary* backreaction effects of dual fields. We are not dealing anymore with a *direct* formulation of gravity where the spacetime structure is coupled to the energy-momentum tensor of the fields, but with a holographic formulation of gravity, where the dual fields are propagating on a fixed auxiliary background, and the *real* backreaction is already encoded in the evolution of the dual fields. Truth be told, we have simplified the problem considerably by investigating a free scalar field instead of the much more complicated field theories that arise in dual models for singularities (e.g. [107]). Therefore our investigations in chapter 12 don't permit us to derive immediate conclusions about the gravitational physics of a specific dual toy-model for a singularity, but the aim of that chapter is rather to investigate the use of a geometrical resolution to derive field evolution across a singularity in a particular example.

Another reason why we have investigated the evolution of a field across the singular parabolic orbifold in chapter 13, is that it allows us to compare with the literature: to define the evolution of the free scalar field we derive mode functions that also appear in the formulation of string theory. The parabolic orbifold [84], the nullbrane [86] as well as the compactified Milne spacetime have been studied as (a part of) backgrounds of gravitational theories, in particular as (a part of) string theory spacetimes. Most attempts to study the singular spacetimes among those just mentioned in string perturbation theory have failed because both the parabolic orbifold and the Milne orbifold exhibit divergences signaling large gravitational backreaction [84, 85, 87, 89]. With respect to a *real* spacetime singularity, the probe limit approximation is only valid if the field under consideration does not become too energetic (when there is a "blueshift" of field modes), which turned out to be problematic indeed for some of the earlier approaches listed above. With respect to the issue of backreaction, we will see in chapter 13 that the total excitation energy of a string propagating through a plane wave singularity becomes unbounded for a large subset of the plane wave profiles. In chapter 12 we will simply limit ourselves to the study of free field propagation and we neglect interesting questions related to possible non-gravitational backreaction because the main focus of the chapter is on the implementation of the geometrical resolution prescription to define the transition of a scalar field through the singularity, and we can view it as an approximation for the more complicated field theories that appear in the dual formulation of spacetime singularities. So in the model we will discuss in chapter 12, the singular spacetime hosts a holographically dual quantum field theory, *not* a gravitational theory as in the approach of [84, 86].

## 10.6 Comparison of resolution prescriptions

As already mentioned, prescribing propagation across a singularity is ambiguous and there are various ways to approach the issue. In this section I will compare the geometrical resolution prescription with a resolution prescription devised by Ben Craps and Oleg Evnin in a paper that preceded my collaboration with them, in order to better motivate the choice for a geometrical resolution. In [94] they noticed that evolution across spacelike or lightlike singularities appears to be often described by time-dependent Hamiltonians with an isolated singularity in their time dependence. They then exposed the most conservative way to define a unitary quantum evolution corresponding to such Hamiltonians by modifying the singular time dependences to become distributions while keeping the operator structure of the Hamiltonian unchanged. This approach is relevant when the transition through the singularity is dominated by a single term (single operator structure) in the Hamiltonian, and one can think of this way to define the transition through the singularity as a sort of "minimal subtraction". To clarify this procedure it is perhaps instructive to describe a specific case. A typical example would be the minimal subtraction procedure for a free scalar field propagating on the Milne orbifold. Then I will argue why the minimal subtraction prescription does not admit a geometrical interpretation. In the light of the importance of the parabolic orbifold in the chapter 12 I will also apply the "minimal subtraction" prescription of [94] to the parabolic orbifold.

### 10.6.1 Minimal substraction on the Milne orbifold

As mentioned above, in [94], Ben Craps and Oleg Evnin divised a kind of "minimal subtraction" prescription and described its particular implementation for the case of a free scalar field propagating on the compactified Milne universe. Here, I would like to argue that the approach of [94] does not lend itself to a geometrical interpretation, and, therefore, should we be interested in geometrical resolutions of space-time singularities, a more general framework is required.

To recapitulate briefly, the metric of the compactified Milne universe (we consider the past and the future cone) is

$$ds^2 = -dt^2 + t^2 dx^2, \qquad x \sim x + 2\pi, \tag{10.25}$$

and the corresponding free scalar field Hamiltonian is

$$H = \frac{1}{2|t|} \int dx \, \left(\pi_\phi^2 + {\phi'}^2\right) + \frac{m^2|t|}{2} \int dx \, \phi^2. \tag{10.26}$$

With this form of the Hamiltonian, the Schrödinger equation cannot be integrated through $t = 0$ on account of the singularity of $1/|t|$.

The idea of the "minimal subtraction" scheme of [94] is to keep the operator structure of the Hamiltonian unchanged and to modify the singular time dependences in (10.26) locally at $t = 0$ by subtracting terms proportional to (possibly) resolved $\delta$-functions and its derivatives such that the time dependences become well-defined in the sense of distributions[1]. Then, the Schrödinger equation can be integrated. Put differently, one can replace $1/|t|$ in (10.26) by its regulated version $f_{1/|t|}(t, \varepsilon)$ (with $\varepsilon$ being a regularization parameter), in such a way that, as $\varepsilon$ is taken to 0, $f_{1/|t|}(t, \varepsilon)$ converges to a distribution $\mathcal{F}_{1/|t|}(t)$, and this distribution $\mathcal{F}_{1/|t|}(t)$ equals $1/|t|$ everywhere away from $t = 0$. A possible choice is

$$f_{1/|t|}(t, \varepsilon) = \frac{1}{\sqrt{t^2 + \epsilon^2}} + 2\ln(\mu\epsilon)\frac{\epsilon}{\pi(t^2 + \epsilon^2)}, \tag{10.27}$$

with $\mu$ an arbitrary mass scale.

With this approach, one obtaines a regularized version of the Hamiltonian (10.26), namely

$$H = \frac{1}{2} \, f_{1/|t|}(t, \varepsilon) \int dx \, \left(\pi_\phi^2 + {\phi'}^2\right) + \cdots \tag{10.28}$$

[1] This procedure bears a strong formal resemblance to the conventional renormalization of local field theories by subtracting local counter-terms, and it can be thought of (see [94] for further discussion) as renormalizing the time dependence of (10.26).

such that, as $\varepsilon$ is taken to 0, the evolution away from $t = 0$ becomes identical to that arising from (10.26), and, furthermore, the system displays a well-defined (unitary) transition through $t = 0$.

### 10.6.2 The demand for a geometrical interpretation

The "minimal subtraction" procedure we have just briefly re-stated, is a consistent evolution prescription in itself. However, a direct inspection of (10.28) shows that the regularized version of our dynamics does not admit a geometrical interpretation (nor should one think of its singular limit, albeit well-defined, as being geometrical).

The problem with constructing a geometrical interpretation of (10.28) is that, since $f_{1/|t|}(t,\varepsilon)$ has an $\varepsilon \to 0$ limit as a distribution, the $\varepsilon \to 0$ limit of

$$\int_{-t_0}^{t_0} dt\, f_{1/|t|}(t,\varepsilon) \tag{10.29}$$

must exist. Furthermore, as stated above, the $\varepsilon \to 0$ limit of $f_{1/|t|}(t,\varepsilon)$ must equal $1/|t|$ everywhere away from $t = 0$. For that reason, in order for the limit of (10.29) to exist, $f_{1/|t|}(t,\varepsilon)$ should be very large and *negative* somewhere in the $\varepsilon$-neighborhood of $t = 0$ so that the positive divergence from integrating $1/|t|$ is compensated in (10.29). This is clearly apparent in figure 10.1. However, the coefficient of the kinetic term in the Hamiltonian of a field in a geometrical background comes from the square root of the determinant of the metric (and the coefficients of the inverse metric), and it needs to be *positive* (as is the function $1/|t|$ appearing in (10.26)).

Nevertheless, in a geometrical context as the propagation of a free field on a curved spacetime in the probe limit approximation, a geometrical interpretation of the singular limit may be desirable. In that case the spacetime is treated as a classical background, and precisely because of the clear geometrical meaning of the spacetime background, it is rather natural to define the transition across the singularity in a geometrical manner. One may want to resolve the singular geometry into a smooth space, and then try to take the singular limit in such a way that the dynamical evolution remains well-defined. It is non-trivial, because ad-hoc resolutions of a singular spacetime will generically not lead to a well-defined dynamics in the singular limit.

Because of the conflict between the minimal subtraction approach and a purely geometrical interpretation of the Hamiltonian, we generally need to relax the specifications of the minimal subtraction approach if we want to construct a geometrical resolution of dynamics on a singular spacetime background. That is, we should permit modifications in the operator structure of the Hamiltonian, as well as in its time dependence, in the vicinity of the singular region.

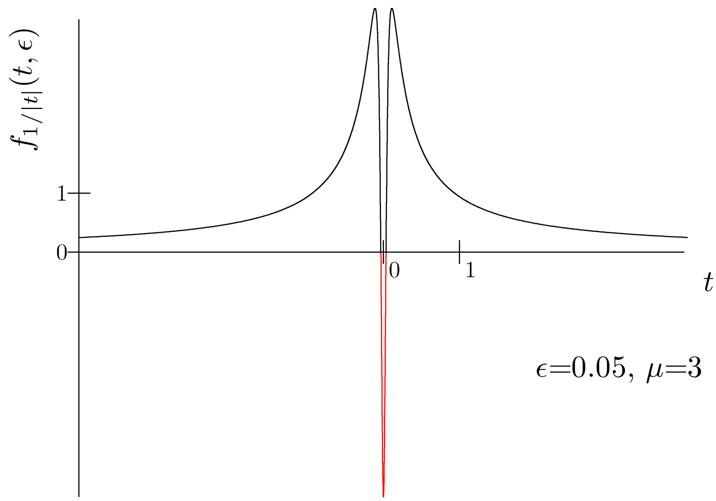



Figure 10.1: Negative contribution around $t = 0$ in regularized $f_{1/|t|}(t,\epsilon)$ (10.27)

We will then typically end up with a situation where a few different operator structures in the Hamiltonian essentially contribute to the transition to the singular region:

$$H(t) = \sum_i f_i(t,\varepsilon) H_i, \tag{10.30}$$

where $H_i$ are time-independent operators and $f_i$ are time-dependent number-valued functions. $\varepsilon$ is a regularization parameter, and the implication is that, as $\varepsilon$ is taken to 0, some of the $f_i$'s may develop isolated singularities at a certain value of $t$, which we choose to be $t = 0$. It is the commutation properties of those different terms in the Hamiltonian that are responsible for divergence cancellation (rather than explicit negative contributions introduced through the "minimal subtraction" scheme of [94]).

### 10.6.3 Minimal subtraction on the parabolic orbifold

In this section we will derive the propagation of a free scalar field across the singularity of the parabolic orbifold by means of a minimal subtraction procedure, which, as mentioned before, does not permit a geometrical interpretation. The readers who are interested to compare this procedure with the geometrical resolution prescription of the next chapter are invited to continue reading. If they prefer to concentrate on the main line of the thesis, it may be better to skip this subsection and to resume reading at the beginning section 10.7 where I will consider some dynamical aspects of Hamiltonians that exhibit a multiple operator structure.

We write the action for a massive scalar field in the parabolic orbifold using the metric (10.10):

$$S = \int \mathrm{d}y\,\mathrm{d}y^+\,\mathrm{d}y^-\,\mathrm{d}z\,|y^+|\left(\partial_+\phi\partial_-\phi - \frac{(\partial_y\phi)^2}{2\,(y^+)^2} - \frac{m^2\phi^2}{2}\right). \tag{10.31}$$

We decompose $\phi$ into Fourier modes along $y^-$ and $y$ (with the condition $\phi^*_{\ell,k_y} = \phi_{-\ell,-k_y}$):

$$\phi = \frac{1}{2\pi}\sum_{k_y}\int \mathrm{d}\ell\phi_{\ell,k_y}\exp(i\ell y^- + ik_y y). \tag{10.32}$$

Now the action can be rewritten as

$$S = \sum_{k_y}\int \mathrm{d}y^+\,\mathrm{d}\ell\,|y^+|\Big[i\ell\left(\phi_{\ell,k_y}\partial_+\phi^*_{\ell,k_y} - \phi^*_{\ell,k_y}\partial_+\phi_{\ell,k_y}\right) - \left(\frac{k_y^2}{(y^+)^2} + m^2\right)\phi_{\ell,k_y}\phi^*_{\ell,k_y}\Big] \tag{10.33}$$

The equations of motion are

$$2i\ell\partial_+\phi_{\ell,k_y} + \frac{i\ell\phi_{\ell,k_y}}{y^+} + \left(\frac{k_y^2}{(y^+)^2} + m^2\right)\phi_{\ell,k_y} = 0. \tag{10.34}$$

One can deal with the constraints due to the first order nature of the lightcone formalism by choosing $\tilde{\phi}_{\ell,k_y} = \sqrt{i\ell y^+}\phi_{\ell,k_y}$ as the canonical coordinate and $\tilde{\pi}_{\ell,k_y} = \sqrt{i\ell y^+}\phi^*_{\ell,k_y}$ as its conjugate momentum (for more details in a similar case, see appendix F). We obtain the Hamiltonian

$$H = \sum_{k_y}\int \mathrm{d}y^+\,d\ell\,\frac{1}{2i\ell}\left(\frac{k_y^2}{(y^+)^2} + m^2\right)\tilde{\pi}_{\ell,k_y}\tilde{\phi}_{\ell,k_y}. \tag{10.35}$$

We now apply the "minimal subtraction" scheme of [94] to the singular time ($y^+$) dependence in (10.35):

$$\frac{1}{(y^+)^2} \to \frac{(y^+)^2 - \epsilon^2}{((y^+)^2 + \epsilon^2)^2}. \tag{10.36}$$

(One could in principle add a (resolved) $\delta$-function with an arbitrary coefficient on the right hand side, but we will not make use of this freedom for the sake of brevity.) At this point we can clearly see the appearance of a negative contribution in the Hamiltonian near the singularity at $t = 0$ due to the $-\epsilon^2$ term in the numerator in (10.36).

To resume the derivation of the scalar field mode functions on the parabolic orbifold, the solution for $\tilde{\phi}$ reads:

$$\tilde{\phi}_{\ell,k_y} \propto \exp\left(-\frac{m^2}{2i\ell}y^+ + \frac{k_y^2}{2i\ell}\frac{y^+}{(y^+)^2+\epsilon^2}\right). \tag{10.37}$$

We can return to the original $\phi_{\ell,k_y}$ and write the scalar field mode functions as:

$$\psi_{\ell,k_y,m^2}(y^+, y^-, y) \propto \frac{1}{\sqrt{2i\ell y^+}}\exp\Big(-\frac{m^2}{2i\ell}y^+ + \frac{k_y^2}{2i\ell}\frac{y^+}{(y^+)^2+\epsilon^2} + ik_y y + i\ell y^-\Big). \tag{10.38}$$

In order to compare our mode functions of (10.38) obtained by a minimal subtraction prescription with the mode functions of [86], we identify $k_y = J$ and $p^+ = -p_- = -\ell$. Furthermore, the mode functions of [86] are derived for the parabolic orbifold times a line, whereas the mode functions (10.38) refer to the parabolic orbifold proper. To compensate for this difference, one should set the momentum along the extra line in [86] to zero (which amounts to imposing $n = J$ in the notation of that paper). Thereafter, the two sets of mode functions agree. The agreement is largely by coincidence. The mode function of [86] are derived via a geometrical regularization (the nullbrane), but they are written in the singular $y$-coordinates (10.15). The minimal subtraction mode functions are obtained through a regularization procedure that does not admit a geometrical interpretation.

## 10.7 Geometrically resolved Hamiltonians

In the absence of further physical specifications, the minimal subtraction procedure of [94] appears to be the most natural way to define evolution across singularities, because the distributions have no effect on the wavefunction away from the singularity. However, the minimal subtraction approach introduces negative contributions to the Hamiltonian at the singularity, apparent in (10.28) or (10.35). On the other hand, for the propagation of a free field, the coefficients of the operator terms in the Hamiltonian are related to the components of the inverse metric. The negative contributions to the Hamiltonian at the singularity is then conflict with the geometrical interpretation of the coefficients of the operator terms in the Hamiltonian [95]. Therefore, in order to maintain the positivity of the coefficients in the Hamiltonian related to the components of the inverse metric, a more geometrical approach to resolve the singularity appears to be desirable, which will be developed in the next chapter.

In preparation for our analysis of geometrical resolutions and their singular limits in chapter 12, we now review the quantum dynamics described by Hamiltonians of the form

$$H(t) = \sum_i f_i(t,\varepsilon)\mathcal{O}_i, . \tag{10.39}$$

Our ultimate question will be whether the $\varepsilon \to 0$ limit of the evolution operator corresponding to (10.39) exists. In chapter 13 we also consider a geometrical resolution, but because of the specific nature the singular metric (plane wave in Brinkmann coordinates) the considerations presented in this section regarding multi-operator Hamiltonians do not appear.

It is in general impossible to solve the Schrödinger equation corresponding to the Hamiltonian (10.39). The familiar symbolic solution for the evolution operator $U(t_1, t_2)$ involves the time-ordering symbol T:

$$U(t_1,t_2) = \mathrm{T}\left[-i\int\limits_{t_1}^{t_2} dt\, H(t)\right]. \tag{10.40}$$

The above representation can be further transformed in an instructive way using a technique known as the Magnus expansion. For possible convenience I discuss the Magnus expansion in more detail in appendix E. The operator $U$ belongs to the group of unitary operators on the Hilbert space, and the Magnus expansion can be thought of as an analog of the Baker-Campbell-Hausdorff formula (the latter is a formula valid in the context of finite-dimensional Lie groups, and discussed in many textbooks on group theory, for example, in [7]). The expansion can be symbolically written as:

$$U(t_1,t_2) = \exp\left[-i\int\limits_{t_1}^{t_2} dt\, H(t) + \eta_1\int dt\, dt'\,[H(t),H(t')] \right.$$
$$\left. + i\,\eta_2\int dt\, dt'\, dt''\,[H(t),[H(t'),H(t'')]] + \cdots\right], \tag{10.41}$$

with some numerical coefficients $\eta_1$, $\eta_2, \ldots$ (their values will not be important for us, and it appears they can only be derived recursively [140]). The key property of the above expression is that the higher order terms are entirely expressed through higher order nested commutators of $H(t)$ at different moments of time.

Even though, in a completely general setting, the Magnus expansion is hopelessly intractable, it displays the broad range of opportunities for divergence cancellation in a singular limit of the dynamics described by (10.39). Namely, for the case of (10.39), the Magnus expansion (10.41) will contain all kinds of combinations of the $f_i$ and their products, in such a way that, even if $f_i$ develop very strong singularities as $\varepsilon$ is taken to 0, the limit of $U(t_1,t_2)$ may still exist. For example, even if all $f_i$ are positive, cancellations may still occur on account of the commutation properties of $H_i$.

Should such cancellations take place, one may think of the $\varepsilon \to 0$ limit of (10.39) as an operator-valued generalization of conventional distributions: just as ordinary distributions may contain singularities in a way that permits evaluating ordinary integrals, the Hamiltonian (10.39) will contain singularities in a way that permits evaluating the time-ordered exponential integral in (10.40).

There is a special case when the above analysis can be taken significantly further. Namely, it may turn out that, for all moments of time, the operator $U$ of (10.40) belongs to a finite-dimensional subgroup of the unitary group of the Hilbert space. This situation has been described as a presence of a *dynamical group* (see [115, 11] and references therein). For the Hamiltonians of the form (10.39), there will exist a finite-dimensional dynamical group if the set of nested commutators of $H_i$'s closes on a finite-dimensional linear space of operators (which would serve as the Lie algebra of the dynamical group). Should that happen, one would be able to use the closed resummed version of the Baker-Campbell-Hausdorff formula for finite-dimensional Lie groups (see, for example, [7]) to treat the Magnus expansion, or, alternatively, the Schrödinger equation can be reduced to a finite number of ordinary differential equations describing the evolution on the finite-dimensional dynamical group manifold [115, 11, 116]. In practical terms, one can choose a particular low-dimensional faithful linear representation of the dynamical group furnished by matrices $M$, and write down the Schrödinger equation in this representation:

$$i\frac{dM(t,t_0)}{dt} = \varphi(H(t))M, \qquad M(t_0,t_0) = 1, \tag{10.42}$$

where $\varphi$ is a homomorphism from Hilbert space operators onto the representation furnished by $M$. (This is a finite-dimensional system of ordinary differential equations.) Given the solution for $M(t,t_0)$, one can reconstruct the original evolution operator as $\varphi^{-1}(M(t,t_0))$.

The analytic power of the dynamical group approach does not appear completely clear or fully explored. It certainly does apply to all linear quantum systems; however, in that case, the conventional WKB analysis would suffice. Beyond linear systems, the relevant finite-dimensional subalgebras of Hermitean operators may be difficult to construct and/or classify. Nevertheless, some nontrivial examples of dynamical groups for quantum-mechanical systems do exist (see, for example, [117]).

While the specific examples of quantum dynamics discussed in chapter 12 will be constructed using a somewhat unconventional application of WKB methods, the fact this "double-semiclassical" analysis is possible reflects an underlying dynamical group structure inherent to the systems we are working with. After explicitly computing the mode functions encoding the dynamics, we will investigate (in this greatly simplified setting) the existence of singular limits. It will then be possible to circumnavigate the formal complications introduced by the non-commuting structures in (10.39), and examine the limiting case of evolution on a singular spacetime background.

# Chapter 11

# Matrix theory

*"I apologize for introducing a difficult formula,"*

*(The slide referred to* $\mathrm{Tr} A_{ij} = \sum_i A_{ii}$*)*

*Anonymous visiting speaker (not in our department).*

Although matrix theory does not literally appear in the work that I will present in the part "Research", it is an important motivation for the projects that I have investigated. Furthermore, matrix theory is a non-perturbative formulation of string theory, so it allows to investigate the quantum structure of spacetime irrespective of the strength of the coupling between strings. Therefore it is ideally suited to investigate quantum gravitational effects near spacetime singularities. In this chapter I will review the conjecture of Banks-Fishler-Shenker-Susskind [98], how a matrix model action can describe M-theory on an eleven-dimensional asymptotically Minkowski spacetime. Here, M-theory is understood as the strong coupling description of type IIA superstring theory (in the strong coupling limit an eleventh direction appears). The low-energy description of M-theory can be given in terms of eleven-dimensional supergravity.

I will also discuss some aspects of the matrix big bang model of Craps, Sethi and Verlinde [107] that provides a toy model for a spacetime with a big bang type singularity, because it is a motivation for our study of D$p$-branes in an asymptotically plane wave background in chapter 14. To arrive at these dual matrix models, I will illustrate how D-branes can be understood as the fundamental degrees of freedom at strong coupling. Then I will illustrate the discrete lightcone quantisation procedure to arrive at the BFSS matrix model in section 11.3. The BFSS matrix model describes eleven-dimensional M-theory, but the matrix big bang model describes ten-dimensional superstrings in the presence of a spacetime singularity. In section 11.5 I will therefore also introduce matrix string theory, which provides a dual and non-perturbative description of superstring theory, to bridge the gap in between the matrix big bang model and the BFSS matrix model for M-theory.

One of the virtues of matrix theory is that it allows to describe spacetime in a quantum mechanical manner, where the spacetime coordinates reveal an intrinsic noncommutative nature. I will avoid technical details and refer the interested reader to some introductions to and reviews of matrix theory [102, 103, 105, 104, 106] where further references can be found. A convenient summary can also be found in [111] or in the first sections of [112]. Throughout the present chapter, I have used information especially from [102, 106, 111, 112].

## 11.1 D-branes as effective degrees of freedom

In section 8 I have already remarked that type IIA superstring theory contains the even D$p$-branes. I recall that the effective action for $N$ D$p$-branes of bosonic string theory in flat spacetime was given in (8.11). For $N$ D0-branes of IIA superstring theory the effective action can be written as

$$S_{N\mathrm{D0}} = \frac{1}{g_s\sqrt{\alpha'}}\int dt\ \mathrm{Tr}\left\{\frac{1}{2}\left(D_0X^i\right)^2 + \left(\frac{[X^i,X^j]}{4\pi\alpha'}\right)^2 - g_s i\Theta^T\partial_t\Theta + \frac{g_s\Theta^T\gamma^j}{2\pi\alpha'}[X_j,\Theta]\right\}. \tag{11.1}$$

Upon closer inspection, formula (11.1) reveals that a collection of $N$ D0-branes can be described by a Super-Yang-Mills theory with gauge group $U(N)$ in $0+1$ dimensions (i.e. along a worldline). In other words, we are dealing with the quantum mechanics of matrices of order $N \times N$ (the matrices only depend on time). The commutator term in the potential

$$V = -\frac{1}{16\pi^2\alpha'^2 g_s}([X^i,X^j])^2 \tag{11.2}$$

is non-negative, because it is the square of the Hermitian matrix $i[X^i,X^j]$. The $X^i$ are Hermitian matrices. I will come back to this action shortly.

Referring to formula (8.15), for a D0-brane the mass is equal to the tension,

$$M_{\mathrm{D0}} = \frac{1}{g_s\sqrt{\alpha'}}\,. \tag{11.3}$$

Thus, in the limit of weak string coupling $g_s \to 0$ the D0-brane has infinite mass. It is expected [14] that for $N$ D0-branes, there are bound states with a mass given by

$$M_{N\mathrm{D0}} = \frac{N}{g_s\sqrt{\alpha'}}\,, \tag{11.4}$$

which is an evenly spaced spectrum with respect to the number of D0-branes, separated by the mass difference $1/(g_s\sqrt{\alpha'})$. Now, as the string coupling becomes large ($g_s \to \infty$), the mass difference between the bound states becomes small and the spectrum (11.4) approaches a continuum. Because of the even spacing with the integer $N$, the spectrum of bound D0-branes actually matches the discrete spectrum of Kaluza-Klein modes for a periodic dimension of radius $R_{11}^{(g)} = g_s\sqrt{\alpha'}$. Kaluza-Klein modes are states that appear when a higher dimensional theory is compactified.

When the radius of compactification becomes large, the mass difference between the Kaluza-Klein modes becomes smaller, and in the limit of decompactification $R_{11}^{(g)} \to \infty$, we reobtain the continuous spectrum of the uncompactified spacetime. Thus, from a more geometrical point of view, we can view the bound states of $N$ D0-branes as the Kaluza-Klein modes of an eleven-dimensional theory that is compactified along a new spatial dimension on a circle with radius given by

$$R_{11}^{(g)} = g_s\sqrt{\alpha'}, \tag{11.5}$$

where the superscript $(g)$ indicates that the radius is actually being measured with respect to the ten-dimensional string frame metric. In the limit of vanishing string coupling, the compactification radius $R_{11}^{(g)}$ becomes zero, which is the reason why it could not be observed in string perturbation theory. However, in the opposite limit of decompactification, an eleventh dimension appears. The eleven dimensional metric is related to the ten dimensional metric and dilaton by

$$ds_{11}^2 = e^{-2\phi/3}ds_{st}^2 + e^{4\phi/3}d\bar{y}^2. \tag{11.6}$$

Still, in the limit $g_s \to \infty$, string theory cannot be defined in terms of a perturbation expansion in the coupling constant $g_s$. However, precisely at strong coupling, the states of bound D0-branes become light (11.4) and it is tempting to call them the fundamental degrees of freedom of string theory at strong coupling. Because of the dualities between the different theoretical descriptions of string theory at different limits of the parameters like $g_s$, it would be more correct to call D0-branes the effective degrees of freedom at strong coupling.

## 11.2 Type IIA superstrings at strong coupling

The existence of an eleven dimensional theory ("M-theory") with the different critical (ten dimensional) superstring theories as effective descriptions in certain regions (e.g. by specifying certain compactifications and limits like weak coupling), was conjectured in [97, 34]. In the previous section we have seen that D0-branes become the effective degrees of freedom in case we are considering the strong coupling limit of type IIA superstring theory.

The parameters of type IIA superstring theory are the string coupling $g_s$ and the string length $\ell_s$, and these are related to the M-theory parameters. The latter parameters are the eleven-dimensional Planck length $\ell_P^{(11)}$ and the compactification radius of the eleventh dimension $R_{11}$, the latter already introduced in the previous section:

$$g_s = \left( \frac{R_{11}}{\ell_P^{(11)}} \right)^{3/2} , \quad \ell_s = \sqrt{\frac{\ell_P^{(11)}}{R_{11}}} \ell_P^{(11)} . \tag{11.7}$$

The eleven-dimensional Planck mass is given by $M_P = 1/\ell_P^{(11)}$.

In [98] Banks, Fishler, Shenker and Susskind (BFSS) argued that it is possible to extend the validity of the description of type IIA superstring theory in terms of D0-branes to certain regions of the parameter space of M-theory. The description of M-theory in terms of D0-brane quantum mechanics is called "matrix theory". In the case of the BFSS matrix model (see the next section) matrix theory gives a description of M-theory in a asymptotically eleven-dimensional Minkowski space. There do exist matrix theory formulations of certain toroidal compactifications of M-theory (see e.g. [104]), and some (lightcone) time-dependent backgrounds, for example the matrix big bang which will be described in section 11.6. It has been shown that the BFSS matrix theory (for the action see (11.1) or (11.25) in section 11.4) contains the Fock space of an arbitrary number of supergravitons and that it describes the scattering of two gravitons in the same way as eleven dimensional supergravity. In the limit $N \to \infty$, the matrix model Hamiltonian reduces to the Hamiltonian of the eleven dimensional supermembrane in the lightcone gauge [96]. The tension of the matrix model membranes agrees with the tension of the membranes in eleven dimensional supergravity.

## 11.3 Discrete lightcone quantisation

To introduce the reader to the BFSS matrix model we will make use of discrete lightcone quantisation (DLCQ) which we will apply to M-theory [101, 100].

### 11.3.1 A lightlike compactification

We specify the eleven-dimensional Minkowski metric as

$$ds^2 = -(dx^0)^2 + \sum_{i=1}^{9} (dx^i)^2 + (dx^{11})^2 . \tag{11.8}$$

Let us now consider the lightlike compactification,

$$\begin{pmatrix} x^0 \\ x^{11} \end{pmatrix} \sim \begin{pmatrix} x^0 + 2\pi n R/\sqrt{2} \\ x^{11} + 2\pi n R/\sqrt{2} \end{pmatrix} , \quad n \in \mathbb{Z} , \tag{11.9}$$

or, equivalently,

$$\begin{pmatrix} x^+ \\ x^- \end{pmatrix} \sim \begin{pmatrix} x^+ \\ x^- + 2\pi n R \end{pmatrix}, \quad n \in \mathbb{Z}, \tag{11.10}$$

where we have made use of lightcone coordinates

$$x^+ = (x^0 - x^{11})/\sqrt{2}, \quad x^- = (x^0 + x^{11})/\sqrt{2}. \tag{11.11}$$

Because of the identification, the lightcone momentum $p^+ = -p_-$, conjugate to $x^-$, is quantized as $p^+ = N/R$. Because of translation invariance in $x^-$, the total lightcone momentum is conserved. We will focus on one sector of $p^+$ with a fixed amount of $N$ units of lightcone momentum (with $N > 0$). The lightlike compactification (11.9) can be defined as a limit of spacelike identifications, more precisely as

$$\begin{pmatrix} x^+ \\ x^- \end{pmatrix} \sim \begin{pmatrix} x^+ - n\pi R_s^2/R \\ x^- + 2\pi n R \end{pmatrix}, \quad n \in \mathbb{Z}, \tag{11.12}$$

in the limit $\lim_{R_s \to 0}$. To show the spacelike nature of (11.12) for nonzero $R_s$, we can consider a Lorentz boost. The relation beween the boosted frame (indicated by a prime) and the original frame is expressed as

$$\begin{pmatrix} x'^+ \\ x'^- \end{pmatrix} = \begin{pmatrix} e^\beta & 0 \\ 0 & e^{-\beta} \end{pmatrix} \begin{pmatrix} x^+ \\ x^- \end{pmatrix}, \tag{11.13}$$

with the boost parameter determined as

$$e^\beta = \sqrt{2} R/R_s. \tag{11.14}$$

In the boosted coordinates (indicated by a prime) it is manifest that the identification is spacelike for $R_s > 0$,

$$\begin{pmatrix} x'^+ \\ x'^- \end{pmatrix} \sim \begin{pmatrix} x'^+ - \sqrt{2} n\pi R_s \\ x'^- + \sqrt{2} n\pi R_s \end{pmatrix}, \quad n \in \mathbb{Z}, \tag{11.15}$$

which becomes even more apparent if we rewrite the identification (11.15) as,

$$\begin{pmatrix} x'^0 \\ x'^{11} \end{pmatrix} \sim \begin{pmatrix} x'^0 \\ x'^{11} + 2\pi n R_s \end{pmatrix}, \quad n \in \mathbb{Z}. \tag{11.16}$$

This means that the lightlike compactification of M-theory (on a circle with finite coordinate radius $R$) is related to a spacelike compactification on a circle with vanishing radius $R_s \to 0$. Therefore, it appears possible to describe M-theory with a lightlike identification in terms of weakly coupled type IIA superstring theory (i.e. $g_s \ll 1$). After the boost, we focus on a sector with momentum on a spacelike circle

$$p'_{11} = N/R_s. \tag{11.17}$$

In the lightlike limit it will corresponds to $p^+ = N/R$. This momentum corresponds to the charge of $N$ D0-branes in type IIA superstring theory, of which the worldvolume effective action is given by (11.1).

Let us now investigate the compactification of M-theory on a lightlike circle more closely. With $R_s$ playing the role of $R_{11}$ in (11.7), we find that in the limit $R_s \to 0$, the string coupling becomes small and the string length becomes infinite. An infinite string length implies a vanishing string tension, see (4.3). In string theory, the appearance of $g_s$ is related to (quantum) loop corrections, and $\alpha'$ corrections (derivative corrections) are related to the finite length of the string. If the length scales of interest are large with respect to $\sqrt{\alpha'}$, then a point particle approximation is reasonable (low-energy limit). So, when $R_s \to 0$ in the lightlike compactification it follows that $\alpha' = (\ell_s)^2 \to \infty$. So this would seem to conflict with our description as lightlike compactified M-theory in terms of weakly coupled string theory, because of the vanishing string tension. However, the parameter $\ell_s$ is a dimensionful quantity, and we should compare it with a typical length scale. In the next section we will verify that the string length remains smaller than the length scales of the phenomena we are interested in.

### 11.3.2 Energy in the lightlike frame

We will compare the inverse string length $1/\ell_s$ with the energy of the states we are interested in. We focus on a sector with $N$ units of lightcone momentum and we keep the eleven-dimensional Planck scale (11.7) fixed. We are interested in the states that have finite energy $E_N^{\ell\ell}$ in the lightlike frame (11.8, 11.11). We define the lightlike Hamiltonian $H_N^{\ell\ell}$ of the discrete lightcone quantisation procedure as

$$H_N^{\ell\ell} = i\partial_{x^+} \tag{11.18}$$

in the lightlike limit $R_s \to 0$. It is related to the boosted frame (11.13) by

$$i\partial_{x^+} = \frac{1}{\sqrt{2}} e^{\beta} \left( i\partial_{x'^0} - i\partial_{x'^{11}} \right) . \tag{11.19}$$

The energy in the spacelike compactification (in the boosted coordinates 11.13) is

$$E' = \frac{N}{R_s} + \Delta E' , \tag{11.20}$$

where the first term is the mass term of $N$ D0-branes, which corresponds to the momentum in the compactified direction $x^{11}$. Thus, making use of (11.14) and substituting (11.20) in the right hand side of (11.19), with $p'_{11} = N/R_s$, the energy in the lightlike frame is related to the energy in the boosted frame by

$$E_N^{\ell\ell} = \frac{R}{R_s} \left( E' - p'_{11} \right) = \frac{R}{R_s} \Delta E' . \tag{11.21}$$

If we now compare $\Delta E'$ with $\ell_s$ (we obtain $\ell_s$ from (11.7) with $R_s$ playing the role of $R_{11}$)

$$\frac{\Delta E'}{1/\ell_s} = \sqrt{R_s \left(\ell_P^{(11)}\right)^3 \frac{E_N^{\ell\ell}}{R}}\,, \tag{11.22}$$

we find that the energy of the system $\Delta E'$ vanishes faster than $1/\ell_s$ in the lightlike limit $R_s \to 0$, while we keep the eleven dimensional Planck length and the energy in the lightlike frame fixed. So for the states that we are interested in, the energy $\Delta E'$ remains much smaller than the energy scale that is related to the string tension (even though the tension vanishes in the lightlike limit). In other words, the string length remains small with respect to length scales of the states we are interested in. Therefore we can reasonably study the lightlike compactification in the limit of spacelike compactifications.

The relation between the lightlike compactification and the spacelike compactification can be formalized by a rescaling of the length and mass scales by a factor $\epsilon = R_s/R$, as $\tilde{L} = \epsilon L$ and $\tilde{M} = M/\epsilon$. Let us write the Hamiltonian of $N$ D0-branes in the boosted coordinate system as $H_N(M, L)$, which will scale with a factor $1/\epsilon$. Note that $p^+ = \epsilon p'_{11}$. If we look back at the lightcone Hamiltonian (11.18) it is then determined by

$$H_N^{\ell\ell} = \lim_{R_s \to 0} \frac{R}{R_s} H_N(M, L) = \lim_{\tilde{R}_s \to 0} \tilde{H}_N(\tilde{M}, \tilde{L})\,. \tag{11.23}$$

### 11.3.3 D-brane worldvolume theory and decompactification limit

The $N$ units of lightcone momentum $p^+$ in M-theory (in the lightlike frame) correspond to $N$ units of D0-brane charge in type IIA string theory. In the boosted coordinate system this corresponds to the effective action (11.1). Thus, in the limit of vanishing $R_s$, we can find a description of the system in terms of D0-brane interactions: the degrees of freedom are captured by the $N \times N$ matrices of the D0-brane worldvolume theory. Eventually, one recovers uncompactified M-theory by taking the limit $R \to \infty$ in a sector with fixed lightcone momentum $p^+$. Therefore the decompactification limit corresponds to a large $N$ limit in which the matrices become infinite dimensional:

$$R \to \infty\,, \quad N \to \infty\,, \quad p^+ = N/R \text{ fixed.} \tag{11.24}$$

### 11.3.4 Summary of DLCQ of M-theory

Recapitulating, a lightlike compactification of M-theory (compactified on a lightlike circle with radius $R$) in a sector with $N$ units of lightcone momentum $p^+ = N/R$, is described by type IIA superstring theory in the limit of weak string coupling ($g_s \ll 1$) and in the limit $\sqrt{\alpha'}\Delta E' \to 0$ with $\Delta E'$ the energies of the states of interest and $\sqrt{\alpha'}$ the string length. Because the string length remains small with respect to the length scale related to $\Delta E'$, it is reasonable to study the lightlike compactification as a limit of spacelike identifications with $R_s \to 0$. We limit our attention to the states that remain light after a boost and rescaling of the system. In the limit $R_s \to 0$ the effective states are D0-branes. The worldvolume theory of D0-branes is described by dimensional reduction of ten-dimensional Super-Yang-Mills theory and is described by quantum mechanics of $N \times N$ matrices. All the higher terms in the D0-brane action become suppressed. To obtain uncompactified M-theory (on flat Minkowski space) we have to consider the decompactification limit $N \to \infty$.

## 11.4 Non-commutative space from D0-branes

The Banks-Fishler-Shenker-Susskind conjecture [98] advocates that M-theory in asymptotically flat spacetime can be described by a theory with only D0-branes as dynamical degrees of freedom. Therefore the system is determined by the effective action of $N$ D0-branes in the $N \to \infty$ limit, with a Hamiltonian that follows from reducing $9+1$ dimensional $U(N)$ Super-Yang-Mills theory to 0+1 dimensions, given by matrix quantum mechanics:

$$S = \frac{1}{\sqrt{\alpha'}} \int dt \mathrm{Tr} \left\{ \frac{1}{2g_s} \left(\partial_t X^i\right)^2 + \frac{([X^i, X^j])^2}{16\pi^2 \alpha'^2 g_s} - i\Theta^T \partial_t \Theta + \frac{\Theta^T \gamma^j}{2\pi\alpha'} [X_j, \Theta] \right\} . \tag{11.25}$$

In the action we find nine bosonic Hermitian $N \times N$ matrices $X^i$ and an $N \times N$ matrix $\Theta$ whose elements are sixteen-component Majorana spinors. In principle there should have been a covariant derivative $D_0$ that contains the gauge field $A_0$, but we can make the gauge choice $A_0 = 0$ to reduce it to $\partial_t$.

The $N$ eigenvalues of the $X^i$ can be interpreted as the position vectors of $N$ D0-branes. The commutator potential,

$$V = -\frac{1}{16\pi^2 \alpha'^2 g_s} ([X^i, X^j])^2 , \tag{11.26}$$

expresses the interaction between the D0-branes. The potential has flat directions, where the matrices commute,

$$[X^i, X^j] = 0 , \tag{11.27}$$

and along which the matrices $X^i$ can be simultaneously diagonalized. In that case the eigenvalues indicate the position of the branes in the $i$'th dimension. The flat directions imply a continuous spectrum. Matrix theory is supposed to describe multi-particle states.

To see how the commutator potential is influenced by the off-diagonal modes, let us consider the following simple example for two branes ($N = 2$)

where for simplicity we also select only two of the nine matrices (say, $X = X^1$ and $Y = X^2$):

$$X = \begin{pmatrix} x_a & x^* \\ x & x_b \end{pmatrix} , \quad Y = \begin{pmatrix} y_a & y^* \\ y & y_b \end{pmatrix} . \tag{11.28}$$

For $N = 2$ it is sometimes convenient to write these $2 \times 2$ Hermitian matrices in terms of the Pauli matrices (5.2). Up to a prefactor, we can rewrite the commutator potential as

$$-[X, Y]^2 = \begin{pmatrix} \mathcal{V} & 0 \\ 0 & \mathcal{V} \end{pmatrix} , \tag{11.29}$$

with

$$\begin{aligned} \mathcal{V} =& |x|^2 (y_a - y_b)^2 + |y|^2 (x_a - x_b)^2 \\ & - 2 \cos(\widehat{x, y})\, |x||y|(x_a - x_b)(y_a - y_b) + 4 \sin^2(\widehat{x, y})\, |x|^2 |y|^2 . \end{aligned} \tag{11.30}$$

If the two branes coincide, then $x_a = x_b$ and $y_a = y_b$. Neglecting the flat directions for which $\mathcal{V} = 0$, the potential behaves as $|x|^2|y|^2$. When the branes are distant, then $x_a \neq x_b$ and $y_a \neq y_b$. For small off-diagonal modes ("small" with respect to $(x_a - x_b)$ and $(y_a - y_b)$) the potential behaves only quadratically in the off-diagonal modes and the off-diagonal modes are therefore more suppressed (by the prefactors $(x_a - x_b)$ and $(y_a - y_b)$ instead of $|x|$ and $|y|$) than when the branes coincide. For very large off-diagonal modes, the dominant term is always the quartic potential $|x|^2|y|^2$.

- Thus if the diagonal terms in the $X^i$ are different from each other and the branes are far away with respect to each other, the commutator term would become very large if the off-diagonal modes in the matrices $X^i$ were nonzero. Thus, the low energetic states are then given by commuting $X^i$, which permits a clear interpretation of the position of the D0-branes in terms of the eigenvalues of the matrices $X^i$.

- However, when the diagonal terms in the $X^i$ have roughly the same value, the commutator term does not become very large in the case of non-commuting matrices $X^i$ with nonzero off-diagonal modes and it cannot be argued that the matrices are diagonal. Consequently, it is not possible to clearly distinguish the position of the individual D0-branes anymore once they become close.

Another way to express the observation that for close D0-branes the commutator term is nonzero and that the matrices $X^i$ are far from diagonal, is that space is intrinsically non-commutative with ordinary commutative space only emerging at long distances. In a more stringlike picture one can say that for distant D0-brane configurations, only the strings with endpoints on the same brane (the diagonal terms) are important, the configurations with very long strings stretching between far D0-branes have high energy. When the D0-branes are close, the strings stretching in between different branes become important and these correspond to the off-diagonal modes in the matrices $X^i$.

To clarify this with an example, suppose the matrix $X^i$ can be written in block-diagonal form with the diagonal blocks representing different D-brane clusters. Then the distance between two clusters $a$ and $b$ can be written as

$$r_{ab} = \left( \sum_{i=1}^{9} \left[ \mathrm{Tr} \left\{ \frac{1}{N_a} X_a^i - \frac{1}{N_b} X_b^i \right\} \right]^2 \right)^{1/2} , \tag{11.31}$$

with $N_a$ and $N_b$ the size of cluster $a$ and $b$. Suppose we can write (at fixed $N$)

$$X^i = \begin{pmatrix} X_A^i & & \\ & X_B^i & \\ & & \ddots \end{pmatrix} , \; X_A^i = \begin{pmatrix} X_{11}^i & X_{12}^i & X_{13}^i \\ X_{21}^i & X_{22}^i & X_{23}^i \\ X_{31}^i & X_{32}^i & X_{33}^i \end{pmatrix} , \; X_B^i = \begin{pmatrix} X_{44}^i & X_{45}^i \\ X_{54}^i & X_{55}^i \end{pmatrix} , \tag{11.32}$$

with the block-diagonal form appearing when clusters of D0-branes are widely separated in the dimension $x^a$ and their off-diagonal crossterms are small. In the example (11.32) we can identity a cluster of three D0-branes, and another cluster of two D0-branes. Classical gravitational interactions between separate clusters of D0-branes arise from quantum corrections in matrix theory.

## 11.5 Matrix (string) theory

The large $N$ limit of the BFSS matrix model provides a description for uncompactified eleven-dimensional M-theory. Now suppose we want to find a non-perturbative description (valid at whatever value of the string coupling $g_s$) of type IIA superstring theory in ten dimensions instead. It will turn out that the latter is described by matrix (string) theory, in terms of the worldvolume theory of $N$ D1-branes. The effective action of the matrix string [99] is described by the dynamics of D1-branes instead of D0-branes. This leads to a 1+1 dimensional quantum field theory, with one spatial dimension, instead of the 0+1 quantum mechanics of the BFSS matrix model. It can also be seen as a second-quantized generalization of the Green-Schwarz action for superstrings, in which the spacetime coordinates have now become matrix valued fields on the worldsheet.

The matrix big bang model that is presented in the next section will be resolved by a matrix string model, therefore it is necessary to illustrate the derivation of matrix string theory. But let me first quickly recapitulate the steps in the construction of the BFSS matrix model.

### 11.5.1 Correspondence between M- and matrix theory

As we have seen in section 11.3, the lightlike compactification of M-theory on a circle with radius $R$ in the direction $x^-$ can be defined as a limit of spacelike compactifications on a circle with vanishing radius $R_s$. As in section 11.3 we write the lightcone coordinates as a combination of $x^0$ and $x^{11}$ with

$$x^{\pm} = (x^0 \mp x^{11})/\sqrt{2} \tag{11.33}$$

and we consider a sector with $N$ units of lightcone momentum. In the limit $R_s \to 0$ (with the eleven-dimensional Planck length $\ell_P$ kept fixed) a dual description of M-theory with $N$ units of lightcone momentum is given by a worldvolume theory of $N$ D0-branes in type IIA superstring theory. The latter D0-brane worldvolume theory is given by the dimensional reduction of ten dimensional Super-Yang-Mills theory down to 0+1 dimensions, i.e. the quantum mechanics of $U(N)$ matrices. In the large $N$ limit (with $N/R$ constant) one obtains uncompactified M-theory.

### 11.5.2 Compactification of the correspondence

Let us now consider the dual description of ten-dimensional type IIA superstring theory in terms of matrix (string) theory. Type IIA superstring theory is given by the compactification of M-theory along a spacelike circle with radius $R_9$, let us say in the direction $x^9$. The remaining coordinates are $x^i$ $(i = 1 \ldots 8)$ and $x^{\pm}$ defined as in (11.33).

In the previous paragraph we have recapitulated the correspondence between lightlike compactified M-theory (in a sector with $N$ units of lightcone momentum) and the worldvolume theory of $N$ D0-branes in an auxiliary type IIA superstring theory (i.e. the BFSS matrix theory). Therefore, if we compactify both sides of this correspondence along a spacelike circle with radius $R_9$, we obtain a relation between type IIA superstring theory (with $N$ units of lightcone momentum) and a worldvolume theory of $N$ D0-branes in an auxiliary type IIA superstring theory compactified on a circle with radius $R_9$.

Making use of (11.7) we find that the original type IIA superstring theory (obtained from M-theory by a compactification along $x^9$) has the following parameters:

$$g_s = \left(\frac{R_9}{\ell_P}\right)^{3/2} , \quad \ell_s = \sqrt{\frac{\ell_P}{R_9}}\ell_P \, . \tag{11.34}$$

On the other hand, the auxiliary type IIA superstring theory (compactified along $x^9$ on a circle with radius $R_9$) is a lightlike compactification of M-theory along the direction $x^-$. The lightlike compactification along the direction $x^-$ is obtained as a limit of spacelike compactification on a circle with radius $R_s$ in the direction $x^{11}$. Therefore the auxiliary type IIA superstring theory has the following parameters (we use a prime for clarity):

$$g'_s = \left(\frac{R_s}{\ell_P}\right)^{3/2} \ , \quad \ell'_s = \sqrt{\frac{\ell_P}{R_s}}\ell_P \, . \tag{11.35}$$

But the lightlike compactification in the direction $x^-$ is obtained in the limit $R_s \to 0$. Yet in this limit the string length $\ell'_s$ in the auxiliary theory becomes much larger than the radius $R_9$ of the compactified direction. Because $\ell'_s \gg R_9$ it is convenient to T-dualize in the direction $x^9$ such that $R_9 \to \alpha'/R_9$ as described in section 8.2. In this way, the worldvolume theory of the $N$ D0-branes is mapped into a worldvolume theory of $N$ D1-branes in an auxiliary type IIB superstring theory compactified on a circle with radius $\alpha'/R_9$. The D1-branes are wrapped around the compact direction $x^9$.

### 11.5.3 Summary of DLCQ of type IIA superstring theory

Let me summarize the dual description with the following schematic representation (the scheme bears a strong resemblance with Fig. 15 of [111]):

$$\begin{array}{ccccc} & & c^- & & \\ & \mathcal{T}_M & \rightarrow & \mathcal{T}_{N\mathrm{D0}} & \\ c^9 & \downarrow & & \downarrow & \\ & \mathcal{T}_{IIA} & \rightarrow & \mathcal{T}^9_{N\mathrm{D0}} \stackrel{T^9}{\Leftrightarrow} & \mathcal{T}_{ND1} \end{array} \tag{11.36}$$

with the abbreviations in (11.36) given by

- $\mathcal{T}_M$: uncompactified M-theory in eleven dimensions;
- $\mathcal{T}_{IIA}$: type IIA superstring theory in ten dimensions (obtained by compactifying $\mathcal{T}_M$ along $x^9$ with radius $R_9$), we are interested to find a dual description for this theory if the string coupling becomes large;
- $\mathcal{T}_{N\mathrm{D0}}$: a worldvolume theory of $N$ D0-branes in an auxiliary type IIA superstring theory, obtained by applying the discrete lightcone quantisation procedure to $\mathcal{T}_M$, in the limit $R_s \to 0$ of a spacelike compactification along $x^{11}$;
- $\mathcal{T}^9_{N\mathrm{D0}}$: a worldvolume theory of $N$ D0-branes in an auxiliary type IIA superstring theory compactified on a circle with radius $R_9$;
- $\mathcal{T}_{N\mathrm{D1}}$: a worldvolume theory of $N$ D1 branes in an auxiliary type IIB superstring theory compactified on a circle with radius $\alpha'/R^9$;
- $c^-$: lightlike compactification along $x^-$ on a circle with radius $R$, corresponding to the limit $R_s \to 0$ of a spacelike compactification along $x^{11}$;

- $c^9$: compactification along $x^9$ on a circle with radius $R_9$;
- $T^9$: T-duality along the compactified direction $x^9$ such that $R_9 \to \alpha'/R_9$.

To recapitulate, type IIA superstring theory is given by the compactification of M-theory along a spacelike circle with radius $R_9$ (say, in the direction $x^9$). We can now compactify the type IIA superstring theory on a lightlike circle (defined as the limit of a spacelike compactification along the direction $x^{11}$). Through a boost and a scaling as described in section 11.3, the lightlike compactification of $\mathcal{T}_{IIA}$ corresponds to a D0-brane worldvolume theory in an auxiliary type IIA superstring theory compactified along the direction $x^9$. An equivalent description is given by a D1-brane worldvolume theory in an auxiliary type IIB superstring theory with $N$ D1-branes wrapped around the compactified direction $x^9$. This is matrix string theory. If we are interested in describing an uncompactified type IIA superstring theory, we have to take the large $N$ limit.

### 11.5.4 9-11 flip, TST-duality and TS-duality

The relation between the theories $\mathcal{T}_{IIA}$ and $\mathcal{T}^9_{N\mathrm{D}0}$ is called the "9-11 flip" because each of them is derived from a compactification of M-theory (the former theory as a compactification along $x^9$ and the latter as a compactification along $x^{11}$). The 9-11 flip is equivalent to a sequence of a T-duality, an S-duality[1] and another T-duality, which together are called the "TST-duality".

By the TST-duality we can thus relate a type IIA superstring theory with $N$ units of lightcone momentum ($\mathcal{T}_{IIA}$) to an auxiliary type IIA superstring theory with $N$ units of D0-brane charge ($\mathcal{T}^9_{N\mathrm{D}0}$). By applying only a TS-duality instead, we can relate the type IIA superstring theory with $N$ units of lightcone momentum to an auxiliary type IIB superstring with $N$ D1-branes wrapped around the compact direction $x^9$ ($\mathcal{T}_{N\mathrm{D}1}$). This is how we will implement the matrix big bang in the following section.

## 11.6 The matrix big bang

The matrix big bang [107, 110] is a toy model for a cosmological big bang like singularity that can be resolved by string theoretical methods. The properties of the *real* spacetime near the singularity are described by a dual model, which is essentially a quantum field theory defined on an *auxiliary* singular spacetime. Hence, if we want to investigate the behaviour of the *real* spacetime near the *real* singularity, we are naturally led to the study of quantum field theory on singular (auxiliary) spacetimes. In addition, to investigate whether spacetime

[1] An S-duality is a duality that relates a string theory with weak coupling to a string theory with strong coupling.

may have bounced through the singularity we should study the evolution of quantum field theory across singularities.

The matrix big bang is a (lightcone) time-dependent model and because of the time-dependence the discrete lightcone quantisation procedure has to be adapted slightly.

### 11.6.1 A string theory model with a lightlike singularity

Let us consider Minkowski spacetime (the lightcone coordinates are written in terms of $x^0$ and $x^9$) in type IIA superstring theory, with a lightlike linear dilaton,

$$ds_{st}^2 = -2dx^+dx^- + \sum_{i=1}^{8}(dx^i)^2\,, \quad \phi = -Qx^+\,. \tag{11.37}$$

In Einstein frame, given by

$$ds_E^2 = e^{-\phi/2}ds_{st}^2\,, \tag{11.38}$$

the metric becomes singular for $x^+ \to -\infty$. After a reparametrization,

$$d\chi^+ = \exp\left(Qx^+/2\right)dx^+\,, \tag{11.39}$$

the Einstein frame metric becomes

$$ds_E^2 = -2d\chi^+dy^- + \frac{1}{2}Q\chi^+(dx^i)^2\,, \quad \phi = -2\log\left(Q\chi^+/2\right)\,, \tag{11.40}$$

and the singularity is located at $\chi^+ = 0$ at a finite distance (for instance, take $\chi^+$ as an affine parameter). The singularity is reminiscent of a big bang singularity, because the scale factor $\sqrt{Q\chi^+/2}$ goes to zero (but it is a lightlike singularity instead of a spacelike singularity). The singularity is also present in string frame because the string coupling blows up for $x^+ \to -\infty$. As one approaches the singularity the string coupling $g_s = e^\phi$ becomes unboundedly large. As discussed in [112], the lightlike dilaton background (11.37) corresponds to an eleven dimensional plane wave.

As we will see, the final result of the matrix big bang model is that the effective dynamics near the singularity is described by a dual matrix (string) model. Stated more precisely, type IIA superstring theory on the lightlike dilaton background (11.37) can be described by a matrix model which is given by Super-Yang-Mills theory with gauge group $U(N)$ and time-dependent Yang-Mills coupling constant.

### 11.6.2 Adaptation of the discrete lightcone quantisation

Because of the (lightcone) time-dependence in the dilaton, the discrete lightcone quantization procedure (in the matrix string theory setting) cannot be applied literally. Instead, the boost that relates the compactification on a lightlike circle with a compactification on a (vanishing) spacelike circle involves one of the transverse coordinates, which we label as $y^1$. So there are two steps: a null rotation and a boost. If we perform the null rotation

$$x^+ = \tilde{x}^+ \,, \quad x^- = \tilde{x}^- - \frac{R}{R_s}\tilde{x}^1 + \frac{R^2}{2R_s^2}\tilde{x}^+ \,, \quad y^1 = \tilde{x}^1 - \frac{R}{R_s}\tilde{x}^+ \,, \tag{11.41}$$

the identification in the $\tilde{x}$-coordinates becomes

$$\begin{pmatrix} \tilde{x}^+ \\ \tilde{x}^- \\ \tilde{x}^1 \end{pmatrix} \sim \begin{pmatrix} \tilde{x}^+ \\ \tilde{x}^- - 2\pi R \\ \tilde{x}^1 + 2\pi R_s \end{pmatrix} . \tag{11.42}$$

Then we make an additional Lorentz boost

$$\begin{pmatrix} x'^+ \\ x'^- \\ x'^1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} R/R_s & 0 & 0 \\ 0 & 2R_s/R & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \tilde{x}^+ \\ \tilde{x}^- \\ \tilde{x}^1 \end{pmatrix} . \tag{11.43}$$

In the limit $R_s \to 0$ the states that have $p'_1 = N/R_s$ are mapped to the states with $p^+ = N/R$ and the arguments of the discrete lightcone quantisation procedure can be carried through (especially the argument in subsection 11.3.2 why the energy of interest remains small with respect to the energy related to the string length).

After applying a T-duality (along $x'^1$) and an S-duality we arrive at a collection of $N$ D1-branes in type IIB superstring theory, wrapped around the compact $x'^1$ direction:

$$ds^2 = \frac{R_s}{\sqrt{\alpha'}} e^{Qx'^+ R_s/R} \left\{ -2dx'^+ dx'^- + \sum_{i=1}^{8} \left(dx'^i\right)^2 \right\} , \tag{11.44a}$$

$$\phi = \frac{R_s}{R} Q x'^+ + \log\left(\frac{R_s}{\sqrt{\alpha'}}\right) , \tag{11.44b}$$

$$x'^1 \sim x'^1 + 2\pi n \alpha'/R_s \,, \quad n \in \mathbb{Z} \,. \tag{11.44c}$$

This configuration represents a dilaton-gravity plane wave (to obtain the metric in standard Rosen coordinates, we have to redefine the lightcone time). The collection of D1-branes in the background (11.44) is one of the main motivations for my study of supergravity solutions in chapter 14. In fact, the solutions presented there are limited to D$p$-branes whose worldvolume is aligned along the propagation direction of the plane wave, while the D1-branes relevant for the matrix big bang are wrapped around the $x'^1$ direction and therefore perpendicular to the propagation direction of the dilaton-gravity plane wave (the latter problem is still under study).

### 11.6.3 Effective action for the matrix big bang

The result of the derivation in [107] is that the effective action of the matrix big bang is given by $\mathcal{N} = 8$ Super-Yang-Mills theory on a cylindrical worldsheet and with the time-dependent Yang-Mills coupling

$$g_{YM} = \frac{1}{\sqrt{\alpha'}} \exp\left(\sqrt{\frac{\alpha'}{2}}\frac{Q\tau}{R}\right) . \tag{11.45}$$

The worldsheet metric is given by

$$ds^2 = -\tau^2 + \sigma^2 , \quad \sigma \sim \sigma + 2\pi n\sqrt{\alpha'} , \tag{11.46}$$

and the terms in the bosonic part of the action are

$$S_{ND1} \approx \frac{1}{4\pi\alpha'} \int d\sigma d\tau \mathrm{Tr} \left\{ -(\partial_\alpha Y^i)^2 - \frac{2\pi^2\alpha'}{g_{YM}^2} F_{\alpha\beta}^2 + \frac{g_{YM}^2}{2\pi^2\alpha'} [Y^i, Y^j]^2 \right\} , \tag{11.47}$$

with the $Y^i(\sigma, \tau)$ matrix valued (string) coordinates. We can see that the Yang-Mills coupling grows with time and that the commutator term, proportional to the coupling, will generically becomes very large at late times. The off-diagonal modes of the matrices $Y^i$ become very massive near $\tau \to +\infty$, hence they will not be excited for configurations near the ground state of the system. Thus at late times ("far away" from the singularity) the matrices $Y^i$ are diagonal and spacetime effectively commutes. At early times, the off-diagonal terms are important, and near the singularity the non-commutative nature of spacetime cannot be neglected. An important feature of the model is that the field theory is weakly coupled (11.45) near the singularity at $\tau \to -\infty$, hence the gravitational physics near the singularity is tractable by means of a perturbation expansion in the dual Super-Yang-Mills field theory.

Some recent research focuses on the strength of the commutator term near the singularity [113].

### 11.6.4 Time-dependent worldsheet description

An alternative description of the matrix big bang model is obtained when one rescales the worldsheet metric to absorb the time-dependent coupling of the Super-Yang-Mills theory. The Super-Yang-Mills theory on a cylindrical worldsheet with a time-dependent coupling constant of (11.47-11.45) is equivalent to a Super-Yang-Mills theory with a constant coupling constant on the future cone of the Milne orbifold

$$ds^2 = e^{2Q\tau/R}\left(-d\tau^2 + d\sigma^2\right) , \quad \sigma \sim \sigma + 2\pi n\sqrt{\alpha'} . \tag{11.48}$$

The Milne background becomes singular for $\tau \to -\infty$. So we find that the physics near the singularity is described by a quantum field theory on a singular spacetime. This provides additional motivation for another research project: in chapter 12 we will investigate the geometrical resolution of a field theory on a singular spacetime background. Actually, we have simplified the problem considerably by investigating a free scalar field (on the parabolic orbifold) instead of the much more complicated Super-Yang-Mills theory on the Milne orbifold as it appears in the dual formulation of the matrix big bang.

# Part III

# Research

# Chapter 12

# Scalar field on the parabolic orbifold

*He first took my altitude by a quadrant, and then with a rule and compasses described the dimensions and outlines of my whole body, all which he entered upon paper, and in six days brought my clothes very ill made, and quite out of shape, by happening to mistake a figure in his calculation.*

*"Gulliver's travels," Jonathan Swift*

In this chapter, we study the quantum dynamics of a free scalar field propagating on the parabolic orbifold, based on the publication [95]. The parabolic orbifold is a singular spacetime and we will carry out our study of the field evolution by considering a geometrical resolution procedure, as anticipated in chapter 10. The geometrical resolution procedure means that we consider the propagation of the field on a regularized spacetime and then take the singular limit. We will immediately consider a whole class of geometrical resolutions of the parabolic orbifold. We have coined this class of resolved geometries the "generalized nullbrane" (which involves two additional parameters). The parabolic orbifold, and (generalized) nullbrane were already introduced in chapter 10. In addition, other important background material concerning geometrical resolutions has been provided there, especially section 10.7 that deals with Hamiltonians with a multiple-operator structure. I refer to chapter 6 for the discussion of a free scalar field in a curved spacetime.

The evolution of a free scalar field on the generalized nullbrane spacetime can be described by means of its mode functions, which are classical solutions to the wave equation. We compute the mode functions by solving the wave equation exactly using WKB methods. Then we discuss properties of the mode functions of a free scalar field in the singular limit in which the generalized nullbranes turn back into the parabolic orbifold. We find that the limit of the mode functions exists for certain discrete values of the parameters $\alpha$ and $\beta$ in the metric of the generalized nullbrane. We compare with the literature in section 12.5 and give a qualitative discussion of the results in section 12.7.

The singular limit of the free scalar field involves the Maslov phase, a physical concept established in the context of caustics. The Maslov phase can be described most easily in the context of the quantum harmonic oscillator, which is summarized in appendix G. The derivation of the Hamiltonian for the free scalar field on the generalized nullbrane in the context of constrained systems, is worked out in appendix F. Those two topics are not included here because they would distract from the main line of the chapter.

## 12.1 Action and Hamiltonian of the free scalar field

We consider a free scalar field $\Phi(X^+, X^-, X, \Theta)$ with the action

$$S = -\frac{1}{2}\int \sqrt{-g}g^{\mu\nu}\left[\partial_\mu\Phi\partial_\nu\Phi + m^2\Phi^2\right] \mathrm{d}X^+\mathrm{d}X^-\mathrm{d}X\mathrm{d}\Theta. \tag{12.1}$$

The scalar field is propagating on the generalized nullbrane metric of which the metric was given in formula (10.21). We Fourier transform with respect to $X^-$ and write a Fourier series with respect to $\Theta$ (the coordinate $\Theta$ is compact),

$$\Phi(X^+, X^-, X, \Theta) = \frac{1}{2\pi}\sum_{k_\Theta}\int dk_-\phi_{k_-,k_\Theta}(X^+, X)\mathrm{exp}\left(ik_-X^- + ik_\Theta\Theta\right). \tag{12.2}$$

Here we have explicitly written

$$\phi_{k_-,k_\Theta}(X^+, X) \tag{12.3}$$

but from now on we will suppress the indices $k_-$ and $k_\Theta$ and omit the arguments $X^+$ and $X$. Because of the difference in notation between the original field $\Phi(X^+, X^-, X, \Theta)$ and the field modes $\phi_{k_-,k_\Theta}(X^+, X)$ that appear in the Fourier transform of the field, the suppression of the indices should not cause any confusion. We use a standard expression for the $\delta$-function

$$\delta(k - k') = \frac{1}{2\pi}\int \exp\left(i(k - k')x\right)\mathrm{d}x, \tag{12.4}$$

and rewrite the action as,

$$S = \sum_{k_\Theta} \int \mathrm{d}X^+ \, \mathrm{d}k_- \, \mathrm{d}X \ \sqrt{R^2 + (X^+)^2} \Bigg[ -ik_- \phi^* \partial_{X^+} \phi - \frac{\partial_X \phi \partial_X \phi^*}{2} - \left( \frac{m^2}{2} + \frac{k_\Theta^2}{2(R^2 + (X^+)^2)} + \frac{\alpha X^2 R^2 k_-^2}{2(R^2 + (X^+)^2)^2} + \frac{k_\Theta k_- \beta X R}{(R^2 + (X^+)^2)^{3/2}} \right) \phi \phi^* \Bigg]. \tag{12.5}$$

Denoting $\partial_{X^+}\phi$ as $\dot{\phi}$, the wave equation reads

$$-i\dot{\phi} = \frac{iX^+}{2\left(R^2 + (X^+)^2\right)} \phi - \frac{\partial_X^2 \phi}{2k_-} + \frac{\beta X R k_\Theta}{\left(R^2 + (X^+)^2\right)^{3/2}} \phi + \frac{k_\Theta^2}{2k_- \left(R^2 + (X^+)^2\right)} \phi + \frac{\alpha}{2} \frac{X^2 R^2 k_-}{\left(R^2 + (X^+)^2\right)^2} \phi + \frac{m^2}{2k_-} \phi. \tag{12.6}$$

In principle we have to deal with constraints when deriving the Hamiltonian that corresponds to the Lagrangian of the action (12.5) because the Lagrangian is first order in time-derivatives. However, we can take the following shortcut. The field $\Phi$ is real, so the following equality holds for the Fourier transform,

$$\phi_{-k_-, -k_\Theta} = \phi^*_{k_-, k_\Theta}, \tag{12.7}$$

If we consider $\phi^*_{k_-,k_\Theta}$ as nondynamical, and only $\phi_{k_-,k_\Theta}$ as a canonical coordinate for all $(k_-, k_\Theta)$, we can interprete $\pi \equiv -ik_- \sqrt{R^2 + (X^+)^2}\phi^*$ as its conjugate momentum. The Hamiltonian then reads

$$H = \sum_{k_\Theta} \int \mathrm{d}k_- \, \mathrm{d}X \, \pi \Bigg[ -\frac{X^+}{2\left(R^2 + (X^+)^2\right)} + \frac{im^2}{2k_-} - \frac{i}{2k_-} \partial_X^2 + \frac{i\beta X R k_\Theta}{\left(R^2 + (X^+)^2\right)^{3/2}} + \frac{i}{2k_-} \frac{k_\Theta^2}{R^2 + (X^+)^2} + \frac{i\alpha}{2} \frac{X^2 R^2 k_-}{\left(R^2 + (X^+)^2\right)^2} \Bigg] \phi. \tag{12.8}$$

For completeness I will derive the Hamiltonian (12.8) more carefully in appendix F (it has been moved to an appendix because it is long enough to distract the reader's attention from the main line developed here, although it is of no further importance for the remainder of the chapter). The Hamiltonian (12.8) is manifestly of the form $H = \sum_i f_i(t, R) H_i$; in other words, it belongs to the class of Hamiltonians we singled out in section 10.7.

## 12.2 Dynamical group and auxiliary Hamiltonian

We now show that the Hamiltonian (12.8) leads to a finite-dimensional *dynamical group* structure of the type discussed in section 10.7. The canonical variables $\pi(X,t)$ and $\phi(X,t)$ appear in four combinations $\int \pi\, \partial_X^2 \phi$, $\int \pi\phi$, $\int \pi X\phi$ and $\int \pi X^2\phi$. The commutation relations for such operators are given by

$$\left[\int \pi\, \hat{A}(X)\, \phi\, dX\, ,\ \int \pi\, \hat{B}(X)\, \phi\, dX\right] = \int \pi[\hat{A}, \hat{B}]\phi\, dX, \tag{12.9}$$

reducing the commutator algebra to that of $\{\partial_X^2,\ 1,\ X,\ X^2\}$, which closes after the addition of $\{X\partial_X,\ \partial_X\}$. Equivalently, we can form the standard (single degree of freedom) creation and annihilation operators $a$ and $a^\dagger$ out of $X$ and $\partial_X$, to conclude that the Lie algebra of the dynamical group is spanned by

$$n = a^\dagger a + 1/2, a^{\dagger 2}, a^2, a^\dagger, a \text{ and } I. \tag{12.10}$$

Given the inclusion of powers of the creation operator up to $a^{\dagger 2}$, it is not surprising that the corresponding algebra has become known as the *two-photon algebra*, or $h_6$, and has been featured in discussions of quantum optics, and squeezed states in particular (see, for example [115]). A complete formal analysis of quantum dynamics on the two-photon group has been given in [116].

Following the picture presented in the previous chapter in section 10.7, we could use the two-photon group considerations of [116] to reduce the question of free scalar field dynamics on the generalized nullbrane to ordinary differential equations. In our present setting, however, we can perform these operations in a considerably more familiar guise. Namely, since the free scalar field is linear, solving for its quantum dynamics amounts to constructing a complete set of solutions to the classical wave equation (12.6). Furthermore, the classical wave equation turns out to be equivalent to the Schrödinger equation for a *linear* auxiliary one-dimensional quantum system. Because the auxiliary system is linear, its Schrödinger equation (i.e. the wave equation of the original scalar field) can be solved exactly by WKB methods. The latter effectively reduce the problem to ordinary differential equations (the classical equations of motion of the auxiliary linear system). Thus, one attains the same level of simplification through this method as through performing the analysis of [116]. We can refer to the above procedure as "double-semiclassical" analysis (there is an (exact) WKB procedure leading from a free quantum scalar field to the wave equation for the mode functions, and an (exact) WKB procedure leading from the wave equation for the mode functions to a one-dimensional auxiliary classical system). Note that both the "double-semiclassical" approach and the general dynamical group approach of [116] (which are essentially one and the same thing) are made possible by the fact that the metric of the generalized nullbrane is a quadratic polynomial in the $X$-variable.

For linear quantum systems, it is most common to work in the Heisenberg picture, instead of (equivalently) deriving WKB wavefunctions in the Schrödinger picture. We obtain the solution for the Heisenberg field operator as an expansion in terms of a complete set of mode functions $u_y(X, X^+)$ (with $y$ being a generic basis label) satisfying the classical equations of motion:

$$\phi_{k_-,k_\Theta} = \int \mathrm{d}y \; u_y(X, X^+) a(y). \tag{12.11}$$

The corresponding conjugate momentum is

$$\pi_{k_-,k_\Theta} = ik_- \sqrt{R^2 + (X^+)^2} \int \mathrm{d}y \; u_y^*(X, X^+) a^\dagger(y). \tag{12.12}$$

If we demand the standard commutation relations for the creation-annihilation operators $a^\dagger$ and $a$, the canonical commutation relation between $\pi$ and $\phi$ determine the normalisation of the mode functions (this is the analog for first order systems of the Klein-Gordon norm for second order systems, see equation (6.13)):

$$\delta(X - \tilde{X}) = k_- \sqrt{R^2 + (X^+)^2} \int \mathrm{d}y \; u_y^*(X, X^+) u_y(\tilde{X}, X^+). \tag{12.13}$$

## 12.3 Solution of the wave equation

To recapitulate, we investigate a free scalar field $\Phi(X^+, X, X^-, \Theta)$ on the generalized nullbrane. Because the field does not interact, the solution for the field evolution in the quantum theory is completely determined by classical solutions to the wave equation (i.e. the mode functions). We performed a Fourier transform to the field $\Phi$ (with respect to the coordinates $X^-$ and $\Theta$) and we composed the field in modes $\phi_{k_-,k_\Theta}(X^+, X)$, which satisfy the wave equation (12.6), for convenience repeated here,

$$\begin{aligned} -i\dot{\phi} \;\; = \;\; & \frac{iX^+}{2\left(R^2 + (X^+)^2\right)}\phi - \frac{\partial_X^2 \phi}{2k_-} + \frac{\beta X R k_\Theta}{\left(R^2 + (X^+)^2\right)^{3/2}}\phi \\ & + \frac{k_\Theta^2}{2k_-\left(R^2 + (X^+)^2\right)}\phi + \frac{\alpha}{2}\frac{X^2 R^2 k_-}{\left(R^2 + (X^+)^2\right)^2}\phi + \frac{m^2}{2k_-}\phi. \end{aligned} \tag{12.14}$$

The wave equation (12.14) has the form of a Schrödinger equation and is quadratic in $X$ and $P_X$, and can therefore be solved exactly by WKB methods. Once we find a set of classical solutions to (12.14) (superficially, these classical solutions will have the "appearance" of a quantum wavefunction because the wave equation resembles a Schrödinger equation), we can promote these to mode functions.

Denoting $X^+$ by $t$, (12.14) takes the form of an auxiliary Schrödinger equation with Hamiltonian

$$\mathcal{H} = \frac{it}{2\left(R^2+t^2\right)} + \frac{P^2}{2k_-} + \frac{\beta X R k_\Theta}{\left(R^2+t^2\right)^{3/2}} + \frac{k_\Theta^2}{2k_-\left(R^2+t^2\right)} + \frac{\alpha}{2}\frac{X^2R^2k_-}{\left(R^2+t^2\right)^2} + \frac{m^2}{2k_-} \tag{12.15}$$

(up to a sign difference in the left hand side of (12.14)). As the corresponding Hamiltonian (12.15) is quadratic in $X$, (12.6) can be solved exactly by WKB methods. The starting point is the observation that the ansatz

$$\phi(X_1,t_1|X_2,t_2) = \mathcal{A}(t_1,t_2)\exp\left(-iS_{cl}\left[X_1,t_1|X_2,t_2\right]\right) \tag{12.16}$$

solves (12.6) (with $t_2 \to t$) if

$$S_{cl} = \int_{t_1}^{t_2} \mathrm{d}t\,\left(P\dot{X} - \mathcal{H}\right)\Big|_{X=X_{cl}(X_1,t_1|X_2,t_2)}; \tag{12.17}$$

$$-2k_-\,\frac{\partial\mathcal{A}(t_1,t)}{\partial t} = \mathcal{A}(t_1,t)\,\frac{\partial^2 S_{cl}\left[X_1,t_1|X,t\right]}{\partial X^2}. \tag{12.18}$$

Here, $S_{cl}$ is the classical action with boundary conditions $X(t_1) = X_1,\ X(t_2) = X_2$. More general solutions to (12.6) are obtained by integrating (12.16) over $X_1$, weighted by an arbitrary smooth wavepacket.

A subtlety arises in this ansatz when the dynamical evolution reaches a *focal point* $t_2 = t^*$, where the classical action diverges, unless a certain relation between $X_1$ ("the source") and $X_2$ ("the image") is met. At such focal points, the differential equation for $\mathcal{A}(t_1,t_2)$ becomes singular. In that case, one solves the WKB equations away from $t^*$ and connects the solution by a phase jump at the focal point. The phase jump should be chosen such that convolutions of (12.16) with a smooth wavepacket are continuous across the focal point. The general guidelines for this procedure are best familiar in the context of *caustic* submanifolds in geometrical optics (see, for example, [16, 12]) and the above-mentioned correction pre-factors have become known as the *Maslov phases*. We will give some further details in the next section. For pedagogical reasons we will also illustrate the appearance of the Maslov phase in a simple system in appendix G. In that appendix we comment on the path-integral description of the quantum harmonic oscillator, where the Maslov phase appears naturally (without having to refer to the full theory of Maslov).

In order to compute the classical action we first need to consider the classical motion and its solution. We use the Hamilton equations

$$\dot{X} = \frac{\partial\mathcal{H}}{\partial P}, \qquad \dot{P} = -\frac{\partial\mathcal{H}}{\partial X}, \tag{12.19}$$

to obtain the classical equation of motion:

$$\ddot{X} + \alpha \frac{R^2}{\left(R^2+t^2\right)^2} X = -\frac{\beta k_\Theta}{k_-} \frac{R}{\left(R^2+t^2\right)^{3/2}}, \tag{12.20a}$$

$$X(t_1) = X_1, \qquad X(t_2) = X_2. \tag{12.20b}$$

This equation is actually exactly solvable, and it has become known as the equation for "bending of a double-walled compressed bar with a parabolic cross-section" [15]. It can be reduced to a driven harmonic oscillator with constant frequency via substitution $X = \sqrt{R^2+t^2}\chi(\eta(t))$, taking $\eta = \arctan(t/R)$:

$$\frac{d^2\chi}{d\eta^2} + (1+\alpha)\,\chi = -\frac{\beta k_\Theta}{R k_-}. \tag{12.21}$$

In order to give a transparent derivation of the solution to (12.20) and the corresponding value of the classical action, we first consider the two independent solutions to the homogeneous version of (12.20a):

$$f(t) = \sqrt{R^2+t^2}\,\sin\left(\sqrt{1+\alpha}\arctan\frac{t}{R}\right), \tag{12.22a}$$

$$h(t) = \sqrt{R^2+t^2}\,\cos\left(\sqrt{1+\alpha}\arctan\frac{t}{R}\right). \tag{12.22b}$$

A useful object to consider is the Dirichlet Green function of the operator

$$\mathcal{D} = \partial_t^2 + \frac{\alpha R^2}{\left(R^2+t^2\right)^2}. \tag{12.23}$$

The Green function is given by

$$G(t,t'|t_1,t_2) = \frac{\left(f_1 h(t_<) - h_1 f(t_<)\right)\left(f_2 h(t_>) - h_2 f(t_>)\right)}{W[f,h](f_1 h_2 - h_1 f_2)}, \tag{12.24}$$

and satisfies

$$\left(\partial_t^2 + \frac{\alpha R^2}{\left(R^2+t^2\right)^2}\right) G(t,t'|t_1,t_2) = \delta(t-t'), \tag{12.25}$$

$$G(t_1,t'|t_1,t_2) = 0, \qquad G(t_2,t'|t_1,t_2) = 0 \tag{12.26}$$

with $W[f,h] = f\dot{h} - h\dot{f}$ being the Wronskian of $f(t)$ and $h(t)$ (independent of $t$), $t_2 > t_1$, $t_< = \min(t,t')$, $t_> = \max(t,t')$ and $f_1 = f(t_1)$, $h_1 = h(t_1)$, etc.

With the Green function given by (12.24), and $b(t)$ denoting the right hand side of equation (12.20a),

$$b(t) = -\frac{\beta k_\Theta}{R k_-} \frac{R}{\left(R^2+t^2\right)^{3/2}}, \tag{12.27}$$

we can write down the solution to (12.20) as

$$X_{cl}(t|X_1, t_1; X_2, t_2) = -X_1 \partial_{t'} G(t, t'|t_1, t_2)\Big|_{t'=t_1} + X_2 \partial_{t'} G(t, t'|t_1, t_2)\Big|_{t'=t_2} + \int_{t_1}^{t_2} dt'\, G(t, t'|t_1, t_2) b(t') \tag{12.28}$$

Given the above formulas, the classical action can be written in a relatively general form that will turn out to be useful later. For a more general Hamiltonian

$$\mathcal{H} = \frac{P^2}{2\mu} + \Omega^2(t)\frac{X^2}{2} - b(t)X + \gamma(t), \tag{12.29}$$

which leads to a differential operator $\mathcal{D} = \partial_t^2 + \Omega^2(t)$, we can re-use the Green function (12.24) with $f(t)$ and $h(t)$ solutions to the homogeneous equation of motion $\mathcal{D}X = 0$. We can then write the classical action corresponding to (12.29) as

$$S_{cl} = -\frac{\mu}{2}\left[\frac{h_2\dot{f}_1 - f_2\dot{h}_1}{f_1h_2 - h_1f_2}\right]X_1^2 + \frac{\mu}{2}\left[\frac{f_1\dot{h}_2 - h_1\dot{f}_2}{f_1h_2 - h_1f_2}\right]X_2^2 - \mu\left[\frac{W[f,h]}{f_1h_2 - h_1f_2}\right]X_1X_2 \tag{12.30a}$$

$$-\mu\int_{t_1}^{t_2} \mathrm{d}t\; b(t)\left(\frac{h_2f(t) - f_2h(t)}{f_1h_2 - f_2h_1}\right)X_1 - \mu\int_{t_1}^{t_2} \mathrm{d}t\; b(t)\left(\frac{f_1h(t) - h_1f(t)}{f_1h_2 - f_2h_1}\right)X_2 \tag{12.30b}$$

$$+\mu\int_{t_1}^{t_2} \mathrm{d}t' \int_{t_1}^{t'} \mathrm{d}t\; b(t)\frac{\big(f_1h(t) - h_1f(t)\big)\big(f_2h(t') - h_2f(t')\big)}{W[f,h](f_1h_2 - h_1f_2)}b(t') \tag{12.30c}$$

$$-\int_{t_1}^{t_2} \gamma(t)\mathrm{d}t\,. \tag{12.30d}$$

Equivalently, we can also solve the equations of motion (12.20) explicitly:

$$X = X_1\frac{\sqrt{R^2+t^2}}{\sqrt{R^2+t_1^2}}\frac{\sin 2\Delta_{t2}}{\sin 2\Delta_{12}} + X_2\frac{\sqrt{R^2+t^2}}{\sqrt{R^2+t_2^2}}\frac{\sin 2\Delta_{1t}}{\sin 2\Delta_{12}} - \frac{\beta k_\Theta\sqrt{R^2+t^2}}{Rk_-(1+\alpha)}\left[1 - \frac{\sin 2\Delta_{t2}}{\sin 2\Delta_{12}} - \frac{\sin 2\Delta_{1t}}{\sin 2\Delta_{12}}\right] \tag{12.31a}$$

where we have used abbreviations for the following arguments

$$\Delta_{12} = \frac{\sqrt{1+\alpha}}{2}\left(\arctan\frac{t_2}{R} - \arctan\frac{t_1}{R}\right) \tag{12.31b}$$

$$\Delta_{t2} = \frac{\sqrt{1+\alpha}}{2}\left(\arctan\frac{t_2}{R} - \arctan\frac{t}{R}\right) \tag{12.31c}$$

$$\Delta_{1t} = \frac{\sqrt{1+\alpha}}{2}\left(\arctan\frac{t}{R} - \arctan\frac{t_1}{R}\right). \tag{12.31d}$$

The classical action can now be evaluated either by brute force using the explicit classical solution (12.31), or, with less work, from (12.30):

$$S_{cl}[X_1, t_1|X_2, t_2] = -k_-\left[\frac{t_1}{2(R^2+t_1^2)} - \frac{R\sqrt{1+\alpha}}{2(R^2+t_1^2)}\cot 2\Delta_{12}\right]X_1^2 \tag{12.32a}$$

$$+k_-\left[\frac{t_2}{2(R^2+t_2^2)} + \frac{R\sqrt{1+\alpha}}{2(R^2+t_2^2)}\cot 2\Delta_{12}\right]X_2^2 \tag{12.32b}$$

$$-\left[\frac{k_-\sqrt{1+\alpha}\,R}{\sqrt{R^2+t_1^2}\sqrt{R^2+t_2^2}\,\sin 2\Delta_{12}}\right]X_1X_2 \tag{12.32c}$$

$$-\left[\frac{\beta\,k_\Theta}{\sqrt{1+\alpha}\sqrt{R^2+t_1^2}}\tan\Delta_{12}\right]X_1 \tag{12.32d}$$

$$-\left[\frac{\beta\,k_\Theta}{\sqrt{1+\alpha}\sqrt{R^2+t_2^2}}\tan\Delta_{12}\right]X_2 \tag{12.32e}$$

$$-\frac{\beta^2\,k_\Theta^2}{k_-\,(1+\alpha)^{3/2}\,R}\Big(\tan\Delta_{12} - \Delta_{12}\Big) \tag{12.32f}$$

$$-\frac{m^2}{2k_-}(t_2-t_1) - \frac{i}{2}\ln\frac{\sqrt{R^2+{t_2}^2}}{\sqrt{R^2+{t_1}^2}} - \frac{k_\Theta^2\Delta_{12}}{k_-\,R\sqrt{1+\alpha}}. \tag{12.32g}$$

Next we consider the "quantum-mechanical" prefactor $\mathcal{A}(t_1, t_2)$ that appeared in (12.16). With the expression for $S_{cl}$ given by (12.30), equation (12.18) becomes:

$$\frac{\partial \mathcal{A}(t_1,t)}{\partial t} = -\frac{1}{2}\frac{f_1\dot{h} - h_1\dot{f}}{f_1 h - h_1 f}\,\mathcal{A}(t_1,t). \tag{12.33}$$

This leads to the solution

$$\mathcal{A}(t_1,t_2) = \mathcal{N}\left(R^2+t_1^2\right)^{-1/4}\left(R^2+t_2^2\right)^{-1/4}|\sin 2\Delta_{12}|^{-1/2}\,\phi_M, \tag{12.34}$$

which contains the Maslov phase $\phi_M$ and a constant normalization factor $\mathcal{N}$. The Maslov phase is piecewise constant away from the focal points (the positions of focal point for $t_2$ are functions of $t_1$). Its value is worked out in the next section 12.4.

We can now fix the normalization of $\mathcal{A}(t_1, t_2)$ by imposing (12.13):

$$\mathcal{N} = \sqrt{\frac{R\sqrt{1+\alpha}}{2\pi\sqrt{R^2+t_1^2}}} \tag{12.35}$$

## 12.4 Focusing properties of the wave equation

In the evolution of classical dynamical systems, it often happens that all the classical trajectories that start at $(X_1, t_1)$ will reach the same point $X^*(X_1)$ at the same moment $t^*(t_1)$, irrespectively of their initial velocity $V_1$. Under such circumstances, we say that $t^*(t_1)$ is classical focal point (or caustic) of the evolution. If $t^*(t_1)$ is such a focal point, the classical action $S_{cl}[X_1, t_1|X_2, t^*(t_1)]$ will diverge unless $X_2 = X^*(X_1, t_1)$. Basically, if it did not, there would have been classical trajectories connecting $(X_1, t_1)$ and $(X_2, t^*(t_1))$ for $X_2 \neq X^*(X_1, t_1)$,in contradiction with the definition of a focal point. A general recipe for handling these divergences can be given [16, 12] and introduces the concept of the "Maslov phase".

As we already remarked in section 12.3, if we pursue a semiclassical construction of the quantum-mechanical mode functions, the singular behavior of the classical action near focal points introduces formal complications in (12.18). We will analyze the Maslov phase for our rather general form of the classical action (12.30). To this end we rewrite the classical action in the following form:

$$S_{cl}[X_1, t_1|X, t] = \frac{k_-}{2}\left[\frac{f_1\dot{h} - h_1\dot{f}}{f_1 h - h_1 f}\right](X - X^*(X_1, t_1, t))^2 + \cdots \tag{12.36}$$

$$= \frac{k_-}{2}\frac{\partial}{\partial t}\ln\left[f_1 h - h_1 f\right](X - X^*(X_1, t_1, t))^2 + \cdots, \tag{12.37}$$

where the dots represent contributions non-singular at $t = t^*(t_1)$. A focal point

$$X^*(X_1, t_1, t^*(t_1)) \tag{12.38}$$

is reached whenever

$$f_1 h(t^*) - h_1 f(t^*) \equiv f(t_1)h(t^*) - h(t_1)f(t^*) = 0. \tag{12.39}$$

At the same time, equation (12.33) for the prefactor $\mathcal{A}(t_1, t)$ can be solved on the left and on the right of the focal point $t^*(t_1)$ (even though constructing a solution at $t = t^*(t_1)$ naively would be problematic on account of the singularity on the right hand side of (12.33)):

$$\begin{aligned} \mathcal{A}(t_1, t) &= \mathcal{N}_< \left|f_1 h(t) - h_1 f(t)\right|^{-1/2} \qquad (t < t^*(t_1)), \\ \mathcal{A}(t_1, t) &= \mathcal{N}_> \left|f_1 h(t) - h_1 f(t)\right|^{-1/2} \qquad (t > t^*(t_1)) \end{aligned} \tag{12.40}$$

(with $\mathcal{N}_<$ and $\mathcal{N}_>$ being complex constants).

Armed with these relations, we can examine the behavior of the entire wavefunction $\phi(X_1, t_1|X, t) = \mathcal{A}(t_1, t)\exp\left(-iS_{cl}\left[X_1, t_1|X, t\right]\right)$ in the vicinity of a focal point $t^*(t_1)$. Generically assuming that $f_1 h(t) - h_1 f(t)$ has a simple zero at the focal point, $f_1 h(t) - h_1 f(t) \propto t - t^*(t_1)$, and keeping in mind that

$$\lim_{\lambda\to\infty} \sqrt{\frac{|\lambda|}{\pi}} \exp\left(-\frac{i\pi}{4}\mathrm{sign}(\lambda)\right) \exp\left(i\lambda x^2\right) \;=\; \delta(x), \tag{12.41}$$

we conclude that

$$\begin{aligned} \lim_{t\to(t^*(t_1))_-} \phi(X_1, t_1|X, t) &= \mathcal{A}_< \delta(X - X^*(X_1, t_1)), \\ \lim_{t\to(t^*(t_1))_+} \phi(X_1, t_1|X, t) &= \mathcal{A}_> \delta(X - X^*(X_1, t_1)), \end{aligned} \tag{12.42}$$

with

$$\frac{\mathcal{A}_<}{\mathcal{A}_>} = \frac{\mathcal{N}_< \exp\left(i\pi\,\mathrm{sign}(k_-)/4\right)}{\mathcal{N}_> \exp\left(-i\pi\,\mathrm{sign}(k_-)/4\right)} \tag{12.43}$$

If we further demand that the limits in (12.42) should be the same (this automatically ensures that any convolution of $\phi(X_1, t_1|X, t)$ with a smooth wavepacket is continuous across the focal point), we conclude that

$$\frac{\mathcal{N}_>}{\mathcal{N}_<} = \exp\left(\frac{i\pi}{2}\mathrm{sign}(k_-)\right). \tag{12.44}$$

When there are many focal points $t^*_\ell(t_1)$, each of them will give a contribution, and the resulting wavefunction can be written as

$$\phi(X_1, t_1|X, t) = \mathcal{N}\phi_M \left|f_1 h(t) - h_1 f(t)\right|^{-1/2} \exp\left(-iS_{cl}\left[X_1, t_1|X, t\right]\right) \tag{12.45}$$

with a constant normalization factor $\mathcal{N}$ and the *Maslov phase* $\phi_M$ of the form

$$\phi_M = \exp\left(\frac{i\pi}{2}\mathrm{sign}(k_-) \sum_\ell \theta(t - t^*_\ell)\right) \tag{12.46}$$

($\theta(t)$ being the Heaviside step function).

We now turn to our specific case for which the relevant term in the classical action (12.32) near a focal point is given by:

$$S_{cl}[X_1, t_1|X, t] \simeq k_- \left[\frac{t}{2\left(R^2 + t^2\right)} + \frac{R\sqrt{1+\alpha}}{2\left(R^2 + t^2\right)} \cot 2\Delta_{1t}\right] X^2 + \cdots \tag{12.47}$$

The focal points are the poles of $\cot 2\Delta_{1t}$:

$$t^* = \frac{t_1 + R\tan(\pi\ell/\sqrt{1+\alpha})}{1 - \tan(\pi\ell/\sqrt{1+\alpha})\, t_1/R}\,, \qquad \ell \in \mathbb{Z} \tag{12.48}$$

The value of $\alpha$ determines the number of focal points. We will restrict our attention to the case $\alpha = (2N)^2 - 1$ relevant for our present investigation, and we obtain $\ell \in \{1, \ldots, 2N-1\}$, i.e. $2N-1$ focal points.

In the $R \to 0$ limit, all the $2N-1$ focal points are squeezed into $t = 0$. This means that there will be one large phase jump from $t < 0$ to $t > 0$. We thus obtain the following expression for the Maslov phase in the $R \to 0$ limit:

$$\phi_M = \exp\left(\frac{i\pi}{2}\mathrm{sign}(k_-)\,(2N-1)\,\theta(t)\right). \tag{12.49}$$

## 12.5 Comparison with earlier work

To facilitate comparison with earlier work, we now derive momentum basis mode functions using the position basis mode functions $\phi\,(X_1, t_1|X_2, t_2)$ of section 12.3. The existence of an $R \to 0$ limit will not be affected by such conversion. To obtain the momentum basis mode functions, we have to manipulate our "propagator" $\phi\,(X_1, t_1|X_2, t_2)$. First, we take a Fourier transform (with respect to $X_1$) of the "propagator" to convert it to an incoming plane wave basis. The Fourier transform introduces an exponential phase $\exp\left[-ip^2 t_1/(2k_-)\right]$ which only depends on the initial time $t_1$ and on the wave number of the Fourier transform. This term is present already for "free evolution". Another such term is $\exp\left[-im^2 t_1/(2k_-)\right]$ which appears through (12.32g). We cancel these terms by multiplying with

$$\exp\left(\frac{i\left(m^2+p^2\right)t_1}{2k_-}\right). \tag{12.50}$$

This is possible because we are simply using the freedom we have in defining the basis of mode functions. We also omit other time-independent overall phase factors. Finally, we will take the limit $t_1 \to -\infty$, which refers our momentum labels to incoming waves in the infinite past:

$$\begin{aligned} V_{k_-,k_\ominus,p,m} = \lim_{t_1\to-\infty}\int_{-\infty}^{\infty} \mathrm{d}X_1\,\mathcal{A}(t_1,t_2)\exp\left(-iS_{cl}\right) \\ \times\ \exp\left(ipX_1\right)\exp\left(\frac{i\left(m^2+p^2\right)t_1}{2k_-}\right) \end{aligned} \tag{12.51}$$

To be consistent with our notation in the beginning of this section, we now switch back to writing $X^+$ instead of $t$ for all spacetime quantities. We finally obtain the mode functions

$$V_{k_-,k_\Theta,p,m} = \sqrt{\frac{2RN}{|k_-|\,(R^2+(X^+)^2)\,|\,\sin 2\Delta\,|}}\,\phi_M \times \tag{12.52}$$

$$\exp\Big[-ik_-\,X^2\frac{(2NR\cot 2\Delta\,+X^+)}{2\,(R^2+(X^+)^2)} + \frac{i\,X}{\sqrt{R^2+(X^+)^2}}\left(k_\Theta \tan\Delta + \frac{2RNp}{\sin 2\Delta}\right)$$

$$+\frac{ipk_\Theta}{k_-}\tan\Delta - \frac{ip^2RN}{k_-}\cot 2\Delta + \frac{ik_\Theta^2}{2k_-NR}\tan\Delta + \frac{im^2X^+}{2k_-} + ik_-X^- + ik_\Theta\Theta\Big]$$

where $\Delta = N\arctan(X^+/R) + \pi N/2$.

For $R \to 0$ we obtain:

$$V_{k_-,k_\Theta,p,m} = \frac{1}{\sqrt{|k_-X^+|}}\exp\left(\frac{i\pi}{2}\mathrm{sign}(k_-)(2N-1)\theta(X^+)\right) \times \tag{12.53}$$

$$\exp\Big[-ipX\,\mathrm{sign}(X^+) + \frac{ip^2X^+}{k_-} - \frac{ik_\Theta^2}{2k_-X^+} + \frac{im^2X^+}{2k_-} + ik_-X^- + ik_\Theta\Theta\Big],$$

where $\theta(t)$ denotes the Heaviside step function.

In order to compare our mode functions (12.52) with those derived in [86], we write out the mode functions for $N = 1$ explicitly:

$$V^{(N=1)}_{k_-,k_\Theta,p,m} = \frac{1}{\sqrt{|k_-X^+|}}\,\phi_M\,\exp\Big[\frac{-ik_-\,R^2X^2}{2X^+\,(R^2+(X^+)^2)} - \frac{ipk_\Theta R}{k_-X^+}$$

$$- \frac{iX}{\sqrt{R^2+(X^+)^2}}\left(k_\Theta\frac{R}{X^+} + p\frac{R^2+(X^+)^2}{X^+}\right)$$

$$-\frac{ip^2}{k_-}\left(\frac{R^2-(X^+)^2}{2X^+}\right) + ik_-X^- + ik_\Theta - \frac{ik_\Theta^2}{2k_-X^+} + \frac{im^2X^+}{2k_-}\Theta\Big]. \tag{12.54}$$

We have already given the general expression for the Maslov phase $\phi_M$ is given in section 12.4. For $N = 1$, there is only one focal point at $X^+ = 0$, and the Maslov phase becomes:

$$\phi_M = \exp\left(\frac{i\pi}{2}\mathrm{sign}(k_-)\,\theta(X^+)\right). \tag{12.55}$$

Written in our interpolating coordinates from (10.19), the mode functions of [86] can be re-expressed as

$$\frac{1}{\sqrt{iX^+}}\exp\Big[\frac{ip^+R^2X^2}{2X^+\,(R^2+(X^+)^2)} - \frac{i\,X}{\sqrt{R^2+(X^+)^2}}\left(J\frac{R^2+(X^+)^2}{RX^+} - n\frac{X^+}{R}\right)$$

$$-\frac{iX^+}{2p^+}\left(\frac{n-J}{R}\right)^2 + \frac{iJ^2}{2p^+X^+} - \frac{im^2X^+}{2p^+} - ip^+X^- + in\Theta\Big], \tag{12.56}$$

with $J$, $n$ being the labels used in [86].

The equality of the two expressions (up to normalization conventions) is established by recognizing the following identifications:

$$\begin{aligned} k_- &= -p^+, \\ p &= \frac{J-n}{R}, \\ k_\Theta &= n. \end{aligned} \tag{12.57}$$

Our results thus agree with those of [86] for non-zero values of $R$ in the particular case of the nullbrane ($\alpha = 3$, $\beta = 2$). A further comparison between the modefunctions of [86] and a minimal subtraction procedure applied to the nullbrane was already made in section 10.6.3 of chapter 10. Now we are ready to examine the $R \to 0$ limit (where our choice of coordinates will reveal a peculiar reflection property at the singularity) for the original as well as the generalized nullbrane.

## 12.6 Construction of the singular limit

Now that we have constructed the "propagator" $\phi(X_1, t_1|X_2, t_2)$, which provides a basis of mode functions labeled by $X_1$, we can investigate its $R \to 0$ limit, which will only exist for special values of $\alpha$ and $\beta$.

The first non-trivial condition for the existence of an $R \to 0$ limit comes from the prefactor $\mathcal{A}(t_1, t_2)$. This will vanish identically for $t_1 < 0$, $t_2 > 0$ as $R$ is sent to 0 (which would make the field operator vanish identically and manifestly destroy unitarity), unless

$$\alpha = N^2 - 1, \tag{12.58}$$

with an integer $N$, on account of the structure $\sqrt{R/\sin 2\Delta_{12}}$ in (12.34) combined with (12.35).

This behavior of the prefactor $\mathcal{A}(t_1, t_2)$ can be naturally understood by inspecting the classical homogeneous solutions (12.22). For generic values of $\alpha$ and $\beta$, in the $R \to 0$ limit, those behave as $f(t) \to t$ and $h(t) \to |t|$. Since these two functions are not linearly independent on the negative real axis, we will not be able to specify arbitrary initial conditions ($X_1$,$V_1$) for the classical solution at $t_1 < 0$. Should we try to do so, in the $R \to 0$ limit, the classical trajectory will be kicked away to infinity for all $t > 0$. Correspondingly, all quantum wavepackets will be kicked away to infinity, and the wavefunction will vanish at all finite values of $X$ for $t > 0$, as manifested by the behavior of the prefactor $\mathcal{A}(t_1, t_2)$. This problem is avoided, however, for special values of $\alpha$: if $\alpha = (2N)^2 - 1$ (with an integer $N$), the $R \to 0$ limit of the two solutions is $f(t) \to \mathrm{sign}(t)$ and $h(t) \to |t|$; if $\alpha = (2N+1)^2 - 1$, it is $f(t) \to t$ and $h(t) \to 1$.

A further condition on $\alpha$ and $\beta$ arises from considering the $R \to 0$ limit of the classical action $S_{cl}[X_1, t_1 | X_2, t_2]$. The problematic terms in the action (12.32) are those with coefficients containing $k_\Theta^2$, namely:

$$-\frac{\beta^2\, k_\Theta^2}{k_-(1+\alpha)^{3/2}\, R}\Big(\tan\Delta_{12} - \Delta_{12}\Big) - \frac{k_\Theta^2 \Delta_{12}}{k_-\, R\sqrt{1+\alpha}}. \tag{12.59}$$

In order to cancel the divergences, we find the following equation for $\beta$:

$$\beta^2 = \frac{1+\alpha}{1 - \dfrac{\tan\left(\pi\sqrt{1+\alpha}/2\right)}{\pi\sqrt{1+\alpha}/2}}. \tag{12.60}$$

If $\alpha = (2N+1)^2 - 1$, this condition would make us naively conclude that $\beta = 0$. However, a direct inspection of (12.59) reveals that it wouldn't make the divergence cancel. The only other option remaining is

$$\alpha = (2N)^2 - 1, \qquad \beta = 2N \tag{12.61}$$

($\beta = -2N$ corresponds to the same space written in different coordinates). These are the conditions for existence of an $R \to 0$ limit for a free scalar field dynamics on the generalized nullbrane. As we would expect, the values $\alpha = 3$ and $\beta = 2$ corresponding to the original nullbrane do meet these conditions (for $N = 1$).

## 12.7 Discussion of the singular limit

To recapitulate, we have examined the dynamics of a free scalar field on the following three-parameter ($\alpha$, $\beta$, $R$) family of backgrounds ("generalized nullbrane"):

$$\begin{aligned} ds^2 =& -2dX^+dX^- + \frac{X^2R^2(\beta^2-\alpha)}{(R^2+(X^+)^2)^2}(dX^+)^2 + \frac{2\beta XR}{\sqrt{R^2+(X^+)^2}}dX^+d\Theta \\ &+ \left(R^2+(X^+)^2\right)d\Theta^2 + dX^2\,. \end{aligned} \tag{12.62}$$

As $R$ goes to 0, all of these geometries (irrespectively of the values of $\alpha$ and $\beta$) reduce (away from $X^+ = 0$) to the parabolic orbifold times a line:

$$ds^2 = -2dX^+dX^- + (X^+)^2 d\Theta^2 + dX^2 \tag{12.63}$$

What we have found is that the $R \to 0$ limit of the scalar field mode functions exists only when $\alpha = (2N)^2 - 1$ and $\beta = 2N$, with $N$ being an integer.

In terms of the limiting expression for the mode functions (12.53), we find few surprises. The result is essentially independent of the values of $\alpha$ and $\beta$

(for those values for which the limit exists) and bears a close resemblance to the mode functions obtained in [86]. However, the minor discrepancy between our results and those of [86] deserves some clarification.

For any $\alpha$ and $\beta$, the metric of the generalized nullbrane converges to the metric of the parabolic orbifold times a line (12.63), which is formally the same as the metric of the parabolic orbifold written in the $y$-coordinates of [86]. Given only the $R = 0$ expressions, it may therefore be tempting to identify $(y^+, y^-, y, u) \leftrightarrow (X^+, X^-, \Theta, X)$. With this identification, however, the mode functions are not exactly the same, even for the case of the original nullbrane ($\alpha = 3$, $\beta = 2$). The difference between the two is the factor of $\text{sign}(X^+)$ in front of the $ipX$ term in the exponential of (12.53). It is important to realize that the difference between the two sets of mode function does not represent any dynamical distinction. Rather, it is explained by the difference in the choice of coordinates.

To construct the parabolic orbifold (12.63) as an $R \to 0$ limit of the nullbrane, the authors of [86] employ their singular $y$-coordinates (this coordinate system fails at $X^+ = 0$ even for smooth spaces at non-zero $R$). As a result, they obtain mode functions without the aforementioned factor $\text{sign}(X^+)$. Incidentally, as I have anticipated in section 10.6.3, the same mode functions are obtained by applying the non-geometrical "minimal subtraction" prescription of [94] directly to a free scalar field on the parabolic orbifold, without any recourse to the nullbrane or its generalizations.

On the other hand, we construct the parabolic orbifold metric (12.63) and the corresponding coordinates as an $R \to 0$ limit of smooth coordinate systems parametrizing smooth geometries. In this case, the factor of $\text{sign}(X^+)$ is present. Its effect is that the position and velocity in the $X$-direction for all particles are reflected as they pass through $X^+ = 0$.

Even though the two sets of mode functions are essentially equivalent (and only differ by a coordinate choice), we may think of our present parametization as being more accurate. Indeed, it is very natural to demand that, since the singular space is constructed as an $R \to 0$ limit of smooth resolved geometries, the coordinates on the singular space should be constructed as an $R \to 0$ limit of smooth coordinate systems on the smooth resolved geometries (even though, with our present theoretical understanding of spacetime singularities, it is not possible to give a systematic justification to this treatment of coordinate systems). Note that the flip of the $X$-direction for positive $X^+$ cannot be undone by a smooth coordinate transformation, so it will always be present if the parabolic orbifold metric (12.63) is constructed as a limit of a smooth coordinate frame on the (generalized) nullbrane.

We find considerably more surprises if we contemplate the properties of those geometrical resolutions for which the singular limit of the scalar field dynamics exists (rather than merely examining the limiting expressions for the mode functions).

Firstly, looking at the nullbrane example of [86], we could get the impression that the singular limit exists because the curvature is identically zero for any finite $R$ (in a way, one can say that the singularity is never really "there"). This is different for the generalized nullbrane. The non-vanishing components of the Riemann tensor for our generalized nullbrane geometries are

$$R_{+X+X} = \frac{R^2(4\alpha - 3\beta^2)}{4(R^2 + (X^+)^2)^2}, \qquad R_{+\Theta+\Theta} = \frac{R^2(\beta^2 - 4)}{4(R^2 + (X^+)^2)}. \tag{12.64}$$

We will now show that "the curvature generically blows up at $X^+ = 0$" in the singular limit $R \to 0$. To make this statement more precise, we will construct a null geodesic that approaches $X^+ = 0$ in finite affine parameter, and we will show that one of the frame components of the Ricci tensor diverges when we reach $X^+ = 0$. Let us consider the geodesic given by

$$X^-(t) = \frac{\xi t}{2} + \frac{\sin\tau(t)}{2}\left(R\sqrt{\alpha+1}\cos\tau(t) + t\sin\tau(t)\right), \tag{12.65a}$$

$$\Theta(t) = \frac{\beta}{\sqrt{\alpha+1}}\cos\tau(t), \quad X(t) = \sqrt{R^2+t^2}\sin\tau(t), \tag{12.65b}$$

$$X^+(t) = t, \quad \tau(t) = \sqrt{\alpha+1}\arctan\frac{t}{R}, \tag{12.65c}$$

with $\xi = 0$ for a lightlike geodesic and $\xi > 0$ for a timelike geodesic. Events on the generalized nullbrane for which $X^+(t) = 0$ are reached in finite parameter $t$. The tangent vector along the geodesic (12.65) is given by

$$\begin{aligned}\mathbf{t} &= \partial_{X^+} + \frac{1}{2}\left(\xi + \sin^2\tau(t) + \frac{R\sqrt{\alpha+1}}{R^2+t^2}\left[t\sin 2\tau(t) + R\sqrt{\alpha+1}\cos 2\tau(t)\right]\right)\partial_{X^-} \\ &\quad - \frac{\beta R}{R^2+t^2}\sin\tau(t)\,\partial_\Theta + \frac{1}{\sqrt{R^2+t^2}}\left(R\sqrt{\alpha+1}\cos\tau(t) + t\sin\tau(t)\right)\partial_X\,.\end{aligned} \tag{12.66}$$

The only nonzero component of the Ricci tensor is

$$R_{X^+X^+} = \frac{R^2\left(2\alpha - 2 - \beta^2\right)}{2\left(R^2 + (X^+)^2\right)^2} \tag{12.67}$$

and therefore the component $R_{11}$ in a parallelly propagated frame along the geodesic (with $\mathbf{t} = \mathbf{e}_1$) is given by

$$R_{11} = R_{X^+X^+}e_1^+e_1^+ = \frac{R^2\left(2\alpha - 2 - \beta^2\right)}{2\left(R^2+t^2\right)^2}\,. \tag{12.68}$$

Let us assume that $\alpha \neq 1 + \beta^2/2$. We will consider the frame component of the Ricci curvature (12.68) at the spacetime event that corresponds to $t = 0$ along the geodesic (12.65). If we now consider the singular limit $R \to 0$, the frame curvature at that event will blow up. This divergence is also present for those values of $\alpha$ and $\beta$ for which the singular limit of the scalar field dynamics exists, except for the value of the couple ($\alpha$, $\beta$) that corresponds to the original nullbrane. Note that the Ricci scalar vanishes, so our results do not depend on the choice of the Ricci scalar coupling of the scalar field (minimal, conformal, etc., [2], see also chapter 6).

The curvature components will obviously vanish for the specific value of $\alpha = 3$ and $\beta = 2$ that corresponds to the original nullbrane spacetime, because its curvature is zero. In fact, if the $\Theta$ coordinate is decompactified, the original nullbrane becomes Minkowski space. Meanwhile (in the decompactification limit) the generalized nullbrane metrics become pp-waves (though in a rather awkward coordinate system). Therefore, string theory sigma-models should be exactly solvable on all the generalized nullbrane spacetime. We have not investigated sigma models on the generalized nullbrane, but a sigma model on a singular plane wave will be considered in the next chapter.

Furthermore, we can examine the Weyl tensor:

$$C_{+X+X} = \frac{R^2(\alpha + 1 - \beta^2)}{2(R^2 + (X^+)^2)^2}, \qquad C_{+\Theta+\Theta} = -\frac{R^2(\alpha + 1 - \beta^2)}{2(R^2 + (X^+)^2)} \tag{12.69}$$

and notice that it actually *does* vanish for all those cases where the limit exists (within the particular family of geometries we have been considering). However, there are many values of $\alpha$ and $\beta$ for which the Weyl tensor (12.69) vanishes, yet no $R \to 0$ limit of the scalar field dynamics exists. For that reason, conformal flatness is not likely to constitute an important part in possible explanations for the existence of the singular limit.

Perhaps the most puzzling feature of our results is that the limit appears to exist for a discrete subset of the possible parameter values within our family of geometries. One could think of this as being an artifact of choosing our particular slice in the space of all possible geometries (this, however, would obviously require a fairly delicate coincidence). If, on the other hand, the feature is generic, it would point to an interesting sort of discreteness inherent to the dynamics in the near-singular region. This question would certainly deserve further investigation, even though that would require mathematical machinery going beyond what has been employed in our present considerations.

# Chapter 13

# String modes in singular plane waves

*The third law of attribution:*

*"Everything of importance has been said before by someone who did not discover it,"*

*Alfred North Whitehead*

In this chapter we study the propagation of a free string across a plane wave singularity. The singular plane waves we investigate have a scale-invariant and isotropic profile. The material of this chapter is based on [120, 121]. The structure of the chapter is as follows: we will first comment on the geometrical resolution of this type of plane waves. Then we derive the Hamiltonian for a free string in the plane wave background. The Hamiltonian is a set of time-dependent harmonic oscillators, related to the center-of-mass mode of the string and the excited modes. We will quickly recapitulate the main results of [119] for the evolution of the center-of-mass motion across the plane wave singularity. I did not participate in the collaboration that led to the publication [119], but its content is necessary for the development of the chapter. We extend this analysis to the evolution of excited string modes, essentially by means of Gronwall's inequality, a mathematical technique which can be used to bound the solutions to perturbed differential equations. We state this rather mathematically oriented material in section 13.3.

Next, we discuss the propagation of the string across the singularity. We consider the propagation of the excited modes in section 13.5. We find that the propagation of the excited modes across the singularity is possible if and only if the center-of-mass mode can propagate across the singularity.

In section 13.6 we investigate the issue of string mode creation, which imposes stringent conditions on the possibility of free string evolution across a singularity. These restrictions arise if we demand the total mass of the string to remain finite after it crosses the singularity, which is a natural demand for consistent propagation of the whole string. Let me immediately refer to chapter 6 where I discuss some aspects of particle creation in the context of time-dependent harmonic oscillators. This material may be useful in view of section 13.6.

For completeness we consider the singular limit of the dilaton in section 13.7 because it supplements the string to satisfy the background consistency conditions in string theory. We have chosen to supplement the plane wave metric with a time-dependent dilaton to establish a classical string theory solution. In order for the singular limit of the dilaton to exist, we obtain two conditions on the regularization profile. We construct a simple theorem in section 13.8 to prove that it is always possible to find a geometrical resolution that satisfies these conditions. We conclude the chapter with a discussion of the singular limit of a free string with respect to the entire string theory solution (which has a time-dependent string coupling due to the dynamical dilaton).

## 13.1 Geometrical resolution of singular plane waves

String propagation in strong gravitational waves has attracted a considerable amount of attention on account of a few highly special properties of such spacetimes which we summarized in chapter 9. See also [59, 66] among other publications. For one thing, the structure of the curvature tensor in plane gravitational waves implies that these solutions to Einstein's equations (coupled to appropriate matter fields, if necessary) remain uncorrected [59, 58] in a number of higher derivative extensions of general relativity. In particular, they do not receive any $\alpha'$-corrections when introduced as backgrounds in perturbative string theories. Furthermore, the corresponding lightcone Hamiltonian of string $\sigma$-models turns out to be quadratic and admits a fairly thorough analytic treatment. This class of backgrounds also admits a natural formulation of the matrix theory description of quantum gravity [107, 112].

In this chapter, we will concentrate on exact plane waves with an isotropic profile, written in Brinkmann form as,

$$ds^2 = -2dx^+dx^- - F(x^+)\sum_{i=1}^{d}(x^i)^2(dx^+)^2 + \sum_{i=1}^{d}(dx^i)^2. \tag{13.1}$$

The case of constant $F(x^+)$ corresponds to supersymmetric plane waves studied in [62], and it is quite different from the rapidly varying $F(x^+)$ we intend to consider. As mentioned in chapter 9, a coordinate transformation can be performed into Rosen coordinates, eliminating the dependence of the metric on the transverse coordinates $x^i$. The resulting metric depends on $x^+$ only and displays manifestly a plane-fronted spacetime wave propagating at the speed of light. However, the Rosen parametrization tends to suffer from coordinate singularities, and we will work with the Brinkmann form.

The function $F(x^+)$ contained in (13.1) is completely arbitrary, and one may ask, for example, what happens to quantum strings propagating in such spacetimes when the wave profile $F(x^+)$ develops an isolated singularity. This question is of some interest per se, since studies of string theory in the presence of spacetime singularities have played a pivotal role in the development of the subject. In this particular case, we are dealing with singularities in time-dependent backgrounds. Additional heuristic justification for our studies is provided by the observation that plane waves of the type (13.1) with

$$F(x^+) = \frac{\lambda k}{(x^+)^2}\,, \quad \lambda k = \text{const}\,, \tag{13.2}$$

arise as Penrose limits of a broad class [69] of spacetime singularities, including the Robertson-Walker cosmological singularities. With $F(x^+)$ of (13.2), the metric (13.1) is invariant under scaling transformations $x^+ \to \alpha x^+$, $x^- \to x^-/\alpha$ (which are identical to Lorentz boosts in flat spacetime). Note that this type of singularities is considerably stronger than the so-called "weak singularities" of [90].

Free string propagation on (13.1) with $F(x^+)$ given by (13.2) has been previously studied in [66]. In particular, it was suggested in that publication that the question of propagation across the $1/(x^+)^2$ singularity in the metric can be addressed by employing analytic continuation in the complex $x^+$-plane. We believe that this issue merits further elucidation.

In the context of string theory and related approaches to quantum gravity, there is a general expectation that the spacetime background used for formulating the theory should satisfy some stringent consistency conditions. For perturbative string theories, these conditions take the form of the appropriate supergravity equations of motion together with an infinite tower of $\alpha'$-corrections. For non-singular plane waves, all the $\alpha'$-corrections vanish automatically on account of the special properties of the Riemann tensor corresponding to these spacetimes. For singular spacetimes, the question of background consistency conditions *at* the singular point appears to be extremely subtle. Indeed, what should replace the supergravity equations of motion at the singular point where they obviously break down? Ad hoc prescriptions are not likely to produce meaningful results under these circumstances.

One approach to formulating string theory in backgrounds (13.1-13.2) is to resolve the singular plane wave profile into a non-singular function, perform the necessary computations and see if the result has a meaningful singular limit. This approach was advocated in [59], where a conjecture was made that for certain choices of the plane wave profile, taking a singular limit may result in well-defined transition amplitudes. We intend to consider this question quantitatively. Note that, for the resolved spacetimes of this sort, perturbative string background consistency conditions are automatically satisfied to all orders in $\alpha'$. The only non-trivial question is the existence of a singular limit.

But how should one resolve? We want to construct a function $F(x^+,\epsilon)$ in such a way that

$$\lim_{\epsilon\to 0} F(x^+,\epsilon) = \frac{\lambda k}{(x^+)^2} \tag{13.3}$$

everywhere away from $x^+ = 0$. There is in principle a large amount of ambiguity associated with such resolutions. One class appears to be very special however. The background (13.1-13.2) possesses a scaling symmetry and does not depend on any dimensionful parameters. It is natural to demand that this symmetry should be recovered when the resolution is removed. This will happen if the resolved profile $F(x^+,\epsilon)$ does not depend on any dimensionful parameters other than the resolution parameter $\epsilon$. In this case, on dimensional grounds,

$$F(x^+,\epsilon) = \frac{\lambda}{\epsilon^2}\Omega(x^+/\epsilon). \tag{13.4}$$

The limit (13.3) will be recovered if

$$\Omega(\eta) \to \frac{k}{\eta^2} + O\left(\frac{1}{\eta^b}\right) \tag{13.5}$$

for large values of $\eta$, with some $b > 2$. The appearance of $\lambda k$ seems redundant, but it is necessary to incorporate the case $k = 0$. Note that the fact that the original background possesses a certain symmetry (away from $x^+ = 0$!) in no way implies that we *must* resolve in a way consistent with this symmetry. For resolved profiles different from (13.4), the limit of the metric may still be given by (13.3) away from $x^+ = 0$ (and thus be scale invariant), but additional dimensionful scales may become buried inside the singularity at $x^+ = 0$ (in a way that only affects processes involving singularity crossing). One would need some strong physical rationale for introducing such scales buried at the singular locus, and in the present chapter we will simply study the "scale-invariant" resolutions (13.4), written out in full as

$$ds^2 = -2dx^+dx^- - \frac{\lambda}{\epsilon^2}\Omega(x^+/\epsilon)\sum_{i=1}^{d}(x^i)^2(dx^+)^2 + \sum_{i=1}^{d}(dx^i)^2. \tag{13.6}$$

For the resolved spacetime (13.6) which approximates a scale-invariant plane wave near infinity by (13.5), we can use (9.21) to specify the Kasner exponents; although the wave profile is isotropic in the transverse coordinates

$x^i$, there are two Kasner exponents because we can allow for different asymptotic profiles near $x^+ \to \pm\infty$ respectively. We denote these Kasner exponents by $m_\pm$, related to the plane wave profile by,

$$m_\pm^2 - m_\pm + (\lambda k_\pm) = 0\,, \tag{13.7}$$

where $k_\pm$ expresses the asymptotic behaviour of the resolved profile $\Omega(\eta)$ (more general than (13.5)):

$$\Omega(\eta) \to \frac{k_\pm}{\eta^2} + O\left(\frac{1}{\eta^b}\right)\,, \quad b > 2\,, \quad \eta \to \pm\infty\,. \tag{13.8}$$

Because the Kasner exponent can be transformed according to $m_\pm \to 1 - m_\pm$ (at least, this is the case for a plane wave without a dilaton), we will select the exponent $m_\pm$ (the largest value of those two) and denote it in this chapter as $a_\pm$, for the asymptote at $-\infty$ and $+\infty$ respectively:

$$a_\pm = \frac{1}{2} + \sqrt{\frac{1}{4} - (\lambda k_\pm)}. \tag{13.9}$$

## 13.2 Free strings in plane waves

Due to the presence of covariantly constant null vectors in plane wave geometries, the string theory $\sigma$-model can be analyzed explicitly in such backgrounds, and reduces to a set of independent classical time-dependent harmonic oscillators. In this section, we re-state this familiar material in a way convenient for our present investigations.

### 13.2.1 The lightcone gauge

In the Green-Schwarz formulation of the superstring, the fermion part of the action in lightcone gauge is always quadratic in the fermions, for any pp-wave background [64]. An exception occurs for plane wave backgrounds that include Ramond-Ramond fields. In that case the fermions can couple to the background through a generalized covariant derivative that includes the gauge field. An example is given by the dilaton-gravity backgrounds with D-branes of the next chapter, where the D-brane sources the Ramond-Ramond fields.

In the plane wave backgrounds that we consider in the present chapter, the string worldsheet fermions are free in lightcone gauge. We will therefore concentrate on the bosonic part of the string action, given by

$$I = -\frac{1}{4\pi\alpha'}\int d\tau \int_0^{2\pi} d\sigma\sqrt{-\gamma}\left(\gamma^{ab}g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\alpha' R^{(\gamma)}\Phi\right)\,, \tag{13.10}$$

with $\gamma_{ab}$ the worldsheet metric and $g_{\mu\nu}$ the spacetime metric. We choose light-cone gauge $X^+ = \alpha' p^+ \tau$ and gauge-fix the worldsheet metric,

$$\det(\gamma_{ab}) = -1, \qquad \partial_\sigma \gamma_{\sigma\sigma} = 0, \tag{13.11}$$

to obtain the following Lagrangian, where we have solved for $\gamma_{\tau\tau}$:

$$\begin{aligned} L = -\frac{1}{4\pi\alpha'} \int_0^{2\pi} \mathrm{d}\sigma \Bigg( & 2\gamma_{\sigma\sigma} p^+ \alpha' \partial_\tau X^- - 2\gamma_{\tau\sigma} \left(\alpha' p^+ \partial_\sigma X^- - \partial_\tau X^i \partial_\sigma X^i\right) \\ & - \gamma_{\sigma\sigma} \sum_{i=1}^{8} \left( (\partial_\tau X^i)^2 + \frac{(\alpha' p^+)^2}{\epsilon^2} \Omega(\alpha' p^+ \tau/\epsilon)(X^i)^2 \right) \\ & + \gamma_{\sigma\sigma}^{-1}(1 - \gamma_{\tau\sigma}^2) \sum_{i=1}^{8} (\partial_\sigma X^i)^2 - \frac{1}{2}\alpha' R^{(\gamma)} \Phi \Bigg). \end{aligned} \tag{13.12}$$

We rescale $\epsilon = \epsilon' \alpha' p^+$,

$$\frac{(\alpha' p^+)^2}{\epsilon^2} \Omega(X^+/\epsilon) = \frac{1}{\epsilon'^2} \Omega(\tau/\epsilon'), \tag{13.13}$$

and from here on, we will denote worldsheet time $\tau = t$ and write $\epsilon$ instead of $\epsilon'$. The $\sigma$-dependent part of the oscillator $X^-$ is non-dynamical and enforces $g_{\tau\sigma} = 0$. The $\sigma$-independent part of the oscillator $X^-$ can be eliminated as a constraint ($g_{\sigma\sigma} = 1$), the (dynamically non-trivial) coupling to the dilaton disappears (see, e.g., [66]), and we can write the following worldsheet Hamiltonian

$$H = \frac{1}{4\pi\alpha'} \int \mathrm{d}\sigma \sum_{i=1}^{d} \left( \pi^2 (P_i)^2 + \frac{\lambda}{\epsilon^2} \Omega(\tau/\epsilon)(X^i)^2 + \left(\partial_\sigma X^i\right)^2 \right), \tag{13.14}$$

where $P_i$ are momenta conjugate to $X^i$. We now choose units in which $\alpha' = 1$. If we Fourier transform the $\sigma$-coordinate,

$$X^i(t,\sigma) = X_0^i(t) + \sqrt{2} \sum_{n>0} \left( \cos(n\sigma)\, X_n^i(t) + \sin(n\sigma)\, \tilde{X}_n^i(t) \right), \tag{13.15}$$

we obtain a set of time-dependent harmonic oscillator Hamiltonians

$$H = \sum_{n=0}^{\infty} \sum_{i=1}^{d} H_{ni}, \tag{13.16}$$

$$H_{0i} = \frac{(P_{0i})^2}{2} + \frac{\lambda}{\epsilon^2} \Omega(t/\epsilon) \frac{(X_0^i)^2}{2}, \tag{13.17}$$

$$H_{ni} = \frac{(P_{ni})^2 + (\tilde{P}_{ni})^2}{2} + \left( n^2 + \frac{\lambda}{\epsilon^2} \Omega(t/\epsilon) \right) \frac{(X_n^i)^2 + (\tilde{X}_n^i)^2}{2}. \tag{13.18}$$

### 13.2.2 WKB solution for time-dependent harmonic oscillator

The Hamiltonian (13.16) is quadratic and the solution to the corresponding Schrödinger equation,

$$i\frac{\partial}{\partial t}\phi(t;X_n^i) = \left(\sum_n\sum_{i=1}^d H_{ni}\right)\phi(t;X_n^i), \tag{13.19}$$

can be found using WKB techniques, which are exact for quadratic Hamiltonians. From (13.19) it follows that

$$i\frac{\partial}{\partial t}\phi_n^i(t;X_n^i) = -\frac{1}{2}\frac{\partial^2}{(\partial X_n^i)^2}\phi_n^i(t;X_n^i) + \frac{1}{2}\left(n^2 + \frac{\lambda}{\epsilon^2}\Omega(t/\epsilon)\right)\left(X_n^i\right)^2\phi_n^i(t;X_n^i), \tag{13.20}$$

if we separate variables as

$$\phi(t;\mathbf{X}) = \prod_n\prod_{i=1}^8 \phi_n^i(t;X_n^i). \tag{13.21}$$

We then take the WKB ansatz

$$\phi_n^i(t;X) = \mathcal{A}_n(t_1,t)\exp\left(iS_{cl;n}[X_{1,n}^i,t_1|X_n^i,t]\right), \tag{13.22}$$

where $S_{cl;n}[X_{1,n}^i,t_1|X_n^i,t]$ is the "classical action" evaluated for the path going from $X_{1,n}^i$ at the time $t_1$ to $X_n^i$ at the time $t$,

$$S_{cl}[X_{1,n}^i,t_1|X_n^i,t] = \int_{t_1}^t \mathrm{d}t'\left(\frac{(\dot{X}_n^i)^2}{2} - \left(n^2 + \frac{\lambda}{\epsilon^2}\Omega\left(\frac{t'}{\epsilon}\right)\right)\frac{(X_n^i)^2}{2}\right). \tag{13.23}$$

If $\mathcal{A}_n(t_1,t)$ satisfies

$$-2\frac{\partial}{\partial t}\mathcal{A}_n(t_1,t) = \mathcal{A}_n(t_1,t)\frac{\partial^2}{\partial(X_n^i)^2}S_{cl}[X_{1,n}^i,t_1|X_n^i,t], \tag{13.24}$$

then (13.22) satisfies the original Schrödinger equation exactly.

Up to normalization, a basis of solutions, labelled by the initial condition $X_n^i(t_1) = X_{1,n}^i$, is given by

$$\phi(t;\mathbf{X}) \sim \prod_{nk}\frac{1}{\sqrt{\mathcal{C}(t_1,t)}}\exp\left(-\frac{i}{2\mathcal{C}}\sum_{k=1}^d\left[(X_{1,n}^k)^2\partial_{t_1}\mathcal{C} - (X_n^k)^2\partial_{t_2}\mathcal{C} + 2X_{1,n}^kX_n^k\right]\right) \tag{13.25}$$

where $\mathcal{C}(t_1, t_2)$ (suppressing the index $n$) is a solution to the "classical equation of motion" for the time-dependent harmonic oscillator Hamiltonian (13.18):

$$\partial_{t_2}^2 \mathcal{C}(t_1, t_2) + \left(n^2 + \frac{\lambda}{\epsilon^2}\,\Omega(t_2/\epsilon)\right) \mathcal{C}(t_1, t_2) = 0, \tag{13.26}$$

with initial conditions specified as

$$\mathcal{C}(t_1, t_2)|_{t_1=t_2} = 0, \qquad \partial_{t_2}\mathcal{C}(t_1, t_2)|_{t_1=t_2} = 1. \tag{13.27}$$

We will refer to $\mathcal{C}(t_1, t_2)$ as "compression factor", since it describes convergence of solutions to the corresponding harmonic oscillator equation starting at the same point at the moment $t_1$. (If $\mathcal{C}(t_1, t_2)$ vanishes, then $t_2$ is a focal point, as the difference between any two solutions with the same initial position $X(t_1)$ is proportional to $\mathcal{C}(t_1, t_2)$.) A useful representation of $\mathcal{C}(t_1, t_2)$ is given by

$$\mathcal{C}(t_1, t_2) = \frac{f(t_1)h(t_2) - f(t_2)h(t_1)}{W[f, h]}, \tag{13.28}$$

where $f(t)$ and $h(t)$ are two independent solutions to the differential equation under consideration, and the Wronskian $W$ is given by

$$W[f, h] = f\dot{h} - h\dot{f}. \tag{13.29}$$

To derive the singular limit of the wavefunction (13.25) it is sufficient to study the singular limit of (13.26-13.27).

## 13.3 Solutions to perturbed differential equations

In view of the subsequent application to the singular limit analysis, we would like to bound the difference $\delta X$ between the solution $X(t)$ of a perturbed differential equation,

$$\frac{\partial^2}{\partial t^2} X + (\Upsilon + \delta\Upsilon)\, X = 0, \tag{13.30}$$

and the solution $\bar{X}(t)$ of an unperturbed differential equation,

$$\frac{\partial^2}{\partial t^2} \bar{X} + \Upsilon \bar{X} = 0, \tag{13.31}$$

where we take

$$X = \bar{X} + \delta X \tag{13.32}$$

and demand that the initial conditions remain unchanged:

$$X(t_0) = \bar{X}(t_0), \qquad \partial_t X(t_0) = \partial_t \bar{X}(t_0). \tag{13.33}$$

If we substitute (13.32) and (13.31) into (13.30) we obtain a differential equation for the perturbation on the solution.

$$\frac{\partial^2}{\partial t^2}\delta X + \Upsilon(t)\delta X = -\delta\Upsilon(t)\left(\bar{X} + \delta X\right). \tag{13.34}$$

A formal solution to (13.34) is given by

$$\delta X(t) = -\int_{-\infty}^{\infty} G_r(t,t')\delta\Upsilon(t')\left(\bar{X}(t') + \delta X(t')\right)dt', \tag{13.35}$$

with the Green function $G_r(t,t')$ satisfying

$$\left(\frac{\partial^2}{\partial t^2} + \Upsilon(t)\right)G_r(t,t') = \delta(t-t') \tag{13.36}$$

and initial conditions

$$G_r(t,t')|_{t=t_0} = 0, \qquad \partial_t G_r(t,t')|_{t=t_0} = 0. \tag{13.37}$$

Therefore, we can write the Green function in terms of the "compression factor" $\bar{\mathcal{C}}$ of the unperturbed equation (13.31), where $\bar{\mathcal{C}}$ obeys the same initial conditions as in (13.27):

$$G_r(t,t') = \begin{cases} \bar{\mathcal{C}}(t',t) & t_0 < t' < t, \\ 0 & \text{otherwise.} \end{cases} \tag{13.38}$$

To obtain a bound on $\delta X$ we will invoke the so-called Gronwall inequality [26].

### 13.3.1 The Gronwall inequality

*Let $I = [A,B]$. Assume $\beta$ and $\alpha$ real valued and continuous on $I$ and $\beta \geq 0$. If $u$ is continuous, real valued on $I$ and satisfies the integral inequality*

$$u(t) < \alpha(t) + \int_A^t \beta(s)u(s)ds, \qquad t \in I, \tag{13.39}$$

*then*

$$u(t) < \alpha(t) + \int_A^t \beta(s)\alpha(s)\exp\left(\int_s^t \beta(r)dr\right)ds, \qquad t \in I. \tag{13.40}$$

**Proof**: First we define

$$z(t) = \int_A^t \beta(s)u(s)\mathrm{d}s, \qquad t \in I. \tag{13.41}$$

Then, after differentiation and using the initial assumption (13.39), we obtain

$$z'(t) = \beta(t)u(t) \leq \beta(t)\alpha(t) + \beta(t)z(t). \tag{13.42}$$

Using the line above we write

$$\left[\exp\left(-\int_A^s \beta(u)\mathrm{d}u\right) z(s)\right]' = \exp\left(-\int_A^s \beta(r)\mathrm{d}r\right)(z'(s) - \beta(s)z(s)) \tag{13.43}$$

$$\leq \beta(s)\alpha(s)\exp\left(-\int_A^s \beta(u)\mathrm{d}u\right), \ s \in I. \tag{13.44}$$

We integrate from $a$ to $t$ and obtain

$$\exp\left(-\int_A^t \beta(s)\mathrm{d}s\right) z(t) \leq \int_A^t \beta(s)\alpha(s)\exp\left(-\int_A^s \beta(u)\mathrm{d}u\right)\mathrm{d}s\,, \ t \in I. \tag{13.45}$$

From assumption (13.39) and (13.45) we now derive the desired inequality,

$$u(t) \leq \alpha(t) + z(t) \tag{13.46}$$

$$\leq \alpha(t) + \exp\left(\int_A^t \beta(r)\mathrm{d}r\right)\int_A^t \beta(s)\alpha(s)\exp\left(-\int_A^s \beta(u)\mathrm{d}u\right)\mathrm{d}s \tag{13.47}$$

$$= \alpha(t) + \int_A^t \beta(s)\alpha(s)\exp\left(\int_s^t \beta(u)\mathrm{d}u\right)\mathrm{d}s, \qquad t \in I. \tag{13.48}$$

### 13.3.2 Bounds on the perturbations $\delta X$

From (13.35) we derive the following bound on the formal solution $\delta X$

$$|\delta X(t)| < \int_{-\infty}^{\infty} |G_r(t,t')\delta\Upsilon(t')\bar{X}(t')|dt' + \int_{-\infty}^{\infty} |G_r(t,t')\delta\Upsilon(t')\delta X(t')|dt'. \tag{13.49}$$

We now make use the fact that, by virtue of (13.38), where nonzero, $G_r(t,t') = \bar{C}(t',t)$. Therefore, making use the expression (13.28) for the compression factor of the unperturbed equation (13.31), we obtain the bound,

$$|G_r(t,t')| < \frac{1}{|W|}\left(|f|_M|h(t')| + |f(t')||h|_M\right) \equiv g(t'), \tag{13.50}$$

with $|f|_M$ and $|h|_M$ being the absolute value maxima of these functions on the integration domain ($f$ and $h$ are solutions of the unperturbed differential equation (13.31)). The integration regions are in fact finite, since the Green function (cf. (13.38)) vanishes unless $t_0 < t' < t$:

$$|\delta X(t)| < \int_{t_0}^{t} |g(t')\delta\Upsilon(t')\bar{X}(t')|dt' + \int_{t_0}^{t} |g(t')\delta\Upsilon(t')\delta X(t')|dt'. \tag{13.51}$$

Since $g(t')$ is independent of $t$ we can now apply Gronwall's inequality to obtain

$$|\delta X(t)| < \int_{t_0}^{t} |g(t')\delta\Upsilon(t')\bar{X}(t')|dt' + \int_{t_0}^{t} \left( \int_{t_0}^{t'} |g(t'')\delta\Upsilon(t'')\bar{X}(t'')|dt'' \right) \times |g(t')\delta\Upsilon(t')| \exp\left( \int_{t'}^{t} |g(t'')\delta\Upsilon(t'')|dt'' \right) dt'. \quad (13.52)$$

On the interval $(t_0, t)$ we assume the existence of a maximum of $|\bar{X}|$ and of $|\delta\Upsilon|$ and we call these $|\bar{X}|_M$ and $|\delta\Upsilon|_M$ respectively. We also assume the integral $\int_{t_0}^{t} |g(t')|dt'$ can be bounded by a number $M$. If

$$\int_{t_0}^{t} |g(t')|dt' < M, \quad (13.53)$$

then it follows that also

$$\int_{t'}^{t} |g(t'')|dt'' < M. \quad (13.54)$$

We thus find

$$|\delta X(t)| < |\bar{X}|_M \left( M|\delta\Upsilon|_M + M^2|\delta\Upsilon|_M^2 \exp\left(M|\delta\Upsilon|_M\right)\right). \quad (13.55)$$

The second term on the right-hand side is negligible compared to the first one for sufficiently small $|\delta\Upsilon|$.

## 13.4 The singular limit for the center-of-mass mode

In [119], the singular limit of a scalar field on a class of scale-invariant plane waves was investigated, which is fundamental for our present work (the center-of-mass mode of the string exhibits the same behaviour as a scalar field). For the convenience of the reader I will restate some of the material that appeared in [119] in the following subsection, reformulated in a language more convenient for the remainder of the chapter. One of the conclusions of [119] is that (generically) a discrete spectrum will appear. A particular exactly solvable example for such a discrete spectrum has been given in [95] (we called it the "lightlike reflector plane") and we will inspect it in section 13.4.2.

### 13.4.1 Consistent propagation across the singularity

We consider equations (13.26-13.27) for the $n = 0$ mode of the string and we obtain as the "classical equation of motion"

$$\ddot{X} + \frac{\lambda}{\epsilon^2}\Omega(t/\epsilon)X = 0. \quad (13.56)$$

We need to study the $\epsilon \to 0$ limit of the solution that obeys the initial conditions

$$X(t_1) = 0, \qquad \dot{X}(t_1) = 1, \qquad t_1 < 0. \tag{13.57}$$

The singular limit of solutions to this equation has been analyzed in [119]. Performing a scale transformation $Y(\eta) = X(\eta\epsilon)$, with $\eta = t/\epsilon$, removes the $\epsilon$-dependence from the equation, leaving

$$\frac{\partial^2}{\partial\eta^2}Y + \lambda\Omega(\eta)Y = 0. \tag{13.58}$$

This scale transformation is possible because our initial singular metric was scale-invariant and we have resolved it according to (13.4) without introducing any dimensionful parameters besides $\epsilon$. The existence of a singular limit is then translated [119] into constraints on the asymptotic behavior of solutions to (13.58). These "boundary conditions at infinity" are strongly reminiscent of a Sturm-Liouville problem, and it is natural that a discrete spectrum of $\lambda$ is singled out by imposing the existence of a singular limit.

For the specific asymptotic behaviour of our resolved profile (13.5), it can be shown [119] that, in the infinite past and infinite future, the solutions approach a linear combination of two powers (denoted below $a$ and $1-a$, with $a$ being a function of $k\lambda$, cf. (13.4-13.5)). This power law behavior simply corresponds to the regime when the second term on the right hand side of (13.5) can be neglected compared to the first. It is then convenient to form two bases of solutions, one asymptotically approaching the two powers (dominant and subdominant) at $\eta \to -\infty$,

$$Y_{1-}(\eta) = |\eta|^{a_-} + o(|\eta|^{a_-}), \qquad Y_{2-}(\eta) = |\eta|^{1-a_-} + o(|\eta|^{1-a_-}), \tag{13.59}$$

and another behaving similarly at $\eta \to +\infty$

$$Y_{1+}(\eta) = |\eta|^{a_+} + o(|\eta|^{a_+}), \qquad Y_{2+}(\eta) = |\eta|^{1-a_+} + o(|\eta|^{1-a_+}), \tag{13.60}$$

where $a_\pm$ is given by

$$a_\pm = \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda k_\pm}. \tag{13.61}$$

(We are temporarily assuming that $k$ can take two different values $k_\pm$ for the positive and negative time asymptotes, a possibility that will be discarded shortly.) The two bases need, of course, to be related by a linear transformation:

$$\begin{bmatrix} Y_{1-}(\eta) \\ Y_{2-}(\eta) \end{bmatrix} = Q(\lambda) \begin{bmatrix} Y_{1+}(\eta) \\ Y_{2+}(\eta) \end{bmatrix}, \tag{13.62}$$

where $Q(\lambda)$ is a $2 \times 2$ matrix whose determinant is constrained by Wronskian conservation as

$$W[Y_{1-}, Y_{2-}] = W[Y_{1+}, Y_{2+}] \det Q. \tag{13.63}$$

The singular limit has been rigorously considered in [119], but the results can be understood heuristically from the following argument. Imagine one is trying to construct a solution $\tilde{Y}$ to (13.58) satisfying some ($\epsilon$-independent) initial conditions at $\eta_1 = t_1/\epsilon < 0$. This solution can be expressed in terms of $Y_{1-}$ and $Y_{2-}$ (a complete basis) as

$$\tilde{Y} = C_1 Y_{1-} + C_2 Y_{2-}. \tag{13.64}$$

Since the initial conditions are specified at $\eta_1 = t_1/\epsilon$, the asymptotic expansions (13.59) are valid. There needs to be a non-trivial contribution from both $Y_{1-}$ and $Y_{2-}$ in the above formula in order to satisfy general initial conditions. Hence, the two terms on the right hand side should be of order 1. Therefore, we should have

$$C_1 = O(\epsilon^{a_-}), \qquad C_2 = O(\epsilon^{1-a_-}). \tag{13.65}$$

If we now apply (13.62) and (13.60) to evaluate $\tilde{Y}$ at a large positive $\eta = t_2/\epsilon$, the powers of $\epsilon$ in $C_1$ and $C_2$ will combine with the powers of $\epsilon$ originating from $Y_{1+}$ and $Y_{2+}$ and yield

$$\begin{aligned}\tilde{Y}(t_2/\epsilon) = {} & Q_{11}(\lambda) t_2^{a_+} O(\epsilon^{a_- - a_+}) + Q_{12}(\lambda) t_2^{1-a_+} O(\epsilon^{a_- + a_+ - 1}) \\ & + Q_{21}(\lambda) t_2^{a_+} O(\epsilon^{1-a_- - a_+}) + Q_{22}(\lambda) t_2^{1-a_+} O(\epsilon^{a_+ - a_-}).\end{aligned} \tag{13.66}$$

Since $a_+$ and $a_-$ are greater than $1/2$, this expression can only have an $\epsilon \to 0$ limit if $a_+ = a_-$ (i.e. $k_+ = k_-$ and we can set both equal to 1 by redefining $\lambda$) and $Q_{21}(\lambda) = 0$. The latter condition implies that the absolute normalization $\lambda$ of the plane wave profile $\Omega(\eta)$ will generically lie in a discrete spectrum, dependent on the specific way the singularity is resolved, i.e. the shape of $\Omega(\eta)$. Conversely, we can expect that for a given $\lambda$, the shape of $\Omega$ lies in a discrete spectrum, as we will illustrate in the next subsection (for $k_\pm = 0$ this has been proven in [119]). With $Q_{21}(\lambda) = 0$ and $\det Q = -1$, the matrix $Q$ can be written as

$$Q = \begin{bmatrix} q & \tilde{q} \\ 0 & -1/q \end{bmatrix}, \tag{13.67}$$

with $q$ being a real nonzero number ($\tilde{q}$ does not affect the singular limit). This means that the subdominant solution $Y_{2-}(\eta)$ is related to the subdominant solution $Y_{2+}(\eta)$ without receiving admixture from the dominant solution, i.e. it is subdominant at both $\pm\infty$. In the singular limit, a basis of solutions is given by

$$\begin{aligned} & Y_1(t) = (-t)^a, \qquad Y_2(t) = (-t)^{1-a}, \qquad t < 0, \\ & Y_1(t) = q\, t^a, \qquad Y_2(t) = -\frac{1}{q} t^{1-a}, \qquad t > 0. \end{aligned} \tag{13.68}$$

In case the resolved profile satisfies $\Omega(\eta) = \Omega(-\eta)$, we have

$$Q = Q^{-1} \quad \Leftrightarrow \quad q^2 = 1\,. \tag{13.69}$$

As a specific example we find $q = -1$ for Minkowski spacetime and $q = 1$ for the "lightlike reflector plane" of [95] (see the next section). Both Minkowski spacetime and the lightlike reflector plane have $k = 0$. The basis with the asymptotics

$$Y_{1-}(t) \to |t|\,, \quad Y_{2-}(t) \to 1\,, \quad t \to -\infty\,, \tag{13.70}$$

is given by

$$Y_{1-}(t) = -t\,, \quad Y_{2-}(t) = 1\,, \quad \forall\, t\,. \tag{13.71}$$

The basis functions with the asymptotics

$$Y_{1+}(t) \to |t| \quad Y_{2+}(t) \to 1\,, \quad t \to +\infty\,, \tag{13.72}$$

are given by

$$Y_{1+}(t) = t\,, \quad Y_{2+}(t) = 1\,, \quad \forall\, t\,. \tag{13.73}$$

Combining (13.62) and (13.67) it is clear that $q = -1$ for Minkowski spacetime. We will explain why $q = 1$ for the lightlike reflector plane in the following section.

### 13.4.2 Example: the lightlike reflector plane

A natural simplification (which, by the way, is not an orbifold) of the generalized nullbrane (10.21) discussed in the previous chapter, is given by the metric

$$ds^2 = -\frac{\alpha\epsilon^2 x^2}{(t^2+\epsilon^2)^2}dt^2 \; - 2dtdx^- + dx^2\,. \tag{13.74}$$

This spacetime is called the "lightlike reflector plane" for $\alpha = (2N)^2 - 1$ with $N$ integer. The lightlike reflector spacetime can be classified as a plane wave geometry in Brinkmann coordinates (see chapter 9, or also [68]), and, in particular, it can be extended to a 10-dimensional background satisfying Einstein's equation by inclusion of the appropriate antisymmetric tensor field and dilaton. What makes this particular plane wave interesting is that it develops a strong singularity when $\epsilon$ is sent to 0 (for example, the singularity is much more dangerous than the "weak pp-wave singularities" of [90]). Furthermore, the $\epsilon \to 0$ limit of the wave equation in this geometry can be explicitly analyzed. Thus the parameter $\epsilon$ can be regarded as a regularization parameter.

We will not present a detailed derivation of the mode functions for (13.74), but simply notice that the wave equation in this background is formally analogous to that on the generalized nullbrane (12.6), with $\beta = 0$, $k_\Theta = 0$ and the first term on the right hand side omitted (this term comes from the determinant of the generalized nullbrane metric). Also we have renamed the coordinates $(X^+, X^-, X)$ as $(t, x^-, x)$ and we wrote $\epsilon$ instead of $R$. The expression for the position basis mode functions $\phi(x_1, t_1|x_2, t_2) = \mathcal{A}(t_1, t_2) \exp[-iS_{cl}(x_1, t_1|x_2, t_2)]$ then follows from (12.32, 12.34, 12.35):

$$\begin{aligned} S_{cl}[x_1, t_1|x_2, t_2] = &- k_- \left[\frac{t_1}{2\,(t_1^2+\epsilon^2)} - \frac{\epsilon\sqrt{1+\alpha}}{2\,(t_1^2+\epsilon^2)}\,\mathrm{cot}2\Delta_{12}\right] x_1^2 \\ &+ k_- \left[\frac{t_2}{2\,(t_2^2+\epsilon^2)} + \frac{\epsilon\sqrt{1+\alpha}}{2\,(t_2^2+\epsilon^2)}\,\mathrm{cot}2\Delta_{12}\right] x_2^2 \\ &- \left[\frac{k_-\sqrt{1+\alpha}\,\epsilon}{\sqrt{t_1^2+\epsilon^2}\sqrt{t_2^2+\epsilon^2}\;\mathrm{sin}2\Delta_{12}}\right] x_1 x_2 - \frac{m^2}{2k_-}(t_2 - t_1) \end{aligned} \tag{13.75}$$

$$\mathcal{A}(t_1, t_2) = \left(\frac{2\pi}{\epsilon\sqrt{1+\alpha}}\sqrt{t_1^2+\epsilon^2}\sqrt{t_2^2+\epsilon^2}\,|\,\mathrm{sin}2\Delta_{12}\,|\right)^{-\frac{1}{2}} \phi_M \tag{13.76}$$

(with $\Delta_{12} = \sqrt{1+\alpha}\,(\arctan(t_2/\epsilon) - \arctan(t_1/\epsilon))\,/2$ and $\phi_M$ being the appropriate Maslov phase).

The $\epsilon \to 0$ limit of $\phi(x_1, t_1|x_2, t_2)$ exists if $\alpha = K^2 - 1$ (with $K$ being an integer) and equals

$$\begin{aligned} \phi(x_1, t_1|x_2, t_2) = &\frac{1}{\sqrt{2\pi|t_2 - t_1|}}\,\phi_M\,\exp\left[\frac{im^2}{2k_-}(t_2 - t_1)\right] \times \\ &\exp\left[\frac{ik_-}{2}\left(x_2 - (\mathrm{sign}(t_1)\mathrm{sign}(t_2))^{K+1}\,x_1\right)^2\right]. \end{aligned} \tag{13.77}$$

If $K$ is odd, the above expression merely represents free motion on Minkowski space. To verify this statement, one can simply check that $\phi(x_1, t_1|x_2, t_2)$ solves the Minkowski space wave equation written in light cone coordinates:

$$-i\dot{\phi} = -\frac{\partial_x^2\phi}{2k_-} + \frac{m^2}{2k_-}\phi \tag{13.78}$$

Despite the strength of the singularity in the $\epsilon \to 0$ limit, the free scalar field dynamics actually becomes identical to that on a completely flat space.

If $K$ is even, the motion is still free for all positive and all negative $t$. However, as the particles pass through $t = 0$, their positions and velocities in the $x$-direction are reflected: note the

$$(\mathrm{sign}(t_1)\mathrm{sign}(t_2))^{K+1} \tag{13.79}$$

structure in (13.77). This reflection is similar to the one happening for the case of the generalized nullbrane, but it occurs on a simpler spacetime geometry.

The mode functions corresponding to (13.77) can be easily derived by means of a Fourier transformation, analogously to section 12.5.

Because of the property we have just described, we have called the spacetime (13.74) with $\alpha = (2N)^2 - 1$, where $N$ is an integer, the lightlike reflector plane. It is an extremely simple family of pp-wave geometries developing a strong singularity at $t = 0$ when $\epsilon$ is sent to 0. Furthermore, at least for free propagation in this background, the singular limit is manifestly well-defined, and includes a peculiar lightlike object reflecting the positions and velocities of all particles passing through it (hence $q = 1$, see also section 12.6).

Thus we can interprete the lightlike reflector plane as a regularized plane wave spacetime *a la* (13.5) with $b = 4$ and $k = 0$, and we conclude that the regularized profile exhibits a discrete spectrum given by $\alpha = (2N)^2 - 1$, $N \in \mathbb{N}$.

## 13.5 The singular limit for the excited string modes

Following our general discussion of free strings in plane wave backgrounds in section 13.2, the evolution of excited string modes is described by time-dependent harmonic oscillator equations

$$\frac{\partial^2}{\partial t^2} X(t) + \left(n^2 + \frac{\lambda}{\epsilon^2}\Omega(t/\epsilon)\right) X(t) = 0. \tag{13.80}$$

Solutions for the wavefunctions of the excited string modes can be expressed in terms of a particular solution to this equation $\mathcal{C}(t_1, t_2)$ defined by (13.26-13.27). Hence, to analyze the singular ($\epsilon \to 0$) limit of the excited modes dynamics, it should suffice to analyze the singular limit of $\mathcal{C}(t_1, t_2)$. Because $n^2$ is finite, it is natural to expect that it does not affect the existence of the singular limit (which is governed by the singularity emerging from $\Omega(t/\epsilon)$). We will prove that it is indeed the case for positive $\lambda$. For negative $\lambda$ unstable motion of the inverted harmonic oscillator leads to divergences. More specifically, the divergences arise from subleading infinities in the position of the inverted harmonic oscillator, while the leading infinities cancel. Such sub-leading infinities are absent for the center-of-mass motion analyzed in section 13.4, hence no analogous divergences in that case. Further details are given in subsections 13.5.3 and 13.5.4. Nevertheless, $\lambda$ influences the string coupling and a negative $\lambda$ corresponds to a strong string coupling near the singularity, so we cannot expect free strings to be a valid first-order approximated to interacting strings (see also the discussion at the end of the chapter in section 13.9) and the divergences related to the inverted harmonic oscillator are of limited physical relevance for our present investigations.

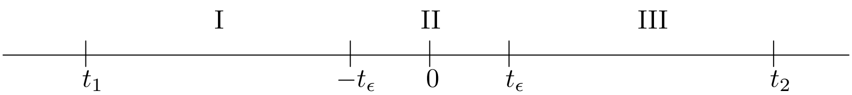
To derive $\mathcal{C}(t_1, t_2)$ for equation (13.80) we use the following strategy: the differential equation (13.80) is linear and any solution $X(t_2)$ at $t = t_2$ can be written in terms of a "transfer matrix" $T$ that only depends on the initial and final times,

$$\begin{bmatrix} X(t_2) \\ \dot{X}(t_2) \end{bmatrix} = T(t_1, t_2) \begin{bmatrix} X(t_1) \\ \dot{X}(t_1) \end{bmatrix}. \tag{13.81}$$

The transfer matrix can be expressed as

$$T(t_1, t_2) = \begin{bmatrix} -\partial_{t_i}\mathcal{C}(t_1, t_2) & \mathcal{C}(t_1, t_2) \\ -\partial_{t_i}\partial_{t_f}\mathcal{C}(t_1, t_2) & \partial_{t_f}\mathcal{C}(t_1, t_2) \end{bmatrix}, \tag{13.82}$$

where $\partial_{t_i}$ and $\partial_{t_f}$ indicate differentiation with respect to the first and second argument respectively. The transfer matrix is completely determined once $\mathcal{C}(t_1, t_2)$ has been determined, and vice versa. We now use the fact that transfer matrices of subintervals are combined by ordinary matrix multiplication. Dividing the solution region into three sub-intervals, we calculate the transfer matrices $T_k$ for each sub-interval $k$ and apply multiplication to construct the total transfer matrix. The sub-intervals are chosen as indicated in the following figure:




We use $t_\epsilon$ to indicate a time that will approach zero in the singular limit as

$$t_\epsilon = \epsilon^{1-c}\tilde{t}^c, \tag{13.83}$$

with $\tilde{t}$ staying finite in relation to the "moments of observation" $t_1$ and $t_2$. The number $c$ (between 0 and 1) will be chosen later as needed for our proof. On each interval, we can write the transfer matrix $T_k$ in terms of the "compression factor" $\mathcal{C}_k$. Using matrix multiplication to construct the full transfer matrix $T$, we can now deduce an expression for the "compression factor" of the complete interval

$$\begin{aligned} \mathcal{C}(t_1, t_2) = {} & \mathcal{C}_I(t_1, -t_\epsilon)\partial_{t_i}\mathcal{C}_{II}(-t_\epsilon, t_\epsilon)\partial_{t_i}\mathcal{C}_{III}(t_\epsilon, t_2) \\ & - \partial_{t_f}\mathcal{C}_I(t_1, -t_\epsilon)\mathcal{C}_{II}(-t_\epsilon, t_\epsilon)\partial_{t_i}\mathcal{C}_{III}(t_\epsilon, t_2) \\ & - \mathcal{C}_I(t_1, -t_\epsilon)\partial_{t_i}\partial_{t_f}\mathcal{C}_{II}(-t_\epsilon, t_\epsilon)\mathcal{C}_{III}(t_\epsilon, t_2) \\ & + \partial_{t_f}\mathcal{C}_I(t_1, -t_\epsilon)\partial_{t_f}\mathcal{C}_{II}(-t_\epsilon, t_\epsilon)\mathcal{C}_{III}(t_\epsilon, t_2), \end{aligned} \tag{13.84}$$

in terms of the "compression factors" of the three sub-intervals. Once again, $\partial_{t_i}$ and $\partial_{t_f}$ differentiate $\mathcal{C}$ with respect to its first and second argument (initial and final time).

To study the existence of the singular limit of $\mathcal{C}(t_1, t_2)$, we will use the following strategy: for two linear differential equations related by a small perturbation we will establish a bound on the difference between perturbed and unperturbed solutions with the same initial conditions. This bound will, of course, apply to $\mathcal{C}_k$. For each of the three sub-intervals introduced above, we will consider a simplified differential equation that is a good approximation to equation (13.80) on the corresponding interval:

- Region I and III: $\ddot{X}(t) + \left(n^2 + \lambda/t^2\right) X(t) = 0$ (related to Bessel's equation);
- Region II: $\ddot{X}(t) + \lambda/\epsilon^2 \Omega(t/\epsilon) X(t) = 0$ (equation of motion for the zero mode).

Then, on each sub-interval, $\mathcal{C}_k$ can be written as the sum of a simplified "compression factor" $\bar{\mathcal{C}}_k$ satisfying the simplified differential equation on this sub-interval, plus a small perturbation $\delta\mathcal{C}_k$. We will prove that, in the singular limit, the $\delta\mathcal{C}_k$ will drop out of the expression for the total "compression factor" $\mathcal{C}(t_1, t_2)$.

Most of this section is dedicated to implementing the proof we have just outlined. The reader primarily interested in the discussion of the singular limit and content with the general sketch given above can jump to section 13.5.3.

### 13.5.1 Solutions away from the singularity

In regions I and III we will take

$$\Upsilon = n^2 + k/t^2, \qquad \delta\Upsilon = \frac{1}{\epsilon^2} O\left(\frac{\epsilon^b}{t^b}\right), \tag{13.85}$$

with $b$ defined in (13.5). The solutions to the unperturbed differential equation (13.31) are given by

$$\sqrt{|t|} J_\alpha(|nt|), \qquad \sqrt{|t|} J_{-\alpha}(|nt|), \qquad \alpha = a - \frac{1}{2}, \tag{13.86}$$

where the Bessel functions, $J_\alpha(x)$ and $J_{-\alpha}(x)$, satisfy the differential equation

$$x^2 \frac{\partial^2}{\partial x^2} J_\alpha(x) + x \frac{\partial}{\partial x} J_\alpha + \left(x^2 - \alpha^2\right) J_\alpha(x) = 0. \tag{13.87}$$

(This Bessel-negative-order-Bessel basis is more convenient for our purposes than the often-used Bessel-Neumann basis, as it approaches $|t|^a$ and $|t|^{1-a}$ for small values of $t$ without mixing the two powers.)

The unperturbed "compression factor" in region I is then

$$\bar{\mathcal{C}}_I(t_1, t) = \sqrt{|t_1|}\sqrt{|t|} \frac{J_\alpha(-nt_1) J_{-\alpha}(-nt) - J_\alpha(-nt) J_{-\alpha}(-nt_1)}{W[\sqrt{|t|} J_\alpha(-nt), \sqrt{|t|} J_{-\alpha}(-nt)]}. \tag{13.88}$$

Using the series expansion of the Bessel function for small arguments (they will be evaluated at $t = -t_\epsilon$),

$$J_\alpha(x) \sim \left(\frac{x}{2}\right)^\alpha \frac{1}{\Gamma(\alpha+1)}, \qquad \alpha \neq -1, -2, -3, \ldots. \tag{13.89}$$

we can estimate the various contributions to (13.55), thereby constraining the correction to the unperturbed "compression factor". One can distinguish three cases:

1) $a > 1, J_\alpha(-nt_1) \neq 0$, which yields

$$|\bar{\mathcal{C}}(t_1, t_\epsilon)| \propto \epsilon^{(1-c)(1-a)},\ |\bar{\mathcal{C}}|_M \propto \epsilon^{(1-c)(1-a)},\ |\delta\Upsilon|_M \propto \epsilon^{bc-2},\ M \propto \epsilon^{(1-c)(1-a)}. \tag{13.90}$$

From (13.55), $\delta\mathcal{C}(t_1, t_\epsilon)$ is negligible compared to $\mathcal{C}(t_1, t_\epsilon)$ if

$$c > \frac{a+1}{a+b-1}. \tag{13.91}$$

2) $a < 1, J_\alpha(-nt_1) \neq 0$, which yields

$$|\bar{\mathcal{C}}(t_1, t_\epsilon)| \propto \epsilon^{(1-c)(1-a)}, \qquad |\bar{\mathcal{C}}|_M \propto \epsilon^0, \qquad |\delta\Upsilon|_M \propto \epsilon^{bc-2}, \qquad M \propto \epsilon^0. \tag{13.92}$$

From (13.55), $\delta\mathcal{C}(t_1, t_\epsilon)$ is negligible compared to $\mathcal{C}(t_1, t_\epsilon)$ if

$$c > \frac{3-a}{b+1-a}. \tag{13.93}$$

3) $J_\alpha(-nt_1) = 0$, which yields

$$|\bar{\mathcal{C}}(t_1, t_\epsilon)| \propto \epsilon^{(1-c)a}, \qquad |\bar{\mathcal{C}}|_M \propto \epsilon^0, \qquad |\delta\Upsilon|_M \propto \epsilon^{bc-2}, \qquad M \propto \epsilon^0. \tag{13.94}$$

From (13.55), $\delta\mathcal{C}(t_1, t_\epsilon)$ is negligible compared to $\mathcal{C}(t_1, t_\epsilon)$ if

$$c > \frac{2+a}{b+a}. \tag{13.95}$$

In any of the three cases, it suffices for $c$ to be greater than a number less than 1, in order for the corrections to the unperturbed "compression factor" to be negligible for small values of $\epsilon$. The discussion of interval III is completely parallel to what we have just presented.

### 13.5.2 Solutions in the near-singular region

The "unperturbed" equation in region II,

$$\frac{\partial^2}{\partial t^2}\bar{X}(t) + \frac{\lambda}{\epsilon^2}\Omega(t/\epsilon)\bar{X}(t) = 0, \tag{13.96}$$

is precisely that of the string center-of-mass motion. In order to simplify derivations, we will assume $a_+ = a_-$, as required for well-defined zero-mode propagation (see section 13.4). The unperturbed "compression factor" in region II takes the form

$$\bar{\mathcal{C}}_{II}(t_i, t_f) = \frac{Q_{22}(\lambda)|t_i|^a t_f^{1-a} - Q_{11}(\lambda)|t_i|^{1-a} t_f^a}{2a-1} + \frac{Q_{21}(\lambda)|t_i|^a t_f^a \epsilon^{1-2a} - Q_{12}(\lambda)|t_i|^{1-a} t_f^{1-a} \epsilon^{2a-1}}{2a-1}, \tag{13.97}$$

where the $2\times2$ matrix $Q$ is defined by (13.67), and we have used $W[|t|^a, |t|^{1-a}] = 2a-1$. Via the expression for the compression factor given by equation (13.28), formula (13.97) follows from the asymptotic behaviour of the zero mode solution at infinity given by expressions (13.59) and (13.60), and from the relation between the two bases given by (13.62). Formula (13.97) is also the same expression as formula (44) of [119]. The given expression corresponds to small values of $\epsilon$. The corrections are suppressed by powers of $\epsilon$ and do not contribute to the singular limit.

To study the perturbation we will first perform the scaling transformation $\eta = t/\epsilon$, $Y(\eta) = X(\eta\epsilon)$, which yields

$$\frac{\partial^2}{\partial \eta^2} Y(\eta) + \left(\epsilon^2 n^2 + \lambda\Omega(\eta)\right) Y(\eta) = 0. \tag{13.98}$$

We now take

$$\Upsilon = \lambda\Omega(\eta), \qquad \delta\Upsilon = \epsilon^2 n^2. \tag{13.99}$$

If we now choose $f \sim \eta^a$, $g \sim \eta^{1-a}$ in (13.50), $M$ of (13.53) for the region $(-t_\epsilon/\epsilon, t_\epsilon/\epsilon)$ (whose size, in $\eta$, is proportional to $\epsilon^{-c}$) becomes (with the three factors derived from $f$, $h$ and the size of the integration region):

$$M \propto \epsilon^{-ac} \epsilon^{-(1-a)c} \epsilon^{-c} = \epsilon^{-2c}. \tag{13.100}$$

Because there are only power laws involved in (13.97), the maximal value $\bar{\mathcal{C}}_M$ is of the same order as $|\bar{\mathcal{C}}(-t_\epsilon, t_\epsilon)|$. Furthermore, $|\delta\Upsilon|_M \propto \epsilon^2$ by construction. It then follows from (13.55) that

$$|\delta\mathcal{C}_{II}| < \left(O\left(\epsilon^{2-2c}\right) + O\left(\epsilon^{4-4c} \exp\left(\epsilon^{2-2c}\right)\right)\right) |\bar{\mathcal{C}}_{II}|. \tag{13.101}$$

The correction is negligible for any $c < 1$.

A subtlety in our above derivation deserves a comment: one might have thought that the factor of $n^2$ in $\delta\Upsilon$ of (13.99) competes with the smallness of $\epsilon$ and undermines the validity of our considerations for sufficiently large mode numbers. It is indeed true that, for each value of $\epsilon$ (i.e. for each fixed resolved space), our analysis is only valid for modes with sufficiently small mode numbers. But this range of validity increases infinitely as $\epsilon$ is taken to 0. However, since the modes are completely independent, the limit for the motion of the entire string (if it exists) is exactly the same as if it were computed mode-by-mode. For that reason, $n$ can be kept fixed in the derivations of this section, and the problem of $n^2$ competing with the smallness of $\epsilon$ does not arise. This attitude guarantees reproducing the $\epsilon \to 0$ limit correctly for the entire set of modes, though it does not allow to draw conclusions on the uniformity of this limit with respect to $n$.

### 13.5.3 Effective matching conditions

Having analyzed the "compression factors" on subintervals I, II and III, we can combine them into the total "compression factor" by applying (13.84). As has been shown above, there exist a number $c$ in (13.84) between 0 and 1, such that the "compression factors" on subintervals I, II and III can be well approximated by the simplified expressions (13.88) and (13.97), with corrections suppressed by positive powers of $\epsilon$. One can then substitute (13.88) and (13.97) into the right-hand-side of (13.84).

For $a > 1$ ($\lambda k < 0$), the Bessel functions featured in (13.88) blow up near the origin (the inverted harmonic oscillator is propelled off to infinity). This threatens the existence of an $\varepsilon \to 0$ limit. In section 13.5.4, we display the divergences arising for $a > 3/2$. For $1 < a < 3/2$, the limit may exist for individual string modes, but a consideration along the lines of section 13.6 would still indicate no consistent propagation for the entire string. In any case, we do not explore this case further since, as will be explained in section 13.9, free strings are not a good approximation to motion in such plane waves.

For $a < 1$ ($\lambda k > 0$), substituting (13.88) and (13.97) in (13.84) yields

$$\begin{aligned}\bar{\mathcal{C}}(t_1,t_2) = \frac{\sqrt{-\pi t_1 t_2}}{2\sin\alpha\pi}\Big(&Q_{22}(\lambda)J_{a-1/2}(-nt_1)J_{1/2-a}(nt_2)\\ - Q_{11}(\lambda)J_{1/2-a}(-nt_1)J_{a-1/2}(nt_2) &+ Q_{21}(\lambda)\epsilon^{1-2a}J_{a-1/2}(-nt_1)J_{a-1/2}(nt_2)\gamma_n\\ - Q_{12}(\lambda)\epsilon^{2a-1}J_{1/2-a}(-nt_1)&J_{1/2-a}(nt_2)\gamma_n^{-1}\Big), \qquad t_1<0, t_2>0, \qquad (13.102)\end{aligned}$$

where $\gamma_n$ are numbers originating from the coefficients of the power law expansion of the Bessel functions.

Note that the expression (13.102) has the same algebraic structure as the one derived for the center-of-mass motion in [119], except that $|t|^a$ and $|t|^{1-a}$ are replaced by $\sqrt{|t|}J_\alpha(|t|)$ and $\sqrt{|t|}J_{-\alpha}(|t|)$. Requiring the existence of the $\epsilon \to 0$ limit results in the condition

$$Q_{21}(\lambda) = 0. \qquad (13.103)$$

It is exactly the same condition as the one for the existence of a singular limit of the center-of-mass motion (generically leading to a discrete spectrum for $\lambda$).

Under the assumption of (13.103) we obtain in the singular limit

$$\begin{aligned}\mathcal{C}(t_1,t_2) =&\sqrt{-t_1t_2}\,\frac{Q_{22}(\lambda)J_{a-1/2}(-nt_1)J_{1/2-a}(nt_2)}{W[\sqrt{-t_1}J_{a-1/2}(-nt_1),\sqrt{-t_1}J_{1/2-a}(-nt_1)]}\\ &-\sqrt{-t_1t_2}\frac{Q_{11}(\lambda)J_{1/2-a}(-nt_1)J_{a-1/2}(nt_2)}{W[\sqrt{-t_1}J_{a-1/2}(-nt_1),\sqrt{-t_1}J_{1/2-a}(-nt_1)]},\\ &\qquad t_1<0,t_2>0. \qquad (13.104)\end{aligned}$$

The matching conditions across the singularity can now be derived rigorously by constructing two independent solutions to (13.80). Note that all the information necessary for such construction is encoded (cf. (13.82)) in the "compression factor" given by (13.104). A convenient shortcut for this procedure is to recall the representation (13.28) of $\mathcal{C}(t_1,t_2)$ in terms of two arbitrary independent solutions $f(t)$ and $h(t)$, and to read off the corresponding singular limit of the two solutions directly from (13.104). Writing $Q_{11}(\lambda)=q$ and $Q_{22}(\lambda)=-1/q$, we obtain as a basis of solutions,

$$\begin{aligned}&Y_1(t)=\sqrt{-t}J_{a-1/2}(-nt), \qquad Y_2(t)=\sqrt{-t}J_{1/2-a}(-nt), \qquad t<0,\\ &Y_1(t)=q\sqrt{t}J_{a-1/2}(nt), \qquad Y_2(t)=-\frac{\sqrt{t}}{q}J_{1/2-a}(nt), \qquad t>0. \qquad (13.105)\end{aligned}$$

### 13.5.4 Divergences for the case of the inverted harmonic oscillator

As remarked in section 13.5.3, for the case of $k\lambda<0$ (inverted harmonic oscillator), divergences may arise in the evolution of excited string modes. These divergences may be seen via a blunt application of (13.84), but it will be more instructive to make their algebraic structure more explicit.

To this end, we will derive a slightly different representation for the total "compression factor" in place of (13.84). We can start by rewriting (13.28) as

$$\mathcal{C}(t_1,t_2)=\frac{1}{W[f,h]}\begin{pmatrix}f(t_1) & h(t_1)\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}f(t_2)\\ h(t_2)\end{pmatrix}. \qquad (13.106)$$

For any two sets of solutions $\{f,h\}$ and $\{F,H\}$, the following relation holds:

$$\begin{pmatrix}f(t)\\ h(t)\end{pmatrix}=\frac{1}{W[F,H]}\begin{pmatrix}W[f,H] & -W[f,F]\\ W[h,H] & -W[h,F]\end{pmatrix}\begin{pmatrix}F(t)\\ H(t)\end{pmatrix}. \qquad (13.107)$$

We can then take four sets of solutions: one approximated by

$$\{\sqrt{-t}J_\alpha(-nt),\sqrt{-t}J_{-\alpha}(-nt)\} \qquad (13.108)$$

in region I, two approximated by

$$\{Y_{1-}(t/\epsilon),Y_{2-}(t/\epsilon)\} \qquad (13.109)$$

and

$$\{Y_{1+}(t/\epsilon), Y_{2+}(t/\epsilon)\} \tag{13.110}$$

in region II, and one approximated by

$$\{\sqrt{t}J_{\alpha}(nt), \sqrt{t}J_{-\alpha}(nt)\} \tag{13.111}$$

in region III. We can then start with (13.106) written with the first of these four sets of solutions. In this representation, the functions featured in (13.106) are easily evaluated at $t_1 < 0$, but not at $t_2 > 0$. One then consecutively applies (13.107), (13.62) and (13.107) again to insert the remaining three sets of solutions, with the Wronskians in (13.107) being evaluated at the boundaries of sub-regions. In the resulting expression, all the four sets of solutions occur only with the values of the arguments for which we have convenient approximations to these solutions, and the total compression factor can be evaluated. As a matter of fact, this is simply another way to write (13.84).

The divergent contributions to the total "compression factor" can be identified with particular Wronskians emerging from (13.107), when one constructs the total "compression factor" with the procedure outlined in the previous paragraph. For example, at the boundary of regions I and II, the following Wronskian occurs:

$$W[\sqrt{-t}J_{-\alpha}(-nt), Y_{2-}(t/\epsilon)]\Big|_{t=-t_\epsilon}. \tag{13.112}$$

The leading terms of both functions featured in the Wronskian are proportional to $|t|^{1-a}$, and therefore cancel by virtue of antisymmetry of the Wronskian. However, the sub-leading contributions have a different functional form and do not have to cancel. For example, for $a > 3/2$, one may consider the contribution from the first sub-leading power-law correction to the Bessel function, and the leading term in $Y_{2-}$. This term is proportional to

$$W[|t|^{3-a}, |t|^{1-a}] \sim t^{3-2a}, \tag{13.113}$$

and furthermore it is not accompanied by any power of $\epsilon$ in the total expression for $\mathcal{C}(t_1, t_2)$. For that reason, evaluating this term at $t = -t_\epsilon$ and taking the $\epsilon \to 0$ limit will produce a divergence.

## 13.6 The singular limit for the entire string

As we have seen in the previous section, for $k\lambda > 0$, consistent propagation of the string center-of-mass across the singularity guarantees that all excited string modes also propagate in a consistent fashion. This is not sufficient, however, to define a consistent evolution for the whole string, since even small excitations of higher string modes can sum up to yield an infinite total energy

[59]. As we will see below, the condition of finite total string energy (after the singularity crossing) turns out to be very restrictive.

The total string excitation energy can be conveniently expressed in terms of the Bogoliubov coefficients for the higher string modes. To compute the latter, we form two different bases of solutions from (13.105) corresponding to purely positive and negative frequencies at large negative and large positive times. More specifically, using the asymptotic expansion for the Bessel functions

$$J_{\pm\alpha}(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x \mp \alpha\frac{\pi}{2} - \frac{\pi}{4}\right), \qquad x \to \infty, \tag{13.114}$$

we construct

$$\begin{bmatrix} \phi_1^- \\ \phi_2^- \end{bmatrix} = \frac{i}{\sin(\alpha\pi)} \begin{bmatrix} -\exp(i\alpha\pi/2 - i\pi/4) & \exp(-i\alpha\pi/2 - i\pi/4) \\ \exp(-i\alpha\pi/2 + i\pi/4) & -\exp(i\alpha\pi/2 + i\pi/4) \end{bmatrix} \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix}, \tag{13.115}$$

such that,

$$\phi_1^-(t) \sim \sqrt{\frac{2}{\pi n}} \exp(int)\,, \qquad \phi_2^-(t) \sim \sqrt{\frac{2}{\pi n}} \exp(-int)\,, \qquad t \to -\infty\,. \tag{13.116}$$

Analogously, we introduce

$$\begin{bmatrix} \phi_1^+ \\ \phi_2^+ \end{bmatrix} = \frac{i}{q\sin(\alpha\pi)} \begin{bmatrix} \exp(-i\alpha\pi/2 + i\pi/4) & q^2\exp(i\alpha\pi/2 + i\pi/4) \\ -\exp(i\alpha\pi/2 - i\pi/4) & -q^2\exp(-i\alpha\pi/2 - i\pi/4) \end{bmatrix} \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix}, \tag{13.117}$$

such that

$$\phi_1^+(t) \sim \sqrt{\frac{2}{\pi n}} \exp(int)\,, \qquad \phi_2^+(t) \sim \sqrt{\frac{2}{\pi n}} \exp(-int)\,, \qquad t \to +\infty\,. \tag{13.118}$$

The two bases are related by a matrix made of Bogoliubov coefficients $\alpha_n$ and $\beta_n$:

$$\begin{bmatrix} \phi_2^+ \\ \phi_1^+ \end{bmatrix} = \begin{bmatrix} \alpha_n & \beta_n \\ \beta_n^* & \alpha_n^* \end{bmatrix} \begin{bmatrix} \phi_2^- \\ \phi_1^- \end{bmatrix} \tag{13.119}$$

For the Bogoliubov coefficients, we obtain the following expressions, independent of $n$:

$$\alpha_n = -\frac{1+q^2}{2q\sin(\alpha\pi)}, \tag{13.120}$$

$$\beta_n = -i\frac{\exp(i\pi\alpha) + q^2\exp(-i\pi\alpha)}{2q\sin(\alpha\pi)}. \tag{13.121}$$

Here, $\alpha = \sqrt{1-4k\lambda}/2$. The total mass of the string after crossing the singularity is given by [60]

$$M = \sum_n n|\beta_n|^2. \tag{13.122}$$

Since the $\beta_n$ are $n$-independent, $M$ can only be finite if $\beta_n = 0$ for all $n$. In general, one needs the uniformity of the $\epsilon \to 0$ limit of $\beta_n$ with respect to $n$ in order to analyze infinite sums as in (13.122). As remarked at the end of section 13.5.2, our considerations allow to draw immediate conclusions on the existence of the limit, but not on its uniformity. However, since $M$ is a sum of positive numbers, it is obvious that it will diverge when the $\beta_n$ approach an $n$-independent non-zero value (in the $\epsilon \to 0$ limit), irrespectively of whether this approach is uniform in $n$. For that reason, no further considerations are needed to draw our conclusions.

For $k\lambda > 0$, finiteness of $M$ cannot be achieved, since $0 < \alpha < 1/2$ and $q$ is real. As a matter of fact, expression (13.122) can only be finite if $q^2 = 1$ and $k\lambda = 1/4 - (N + 1/2)^2$, that is for $k\lambda \leq 0$. For an asymptotical profile with $k = 0$, all $\beta_n$ will vanish if $q^2 = 1$, which is satisfied automatically for any reflection-symmetric $\Omega(\lambda)$. The case $k\lambda = 0$ is not only the trivial case of flat Minkowski space, for example it is also the case of the "lightlike reflector plane" of [95] which is a simplification of the generalized nullbrane.

## 13.7 Background consistency and singular limit for the dilaton

As we have seen in the course of main exposition, consistent free string propagation turns out to impose extremely stringent constraints on the treatment of scale-invariant dilaton-gravity plane wave backgrounds. For that reason, it was not crucial for our picture to explore further conditions arising from supergravity equations of motion imposed on the background. However, for methodological completeness, we will present considerations for the singular limit of the dilaton field, and examine how this condition combines with propagation of individual string modes. These derivation will not have much bearing on the outcome of the analysis in the main text, but they may be useful for pursuing various modifications of our present set-up.

If a time-dependent dilaton is used to support the curvature of the metric (13.1-13.2) in the context of string theory, the condition for conformal invariance of the world-sheet theory is given by [66] (for details see [118])

$$R_{\mu\nu} = -2D_\mu D_\nu \phi. \tag{13.123}$$

We impose this equation for all $X^+$ in the resolved plane wave profile, and then examine the singular limit of the solutions for the dilaton. This is in contrast to the approach in [66], where the background consistency conditions at the singular locus were not discussed. The condition for conformal invariance (13.123) leads to the equation

$$\ddot{\phi}(t) = -\frac{\lambda d}{2\epsilon^2}\Omega(t/\epsilon) \tag{13.124}$$

for the dilaton where $d$ is the number of transverse dimensions $X^i$. We want to consider the limit $\epsilon \to 0$ of the solution $\phi$ to this equation. In order for this limit to exist, the regularization $\Omega$ will have to fulfill extra conditions. Since, in the singular limit, the spacetime is regular away from $X^+ = 0$, we can construct a solution $\phi(t)$ to the left of the singularity and another solution $\phi(t)$ to the right. The requirements for the singular limit of $\phi$ to exist then reduce to demanding that the jumps in $\phi(t)$ and in its first derivative $\dot{\phi}(t)$ are finite:

$$\Delta\phi = \int_{t_1}^{t_2} \dot{\phi}(t)dt = \left[t\dot{\phi}(t)\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} t\ddot{\phi}(t)dt \tag{13.125}$$

$$= \left[t\dot{\phi}(t)\right]_{t_1}^{t_2} + \frac{\lambda d}{2}\int_{t_1/\epsilon}^{t_2/\epsilon} \eta\Omega(\eta)d\eta \tag{13.126}$$

and

$$\Delta\dot{\phi} = -\int_{t_1}^{t_2} \frac{\lambda d}{2\epsilon^2}\Omega(t/\epsilon)dt = -\frac{\lambda d}{2\epsilon}\int_{t_1/\epsilon}^{t_2/\epsilon} \Omega(\eta)d\eta \tag{13.127}$$

Thus, $\Delta\dot{\phi}$ can only be finite if

$$\int_{-\infty}^{+\infty} \Omega(\eta)d\eta = 0. \tag{13.128}$$

If that is the case, the first term in (13.126) is automatically finite, and we are left to demand finiteness of the second term

$$\lim_{\epsilon\to 0}\int_{t_1/\epsilon}^{t_2/\epsilon} \eta\Omega(\eta)d\eta < \infty. \tag{13.129}$$

If $\Omega$ is even and satisfies (13.5), this second condition is automatically satisfied.

## 13.8 Explicit example of a geometrical resolution

We now want to show that it is possible to combine the finite dilaton condition (13.128) with consistent propagation of individual string modes. Given the considerations in the main text, this translates into finding $\Omega(\eta)$ such that (13.128) is satisfied and, in addition,

$$\frac{\partial^2}{\partial\eta^2}Y(\eta) + \lambda\Omega(\eta)Y(\eta) = 0 \tag{13.130}$$

has a solution approaching $Y(\eta) \propto \eta^{1-a}$ for $\eta \to \pm\infty$. We apply inverse reconstruction to $\Omega(\eta)$, assuming some shape of this solution and adjusting it so as to satisfy

$$\int_{-\infty}^{+\infty} \frac{Y''(\eta)}{Y(\eta)}\, d\eta = 0. \tag{13.131}$$

This "inverse reconstruction" technique is generally useful for contemplating qualitative properties of various plane wave profiles in relation to the singular limit.

### 13.8.1 No-go theorem for $Y(\eta)$ without zero crossings

In constructing an appropriate $Y(\eta)$, it is important to decide whether it should have zeros. If $Y$ has no zeros, $Y'/Y$ is regular everywhere, and we can rewrite (13.131) as:

$$\int_{-\infty}^{+\infty} \frac{Y''(\eta)}{Y(\eta)}\, d\eta = \left[\frac{Y'(\eta)}{Y(\eta)}\right]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} \frac{Y'^2(\eta)}{Y^2(\eta)}\, d\eta. \tag{13.132}$$

We now use $Y(\eta) \propto \eta^{1-a}$ for $\eta \to \pm\infty$, yielding

$$\int_{-\infty}^{+\infty} \frac{Y''(\eta)}{Y(\eta)}\, d\eta = \int_{-\infty}^{+\infty} \frac{Y'^2(\eta)}{Y^2(\eta)}\, d\eta > 0. \tag{13.133}$$

Therefore, if $Y$ has no zeros, it is impossible to construct an $\Omega(\eta)$ that integrates to zero. One must permit zeros (say $Y(\eta_i) = 0$), and it is necessary to have bending points ($Y''(\eta_i) = 0$) at the same locations due to (13.130). We will aim at constructing a symmetric $\Omega$, assuming that $Y$ is symmetric and restricting our analysis to $\eta > 0$, and we will look for $Y$ that has only one zero for $\eta > 0$.

### 13.8.2 Piece-wise construction

We prove that it is possible to construct an $\Omega$ that integrates to zero for a $Y$ that has one zero-crossing. $\Omega$ can be made arbitrarily smooth but for the simplicity of the proof we will allow $\Omega$ to have discontinuities. The main idea is to split the contributions to the integral

$$\int_0^{\infty} \Omega(\eta) \mathrm{d}\eta, \tag{13.134}$$

into two parts, separated by $\eta = \eta_M$. The part

$$\int_{\eta_M}^{\infty} \Omega(\eta) \mathrm{d}\eta, \tag{13.135}$$

will be chosen to be always positive. Then we prove that the contribution

$$\int_0^{\eta_M} \Omega(\eta) \mathrm{d}\eta, \tag{13.136}$$

can be made equal to any negative number while keeping the $\eta > \eta_M$ region intact. Therefore the total sum (13.134) can always be taken zero by adjusting the $\eta < \eta_M$ contribution.

We rewrite equation (13.130) as

$$\Omega = -\frac{1}{\lambda}\frac{Y''}{Y}, \tag{13.137}$$

and we take a piecewise $Y(\eta)$ (with a continuous first derivative),

$$Y(\eta) = \begin{cases} Y_1(\eta) & -\eta_M < \eta < \eta_M \\ Y_2(\eta) & |\eta| > \eta_M. \end{cases} \tag{13.138}$$

The function $Y_2$ is fixed throughout our considerations, and we demand that it asymptotes to the subdominant solution for large $\eta$: $Y_2 \to \eta^{1-a}$ with $2a = 1 + \sqrt{1-4\lambda}$. As mentioned above, because of the denominator $Y$ in $\Omega$ there needs to be a bending point for each crossing of the $\eta$-axis. $Y_2''/Y_2$ is *negative* everywhere at $\eta > \eta_M$. The splicing point $\eta_M$ is taken to be a minimum, and we demand that $Y_1(\eta_M) = Y_2(\eta_M) \equiv Y(\eta_M)$. We take the following ansatz:

$$Y_1(\eta) = (C - Y(\eta_M))\left(\frac{\eta^4}{\eta_M^4} - 2\frac{\eta^2}{\eta_M^2}\right) + C. \tag{13.139}$$

A pictorial representation of our assumed solution is given on Fig. 13.1. Due to the piecewise construction of $Y$ it is clear that $\int \Omega(\eta) d\eta$ consists of a separate $Y_1$ and $Y_2$ contribution. The contribution of $Y_2$ (i.e. -$\int_{\eta_M}^{\infty} Y_2''/Y_2 d\eta$) will always be positive. It remains to be proven that $Y_1$ can contribute an arbitrarily negative value for fixed $Y(\eta_M)$ and $\eta_M$. With $\eta_M > 0$ and $\lambda > 0$, this is equivalent to asking that

$$\int_0^1 \frac{3y^2 - 1}{y^4 - 2y^2 + \frac{C}{C - Y(\eta_M)}} dy \tag{13.140}$$

can be set equal to an arbitrarily positive number. We know that $Y(\eta_M) \leq C < 0$, since $Y_1$ should not cross the $\eta$-axis and $\eta = \eta_M$ is a minimum. First, if $C = Y(\eta_M)$, the integral above is 0. Then, for $C \to 0^-$, with $\delta = -C/(C - Y(\eta_M)) > 0$, we find in the limit of $\delta \to 0$:

$$\int_0^1 \frac{3y^2 - 1}{y^4 - 2y^2 - \delta} dy \sim \frac{\pi}{2\sqrt{2\delta}}. \tag{13.141}$$

For $C \to 0^-$ or $\delta \to 0$ this becomes arbitrarily large and positive. As a consequence (13.136) can be made equal to any negative number (between 0 and $-\infty$), and (13.128) can be satisfied by appropriately adjusting $Y_1(\eta)$.

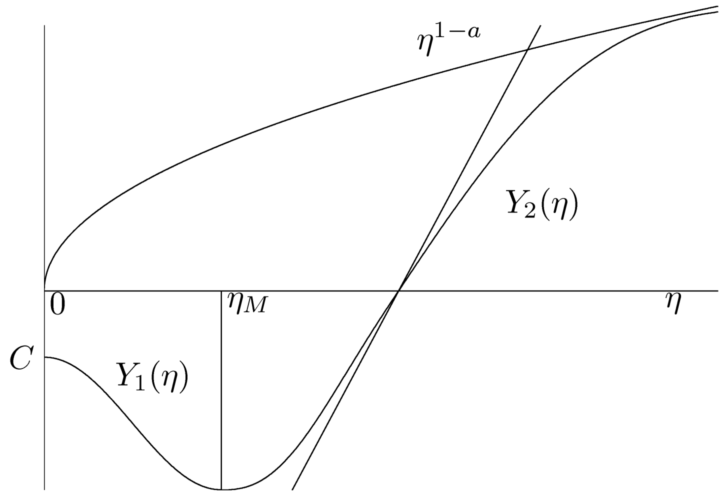



Figure 13.1: Piece-wise construction of $Y(\eta)$.

## 13.9 Discussion of the singular limit

Before we recapitulate our main results, it is appropriate to make two observations. First, one can ask what kind of cosmological singularities gives rise, when the Penrose limit is taken, to the plane wave singularities we have been considering. According to [69], if one starts with isotropic homogeneous cosmology of the type

$$ds^2 = -dt^2 + t^{2h} dx^i dx^i, \tag{13.142}$$

and performs a Penrose limit, one obtains a plane wave of the form (13.1-13.2) with

$$k\lambda = \frac{h}{(1+h)^2}. \tag{13.143}$$

Thus, positive values of $k\lambda$ correspond to positive $h$, i.e. Friedmann-like big bang singularities, and negative values of $k\lambda$ correspond to negative $h$, i.e. an infinite-expansion rather than an infinite-contraction singularity. Because (with $p = w\rho$)

$$h = \frac{2}{3(1+w)}, \tag{13.144}$$

such an infinite-expansion singularity has state parameter $w < -1$ and it was called a big rip singularity in chapter 2.

Secondly, the dilaton field (which was discussed in more detail in section 13.7) in the backgrounds of the type (13.1-13.2) takes the form [66]

$$\phi = \phi_0 + cx^+ + \frac{dk\lambda}{2} \ln x^+ \,. \tag{13.145}$$

We can put $c = 0$ if we demand an asymptotically finite string coupling for $x^+ \to \pm\infty$. Expression (13.145) becomes singular at $x^+ = 0$, but if $k\lambda$ is positive the string coupling

$$g_s = (x^+)^{dk\lambda/2}\, e^{\phi_0} \tag{13.146}$$

vanishes at $x^+ = 0$. On the other hand, if $k\lambda$ is negative then the string coupling becomes infinitely large near $x^+ = 0$, invalidating a perturbative string theory approach in the vicinity of the singularity, and invalidating free string propagation as the zeroth order approximation thereto.

### 13.9.1 Case of the inverted harmonic oscillator

We have paid relatively little attention to $k\lambda < 0$ because of the limited validity of the free string approximation in that case. What we could see is that, generically, it is hard to make excited string modes propagate consistently across the singularity (though it may still be possible to arrange such propagation by means of a judicious choice of the resolved profile $\Omega(\eta)$ of the plane wave). The issue, however, cannot be competently addressed within perturbative string theory on account of string coupling blow-up. Our considerations can be seen as a motivation to study these backgrounds in the context of non-perturbative matrix theory descriptions of quantum gravity. Some steps in this direction have been taken in [112]. Alternatively, one could try to construct plane wave backgrounds of the type (13.1-13.2) where the curvature of the metric is compensated by a dynamical $B$-field (or with non-zero $p$-forms), rather than the dilaton, thus avoiding the dilaton blow-up problem, but care should be taken that the configuration of metric and $B$-field (or metric and $p$-forms) is still a classical string theory solution.

### 13.9.2 Case of standard harmonic oscillator

For the case of positive $k\lambda$, i.e. those plane waves that arise as Penrose limits of Friedmann-like cosmologies, it turns out that individual excited string modes propagate consistently across the singularity, whenever the center-of-mass of the string does. In those cases, the dilaton (13.145) is actually very large and *negative* near the singularity, and one can expect that free strings are a good approximation as far as propagation across the singularity is concerned (the string coupling is small in the near-singular region). However, for free strings, we find it impossible to maintain a finite total string energy after the singularity crossing, provided that the (scale-invariant) singularity is resolved in a way that does not introduce new dimensionful parameters. One way out appears to be to allow hidden scales buried at the singular locus (even though the spacetime away from the singularity is scale-invariant). On the other hand, if arbitrary resolutions, more general than (13.4), are allowed, for a given string mode, one should be able to reproduce (virtually) any matching conditions. This can be seen by assuming a particular form of solutions to the harmonic oscillator equation describing string propagation, and then reconstructing the plane wave profile necessary to produce this assumed motion. However, it is non-trivial to fit matching conditions for the entire tower of string modes in a particular geometrical resolution. For example, it is not obvious whether the matching conditions postulated in [66] should have any geometrical interpretation at all.

Another relevant consideration would be the propagation of strings across plane wave singularities stronger than $1/(x^+)^2$. Unfortunately, at present, little can be said about this case, even for the center-of-mass motion.

### 13.9.3 Discrete spectrum and shape of the resolution profile

Finally, let us look more closely at the appearance of a discrete spectrum for the normalization $\lambda$ of the (isotropic) plane wave profile (13.4). Just as it was the case for the generalized nullbrane in the previous chapter, the authors of [119] also found that the consistent propagation of a free scalar field on singular scale-invariant plane waves leads to a discrete spectrum (which is generalized to the propagation of a free string across a scale-invariant singularity in this chapter). Nevertheless, there are a few differences between the resolved plane wave singularities of the present chapter and the generalized nullbrane geometries of the previous chapter. In the previous chapter, the geometrical resolution of the parabolic orbifold in terms of the (generalized) nullbrane introduced an additional (trivial) dimension in the singular limit. The geometrical resolution applied in this chapter only changes the profile of the plane wave without even affecting the other components of the plane wave metric, which is possible because we work in Brinkmann coordinates where the (lightcone) time-dependence of $g_{uu}$ is arbitrary. In addition, the geometrical resolution applied in this chapter is not explicit. In fact, the profile $\Omega(\eta)$ remains arbitrary for finite $\eta$, we only restrict the asymptotics near infinity such that $\Omega(\eta)$ reduces to the original singular scale-invariant plane wave in the singular limit.

We have seen that the specific shape of the resolution profile $\Omega$ will generically lead to a discrete spectrum for $\lambda$ that characterizes the plane wave profile (13.2). In fact, it would be more natural to consider that $\lambda$ is fixed for a given singular scale-invariant plane wave. That is, we start with a specific scale-invariant plane wave (of which the singular profile is determined by $\lambda$) for which we want to find a geometrical resolution. We did not construct a proof that could state that it is always possible to find a resolution profile $\Omega$ that corresponds to a given $\lambda$. Instead we silently assumed that we have chosen $\Omega$ such that its spectrum includes $\lambda$. On the other hand, all the results of the present chapter where derived without relying on the exact shape of $\Omega(\eta)$, though consistent propagation of the dilaton implies that the profile integrates to zero and that it is (for example) even in $\eta$, but these are relatively mild integral assumptions that leave local freedom for $\Omega$.

# Chapter 14

# Supergravity D$p$-brane solutions

*The zeroth law of attribution:*

*"Therefore only false discoveries ... may be truly called original."*

*(Of course the fourth law is self-referential too).*

*F.D.R.*

In this chapter we present supergravity solution that describe extremal $p$-branes embedded in a dilaton-gravity plane wave, based on the publication [136]. Dilaton-gravity plane waves are gravitational waves equipped with a dilaton with a non-constant profile. They play a special role in string theory and related approaches to quantum gravity, since they provide a rare example of tractable strongly curved (possibly singular) time-dependent spacetime backgrounds, essentially because they possess a covariantly constant null vector and their curvature invariants are zero. Furthermore dilaton-gravity plane waves permit a formulation of (time-dependent) matrix theories of quantum gravity [107, 112]. In the context of string theory they have already appeared in the previous chapter, and (for instance) in [59, 66].

With respect to quantum gravity approaches in the context of string theory, a likewise prominent role is accorded to the $p$-brane supergravity solutions (see e.g. [125]). Through their connection with the D-branes of string theory, they lead to the formulation of the AdS/CFT correspondence [127] and its generalizations to different dimensions [128].

Thus, to investigate the time-dependent matrix theories in more detail, and to formulate time-dependent generalizations of the AdS/CFT correspondence, it appears important to derive supergravity solutions describing $p$-branes embedded into dilaton-gravity plane waves. The simplest of these solutions are supersymmetric configurations corresponding to extremal $p$-branes aligned along the propagation direction of the plane wave (the existence of such configurations can be suggested by the DBI worldvolume analysis for the corresponding D-branes). Some considerations of these, and related, configurations have been undertaken in [133, 131, 134, 130] (among other publications) for highly specific choices of the plane wave profile. Our present purpose is to derive this type of solutions without any assumptions regarding the functional shape of the asymptotic plane wave. Nevertheless, we will assume that the plane wave profile is isotropic.

The structure of the chapter is as follows: in order to solve the supergravity equations of motion for a $p$-brane embedded in a dilaton-gravity plane wave in a tractable manner, we limit ourselves to a restricted ansatz that will prove sufficient to find extremal $p$-brane solution. The supergravity equations split into time-independent and time-dependent equations, which we can solve sequentially. We verify that our ansatz for extremal $p$-branes preserves supersymmetry. We then fix the coordinates in which our solution is written, such that it becomes manifest that our solution asymptotically agrees with an isotropic Brinkmann plane wave in string frame.

## 14.1 $p$-branes aligned with the dilaton

Thus we will search for the supergravity solution of a metric that expresses a $p$-brane embedded in an asymptotically time-dependent isotropic plane wave. We select a radial coordinate $r$, transverse to the brane, and define that the dilaton-gravity plane wave is recovered for $r \to \infty$ . The $p$-brane is charged and therefore there is an additional Ramond-Ramond field strength, which vanishes asymptotically. We need a time-dependent dilaton to satisfy the background consistency conditions at $r = \infty$, and it is natural to assume that the dilaton will also depend on the radial distance to the brane.

### 14.1.1 Supergravity action and equations of motion

We assume that the Kalb-Ramond field is zero, and we start by inspecting the ten-dimensional Einstein-frame supergravity equations of motion (see, e.g., [125]) which contain a metric, a dilaton and a Ramond-Ramond form associated to the charged brane:

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\,\partial_\nu\phi + \sum_N \frac{1}{2n_N!}e^{a_N\phi}\left[n_N\left(F^2_{n_N}\right)_{\mu\nu} - \frac{n_N-1}{8}F^2_{n_N}g_{\mu\nu}\right], \quad (14.1)$$

$$\Box\phi = \sum_N \frac{a_N}{2n_N!}e^{a_N\phi}F^2_{n_N}, \quad (14.2)$$

$$\partial_{\mu_1}\left(\sqrt{-g}e^{a_N\phi}F^{\mu_1\cdots\mu_{n_N}}\right) = 0, \quad (14.3)$$

$$\partial_{[\mu}F_{\mu_1\cdots\mu_{n_N}]} = 0, \quad (14.4)$$

where $N$ labels the various form fields of the theory, with field strengths $F_{n_N}$ of rank $n_N$, and $a_N = (5-n_N)/2$ for Ramond-Ramond form fields. For the "square" of the forms we have used the notation,

$$\left(F^2_{n_N}\right)_{\mu\nu} = F_{\mu\mu_2\cdots\mu_{n_N}}F_\nu{}^{\mu_2\cdots\mu_{n_N}}, \quad F^2_{n_N} = F_{\mu_1\cdots\mu_{n_N}}F^{\mu_1\cdots\mu_{n_N}}. \quad (14.5)$$

### 14.1.2 Restricted ansatz for extremal solutions

We will solve the supergravity equations of motions and Bianchi identity (14.1-14.4) for the following ansatz that consists of one $p$-brane with an associated field strength $F$ that is a $p+2$-form:

$$ds^2_E = A(u,r)\left(-2dudv + K(u,r)du^2 + dy^2_\alpha\right) + B(u,r)\,dx^2_a, \quad (14.6)$$

$$\phi = \phi(u,r), \quad (14.7)$$

$$F_{uv\alpha_1\cdots\alpha_{p-1}a} = \frac{x^a}{r}\,\frac{F(u,r)A^{(p+1)/2}e^{\frac{p-3}{2}\phi}}{B^{(7-p)/2}}\epsilon_{\alpha_1\cdots\alpha_{p-1}}\left[\frac{1}{\sqrt{2}}\right]_{p=3} \qquad (p\leq 3), \quad (14.8)$$

$$F_{a_1\cdots a_{8-p}} = \frac{x^a}{r}\,F(u,r)\epsilon_{a_1\cdots a_{8-p}a}\left[\frac{1}{\sqrt{2}}\right]_{p=3} \qquad (p\geq 3). \quad (14.9)$$

Here the transverse radius is written as $r^2 = x^a x^a$, $p$ is the number of spatial dimensions of the $p$-brane, $F$ is the field strength of the corresponding Ramond-Ramond form, $\alpha$ runs from 1 to $p-1$ and $a$ runs from 1 to $9-p$; the factors of $1/\sqrt{2}$ are only inserted into the form field ansatz for the self-dual case $p=3$. It will turn out that the factor $K(u,r)$ is asymptotically related to the (isotropic) plane wave profile in Brinkmann coordinates.

This ansatz is not the most general one allowed by the symmetries (in particular, when the $u$-dependences are non-trivial there is in general no Poincaré symmetry relating $g_{uv}$ and $g_{\alpha\alpha}$), however it will prove sufficiently general for our purposes (i.e. extremal branes) and it simplifies the calculations and analysis considerably. To find more general non-extremal (but still isotropic) time-dependent branes one should add a $g_{ua}(u,r)dudx_a$ term to the line element and alter the metric component $A(u,r)dy^2_\alpha$ to $A(u,r)L(u,r)dy^2_\alpha$.

## 14.2 Step I: equations of motion for our ansatz

In order not to distract the reader from the main line of this chapter (the "solution menu"), I have separated the equations of motion for our ansatz from the "recipe" of the chapter presented in the next section 14.3. If the reader has an appetite for equations and a strong stomach, an explicit "raw" form of the equations of motion for our ansatz is given in [136] which I will not write out here, instead I will present some convenient combinations of the equations of motion for our ansatz below. But even these "pre-cooked" equations can easily be skipped. In that case I recommend the reader to take a look at the strategy to obtain the $p$-brane solution in the next section 14.3.

I now present the manipulated equations of motion. Throughout, prime denotes derivatives with respect to $r$ and dot denotes derivatives with respect to $u$. First of all, the equations for the form (14.3-14.4) can be integrated straightforwardly to yield

$$F(u,r) = \frac{Q}{r^{8-p}}, \tag{14.10}$$

where $Q$ measures the brane charge. With these dependences, the $uv$-component of Einstein's equations (identical to the $\alpha\alpha$-components) can be written as,

$$\left(r^{8-p}A^{(p+1)/2}B^{(7-p)/2}\frac{A'}{A}\right)' = \frac{7-p}{8}Q^2\frac{e^{\frac{p-3}{2}\phi}A^{(p+1)/2}}{r^{8-p}B^{(7-p)/2}}, \tag{14.11}$$

which we will call the "$uv$-equation". The dilaton equation (14.2) gives

$$\left(r^{8-p}A^{(p+1)/2}B^{(7-p)/2}\phi'\right)' = \frac{p-3}{4}Q^2\frac{e^{\frac{p-3}{2}\phi}A^{(p+1)/2}}{r^{8-p}B^{(7-p)/2}}. \tag{14.12}$$

The $ab$-components of Einstein's equations yield (from terms proportional to $\delta_{ab}$) the "$\delta_{ab}$-equation"

$$\left(r^{8-p}A^{(p+1)/2}B^{(7-p)/2}\frac{B'}{B}\right)' + 2r^{7-p}\left(A^{(p+1)/2}B^{(7-p)/2}\right)' = -\frac{p+1}{8}Q^2\frac{e^{\frac{p-3}{2}\phi}A^{(p+1)/2}}{r^{8-p}B^{(7-p)/2}}, \tag{14.13}$$

and (from terms proportional to $x_a x_b$, after (14.11) and (14.13) have been used to eliminate the terms depending on $Q$, i.e. originating from the form field) the "$x_a x_b$-equation"

$$-\left(p\frac{A'}{A} + (8-p)\frac{B'}{B}\right)' + 4\frac{A'}{A}\frac{B'}{B} + \frac{8-p}{r}\left(\frac{A'}{A} - \frac{B'}{B}\right) = \phi'^2. \tag{14.14}$$

The $ua$-component of Einstein's equations (we rearranged to derivatives for later convenience) gives the "$ua$-equation"

$$-\left(p\frac{A'}{A}+(8-p)\frac{B'}{B}\right)^{\cdot}+4\frac{A'}{A}\frac{\dot{B}}{B}=\dot{\phi}\phi'. \tag{14.15}$$

Finally, the $uu$-component of Einstein's equations (combined with the $uv$-component to eliminate the form) yields the "$K$-equation"

$$-\frac{A}{B}\frac{\left(r^{8-p}A^{(p+1)/2}B^{(7-p)/2}K'\right)'}{r^{8-p}A^{(p+1)/2}B^{(7-p)/2}}= \\ =(p-1)\left[\frac{\ddot{A}}{A}-\frac{3}{2}\left(\frac{\dot{A}}{A}\right)^2\right]+(9-p)\left[\frac{\ddot{B}}{B}-\frac{1}{2}\left(\frac{\dot{B}}{B}\right)^2-\frac{\dot{B}}{B}\frac{\dot{A}}{A}\right]+\dot{\phi}^2. \tag{14.16}$$

## 14.3 Step II: solution strategy

In this section I present the recipe to obtain the extremal $p$-brane solutions. Let us first notice that the electric ansatz of the form (14.8) immediately satisfies the Bianchi identity (14.4), and the magnetic ansatz (14.9) satisfies the equation of motion of the form (14.3). Then the remaining equation (the equation of motion for the form for the electric ansatz and the Bianchi identity for the magnetic ansatz) can be integrated straightforwardly (see previous section), which yields one integration constant related to the brane charge.

The remaining equations split into two groups: the equations without time derivatives (14.11-14.14) and the equations with time-derivatives (14.15-14.16). We have six equations for four unknown functions $A(u,r)$, $B(u,r)$, $\phi(u,r)$ and $K(u,r)$. We first solve the time-independent equations, then promote all integration constants to functions of $u$, and finally solve the time-dependent equations. This algebraic structure essentially reduces the $u$-dependent case to the $u$-independent one.

The equations without time-derivatives (14.11-14.14) are identical to those for the static ($u$-independent) problem, and should be solved first. The static $p$-brane has already been considered in the literature, and the techniques we use in section 14.4 have previously appeared in other work. Because $K(u,r)$ does not appear in these equations there are four equations for three unknowns, but it will turn out in section 14.4 that two of the four equations are related, and only impose a condition on the integration constants. Therefore we obtain two integration constants. All this will be performed in step III.

Once the time-independent equations have been solved, all the integration constants should be promoted to functions of $u$. We have three integration constants (one from the integration of the form and two from solution of the time-independent equations). Meanwhile, we will restrict to extremal solutions in step IV and check that these solutions preserve supersymmetry in step V.

In step VI we finally consider the two time-dependent equations. They have a different complexity: equation (14.15) does not contain the $g_{uu}$ prefactor $K(u,r)$, while (14.16) does. Therefore, into the "$ua$-equation" (14.15) we should first substitute the extremal solutions of the time-independent equations (14.11-14.14), with the integration constants promoted to functions of $u$. This will constrain the $u$-dependences of our integration constants. Finally, the "$K$-equation" (14.16) will determine $K(u,r)$.

## 14.4 Step III: time-independent equations

The solution for the $u$-independent case corresponding to our present ansatz has been given in [123]. Essentially, one eliminates the $Q$-dependent terms (coming from the form field) from the "dilaton equation" (14.12) and the "$\delta_{ab}$-equation" (14.13) using the "$uv$-equation" (14.11) to obtain

$$\left(r^{8-p}A^{(p+1)/2}B^{(7-p)/2}\left(\phi' - \frac{2(p-3)}{7-p}\frac{A'}{A}\right)\right)' = 0, \tag{14.17}$$

$$\left(r^{15-2p}\left(A^{(p+1)/2}B^{(7-p)/2}\right)'\right)' = 0. \tag{14.18}$$

These equations are easily integrated, whereupon the "$uv$-equation" (14.11) reduces to a Liouville equation (one-dimensional classical particle moving in an exponential potential) with respect to a new variable $\rho$ defined as $d/d\rho = r^{8-p}A^{(p+1)/2}B^{(7-p)/2}d/dr$. For example, in terms of the new variable $\rho$ the dilaton equation (14.17) turns into,

$$\frac{d^2}{d\rho^2}\left(\phi - 2\frac{p-3}{7-p}\log A\right) = 0. \tag{14.19}$$

All the non-linearity of the problem becomes concentrated in this simple non-linear equation, which can be solved explicitly in terms of hyperbolic functions. Furthermore, as it turns out, the "$x_a x_b$-equation" (14.14) can be equivalently rewritten as an energy value specification for the above-mentioned Liouville equation and simply reduces to one constraint on the integration constants. We refer the reader to [123] for explicit expressions.

Even though the static ($u$-independent) problem can be solved explicitly for our ansatz, it appears to be of limited use for general *non-extremal* $p$-branes. The ansatz we have chosen was not the most general one allowed by the symmetries of the problem (though it will suffice for constructing the *extremal* solutions we are aiming at, and help us to keep the derivations reasonably compact), and in the presence of strong non-linearities, one should expect all types of motion permitted by the symmetry constraints to mix. In particular, as I mentioned before, one should relax the equality of $g_{uv}$ and $g_{\alpha\alpha}$ (some of related static non-extremal solutions have been constructed in [122], and a rather general analysis has been presented in [132]), and add a non-zero $g_{ua}$. Our present investigations will not pursue this computation-extensive program but concentrate instead on the case of extremal $p$-branes, which can be completely analyzed using the ansatz (14.6-14.9).

## 14.5 Step IV: restriction to extremal solutions

To obtain extremal $p$-brane solutions, we take particular integrals of (14.17) and (14.18), namely:

$$\frac{d}{dr}\left(A^{(p+1)/2}B^{(7-p)/2}\right) = 0, \qquad \frac{d}{dr}\left(\phi - \frac{2(p-3)}{7-p}\log A\right) = 0. \tag{14.20}$$

(These particular integrals are known to correspond to extremal $p$-branes for the $u$-independent case.) One can then take

$$A \propto \left(1 + \frac{R^{7-p}}{r^{7-p}}\right)^{(p-7)/8} \tag{14.21}$$

(where $R$ will turn out to be simply another parametrization for the brane charge $Q$; we restore the expressions for the form field explicitly in our final results), compute the corresponding $B(u,r)$ and $\phi(u,r)$ using (14.20), and check that the resulting $A(u,r)$, $B(u,r)$ and $\phi(u,r)$ solve both the remaining "$uv$-equation" (14.11) and the "$x_a x_b$-equation" (14.14). Equations (14.11-14.14) have now been satisfied. We will check that these branes are extremal in subsection 14.6.

### 14.5.1 Ansatz for the time-dependent equations

As explained in the previous section, one needs to further promote all the integration constants to functions of $u$ and solve (14.15) and (14.16). The $u$-dependent prefactor in $g_{uv}$ can be changed arbitrarily by a redefinition of $u$, and we can use this freedom to relate the $u$-dependent prefactor of $A$ to the $u$-dependence of the dilaton. We thus introduce the following expressions to be substituted into (14.15) and (14.16):

$$\begin{aligned} A &= e^{-f(u)/2}\left(1+h(u)\frac{R^{7-p}}{r^{7-p}}\right)^{(p-7)/8}, \\ B &= \mu(u)e^{-f(u)/2}\left(1+h(u)\frac{R^{7-p}}{r^{7-p}}\right)^{(p+1)/8}, \\ \phi &= f(u)+\frac{3-p}{4}\ln\left(1+h(u)\frac{R^{7-p}}{r^{7-p}}\right). \end{aligned} \tag{14.22}$$

This ansatz is designed to make the large $r$ asymptotics in *string frame* ($ds^2 \equiv e^{\phi/2}ds_E^2$) look simple, as we choose to parametrize our solutions by this asymptotics.

### 14.5.2 Transition to string frame and quasi-harmonic function

More specifically, if we transform our metric to string frame, we obtain the following "intermediate (string-frame) ansatz", which we can use to verify the supersymmetry properties of our claimed extremal solutions. The metric is written as,

$$ds^2 = H(u,r)^{-1/2}(-2dudv + K(u,r)du^2 + dy_\alpha^2) + \mu(u)H(u,r)^{1/2}dx_a^2\,, \tag{14.23a}$$

with (what I would call the "quasi-harmonic" function) $H(u,r)$ given by,

$$H(u,r) = 1 + h(u)\frac{R^{7-p}}{r^{7-p}}\,, \tag{14.23b}$$

and the dilaton and form become

$$\phi = f(u) + \frac{3-p}{4}\ln\left(H(u,r)\right) \tag{14.23c}$$

$$F_{uv\alpha_1.....\alpha_{p-1}a} = -e^{-f(u)}H'H^{-2}\frac{x^a}{r}\epsilon_{\alpha_1...\alpha_{p-1}}\,. \tag{14.23d}$$

The function $H(u,r)$ approaches the identity for $r\to\infty$, thus from this form of the metric in string frame, it is clear that $g_{uv}$ is set to go to 1 for large $r$ (in string frame) as a matter of gauge choice; $g_{\alpha\alpha}$ is forced to go to 1 for large $r$ by hand (recall that we have chosen to impose Poincaré symmetry between $g_{uv}$ and $g_{\alpha\alpha}$ on the brane worldvolume).

## 14.6 Step V: Supersymmetry analysis

The fact that, in constructing our solutions, we have relied on the particular integrals (14.20) of the equations of motion (which, for the $u$-independent case, correspond to extremal BPS solutions) makes it natural to expect that our $u$-dependent solutions will likewise be supersymmetric (and thus related to the D-branes of string theory), which we will verify in this section. Readers primarily interested in the solution for the time-dependent $p$-brane, can skip this section and jump to section 14.7 where we solve the remaining time-dependent equations.

There are two types of maximal supergravity theories in ten dimensions: type IIA and type IIB, whose multiplet structure agrees with the massless spectrum of IIA and IIB superstring theory, respectively. We have to consider both IIA and IIB supergravity because the even $p$-branes appear in IIA supergravity, and the odd branes in type IIB. The massless multiplet structure includes the supersymmetric partners of the graviton, these are the gravitino and the dilatino.

Although the $p$-brane background (14.35) that we want to derive is purely bosonic (there is a dilaton, a metric and a form) it can nevertheless be supersymmetric. In fact, in order for our $p$-brane solution to be identifiable with a D$p$-brane in string theory, we have to check that it is (partially) supersymmetric and that it preserves (some part of) the supersymmetry transformations of the dilatino and the gravitino. The dilatino and gravitino fields are assumed to be absent in our ansatz, but under an infinitesimal supersymmetry transformation their field values will deviate from zero. More specifically, they will transform according to the values of bosonic fields that are present in the background. If these variations are different from zero, our $p$-brane background breaks supersymmetry. Should that be the case, the $p$-brane background would still make sense as an supergravity spacetime, but it will not be sypersymmetric, and it will not be possible to identify it with a D-brane solution.

The supersymmetry transformations of the dilatino and the gravitino in string frame are given by [124, 126]

$$\delta\lambda = (\partial_\mu \phi)\Gamma^\mu \varepsilon + \frac{3-p}{4(p+2)!} e^\phi F_{\mu_1 \dots \mu_{p+2}} \Gamma^{\mu_1 \cdots \mu_{p+2}} \varepsilon'_{(p)} \,, \tag{14.24}$$

$$\delta\psi_\mu = \left(\partial_\mu + \frac{1}{4}\omega_{\mu\, ab}\gamma^{ab}\right)\varepsilon + \frac{(-1)^p}{8(p+2)!} e^\phi F_{\mu_1 \dots \mu_{p+2}} \Gamma^{\mu_1 \cdots \mu_{p+2}} \Gamma_\mu \varepsilon'_{(p)} \,, \tag{14.25}$$

where $\gamma^a$ are the Minkowski space $\gamma$-matrices and $\Gamma^\mu = e^\mu_a \gamma^a$ are the curved space gamma matrices, with $e^\mu_a$ as the (inverse) vielbein (see section B.4). $\varepsilon$ is a 32-dimensional Majorana spinor for type IIA supergravity and a 32-dimensional complex Weyl spinor for type IIB supergravity. The spinor $\varepsilon'$ that appears in the equations (14.24-14.25) is defined as:

$$\varepsilon'_{(p=1,5)} = i\varepsilon^*, \qquad \varepsilon'_{(p=3)} = i\varepsilon, \qquad \varepsilon'_{(p=2,6)} = \gamma_{11}\varepsilon, \qquad \varepsilon'_{(p=4)} = \varepsilon \,. \tag{14.26}$$

These supersymmetry variations are written in a formalism where both form fields and their duals are explicitly present, and we should use the duals of the forms of (14.6-14.9) for $p > 3$ (we do not consider explicitly the $p = 3$ self-dual case for the sake of compactness). We check the supersymmetry variations for the intermediate solution in string frame (14.23a-14.23d). This is considerably more general than our full solution ($K(u,r)$ is unconstrained and the functions $\mu(u)$, $f(u)$ and $h(u)$ are unrelated).

In [136] it was shown that the variations of dilatino and gravitino vanish if

$$\varepsilon = H^{-1/8}\tilde{\varepsilon}\,, \tag{14.27}$$

with the "quasi-harmonic" function $H(u,r)$ defined in formula (14.23b), and where $\tilde{\varepsilon}$ is a constant spinor such that

$$\gamma^u\tilde{\varepsilon} = 0\,, \tag{14.28}$$

and

$$\frac{x^a}{r}\gamma^a\tilde{\varepsilon} - \frac{\epsilon_{\alpha_1\ldots\alpha_{p-1}}}{(p-1)!}\gamma^{uv\alpha_1\ldots\alpha_{p-1}a}\frac{x^a}{r}\tilde{\varepsilon}' = 0\,, \tag{14.29}$$

with $\tilde{\varepsilon}'$ defined similarly to $\varepsilon'$. These two conditions restrict the components of the Majorana (or complex Weyl) spinor down to 8 real components, which makes 8 supersymmetries manifest for our solutions and establishes them as the BPS $p$-branes. Note that the presence of these supersymmetries is insensitive to whether the time-dependent equations of motion are satisfied or not, since supersymmetry is preserved as long as the field configuration is of the form (14.23a-14.23d).

## 14.7 Step VI: time-dependent equations

To recapitulate, we have obtained solutions for the time-independent equations in section 14.4. We have restricted these solutions, corresponding to extremal branes, and we have promoted the integration constants in these solutions to three functions $f(u)$, $\mu(u)$ and $h(u)$. We have verified that the restriction corresponds to extremal branes (they preserve supersymmetry). We now solve the time-dependences. We find the relation between $f(u)$, $\mu(u)$ and $h(u)$, and we obtain an expression for $K(u,r)$. To write out the explicit solution to $K(u,r)$ we will fix our coordinate system such that the asymptotic plane wave is expressed in Brinkmann coordinates.

### 14.7.1 Analysis of the remaining equations

Plugging the "time-dependent ansatz" (14.22) into equation (14.15) yields a relation between $f(u)$, $\mu(u)$ and $h(u)$:

$$\frac{\dot{h}}{h} = \dot{f} - \frac{7-p}{2}\frac{\dot{\mu}}{\mu}, \qquad\qquad h = \frac{e^f}{\mu^{(7-p)/2}} \tag{14.30}$$

(the integration constant can always be absorbed into $R$). Equation (14.16) becomes

$$\frac{\left(r^{8-p}K'\right)'}{\mu r^{8-p}} = \left[4\ddot{f} - (9-p)\left(\frac{\ddot{\mu}}{\mu} - \frac{\dot{\mu}^2}{2\mu^2}\right)\right] + \frac{2e^f R^{7-p}}{\left(\sqrt{\mu}r\right)^{7-p}}\left[\ddot{f} - \dot{f}\frac{\dot{\mu}}{\mu} - \frac{\ddot{\mu}}{\mu} + \frac{9-p}{4}\frac{\dot{\mu}^2}{\mu^2}\right], \tag{14.31}$$

which is easily integrated to obtain a specific combination of $r^2$ and $1/r^{5-p}$ dependences. It is always possible to add terms solving the homogeneous version of (14.31), i.e. $r^0$ and $1/r^{7-p}$ with *arbitrary* $u$-dependent coefficients. The $r$-independent term can be absorbed into a redefinition of $v$. The $1/r^{7-p}$ term describes a peculiar singular pp-wave that propagates parallel to the brane essentially not interacting with it (in the sense that the shape of this wave does not affect the metric apart from its $uu$-component). We will ignore these terms in our present considerations.

### 14.7.2 Plane wave asymptotics in Brinkmann coordinates

If we now examine the large $r$ asymptotics of our solutions in *string frame*, we obtain:

$$ds^2 \equiv e^{\phi/2}ds_E^2 = -2dudv + K(u,r)du^2 + dy_\alpha^2 + \mu(u)dx_a^2. \tag{14.32}$$

As indicated above, $K(u,r)$ contains an $r^2$ term, so the asymptotics indeed look like a plane wave. It is known, however, that, by redefining $v$ and $x^a$, plane wave metrics can always be put into a form that makes the $r^2du^2$ term in the metric vanish, with the wave profile encoded in $\mu(u)$ (the Rosen form), or into a form that makes $\mu(u) = 1$, with the wave profile encoded in the coefficient of the $r^2du^2$ term in the metric (the Brinkmann form). Not surprisingly, this kind of transformations can be extended to our entire $p$-brane solutions (at all values of $r$).

More specifically, one can check that the transformation

$$v = \tilde{v} + \mu(u)\eta(u)\dot{\eta}(u)\left(\frac{\tilde{r}^2}{2} + h(u)\left(\frac{R}{\eta(u)}\right)^{7-p}\frac{\tilde{r}^{p-5}}{p-5}\right), \qquad x^a = \eta(u)\tilde{x}^a \tag{14.33}$$

preserves the algebraic form of our ansatz given by (14.6) and (14.22), while multiplying $\mu$ by $\eta^2$. Since $\eta$ is an arbitrary function of $u$, it can be used to set $\mu$ to 1, in which case our $p$-brane solution is parametrized in a way that approaches the Brinkmann form of the plane wave in the asymptotic region.

### 14.7.3 Solution for the profile $K(u,r)$

If we choose the particular coordinate system for our solution that specifies the asymptotic plane wave that our solution approaches to as a Brinkmann plane wave, then (14.31) simplifies further and it can be easily integrated to yield

$$K = \ddot{f} r^2 \left( \frac{2}{9-p} - \frac{e^f}{5-p} \frac{R^{7-p}}{r^{7-p}} \right) . \tag{14.34}$$

(Of course, other parametrization choices can be made, with (14.22)-(14.31) giving the appropriate solutions; also, as already mentioned, we do not include the homogeneous solutions of (14.31) into our expressions.)

## 14.8 Solution for branes aligned with the dilaton

With all the ingredients assembled together, our extremal plane-wave-$p$-brane solutions can be written in string frame, in the form that approaches asymptotically Brinkmann plane waves, as follows:

$$\begin{aligned}
ds^2 \equiv e^{\phi/2} ds_E^2 &= \left( 1 + e^{f(u)} \frac{R^{7-p}}{r^{7-p}} \right)^{1/2} dx_a^2 \\
&+ \left( 1 + e^{f(u)} \frac{R^{7-p}}{r^{7-p}} \right)^{-1/2} \left[ -2du dv + \ddot{f}(u)\, r^2 \left( \frac{2}{9-p} - \frac{e^{f(u)}}{5-p} \frac{R^{7-p}}{r^{7-p}} \right) du^2 + dy_\alpha^2 \right], \\
\phi &= f(u) + \frac{3-p}{4} \ln \left( 1 + e^{f(u)} \frac{R^{7-p}}{r^{7-p}} \right), \\
F_{uv\alpha_1 \cdots \alpha_{p-1} a} &= \frac{x^a}{r}\, e^{-f(u)} \frac{\partial}{\partial r} \left( 1 + e^{f(u)} \frac{R^{7-p}}{r^{7-p}} \right)^{-1} \epsilon_{\alpha_1 \cdots \alpha_{p-1}} \left[ \frac{1}{\sqrt{2}} \right]_{p=3} \qquad (p \leq 3), \\
F_{a_1 \cdots a_{8-p}} &= \frac{x^a}{r}\, e^{-f(u)} \frac{\partial}{\partial r} \left( 1 + e^{f(u)} \frac{R^{7-p}}{r^{7-p}} \right) \epsilon_{a_1 \cdots a_{8-p} a} \left[ \frac{1}{\sqrt{2}} \right]_{p=3} \qquad (p \geq 3).
\end{aligned} \tag{14.35}$$

For large values of $r$, this metric takes the form (ignoring the infrared problems for branes with a small number of transverse dimensions)

$$ds^2 = -2du dv + \frac{2}{9-p} \ddot{f}(u)\, r^2 du^2 + dy_\alpha^2 + dx_a^2, \qquad \phi = f(u), \tag{14.36}$$

which is indeed the most general Brinkmann-coordinate plane wave (isotropic with respect to $x_a$-directions and with flat $y_\alpha$-directions), written in string frame.

It could be very interesting and important to generalize our results to the case of 0-branes. In that case, there is no worldvolume to be aligned with the propagation direction of the wave, and the 0-brane is subject to forces induced by the plane wave. However, it can be seen from the corresponding D0-brane DBI analysis that there are configurations for which the gravity and dilaton forces balance each other and the 0-brane does not move. One could expect relatively simple supergravity solutions for these cases, and they are also precisely the solutions whose near-horizon geometry may have significance within the context of time-dependent matrix models. Unfortunately, our present derivations do not allow to construct such solutions. It may also be worth to investigate generalizations to the non-extremal case.

## 14.9 Comparison with the literature

Now that we have derived our time-dependent $p$-brane solutions, it is appropriate to compare them with the literature. Our solutions are new, but other papers investigate related issues, such as configurations that consist of multiple branes. For the reader who is satisfied with the taste of our solution presented in the previous section, the following comments are relatively detailed, and can be skipped, in which case the reader has come to the end of this chapter.

Comparing our result to the previously published derivations, one can note that (22) of [133] becomes identical to our (14.35) for a specific choice of $f(u)$ in the dilaton profile as a linear function of $u$.

For $p = 1$, (21) of [133] corresponds to a special choice of $f(u)$ in the dilaton profile (logarithmic in $u$, if the definition of $u$ is changed to agree with the one we are using), for which (in the asymptotically Rosen frame, different from the one used in (14.35) and related to it by transformations of the form (14.33)), the $du^2$ term disappears from the metric and the $u$- and $r$-dependences factorize throughout. Incidentally, (39) of [134] presents a family of intersecting $p1$-$p5$-solutions that should reduce to (21) of [133] when the 5-brane charge is set to 0. [134] suggests that this family of solutions should have two free parameters (three numbers, $a$, $b$ and $c$ with one quadratic constraint). However, we believe that there is in fact only a one-parameter family in (39) of [134], corresponding to the single parameter $Q$ of [133] (when the 5-brane charge is set to 0). An additional constraint on $a$, $b$ and $c$ of [134] (restoring the correspondence between (39) of [134] and (21) of [133]) can be derived by considering the $ur$-component of the Einstein equations.

For $p > 1$, (21) of [133] corresponds to a plane wave asymptotics different from (14.36), with non-trivial $y_\alpha$ polarizations present in the asymptotic plane wave (there is a $u$-dependent function multiplying $dy_\alpha^2$ in the asymptotic expression for the metric). We have not considered such asymptotic plane waves here for the sake of compactness, but one should not expect any considerable complications in including them (the brane geometry is trivial in the longitudinal directions, so superposing plane waves polarized in $y_\alpha$-directions on it should be even simpler than for the case of $x_a$-directions). The reason why only special choices of the functional shape of the asymptotic plane wave appeared in the previous publications is that assumptions have been made about $u$- and $r$-dependence factorization, or about the absence of $du^2$ terms in the metric. By relaxing these assumptions, we have restored the functional arbitrariness of the asymptotic plane wave profile.

After the work that I described in this chapter was completed and published online, a preprint [135] addressing very similar issues came to our attention. In that publication, a somewhat more general ansatz (compared to the one we have used here) is examined (non-trivial asymptotic plane wave polarizations in the directions parallel to the brane are added); considerations are also given to intersecting brane solutions. The advantage of our present treatment is that all the light-cone time dependences are derived explicitly (in [135], the problem is reduced to ordinary differential equations, which are not solved), the equation determining the $uu$-component of the metric (14.6) is analyzed without any assumptions. Indeed, this analysis does not confirm the suggestions of [135]. In (2.42) of [135] it was assumed that $K$ of (14.6) is a combination of $r^0$ and $1/r^{7-p}$ dependences on $r$. As is evident from our analysis in section 3, however, an inclusion of $r^2$ and $1/r^{5-p}$ dependences is essential for maintaining the functional arbitrariness of the plane wave profile. The inclusion of $r^0$ and $1/r^{7-p}$ terms is optional, as far as the construction of plane-wave-$p$-brane solutions is concerned, cf. the remark at the end of subsection 14.7.1. In a more recent paper [137] the authors of [135] enlarged their previous solutions by applying a similar technique to construct $K(u, r)$ as in this chapter.

# Chapter 15

# Conclusions

*"If I have seen less far than others*
*it is because I have stood behind giants,"*

*Edoardo Specchio*

In this final chapter I will briefly recapitulate the results of the previous chapters and add a some final comments. But let me first of all illustrate the main idea behind the thesis again.

## 15.1 Geometrical resolution of spacetime singularities

In general relativity, spacetime becomes curved in the presence of matter sources. Because matter moves on a curved spacetime, the spacetime geometry encodes the gravitational interaction, at least at the classical level. But all matter obeys quantum-mechanical laws, and thus a quantum theory of general relativity is needed for theoretical consistency. String theory is an approach to quantum gravity (and also to a unified description of all forces) that has general relativity as its limit at low energies compared to the Planck scale. A key concept in string theory is supersymmetry, a symmetry that relates bosons (force carriers) to fermions (matter particles). One of the modern developments in string theory is that it can be formulated in terms of different *dual* descriptions, which are derived by making use of supersymmetry. One class of dual theories are matrix models, which can be used to study the strong coupling behaviour of strings, for example near spacetime singularities.

General relativity predicts spacetime singularities like the big bang singularity or black hole singularities, which are boundary points of the spacetime manifold where the curvature becomes unbounded or ill-defined. Given that these singularities appear at the classical level, it is expected that a quantum theory may perhaps clarify their nature. At present, string theory is well understood in static spacetimes (for example, it is possible to formulate the microscopic degrees of freedom of certain static black holes), but further research is necessary to develop string theory in singular and time-dependent backgrounds relevant to the spacetime that corresponds to our universe.

The presence of spacetime singularities raises the question whether propagation across singularities is possible and how it should be described theoretically. In certain models, propagation of string theory on singular spacetimes leads to dual descriptions in terms of matrix models. The singularity in the spacetime is then mapped into singular terms in the Hamiltonian that describes the matrix model. In other cases, on can directly relate the singular terms in Hamiltonians that describe the propagation across the singularity to the original singularity in the spacetime metric. In such a case, a geometrical resolution is a meaningful resolution prescription. In order to investigate the question of field propagation across a singularity, we first regularize the spacetime with a regularization parameter. When this parameter is sent to zero, we reobtain the singular spacetime. We then derive the field evolution on the regularized spacetime, and consider the singular limit. If the limit for the field evolution exists, we have found a geometrical resolution to describe the field propagation across the singularity.

I have used a geometrical resolution to study the propagation of free fields across singularities. In chapter 10, the relevance of Hamiltonians involving multiple operator structures (with singular time-dependent prefactors) for the problem of geometrical resolution of singular spacetimes was stressed. A general review of the quantum dynamics corresponding to this type of Hamiltonians was given, with an emphasis on important simplifications that can occur if the Hamiltonian possesses a finite dimensional dynamical group.

## 15.2 Scalar field on the parabolic orbifold

I started my research with the study of a geometrical resolution of a specific toy-model: the propagation of a free scalar field on the singular parabolic orbifold spacetime [95]. A geometrical regularization of the parabolic orbifold is given by the nullbrane spacetime. We have considered a two-parameter generalization $(\alpha,\, \beta)$ of the nullbrane spacetime,

$$\begin{aligned} ds^2 = -2dX^+dX^- &+ \frac{X^2R^2(\beta^2-\alpha)}{(R^2+(X^+)^2)^2}(dX^+)^2 + \frac{2\beta XR}{\sqrt{R^2+(X^+)^2}}dX^+d\Theta \\ &+ \left(R^2+(X^+)^2\right)d\Theta^2 + dX^2\,, \end{aligned} \tag{15.1}$$

and addressed the question of the singular limit $(R \to 0)$ of the dynamics of a free scalar field on this regular background. The evolution of a free scalar field on the parabolic orbifold is governed by a dynamical group, more specifically the two-photon group $H6$. The wave equation of the free field is related to an auxiliary quantum mechanical system, expressed by the auxiliary Hamiltonian

$$\mathcal{H} = \frac{it}{2\left(R^2+t^2\right)} + \frac{P^2}{2k_-} + \frac{\beta X R k_\Theta}{\left(R^2+t^2\right)^{3/2}} + \frac{k_\Theta^2}{2k_-\left(R^2+t^2\right)} + \frac{\alpha}{2}\frac{X^2R^2k_-}{\left(R^2+t^2\right)^2} + \frac{m^2}{2k_-}\,. \tag{15.2}$$

We can find an exact solution to the auxiliary system by means of semiclassical methods, leading to the "propagator" in position space

$$\phi(X_1,t_1|X_2,t_2) = \mathcal{A}(t_1,t_2)\exp\left(-iS_{cl}[X_1,t_1|X_2,t_2]\right)\,, \tag{15.3}$$

where $t$ corresponds to coordinate $X^+$ in the metric 15.1. The non-standard minus sign in front of the classical action is a consequence of our choice to write $-2dX^+dX^-$ in the line element (15.1). The classical action $S_{cl}$ in (15.3) can be evaluated as

$$\begin{aligned} S_{cl}[X_1,t_1|X_2,t_2] &= -\left[\frac{k_-\sqrt{1+\alpha}\,R}{\sqrt{R^2+t_1^2}\sqrt{R^2+t_2^2}\,\sin 2\Delta_{12}}\right]X_1X_2 \qquad (15.4a)\\ &- k_-\left[\frac{t_1}{2\left(R^2+t_1^2\right)} - \frac{R\sqrt{1+\alpha}}{2\left(R^2+t_1^2\right)}\cot 2\Delta_{12}\right]X_1^2\\ &+ k_-\left[\frac{t_2}{2\left(R^2+t_2^2\right)} + \frac{R\sqrt{1+\alpha}}{2\left(R^2+t_2^2\right)}\cot 2\Delta_{12}\right]X_2^2\\ &- \left[\frac{\beta\,k_\Theta}{\sqrt{1+\alpha}\sqrt{R^2+t_1^2}}\tan\Delta_{12}\right]X_1 - \left[\frac{\beta\,k_\Theta}{\sqrt{1+\alpha}\sqrt{R^2+t_2^2}}\tan\Delta_{12}\right]X_2\\ &- \frac{\beta^2\,k_\Theta^2}{k_-\left(1+\alpha\right)^{3/2}\,R}\Big(\tan\Delta_{12} - \Delta_{12}\Big) - \frac{m^2}{2k_-}(t_2-t_1)\\ &- \frac{i}{2}\ln\frac{\sqrt{R^2+{t_2}^2}}{\sqrt{R^2+{t_1}^2}} - \frac{k_\Theta^2\Delta_{12}}{k_-\,R\sqrt{1+\alpha}}\,. \end{aligned}$$

We have abbreviated the arguments according to

$$\Delta_{12} = \frac{\sqrt{1+\alpha}}{2}\left(\arctan\frac{t_2}{R} - \arctan\frac{t_1}{R}\right) \tag{15.4b}$$

$$\Delta_{t2} = \frac{\sqrt{1+\alpha}}{2}\left(\arctan\frac{t_2}{R} - \arctan\frac{t}{R}\right) \tag{15.4c}$$

$$\Delta_{1t} = \frac{\sqrt{1+\alpha}}{2}\left(\arctan\frac{t}{R} - \arctan\frac{t_1}{R}\right)\,. \tag{15.4d}$$

The quantum-mechanical prefactor in (15.3) is given by,

$$\mathcal{A}(t_1,t_2) = \sqrt{\frac{R\sqrt{1+\alpha}}{2\pi\sqrt{R^2+t_1^2}}}\left(R^2+t_1^2\right)^{-1/4}\left(R^2+t_2^2\right)^{-1/4}\,|\,\mathrm{sin}2\Delta_{12}|^{-1/2}\,\phi_M\,, \tag{15.5a}$$

and the "Maslov phase" (which determines the phase jumps across the focal points) is

$$\phi_M = \exp\left(\frac{i\pi}{2}\mathrm{sign}(k_-)\sum_\ell \theta(t-t_\ell^*)\right), \tag{15.5b}$$

with $\theta(t)$ being the Heaviside step function and $t_\ell^*$ the focal points of $\mathrm{cot}2\Delta_{1t}$, located at

$$t_\ell^* = \frac{t_1 + R\tan(\pi\ell/\sqrt{1+\alpha})}{1-\tan(\pi\ell/\sqrt{1+\alpha})\,t_1/R}\,, \qquad \ell\in\mathbb{Z}. \tag{15.5c}$$

Now we have found the solution for the propagation on the generalized nullbrane, we can study the singular limit. The limit must be taken carefully, because of focussing properties of the wave equation. Surprisingly, the limit happened to exist for a *discrete* subset of the possible values of the two parameters,

$$\alpha = (2N)^2-1\,, \quad \beta = 2N\,, \quad N\in\mathbb{N}\,. \tag{15.6}$$

The limiting mode functions are closely related to those previously obtained for the nullbrane by Liu *et al* [86]. We have opted for an accurate coordinatization of the singular limit of our spaces, based on taking a limit of smooth coordinate systems on the smooth geometrical regularized spacetime. In contrast to the coordinates employed in [86], our coordinate system reveals a peculiar "reflection" property of the generalized (as well as the original) nullbrane spacetimes.

## 15.3 String modes in singular plane waves

During my second project [120] I have investigated the propagation of free strings across the singularity of a scale-invariant and isotropic plane wave. These plane waves are first approximations to realistic spacetime singularities, the scale-invariance of a plane wave naturally follows from the Penrose limit procedure that associates a plane wave to a generic (power-law) spacetime singularity [70]. Again we employ a geometric resolution prescription to investigate the free field propagation across the singularity. The resolved metric is given by

$$ds^2 = -2dx^+dx^- - \frac{\lambda}{\epsilon^2}\Omega(x^+/\epsilon)\sum_{i=1}^d (x^i)^2(dx^+)^2 + \sum_{i=1}^d (dx^i)^2. \tag{15.7}$$

and the resolution profile $\Omega$ has the following asymptotic profile,

$$\Omega(\eta) \to \frac{1}{\eta^2} + O\left(\frac{1}{\eta^b}\right), \quad b > 2\,, \tag{15.8}$$

such that when the resolution parameter $\epsilon$ is taken to zero, we obtain the singular metric

$$ds^2 = -2dx^+dx^- - \lambda \sum_{i=1}^{d} \left(\frac{x^i}{x^+}\right)^2 (dx^+)^2 + \sum_{i=1}^{d}(dx^i)^2\,. \tag{15.9}$$

Thus we have resolved the singularity in a scale-invariant manner, without introducing dimensionful parameters except for $\epsilon$.

To satisfy the background consistency conditions in string theory, we have supplemented the metric with a time-dependent dilaton to compensate the non-zero Ricci tensor of the isotropic plane wave profile. Then the string coupling becomes time-dependent, and near the singularity $x^+ = 0$ it can be written as

$$g_s = e^{\phi_0} + \left(x^+\right)^{\lambda d/2}\,. \tag{15.10}$$

The higher curvature invariants of plane wave metrics are zero and therefore plane waves are an exact classical solution in string theory. Due to the geometrical resolution prescription, the background consistency conditions hold for all $x^+$, for every spacetime in our regular class (15.7).

The string theory worldsheet sigma model is exactly solvable in plane wave backgrounds and the presence of a covariantly constant null vector in our background permits us to use lightcone gauge. Then the worldsheet Hamiltonian can be obtained from the bosonic part of action solely, because in lightcone gauge the fermionic superpartners decouple. In addition, in lightcone gauge we obtain a decoupled set of Hamiltonians for each oscillation mode of the string. Thus the string motion splits into the evolution for all modes separately. The evolution for each oscillation mode corresponds to a time-dependent harmonic oscillator, therefore the semiclassical WKB analysis is exact.

The behaviour of the center-of-mass mode was studied previously in [119]. The scale-invariance of the resolution permits to perform a scale transformation that removes the $\epsilon$-dependence from the problem. In the rescaled formulation the evolution of a string mode across the singulatiry then resembles a Sturm-Liouville problem, and one is led to a discrete set of solutions. Practically this means that the resolution profile $\Omega$ should be chosen such that the number $\lambda$ that appears in the plane wave profile falls into its discrete spectrum. We can relate the behaviour of the excited modes to the zero mode by means of a mathematical technique, called the Gronwall inequality. We find that the excited modes can propagate through the singularity if and only if the zero-mode can propagate through. We then construct a basis of solutions in the $\epsilon \to 0$ limit.

It is not sufficient that all string modes can propagate through the singularity. We also have to demand that the total energy related to the oscillations of the modes remains finite. To specify the excitation of the string modes at $x^+ \to +\infty$ we calculate the Bogoliubov coefficients that express the mode creation. In lightcone gauge the worldsheet theory is also scale-invariant, and the Bogoliubov coefficients are independent of the mode number. Therefore the energy related to the string oscillations blows up when the string crosses the singularity, unless

$$\lambda = \frac{1}{4} - \left(N + \frac{1}{2}\right)^2 , \quad N \in \mathbb{Z} . \tag{15.11}$$

We see we can only obtain a solution for $\lambda \leq 0$. However, in that case free strings are not a good approximation because the string coupling (15.10) near the singularity becomes large. This is an encouragement to look at matrix models that can provide a strong coupling prescription of string theory. To investigate certain aspects of matrix models for singular plane waves (cf. [107, 112]) in detail, it would be interesting to obtain supergravity solutions that describe D0-branes embedded in plane waves.

## 15.4 Supergravity D$p$-brane solutions

During my third collaboration [136] I have constructed a family of ten-dimensional supergravity solutions describing extended extremal $p$-branes embedded into a dilaton-gravity plane wave, with the brane worldvolume aligned along the propagation direction of the wave. We have assumed an isotropic plane wave polarization in the directions transverse to the brane worldvolume, and the absense of polarization components along the brane worldvolume. No assumptions have been made about the functional shape of the plane wave profile, which is contained in our family of solutions as an arbitrary function of the lightcone time. We present the solution in string frame. First of all, the line element of the metric is given by

$$\begin{aligned} ds^2 = {} & \mathcal{H}(u,r)^{1/2} dx_a^2 \\ & + \mathcal{H}(u,r)^{-1/2} \left[ -2du dv + \ddot{f}(u)\, r^2 \left( \frac{2}{9-p} - \frac{e^{f(u)}}{5-p} \frac{R^{7-p}}{r^{7-p}} \right) du^2 + dy_\alpha^2 \right] . \end{aligned} \tag{15.12a}$$

The dilaton is written as

$$\phi = f(u) + \frac{3-p}{4} \ln\bigl[\mathcal{H}(u,r)\bigr] , \tag{15.12b}$$

and the field strength is determined by

$$F_{uv\alpha_1\cdots\alpha_{p-1}a} = \frac{x^a}{r}\, e^{-f(u)}\, \frac{\partial}{\partial r}\mathcal{H}(u,r)^{-1}\epsilon_{\alpha_1\cdots\alpha_{p-1}}\hat{\delta}(p) \qquad (p \leq 3)\,, \tag{15.12c}$$

$$F_{a_1\cdots a_{8-p}} = \frac{x^a}{r}\, e^{-f(u)}\, \frac{\partial}{\partial r}\mathcal{H}(u,r)\,\epsilon_{a_1\cdots a_{8-p}a}\hat{\delta}(p) \qquad (p \geq 3)\,, \tag{15.12d}$$

with the factor $\hat{\delta}(p) = 1/\sqrt{2}$ for $p = 3$ and $\hat{\delta}(p) = 1$ otherwise. Finally, the function $\mathcal{H}(u,r)$ is defined as

$$\mathcal{H}(u,r) = 1 + e^{f(u)}\,\frac{R^{7-p}}{r^{7-p}}\,. \tag{15.12e}$$

The extension of this class of supergravity solutions to D0-branes, in which case there is no worldvolume to align the propagation direction of the plane wave with, is still under study.

## 15.5 Discussion

I have investigated free field propagation across singularities by means of a geometrical resolution prescription, in some special classes of lightcone time-dependent spacetimes. I have mainly concentrated on singular plane waves. I have also studied the formulation of branes embedded in (lightcone) time-dependent (and possibly singular) isotropic plane waves.

### 15.5.1 Geometrical resolution prescription

The main line of the thesis is that a geometrical resolution prescription provides a natural method to define and investigate evolution across a singularity. Of course, one is naturally tempted to prefer the regularization prescription that one has used himself/herself, but the geometrical resolution prescription is also relatively attractive from a more objective point of view.

We regularize the singular metric components by writing out explicit expressions for the regularized metric components in terms of a regularization parameter $\epsilon$. The field propagation on the singular spacetime is then derived as the limit of propagation on a class of regular spacetimes. The singular limit $\epsilon \to 0$ is taken with respect to the coordinate system in which we wrote out the regularization of the metric components.

The field propagation on the regularized spacetimes admits a physical interpretation. Yet we shouldn't carry this physical interpretation too far: we primarily utilize the regularized spacetimes as mathematical objects that allow us to define the field evolution on the singular spacetime and they disappear once we take the singular limit. Perhaps I could compare the appearance of the regularized spacetimes with the ghost fields that appear in the calculation of Yang-Mills-amplitudes to ensure unitarity. The ghosts aren't physical particles. Similarly, in our case, it is the field evolution on the singular spacetime that should be interpreted physically, the field evolution on the regularized spacetimes is a tool to obtain the former.

We have seen hints that there is a certain discrete feature related to the geometrical resolution prescription. For example, in the case of the generalized nullbrane the singular limit exists only for a discrete subset of parameter values of the generalized nullbrane geometries. In case the discreteness manifests itself in the expressions for the field evolution on the singular spacetime, it can be considered as a (perhaps essential) physical characteristic related to field propagation across a singularity. In the geometrical resolution of the parabolic orbifold the physical consequence of the discreteness (thus after the singular limit has been taken) manifests itself in a global phase jump across the singularity. In principle, if one could make a comparison between the phase of the field before and after the singularity, one could determine if the discreteness really existed and if so, which of the regularized spacetimes were preferred in terms of the phase jump. If the discreteness had been solely related to the specific choice of the regularized spacetimes it should have rather been considered as an (interesting) mathematical curiosity.

The geometrical resolution prescription is useful for plane wave singularities. Its extension to other singular spacetimes, e.g. cosmological spacetimes or black holes, is certainly not straightforward. However, in a first approximation, certain aspects of spacetime singularities can be studied in terms of the singular plane waves that correspond to them through a Penrose limit.

### 15.5.2 Backreaction

The aim of the research presented in chapter 12 was to investigate the geometrical resolution prescription in a simple toy model. The toy model appears at two levels: the parabolic orbifold is a toy model singularity, and we considered a scalar field without investigating possible backreaction on the geometry. The issue of backreaction is unavoidable if one is considering a geometrical resolution of a physical spacetime because matter does gravitate. From the point of view of backreaction, one can consider chapter 13 about string propagation across a plane wave singularity as a simple investigation of this issue: closed strings describe the gravitational interaction and if the string coupling grows large, backreaction cannot be neglected. Admittedly, the propagation of one free string across a plane wave singularity is a rather naive picture (and it could have been investigated earlier, or even stronger: it should have been investigated earlier, as it doesn't rely on the more recent string theoretical developments). Nevertheless, I would like to remark that it remains important to find out how string theory can be used in a (more) realistic setting. A plane wave singularity is only an approximation to a real cosmological singularity, but at least one has to investigate such a toy-model problem in order to get some feeling for the present status of the difficulties of applying string theory to (more) realistic models, which are necessary to be able to elucidate some of the remaining puzzles in quantum gravity and cosmology.

### 15.5.3 Background spacetime

In chapter 13 we have considered the propagation of a free string on a fixed classical background spacetime. Still, even if we would consider interacting strings on a singular plane wave, this remains a perturbative approach to quantum gravitational interactions, because we still treat our background geometry as classical. In general relativity the gravitational interaction is encoded in the spacetime structure, so in a more complete model the spacetime structure should also be described by means of string theory. For example, one clear realisation is in Maldacena's conjecture that relates a superstring theory on an Anti-deSitter background to a holographic gauge theory, but such a holographic picture is not yet applicable for all spacetimes. The treatment of spacetime as a classical background is unavoidable when we investigate a string theory sigma model on a particular background spacetime. In order to treat the background spacetime as a fluctuating entity, we can consider matrix models for such a background spacetime. In that case only the asymptotical structure of the spacetime is kept fixed and the quantum mechanical degrees of freedom in the bulk spacetime are allowed to fluctuate.

As I have already remarked during the thesis, in order to resolve the issue of string propagation across a plane wave singularity it appears necessary to investigate matrix models in plane wave spacetimes (and thereby considering the quantum-mechanical nature of the spacetime). In this sense, the results of chapter 14 are perhaps the most important for other researchers in the string theory community. Although we haven't yet derived the supergravity configuration that describes D0-branes, which would be of most interest for investigating matrix models, that chapter contains the supergravity solutions that describe time-dependent D$p$-branes which may also be useful for future research.

### 15.5.4 Lightcone time-dependent models

In this thesis we have investigated (lightcone) time-dependent models. Models on (lightcone) time-dependent spacetime are very different from models on globally static spacetimes. Yet there is also a considerable difference between the lightcone time-dependent spacetimes we have investigated and more general time-dependent spacetimes. Although one is free to choose whatever direction within the lightcone as "time", a generic time-dependence would lead to a dependence on the two lightcone coordinates. For example, a function $f(t)$

would become $f(u+v)$, with $u=(t-x)/\sqrt{2}$ and $v=(t+x)/\sqrt{2}$. Of course, there are exceptions: for example a wave travelling in the $x$ direction will exhibit a special time dependence $f(t-x)$ and will therefore only depend on the lightcone coordinate $u$.

Because we have investigated time dependence with respect to only one of the lightcone coordinates, our spacetimes automatically possessed a lightlike Killing vector $\partial_v$, which is a necessary condition for supersymmetry. It seems an outstanding question how to generalize the results, derived in (lightcone) time-dependent models with a dependence on only one of the lightcone time coordinates, to fully time-dependent models without a lightlike Killing vector and therefore certainly without unbroken supersymmetry.

Nevertheless, string theory models on lightcone time-dependent spacetimes do already provide some information about more general time-dependent spacetimes. The argument is based on the relation between time-dependent spacetimes and (lightcone) time-dependent spacetimes through the Penrose limit: every time-dependent spacetime can be approximated by a lightcone time-dependent plane wave spacetime with a Killing vector $\partial_v$. Of course, the Penrose limit yields only a first approximation (though, for example in the case of a singular spacetime, the plane wave profile already captures the diverging tidal forces near the singularity, a prominent characteristic of some singularities) but it makes it possible to obtain information about string theory in a general time-dependent spacetime. To extend the information beyond this approximation probably requires a more direct formulation of string theory (or matrix theory) in general spacetimes.

# Part IV

# Appendices

# Appendix A

# An introduction to gravitational singularities

*Lasciate ogne speranza, voi ch'intrate*

*"Divina Commedia," Dante Alighieri*

In this appendix I introduce the notion of gravitational singularities in general relativity at a very introductory level. In fact, it would be more rigourous to stick to the concept of "singular spacetime" of appendix C. Roughly speaking, with the term "spacetime singularity" we are referring to a "singular point" that is related to a singular spacetime. Notice that we can just switch names between "spacetime singularity" and "gravitational singularity" because in Einstein's classical theory of gravity the gravitational interaction is encoded in the geometry of the spacetime.

The aim of the appendix is to elucidate the concept of a "singularity" in general relativity. But meanwhile I can also mention a few concepts that appear in general relativity such as the metric and the line element. These concepts will become more clear when I discuss them further in chapter 3 and appendix B.

In general relativity free test particles follow geodesics, which extremize their path length in curved spacetime, as expressed by an action that is simply equal to the path length:

$$S = \int_{\mathcal{P}} \mathrm{d}\sigma. \tag{A.1}$$

The structure of spacetime is governed by the metric tensor $\mathrm{g}_{\mu\nu}$. It is related to the path length. With respect to a certain set of coordinates $\{x^\mu\}$ this relation is expressed as ($\mathrm{d}\sigma^2 = -\mathrm{d}s^2$, with $\mathrm{d}s^2$ the line element)

$$\mathrm{d}s^2 = \sum_{\mu\nu} g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{A.2}$$

The coefficients of the metric tensor depend on the coordinate basis, but the signature of its eigenvalues is a coordinate-independent property. Physically, it turns out that the “temporal eigenvalue” has the opposite sign with respect to the “spatial eigenvalues”. We will choose the ”mostly plus” convention, which is common in general relativity. The signature of the metric is then determined as $(- + ++)$ which means that we choose the temporal eigenvalue to be negative.

Causal geodesics are either null (lightlike) or timelike. The equation for geodesics immediately follows when we parametrize the path $\mathcal{P}$ in terms of an affine parameter $\lambda$ as

$$\mathcal{P} : \lambda \to x^\mu(\lambda)\,, \tag{A.3}$$

in the case of a null geodesic (e.g. for a light ray), or in terms of an eigentime $\tau$ as

$$\mathcal{P} : \tau \to x^\mu(\tau)\,, \tag{A.4}$$

in the case of a timelike curve (e.g. the worldline of a massive particles), and when we vary the action with respect to the coordinates $x^\mu$:

$$\ddot{x}^\kappa + \sum_{\mu\nu} \Gamma_{\mu\nu}{}^{\kappa} \dot{x}^\mu \dot{x}^\nu = 0. \tag{A.5}$$

The dot is the derivative with respect to the affine parameter or eigentime along the path. In the case of a null geodesic we have

$$\sum_{\mu\nu} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0\,, \tag{A.6}$$

and for a timelike curve

$$\sum_{\mu\nu} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu < 0\,. \tag{A.7}$$

The “Christoffel symbols” $\Gamma_{\mu\nu}{}^{\kappa}$ in equation (A.5) are derived from the metric tensor $g_{\mu\nu}$ (see also appendix B).

Out of the Christoffel symbols one can construct the Riemann curvature tensor, which determines the tidal gravitational forces on nearby particles. In general relativity the gravitational force is only an “apparent” force that depends on the movement, like the Coriolis force or centrifugal force in Newtonian mechanics [13]. Of course, this does not mean that there exists no gravitational force: consider, for example, the attraction towards the center of the earth. More specifically, it is only when there is time translation invariance in the physical problem, that it is possible to define a gravitational force by comparing geodesics with the natural “static” curves that correspond to the time translation symmetry [21]. In more general cases like the gravitational attraction caused by several stellar bodies, it is only the tidal forces between nearby geodesics that are well-defined. In such a general case there are simply no static curves to compare the geodesics with, in order to be able to define an absolute gravitational force. Instead the relative gravitational force (i.e. the tidal force) still makes sense in a general setting without time translation invariance. Thus the tidal force encodes the physical effects of gravitation that are independent of the movement.

We now define a spacetime singularity in terms of the incompleteness of causal (i.e. timelike or lightlike) geodesics. If the geodesic $\mathcal{P}(\lambda)$ cannot be extended for all finite values of the affine parameter $\lambda$, the worldline of a particle that follows this particular geodesic will have a beginning or an end, corresponding to a singularity. In order to have a "physical singularity", test particles travelling along a geodesic that hits a singularity should reach it in finite eigentime (or finite affine parameter). If not, the singularity would be located infinitely far away and would essentially not interact with the rest of spacetime.

The incompleteness of the geodesic can appear because of various reasons: for example the scalar curvature becomes unbounded along the geodesic as one approaches the singular point. It may also occur that the scalar curvature remains finite, but that certain components of the Riemann tensor become unbounded at a certain value of the affine parameter along the geodesic, or that certain components of the metric tensor become ill-defined. For example, in the case of conical singularities that appear in orbifolds (see chapter 10), the manifold becomes ill-defined at the singular point while the curvature remains bounded.

I would like to remark that the divergence (or the ill-definition) of certain components of the metric of the curvature tensor by itself does not necessarily represent a "physical singularity". It may just as well reflect an artifact of the particular coordinate system that we have used to parametrize the spacetime. If the singularity is due to the locally bad behaviour of the coordinate system and therefore disappears in a more regular coordinate system, we call it a "coordinate singularity". For example, the metric for a Schwarzschild black hole (in spherical coordinates) is given by

$$\mathrm{d}s^2 = -\left(c^2 - \frac{2MG_N}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-2MG_N/(rc^2)} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) \quad \text{(A.8)}$$

where $M$ is the mass located at $r = 0$. There is a coordinate singularity at $r = 2MG_N/c^2$, which can be removed by a change of coordinates, for example to Kruskal coordinates. Because true physical observables are expressed by constructing "scalar" quantities which do not transform under a change of coordinates, a physical singularity would remain present in any coordinate system. For example, the Schwarzschild metric has another singularity at $r = 0$, which cannot be removed by a change of coordinate system.

Thus, recapitulating, we define physical singularities as singular points (these singular points are excluded from the spacetime manifold) where causal geodesics end within finite eigentime or finite affine parameter. A more rigourous definition of spacetime singularities is given, for example, by “essential singularities” in the abstract boundary formalism, see appendix C.

# Appendix B

# Some mathematics for general relativity

*But gravity always wins.*
*"Fake plastic trees," Radiohead*

The main purpose of this appendix is to refresh some of the elementary mathematical material of general relativity, i.e. differential geometry. I expect that most readers are acquainted with differential geometry, but maybe some haven't been recently exposed to this anymore. Those who are not acquainted, may perhaps benefit from a quick introduction to the basics of differential geometry, whereas the introduction of other mathematical material would complicate matters unnecessarily. Why only some basic mathematics for general relativity? The knowledge of the basics of differential geometry roughly suffices to understand the major part of the thesis, the full mathematical machinery of string theory is not needed, as I already mentioned in the outline. Nothwithstanding the word "mathematical" in the title of this appendix, this appendix is (very) far from mathematically rigorous. Therefore, the lay reader who wants to be well-prepared is referred to the literature, e.g. the initial chapters of [21].

In general relativity, spacetime appears as a (four-dimensional) Lorentzian manifold. In the first section of this chapter I will explain the notion of "manifold", and of "Lorentzian". In the second section I will introduce covariant differentiation. In the third section I will comment on Killing vectors and in the final section I will discuss the vielbein.

# B.1 Mathematical preliminaries

## B.1.1 Manifolds

A manifold is a mathematical structure that can be differentiated, because the local neighbourhood of each point is diffeomorphic to $\mathbb{R}^n$, with $n$ the dimension of the manifold, to be identified with the dimension of the spacetime $D$. Loosely speaking, a manifold can be "visualized" as a smooth surface in a higher number of dimensions (which is generally larger than the dimensionality $n$ of the manifold itself and which has no related with the dimension of the physical spacetime). Of course, the definition of a manifold is independent of such an embedding.

As is the case with an ordinary surface, it is possible to parametrize a manifold in terms of different coordinate systems, although this does not change anything about the manifold. A change of coordinates (a diffeomorphism) is not a physical transformation, and in general relativity it will just reflects the "gauge freedom" in the metric field (which is needed because the metric field is massless). Because changes of coordinates do not affect the outcome of any physical experiment in general relativity, therefore all general relativistic laws have to be formulated in a "generally covariant" manner, which reflects the fact that physical observables are independent of coordinate changes (diffeomorphisms). To achieve this we have to introduce the notion of vectors and tensors that have specific transformation properties under the diffeomorphism group. More precisely we introduce vectors and tensors with respect to diffeomorphisms on the manifold. All physical ingredients for a theory are then formulated in terms of vectors and tensors (so they do transform under diffeomorphisms according to a strict rules) but true physical observables (which can be measured experimentally) are constructed out of these vectors and tensors in such a way that they are invariant with respect to the coordinate transformations.

A priori, the manifold is a mathematical construction and points do not have physical meaning because of diffeomorphism invariance in general relativity. Yet we observe that objects at different spacetime locations are clearly physically inequivalent. In order to give physical meaning to the points on the mathematical manifold ("events" in the case of Lorentzian spacetimes), they have to be labeled, by fixing a coordinate system and by a physical measurement (see e.g. chapter 19 of [20]).

## B.1.2 Vectors and tensors

Suppose we parametrize the manifold by coordinates $x^\mu$. By following a specified path $P$ on the manifold described by a parameter $\tau$: $P : \tau \to x^\mu(\tau)$, we can define vector fields as the set of tangent vectors along this path. At each point of the manifold one can find a basis of tangent vectors that are tangent to the coordinate lines. This basis spans up the tangent space $V_p$ at that point, which has the same dimensionality as the manifold. In each tangent space one can now define (tangent) vectors. Vectors are associated to directional derivatives, and are defined as the map from functions on the manifold to scalars. Thus, if we call $\mathcal{F}$ the space of $C^\infty$ real-valued functions on the manifold $\mathcal{M}$, then for a vector $\mathbf{V}$ at the point $p \in \mathcal{M}$:

$$\mathbf{V} : \mathcal{F} \to \mathbb{R} \,, \tag{B.1}$$

where the map $\mathbf{V}$ is linear and obeys the Leibnitz rule. A vector field on the manifold $\mathcal{M}$ is defined as an assigment of vectors $\mathbf{V}|_p \in V_p$ at each point $p \in \mathcal{M}$. With respect to the basis vectors $\partial_\mu$ (which is shorthand for $\partial/\partial x^\mu$) that are associated to a coordinate system $\{x^\mu\}$, each vector (field) $\mathbf{V}$ can be decomposed in its "contravariant" components $V^\mu$, according to

$$\mathbf{V} = V^\mu \partial_\mu . \tag{B.2}$$

In most formulas the Einstein summation convention is implicit, which means that one sums over repeated upper and lower indices (so in the formula above a summation $\sum_{\mu=1}^{n}$ is implicitly understood). The "contravariant" components means that the indices are "up". One can also write out the "covariant" components of a vector (these have the indices "down"). For this, one has to associate to each vector a dual vector. Dual vectors are defined as the maps that map a vector back to a number. The basis of dual vectors spans up the cotangent space (or dual vector space) at a point on the manifold and is given by $\{\mathrm{d}x^\mu\}$. By making a coordinate transformatie from the original coordinates $x^\mu$ to another set of coordinates $x'^\nu$ (the coordinate transformation being given by the functions $x'^\nu(x^\mu)$), the components of a vector and a dual vector transform as

$$V'^\nu(x') = \frac{\partial x'^\nu}{\partial x^\mu} V^\mu(x), \tag{B.3}$$

$$W'_\nu(x') = \frac{\partial x^\mu}{\partial x'^\nu} W_\mu(x), \tag{B.4}$$

where $x$ represents the set of coordinates $\{x^\mu\}$. One can also define an inner product between two vectors in function of their components $\mathbf{V} \cdot \mathbf{W} = V_\mu W^\mu = W_\mu V^\mu$. The relation between the covariant components $W_\mu$ and contravariant components $W^\mu$ of a vector $\mathbf{W}$ will be given in the next subsection after I have introduced the metric. It is good to keep in mind that there is often (also in this thesis) abuse of language between vectors and their components.

Then, with $V$ an $n$-dimensional vector space and $V^*$ its dual vector space, we can define tensors of rank $(k{,}l)$ as the multilinear map from a collection of vectors and dual vectors to the real numbers

$$T : \underbrace{V^* \times \cdots \times V^*}_{k} \times \underbrace{V \times \cdots \times V}_{l} \to \mathbb{R} \,. \tag{B.5}$$

Tensors are reminiscent of vectors and dual vectors but they have additional upper and/or lower indices. Each tensor has a rank $(k,l)$ that is associated to the number of contravariant indices $k$ and the number of covariant indices $l$ when the tensor is decomposed with respect to the basis vectors $\partial_\mu$ and $dx^\mu$. The transformation of tensors with respect to a coordinate transformation is essentially a combination of the transformations for all the vector indices as in B.3 and dual vector indices as in B.4,

$$T'_{\pi\cdots\sigma}{}^{\mu\cdots\nu} = \frac{\partial x'^\mu}{\partial x^\kappa} \cdots \frac{\partial x'^\nu}{\partial x^\lambda} \frac{\partial x^\rho}{\partial x'^\pi} \cdots \frac{\partial x^\tau}{\partial x'^\sigma} T_{\rho\cdots\tau}{}^{\kappa\cdots\lambda} . \tag{B.6}$$

### B.1.3 The metric and Lorentzian spacetimes

We define a "line element" $ds^2$ on a manifold that is an indication of the distance between two points. The distance between two points along a certain path is then obtained by integrating the "path length" d$s$ between these points, given by the square root of the line element. As a physical observable, the line element depends on the coordinates in such a way that it is invariant with respect to coordinate transformations. Therefore the metric function itself is usually written as,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{B.7}$$

where the metric $\mathbf{g}$ is a covariant tensor of rank (0,2) with components $g_{\mu\nu}$. The metric $g_{\mu\nu}$ can be inverted and the inverse is $g^{\mu\nu}$, with

$$g_{\mu\nu} g^{\mu\pi} = \delta^\pi_\nu, \tag{B.8}$$

where I have introduced the usual Kronecker delta $\delta^\pi_\nu$. Because the metric is symmetric the order of the indices doesn't matter. The metric tensor and its inverse are used to lower and raise the indices of tensor components, so contravariant and covariant components of a tensor are related to each other by the metric tensor. For example, the components of the following tensor can be raised and lowered according to

$$T_{\mu\cdots\nu}{}^{\pi\cdots\sigma} = g^{\pi\rho} \ldots g^{\sigma\tau} g_{\mu\kappa} \ldots g_{\nu\lambda} T^{\kappa\cdots\lambda}{}_{\rho\cdots\tau} . \tag{B.9}$$

The sign of the eigenvalues of the metric tensor $g_{\mu\nu}$, called the "signature" of the metric, is independent of the coordinates used to describe the manifold. In the case of a Euclidean manifold all the eigenvalues have the same sign, conventionally noted as $(++++)$ in the case of a four-dimensional manifold. In the case of a Lorentzian manifold, there is one "timelike" direction which has a negative sign with respect to the others. We will use the mostly plus convention and write the signature of a Lorentzian spacetime metric as $(-+++)$.

### B.1.4 Forms and wedge products

We can also introduce forms, these are tensor fields whose components are antisymmetric in all their indices (dual vectors are one-forms). With respect to the standard basis of one-forms, a $p$-form is decomposed as

$$T^{(p)} = T_{\mu_1 \cdots \mu_p} dx_1^{\mu} \dots dx_p^{\mu}, \tag{B.10}$$

with the following condition on the components of the form,

$$T^{(p)}_{\mu_1 \cdots \mu_p} = T^{(p)}_{[\mu_1 \cdots \mu_p]}, \tag{B.11}$$

where bracketed indices $[\mu_1 \cdot \mu_p]$ indicate the total antisymmetrization of all the indices, with a prefactor $1/(p!)$. Between a $p$-form and a $q$-form we can now define a "wedge product" $\wedge$ according to,

$$\left(T^{(p)} \wedge T^{(q)}\right)_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_q} = \frac{(p+q)!}{p!q!} T^{(p)}_{[\mu_1 \cdots \mu_p} T^{(q)}_{\nu_1 \cdots \nu_q]}. \tag{B.12}$$

## B.2 Covariant differentiation

The ordinary partial derivative of a scalar field yields a vector, but the ordinary partial derivative of a vector is not a tensor because it does not possess the correct transformation property under diffeomorphisms. Nevertheless, in order to be able to write generally covariant expressions we have to be able to write dynamics in terms of tensors (and there is no dynamics without derivatives). Therefore we have to introduce the concept of the covariant derivative $D_\mu$, which is a modification of the ordinary derivative $\partial_\mu$, such that the (covariant) derivative of a vector will yield a tensor. This procedure is of broad generality: in general relativity the covariant derivative is defined with respect to coordinate transformations, but in electromagnetism there is also a covariant derivative with respect to infinitesimal transformations in the $U(1)$ gauge field, which is the vectorpotential of electromagnetism.

The covariant derivative $D_\mu$ can be written in terms of the ordinary derivative $\partial_\mu$ by means of a "connection". The connection, say $\Gamma_{\mu\nu}{}^{\kappa}$, is not a tensor by itself. How the ordinary derivative and the covariant derivative are precisely related actually depends on the nature of the tensor that is being differentiated, for example

$$\begin{aligned} D_\kappa T_{\pi\cdots\sigma}{}^{\mu\cdots\nu} =& \partial_\kappa T_{\pi\cdots\sigma}{}^{\mu\cdots\nu} - \Gamma_{\kappa\pi}{}^{\lambda} T_{\lambda\cdots\sigma}{}^{\mu\cdots\nu} - \cdots - \Gamma_{\kappa\sigma}{}^{\lambda} T_{\pi\cdots\lambda}{}^{\mu\cdots\nu} \\ &+ \Gamma_{\kappa\lambda}{}^{\mu} T_{\pi\cdots\sigma}{}^{\lambda\cdots\nu} + \cdots + \Gamma_{\kappa\lambda}{}^{\nu} T_{\pi\cdots\sigma}{}^{\mu\cdots\lambda}. \end{aligned}$$

There are several ways to define a connection, but in the metrical notation of general relativity the most "natural" way is by making use of the Christoffel connection which we already anticipated by the notation ${\Gamma_{\mu\nu}}^{\kappa}$. The Christoffel connection is symmetric in its lower indices and preserves the metric ($D_\alpha g_{\mu\nu} = 0$). In a coordinate representation it can be written as

$$ {\Gamma_{\alpha\nu}}^{\beta} = \frac{1}{2} g^{\beta\mu} \left( \partial_\alpha g_{\mu\nu} + \partial_\nu g_{\mu\alpha} - \partial_\mu g_{\alpha\nu} \right). \tag{B.13} $$

On a curved manifold covariant differentiation is commonly introduced by the notion of "parallel displacement". Vectors that are located at different point on a manifold cannot be directly compared with one another. One of the vectors has to be parallelly displaced till they are both located at the same point on the manifold. A vector $V^\mu$ is parallelly displaced along a curve $\mathcal{C}$ if the following identity holds

$$ T^\nu D_\nu V^\mu = 0 \,, \tag{B.14} $$

where $T^\nu$ is the tangent vector along the curve $\mathcal{C}$. A curve is called a "geodesic" if the parallel transport of its tangent vector is zero:

$$ T^\nu D_\nu T^\mu = 0. \tag{B.15} $$

## B.3 Isometries and Killing vectors

Isometries are coordinate transformations that leave the metric invariant. The most simple example is when the metric coefficients are independent of a certain coordinate $y$, in which case the transformation $y \to y + y_0$ obviously leaves the metric invariant. For a general coordinate transformation $x \to x'$ the metric tensor transforms as

$$ g'_{\mu\nu}(x') = \frac{\partial x^\pi}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\pi\sigma}(x) \tag{B.16} $$

and isometries are expressed by the condition

$$ g_{\mu\nu}(x) = g_{\mu\nu}(x'). \tag{B.17} $$

Isometries are generated by Killing vectors. The condition for a Killing vector is

$$ D_\mu \xi_\nu + D_\nu \xi_\mu = 0 \,. \tag{B.18} $$

If the metric is independent of $x$, then $\xi = \partial/\partial x$ ($\xi^\mu = \delta^\mu_x$) is a Killing vector. A Killing vector that satisfies

$$ D_\mu \xi^\nu = 0 \tag{B.19} $$

is called covariantly constant. In a $D$-dimensional spacetime there are at most $D(D+1)/2$ independent Killing vectors. For example, in four-dimensional Minkowski spacetime there are four translations, three boosts and three spatial rotations.

## B.4 Vielbein and spin connection

The formulation of general relativity in terms of a metric $g_{\mu\nu}$ does not allow to accomodate the presence fermions in curved spacetimes. To be able to formulate fermions, the concepts of "vielbein" (also called "tetrad" or "vierbien" in four dimensions) and spin connection need to be introduced. In addition, the spin connection allows for a more efficient computation of the Riemann tensor. Nevertheless, in this thesis the vielbein formalism is only needed to calculate the supersymmetry variations in chapter 14.

At each point of the manifold we can construct an orthonormal basis of vectors $e_\mu^a$ called the "vielbein", which satisfies

$$e_\mu^{\ a} e_\nu^{\ b} \eta_{ab} = g_{\mu\nu}. \tag{B.20}$$

$a$ is a so-called "flat" or "Lorentz" index from the tangent space and runs up to the number of dimensions of the manifold. The Lorentz indices are raised and lowered with the Minkowski tensor $\eta$ and we define the inverse vielbein $e_b^\mu$ as

$$e_\mu^{\ a} e^\mu_{\ b} = \delta^a_{\ b}. \tag{B.21}$$

We will continue to use Latin letters for the tangent space and Greek letters to denote the "curved" or "Riemann" indices. Next, we define $D$ vielbein one-forms and $D^2$ spin connection one-forms,

$$e^a = e_\mu^{\ a} dx^\mu, \tag{B.22}$$

$$\omega^a_{\ b} = \omega_{\mu\ b}^{\ a} dx^\mu. \tag{B.23}$$

From these the torsion two-form and the curvature two-form can be deduced, introducing a covariant derivative $D$ that acts on Lorentz vectors,

$$T^a = De^a = de^a + \omega^a_{\ b} \wedge e^b \tag{B.24}$$

$$R^a_{\ b} = d\omega^a_{\ b} + \omega^a_{\ c} \wedge \omega^c_{\ b} \tag{B.25}$$

If no torsion is present, the tangent space covariant derivative of the vielbein is zero. In that case one can write the spin connection uniquely in terms of the vielbein,

$$\omega_\mu^{\ ab} = 2e^{\nu[a}\ \partial_{[\mu} e_{\nu]}^{\ b]} + e^{\nu a} e^{\sigma b} e_{\mu c} \partial_{[\sigma} e_{\nu]}^{\ c}, \tag{B.26}$$

where the forms $\omega^{ab}$ are antisymmetric in the tangent indices.

# Appendix C

# Singular spacetimes

> *Three Countrymen were pursuing a* Wiltshire *Thief through* Brentford. *The simplest of them, seeing the* Wiltshire House *written under a Sign, advised his Companions to enter it, for there most probably they would find their Countryman. The second, who was wiser, laughed at this Simplicity; but the third, who was wiser still, answered, 'Let us go in, for he may think we should not suspect him of going amongst his Countrymen.' They accordingly went in and searched the House, and by that Means missed overtaking the Thief, who was at that Time, but a little ways before them; and who, as they all knew, but had never once reflected, could not read.*
>
> *Henry Fielding*

The material of this appendix on the nature of singularities aims to give the interested reader a little more background information about this subject. In appendix A we have defined a spacetime singularity in terms of incompleteness of causal geodesics. We will formalize the concept of singularities in terms of the incompleteness of more general curves. In addition, a more rigorous definition of singularities can be given in terms of a boundary construction, of which the "abstract boundary" of Scott and Szekeres [138] is a modern example. This appendix is largely based on [8, 138, 139] and should be read in the context of (classical) general relativity.

## C.1 Boundary constructions

A spacetime manifold is represented by a pair $(\mathcal{M}, g_{\mu\nu})$ of a manifold $\mathcal{M}$ with a metric $g_{\mu\nu}$. Singularities can be regarded as "failed" boundary points, either due to the incompleteness of physically important curves, or because a certain notion of the spacetime curvature becomes unbounded or non-smooth in the limit as we approach the singular point. A particular case of curve incompleteness, i.e. incompleteness of causal geodesics, was used in appendix A. However, there are mathematical complications associated to a description of singularities solely in terms of curve incompleteness.

Singularities may be realised as some boundary set of (an open embedding of) the spacetime manifold. In order to do so, we need to be able to attach some sort of boundary $\partial\mathcal{M}$ to the manifold $\mathcal{M}$, which is uniquely determined by the non-singular structure of $(\mathcal{M}, g_{\mu\nu})$. Several boundary prescriptions have already been developed. Historically, the most important of these prescriptions were the g-boundary of Geroch, the b-boundary of Schmidt and the c-boundary of Geroch, Kronheimer and Penrose. A suitable and more modern boundary construction is given by the abstract boundary (or a-boundary). We will discuss it below. All boundary constructions are problematic or counterintuitive in some specific cases. For a discussion, see [139].

A boundary prescription can be constructed in the context of curve incompleteness. For example, in the g-boundary prescription of Geroch it is investigated whether two (causal) curves limit to the same point. But we can consider different classes of curves. If the choice of curve family is fixed in advance, then the concept of singular point becomes crucially dependent on the chosen curve family. Certain physically relevant phenomena may be missed if we restrict our attention to a too limited set of curves such as causal geodesics. In some cases timelike curves with bounded acceleration can reach singularities that are not reached by causal geodesics. In some other cases it should also be possible to consider spacelike geodesics or even other curves with possible geometrical significance. Therefore in order to have a useful boundary construction, there should be some flexibility in the choice which curve-family to consider.

## C.2 Incompleteness of general curves

The restriction to incompleteness of causal geodesics is not sufficient to characterize all kinds of singularities. To consider more general curves we need to generalize the concept of affine parameter to all $C^1$ curves, whether geodesic or not. To make this more precise, let us follow [8]. Let us consider the $C^1$ curve $\mathcal{P}(t)$ that passes through $p \in \mathcal{M}$ and we choose $\{\mathbf{e}_i\}$ as a basis for the tangent space $T_p$. Along the curve $\mathcal{P}(t)$, the basis $\{\mathbf{e}_i\}$ can be parallelly propagated to obtain a basis for $T_{\mathcal{P}(t)}$ at each value of $t$. The tangent vector $\mathbf{V} = (\partial/\partial t)_{\mathcal{P}(t)}$ can be expressed in terms of the basis as $\mathbf{V} = V^i(t)\mathbf{e}_i$ and a generalized affine parameter $u$ can be defined on the curve $\mathcal{P}(t)$ by

$$u = \int_p \sqrt{\delta_{ij} V^i V^j} \mathrm{d}t. \tag{C.1}$$

The parameter $u$ depends on the point $p$ and the basis $\{\mathbf{e}_i\}$ at $p$. But it can be shown that the length of a curve $\mathcal{P}$ is finite in the parameter $u$ if and only if it is finite in the parameter $u'$, obtained by choosing the basis $\{\mathbf{e}'_i\}$ instead of $\{\mathbf{e}_i\}$. If $\mathcal{P}$ is a geodesic then $u$ corresponds to an affine parameter. We can now define a spacetime $(\mathcal{M}, g_{\mu\nu})$ to be b-complete if there is an endpoint for every $C^1$ curve of finite length as measured by a generalized affine parameter. In [8] a spacetime is called singularity free if it is b-complete. As a side remark, b-completeness can also be defined without a metric, it is sufficient that a connection is defined on the spacetime manifold.

The b-boundary of Schmidt is a boundary construction that considers all general curves by making use of the definition of b-completeness. Yet it is problematic in other aspects: because of the generality of curves the b-boundary is very difficult to generate, and when it can be done it has some drawbacks, e.g. in the case of the closed Friedmann model the initial and final singularity are identified. For further comments and references, see [139].

## C.3 The abstract boundary

Besides the flexibility in the choice of what curves to consider (flexibility in the sense of not to be fixed in advance) it would also be preferable to have a boundary construction that makes use of the principle of general covariance by providing information that is invariant under any re-embedding of the spacetime. For these issues the abstract boundary construction provides a solution.

The abstract boundary is a rigorous classification scheme for boundary points of pseudo-Riemannian manifolds, thus including Lorentzian manifolds relevant for spacetimes. The aim of the abstract boundary construction is to produce a formal description of a singularity as a place with respect to the spacetime manifold. As such it provides a model for "essential singularities".

In contrast to the previous attempts of defining singularities as failed boundary points of the manifold, the abstract boundary [138] provides an algorithm for classifying topological boundary points of some manifold once they are obtained in a particular embedding of the spacetime. The construction of the abstract boundary involves a method to identify equivalent boundary sets of the same manifold from different embeddings. Thus the main advantage of the abstract boundary is that many physically important and topologically essential concepts are defined in a way that is invariant under the choice of the boundary representative.

Once a manifold is provided with an affine connection (or a metric, but an affine connection is sufficient) and some class of curves that satisfy a bounded parameter property (affinely parametrized curves satisfy this property) then abstract boundary points naturally fall into various categories, which are independent of the affine connection and the chosen family of curves. The boundary points can be classified in the following categories: unapproachable points, indeterminate points, points at infinity and essential singularities. Essential singularities are further subdivided into directional singularities and pure singularities. A point is called a directional singularity if in some particular embedding it also covers regular points and points at infinity.

Armed with theorems derived within the abstract boundary formalism, it is then possible to rigorously describe situations under which a spacetime is singularity free. It is important to remark that geodesic incompleteness is not a sufficient condition for a manifold to be considered singular within the abstract boundary prescription. This is because geodesic incompleteness is not always due to the presence of essential singularities. Therefore, one has to focus on additional requirements to enforce equivalence between the existence of essential singularities and the incompleteness of causal geodesics. The latter concept was used in the singularity theorems of Hawking and Penrose which were briefly mentioned in the introduction. In the case of strongly causal, maximally extended spacetimes, this equivalence has been proven in [139].

# Appendix D

# Gravitons

*"The gauge condition always strikes twice,"*

*Marc Henneaux*

In the context of quantum gravity, one often speaks about gravitons, the quanta of the gravitational force. I will first introduce gravitational perturbations, then show that these perturbations have spin-two helicity, and finally present an argument (originally due to Steven Weinberg) why the gravitational force naturally couples to the energy-momentum tensor.

## D.1 Gravitational perturbations

Let us consider a metric perturbation $\breve{g}_{\mu\nu}$ upon a background metric $\bar{g}_{\mu\nu}$ that is a solution to Einstein's equation in vacuum, such that the total spacetime metric can be written as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \breve{g}_{\mu\nu}. \tag{D.1}$$

The (linearized) inverse metric should be written as

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \breve{g}^{\mu\nu}, \tag{D.2}$$

where raising and lowering of the perturbation is performed with the background metric. Consequently, to the metric $g_{\mu\nu}$ a Ricci tensor is associated that is slightly different from the background Ricci tensor

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \breve{R}_{\mu\nu}, \tag{D.3}$$

where the perturbation of the Ricci tensor is given by

$$\breve{R}_{\mu\nu} = \frac{1}{2}\left(\bar{D}^{\alpha}\bar{D}_{\mu}\breve{g}_{\alpha\nu} + \bar{D}^{\alpha}\bar{D}_{\nu}\breve{g}_{\alpha\mu} - \bar{D}_{\mu}\bar{D}_{\nu}\breve{g}^{\alpha}_{\alpha} - \bar{D}_{\alpha}\bar{D}^{\alpha}\breve{g}_{\mu\nu}\right). \tag{D.4}$$

The "bars" indicate that the covariant derivative is taken with respect to the background. But the new spacetime metric $g_{\mu\nu}$ would not necessarily represent a different spacetime. We could consider an infinitesimal coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x), \tag{D.5}$$

which would lead to the transformation (up to first order) of the metric perturbation

$$\begin{aligned}\breve{g}_{\mu\nu} \to \breve{g}'_{\mu\nu} &= \breve{g}_{\mu\nu} + \frac{\partial \bar{g}_{\mu\nu}}{\partial x^\alpha}\xi^\alpha + \bar{g}_{\alpha\nu}\frac{\partial \xi^\alpha}{\partial x^\mu} + \bar{g}_{\alpha\mu}\frac{\partial \xi^\alpha}{\partial x^\nu}, \\ &= \breve{g}_{\mu\nu} + \bar{D}_\mu \xi_\nu + \bar{D}_\nu \xi_\mu. \end{aligned} \tag{D.6}$$

Any metric perturbation $\breve{g}'_{\mu\nu}$ that is related to another $\breve{g}_{\mu\nu}$ by this kind of transformation is physically the same (and if a metric perturbation can be made zero by this kind of coordinate transformation the perturbation is simply "pure gauge"). To separate the physical perturbation from the variation due to the coordinate freedom, we can gauge-fix the coordinates (or construct gauge-invariant quantities). We will choose coordinates such that the perturbed metric (we drop the primes) satisfies the condition

$$\bar{D}^\mu \left( \breve{g}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\breve{g} \right) = 0, \tag{D.7}$$

with $g$ the trace of the perturbation (with respect to the background metric). In order to do so we need to find the vector field $\xi^\mu$ (which generates the coordinate transformation (D.5)) that satisfies

$$\bar{D}^\nu \bar{D}_\nu \xi_\mu + \bar{R}_\mu{}^\nu \xi_\nu = -\bar{D}^\nu \breve{g}_{\mu\nu} + \frac{1}{2}\bar{D}_\mu \breve{g}, \tag{D.8}$$

and it can be proven that it is always possible to find such a $\xi^\mu$. Still, even after imposing the gauge condition (D.7) we have some residual freedom left. We can use this residual freedom to impose a condition on the initial value of the perturbation coefficients. By making a coordinate transformation

$$x' \to x''^\mu = x'^\mu + \chi^\mu, \tag{D.9}$$

where $\chi^\mu$ satisfies

$$\bar{D}^\nu \bar{D}_\nu \chi_\mu + \bar{R}_\mu{}^\nu \chi_\nu = 0, \tag{D.10}$$

we will set $\breve{g} = 0$ (again without proof, this is possible in the case considered here). Combining the conditions, we conclude that for an arbitrary vacuum perturbation $\breve{g}_{\mu\nu}$ of an arbitrary vacuum solution $\bar{g}_{\mu\nu}$, we can always choose the transverse, traceless gauge where

$$\bar{D}^\mu \breve{g}_{\mu\nu} = 0, \tag{D.11}$$

$$\breve{g} = 0. \tag{D.12}$$

The perturbation of Einstein's equation follows by relating the pertubation of the Ricci tensor (D.4) to the perturbation of the energy-momentum tensor, which is zero in vacuum. In the gauge (D.11-D.12) the perturbation of Einstein's equation becomes

$$\bar{D}^{\kappa}\bar{D}_{\kappa}\breve{g}_{\mu\nu} - 2\bar{g}^{\kappa\pi}\bar{R}_{\kappa\mu\nu}{}^{\lambda}\breve{g}_{\lambda\pi} = 0\,. \tag{D.13}$$

This is the wave equation for the perturbation $\breve{g}_{\mu\nu}$ that we want to solve. These are vacuum perturbations with respect to a background that is a vacuum solution. In case we want to consider perturbations with respect to a non-vacuum background that is related to a certain energy-momentum tensor, we should investigate the transformation of the components of the energy-momentum tensor under the infinitesimal coordinate changes (D.5,D.9).

We will write out a solution to equation (D.13), subject to the gauge conditions (D.11-D.12). For simplicity, we will assume that the background is Minkowski spacetime. With hindsight, we will single out a certain direction $x$ and write the background in lightcone coordinates (with $u = (t-x)/\sqrt{2}$ and $v = (t+x)/\sqrt{2}$):

$$ds^2 = -2dudv + \sum_{i=1}^{D-2}\left(dx^i\right)^2 \tag{D.14}$$

which is essentially the same as (3.25) except that we allow for more than four dimensions (the index $i$ runs over the $D-2$ transverse dimensions) and that the transverse coordinates are Cartesian instead of spherical. The wave equation is simply

$$\left(-2\partial_u\partial_v + \sum_i \partial_i^2\right)\breve{g}_{\mu\nu} = 0 \tag{D.15}$$

We will split the perturbation in modes propagating in the $x$-direction (dependent on $u$), and propagating in the negative $x$-direction (dependent on $v$) which we collect in two "polarization matrices" $A$ and $B$ (in what follows we use $(i,j,k)$ to denote the transverse indices):

$$\breve{g}_{\mu\nu} = A_{\mu\nu}(u,x^i) + B_{\mu\nu}(v,x^i). \tag{D.16}$$

We set the components of $A$ and $B$ that have indices in the lightcone directions $u$ or $v$ equal to zero. The gauge conditions and the wave equation become

$$\sum_i \partial_i A_{ij} = 0\,, \qquad \sum_i A_{ii} = 0\,, \quad \sum_k \partial_k^2 A_{ij} = 0\,. \tag{D.17}$$

The easiest way to solve the transversality condition and the (transverse) Laplace equation is to remove the dependence of the polarization matrices on the transverse coordinates. So the solution is given by $A_{ij}(u)$ and $B_{ij}(v)$

where $\sum_i A_{ii} = \sum_i B_{ii} = 0$. The solution represents a superposition[1] of plane gravitational waves that travelling to the right or to the left, respectively, with waves vectors $k^R_\mu = k_R \delta^u_\mu$ and $k^L_\mu = k_L \delta^v_\mu$. The polarization profile is encoded in the matrix structure, which is traceless and symmetric, so the matrices $A$ and $B$ have each $D(D-3)/2$ components. In four dimensions, we can write out the perturbation as

$$\breve{g}_{\mu\nu} dx^\mu dx^\nu = (A_{11}(u) + B_{11}(v)) \left(dx^1 dx^1 - dx^2 dx^2\right) + 2\,(A_{12}(u) + B_{12}(v))\, dx^1 dx^2 \tag{D.19}$$

The perturbation is assumed to have a small amplitude, but we will see later that these gravitational waves can actually have arbitrary strength, but only if we restrict the solution to either right-moving (with polarization $A_{ij}$) or left-moving waves (with polarization $B_{ij}(v)$).

For a long time there has been the question whether these perturbative solutions really represent physical waves, which was clarified when nonlinear gravitational wave solutions were discovered. These perturbative solutions do represent fluctuations in the metric, just as Maxwells equations contain electromagnetic waves. And just as Maxwell's waves resemble a collection of massless spin one particles (photons), we will see in the next section that these gravitational waves resemble the characteristics of a massless spin two particle (graviton). Yet, we should remark that this resemblance of the metric perturbation is only with respect to the (fixed) Minkowski background, and the split of the full metric into a Minkowski background plus a perturbation is a particular approximation (that is a priori only valid under the assumption that the perturbations remain small) and from the stanpoint of the equivalence between the gravitational force and the spacetime structure, it is not natural to make split between the spacetime structure (the Minkowski background) and the gravitational force (the perturbations), except when we assume that the perturbations are to remain small.

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[1] If desired, one can decompose the solution into a basis of plane waves, e.g.

$$A_{ij}(u) = \frac{1}{\sqrt{2\pi}} \int \hat{A}_{ij}(k_R) \exp\left(i k_R u\right) \mathrm{d}k_R, \tag{D.18}$$

and likewise for $B_{ij}(v)$.

## D.2 Helicity of the gravitational perturbations

Let us return to the equation of motion (D.4) for the metric perturbation on a Minkowski background, subject to the first gauge condition (D.7),

$$\partial_\alpha \partial^\alpha \breve{g}_{\mu\nu} = 0 \tag{D.20}$$

$$\partial^\mu \breve{g}_{\mu\nu} - \frac{1}{2}\partial_\nu \breve{g} = 0 \tag{D.21}$$

We do not immediately choose the transverse traceless gauge (D.11-D.12) in order to impose constraints on the components of the metric perturbation in a clearer manner. If we take the (single) plane wave ansatz

$$\breve{g}_{\mu\nu} = a_{\mu\nu} \exp\left(ik_\lambda x^\lambda\right) + a^*_{\mu\nu} \exp\left(-ik_\lambda x^\lambda\right), \tag{D.22}$$

it will solve (D.20-D.21) if the wave vector $k^\mu$ obeys

$$k_\mu k^\mu = 0 \tag{D.23}$$

$$k^\mu a_{\mu\nu} - \frac{1}{2} k_\nu a = 0, \tag{D.24}$$

meaning that the gravitational wave travels at the speed of light and that the polarization tensor $a_{\mu\nu}$ is subject to $D$ conditions. But the polarization tensor is not gauge-invariant yet, if we make the coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu$, with $\epsilon^\mu$ given by

$$\epsilon^\mu(x) = i\epsilon^\mu \exp\left(ik_\lambda x^\lambda\right) - i\epsilon^{\mu *} \exp\left(-ik_\lambda x^\lambda\right), \tag{D.25}$$

then the polarization tensor will change according to,

$$a_{\mu\nu} \to a_{\mu\nu} + k_\mu \epsilon_\nu + k_\nu \epsilon_\mu, \tag{D.26}$$

without violation of the conditions (D.24). So we have another $D$ components of the polarization tensor that are fixed. In total there are $D(D-3)/2$ free components to characterize the plane wave travelling along $k_\mu$.

For the clarity of the exposition we will assume that $D$ is even, and without loss of generality we can write out the Minkowski background in lightcone coordinates, and combine the transverse coordinates in complex pairs (e.g. $\zeta_1 = x_1 + ix_2$), such that

$$\bar{g}_{\mu\nu} dx^\mu dx^\nu = -2du dv + \sum_i d\zeta_i d\zeta_i^* \tag{D.27}$$

and that $k_\mu = k_u \delta^u_\mu$. We use the conditions (D.26) to remove all $u$-components of the polarization tensor. Then we use the remaining conditions (D.24) to remove all the $v$-components of the polarization tensor and the trace $a$, now determined by

$$a = \sum_i a_{\zeta_i \zeta_i^*}. \tag{D.28}$$

We now split the remaining components of the (transverse and traceless) polarization tensor, according to their transformation under a Lorentz transformation $\Lambda^\mu{}_\nu$ (which is a global symmetry of the Minkowski background spacetime). More specifically we choose a rotation with angle $\theta$ in the complex plane $\zeta_j$ (in $D = 4$ this corresponds a rotation along the wave vector $k_\mu$). Under this rotation $J^\mu{}_\nu$ the coordinates transform as

$$x^\mu \to J^\mu{}_\nu x^\nu, \qquad J^\mu{}_\nu = e^{i\theta}\delta^\mu_{\zeta_j}\delta^{\zeta_j}_\nu + e^{-i\theta}\delta^\mu_{\zeta_j^*}\delta^{\zeta_j^*}_\nu \tag{D.29}$$

and the polarization tensor according to

$$\begin{aligned} a_{\mu\nu} &\to J_\mu{}^\rho J_\nu{}^\sigma a_{\rho\sigma} \\ &\to \Big[e^{-2i\theta}\delta^\rho_{\zeta_j}\delta^\sigma_{\zeta_j} + e^{2i\theta}\delta^\rho_{\zeta_j^*}\delta^\sigma_{\zeta_j^*} + 2\delta^{(\rho}_{\zeta_j}\delta^{\sigma)}_{\zeta_j^*} \\ &\qquad + 2e^{-i\theta}\delta^{(\rho}_{\zeta_j}\sum_{k\neq j}\delta^{\sigma)}_{\zeta_k} + 2e^{i\theta}\delta^{(\rho}_{\zeta_j^*}\sum_{k\neq j}\delta^{\sigma)}_{\zeta_k}\Big] a_{\rho\sigma}. \end{aligned} \tag{D.30}$$

Under the rotation elements of the little group $ISO(D-2)$ that preserves the propagation direction of a massless wave, a wave component with helicity $h$ transforms as

$$\psi' = e^{ih\theta}\psi. \tag{D.31}$$

We can thus separate the polarization vector in helicity components $h = 0, \pm1, \pm2$. In arbitrary dimensions there are two components with helicities $h = \pm 2$, $2(D-4)$ components with $h = \pm 1$, and $(D-4)(D-3)/2$ components with $h = 0$. In four dimensions we only have the $h = \pm 2$ components with maximal (absolute) helicity because the traceless condition (D.28) removes the $\zeta\zeta^*$ component. The positive and negative helicities are related to each other by an inversion of space.

## D.3 Interacting spin-two particles

If we specify our background as Minkowski space, (D.13) becomes

$$\partial_\kappa \partial^\kappa \breve{g}_{\mu\nu} = 0 \tag{D.32}$$

which is the classical equation of motion for a massless spin-two particle, also known as a graviton, and it was first written by Pauli and Fierz. (Suppose we consider the same equation for an arbitrary rank-two tensor, we can split it into parts that are symmetric or antisymmetric in their indices. Then we can still consider only the symmetric part $\breve{g}_{\mu\nu}$, because the antisymmetric part of the tensor field would rather behave like two photon fields, see e.g. [4]. We can decompose the field $\breve{g}_{\mu\nu}$ into positive and negative frequency modes to which we associate creation and annihilation operators. From these creation and annihilation operators for gravitons, we can construct (see e.g. [4, 24]) a tensor $\breve{R}_{\mu\nu\rho\sigma}$ which has the same algebraic properties of the Riemann curvature tensor (antisymmetric within the pairs $(\mu\nu)$ and $(\rho\sigma)$, and symmetric between the pairs).

Because the graviton is a massless spin-two particle, its the number of degrees of freedom in $D$ dimensions is only $D(D-3)/2$. Therefore we have to remove the $2D$ unphysical states (e.g. in four dimensions only 2 polarizations of the graviton are physical) that appear because generically a symmetric (spacetime) tensor with two indices has $D(D+1)/2$ components. The only way to do this, is to include a gauge symmetry of $D$ elements, under which the unphysical states are pure gauge, and can be removed by gauge fixing.

Then, if we want to construct a Lorentz-invariant physical theory in which the field $\breve{g}_{\mu\nu}$ interacts with other fields, it is not sufficient that the couplings of $\breve{g}_{\mu\nu}$ are only invariant under transformations of the type,

$$\breve{g}_{\mu\nu} \to {\Lambda_\mu}^\rho {\Lambda_\nu}^\sigma \breve{g}_{\mu\nu}, \tag{D.33}$$

which would imply a formal Lorentz invariance, but the couplings of $\breve{g}_{\mu\nu}$ also have to be invariant under the gauge transformations

$$\breve{g}_{\mu\nu} \to \breve{g}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \tag{D.34}$$

The latter condition can be accomplished by constraining the interactions of $\breve{g}_{\mu\nu}$ with other fields to be of the form $\breve{g}_{\mu\nu} T^{\mu\nu}$, where $T_{\mu\nu}$ is a conserved tensor current that satisfies $\partial_\mu T^{\mu\nu}$. As anticipated in the notation, the only possible such tensor is the energy-momentum tensor.

In principle one could avoid this conclusion by using e.g. $\breve{R}_{\mu\nu\rho\sigma}$ instead of $\breve{g}_{\mu\nu}$ when one constructs interacting theories of massless particles of spin two. An interaction density that is constructed solely from tensors related to $\breve{R}_{\mu\nu\rho\sigma}$ will have matrix elements that vanish more rapidly for small energy and momentum (of a massless particle) than an interaction density that also uses $\breve{g}_{\mu\nu}$, because $\breve{R}_{\mu\nu\rho\sigma}$ is obtained by taking derivatives of $\breve{g}_{\mu\nu}$. So interactions in the former theory without $\breve{g}_{\mu\nu}$ will have a rapid fall-off at large distances, faster than the inverse-square gravitational law discovered by Newton. In other words, in order to incorporate the usual inverse-square law of gravitational interactions, we need a field $\breve{g}_{\mu\nu}$ that transforms as a symmetric tensor, up to gauge transformations that are associated with coordinate transformations in general relativity. Then, when considering a quantum theory of massless particles of helicity $\pm 2$ that can incorporate long-range gravitational interactions, it is necessary for those gravitons to have a gauge symmetry that is reminiscent of general covariance.

# Appendix E

# The Magnus expansion

*A diferencia de Newton y de Schopenhauer, su antepasado no creía en un tiempo uniforme, absoluto. Creía en infinitas series de tiempos, en una red creciente y vertiginosa de tiempos divergentes, convergentes y paralelos. Esa trama de tiempos que se aproximan, se bifurcan, se cortan o que secularmente se ignoran, abarca* todas *las posibilidades. No existimos en la mayoría de esos tiempos; en algunos existe usted y no yo; en otros, yo, no usted; en otros, los dos.*

*"El jardín de senderos que se bifurcan," Jorge Luis Borges.*

In this appendix we will review the formulation of the Magnus expansion [140, 141]. The Magnus expansion is an (approximate) formula for the exponential representation of the operator solution to the Schrödinger equation with a time-dependent Hamiltonian. Its main virtue is that it gives a unitary time-displacement operator in every order of the approximation.

Let us consider a system that obeys the time-dependent Schrödinger equation. If the system is describet at time $t_0$ by the wavefunction $\psi(t_0)$, it will be described at time $t$ by the wavefunction $\psi(t)$ that is generated by the unitary time-displacement operator $U$,

$$\psi(t) = U(t, t_0)\psi(t_0), \tag{E.1}$$

where the operator $U$ satisfies

$$i\hbar\frac{dU(t,t_0)}{dt} = H(t)U(t,t_0), \qquad U(t_0,t_0) = 1. \tag{E.2}$$

The Hamiltonian $H$ will be time-dependent if the system is not isolated. The formal solution

$$U(t,t_0) = \exp\left[-i\int_{t_0}^{t}\frac{dt'}{\hbar}H(t')\right] \tag{E.3}$$

is only correct if all the $H(t)$ commute with each other for all $t$, unless it is interpreted symbolically (by inserting a time-ordered product). Then the solution to equation (E.1) is written iteratively as

$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt_1 H(t_1)U(t_1,t_0) \tag{E.4}$$

$$= 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt_1 H(t_1) + \left(\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H(t_1)H(t_2) + \cdots . \tag{E.5}$$

But this approximation to $U(t,t_0)$ is not manifestly unitary at each order of the expansion, and it can only be expected to hold for small $H$ and small time differences between $t_0$ and $t$. However, precisely because of its exponential form, $U(t,t_0)$ can be approximated by an expansion that is manifestly unitary order-by-order. Suppose we can write

$$U(t,t_0) = \exp\left[W(t)\right], \tag{E.6}$$

then any approximation to $W(t)$ that is anti-Hermitian, will result in a unitary approximation for $U(t,t_0)$. The Magnus formula now presents an expansion for the operator $W(t)$, and it is essentially just a rearrangement of the perturbation series (E.5).

The Magnus expansion provides an expression for the operator $W(t)$ as a functional of $H(t')$. It is related to the Baker-Campbell-Hausdorff formula, that gives the expansion of a matrix $C$ defined in terms of the matrices $A$ and $B$ by the expression

$$\exp(A)\exp(B) = \exp(C), \tag{E.7}$$

in terms of a series of nested commutators in $A$ and $B$:

$$C = A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] + \frac{1}{12}[[A,B],B] + \frac{1}{24}[[A,[A,B]],B] + \cdots , \tag{E.8}$$

where we have omitted the nested commutators with an order higher than three. Obviously, this formula has special importance when the commutator algebra of $A$ and $B$ closes at a certain order, for example $[[A,B],A] = [[A,B],B] = 0$.

In the case of the Magnus expansion we combine the equations (E.2) and (E.6). The derivative of the evolution operator

$$\frac{d}{dt}\exp(W(t)), \tag{E.9}$$

is expanded into commutators of $W(t)$ and $dW(t)/dt$ and equated to the expansion of

$$-\frac{i}{\hbar}H(t)\exp(W(t)). \tag{E.10}$$

We will not derive the full Magnus formula, but it can be obtained by writing $W(t)$ as a series

$$W(t) = \sum_{n=1}^{\infty} W_n(t), \tag{E.11}$$

where the first term is given by

$$W_1'(t) = -\frac{i}{\hbar} H(t), \tag{E.12}$$

and then solving iteratively for the $W_n(t)$. These terms will be expressed as a sum of integrals of $(n-1)$-fold multiple commutators of $H(t)$. Each $W_n(t)$ is the commutator of two anti-Hermitian operators and as such it is again anti-Hermitian. Therefore the Magnus expansion of $W(t)$ may be truncated at any order without affecting the unitarity of the operator $U(t)$. Each term of the Magnus expansion is in principle no more difficult to solve than term of corresponding order in equation (E.5), but there does not exist a general nonrecursive expression for the $n$-th term. Therefore the Magnus expansion is mainly useful when it is possible to truncate the series after the lowest orders (and when one wishes to obtain an approximation for the exponential form of the evolution operator).

The commutators in the first terms of the Magnus expansion can be rearranged to give,

$$W_1(t) = -\frac{i}{\hbar} \int_{t_0}^{t} \mathrm{d}t_1 H(t_1); \tag{E.13}$$

$$W_2(t) = \frac{1}{2\hbar^2} \int_{t_0}^{t} \mathrm{d}t_2 \int_{t_0}^{t_2} \mathrm{d}t_1 \, [H(t_1), H(t_2)]\,; \tag{E.14}$$

$$\begin{aligned} W_3(t) = & \frac{i}{6\hbar^3} \int_{t_0}^{t} \mathrm{d}t_3 \int_{t_0}^{t_3} \mathrm{d}t_2 \int_{t_0}^{t_2} \mathrm{d}t_1 \, [H(t_1)[H(t_2), H(t_3)]] \\ & + \frac{i}{6\hbar^3} \int_{t_0}^{t} \mathrm{d}t_3 \int_{t_0}^{t_3} \mathrm{d}t_2 \int_{t_0}^{t_2} \mathrm{d}t_1 \, [[H(t_1), H(t_2)], H(t_3)]. \end{aligned} \tag{E.15}$$

The minus sign difference between the second order contribution in (E.5) and (E.14) is easily explained by noticing that in the Magnus expansion $t_2$ is later than $t_1$, but in the common expansion $t_2$ is earlier than $t_1$. Of course, we should immediately remark that the Magnus formula gives an expansion for $W(t, t_0)$, which is the logarithm of $U(t, t_0)$, while the common Dyson series (E.5) is the expansion for $U(t, t_0)$ itself.

# Appendix F

# Field theory with constraints: an example

*There is certainly a gauge* fixation *in this book,*

*Anonymous physicist upon reading [9]*

In this appendix I will derive the Hamiltonian (12.8) of the free scalar field on the generalized nullbrane of chapter 12 more carefully, while taking care of the constraints. For the readers who would like to read more about the subject of constrainted systems: a classic introduction is [3]. For constraints in the context of quantum field theory see chapter 7 of [23], and for a detailed treatment see [9].

We will consider the action (12.5) rewritten in the form

$$S = \sum_{k_\Theta} \int \mathrm{d}X^+ \, \mathrm{d}k_- \, \mathrm{d}X \, \sqrt{R^2 + (X^+)^2} \Bigg[ \frac{ik_-}{2} \left( \phi \partial_{X^+} \phi^* - \phi^* \partial_{X^+} \phi \right) - \frac{\partial_X \phi \partial_X \phi^*}{2} - \left( \frac{m^2}{2} + \frac{k_\Theta^2}{2(R^2 + (X^+)^2)} + \frac{\alpha X^2 R^2 k_-^2}{2(R^2 + (X^+)^2)^2} + \frac{k_\Theta k_- \beta X R}{(R^2 + (X^+)^2)^{3/2}} \right) \phi \phi^* \Bigg], \tag{F.1}$$

to bring the fields $\phi$ and $\phi^*$ on equal footing. Fields $\phi$ and $\phi^*$ with opposite $k_-$ and $k_\Theta$ wavenumbers are still related through the reality condition on the original field $\Phi$ (the reality condition on the Fourier decomposition yields $\phi^*_{k_-,k_\Theta} = \phi_{-k_-,-k_\Theta}$), but in what follows we consider each wavenumber $(k_-, k_\Theta)$ separately and we will consider them to be functionally independent (for the time being, because they will turn out to be related through the first-order formalism). We can obtain the Lagrangian and the Hamiltonian by

$$S = \int \mathrm{d}X^+ \, L \,, \tag{F.2}$$

$$H = \sum_{k_\ominus} \int \mathrm{d}k_- \int \mathrm{d}X \ \left[\pi_{k_-,k_\ominus} \dot{\phi}_{k_-,k_\ominus} + \pi^*_{k_-,k_\ominus} \dot{\phi}^*_{k_-,k_\ominus}\right] \ - \ L \,, \tag{F.3}$$

where the conjugate momenta $\pi_{k_-,k_\ominus}(X, X^+)$ and $\pi^*_{k_-,k_\ominus}(X, X^+)$ are defined as usual by

$$\pi = \frac{\delta L}{\delta \dot{\phi}} \,, \qquad \pi^* = \frac{\delta L}{\delta \dot{\phi}^*} \,. \tag{F.4}$$

As mentioned in chapter 12, there are constraints because the Lagrangian is linear in the time derivatives of the fields $\dot{\phi}$ and $\dot{\phi}^*$. This means that the conjugate momenta (from now on the labels $k_-$ and $k_\ominus$ are implicit)

$$\pi = -\frac{ik_-}{2}\sqrt{R^2 + (X^+)^2}\phi^* \tag{F.5}$$

$$\pi^* = \frac{ik_-}{2}\sqrt{R^2 + (X^+)^2}\phi \tag{F.6}$$

are determined by the fields only (and not by their derivatives), so the transformation of the Langrangian to the Hamiltonian is not well defined: the time derivatives of the fields cannot be expressed in terms of the fields and their conjugate momenta. We can write that the action 12.5 is supplemented by two primary constraints:

$$\chi_1 = \pi + \frac{ik_-}{2}\sqrt{R^2 + (X^+)^2}\phi^* \approx 0, \tag{F.7a}$$

$$\chi_2 = \pi^* - \frac{ik_-}{2}\sqrt{R^2 + (X^+)^2}\phi \approx 0. \tag{F.7b}$$

The weak equality sign "$\approx$" is standard use in the theory of constrained systems and symbolizes the fact that these equations only hold on the constraint surface (which is determined by these equations), but these equations can be different from zero elsewhere in the phase space. For our present discussion the notion of constraint surface is not very important, but in order to distinguish equalities that hold under the constraints from ordinary equalities, it may still be useful to write the weak equality sign. Because the constraints are weakly zero, it is always possible to add a linear combination with Langrange multipliers $u_m$ to the Hamiltonian (F.3) and obtain

$$H' = H + \sum_{k_\ominus} \int \mathrm{d}k_- \mathrm{d}X \ (u_1\chi_1 + u_2\chi_2) \,. \tag{F.8}$$

At the classical level the time evolution of a field variable is expressed by its Poisson bracket with the total Hamiltonian. The Poisson bracket is defined for functionals that depend on the canonical variables $\phi$ and $\pi$,

$$[\xi, \eta]_{PB} = \frac{\delta \xi}{\delta \phi}\frac{\delta \eta}{\delta \pi} - \frac{\delta \xi}{\delta \pi}\frac{\delta \eta}{\delta \phi} + \frac{\delta \xi}{\delta \phi^*}\frac{\delta \eta}{\delta \pi^*} - \frac{\delta \xi}{\delta \pi^*}\frac{\delta \eta}{\delta \phi^*}. \tag{F.9}$$

We introduce an explicit subscript "PB" to distinguish the Poisson bracket from the commutator that appears in the canonical quantisation prescription

$$[\phi, \pi]_{PB} \to -\frac{i}{\hbar}[\phi, \pi]. \tag{F.10}$$

We should check that the primary constraints (F.7) do not generate secondary constraints. Since the primary constraints have to hold during the time evolution, this means that if we calculate their Poisson bracket with the Hamiltonian (F.8), these should remain weakly zero:

$$[\chi_1, H]_{PB} + u_2[\chi_1, \chi_2]_{PB} = \dot{\pi} + ik_-\sqrt{R^2 + (X^+)^2}u_2 \approx 0, \tag{F.11a}$$

$$[\chi_2, H]_{PB} + u_1[\chi_2, \chi_1]_{PB} = \dot{\pi}^* - ik_-\sqrt{R^2 + (X^+)^2}u_1 \approx 0. \tag{F.11b}$$

There are no secondary constraints because these relations depend on the Lagrange multipliers. Following Dirac, we should classify our constraints as "first class" or "second class". We calculate the Poisson bracket between our constraints,

$$[\chi_1, \chi_2]_{PB} = ik_-\sqrt{R^2 + (X^+)^2}, \tag{F.12}$$

and see that it doesn't vanish, so the constraints are second class. Because the constraints are second-class, the standard theory tells us to introduce the Dirac bracket, related to the Poisson bracket by (in our specific case)

$$[\xi, \eta]_{DB} = [\xi, \eta]_{PB} + \frac{i}{k_-}\frac{\{[\xi, \chi_2]_{PB}[\chi_1, \eta]_{PB} - [\xi, \chi_1]_{PB}[\chi_2, \eta]_{PB}\}}{\sqrt{R^2 + (X^+)^2}}. \tag{F.13}$$

In the standard theory it suffices to continue working with the Dirac bracket instead of the Poisson bracket. However, this would be cumbersome, and it is mentioned in [9] that second-class constraints can be used to eliminate degrees of freedom. So it would be more practical instead to make a change of basis in the canonical variables, such that the constraints become aligned with one of the canonical pairs. In this way, applying the constraints becomes equivalent to omitting one of the canonical pairs. For any set of canonical variables $\Phi^k$ and $\Pi_k$ governed by second class constraints, it is always possible [114] to make a canonical transformation and construct two sets of canonical variables $Q^n$, $\mathcal{Q}^r$ and their respective conjugate momenta $P_n$, $\mathcal{P}_r$ such that the constraints become $\mathcal{Q}^r \approx 0$ and $\mathcal{P}_r \approx 0$.

Thus, we direct our focus back to the Hamiltonian (F.3) and we perform a canonical transformation from the fields $\phi$, $\phi^*$ to the fields $\phi^a$, $\phi^b$,

$$\phi_a = \frac{1}{2}\left(\phi - \frac{2i}{k_-}\frac{\pi^*}{\sqrt{R^2+(X^+)^2}}\right), \tag{F.14a}$$

$$\pi_a = \pi - \frac{ik_-}{2}\sqrt{R^2+(X^+)^2}\phi^*, \tag{F.14b}$$

$$\phi_b = \frac{1}{2}\left(\phi^* - \frac{2i}{k_-}\frac{\pi}{\sqrt{R^2+(X^+)^2}}\right), \tag{F.14c}$$

$$\pi_b = \pi^* - \frac{ik_-}{2}\sqrt{R^2+(X^+)^2}\phi. \tag{F.14d}$$

The new variables satisfy $[\phi_a(x,t), \pi_b(x',t)]_{PB} = \delta_{ab}\delta(x-x')$. These new variables will lead to a new Hamiltonian $H'$ which we will derive below. But let us first notice that if we apply the constraints in the new variables (thus *after* the canonical transformation) we simply obtain

$$\phi_a \approx \phi, \tag{F.15}$$

$$\pi_a \approx -ik_-\sqrt{R^2+(X^+)^2}\phi^*, \tag{F.16}$$

$$\phi_b \approx 0, \tag{F.17}$$

$$\pi_b \approx 0. \tag{F.18}$$

Because the constraints are equivalent to $\phi_b \approx 0$ and $\pi_b \approx 0$ in the new canonical basis, we can directly impose them by simply reducing the dimension of the phase space.

To derive the new Hamiltonian $H'(\phi_a, \pi_a, \phi_b, \pi_b)$, which appears after the canonical transformation (F.14), we will make use of a generating function (see e.g. [10])

$$G(\phi, \phi^*, \phi_1, \phi_2) = ik_-\sqrt{R^2+(X^+)^2}\Big[\phi_b\phi + \phi_a\phi^* - \phi_a\phi_b - \frac{1}{2}\phi^*\phi\Big], \tag{F.19}$$

with the conjugate momenta given by

$$\pi = \frac{\partial G}{\partial \phi}\,,\ \pi^* = \frac{\partial G}{\partial \phi^*}\,,\ \pi_a = -\frac{\partial G}{\partial \phi_a}\,,\ \pi_b = -\frac{\partial G}{\partial \phi_b}, \tag{F.20}$$

and the new Hamiltonian

$$H'(\phi_a, \pi_a, \phi_b, \pi_b) = H(\phi, \pi, \phi^*, \pi^*) + \frac{\partial G}{\partial X^+}. \tag{F.21}$$

In our case the canonical transformation does lead to an additional time-dependent term in the new Hamiltonian (F.21)

$$\frac{\partial G}{\partial X^+} = \frac{ik_- X^+}{\sqrt{R^2 + (X^+)^2}} \left( \phi_b \phi + \phi_a \phi^* - \phi_a \phi_b - \frac{1}{2} \phi \phi^* \right) , \tag{F.22}$$

$$\approx -\frac{X^+}{2\sqrt{R^2 + (X^+)^2}} \pi_a \phi_a \, . \tag{F.23}$$

The equations (F.20) are identical to (F.14) and the new Hamiltonian becomes

$$H' = \sum_{k_\Theta} \int \mathrm{d}X^+ \, \mathrm{d}k_- \, \mathrm{d}X \; \left[ -\frac{X^+}{2\sqrt{R^2 + (X^+)^2}} \pi_a \phi_a - i\pi_a \frac{\partial_X^2 \phi_a}{2k_-} \right. \tag{F.24}$$

$$\left. + \, i\pi_a \left( \frac{m^2}{2k_-} + \frac{k_\Theta^2}{2k_-(R^2 + (X^+)^2)} + \frac{\alpha X^2 R^2 k_-}{2(R^2 + (X^+)^2)^2} + \frac{k_\Theta \beta X R}{(R^2 + (X^+)^2)^{3/2}} \right) \phi_a \right] . \tag{F.25}$$

So we see that our shortcut at the end of section 12.1, to interprete the conjugate momentum in terms of the complex conjugate of the field

$$\pi \equiv -ik_- \sqrt{R^2 + (X^+)^2} \phi^*, \tag{F.26}$$

in order to obtain the Hamiltonian (12.8), agrees with the lengthier procedure illustrated here, once we remove the label $a$.

# Appendix G

# The Maslov phase for the quantum harmonic oscillator

*I believe what characterizes 20th-century physics, so as to distinguish it from the flavor of physics in past centuries, are three concepts: quantization, phase factor, and symmetry.*

*C. N. Yang*

In the context of this thesis the Maslov phase arose in the derivation of the mode functions on the generalized nullbrane. As we already remarked in section 12.3, if we pursue a semiclassical construction of the quantum-mechanical mode functions, the singular behavior of the classical action near focal points introduces formal complications in the equation that determines the quantum-mechanical prefactor that accompanies the exponential of the classical action.

The Maslov phase is a phase factor that is picked up each time a focal point is passed. For example, the phase of light jumps by $-\pi/2$ when it passes through a focal point. Discussing the general theory would lead us too far, but it is straightforward to describe a simple system to show the origin of the Maslov phase. In the quantum harmonic oscillator the Maslov phase appears naturally when the propagator is evaluated by means of the Feynman path integral. The phase factor that accompanies the exponential of the classical action is obtained by integrating the variations with respect the classical path, and it is precisely this integral that yields the Maslov phase.

In this appendix we will derive the Maslov phase for a one-dimensional quantum harmonic oscillator in the context of the Feynman propagator in quantum mechanics. The exposition is based on [142]. We will thereby summarize some general results about the Feynman propagator, but we emphasize the role of focal points. The main purpose of this appendix is to serve as an illustration, from another point of view, to the discussion of the focusing properties of the wave equation on the generalized nullbrane in section 12.4.

We consider two points $x_1$ and $x_2$ and a set of paths $\gamma(t)$ with $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$. According to the principle of least action, the classical motion from $x_1$ to $x_2$ within the time-interval $[t_1, t_2]$ is given by the path $\bar{\gamma}$ that extremizes[1] the action $S(\gamma)$. In quantum mechanics all paths are "possible" and the evolution between $x_1$ and $x_2$ is described by a propagator $K(x_2, t_2|x_1, t_1)$ that is given by a "sum" over all paths. A phase factor is associated to each path $\gamma$ and the propagator is given by

$$K(x_2, t_2|x_1, t_1) = \int_{\mathcal{P}} \exp\left(\frac{i}{\hbar} S(\gamma)\right) \mathcal{D}\gamma, \tag{G.1}$$

with $\mathcal{P}$ the set of all paths and $\mathcal{D}\gamma$ the integration measure. To see that this gives us the correct classical limit (when the action of a path is much larger than $\hbar$) we note that paths that are distinct from the classical path $\bar{\gamma}$ (which is the path with stationary phase) will end up with a rapidly oscillating phase factor, and cancel each other out. To proceed we have to define the integration measure, which is possible for a harmonic oscillator.

We will assume there is a unique classical path $\bar{\gamma}$ and we will decompose each path $\gamma(t)$ as a variation $\eta(t)$ with respect to the classical path $\bar{\gamma}(t)$, i.e. $\gamma(t) = \bar{\gamma}(t) + \eta(t)$ with $\eta(t_1) = \eta(t_2) = 0$. The action along $\gamma(t)$ is then

$$\begin{aligned} S(\gamma) &= \int_{t_1}^{t_2} \frac{m}{2} \left(\dot{\gamma}^2 - \omega^2 \gamma^2\right) \mathrm{d}t \\ &= \int_{t_1}^{t_2} \frac{m}{2} \left(\dot{\bar{\gamma}}^2 - \omega^2 \bar{\gamma}^2\right) \mathrm{d}t + \int_{t_1}^{t_2} \frac{m}{2} \left(\dot{\eta}^2 - \omega^2 \eta^2\right) \mathrm{d}t, \end{aligned} \tag{G.2}$$

where the terms that included both $\eta$ and $\bar{\gamma}$ disappeared after partial integration (we have invoked the fact that the classical path $\bar{\gamma}$ satisfies the equations of motion $\ddot{x} + \omega^2 x = 0$). The equations of motion yield the following solution for the classical path $\bar{\gamma}$:

$$x(t) = \frac{\sin\omega(t_2 - t)}{\sin\omega(t_2 - t_1)}\, x_1 \; + \; \frac{\sin\omega(t - t_1)}{\sin\omega(t_2 - t_1)}\, x_2. \tag{G.3}$$

Therefore the classical action (unless it is evaluated at a focal point, determined by $\omega(t_2 - t_1) = k\pi$) is given by

$$S(\bar{\gamma}) = \frac{m\omega}{2\sin\omega(t_2 - t_1)} \left((x_1^2 + x_2^2)\cos\omega(t_2 - t_1) \; - \; 2x_1 x_2\right). \tag{G.4}$$

[1] In general there is only one such path.

The measure $\mathcal{D}\gamma$ reduces to $\mathcal{D}\eta$ and we can rewrite the propagator as

$$K(x_2,t_2|x_1,t_1) = \mathcal{K}(t_1,t_2) \times \exp\left(\frac{i}{\hbar}S(\bar{\gamma})\right), \tag{G.5}$$

with $\mathcal{K}(t_1,t_2)$ a path integral over all variations $\eta(t)$,

$$\mathcal{K}(t_1,t_2) = \int_{\mathcal{P}} \exp\left[\frac{i}{\hbar}\int_{t_1}^{t_2} \frac{m}{2}\left(\dot{\eta}^2 - \omega^2\eta^2\right) \mathrm{d}t\right] \mathcal{D}\eta\,. \tag{G.6}$$

We now expand the variations $\eta$ on the interval $[x_1,x_2]$ into a Fourier series. More specifically (with the time interval $T = t_2 - t_1$) we write

$$\eta(t) = \sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi}{T}t\right)\,. \tag{G.7}$$

In the Fourier domain the integration over all paths becomes the integration over all fourier coefficients $a_k$ such that

$$\mathcal{K}(t_1,t_2) = \lim_{n\to\infty}\ \mathcal{J}\int\cdots\int_{-\infty}^{\infty}\prod_{j=1}^{n}\mathrm{d}a_j \times \exp\left[\sum_{k=1}^{n} i\frac{m\omega^2}{2\hbar}\left(\frac{\pi^2k^2}{\omega^2(t_2-t_1)^2} - 1\right)a_k^2\right]. \tag{G.8}$$

The Jacobian $\mathcal{J}$ of the linear transformation from the set of paths to the set of fourier coefficients is independent of the parameters of the problem, and together with some formally divergent factors we include it in a prefactor $C_n$. We use the following results for the Fresnel integrals that appeared in expression (G.8)

$$\int_{-\infty}^{\infty} \exp\left(\frac{i\lambda_k}{2}x^2\right) \mathrm{d}x = \begin{cases} \sqrt{\left|\frac{2\pi}{\lambda_k}\right|}\, e^{i\pi/4}, & \lambda_k > 0 \\ \sqrt{\left|\frac{2\pi}{\lambda_k}\right|}\, e^{-i\pi/4}, & \lambda_k < 0 \end{cases}, \tag{G.9}$$

where, in our case, $\lambda_k$ is shorthand for $(\pi k)^2/(\omega T)^2 - 1$. From the formula for the Fresnel integrals we can already see where the Maslov phase will appear. At every focal point (i.e. $t_2 = t_1 + k\pi/\omega$) another $\lambda_k$ will become negative, each time adding a phase $-\pi/2$. Thus we obtain

$$\mathcal{K}(t_1,t_2) = \lim\nolimits_{n\to\infty} C_n \prod_{k=1}^{n}\left|\frac{\pi^2k^2}{\omega^2(t_2-t_1)^2} - 1\right|^{-1/2} \times \exp\left(-i\frac{\pi}{2}N\right). \tag{G.10}$$

The exponential in this expression is the Maslov phase, and $N$ is the number of focal points already crossed. Manipulating the square root in (G.10) and using the Euler formula

$$\prod_{k=1}^{\infty}\left|1 - \frac{\omega^2(t_2-t_1)^2}{k^2\pi^2}\right| = \frac{|\sin\omega(t_2-t_1)|}{\omega(t_2-t_1)}, \qquad t_2 > t_1, \tag{G.11}$$

we obtain

$$\mathcal{K}(t_1, t_2) \propto \sqrt{\frac{\omega(t_2 - t_1)}{|\sin\omega(t_2 - t_1)|}} \tag{G.12}$$

and the (formally divergent) proportionality factor can be chosen such that in the limit $\omega \to 0$ the expression for the propagator of the harmonic oscillator reduces to the propagator of a free particle. So we obtain (away from focal points)

$$K(x_2, t_2|x_1, t_1) = \sqrt{\frac{m\omega}{2\pi\hbar\,|\sin\omega(t_2 - t_1)|}}\; e^{-i\pi/4} \exp\left(-i\frac{\pi}{2}N\right) \times \\ \exp\left\{i\frac{m\omega}{2\hbar\sin\omega(t_2 - t_1)}\left[(x_1^2 + x_2^2)\cos\omega(t_2 - t_1) \;-\; 2x_1x_2\right]\right\}, \tag{G.13}$$

where $N$ is the number of focal points crossed. Finally, at the $N$'th focal point the propagator will be given by

$$K(x_2, t_2|x_1, t_1) = \exp\left(-i\frac{\pi}{2}N\right) \times \delta(x_1 - (-1)^N x_2). \tag{G.14}$$

# Appendix H

# Bessel functions

*Bloch: "Space is the field of linear operators."*

*Heisenberg: "Nonsense, space is blue and birds fly through it."*

*"Heisenberg and the early days of quantum mechanics," Felix Bloch*

For convenience we list some properties of Bessel functions of order $\nu$, obtained from [1]. They solve the differential equation

$$x^2 \frac{d^2y(x)}{dx^2} + x\frac{dy(x)}{dx} + \left(x^2 - \nu^2\right) y(x) = 0. \tag{H.1}$$

A basis of solutions to this differential equation (for non-integer $\nu$) is given by the Bessel functions of the first kind: the Bessel functions $J_\nu(x)$ and $J_{-\nu}(x)$. A different basis is given by $J_\nu(x)$ and the Neumann function (or Bessel functions of the second kind) $Y_\nu(x)$, which are linearly independent for all $\nu$. The Neumann function is written as

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}. \tag{H.2}$$

A series expansion for the Bessel function is given by

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m+\nu}, \tag{H.3}$$

with the Gamma function defined as

$$\Gamma(x) = \int_0^\infty \mathrm{d}t\, t^{x-1}\, e^{-t}. \tag{H.4}$$

For discrete arguments we can write $\Gamma(n+1) = n!$.

The Hankel functions (or Bessel functions of the third kind) are written as

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x) = i\mathrm{cosec}(\nu\pi)\left\{e^{-i\pi\nu}J_\nu(x) - J_{-\nu}(x)\right\}, \tag{H.5}$$

$$H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x) = i\mathrm{cosec}(\nu\pi)\left\{J_{-\nu}(x) - e^{i\pi\nu}J_\nu(x)\right\}, \tag{H.6}$$

and are linearly independent for all $\nu$.

**Asymptotic expansions**

For mall arguments $0 < x < \sqrt{\nu+1}$ we can write:

$$J_\nu(x) \sim \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu, \tag{H.7}$$

$$Y_\nu(x) \sim \begin{cases} \frac{2}{\pi}\left[\ln(x/2)+\gamma\right] & ,\nu = 0 \\ -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu & ,\nu > 0 \end{cases}. \tag{H.8}$$

For large arguments $x \gg |\nu^2 - 1/4|$ we can write:

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \nu\frac{\pi}{2} - \frac{\pi}{4}\right), \tag{H.9}$$

$$Y_\nu(x) \sim \sqrt{\frac{2}{\pi x}}\sin\left(x - \nu\frac{\pi}{2} - \frac{\pi}{4}\right), \tag{H.10}$$

$$H_\nu^{(1)} \sim \sqrt{\frac{2}{\pi x}}\, e^{i\left(x-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)}, \tag{H.11}$$

$$H_\nu^{(2)} \sim \sqrt{\frac{2}{\pi x}}\, e^{-i\left(x-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)}. \tag{H.12}$$

**Wronskians**

$$W[J_\nu(x), J_{-\nu}(x)] = J_{\nu+1}(x)J_{-\nu}(x) + J_\nu(x)J_{-(\nu+1)}(x) \tag{H.13}$$

$$= -\frac{2}{\pi x}\sin(\nu\pi). \tag{H.14}$$

$$W[J_\nu(x), Y_\nu(x)] = J_{\nu+1}(x)Y_\nu(x) - J_\nu(x)Y_{\nu+1}(x) \tag{H.15}$$

$$= \frac{2}{\pi x}. \tag{H.16}$$

Note: in this thesis we have used the convention $W[f,h] = f\dot{h} - h\dot{f}$, in accordance with our reference.

**Recurrence relations**

In the following prime denotes the derivative with respect to $x$ and $\mathcal{C}$ can denote a linear combination of Bessel functions of any kind (but with coefficients independent of $\nu$ and $x$):

$$\mathcal{C}_{\nu-1}(x) + \mathcal{C}_{\nu+1}(x) = \frac{2\nu}{x}\mathcal{C}_\nu(x)\,, \tag{H.17}$$

$$\mathcal{C}_{\nu-1}(x) - \mathcal{C}_{\nu+1}(x) = 2\mathcal{C}'_\nu(x)\,, \tag{H.18}$$

$$\mathcal{C}'_\nu(x) = \mathcal{C}_{\nu-1}(x) - \frac{\nu}{x}\mathcal{C}_\nu(x)\,, \tag{H.19}$$

$$\mathcal{C}'_\nu(x) = -\mathcal{C}_{\nu+1}(x) + \frac{\nu}{x}\mathcal{C}_\nu(x)\,. \tag{H.20}$$

# Appendix I

# Majorana-Weyl spinors in $9+1$ dimensions

*In the early days, such matrices were taken as a literal truth [...] Every morning, day after day, Klein and Nishina would multiply away explicit* $[4 \times 4]$ $\gamma_\mu$ *matrices and sum over* $\mu$*'s. In the afternoon, they would meet in the cafeteria of the Niels Bohr Institute to compare their results.*

*"Group theory: Birdtracks, Lie's and Exceptional Groups," Predrag Cvitanović*

In this appendix I will briefly discuss a few aspects related to the supersymmetry variations in type IIA and IIB supergravity, in view of our derivations in chapter 14. First of all I have to introduce Weyl and Majorana spinors in terms of Dirac spinors. Let us follow the conventions of [14] (adapted to our notation with small Roman letters for tangent space indices) and consider a representation of the Clifford algebra given by the (flat-spacetime) $\gamma$-matrices,

$$\left\{\gamma^a, \gamma^b\right\} = 2\eta^{ab}, \tag{I.1}$$

where the curly brackets indicate anticommutators as usual and $\eta^{ab}$ is the inverse Minkowksi metric in $(D-1,1)$ dimensions, with the mostly plus signature $(-1,1,\ldots,1)$. From the $\gamma$-matrices we can construct the matrices $\Sigma^{\mu\nu}$ by

$$\Sigma^{ab} = \frac{1}{4i}\left[\gamma^a, \gamma^b\right]. \tag{I.2}$$

They satisfy the commutation relations of the Lorentz group generators:

$$i[\Sigma^{mn}, \Sigma^{sr}] = \eta^{ns}\Sigma^{mr} + \eta^{mr}\Sigma^{ns} - \eta^{nr}\Sigma^{ms} - \eta^{ms}\Sigma^{nr}. \tag{I.3}$$

If we now consider an infinitesimal rotation of $SO(D-1,1)$ and store the parameters associated to the rotation in the matrix $\theta_{mn}$, then a spinor is an object with the following transformation property under such a rotation:

$$\delta\psi = \frac{i}{2}\theta_{mn}\Sigma^{mn}\psi. \tag{I.4}$$

We restrict ourselves to $9+1$ dimensions (one timelike and nine spacelike dimensions), where Dirac spinors have 32 components. We define a matrix $\gamma_{11}$ by

$$\gamma_{11} = \gamma^0\gamma^1\dots\gamma^9. \tag{I.5}$$

It has eigenvalues $\pm 1$. The Dirac representation splits into two sixteen-dimensional Weyl representations, defined by the eigenvectors of $\gamma_{11}$ with eigenvalues (chirality) $\pm 1$ respectively. These are called Weyl spinors. If we construct the chiral projectors

$$P_\pm = \frac{1}{2}\left(1 \pm \gamma_{11}\right). \tag{I.6}$$

a Weyl spinor (of chirality $\pm 1$) satisfies

$$P_\pm\psi = \psi. \tag{I.7}$$

In ten dimensions, the Weyl representations are self-conjugate (this is different from four dimensions, where they are conjugate to each other). The anti-symmetrized products of $\gamma$-matrices are abbreviated as

$$\gamma^{m_1 m_2\cdots m_n} = \gamma^{[m_1}\gamma^{m_2}\dots\gamma^{m_n]}. \tag{I.8}$$

Let us define another matrix, following the notation of [14],

$$B = \gamma^3\gamma^5\gamma^7\gamma^9. \tag{I.9}$$

A Majorana spinor (in ten dimensions) satisfies the reality condition

$$\psi^* = B\psi, \tag{I.10}$$

which expresses a relation between the spinor and its conjugate. In ten dimensions it is possible to consider Majorana-Weyl spinors with sixteen real components, which satisfy both Majorana and Weyl conditions.

In chapter 14 the supersymmetry variations in the time-dependent $p$-brane background are written in a formalism that allows to formally consider type IIA and type IIB supergravity simultaneously. To achieve this, the formalism works with 32-dimensional spinors. The two (sixteen-dimensional) Majorana-Weyl spinors of type IIA and type IIB supergravity are combined into a Majorana spinor for type IIA supergravity (the two Majorana-Weyl spinors have opposite chirality), and a complex Weyl spinor for type IIB supergravity (the two Majorana-Weyl spinors have the same chirality).

# Nederlandse samenvatting

*"The production of useful work is limited by the laws of thermodynamics, but the production of useless work seems to be unlimited."*

*Donald Simanek*

## Geometrische resolutie van ruimte-tijd singulariteiten

In algemene relativiteit wordt verondersteld dat de ruimte-tijd geen op voorhand vastgelegde structuur is, maar door middel van een dynamisch metrisch veld wordt beschreven. De gravitationele aantrekking tussen materie manifesteert zich dan op de volgende wijze: de dynamica van het metrische veld is gerelateerd aan de distributie van de materie doorheen het universum, en de propagatie van materie in de ruimte-tijd wordt beïnvloed door het metrische veld dat de voorstelling vormt van het universum. Sinds ontdekt werd dat de materiedeeltjes en de microscopische krachten gehoorzamen aan kwantummechanische wetten, is het dus wegens consistentie noodzakelijk om ook het metrische veld te kwantiseren. Eén van de onderzoeksrichtingen naar een consistente theorie van kwantumgravitatie heeft tot snaartheorie geleid. Snaartheorie beschrijft de gravitationele interactie door middel van gravitonen, die de kwanta zijn van de zwaartekracht. In snaartheorie wordt verondersteld dat, op de kleinste afstandsschalen, elementaire deeltjes geen puntdeeltjes zijn maar daarentegen ééndimensionale snaren zijn. Het gedrag van snaren op tijdsafhankelijke achtergronden zoals ons uitdijend universum is echter nog niet volledig begrepen.

Algemene relativiteit voorspelt het bestaan van gravitationele singulariteiten op het klassieke niveau: ons universum nam een aanvang met de oerknal, en zware sterren kunnen ineenstorten tot zwarte gaten. Een theorie die kwantumgravitationele effecten kan beschrijven zou ons begrip over dit soort singulariteiten moeten verbeteren. Bovendien doet het bestaan van ruimte-tijd-singulariteiten de vraag rijzen of de propagatie van kwantumvelden doorheen

een singulariteit mogelijk is, en indien ja, hoe dat zou moeten worden geformuleerd. Snaartheorie kan reeds sommige tijdsachtige singulariteiten beschrijven maar nog geen ruimteachtige singulariteiten zoals de oerknal. Nabij singulariteiten interageren snaren vaak sterk en een formulering van snaartheorie die toelaat om sterke interacties tussen snaren in rekening te brengen wordt gegeven door matrix theorie. Modellen in matrix theorie die singulariteiten beschrijven hebben vaak een duale beschrijving in de vorm van een kwantumvelden theorie die gedefinieerd is op een supplementaire singuliere ruimte-tijd.

In mijn thesis wordt onderzocht hoe de propagatie van kwantumvelden doorheen een singulariteit te definiëren. We gebruiken een geometrisch regularisatievoorschrift om de evolutie van een vrij scalair veld, alsook van een vrije snaar, doorheen een singulariteit op een niet-ambiguë manier te definiëren. Merkwaardig genoeg suggereert de geometrische regularisatie dat er aan de evolutie doorheen de singulariteit een zeker discreet gedrag gerelateerd is. We onderzoeken eveneens een belangrijke klassie van tijdsafhankelijke achtergronden die onderzocht kunnen worden in snaartheorie. Deze klasse wordt gevormd door de vlakke zwaartekrachtsgolven. Deze vlakke golven kunnen gebruikt worden om de effecten vanwege sterke kromming nabij een singulariteit te onderzoeken. Onze studie toont aan dat het nodig is om in rekening te nemen dat de snaren sterk kunnen interageren nabij de singulariteit. Om een beter begrip te krijgen van matrix theorie op een vlakke golfachtergrond, onderzoeken we oplossingen die D-branen beschrijven in vlakke golfachtergrond. D-branen zijn objecten die voorkomen in snaartheorie naast snaren, en ze zijn belangrijk voor de formulering van matrix theorie.

## Vrij scalair veld op het parabolisch orbifold

In de context van matrixmodellen voor tijdsafhankelijke singulariteiten speelt de evolutie van kwantumvelden op een singuliere ruimte-tijd een belangrijke rol, wat aanleiding geeft tot singuliere tijdsafhankelijke termen in de Hamiltoniaan. Daarom heb ik tijdens mijn eerste project het voorkomen van singuliere tijdsafhankelijke termen in de Hamiltoniaan onderzocht. In een eerder project hadden mijn medewerkers Ben Craps en Oleg Evnin overwogen hoe zulke Hamiltonianen te regulariseren door middel van de meest conservatieve benadering die hen zou toelaten een unitaire evolutie doorheen de singulariteit te definiëren [94]. Deze benadering, die zij “minimale subtractie” noemden, bestaat erin om de singuliere tijdsafhankelijke termen in de Hamiltoniaan in distributionele zin te definiëren terwijl de operatorstructuur van de Hamiltoniaan onveranderd wordt gehouden (deze aanpak is relevant als de transitie doorheen de singulariteit wordt gedomineerd door een enkele term in de Hamiltoniaan). De neutralisatie van de divergentie houdt rechtstreeks verband met de negatieve contributies vanwege de distributies. We ontdekten dat dit voorschrift afweek van een geometrische regularisatie omdat de negatieve functiewaarden geassocieerd met de distributies onverenigbaar zijn met een geometrische interpretatie. Voor een geometrische resolutie van veldendynamica op een singulaire ruimte-tijdachtergrond, moet men in het algemeen de specificaties van de "minimale subtractie" aanpak verzwakken en wijzigingen in de operatorstructuur van de Hamiltoniaan, alsook wijzigingen in de tijdsafhankelijke termen, toelaten in de buurt van de singulariteit.

Als een specifiek voorbeeld bekeken we de propagatie van een massief scalair veld in een singuliere ruimte-tijd. We onderzochten het zogenaamde parabolische orbifold, hetgeen de singuliere limiet van het reguliere nulbraan is. Het parabolische orbifold ontstaat wanneer men in vlakke Minkowski ruimte-tijd een identificatie maakt langs één van de twee richtingen op de lichtkegel (de andere richting wordt geïnterpreteerd als de tijdsrichting). Vanwege deze identificatie wordt een singulariteit gecreëerd die een speelgoedmodel biedt om singulariteiten te onderzoeken (vergelijk bevoorbeeld, bij wijze van eenvoud, met de singuliere tip van een kegel die ontstaat door een vlak oppervlak te vouwen). Het nulbraan is een vierdimensionaal orbifold met een vrije parameter $R$. In de limiet $R \to 0$ reduceert de nulbraan geometrie zich tot een produkt van het parabolische orbifold met de reële as, en in deze zin is het nulbraan een geometrische regularisatie van het parabolische orbifold. Eerder hadden Liu *et al* [84, 86] het parabolische orbifold reeds onderzocht in de context van perturbatieve snaartheorie.

Overeenkomstig met ons geometrische resolutievoorschrift, hebben we eerst de evolutie van het vrije scalair veld op het reguliere nulbraan geanalyseerd alvorens de singuliere limiet te nemen. Om in staat te zijn de singuliere limiet te onderzoeken, introduceren we een nieuw coördinatensysteem op het nulbraan dat globaal gedefinieerd is en een wel-gedefinieerde limiet heeft. We hebben ook een veralgemeende nulbraan metriek beschouwd (dit is in essentie het nulbraan met twee extra vrije parameters). De essentiële stap in de oplossing van het probleem was om de kwantummechanische evolutie op het nulbraan te verlaten in gekende evolutievergelijkingen van een dynamische groep (in dit geval de twee-foton groep bekend in kwantumoptica). Niettegenstaande het ogenschijnlijk sterk singuliere gedrag van de limiterende Hamiltoniaan (de singuliere termen kunnen zelfs niet als distributies geschreven worden) is de kwantummechanische evolutie doorheen de singulariteit goed-gedefinieerd. Het commutatie-gedrag van de verschillende operatortermen in de Hamiltoniaan compenseren het singuliere gedrag precies. Maar we vinden dat de singuliere limiet slechts bestaat voor een discrete deelverzameling van de mogelijke parameterwaarden binnen de familie van veralgemeende nulbraangeometrieën. We kunnen deze deelverzameling labelen door één natuurlijk getal. Zoals verwacht kan worden, maakt het originele nulbraan deel uit van deze deelverzameling.

De evolutie van de modes van het scalaire veld wordt volledig bepaald door zijn modefuncties (die zelfde modefuncties treden ook naar voor in de beschrijving van snaren op het nulbraan). Als we onze resultaten vergelijken met Liu *et al* [84, 86] vinden we dezelfde modefuncties behalve in de expontiële factor die de golf in de $X$-richting karakteriseert waar we een extra sign($t$) factor hebben bij de coördinaat $X$. Het effect ervan is dat de positie en de snelheid in de $X$-richting voor alle deeltjes wordt gereflecteerd als ze doorheen de singulariteit gaan. Het verschil treedt op vanwege ons nieuw coördinatensysteem dat niet faalt in de oorsprong $t = 0$.

Als we naar de discrete deelverzameling kijken waarvoor de singuliere limiet bestaat, merken we op dat hun modefuncties equivalent zijn aan elkaar op een (globale) fasesprong ter hoogte van de singulariteit na. Deze treedt op omdat de veldmodes in de oorsprong een aantal focale punten kruisen, en het aantal is proportioneel met het natuurlijk getal dat de het element uit de deelverzameling karakteriseert.

## Vrije snaar op een singuliere vlakke golf

Vlakke gravitationele golven vormen een analytisch oplosbare achtergrond voor de propagatie van snaren. In samenwerking met Ben Craps en Oleg Evnin heb ik de evolutie van een vrije snaar op een singuliere vlakke golf onderzocht [120]. De benadering van een vrije snaar kan gezien worden als een eerste stap vooraleer perturbatieve snaartheorie op zulk een singuliere achtergrond te onderzoeken. We hebben ons geconcentreerd op vlakke golven met een schaal-onafhankelijk profiel ten opzichte van de coordinaat $x^+$,

$$ds^2 = -2dx^+dx^- - \lambda \sum_i \left(\frac{x^i}{x^+}\right)^2 + \sum_i \left(dx^i\right)^2 . \tag{I.11}$$

Dit soort profiel ontstaat op een natuurlijke wijze via een Penrose limiet van kosmologische singulariteiten. De coördinaten die loodrecht staan op de lichtkegelrichtingen $x^+$ en $x^-$ noteren we met $x^i$ en we gebruiken $x^+$ om de lichtkegeltijd aan te duiden. De schaal-invariantie betekent dat de vlakke golf-metriek invariant is ten opzichte van herschalingen in de lichtkegel-coördinaten

$$(x^+, x^-) \to (Cx^+, x^-/C). \tag{I.12}$$

Vanwege de schaal-onafhankelijkheid van het singuliere profiel zullen we veronderstellen dat onze klasse van geregularizeerde profiel ook schaal-onafhankelijk is, bovenop de beperking dat onze geresolveerde klasse een oplossing vormt van de veralgemeende Einsteinvergelijkingen in snaartheorie (de achtergrond consistentiecondities). Dus regulariseren we het singuliere schaal-onafhankelijke profiel als

$$\lambda \sum_i \left(\frac{x^i}{x^+}\right)^2 \quad \rightarrow \quad \frac{\lambda}{\epsilon^2}\Omega(x^+/\epsilon)\sum_i \left(x^i\right)^2 \ , \ \lim_{\eta\rightarrow\pm\infty}\Omega(\eta) \sim \frac{1}{\eta^2}, \tag{I.13}$$

waar we $\lambda$ de "normalisering" van het golfprofiel noemen, met $\epsilon$ de unieke resolutieparameter en met $\Omega$ het geregulariseerde profiel. Vanwege de schaalonafhankelijkheid is er slechts één dimensionele parameter (i.e. $\epsilon$) die kan optreden in de geregulariseerde metriek. Dit is de resolutieparameter die we zullen verwijderen in de singuliere limiet. De schaalinvariantie laat ons toe de propagatie doorheen de singulariteit op te lossen zonder verdere specificatie van de geregulariseerde metriek.

Opdat de achtergrondsmetriek voldoet aan de consistentie condities in snaartheorie voegen we een dilatonveld toe. Het dilaton is een oscillatiemode van de snaar, zoals het graviton, maar het bepaald ook de snaarkoppeling. De consistentie condities voor de ruimte-tijd achtergrond relateren de kromming van de metriek aan de spatiotemporele variatie van het dilaton. Vanzelfsprekend eisen we ook dat het dilaton doorheen de singulariteit kan propageren en we bewijzen dat dit mogelijk is.

In lichtkegel-ijk is de Schrödinger golfvergelijking voor de snaar bepaald door een Hamiltoniaan die opgedeeld kan worden in een som van kwadratische Hamiltonianen met een tijdsafhankelijke frequentie, waarvan elke deelhamiltoniaan het gedrag van een andere oscillatiemode van de snaar bepaalt. Daarom kunnen we alle snaarmodes initieel als afzonderlijk beschouwen. Wanneer de resolutieparameter wordt verwijderd, divergeren de frequenties op $t = 0$. Vanwege de kwadratische afhankelijkheid van deze Hamiltonian in functie van de positie- en impulsoperatoren, kunnen we met behulp van een semiklassieke benadering een exacte oplossing vinden voor de Schrödinger vergelijking. Dit betekent dat de golffunctie voor de snaar volledig bepaald is door oplossingen van de klassieke bewegingsvergelijkingen met gepaste randvoorwaarden. We merken op dat de bewegingsvergelijking voor de snaaroscillaties gerelateerd zijn aan de propagatie van het massamiddelpunt (of nulmode) van de snaar. Meer in het bijzonder, het enige verschil tussen de vergelijkingen voor de geëxciteerde snaarmodes en de nulmode is het kwadraat van het modegetal dat bijdraagt tot de tijdsafhankelijke frequentie in de Hamiltoniaan.

Maar het modegetal is een eindige term in vergelijking met de (divergerende) tijdsafhankelijke frequentie en we kunnen rigoureus bewijzen dat het modegetal het bestaan van de singuliere limiet niet beïnvloedt. We beschouwen het modegetal als een kleine perturbatie en we bepalen een grens hoeveel de oplossingen van de geëxciteerde modes kunnen verschillen ten opzichte van de nulmode. In de singuliere limiet verdwijnt het verschil tussen de oplossingen en we kunnen bewijzen dat de geëxciteerde modes doorheen de singulariteit kunnen propageren als de nulmode propageert. In een eerdere publicatie hadden

Evnin en Nguyen [119] reeds bewezen onder welke condities de nulmode kan propageren doorheen de singulariteit, wat leidt tot een discreet spectrum in de parameter $\lambda$ die optreedt in het profiel van de vlakke golf.

Dus, net als in het geval van het vrije scalair veld op het parabolische orbifold, bekomen we ook hier een discreet spectrum dat gerelateerd is aan de propagatie doorheen de singulariteit. De nulmode (en de geëxciteerde modes) kunnen slechts doorheen de vlakke golf-singulariteit propageren (in een generiek geval) voor een discrete verzameling in $\lambda$. Het precieze spectrum in $\lambda$ wordt bepaald door de vorm van het geregulariseerde profiel $\Omega(\eta)$. Maar de schaalinvariantie van de resolutie heeft ons toegelaten om de propagatie doorheen de singulariteit te bepalen zonder enige verdere specificatie van het geregulariseerde profiel $\Omega(\eta)$ behalve asymptotisch (I.13).

We hebben ontdekt dat alle oscillatiemodes van de snaar afzonderlijk doorheen de singulariteit kunnen propageren, maar opdat de snaar in haar geheel doorheen de singulariteit kan propagaren, moeten we opleggen dat de excitatie-energie van de snaar eindig blijft gedurende de transitie doorheen de singulariteit. We vinden dat dit alleen het geval kan zijn als de “normalisatie” $\lambda$ van het vlakke golfprofiel voldoet aan de conditie

$$\lambda = \frac{1}{4} - \left(N + \frac{1}{2}\right)^2, \tag{I.14}$$

waar $N$ een natuurlijk getal is ($N = 0$ komt bijvoorbeeld overeen met Minkowski ruimte-tijd of met het lichtachtig reflectorvlak uit [95]). Maar voor $\lambda < 0$ divergeert het dilaton nabij de singulariteit en de snaarkoppeling wordt onbegrensd sterk. Zo wordt perturbatieve snaartheorie ongeldig. Dus het is onmogelijk dat de totale excitatie-energie eindig blijft onder de veronderstelling dat de snaar vrij is (opdat het beschouwen van een vrije snaar een consistente benadering zou zijn is het vereist dat de interactie tussen de snaren klein is). Aangezien perturbatieve snaartheorie op die manier ongeldig wordt nabij de singulariteit, motiveert dit ons om matrixmodellen van singuliere vlakke golven te onderzoeken, want deze matrixmodellen laten toe om sterke interacties tussen snaren te beschouwen.

## Supergravitatie D$p$-braan oplossingen

Matrixmodellen die een beschrijving vormen van snaartheorie in de limiet van sterke snaarkoppeling, worden geformuleerd in functie van de effectieve actie van D0-branen (of D1-branen). Dus, als we de eigenschappen van matrixmodellen van singuliere vlakke golven beter willen onderzoeken (zoals bijvoorbeeld het matrix oerknalmodel van Craps *et al* [107] of the vlakke golf matrixmodellen van Blau en O’Loughlin [112]) dan moeten we de formulering van D-branen in een asymptotisch vlakke golf-achtergrond bestuderen. D-branen zijn branen die voldoen aan specifieke randvoorwaarden (ze karakteriseren de eindpunten van open snaren) en ze spelen een belangrijke rol als de effectieve vrijheidsgraden in matrix theorie. De branen die optreden in snaartheorie zijn dynamische objecten, maar ze kunnen ook beschreven worden als klassieke ruimte-tijd oplossingen in superzwaartekracht. Superzwaartekracht is een uitbreiding van algemene relativiteit die fermionen in zijn spectrum bevat (alle materie die we kennen bestaat uit fermionen). Superzwaartekracht is de laag-energetische benadering voor snaartheorie: het is een geldige benadering wanneer er onvoldoende energie beschikbaar is om de hogere oscillatiemodes van de snaar te exciteren. Daarom kunnen de D-branen op het klassieke niveau, waar ze massief zijn, kunnen beschreven worden door een metriek, een dilaton en een ijkveld (het ijkveld treedt op omdat het D-braan geladen is). De standaard matrix model Hamiltoniaan bescrhijft elf-dimensionale statische Minkowski ruimtetijd. De laag-energetische beschrijving van de matrix model Hamiltoniaan wordt gegeven door snaartheorie in een superzwaartekracht achtergrond van D0-branen [105]. Het matrix oerknalmodel [107] is een tijdsafhankelijk model en wordt geformuleerd in termen van D1-branen in een tijdsafhankelijke vlakke golf-achtergrond.

Dit betekent dat we geïnteresseerd zijn in de klassieke oplossingen in superzwaartekracht die tijdsafhankelijke D-branen beschrijven in een asymptotisch vlakke golf achtergrond beschrijft in superzwaartekracht. Asymptotisch heeft die ruimte-tijd metriek het karakter van een vlakke golf, maar de aanwezigheid van D-branen in de oorsprong zal de metriek veranderen voor eindige afstanden ten opzicht van het braan. Een eenvoudiger probleem is de formulering van D1-branen die gealigneerd zijn met de lichtkegel (in andere woorden, het “wereldvlak van de braan” is evenwijdig met de bewegingsrichting van de vlakke golf). In samenwerking met Ben Craps, Oleg Evnin en Federico Galli heb ik de metriek ontdekt die extremale D1-branen in een asymptotisch vlakke golfachtergrond beschrijft, en we hebben deze oplossingen voor D1-branen in superzwaartekracht uitgebreid naar hoger-dimensionale D$p$-branen (met $p \geq 1$) [136]. Momenteel bestuderen we de uitbreiding van deze $p$-braan oplossingen tot een configuratie van D0-branen.

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*“p-branes on the waves,” Medieval wood carving (anonymous)*